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Daniel Berhanu MAMO

born on 14 October 1992 in Addis Ababa (Ethiopia)

A newform theory for Katz modular forms

Dissertation defence committee

Dr Gabor Wiese, dissertation supervisor

Professor, University of Luxembourg

Dr Antonella Perucca, Vice Chairman

A-Professor, University of Luxembourg

Dr Hugo Parlier, Chairman

Professor, University of Luxembourg

Dr Ian Kiming

Professor, University of Copenhagen

Dr Samuele Anni

A-Professor, University of Aix-Marseille

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Abstract

In this thesis, a strong multiplicity one theorem for Katz modular forms is studied. We show that a cuspidal Katz eigenform which admits an irreducible Galois representation is in the level and weight old space of a uniquely associated Katz newform. We also set up multiplicity one results for Katz eigenforms which have reducible Galois representation.

Contents

Introduction

The study of relations between the coefficients of classical modular forms by L. Atkin and J. Lehner in [2] and by W. Li in [24] led to the invention of the theory of newforms. Atkin and Lehner used the L-functions associated to the newforms for their investigation. W. Li in [24], using the notion of trace operators, obtained the generalization of the Atkin-Lehner theory to the case of modular forms over congruence subgroups parameterized by two variables with characters. In this thesis we will generalize some of these results to Katz modular forms over $\overline{\mathbb{F}}_p$.

Katz modular forms are modular forms defined via algebraic geometry methods by N. Katz in [19]. They are defined over any ring in which the level is invertible. See the first chapter for further explanation. We work with Katz modular forms over $\overline{\mathbb{F}}_p$. Thus we always assume that the prime p does not divide the levels of our modular forms. Katz modular forms admit Hecke operators analogously to holomorphic modular forms. We denote the space of Katz modular forms in level $\Gamma_1(N)$, weight k and Dirichlet character ε by $M_k(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$ and its cuspidal subspace by $S_k(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$. When we do not write the Dirichlet character ε we assume that we did not fix any character.

Let f be a normalised Katz eigenform of level $\Gamma_1(N)$ with $p \nmid N$, weight k and character ε, with coefficients in $\overline{\mathbb{F}}_p$. Let $f(T_l)$ be the eigenvalue of f for the Hecke operator T_l. Then thanks to the works in [11], [16] there exists a unique 2-dimensional semi-simple continuous representation $\rho_f : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, which is unramified outside pN and has the property that $\mathrm{tr}(\rho_f(\mathrm{Frob}_l)) = f(T_l)$ and $\det(\rho_f(\mathrm{Frob}_l)) = \varepsilon(l)l^{k-1}$ for all primes $l \nmid pN$. We prefer to state the results in terms of Galois representations because they shorten the statements. However it would be possible to avoid that language for most statements.

Let us informally introduce the notation of level and weight old spaces. They are defined more precisely in the first chapter. First, like the classical case, we have level degeneracy maps on Katz modular forms. Let $f \in M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$ be any Katz modular form and let $d \geq 1$ be an integer coprime to p. Then we have the d-th degeneracy map $f(q) \mapsto f(q^d)$, which increases the level by a multiple of d. Then the *level old space* of f in the level M divisible by N is the $\overline{\mathbb{F}}_p$ vector space generated by modular forms $f(q^d)$ where d runs through all possible divisors of M/N. Second, we have the following weight degeneracy maps on Katz modular forms. The map α_A defined by $f \mapsto Af$, where $A \in M_{p-1}(\Gamma_1(1), \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$ is the Hasse invariant with

q-expansion at the cusp infinity equal to 1. It adds $p-1$ to the weight but does not change the q-expansion. The Frobenius map takes a form $f(q)$ to its Frobenius $\mathrm{Frob}(f)(q) = f(q^p)$. It multiplies the weight by p but does not change the level. Thus, the missing degeneracy map $q \mapsto q^p$ in the level is provided by the Frobenius.

Then by the *level and weight old space* of f in level M, a multiple of N, and weight $k' \geq k$ we understand the space generated by the images of f under all possible combinations of the level and weight degeneracy maps targeted to the space of modular forms of level M and weight k'.

We will use the following definitions of minimal levels and weights to introduce our newforms. Let $d \geq 2$ be a positive integer and let $f \in M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$ be any Katz eigenform. Then f is said to be in *d-minimal level* if ρ_f does not arise from any non-zero Katz eigenform of level MN/d^m where M and m are any positive integers such that $\gcd(M, d) = 1$. A Katz modular form f is said to be in *minimal weight k* if the associated mod p Galois representation ρ_f does not arise from any non-zero Katz eigenform of weight strictly smaller than k and any level.

Definition 1. *A normalised Katz eigenform $f \in M_k(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{Katz}$ is called a* Katz newform *if f is in l-minimal level for any prime $l \neq p$ and is in minimal weight k.*

The motivation behind the definition of our Katz newform is that it satisfies some of the analog results of classical newform theory.

The aim of the thesis is to prove the following strong multiplicity one theorems.

Theorem 2. *Let $f \in S_k(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{Katz}$ and $g \in S_{k'}(\Gamma_1(N'), \varepsilon', \overline{\mathbb{F}}_p)_{Katz}$ be Katz newforms with $a_l(f) = a_l(g)$ for each l in a set of primes of density 1. Then $f = g, k = k', N = N'$, and $\varepsilon = \varepsilon'$.*

This means that Katz newforms are uniquely characterised by their associated mod p Galois representations.

Theorem 3. *Let $F \in S_{k'}(\Gamma_1(M), \overline{\mathbb{F}}_p)_{Katz}$ be a normalised Katz eigenform. Suppose that there is a Katz newform $f \in S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{Katz}$ such that $\rho_F \cong \rho_f$. Then F is in the level and weight old space of f.*

As a consequence, one can determine all possible coefficients of any normalised Hecke eigenform from its associated Katz newform, provided that it exists. See Corollary 52 for explicit expressions.

The existence of Katz newforms is established in Theorem 46 when the associated mod p Galois representation is irreducible and in Proposition 57 when the associated mod p Galois representation is reducible of a certain type.

Katz modular forms over $\overline{\mathbb{F}}_p$ are "almost" the same as the reductions of classical modular forms. Differences occur in very small levels or in weight 1. It is essential for the newform theory to work that one uses Katz morular forms.

In Chapter 1, we define a Katz newform to get a similar result of classical newform theory for Katz modular forms. The interesting idea is that we introduce weight degeneracy to make up for level degeneracy maps. Thus we treat level and weight of Katz modular form on an equal footing. The definition of Katz newform is purely local we consider every prime in the level separately, and the weight separately.

In classical newform theory, every modular form can be expressed as a linear combination of newforms via level degeneracy maps. There is no weight degeneracy maps. The corresponding result for Katz forms over $\overline{\mathbb{F}}_p$ is wrong. One can consult a counterexample, see Remark 55. The main result states that every Katz Hecke eigenform can be expressed as a linear combination of newforms via level and weight degeneracy maps. This also allows us to explicitly describe all coefficients of all Katz Hecke eigenforms if one knows the coefficients of the corresponding Katz newform.

In Chapter 2, we set up the theory of newforms for the space of Katz Eisenstein series. In the case where cuspidal Katz eigenforms have reducible mod p Galois representations, Eisenstein series come into the picture to describe their associated newforms. We have shown in Theorem 62 that, under some condition, up to a suitable power multiple of the Hasse invariant, any non-ordinary cuspidal Katz eigenform with a reducible mod p Galois representation is in the level old space of an associated Katz eigenform which has an optimal level obtained from an associated mod p Eisenstein series.

Chapter 1

Background

1.1 Holomorphic modular forms

In this first chapter we study the classical and algebraic definitions of modular forms. We will also present some of the background materials that we need later. The Hecke operators for the space of modular forms are studied. We also point out the classical results of Serre, Shimura and Deligne about the existence of a continuous almost everywhere unramified Galois representations associated to a Hecke eigenforms. For this chapter most of the time our reference would be [14].

1.1.1 Classical modular forms

Let $\mathcal{H}$ be an upper half plane and $\mathrm{SL}_2(\mathbb{Z})$ be the modular group of integral matrices of determinant 1. The principal congruence subgroup of level N is the group

$$\Gamma(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

Then we have two special congruence subgroups

$$\Gamma_0(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\},$$

$$\Gamma_1(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

For $k \in \mathbb{Z}_{\geq 0}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$, we define an operator $|_k\gamma$ acting on meromorphic functions $f : \mathcal{H} \to \mathbb{C}$ by

$$(f|_k\gamma)(z) = \det(\gamma)^{k-1}(cz+d)^{-k} f(\gamma \cdot z),$$

where $\gamma \cdot z$ is the fractional linear transformation, $\gamma \cdot z = \frac{az+b}{cz+d}$.

Definition 4. *A function $f : \mathcal{H} \to \mathbb{C}$ is said to be weakly modular of weight k if f is meromorphic and for all $\gamma \in SL_2(\mathbb{Z})$ and $z \in \mathcal{H}$ we have $f|_k\gamma(z) = f(z)$.*

Definition 5. *A modular form of weight k with respect to $\Gamma_1(N)$ is a function $f : \mathcal{H} \to \mathbb{C}$ such that*
(i). f is weakly modular of weight k with respect to $\Gamma_1(N)$,
(ii). f is holomorphic on $\mathcal{H}$, and
(iii). $f[\alpha]_k$ is holomorphic at ∞ for all $\alpha \in SL_2(\mathbb{Z})$.

We denote the space of such functions by $M_k(\Gamma_1(N), \mathbb{C})$.

The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In particularly we can observe that T transforms $z \to z + 1$. Thus every modular form f admits a Fourier series

$$f(z) = \sum_{n=0}^{\infty} a_n q^n, q = e^{2\pi i z}$$

which is called the q-expansion of the modular form f. The series starts from $n = 0$ as f is holomorphic at ∞. Let $a_n(f)$ stand for the nth coefficient of f in its Fourier series expansion.

Definition 6. *A cusp form of weight k with respect to $\Gamma_1(N)$ is a modular form of weight k with respect to $\Gamma_1(N)$ that vanishes at all cusps. Equivalently, $a_0(f|_k\gamma) = 0$ for all $\gamma \in SL_2(\mathbb{Z})$.*

Let Γ be an arbitrary congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and denote by $\overline{\Gamma}$ its projectivization, i.e., its image in $\mathrm{PSL}_2(\mathbb{Z})$. On the space of $S_k(\Gamma_1(N), \mathbb{C})$ of cusp forms we have the Petersson inner product

$$\langle f, g \rangle = \frac{1}{[\overline{\Gamma(1)} : \overline{\Gamma}]} \int_D f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}$$

where $z = x + iy$ and D is the fundamental domain for Γ. The issue of convergence of the integral is granted by the following proposition.

Proposition 7 ([13], Lemma 3.6.1)**.** *If f is a cusp form in $S_k(\Gamma_1(N), \mathbb{C})$, the function $f(z)y^{k/2}$ is bounded on $\mathcal{H}$.*

1.1.2 Hecke operators

We begin by introducing Hecke operators. The reference is [14]. We copied the theorems from [14].

For congruence subgroups Γ_1 and Γ_2 of $SL_2(\mathbb{Z})$ and $\alpha \in GL_2^+(\mathbb{Q})$, the weight-$k$ $\Gamma_1\alpha\Gamma_2$ operator takes functions $f \in M_k(\Gamma_1)$ to

$$f[\Gamma_1\alpha\Gamma_2]_k = \sum_j f[\beta_j]_k$$

where $\{\beta_j\}$ are orbit representatives, i.e., $\Gamma_1\alpha\Gamma_2 = \cup_j \Gamma_1\beta_j$ is a disjoint union.

Let p be prime. The pth Hecke operator T_p of weight k

$$T_p : M_k(\Gamma_1(N), \mathbb{C}) \to M_k(\Gamma_1(N), \mathbb{C})$$

is given by

$$T_p f = f\Big[\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N)\Big]_k.$$

The double coset here is

$$\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N) = \Big\{\gamma \in M_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & p \end{pmatrix} (\mathrm{mod}\ N), \det\gamma = p\Big\},$$

so in fact $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ can be replaced by any matrix in this double coset in the definition of T_p.

We have

Proposition 8. *Let $N \in \mathbb{Z}^+$ and let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ where p is prime. The operator $T_p = \Big[\Gamma_1(N)\alpha\Gamma_1(N)\Big]_k$ on $M_k(\Gamma_1(N), \mathbb{C})$ is given by*

$$T_p f = \begin{cases} \sum_{j=0}^{p-1} f\Big[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\Big]_k & \text{if } p|N, \\ \sum_{j=0}^{p-1} f\Big[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\Big]_k + f\Big[\begin{pmatrix} m & n \\ N & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\Big]_k & \text{if } p \nmid N, \text{ where} \\ mp - nN = 1. \end{cases}$$

For each $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, define the diamond operator acting on $f \in M_k(\Gamma_1(N), \mathbb{C})$ by $\langle d\rangle f = f|_k\gamma$ for any $\gamma = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \in \Gamma_0(N)$ with $\delta \equiv d(\mathrm{mod}\ N)$. For d not invertible mod N, define $\langle d\rangle f = 0$.

For any character $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, let $M_k(\Gamma_1(N), \varepsilon, \mathbb{C})$ denote the $\mathbb{C}$-subspace of $M_k(\Gamma_1(N), \mathbb{C})$ of elements f such that for all $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, $\langle d \rangle f = \varepsilon(d) f$.

The next result describes the effect of T_p on Fourier coefficients.

Proposition 9. *For each prime p, the above linear operator T_p act on the space $M_k(\Gamma_1(N), \varepsilon, \mathbb{C})$ of modular forms, with effect on q-expansion:*
$T_p(f) = \sum_{n \geq 0} a_{np} q^n + p^{k-1} \varepsilon(p) \sum_{n \geq 0} a_n q^{pn}$ *if* $p \nmid N$,
$T_p(f) = \sum_{n \geq 0} a_{np} q^n$ *if* $p \mid N$.

Definition 10. *The operator T_p of the above Proposition is called the pth Hecke operator on $M_k(\Gamma_1(N), \mathbb{C})$.*

For a prime power p^r, we define the Hecke operator T_{p^r} recursively by:

$$\begin{aligned} T_1 &:= 1 \\ T_{p^r} &:= T_p T_{p^{r-1}} - \langle p \rangle p^{k-1} T_{p^{r-2}}, \text{if } p \nmid N \\ T_{p^r} &:= (T_p)^r, \text{if } p \mid N. \end{aligned}$$

For any positive integer n with prime factorisation $n = \prod p_i^{e_i}$, we define the nth Hecke operator by $T_n = \prod T_{p_i}^{e_i}$.

Proposition 11. *The Hecke operators T_n for $n \geq 1$ commute with each other, and with the diamond operators $\langle d \rangle$.*

Definition 12. *A modular form which is a simultaneous eigenvector for all Hecke operators is called a Hecke eigenform, or simply an eigenform. A modular form $f = \sum_{n \geq 1} a_n q^n$, is said to be normalised if $a_1(f) = 1$.*

Proposition 13. *Let f be a normalised eigenform. Then a_n is an algebraic integer for every n, and: $T_n f = a_n(f) f$ for all $n \geq 1$.*

Proposition 14. *Let $f = \sum_{n=0}^{\infty} a_n q^n$ be a modular form of level N weight k and Nebentypus character ε. Then f is a normalised eigenform if and only if*
(i). $a_1(f) = 1$,
(ii). $a_{nm}(f) = a_n(f) a_m(f)$ *for all* $(n, m) = 1$,
(iii). $a_{p^t}(f) = a_p(f) a_{p^{t-1}}(f) - p^{k-1} \varepsilon(p) a_{p^{t-2}}(f)$ *for all* $t \geq 2$.

On $S_k(\Gamma_0(N), \mathbb{C})$ the Hecke operators T_n for $(n, N) = 1$ are self-adjoint with respect to the Petersson inner product. In fact, on $S_k(\Gamma_1(N), \varepsilon, \mathbb{C})$ we have for all n prime to N that

$$\langle T_n f, g \rangle = \varepsilon(n) \langle f, T_n g \rangle.$$

On $S_k(\Gamma_1(N), \mathbb{C})$ the adjoint T_n^* of T_n is for $(n, N) = 1$ is $T_n \circ \langle \overline{n} \rangle$. Thus the operators of the form T_n and $\langle n \rangle$ for n relatively prime to N form a mutually commutative set of normal operators on $S_k(\Gamma_1(N), \mathbb{C})$. Those operators T_n with $(n, N) \neq 1$ on $S_k(\Gamma_1(N), \varepsilon, \mathbb{C})$ need not be normal.

1.1.3 Classical Atkin-Lehner-Li Theory

In this section we will present the classical newform theory. We will closely follow [14]. Let N and M be positive integers such that $N|M$. Then there is an obvious inclusion

$$S_k(\Gamma_1(N), \mathbb{C}) \hookrightarrow S_k(\Gamma_1(M), \mathbb{C})$$

resulting from $\Gamma_1(N) \subset \Gamma_1(M)$.

In addition to the above inclusion, we have the following maps. For $d|M/N$, let $\alpha_d = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$. For $f : \mathcal{H} \to \mathbb{C}$, we have

$$(f|_k \alpha_d)(z) = d^{k-1} f(dz) \text{ for } f : \mathcal{H} \to \mathbb{C}.$$

We have an injective map

$$S_k(\Gamma_1(N), \mathbb{C}) \to S_k(\Gamma_1(M), \mathbb{C}), f \mapsto f|_k \alpha_d.$$

For each $N|M$ and $d|M/N$, $(d \neq 1)$ we have the degeneracy map

$$i_d : (S_k(\Gamma_1(N), \mathbb{C}))^2 \to S_k(\Gamma_1(M), \mathbb{C})$$

given by

$$(f, g) \to f + g|_k \alpha_d.$$

Combining these maps, we get a subspace of $S_k(\Gamma_1(M), \mathbb{C})$ which arises from lower level. The subspace of oldforms at level M is

$$S_k(\Gamma_1(M), \mathbb{C})^{old} = \sum_{\substack{N|M; d|M/N \\ N<M}} i_d((S_k(\Gamma_1(N), \mathbb{C}))^2)$$

and the subspace of newforms at level N is the orthogonal complement with respect to the Petersson inner product,

$$S_k(\Gamma_1(M), \mathbb{C})^{new} = (S_k(\Gamma_1(M), \mathbb{C})^{old})^{\perp}.$$

We have that the spaces $S_k(\Gamma_1(M), \mathbb{C})^{old}$ and $S_k(\Gamma_1(M), \mathbb{C})^{new}$ are stable under the Hecke operators.

Proposition 15. *The subspaces* $S_k(\Gamma_1(N),\mathbb{C})^{old}$ *and* $S_k(\Gamma_1(N),\mathbb{C})^{new}$ *are stable under the Hecke operators* T_n *and* $\langle n\rangle$ *for all* $n\in\mathbb{Z}_+$*.*

Corollary 16. *The spaces* $S_k(\Gamma_1(N),\mathbb{C})^{old}$ *and* $S_k(\Gamma_1(N),\mathbb{C})^{new}$ *have orthogonal bases with respect to the Peterson inner product of eigenforms for the Hecke operators away from the level,* $\{T_n,\langle n\rangle:(n,N)=1\}$*. As we will see, the condition* $(n,N)=1$ *can be removed for the newforms.*

We have a variant map ι_d of i_d,

$$\begin{aligned}\iota_d &:= d^{1-k}|_k\alpha_d : S_k(\Gamma_1(M),\mathbb{C})\to S_k(\Gamma_1(N),\mathbb{C})\\ (\iota_d f)(z) &:= f(dz).\end{aligned}$$

Theorem 17 (Main Lemma)**.** *If* $f\in S_k(\Gamma_1(N),\mathbb{C})$ *has Fourier expansion* $f(\tau)=\sum_n a_n(f)q^n$ *with* $a_n(f)=0$ *whenever* $(n,N)=1$*, then* f *takes the form* $f=\sum_{p|N}\iota_p f_p$ *with each* $f_p\in S_k(\Gamma_1(N/p),\mathbb{C})$*.*

Definition 18. *A nonzero modular form* $f\in M_k(\Gamma_1(N),\mathbb{C})$ *that is an eigenform for the Hecke operators* T_n *and* $\langle n\rangle$ *for all* $n\in\mathbb{Z}_+$ *is called a newform if it is normalized (*$a_1(f)=1$*) and is in* $S_k(\Gamma_1(N),\mathbb{C})^{new}$*.*

Theorem 19. *Let* $f\in S_k(\Gamma_1(N),\mathbb{C})^{new}$ *be a nonzero eigenform for the Hecke operators* T_n *and* $\langle n\rangle$ *for all* n *with* $(n,N)=1$*. Then*

(a) f *is a Hecke eigenform, i.e., an eigenform for* T_n *and* $\langle n\rangle$ *for all* $n\in\mathbb{Z}_+$*. A suitable scalar multiple of* f *is a newform.*

(b) If f' *satisfies the same conditions as* f *and has the same* T_n*-eigenvalues for all* n*, then* $f'=cf$ *for some constant* c*.*

The set of newforms in the space $S_k(\Gamma_1(N),\mathbb{C})^{new}$ *is an orthogonal basis of the space. Each such newform lies in an eigenspace* $S_k(N,\chi,\mathbb{C})$ *and satisfies* $T_nf=a_n(f)f$ *for all* $n\in\mathbb{Z}_+$*. That is, its Fourier coefficients are its* T_n *eigenvalues.*

Theorem 20. *The set* $B_k(N)=\{f(n\tau): f$ *is a newform of level* M *and* $nM|N\}$ *is a basis of* $S_k(\Gamma_1(N),\mathbb{C})$*.*

Theorem 21 (Strong Multiplicity One)**.** *Let* $f\in S_k(\Gamma_1(N),\varepsilon,\overline{\mathbb{F}}_p)_{Katz}$ *and* $g\in S_{k'}(\Gamma_1(N'),\varepsilon',\overline{\mathbb{F}}_p)_{Katz}$ *be newforms with* $a_l(f)=a_l(g)$ *for primes* $l\nmid pNN'$*. Then* $f=g, k=k', N=N'$*, and* $\varepsilon=\varepsilon'$*.*

Strong Multiplicity One also plays a role in the proof of

Proposition 22. *Let $g \in S_k(\Gamma_1(N), \mathbb{C})$ be a normalized eigenform. Then there is a newform $f \in S_k(\Gamma_1(M), \mathbb{C})^{new}$ for some $M|N$ such that $a_p(f) = a_p(g)$ for all $p \nmid N$.*

Next we will present the newform theory for Eisenstein series following [25] and [26].

We define the generalized Bernoulli number B_k^{ε} attached to a complex modulo n Dirichlet character ε by the following infinite series

$$\sum_{j=1}^{n} \frac{\varepsilon(j) x e^{jx}}{e^{nx} - 1} = \sum_{k=0}^{\infty} \frac{B_k^{\varepsilon} x^k}{k!}.$$

Let ε_1 and ε_2 be two Dirichlet characters modulo N_1 and N_2 such that $N_1 N_2 | N$ and let k be a positive integer such that $(\varepsilon_1 \varepsilon_2)(-1) = (-1)^k$. Let t be a positive integer. Then we have the Eisenstein series $E_k^{\varepsilon_1, \varepsilon_2}(q)$ defined by the power series

$$E_k^{\varepsilon_1, \varepsilon_2}(q) := c_0 + \sum_{m \geq 1} \left(\sum_{0 < d | m} \varepsilon_1(d) \varepsilon_2\left(\frac{m}{d}\right) d^{k-1} \right) q^m \in \mathbb{Q}(\varepsilon_1, \varepsilon_2)[[q]]$$

where $c_0 = -\frac{B_k^{\varepsilon_1}}{2k}$ when $\mathrm{cond}(\varepsilon_2) = 1$ and $c_0 = 0$ otherwise. Then, except when $k = 2$ and $\varepsilon_1 = \varepsilon_2 = 1$, the power series $\iota_t E_k^{\varepsilon_1, \varepsilon_2}(q)$ belongs to $M_k(\Gamma_1(tuv), \mathbb{C})$ for all $t \geq 1$. If $k = 2$ and $\varepsilon_1 = \varepsilon_2 = 1$, let $t > 1$, then $E_2^{1,1}(q) - t\iota_t E_2^{1,1}(q)$ is a modular form in $M_2(\Gamma_1(t), \mathbb{C})$. Moreover the modular form $E_k^{\varepsilon_1, \varepsilon_2}(q)$ is a normalized eigenform for all Hecke operators. Analogously, for all positive integers $t > 1$ the series $E_2^{1,1}(q) - tE_2^{1,1}(q^t)$ is a normalised eigenform for all Hecke operators. Let us set $E_k^{\varepsilon_1, \varepsilon_2, t}(q)$ to $E_2^{\varepsilon_1, \varepsilon_2}(q) - t\iota_t E_2^{\varepsilon_1, \varepsilon_2}(q)$ when $k = 2$ and $\varepsilon_1 = \varepsilon_2 = 1$, and to $\iota_t E_k^{\varepsilon_1, \varepsilon_2}(q)$ otherwise.

Sometimes by disregarding the characters $\varepsilon_1, \varepsilon_2$ we write $E_k(\Gamma_1(N_1 N_2), \varepsilon_1 \varepsilon_2, \mathbb{C})$ for the space of the corresponding Eisenstein series.

Definition 23. *We will say that the Eisenstein series $E_k^{\varepsilon_1, \varepsilon_2}$ is a newform if the characters $\varepsilon_1, \varepsilon_2$ are primitive.*

Like the cuspidal setting, Eisenstein series newforms are eigenforms for all the Hecke operators. Here after for this section we assume that $k \geq 2$.

Let $E_k^{\mathrm{new}}(\Gamma_1(N), \varepsilon, \mathbb{C})$ denote the subspace of $E_k(\Gamma_1(N), \varepsilon, \mathbb{C})$ spanned by newforms of exact level N. Then as in the setting of cusp forms the space $E_k(\Gamma_1(N), \varepsilon, \mathbb{C})$ has basis of newforms. In particular we have the decomposition ([26], Theorem 2.2)

$$E_k(\Gamma_1(N), \varepsilon, \mathbb{C}) = \bigoplus_{\mathrm{cond}(\varepsilon)|M|N} \bigoplus_{d|NM^{-1}} \iota_d E_k^{\mathrm{new}}(\Gamma_1(M), \varepsilon, \mathbb{C}).$$

For Eisenstein series, the density of primes that uniquely determine the newforms is smaller than 1. In fact, we have

Theorem 24 ([25], Theorem 5.1)**.** *Let $f \in E_k(\Gamma_1(N), \varepsilon_f, \mathbb{C})$ and $g \in E_{k'}(\Gamma_1(N'), \varepsilon_g, \mathbb{C})$ be newforms such that*

$$a_p(f) = a_p(g)$$

for a set of primes with density greater than 1/2. Then $k = k', N = N', \varepsilon_f = \varepsilon_g$ and $f = g$.

The Theorem is proved from

Lemma 25 ([25], Lemma 5.2)**.** *Let $\chi_1, \chi_2, \psi_1, \psi_2$ be Dirichlet characters modulo M and c be a nonzero complex number. There exists a constant p_0 such that if $p > p_0$ is prime and*

$$\chi_1(p) + \chi_2(p)p^{k-1} = c(\psi_1(p) + \psi_2(p)p^{k-1}),$$

then $\chi_1(p) = c\psi_1(p)$ and $\chi_2(p) = c\psi_2(p)$.

We have the following stronger result

Theorem 26 ([25], Theorem 5.4)**.** *Let $f \in E_k(\Gamma_1(N), \varepsilon_f, \mathbb{C})$ and $g \in E_{k'}(\Gamma_1(N'), \varepsilon_g, \mathbb{C})$ be newforms such that*

$$sgn(a_p(f)) = sgn(a_p(g))$$

for a set of primes S with density greater than 1/2. Then $N = N', \varepsilon_f = \varepsilon_g$ and $f = g$.

Since we liked the shortness of the proof we copied the proof here. The proof make use of the following Lemma.

Lemma 27 ([25], Lemma 5.5)**.** *Let $z_1, \ldots, z_m$ be distinct complex numbers lying on the unit circle. Then there exists an $\delta > 0$ such that for any positive real number r and $i \neq j$ we have*

$$|rz_i - z_j| > \delta.$$

Proof of Theorem 26. Write $f = E_k^{\chi_1,\chi_2}$ and $g = E_{k'}^{\psi_1,\psi_2}$. Let $n_1, \ldots, n_{\phi(NN')}$ represent the residue classes of $(\mathbb{Z}/NN'\mathbb{Z})^\times$ and δ be the constant from Lemma 27 applied to the set

$$\{\chi_i(n_j) : i \in \{1,2\}, 1 \leq j \leq \phi(NN')\} \cup \{\psi_i(n_j) : i \in \{1,2\}, 1 \leq j \leq \phi(NN')\}.$$

Let $S' \subset S$ be the subset of S consisting of primes p for which $2p^{1-k} < \delta$ and $p > NN'$. Where S is a set of primes of density $> 1/2$, as in above Theorem. Note that $density(S') = density(S) > 1/2$. For each prime $p \in S'$, define a positive real number $\delta_p := a_p(f)/a_p(g) = |a_p(f)/a_p(g)|$. We claim that $\delta_p = 1$ for all primes $p \in S'$. If $\delta_p \neq 1$ for some prime p, then by interchanging f and g (if necessary) we may assume $\delta_p < 1$. By definition of δ_p we have,

$$\chi_1(p) + \chi_2(p)p^{k-1} = \delta_p\psi_1(p) + \delta_p\psi_2(p)p^{k-1}.$$

From this identity, it follows that

$$\begin{aligned} |\delta_p\psi_2(p) - \chi_2(p)| &= \frac{|\chi_1(p) - \delta_p\psi_1(p)|}{p^{k-1}} \\ &\leq \frac{1+\delta_p}{p^{k-1}} \\ &< 2p^{1-k} \\ &< \delta. \end{aligned}$$

This contradicts Lemma 27, hence $\delta_p = 1$. Theorem 26 now follows from Theorem 24. ■

1.2 Katz modular forms

In this section we recall the definition of Katz modular forms and some of its most important properties. Let N and k be positive integers and let R be a $\mathbb{Z}[1/N]$ algebra. Let $[\Gamma_1(N)]_R$ be the category of generalised elliptic curves with Γ_1-level structures defined in [17]. For more details we also refer to that article. For a generalised elliptic curve E over a scheme S/R we have the invertible sheaf $\underline{\omega}_{E/S} = 0^*\Omega^1_{E/S}$. A Katz modular form f of level N and weight k over R is a rule that assigns to every object $(E/S/R, \alpha)$ of $[\Gamma_1(N)]_R$, where $\alpha : (\mathbb{Z}/n\mathbb{Z})_S \to E[N]$ an embedding of group schemes, an element $f(E/S/R, \alpha)$ of $\omega^{\otimes k}_{E/S}$, compatible with morphisms in $[\Gamma_1(N)]_R$. The R-module of such modular forms will be denoted by $M_k(\Gamma_1(N), R)_{\text{Katz}}$.

We can obtain the q-expansions of f at the various cusps of $[\Gamma_1(N)]_R$ by evaluating f on pairs $(\text{Tate}(q^d), \alpha)$ where $\text{Tate}(q^d)$ the Tate curve $\mathbb{G}_m/q^{d\mathbb{Z}}$ over $R[[q]](q^{-1})$ and $d|N$ and $\alpha : (\mathbb{Z}/N\mathbb{Z})_S \to \text{Tate}(q^d)[N]$ an embedding of group schemes whose image meets all irreducible components of all geometric fibres. The q-expansion $f_{d,\alpha}(q)$ of f at the cusp $(Tate(q^d), \alpha)$ is the power series $f(Tate(q^d), \alpha)/(dt/t)^{\otimes k}$ in $R[[q]]$. A modular form which vanishes at all cusps is called cusp form. The space of cusp forms on $\Gamma_1(N)$ of wight k is denoted by $S_k(\Gamma_1(N), R)_{\text{Katz}}$.

One can recover the usual definition of a modular form over $\mathbb{C}$. See [[5], §2.3] to see how it follows.

We reinterpret the definition of modular forms using the following.

Proposition 28 ([18], Proposition 2.1)**.** *The functor which assigns to each* $\mathbb{Z}[l/N]$*-scheme* S *the set of isomorphism classes of pairs* (E, α)*, where* E *is a generalized elliptic curve over* S *and* $\alpha : \mu_N \hookrightarrow E[N]$ *an embedding schemes whose image meets every irreducible component in each geometric fibre, is represented by a stack which is proper and smooth over* $\mathbb{Z}[1/N]$*. When* $N > 4$ *this functor is represented by algebraic curve* $X_1(N)$*, which is proper, smooth, and geometrically connected over* $\mathbb{Z}[1/N]$*.*

For next theorem we will assume that $N > 4$, so that the stack classifying pairs (E, α) is a scheme. Let $\underline{\omega} = \underline{\omega}_E$ be the line bundle on the curve $X_1(N)$ defined at the end of section 1 of [18]. Then we have

Theorem 29 ([18], Proposition 2.2)**.** *The space of modular forms of weight* k *for* $\Gamma_1(N)$ *defined over a commutative ring* R *in which* N *is invertible is equal to* $H^0(X_1(N), \underline{\omega}^{\otimes k} \otimes R)$*.*

As for base changes we have

Theorem 30 ([28], Proposition 1.2.11)**.** *If* R' *is a flat* R*-algebra, the canonical mapping*

$$M_k(\Gamma_1(N), R)_{Katz} \otimes_R R' \to M_k(\Gamma_1(N), R')_{Katz}$$

is an isomorphism. Similarly for cusp forms.

Let us consider the problem of lifting modular forms from $\overline{\mathbb{F}}_p$ to $\overline{\mathbb{Z}}_p$.

Theorem 31 ([17], Lemma 1.9)**.** *1. Suppose that* $k \geq 2$*. Then the map* $S_k(\Gamma_1(N), \overline{\mathbb{Z}}_p)_{Katz} \to S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{Katz}$ *is surjective if* $N \neq 1$ *or if* $p > 3$*.*
2. The map $S_k(\Gamma_1(1), \overline{\mathbb{Z}}_2)_{Katz} \to S_k(\Gamma_1(1), \overline{\mathbb{F}}_2)_{Katz}$ *is not surjective if and only if* $k \geq 12$ *and (*$k \equiv 1 (mod\ 2)$ *or* $k \equiv 2 (mod\ 12)$*).*
3. The map $S_k(\Gamma_1(1), \overline{\mathbb{Z}}_p)_{Katz} \to S_k(\Gamma_1(1), \overline{\mathbb{F}}_p)_{Katz}$ *is not surjective if and only if* $k \geq 12$ *and* $k \equiv 2 (mod\ 12)$*.*

Let $f \in S_k(\Gamma_1(N), \mathbb{C})$ be a normalised Hecke eigenform such that $p \nmid N$ and let $f(q) = \sum_{n=1}^{\infty} a_n(f) q^n$ be its q-expansion at ∞. Then by what we cite above we have $S_k(\Gamma_1(N), \mathbb{C}) = S_k(\Gamma_1(N), \mathbb{C})_{\mathrm{Katz}}$. Theorem 30 allows us to identify $S_k(\Gamma_1(N), \mathbb{C})_{\mathrm{Katz}}$ with $S_k(\Gamma_1(N), \overline{\mathbb{Z}})_{\mathrm{Katz}} \otimes_{\overline{\mathbb{Z}}} \mathbb{C}$. By the q-expansion principle Theorem 32 $S_k(\Gamma_1(N), \overline{\mathbb{Z}})_{\mathrm{Katz}}$ is the subset of $S_k(\Gamma_1(N), \mathbb{C})_{\mathrm{Katz}}$ consisting of forms with q-expansions in $\overline{\mathbb{Z}}$. Theorem 30 further allows us to identify $S_k(\Gamma_1(N), \overline{\mathbb{Z}})_{\mathrm{Katz}} \otimes_{\overline{\mathbb{Z}}} \overline{\mathbb{Z}}_p$ with $S_k(\Gamma_1(N), \overline{\mathbb{Z}}_p)_{\mathrm{Katz}}$. We can always map $S_k(\Gamma_1(N), \overline{\mathbb{Z}}_p)_{\mathrm{Katz}}$ to $S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$ via the reduction homomorphism $\overline{\mathbb{Z}}_p \twoheadrightarrow \overline{\mathbb{F}}_p$. Combining the maps, we can map $S_k(\Gamma_1(N), \overline{\mathbb{Z}})_{\mathrm{Katz}}$ to $S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$. This means that we can reduce any modular form in $S_k(\Gamma_1(N), \overline{\mathbb{Z}})_{\mathrm{Katz}}$

and get a Katz form in $S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\text{Katz}}$. If the assumptions of Theorem 31 are satisfied, this reduction map is surjective, i.e. any Katz form in $S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\text{Katz}}$ comes from a classical modular form.

We have a fact that $\mathbb{Q}(a_n(f) : n \geq 1)$ is a number field. See [[14], Theorem 6.5.1]. Thus all the coefficients of a normalised eigenform f, $a_n(f)$ are algebraic integers so that for all $n \geq 1$, all the coefficients of f in the q-expansion at ∞ belongs to $\overline{\mathbb{Z}}$, $a_n(f) \in \overline{\mathbb{Z}}$. So by the q-expansion principle $f \in S_k(\Gamma_1(N), \overline{\mathbb{Z}})_{\text{Katz}}$, so the previous discussion applies to them and we have that the reduction form $\overline{f} \in S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\text{Katz}}$.

Theorem 32 ([13], Theorem 12.3.4(The q-expansion principle))**.** *Let N be at least 5. 1. The map $\phi_{\infty,R}$ taking f to its q-expansion at ∞ is injective.*
2. If $R_0 \subset R$ is a subring, then the commutative diagram

$$\begin{array}{ccc} M_k(\Gamma_1(N), R_0)_{Katz} & \xrightarrow{\phi_{\infty,R_0}} & R_0[[q]] \\ \downarrow & & \downarrow \\ M_k(\Gamma_1(N), R)_{Katz} & \xrightarrow{\phi_{\infty,R}} & R[[q]] \end{array}$$

is Cartesian; i.e., the image of $M_k(\Gamma_1(N), R)_{Katz}$ in $M_k(\Gamma_1(N), R_0)_{Katz}$ is precisely the set of modular forms whose q-expansions at ∞ have coefficients in R_0.
3. The above assertions hold for cusp forms, i.e., M_k replaced by S_k.

Let us give one example of a Katz modular form: Given $(E/S/R, \alpha)$ an element of $[\Gamma_1(N)]_R$ where R is an $\overline{\mathbb{F}}_p$-algebra, let $\eta \in H^1(E, \mathcal{O}_E)$ be the basis dual to $\omega \in H^0(E, \Omega^1_{E/S})$. The pth power endomorphism $x \mapsto x^p$ of $\mathcal{O}_E$ induces an endomorphism of $H^1(E, \mathcal{O}_E)$, which must carry η to a multiple of itself. So we have $\eta^p = A(E, \alpha) \cdot \eta$ in $H^1(E, \mathcal{O}_E)$, for some $A(E, \alpha) \in R$, which is the value of A on (E, α). This defines a modular form $A \in M_{p-1}(\Gamma_1(1), \overline{\mathbb{F}}_p)_{\text{Katz}}$ which is called the Hasse invariant (See also [20]). All its q-expansions are identically equal to 1.

The group $(\mathbb{Z}/N\mathbb{Z})^\times$ acts on $[\Gamma_1(N)]_R$ by:

$$\langle d \rangle^* : (E/S/R, \alpha) \mapsto (E/S/R, d\alpha),$$

for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. This gives an action by $(\mathbb{Z}/N\mathbb{Z})^\times$ on modular forms:

$$(\langle d \rangle f)(E/S/R, \alpha) = f(E/S/R, d\alpha).$$

For any character $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to R^\times$, let $M_k(\Gamma_1(N), \varepsilon, R)_{\text{Katz}}$ denote the R-submodule of $M_k(\Gamma_1(N), R)_{\text{Katz}}$ of elements f such that for all $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, $\langle d \rangle f = \varepsilon(d) f$. If f is a non-zero element of $M_k(\Gamma_1(N), R)_{\text{Katz}}$ which is an eigenform, then there is a unique character $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to R^\times$ such that $f \in M_k(\Gamma_1(N), \varepsilon, R)_{\text{Katz}}$. For any character $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to R^\times$, let

$$S_k(\Gamma_1(N), \varepsilon, R)_{\text{Katz}} := \{f \in S_k(\Gamma_1(N), R)_{\text{Katz}} | \forall d \in (\mathbb{Z}/N\mathbb{Z})^\times, \langle d \rangle f = \varepsilon(d) f\}$$

be the space of Katz cusp forms of weight k for $\Gamma_1(N)$ and character ε.

One can define Hecke operators geometrically. They have the following action on q-expansions of modular form. Let $f \in M_k(\Gamma_1(N), \varepsilon, R)_{\text{Katz}}$ be a Katz modular form. Then we have, see [28] §1.5 for the details.

$T_l f(q) = \sum_{n=0}^{\infty} a_{nl} q^n + l^{k-1} \varepsilon(l) \sum_{n=0}^{\infty} a_n q^{nl}$, for prime $l \nmid N$

$U_l f(q) = \sum_{n=0}^{\infty} a_{ln} q^n$, for prime $l | N$

For the sake of simplicity we write T_l for U_l. The Hecke operators T_n for $n \geq 1$ are defined by multiplication formula $T_1 = 1$, $T_n T_m = T_{nm}$ if $\gcd(n, m) = 1$ and $T_{l^r} = T_l T_{l^{r-1}} - l^{k-1} \langle l \rangle T_{l^{r-2}}$ for $r \geq 2$ where $l \nmid N$ and $T_{l^r} = T_l T_{l^{r-1}}$ when $l | N$. In particular, we note that all Hecke operators commute. They preserve $M_k(\Gamma_1(N), R)_{\text{Katz}}$ and $S_k(\Gamma_1(N), R)_{\text{Katz}}$. A Katz modular form f is called a Katz eigenform if f is an eigenfunction for all Hecke operators $T_n, n \geq 1$ and $\langle d \rangle, d \nmid N$. A Katz eigenform f is said to be normalised if the coefficient $a_1(f)$ in its q expansion at ∞ is 1. If we write $f(q) = \sum_{n=1}^{\infty} a_n q^n$ for the q-expansion at ∞ of $f \in M_k(\Gamma_1(N), R)_{\text{Katz}}$, we have the important formula $a_1(T_n f) = a_n(f)$.

Similarly to the classical modular forms one has the notion of level degeneracy maps on Katz modular forms. See [1] to see how they are induced from degeneracy maps of modular curve. Let $f \in M_k(\Gamma_1(N), R)_{\text{Katz}}$ be any Katz modular form, let M be a positive multiple of N with $1/M \in R$ and let $d \geq 1$ be an integer such that $d | M/N$. Then we have the d-th degeneracy map to level M,

$$B_d^{N,M} : M_k(\Gamma_1(N), R)_{\text{Katz}} \to M_k(\Gamma_1(M), R)_{\text{Katz}}$$

given by $f(q) \mapsto f(q^d)$. $B_d^{N,M}$ commutes with T_n whenever $\gcd(d, n) = 1$.

The *level old space* of f in the level M is given by

$$\mathcal{L}_M^{\text{old}}(f) = \left\langle B_d^{N,M}(f) : d | M/N \right\rangle_R \subset M_k(\Gamma_1(M), R)_{\text{Katz}},$$

the R-module generated by $B_d^{N,M}(f)$ where d runs through all possible divisors of M/N.

One has the following weight degeneracy maps on Katz modular forms which are not present in classical modular forms. See [[18], §4, pg. 457] for the details.

The first one is multiplying a form by the Hasse invariant. We denote this map by A. The other one is first defined for $M_k(\Gamma_1(N), \mathbb{F}_p)_{\text{Katz}}$ by the Frobenius by sending $\sum_n a_n q^n$ to $(\sum_n a_n q^n)^p = \sum_n a_n q^{np}$. The map is extended by linearity to $M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\text{Katz}}$ using Theorem 30. We still call it Frobenius. So taking the Frobenius of a form f, will be $\text{Frob}(f)(q) = f(q^p)$.

Multiplying a form by the Hasse invariant does not change the level and the q-expansion of the form but adds $p-1$ to the weight. Taking the Frobenius of a form multiplies the weight by p but does not change the level. These two degeneracy maps commute with Hecke operators T_n for all n such that $p \nmid n$. This follows from computations on q-expansions.

Let $f \in M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\text{Katz}}$ be any Katz modular form. Then to introduce the notion of weight old space corresponding to a form f let us recursively associate a weight to a word formed by the letters Frob and A. Let the empty word have weight k. Suppose m is a word of length n and weight w. Then set the weight of $A \circ m$ to be $w + p - 1$ and the weight of $\text{Frob} \circ m$ to be pw. Then the *weight old space* of f in the weight $k' \geq k$ is defined by

$$\mathcal{W}_{k'}^{\text{old}}(f) = \langle W(f) : W \text{ is a word in A and Frob such that } W(f) \text{ has weight } k' \rangle_{\overline{\mathbb{F}}_p}.$$

By examining the q-expansions, it is clear that we have the following commutativity properties: $B_d^{N,M} \circ \text{Frob} = \text{Frob} \circ B_d^{N,M}$ and $B_d^{N,M} \circ A = A \circ B_d^{N,M}$. Then the *level and weight old space* of f in the level M and weight k' is the $\overline{\mathbb{F}}_p$ vector space generated by $\left(B_d^{N,M} \circ W\right)(f)$ where $d|M/N$ and W is a word in A and Frob such that $W(f)$ has weight k'.

1.3 Galois representations

In this section we recall Galois representations and state the Chebotarev density theorem.

A Galois representation of G_K, the absolute Galois group of K, where K is any field, over a topological field L, is a finite-dimensional L-vector space V together with a continuous morphism $\rho : G_K \to GL(V)$.

We have the following classification. The representation of G_K is called a global Galois representation if K is a global field. On the other hand the representation ρ is called local Galois representation if K is a local field. Let L be a finite extension of $\mathbb{Q}_l$ and K be a finite extension of $\mathbb{Q}_p$.

Examples of Galois representations: 1. p-adic cyclotomic character. Let K be a number field, $n \geq 1$ integer and $K(\mu_{p^n}) \subset \overline{K}$ the cyclotomic field. Then $K(\mu_{p^n})$ over

K is Galois and there is a natural morphism

$$\chi_{p,n} : Gal(K(\mu_{p^n})/K) \hookrightarrow (\mathbb{Z}/p^n\mathbb{Z})^\times$$

given by

$$\sigma(\zeta) = \zeta^{\chi_{p,n}(\sigma)}$$

where $\sigma \in Gal(K(\mu_{p^n})/K)$ and ζ a primitive p^nth root of unity.

Let $K_\infty = \lim_\rightarrow K(\mu_{p^n})$. Then we have $Gal(K_\infty/K) = \lim_\leftarrow Gal(K(\mu_{p^n})/K)$, so

$$\chi_p : G_K \to Gal(K_\infty/K) \to \mathbb{Z}_p^\times \subset \mathbb{Q}_p^\times.$$

It is called the p-adic cyclotomic character over K.

It enjoy the following properties.

Theorem 33. *χ_p is a 1-dimensional global Galois representation, continuous and unramified at all places of K not dividing p. Moreover, if ν is a finite place of K not dividing p, then $\chi_p(Frob_\nu)$ is well-defined and is equal to the size of the residue field of ν.*

2. Galois representations attached to eigenforms.

Theorem 34. *Let f be a normalized Hecke eigenform, let N be its level, let k be its weight, and let $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be its character. Then the subfield K_f of $\mathbb{C}$ generated over $\mathbb{Q}$ by the $a_n(f)$, $n \geq 1$, and the image of ε is finite over $\mathbb{Q}$. Choose a prime λ of K_f with residue characteristic l. There exists a 2-dimensional $K_{f;\lambda}$-vector space $V_{f;\lambda}$ and a continuous representation $\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(K_{f;\lambda})$ that is unramified outside lN and such that for each prime number p not dividing lN the characteristic polynomial of the Frobenius at p acting on ρ is $\det(1 - x\rho(Frob_p) = 1 - a_p(f)x + p^{k-1}\varepsilon(p)x^2$.*

This is due to Eichler and Shimura [33] for $k = 2$, to Deligne [10] for $k > 2$, and to Deligne and Serre [11] for $k = 1$.

Theorem 35. *Let N and k be positive integers. Let $\mathbb{F}$ be a finite field, and $f : \mathbb{T}_{N,k} \to \mathbb{F}$ a surjective morphism of rings. Then there is a continuous semisimple representation $\rho_f : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F})$ that is unramified outside lN, where l is the characteristic of $\mathbb{F}$, such that for all p not dividing lN we have, in $\mathbb{F}$: $trace(\rho_f(Frob_p)) = f(T_p)$ and $det(\rho_f(Frob_p)) = f(\langle p \rangle)p^{k-1}$. Such a ρ_f is unique up to isomorphism (that is, up to conjugation).*

Concerning traces and characteristic polynomials we have the following important result

Lemma 36. *Let Π be a profinite group, let $F \subset \Pi$ be a subset such that Π is the topological closure of the conjugacy classes of F, and fix a positive integer n. Then any continuous semi-simple representation $\rho : \Pi \to GL_n(L)$ where $L \in \{\mathbb{C}, \overline{\mathbb{F}}_p, \overline{\mathbb{Q}_p}\}$ is uniquely determined by the characteristic polynomials $charpol(\rho(g)) \in L[T]$ for all $g \in F$.*

For $\mathbb{C}$ this is classical representation theory, for $\overline{\mathbb{F}}_p$ this follows from the theorem of Brauer-Nesbitt, see [[9], §30.16]. A proof for $\overline{\mathbb{Q}_p}$ is in [35].

The Frobenius elements play a very special role in the theory. Their images determine the Galois representation uniquely. This is a consequence of Chebotarevs density theorem.

Theorem 37 (Chebotarev density theorem)**.** *Let L/K be a finite Galois extension of number fields with Galois group $G = Gal(L/K)$. Let C be a subset of G which is stable under conjugation. Then*

$$\{\mathcal{P} | \mathcal{P} \text{ a prime of } K, \mathcal{P} \nmid \Delta_{L/K}, \sigma_{\mathcal{P}} \in C\}.$$

has density $\#C/\#G$. In particular, this ratio is greater than zero, so there always exist such primes.

Recall that the norm of an ideal is denoted as $N(\mathcal{P}) = \#(O/\mathcal{P})$. The natural density of S is defined as

$$d(S) := \lim_{x\to\infty} \frac{\#\{\mathcal{P} \in S | N(\mathcal{P}) < x\}}{\#\{\mathcal{P} \text{ prime } | N(\mathcal{P}) < x\}}$$

if the limit exists. If the natural density exists, then it is equal to the analytic (Dirichlet) density

$$\delta(S) := \lim_{s\to 1, s>1} \frac{\sum_{\mathcal{P}\in S} N(\mathcal{P})^{-s}}{\sum_{\mathcal{P} \text{ prime}} N(\mathcal{P})^{-s}}.$$

The existence of the natural density implies the existence of the Dirichlet density, but the converse does not hold in general. However, the Chebotare density theorem is valid with either notion of density.

One result which we need later

Lemma 38. *Let $\rho_1, \rho_2 : Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_2(\overline{\mathbb{F}}_p)$ be two semisimple finite image Galois representations such that $tr(\rho_1(Frob_l)) = tr(\rho_2(Frob_l))$ for all l in the set S of primes of density 1. Then $\rho_1 \cong \rho_2$.*

Proof. Since ρ_1 and ρ_2 have finite image there exists an extension $L/\mathbb{Q}$ finite Galois such that $G_L := Gal(\overline{\mathbb{Q}}/L) \subset \ker(\rho_1)$ and $G_L \subset \ker(\rho_2)$. Thus we have the factorisation

$$\rho_i : G_{\mathbb{Q}} \twoheadrightarrow Gal(L/\mathbb{Q}) = G_{\mathbb{Q}}/G_L \to GL_2(\overline{\mathbb{F}}_p).$$

By Chebotarev density theorem and the density 1 assumption we have that in every conjugacy class there is $Frob_l$ for some $l \in S$. Thus every $g \in Gal(L/\mathbb{Q})$ is a Frobenius from S. Then by Brauer-Nesbitt theorem $\rho_1 \cong \rho_2$. ■

1.4 Level and weight lowering

To state some of the theoretical results that we need later let us set the following notation. Let ρ be a continuous Galois representation

$$\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p).$$

Let *lneqp* be prime. Choose an extension of l-adic valuation of $\mathbb{Q}$, and let

$$G_0 \supset G_1 \supset \cdots \supset G_i \supset \cdots$$

be a sequence of ramification groups of $G_{\mathbb{Q}}$ corresponding to this valuation. Let V_i be a subspace of V which is fixed by G_i. Then write

$$n(l) = \sum_{i=0}^{\infty} \frac{1}{(G_0 : G_i)} \dim V/V_i.$$

We can rewrite as

$$n(l) = \dim V/V_0 + b(V)$$

where $b(V)$ is the wild invariant of G_0-module V.

The formula imply

(a). $n(l)$ is a non negative integer

(b). $n(l) = 0$ if and only if $G_0 = \{1\}$, i.e., ρ is not ramified at l

(c). $n(l) = \dim V/V_0$ if and only if $G_1 = \{1\}$, meaning ρ is tamely ramified.

It follows from (a) and (b) that we can define an integer $N(\rho)$ by the formula

$$N(\rho) = \prod_{l \neq p} l^{n(l)}.$$

And we call this number $N(\rho)$ the conductor of ρ; and by construction $N(\rho)$ is coprime with p.

For any continuous Galois representation $\rho_p : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ we associate an integer $k(\rho_p)$ called the minimal weight as in Definition 4.3 of [15]. See below for the definition. This differs slightly from Serre's definition in [32].

Let ψ, ψ' denote the fundamental characters of level 2, i.e. the characters of the tame inertia with values in $\overline{\mathbb{F}}_p^\times$ induced by the embeddings of fields $\mathbb{F}_{p^2} \hookrightarrow \overline{\mathbb{F}}_p$, see [[32], Section 2 and Proposition 1].

Definition 39 ([15], Definition 4.3)**.** *Let ρ be a continuous 2-dimensional Galois representation and let ρ_p be its restriction to the decomposition group at p. We associate an integer $k(\rho)$ to ρ as follows:*

1. *Suppose that ϕ, ϕ' are characters of $G_p = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of level 2 and we have*

$$\rho_p \cong \begin{pmatrix} \phi & 0 \\ 0 & \phi' \end{pmatrix}.$$

After interchanging ϕ and ϕ' if necessary, we have $\phi = \psi^a \psi'^b$ and $\phi' = \psi'^a \psi^b$ with $0 \le a < b \le p-1$. Then, we set $k(\rho) = 1 + pa + b$.

2. *Suppose that ϕ, ϕ' are characters of $G_p = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of level 1.*

   - *If $\rho_p|_{I_{p,w}}$ is trivial, where $I_{p,w}$ is the wild inertia subgroup, then we have*

$$\rho_p \cong \begin{pmatrix} \chi_p^a & 0 \\ 0 & \chi_p^b \end{pmatrix},$$

   with $0 \le a \le b \le p-2$. Then, we set $k(\rho) = 1 + pa + b$.

   - *If $\rho_p|_{I_{p,w}}$ is not trivial, we have*

$$\rho_p \cong \begin{pmatrix} \chi_p^\beta & * \\ 0 & \chi_p^\alpha \end{pmatrix},$$

   for unique α, β such that $0 \le \alpha \le p-2$ and $1 \le \beta \le p-1$. Then, we set $a = \min\{\alpha, \beta\}$ and $b = \max\{\alpha, \beta\}$. If $\chi_p^{\beta-\alpha} = \rho_p$ and $\rho_p \otimes \chi_p^{-\alpha}$ is not finite at p then we set $k(\rho) = 1 + pa + b + p1$, otherwise $k(\rho) = 1 + pa + b$.

The recipe for $N(\rho)$ depends on the local behavior of ρ at primes l other than p; the recipe for $k(\rho)$ depends on the restriction $\rho|_{I_p}$ of ρ to the inertia group at p. Level lowering was proved by Ribet in the 1990's [30] for $p \ge 2$ and by Buzzard [4] for $p = 2$. Carayol proved in [7] that the level always has to be a multiple of the conductor. Level lowering [[31], Chapter 3], Theorem 43 is the statement that given a modular Galois representation ρ over $\overline{\mathbb{F}}_p$, there is a modular form (in some weight) of level equal to the conductor of ρ, $N(\rho)$.

We have the following very important formula for any cusp form.

Proposition 40. *Let $f \in S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{Katz}$ be a Katz modular form such that $f(q) \in \overline{\mathbb{F}}_p[[q^l]]$ for some prime $l \neq p$. Then there exists a unique cusp form $g \in S_k(\Gamma_1(N/l), \overline{\mathbb{F}}_p)_{Katz}$ such that $f(q) = B_l^{N/l,N} g(q)$. In particular, $g = 0$ if $l \nmid N$.*

Proof. This follows from Lemma 3.6 of [1] when $l|N$. When $l \nmid N$ the statement follows from the claim in the 4th paragraph of page 31 of [29]. ■

Proposition 41 ([19], Corollary 4.4.2)**.** *Let $f \in M_k(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{Katz}$ and $g \in M_{k'}(\Gamma_1(N), \varepsilon', \overline{\mathbb{F}}_p)_{Katz}$ be Katz eigenforms with $\rho_f \cong \rho_g$. Then $k \equiv k' (mod\ p-1)$ and $\varepsilon = \varepsilon'$, provided that they are primitive.*

Proof. Let us set $\tilde{\varepsilon}$ and $\tilde{\varepsilon}'$ to be the corresponding 1-dimensional Galois representations of ε and ε'. Then since $p \nmid N$, $\tilde{\varepsilon}$ is unramified at p and so $\tilde{\varepsilon}(\mathrm{Frob}_p) = \varepsilon(p)$ is well defined. By restricting $\tilde{\varepsilon}\chi_p^{k-1} = \tilde{\varepsilon}'\chi_p^{k'-1}$ to the inertia group I_p we get $\chi_p^{k-1} = \chi_p^{k'-1}$ from which $k \equiv k' (\mathrm{mod}\ p-1)$ follows, so $\tilde{\varepsilon} = \tilde{\varepsilon}'$. We get $\varepsilon = \varepsilon'$, for all primes $l \nmid N$. ■

Let $f \in M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$ and $f_0 \in M_{k'}(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$ be Katz eigenforms with the same q-expansions where $k \geq k'$. Then we have $f = A^t f_0$ where $t = (k-k')/(p-1)$. This is because the q-expansion of A is 1 and multiplication by A^t matches the weights on both sides.

In the literature, one has the following result on weight lowering.

Theorem 42 ([15], Theorem 4.5)**.** *Let p be a prime and let $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ be a continuous, irreducible and odd mod p Galois representation. Suppose that there exists a Katz eigenform $g \in S_k(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{Katz}$ such that ρ is isomorphic to ρ_g. Then there exists a Katz eigenform $f \in S_{k(\rho)}(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{Katz}$ with the same eigenvalues for $T_l (l \neq p)$ as g has, such that ρ is isomorphic to ρ_f. Moreover, there is no other eigenform of level prime to p and of weight less than $k(\rho)$ whose associated Galois representation is isomorphic to ρ.*

Proof. This is [[15], Theorem 4.5] together with the last paragraph of the introduction of [15]. The case $p = 2$ is explained in [6].

■ As a remark, due to the theorems of C. Khare and J.-P. Wintenberger [[21], Theorem 1.1 and Theorem 1.2], [22] and of M. Kisin [23] proving Serre's conjecture there exists a Katz eigenform $F \in S_k(\Gamma_1(N), \mathbb{F}_p)_{\mathrm{Katz}}$ for some integers k and N such that F gives rise to the same Galois representation of the above theorem. The following theorem is level lowering.

Theorem 43. *Let $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ be a continuous, irreducible and odd mod p Galois representation. Suppose that there exists a Katz eigenform $g \in S_k(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{Katz}$*

such that ρ is isomorphic to ρ_g. Then there exists a Katz eigenform $f \in S_k(\Gamma_1(N(\rho)), \varepsilon, \overline{\mathbb{F}}_p)_{Katz}$ such that ρ is isomorphic to ρ_f.

Proof. Let $p > 2$ and let $g \in S_k(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{\text{Katz}}$ be a normalised eigenform giving rise to ρ. If $k = 1$, multiply g by the Hasse invariant so without loss of generality assume that $k \geq 2$. By the discussion after Theorem 31, there exists a Hecke eigenform $\widetilde{g} \in S_k(\Gamma_1(N), \mathbb{C})$ such that ρ_g arises from $\widetilde{g}$. Then by [[12], Theorem 1.1] we have that ρ_g arises from an eigenform $\widetilde{f} \in S_{k'(\rho)}(\Gamma_1(N(\rho)), \mathbb{C})$, where $k'(\rho)$ is Serre's original weight as in [12]. By discussion after Theorem 31, we can reduce $\widetilde{f}$ to get Katz eigenform of level $N(\rho)$. Apply Theorem 42 and multiply by a power of the Hasse invariant to find the desired f.

For $p = 2$ we know that $k(\rho)$ is 1, 2 or 3. If $k(\rho) = 1, 2$, then $k'(\rho) = 2$ and level lowering is possible by [[21], Theorem 1.2]. If $k(\rho) = 3$, then ρ satisfies multiplicity one and ρ is not finite, then level lowering is possible by [[4], Theorem 3.2]. ■

There exists a derivation $\theta : M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\text{Katz}} \to M_{k+p+1}(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\text{Katz}}$ which increases weights by $p + 1$ and whose effect upon each q-expansions is $q\frac{d}{dq}$. See [20] for the details. The Galois representations of f and θf are twists of each other by the mod p cyclotomic character: $\rho_{\theta f} = \chi_p \otimes \rho_f$, see [[15], §3.1]. For a form $f \in M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\text{Katz}}$, f and $\theta^{p-1} f$ have the same Galois representations. By the principle of q-expansions we have that the operator θ maps modular forms to cusp forms since θ always kills the constant term.

Similarly to Proposition 40 we have the following weight version result.

Proposition 44 ([20], Corollary 5 and Corollary 6)**.** *Let $f \in M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{Katz}$ be a Katz modular form such that $\theta f = 0$. Then we can uniquely write $f(q) = A^r g(q^p)$ with $0 \leq r \leq p-1, r+k \equiv 0 (mod\, p)$ and $g \in M_l(\Gamma_1(N), \overline{\mathbb{F}}_p)_{Katz}$ with $pl + r(p-1) = k$. Furthermore, if f is a cusp form, then so is g.*

Proof. This is a combination of Corollary 5 and Corollary 6 of [20]. ■

Chapter 2

Main results

2.1 Strong multiplicity one

Let us start by proving that by moving into higher level we can make some of the inside level coefficients of any Katz eigenform zero. That is we make some of the coefficients at the primes which divide the level of the eigenform zero.

Let $g \in M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$ be a cuspidal Katz eigenform. Then g is called an outside N' eigenform if g is an eigenform for all T_n where $\gcd(n, N') = 1$.

Lemma 45. *Let $f \in M_k(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{Katz}$ be a normalised Katz eigenform and let $N = \prod_{i=1}^{n} l_i^{\alpha_i}$ be the prime factorization of N with $\alpha_i \geq 1$. Let $I_N = \{l_1, l_2, \ldots, l_n\}$ and $S \subset I_N$ be any subset. Then there exists a normalised Katz eigenform $\widetilde{f} \in M_k(\Gamma_1(N \prod_{l_i \in S} l_i), \varepsilon, \overline{\mathbb{F}}_p)_{Katz}$ such that $a_l(\widetilde{f}) = a_l(f)$ for all primes $l \notin S$ and $a_{l^m}(\widetilde{f}) = 0$ for all $l \in S$ and $m \in \mathbb{Z}_{>0}$.*

Proof. Without loss of generality $S = \{l_1, l_2, \ldots, l_t\}$. Then set

$$\begin{aligned}
f_1(q) &:= f(q) - a_{l_1}(f) B_{l_1}^{N, Nl_1} f(q) \\
f_2(q) &:= f_1(q) - a_{l_2}(f_1) B_{l_2}^{Nl_1, Nl_1 l_2} f_1(q) \\
&\vdots \\
\widetilde{f}(q) = f_t(q) &:= f_{t-1}(q) - a_{l_t}(f_{t-1}) B_{l_t}^{N\prod_{i=1}^{t-1} l_i, N\prod_{l_i \in S} l_i} f_{t-1}(q).
\end{aligned}$$

Note that $\tilde{f}$ is normalised. Then by the property of level degeneracy maps $\widetilde{f}$ is an outside $\prod_{l_i \in S} l_i$ Katz eigenform. Let us evaluate $T_{l_i}\widetilde{f}$. For $n \geq 0$, we have $a_n(T_{l_i}\widetilde{f}) = a_{nl_i}(\widetilde{f}) = a_{nl_i}(f_i) = a_{nl_i}(f_{i-1}) - a_{l_i}(f_{i-1}) a_n(f_{i-1}) = a_n(f_{i-1}) a_{l_i}(f_{i-1}) - a_{l_i}(f_{i-1}) a_n(f_{i-1}) = 0$ where $f_0 = f$. Thus $T_{l_i}\widetilde{f} = 0$ so $\widetilde{f}$ is a T_{l_i}-eigenform for all $l_i \in S$. From the definition of $\widetilde{f}$ we observe that $a_l(\widetilde{f}) = a_l(f)$ for all primes $l \notin S$. We prove $a_{l_i^m}(\widetilde{f}) = 0$ for all $l_i \in S$ $(i = 1, 2, 3, \ldots, t)$and $m \in \mathbb{Z}_{>0}$ by induction. For

$i = 1, 2, 3, \ldots, t$, $a_{l_i}(\widetilde{f}) = a_{l_i}(f_i) = a_{l_i}(f_{i-1}) - a_{l_i}(f_{i-1})a_1(f_{i-1}) = 0$. Suppose $a_{l_i^m}(\widetilde{f}) = 0$ for some $l_i \in S$. Then $a_{l_i^{m+1}}(\widetilde{f}) = a_{l_i^{m+1}}(f_i) = a_{l_i^{m+1}}(f_{i-1}) - a_{l_i^m}(f_{i-1})a_{l_i}(f_{i-1}) = a_{l_i^m}(f_{i-1})a_{l_i}(f_{i-1}) - a_{l_i^m}(f_{i-1})a_{l_i}(f_{i-1}) = 0$. ∎

Proof of Theorem 2. Let ρ_f and ρ_g be the associated Galois representations. Then by hypothesis, the traces of ρ_f and ρ_g agree on the Frobenius elements for all primes in a set of primes of density 1. This implies by above Lemma 38 that $\rho_f \cong \rho_g$ so $a_l(f) = a_l(g)$ for all primes $l \nmid NN'p$. Then by definition of cuspidal Katz newforms we have $N = N'$ and $k = k'$. Thus $a_l(f) = a_l(g)$ for all primes $l \nmid pN$. Let $N = \prod_{i=1}^{n} l_i^{\alpha_i}$ be the prime factorization of N. Then by taking $S = I_N$ in Lemma 45 we have normalised eigenforms $\widetilde{f}$ and $\widetilde{g}$ such that $a_{l_i^m}(\widetilde{f}) = 0 = a_{l_i^m}(\widetilde{g})$ for all $l_i \in S$ and all $m \geq 1$. Hence since $\widetilde{f}$ and $\widetilde{g}$ have the same eigenvalues away from p their q expansion coefficients equal away from p. Thus we have that $\widetilde{f} - \widetilde{g}$ is supported with q^p so $\theta(\widetilde{f} - \widetilde{g}) = 0$. If $\widetilde{f} - \widetilde{g} \neq 0$, then by Proposition 44 it must be up to a suitable power multiple of the Hasse invariant in the image of Frobenius of some cusp form of weight smaller than k, which is impossible by the minimality of weight k. Thus, $a_p(f) = a_p(g)$. If $a_{l_i}(f) \neq a_{l_i}(g)$ for some $l_i \in I_N$, then by taking $S = I_N - \{l_i\}$ in above lemma we have cusp forms $\widetilde{f}_i$ and $\widetilde{g}_i$ such that $a_{l_j^m}(\widetilde{f}_i) = 0 = a_{l_j^m}(\widetilde{g}_i)$ for all $l_j \in S$ and $m \geq 1$. Let $\widetilde{G} := \widetilde{f}_i - \widetilde{g}_i$. Similarly here we have that the normalised eigenforms $\widetilde{f}_i$ and $\widetilde{g}_i$ have the same eigenvalues at primes which do not divide the level of the forms N and at the prime p. Furthermore they have the same coefficients at all primes dividing the level except possibly at l_i. Thus the q-expansion of $\widetilde{G}$ is in q^{l_i}, i.e., $a_n(\widetilde{G}) = 0$ unless $l_i | n$. Then by Proposition 40, $\widetilde{G}(q) = B_{l_i}^{\frac{N}{l_i}\prod_{l_j \in S} l_j, N \prod_{l_j \in S} l_j} \widetilde{G}_1(q)$ for some cusp form $\widetilde{G}_1$ of level $\frac{N}{l_i}\prod_{l_j \in S} l_j$ which is impossible by the l_i-minimality of the level $\frac{N}{l_i}\prod_{l_j \in S} l_j$. Thus $f = g$. Furthermore, for all $d \in \mathbb{N}$ such that $\gcd(d, N) = 1$ we have $\langle d \rangle f = \varepsilon(d) \cdot f$ and $\langle d \rangle g = \varepsilon'(d) \cdot g$. Then $f = g$ gives $\varepsilon = \varepsilon'$. ∎

Let us prove the existence of Katz newform for irreducible mod p Galois representation.

Theorem 46. *Let $f \in S_k(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{Katz}$ be a Katz eigenform such that ρ_f is irreducible. Then there exists a unique Katz newform g in level $N(\rho_f)$ and weight $k(\rho_f)$ such that $\rho_f \cong \rho_g$.*

Proof. Since ρ_f is irreducible we have by Theorem 42 that there exists a cuspidal eigenform $h \in S_{k(\rho)}(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{\text{Katz}}$ such that $\rho_f \cong \rho_h$ where $k(\rho)$ is the minimal weight. Let g be the level lowering (See Theorem 43) of h to level $N(\rho)$. Note that by Carayol [7], $N(\rho)$ is l-minimal for all primes l. Then we have $\rho_h \cong \rho_g$, which gives the result as g is uniquely determined by Theorem 2. ∎

Next we prove level degeneracy results, which we use to prove the level part of the main theorem.

Lemma 47. *(i). Let $f \in S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{Katz}$ be a normalised Katz eigenform with p_1-minimal level and $g \in S_k(\Gamma_1(Np_1^{m_1}), \overline{\mathbb{F}}_p)_{Katz}$ where $m_1 \geq 0$ be an outside p_1 eigenform such that for all positive integers n such that $p_1 \nmid n$, $T_n g = a_n(f)g$. Then $g \in \mathcal{L}^{old}_{Np_1^{m_1}}(f)$.*
(ii). Let $M = N \cdot \prod_{i=1}^{t} p_i^{m_i}$ where $m_i \geq 1$. Let $f \in S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{Katz}$ be a normalised Katz eigenform with $p_1, p_2, p_3, \ldots, p_t$-minimal level and $g \in S_k(\Gamma_1(M), \overline{\mathbb{F}}_p)_{Katz}$ be a normalised Katz eigenform satisfying $T_l(g) = a_l(f)g$ for all primes $l \neq p_1, p_2, p_3, \ldots, p_t$. Then $g \in \mathcal{L}^{old}_M(f)$.

Proof. (i). We have $a_n(g - a_1(g)f) = 0$ for all integers $n \geq 1$ such that $p_1 \nmid n$ because $a_n(g - a_1(g)f) = a_1(T_n(g - a_1(g)f)) = 0$. When $m_1 = 0$, by p_1 minimality of level N and by Proposition 40 we have the result $g = a_1(g)f$. Assume $m_1 \geq 1$. Then by Proposition 40, $(g - a_1(g)f)(q) = B_{p_1}^{Np_1^{m-1}, Np_1^m} F(q)$ for some outside p_1 eigenform F of level $Np_1^{m_1-1}$ such that $\rho_g \cong \rho_F$. Then we proceed by induction on m_1. Assume the result holds for $m-1$. Then when $m_1 = m$ we have $(g - a_1(g)f)(q) = B_{p_1}^{Np_1^{m-1}, Np_1^m} G(q)$ for some outside p_1 eigenform G of level Np_1^{m-1} such that $\rho_g \cong \rho_G$, which by induction assumption is in the level old space of f in level Np_1^{m-1}. Thus $g \in \mathcal{L}^{\text{old}}_{Np_1^{m_1}}(f)$.
(ii). The case $t = 1$ follows from (i) above. Assume the result holds for $t \leq r - 1$. Let $f \in S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\text{Katz}}$ be a normalised eigenform with $p_1, p_2, p_3, \ldots, p_r$-minimal level and $g \in S_k(\Gamma_1(M), \overline{\mathbb{F}}_p)_{\text{Katz}}$ be a normalised eigenform with $T_l g = a_l(f)g$ for all primes $l \neq p_1, p_2, p_3, \ldots, p_r$. Then by using Lemma 45 with $S = \{p_1\}$ and assuming by canonical embedding, B_1, that $f \in S_k(\Gamma_1(Np_1), \overline{\mathbb{F}}_p)_{\text{Katz}}$ and $g \in S_k(\Gamma_1(Mp_1), \overline{\mathbb{F}}_p)_{\text{Katz}}$ we have normalised eigenforms $\widetilde{f} \in S_k(\Gamma_1(Np_1^2), \overline{\mathbb{F}}_p)_{\text{Katz}} \subset S_k(\Gamma_1(Np_1^{m_1+2}), \overline{\mathbb{F}}_p)_{\text{Katz}}$ and $\widetilde{g} \in S_k(\Gamma_1(Mp_1^2), \overline{\mathbb{F}}_p)_{\text{Katz}}$ such that $\widetilde{f}$ has $p_2, p_3, \ldots, p_r$-minimal level and $\widetilde{g}$ satisfies $T_l \widetilde{g} = a_l(\widetilde{f})\widetilde{g}$ for all primes $l \neq p_2, p_3, \ldots, p_r$. Then by the induction assumption $\widetilde{g} \in \mathcal{L}^{\text{old}}_{Mp_1^2}(\widetilde{f})$, say $\widetilde{g}(q) = \sum_{d | M/(Np_1^{m_1})} \beta_d B_d^{Np_1^2, Mp_1^2} \widetilde{f}(q)$ for some $\beta_d \in \overline{\mathbb{F}}_p$. Let $h(q) := \sum_{d | M/(Np_1^{m_1})} \beta_d B_d^{N, M/p_1^{m_1}} f(q) \in S_k(\Gamma_1(Np_2^{m_2} \cdots p_r^{m_r}), \overline{\mathbb{F}}_p)_{\text{Katz}}$. Then h is a normalised outside $M/Np_1^{m_1}$ eigenform with p_1-minimal level. On the other hand, since $a_n(\widetilde{g}) = a_n(h)$ for all $n \geq 1$ such that $p_1 \nmid n$ we have $a_n(T_l h - a_l(\widetilde{g}) \cdot h) = 0$ for all $n \geq 1$ such that $p_1 \nmid n$ and prime $l \neq p_1$. So $(T_l h - a_l(\widetilde{g}) \cdot h)(q) \in \overline{\mathbb{F}}_p[[q^{p_1}]]$. But by p_1-minimality $T_l h - a_l(\widetilde{g}) \cdot h = 0$. Hence h is an eigenform at primes $l = p_2, p_3, \ldots, p_r$. Thus h is a normalised Katz eigenform with p_1-minimal level. Then $T_l g = a_l(h)g$ for all primes $l \neq p_1$. Then by part (i) above we have $g \in \mathcal{L}^{\text{old}}_M(h)$, so $g \in \mathcal{L}^{\text{old}}_M(f)$. ∎

Lemma 48. *Let $f \in S_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{Katz}$ be a normalised Katz eigenform with p_1-minimal level for all primes $p_1 | N$. Then any normalised Katz eigenform $g \in S_k(\Gamma_1(M),$*

$\overline{\mathbb{F}}_p)_{Katz}$ *such that* $\rho_f \cong \rho_g$ *and* $a_p(f) = a_p(g)$ *is in the level old space of* f.

Proof. Suppose $a_l(f) \neq a_l(g)$ for some prime $l \nmid M/N$ and $l|M$. Then taking $S = I_M - \{l\}$ in Lemma 45 gives forms $\widetilde{f}$ and $\widetilde{g}$ such that $a_{l'}(\widetilde{f}) = 0 = a_{l'}(\widetilde{g})$ for all primes $l'|M$ and $l' \neq l$. Then $B_1^{N\prod_{l'\in S} l', M\prod_{l'\in S} l'}\widetilde{f}(q) - \widetilde{g}(q) = B_l^{\frac{M}{l}\prod_{l'\in S} l', M\prod_{l'\in S} l'}F(q)$ for some modular form $F \neq 0$ of level $\frac{M}{l}\prod_{l'\in S} l'$ which is impossible by l-minimality. Thus $T_l g = a_l(f)g$ for all primes $l \nmid M/N$. Then since the level N of f is l-minimal for any prime l, in particular it is l-minimal for $l|M/N$. Then by applying Lemma 47 we have $g \in \mathcal{L}_M^{\text{old}}(f)$. ■

Proposition 49. *Let* $f \in S_k(\Gamma_1(N, \overline{\mathbb{F}}_p)_{Katz}$ *be a Katz newform and let* M *be a multiple of* N. *Then any normalised Katz eigenform* $g \in S_k(\Gamma_1(M), \overline{\mathbb{F}}_p)_{Katz}$ *such that* $\rho_f \cong \rho_g$ *is in the level old space of* f.

Proof. By the hypothesis we have $f = B_1^{N,M} f \in S_k(\Gamma_1(M), \overline{\mathbb{F}}_p)_{\text{Katz}}$ as $N|M$. Then by setting $S = I_M$ in Lemma 45 we have forms $\widetilde{f}$ and $\widetilde{g}$ such that $\theta(\widetilde{f} - \widetilde{g}) = 0$, so by applying Proposition 44 we have $(\widetilde{f} - \widetilde{g})(q) = A^r G(q^p)$ for some integer r and an outside p Katz eigenform G of weight smaller than k, which is impossible by the minimality of weight unless $G = 0$, so we have $a_p(f) = a_p(g)$. Then apply above lemma. ■

Let $f \in M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\text{Katz}}$ be a normalised Katz eigenform with minimal weight k. Then for $k' \geq k$, define $V_{f,k'}$ as the $\overline{\mathbb{F}}_p$ vector space generated by $F \in M_{k'}(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\text{Katz}}$ such that F is an outside p Katz eigenform with eigenvalues $\lambda_l(F) = a_l(f)$ for all primes $l \neq p$. Then we have the following

Lemma 50. *The space* $V_{f,k'}$ *is a subspace of the weight old space of* f *in the weight* k'.

Proof. We proceed by induction on k'. Let $k' = k$. Then for every $F \in V_{f,k'}$, by Proposition 44, we can write $(F - a_1(F)f)(q) = A^r \text{Frob}G(q)$ for some integer r and an outside p Katz eigenform G of weight smaller than k', which is impossible by the minimality of weight unless $G = 0$, so we have $V_{f,k'} = \langle f \rangle$. Then suppose the induction hypothesis is correct for all weights less than k'. Then by Proposition 41, $k' = k + m(p-1)$ for some non-negative integer m. Set $f_0 = A^m f \in V_{f,k'}$. Then since $V_{f,k'}$ is a finite dimensional $\overline{\mathbb{F}}_p$-vector space, say of dimension d, we can pick modular forms $f_1, f_2, f_3, \ldots, f_{d-1} \in V_{f,k'}$ such that $f_0, f_1, f_2, \ldots, f_{d-1}$ constitutes a basis for $V_{f,k'}$. Then for all $1 \leq i \leq d-1$, define $g_i := f_i - a_1(f_i)f_0$. Then $a_1(g_i) = 0$ which gives $a_n(g_i) = 0$ for all integers $n \geq 1$ such that $p \nmid n$ as $a_n(g_i) = a_1(T_n g_i) = a_n(f)a_1(g_i) = 0$ for such n. Then by Proposition 44 there exist modular forms $\widetilde{g}_i \in M_{k_i}(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\text{Katz}}$

for $i = 1, 2, 3, \dots, d-1$ such that $g_i(q) = A^{r_i}\mathrm{Frob}\widetilde{g}_i(q)$ and $\widetilde{g}_i \in V_{f,k_i}$ for some integers r_i and $k_i < k'$. Then by the induction assumption $\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3, \dots, \widetilde{g}_{d-1}$ are in the weight old space of f in weights $k_1, k_2, k_3, \dots, k_{d-1}$. This implies that the basis elements $f_1, f_2, f_3, \dots, f_{d-1}$ are in the weight old space of f in weight k'. This gives the result. ■

Corollary 51. *Let $f \in M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{Katz}$ be a normalised Katz eigenform with minimal weight. Then any normalised Katz eigenform $g \in M_{k'}(\Gamma_1(N), \overline{\mathbb{F}}_p)_{Katz}$ such that $\rho_f \cong \rho_g$ and $a_l(f) = a_l(g)$ for all primes $l|N$ is in the weight old space of f.*

In the above corollary one cannot relax the condition that the eigenvalues $a_l(f)$ and $a_l(g)$ for T_l for all primes l dividing the level are the same. To construct a counterexample let $F \in S_k(\Gamma_1(M), \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$ be a Katz newform. Then choose a prime $l \nmid Mp$ such that T_l has two distinct eigenvalues on $\langle F(q), B_l^{M,lM}F(q)\rangle \subseteq S_k(\Gamma_1(Ml), \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$. Then we can produce normalised Katz eigenforms f and g (for example take $F(q)$ and $F(q) + \alpha B_l^{M,Ml}F(q)$ for suitable choose of α) in this subspace such that $\rho_f \cong \rho_g$ and $a_l(f) \neq a_l(g)$ but one is not in the weight old space of the other.

In Theorem 46 we have associated a Katz newform to any Katz eigenform which has an irreducible Galois representation. More generally if we assume the existence of Katz newforms for Katz eigenforms which have reducible Galois representations we have

Proof of Theorem 3. By applying Lemma 45 with $S = I_M$ we have eigenforms $\widetilde{F}$ and $\widetilde{f}$ such that $a_l(\widetilde{F}) = 0 = a_l(\widetilde{f})$ for all primes $l|M$. Then by using Corollary 51 one can write

$$\widetilde{F}(q) = \sum_{\delta \in D_k^{k'}} \alpha_\delta \delta(\widetilde{f}(q))$$

for some $\alpha_\delta \in \overline{\mathbb{F}}_p$ where $D_k^{k'}$ is the set of words W in A and Frob such that W takes weight k forms into weight k' forms. Let us define

$$F_1(q) := \sum_{\delta \in D_k^{k'}} \alpha_\delta \delta(f(q))$$

by replacing the form $\widetilde{f}$ by f. Suppose that $F_1(q) = \sum_{t=0;\ i}^{u} \beta_t A^i \mathrm{Frob}^t f(q)$. Then F_1 is a T_p Katz eigenform since for any positive integer m such that $\gcd(m, p) = 1$ we have $a_{pm}(F_1) = \beta_0 a_{pm}(f) + \beta_1 a_m(f)$, $a_p(F_1) = \beta_0 a_p(f) + \beta_1$ and $a_m(F_1) = \beta_0 a_m(f)$ where $\beta_0 = 1$ so $a_{pm}(F_1) = \beta_0 a_{pm}(f) + \beta_1 a_m(f) = a_m(f)(a_0(f) + \beta_1) = a_m(F_1)a_p(F_1)$. On the other hand since $a_{p^n}(F_1) = a_{p^n}(\widetilde{F})$ for any positive integer n and that $\widetilde{F}$ is a T_p eigenfunction of weight k' we have

$$a_{p^n}(F_1) = a_p(F_1)a_{p^{n-1}}(F_1) - p^{k'-1}a_{p^{n-2}}(F_1)$$

for any positive integer $n \geq 2$. Then since $\rho_F \cong \rho_{F_1}, a_p(F) = a_p(F_1)$ and $\text{level}(F_1) = N$, by applying Lemma 48 we have $F \in \mathcal{L}_M^{\text{old}}(F_1)$, which gives the desired result. ■

In particular, we have determined all possible coefficients of any Katz eigenform which has irreducible mod p Galois representation from the coefficients of the corresponding Katz newform.

Corollary 52. *Let $F \in S_k(\Gamma_1(M), \overline{\mathbb{F}}_p)_{Katz}$ be a normalised Katz eigenform and assume that there exists a Katz newform $f \in S_{k'}(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{Katz}$ such that $\rho_F \cong \rho_f$. Suppose that $F(q) = \sum_{n\geq 1} a_n q^n$ and $f(q) = \sum_{n\geq 1} b_n q^n$ are their q-expansions. Then we have the following identities:*
(i). When prime $l \nmid Mp/N$, $a_l = b_l$.
(ii). When prime $l|M/N$ and $l|N$, $a_l = 0$ or $a_l = b_l$.
(iii). When prime $l|pM/N$ but $l \nmid N$, $a_l = 0$ or $a_l^2 - a_l b_l + \varepsilon(l) l^{k-1} = 0$.

Proof. We have F in the level and weight old space of f. We can write

$$F = \sum_{l|M/N} \alpha_l B_l^{N,M} g(q)$$

for some $g \in \mathcal{W}_k^{\text{old}}(f)$. We can easily observe that for primes $l \nmid Mp/N, a_l = a_l(g) = a_l(f) = b_l$. For a prime $l|M/N$ and $l|N$, if $l^m||M/N$, then

$$a_{l^m} = b_{l^m} + \sum_{t=0}^{m} \alpha_l b_{l^{m-t}}$$

and

$$a_{l^{m+1}} = b_{l^{m+1}} + \sum_{t=0}^{m} \alpha_t b_{l^{m+1-t}}.$$

This yields

$$\begin{aligned} a_{l^m} a_l &= b_{l^m} b_l + \sum_{t=0}^{m} b_{l^{m-t}} b_l \\ &= b_l a_{l^m}. \end{aligned}$$

This reduces to $a_l^m(a_l - b_l) = 0$ so $a_l = 0$ or $a_l = b_l$. If prime $l|M/N$ and $l \nmid N$ then we have

$$a_{l^m} a_l = b_{l^m} b_l - \varepsilon(l) l^{k'-1} b_{l^{m-1}} + \sum_{t=0}^{m-1} \alpha_t (b_{l^{m-t}} b_l - \varepsilon(l) l^{k'-1} b_{l^{m-t-1}}) + \alpha_m b_l.$$

Which reduces to

$$\begin{aligned} a_{l^{m+1}} &= b_l (b_{l^m} + \sum_{t=0}^{m} \alpha_t b_{l^{m-t}}) - \varepsilon(l) l^{k'-1} (b_{l^{m-1}} + \sum_{t=0}^{m-1} b_{l^{m-t-1}}) \\ &= b_l a_{l^m} - \varepsilon(l) l^{k'-1} a_{l^{m-1}}. \end{aligned}$$

Which is equivalent to $a_l^{m+1}(a_l^2 - a_l b_l + \varepsilon(l) l^{k'-1}) = 0$ so $a_l = 0$ or $a_l^2 - a_l b_l + \varepsilon(l) l^{k'-1} = 0$. Similar result for $l = p$ follows by considering that g is generated by $f(q), f(q^p), f(q^{p^2}), \ldots, f(q^{p^t})$ for some finite t. ■

One might ask if the converse is true. That is, given a newform f and some power series F which satisfies the conditions of above lemma, Lemma 52, then is F a modular form? Is F in the level and weight old space of f? The answer is true. In fact, we will prove a more general statement. We will make use of the following Lemma.

Lemma 53. *Any word in Frob and A can be written uniquely in the form*

$$A^{b_0} Frob A^{b_1} Frob \cdots A^{b_s} Frob A^c$$

with $s \in \mathbb{Z}_{\geq 0}, 0 \leq b_0, b_1, \ldots, b_s \leq p-1$ *and* $c \in \mathbb{Z}_{\geq 0}$.

Proof. The result follows from the identity $A^p\mathrm{Frob} = \mathrm{Frob}A$. Let k' be the weight associated to

$$A^{b_0}\mathrm{Frob}A^{b_1}\mathrm{Frob}\cdots A^{b_s}\mathrm{Frob}A^c f$$

where f is of weight k. Then we have

$$\frac{k' - p^{s+1}k}{p-1} = p^{s+1}c + p^s b_s + \cdots + pb_1 + b_0.$$

Here we have a fact that in the p-adic representation the digits are unique. So the integers $b_0, b_1, \ldots, b_s$ and c are unique. ■

Proposition 54. *Let* $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{Katz}$ *be a normalised Katz Hecke eigenform. Let* $N|M$ *and* $k' \geq k, k' \equiv k (mod\, p-1)$. *Let* $\mathcal{W}$ *be the level-weight old space of* f *in level* M *and weight* k'. *Then we have*
(a). For all $n \in \mathbb{N}, T_n\mathcal{W} \subset \mathcal{W}$
(b). The minimal polynomial of the Hecke operator T_l *for prime* l *equals*
(i). $X - a_l$ *if prime* $l \nmid Mp/N$
(ii). $(X - a_l)X^r$ *if for prime* l, $l^r || M/N$ *and* $l|N$
(iii). $(X^2 - a_l X + \varepsilon(l) l^{k-1})X^{r-1}$ *if for prime* l, $l^r||M/N$ *and* $l \nmid N$, $r \geq 1$
(c). If $k \geq 2$, *then the minimal polynomial of* T_p *on* $\mathcal{W}$ *is* $(X - a_l)X^r$, *where* r *is the maximum number of times Frob appears in a word in A and Frob taking from weight* k *into weight* k', *which is in the floor,* $r = \lfloor \frac{\log k'/k}{\log p} \rfloor$.
(d). If $k = 1$ *and* $k' \geq p$, *then the minimal polynomial of* T_p *on* $\mathcal{W}$ *is* $(X^2 - a_l X + \varepsilon(l))X^{r-1}$ *where* r *is the maximum integer such that* $k' \geq p^r$, *i.e.,* $r = \lfloor \frac{\log k'}{\log p} \rfloor$.

Proof. (a). Since T_n is linear it is enough to show that, if

$$g := B_{d_1}^{\alpha_1} B_{d_2}^{\alpha_2} \cdots B_{d_r}^{\alpha_r} A^{n_1} \mathrm{Frob}^{m_1} A^{n_2} \mathrm{Frob}^{m_2} \cdots A^{n_s} \mathrm{Frob}^{m_s} f, \text{ then } T_n g \in \mathcal{W}$$

where α_i, n_j and m_r are non negative integers. Here we drop the level descriptions from the level degeneracy maps. By $B_d^{\alpha_1}$ we mean a composition $B_d B_d \cdots B_d$ of $B_d, \alpha_1 \in \mathbb{Z}_{\geq 1}$ times. Since the Hecke algebra $\mathbb{T}_k(N)$ is generated by T_l and $\langle l \rangle$ for l in a set of primes, it suffices to consider the following relations. One can check them on the q-expansions: Let $h \in S_k(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$ be a cuspidal Katz modular form. Then

- We have $T_l B_l h(q) = T_l B_l (\sum_{n=1}^{\infty} a_n q^n) := T_l \sum_{n=1}^{\infty} b_n q^n = \sum_{n=1}^{\infty} c_n q^n$

  where $c_n \sum_{d|(n,l)} \varepsilon(d) d^{k-1} b_{nl/d^2} = b_{nl} = a_n$, since $l|\mathrm{level}(B_l h)$. Thus $T_l B_l = \mathrm{id}$ on $S_k(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$

- $$T_l B_{l'} = B_{l'} T_l \text{ for primes } l \neq l'.$$

  The formula for $T_l B_{l'} h(q) = T_l \sum_{n=1}^{\infty} a_n q^{nl'} := T_l \sum_{n=1}^{\infty} b_n q^n = \sum_{n=1}^{\infty} c_n q^n$ where $c_n = b_{nl} + \varepsilon(l) l^{k-1} b_{n/l}$, and the formula for $B_{l'} T_l h(q) = B_{l'} T_l \sum_{n=1}^{\infty} a_n q^n = B_{l'} \sum_{n=1}^{\infty} b_n q^n = \sum_{n=1}^{\infty} b_n q^{nl'} := \sum_{n=1}^{\infty} c_n q^n$ where $b_n = a_{nl} + \varepsilon(l) l^{k-1} a_{n/l}$ are the same. Thus, $T_l B_{l'} = B_{l'} T_l$ for primes $l \neq l'$.

- We need to calculate $T_l B_1^{N,M} f$. When $l|M, a_n(T_l B_1^{N,M} f) = a_{nl}(h)$ which is for $l \nmid N$ equal to $a_n(T_l h) - l^{k-1} a_{n/l}(\langle l \rangle h) = a_n(T_l h) - l^{k-1} B_l(\langle l \rangle h)$ and for $l|N, a_{nl}(h) = a_n(T_l h)$. Thus, for $l|M$

  $$T_l B_l^{N,M} h = \begin{cases} B_1^{N,M} T_l h - l^{k-1} B_l(\langle l \rangle) h & \text{if } l \mid N, \\ B_1^{N,M} T_l h & \text{if } l|N. \end{cases}$$

  When $l \nmid M$, for $l \nmid N$ we have $a_n(T_l B_1^{N,M} h) = a_{nl}(h) - l^{k-1} a_{n/l}(\langle l \rangle h) = a_n(T_l h) - l^{k-1} a_{n/l}(\langle l \rangle h) + l^{k-1} a_{n/l}(\langle l \rangle h) = a_n(T_l h)$. And for $l|N$ we have $a_n(T_l B_1^{N,M} h) = a_{nl}(h) = a_n(T_l h)$. Thus, for $l \nmid M, T_l B_1^{N,M} H = B_1^{N,M} T_l h$.

- Since weight of Ah is at least 2 we have

  $a_n(T_p A \sum_{n=1}^{\infty} a_n q^n) = a_{np}(h) = a_n(A T_p h - p^{k-1} a_{n/p}(\langle p \rangle h))$

  which is equal to $a_n(A T_p h) - a_n(\mathrm{Frob}\langle p \rangle h)$ when weight is 1 and equal to $a_n(A T_p h)$ when weight is at least 2.

- $$T_l A = A T_l \text{ for prime } l \neq p.$$

  Computing both side expressions with q-expansion gives the result.

- Since weight of Frob$h$ is at least p we have

  $a_n(T_P\text{Frob}h) = a_{np}(\text{Frob}h) = a_n(h)$. Thus, $T_p\text{Frob}h = \text{id}$.

- $$T_l\text{Frob} = \text{Frob}T_l \text{ for prime } l \neq p.$$

  Here the result follows by computing both side expressions with q-expansion.

- $$\langle l \rangle B_l = 0 \text{ for all prime } l \neq p.$$

  This is because $l|\text{level}(B_l h)$.

- $$\langle l \rangle B_{l'} = B_{l'} \langle l \rangle \text{ for all primes } l \text{ and } l' :$$

  Without loss of generality we can assume $l \neq p$. Then we have $\langle l \rangle B_{l'} \sum_{n=1}^{\infty} a_n q^n = \langle l \rangle \sum_{n=1}^{\infty} a_n q^{nl} = \sum_{n=1}^{\infty} \varepsilon(l) a_n q^{nl}$ on the other hand, $B_{l'} \langle l \rangle \sum_{n=1}^{\infty} a_n q^n = B_{l'} \varepsilon(l) \sum_{n=1}^{\infty} a_n q^n = \sum_{n=1}^{\infty} \varepsilon(l) a_n q^{nl}$.

  Similarly we have

- $$\langle l \rangle A = A \langle l \rangle \text{ for all prime } l.$$

- $$\langle l \rangle \text{Frob} = \text{Frob} \langle l \rangle \text{ for all prime } l.$$

Thus using the above relations we have that the inclusion $T_n g \in \mathcal{W}$ holds. As we said above, the result follows from above relations because the Hecke algebra $\mathbb{T}_k(N)$ is generated by T_l and $\langle l \rangle$ for l in a set of primes.

(b). (i). We have $(T_l - a_l) f = 0$, so for prime $l \nmid Mp/N$

$$(T_l - a_l)(B_{d_1}^{\alpha_1} \cdots \text{Frob}^{m_s}) f = (B_{d_1}^{\alpha_1} \cdots \text{Frob}^{m_s})(T_l - a_l) f = 0.$$

We can see that $B_{d_1}^{\alpha_1} \cdots \text{Frob}^{m_s} f \neq 0$ as long as f is not zero. Thus, $x - a_l$ is the minimal polynomial for this case.

(ii). Let $l|N$ and $l^r||M/N$ for $r \geq 1$. Then consider the space of weight-level old space of f in weight k and level Nl^r, denoted $\mathcal{W}_{k,N}^{k,Nl^r}$. On this space, according to the computations in (a) the matrix of T_l on $\mathcal{W}_{k,N}^{k,Nl^r}$ with respect to the basis $B_1 f, B_l f, \ldots, B_{l^r} f$ is the one in ([37], Proposition 4). That is we have $T_l B_{l^r} f = B_{l^{r-1}} f$ for $r \in \mathbb{Z}_{\geq 1}$. Thus with respect to the basis $B_1 f, B_l f, \ldots, B_{l^r} f$ we have the matrix

$$\begin{bmatrix} a_l(f) & 1 & 0 & 0 & \cdots & 0 \\ -\delta l^{k-1}\varepsilon(l) & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ & & \vdots & & & \\ 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\delta = 1$ if $l \nmid N$ and $\delta = 0$ otherwise. Its minimal polynomial is $(X - a_l)X^r$.

As T_l commutes with A and Frob, $(X - a_l)X^r$ is also the minimal polynomial on $\mathcal{W}_{k,N}^{k',Nl^r}$.

As T_l commutes with B_d for $(d, pl) = 1, d|M/N$, $(X - a_l)X^r$ is also the minimal polynomial of T_l on $\mathcal{W}$.

(iii). Here $\delta = 1$, so the minimal polynomial is $(X^2 - a_l X + \varepsilon(l) l^{k-1})X^{r-1}$.

(c). If we fix a basis of type of above Lemma for the space $\mathcal{W}_k^{k'}$. The matrix of T_p with respect to these basis is the same as in (b). Thus, we get the same type of minimal polynomial.

(d). The same argument as above works. ■

Remark 55. *Let us recall that in classical newform theory, the newspace has a basis consisting of newforms. However, this cannot be generalised to Katz modular forms. A counterexample occurs in $S_1(\Gamma_0(229), \overline{\mathbb{F}}_2)_{Katz}$. The associated Hecke algebra $\mathbb{T}$ is a local 2-dimensional $\overline{\mathbb{F}}_2$-algebra, hence it has a unique attached Katz eigenform, whereas $S_1(\Gamma_0(229), \overline{\mathbb{F}}_2)_{Katz}$ is a 2-dimensional $\overline{\mathbb{F}}_2$-vector space. Hence it does not have a basis of Katz newforms.*

2.2 Reducible case

In his thesis [36], Weisinger and separately Linowitz and Thompson in [25] developed a newform theory for the space of classical Eisenstein series. A classical Eisenstein newform is uniquely determined by the signs of its Hecke eigenvalues with respect to any set of primes with density greater than 1/2. In this section, we will prove a strong multiplicity one result for Katz Eisenstein series. Then later we will show that under some condition a reducible mod p Galois representation arises from a normalised Katz eigenform with optimal level.

In Section 1.1.3 we have defined a classical Eisenstein series. In this section we study their mod p reductions.

Hereafter we will assume that all Dirichlet characters which we consider are primitive and we are not in the situation where our Eisenstein series have weight $k = 2$ and level $N = 1$. Let $\mathrm{Den}\left(\frac{B_k^{\varepsilon_1}}{2k}\right)$ denotes the denominator of $\frac{B_k^{\varepsilon_1}}{2k}$ when it is written in a reduced fractional form.

We can define Katz Eisenstein series by considering the mod p reduction of Eisenstein series. We define the Katz eigenform $E_k^{\varepsilon,\varepsilon'}$ as the mod p reduction of the associated Eisenstein series $E_k^{\varepsilon_1,\varepsilon_2}$ by assuming that prime $p \nmid \mathrm{Den}\left(\frac{B_k^{\varepsilon_1}}{2k}\right)$ in case $\varepsilon_2 = 1$. All coefficients of $E_k^{\varepsilon_1,\varepsilon_2}$ belong to $\mathcal{O}$ where $\mathcal{O}$ is the ring of integers of some finite extension of $\mathbb{Q}_p$. In order to show that it is a Katz modular form over $\mathcal{O}$, we apply Theorem 32. Note that we can do this after possibly considering it in a higher level. So, we have a well defined reduction of $E_k^{\varepsilon_1,\varepsilon_2}$ at some prime above p, $E_k^{\varepsilon,\varepsilon'}$ which is a Katz Eisenstein series. Similarly by taking positive integer $t \geq 1$ we can define a Katz Eisenstein series $E_k^{\varepsilon,\varepsilon',t}$. A normalized Katz eigenform $f(q) = E_k^{\varepsilon,\varepsilon'}(q) \in M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$ is called a *Katz new Eisenstein series* if it satisfies the condition of Definition 1, i.e. it is a Katz newform.

Definition 56. *A normalised Katz Eisenstein series $f \in M_k(\Gamma_1(N), \varepsilon, \overline{\mathbb{F}}_p)_{Katz}$ is called a* Katz new Eisenstein series *if f has l-minimal level for any prime l and has minimal weight k.*

We know that the mod p Galois representations associated to Eisenstein series are reducible. One reference could be [[38], §2, pg. 1415]. Let $f = E_k^{\varepsilon,\varepsilon',t} \in M_k(\Gamma_1(N), \chi, \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$ be any Katz Eisenstein series. Then we have $\rho_f \cong \varepsilon' \oplus \varepsilon\chi_p^{k-1}$ which is unramified outside pN with the property that $\mathrm{tr}(\rho_f(\mathrm{Frob}_l)) = f(T_l)$ and $\det(\rho_f(\mathrm{Frob}_l))$
$= \chi(l)l^{k-1}$ for all primes $l \nmid pN$. In fact the converse also holds. Any semi-simple reducible mod p Galois representation comes from some twist of an Eisenstein series.

Let ε and ε' be primitive Dirichlet characters with values in $\overline{\mathbb{F}}_p$ and let ε_1 and ε_2 be their respective complex liftings with the same conductors and the same orders. Then we start by proving the existence of Katz new Eisenstein series.

Proposition 57. *Let $1 \leq k \leq p-1$ and $N(\rho) = cond(\varepsilon) \cdot cond(\varepsilon')$ be positive integers. Assume $k \neq 2$ if $\varepsilon = \varepsilon' = 1$. Assume also $p \nmid Den\left(\frac{B_k^{\varepsilon_1}}{2k}\right)$ if $\varepsilon_2 = 1$. Then $f = E_k^{\varepsilon,\varepsilon'} \in M_k(\Gamma_1(N(\rho)), \overline{\mathbb{F}}_p)_{Katz}$ is a Katz new Eisenstein series such that $\rho_f \cong \varepsilon' \oplus \varepsilon\chi_p^{k-1}$.*

Proof. This immediately follows from the discussion above. Here $E_k^{\varepsilon,\varepsilon'}$ is Katz new Eisenstein series as it is a normalised eigenform with optimal level and weight. This is because both characters ε and ε' are primitive and the product of their conductors is the conductor of the representation and k is in the range $1 \le k \le p-1$. ■

Corollary 58. *Let $F \in M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{Katz}$ be a normalised Katz eigenform such that $\rho_F \cong \varepsilon' \oplus \varepsilon\chi_p^b$ where $0 \le b \le p-2$. Suppose $N = cond(\rho_F)$. Assume $b \neq 1$ if $cond(\rho_F) = 1$. Also assume $p \nmid Den\left(\frac{B_{b+1}^{\varepsilon_1}}{2(b+1)}\right)$ if $\varepsilon_2 = 1$. Then F is in the weight old space of $E_{b+1}^{\varepsilon,\varepsilon'}$ in the weight k.*

Proof. By Proposition 57 above, $f = E_{b+1}^{\varepsilon,\varepsilon'}$ is a Katz new Eisenstein series such that $\rho_f \cong \rho_F$. By Proposition 41 we can write $k = (b+1)+m(p-1)$ for some non-negative integer m. Then by comparing coefficients of $\theta^{p-1}F$ and $\theta^{p-1}A^m f$ using Lemma 48 we have $a_l(F) = a_l(f)$ for all primes $l|N$ as N is optimal level. Then Corollary 51 completes the proof. ■

Proposition 59. *Let $f(q) = E_k^{\varepsilon,\varepsilon'}(q) \in M_k(\Gamma_1(N), \overline{\mathbb{F}}_p)_{Katz}$ and $g(q) = E_{k'}^{\chi,\chi'}(q) \in M_{k'}(\Gamma_1(N'), \overline{\mathbb{F}}_p)_{Katz}$ be Katz new Eisenstein series with $a_l(f) = a_l(g)$ for each l in a set of primes of density 1. If $k \not\equiv 1(mod\ p-1)$, then $f = g, k = k', N = N', \varepsilon = \chi$ and $\varepsilon' = \chi'$. If $k \equiv 1(mod\ p-1)$, then the same conclusion holds except that the characters could be exchanged: $\varepsilon' = \chi$ and $\varepsilon = \chi'$.*

Proof. From the hypothesis we have $\rho_f \cong \rho_g$ or $\varepsilon' \oplus \varepsilon\chi_p^{k-1} \cong \chi' \oplus \chi\chi_p^{k'-1}$. Then by the definition of Katz new Eisenstein series the levels and weights of the forms are optimal so they are equal. Then from $\varepsilon' \oplus \varepsilon\chi_p^{k-1} \cong \chi' \oplus \chi\chi_p^{k-1}$ we have the following two cases. (i). $\varepsilon' = \chi'$ and $\varepsilon\chi_p^{k-1} = \chi\chi_p^{k-1}$ or (ii). $\varepsilon' = \chi\chi_p^{k-1}$ and $\varepsilon\chi_p^{k-1} = \chi'$. The first case gives $\varepsilon' = \chi'$ and $\varepsilon = \chi$ as a Galois representations while the second case gives $\varepsilon' = \chi$ and $\varepsilon = \chi'$ provided that $k \equiv 1, p(\text{mod } p-1)$. This completes the proof. ■

Remark 60. *It is not always the case to obtain a cuspidal eigenform with both optimal weight and optimal level which gives rise to a given reducible mod p Galois representation. For example, there exists a modular form $f \in S_{28}(\Gamma_1(1), \overline{\mathbb{F}}_7)_{Katz}$ such that $\rho_f \cong \rho_{\overline{E}_4}$ but there is no cuspidal eigenform $g \in S_4(\Gamma_1(1), \overline{\mathbb{F}}_7)_{Katz}$ such that $\rho_g \cong \rho_f$. Here $\overline{E}_4$ is the mod 7 reduction of E_4.*

Consider again Theorem 3 when the mod p representation of the modular form is reducible. To be precise let $F \in S_{k'}(\Gamma_1(M), \overline{\mathbb{F}}_p)_{\text{Katz}}$ be a normalised Katz eigenform with reducible mod p Galois representation $\rho_F \cong \varepsilon'\chi_p^a \oplus \varepsilon\chi_p^b$ where $\det \rho_F = \varepsilon\varepsilon'\chi_p^{k-1}$

for some $a, b \in \mathbb{Z}/(p-1)\mathbb{Z}$ such that $1 \leq k \leq p+1$ and $k-1 \equiv b-a (\text{mod } p-1)$. Assume that ε and ε' are primitive when considered as Dirichlet characters.

Then we show that there exists a normalised Katz cuspidal eigenform g of optimal level such that $\rho_F \cong \rho_g$. The weight may not be optimal.

Lemma 61. *Let $F \in S_{k'}(\Gamma_1(M), \overline{\mathbb{F}}_p)_{Katz}$ be a normalised Katz eigenform such that $\rho_F \cong \varepsilon' \chi_p^a \oplus \varepsilon \chi_p^b$ where $1 \leq a \leq p-1, 0 \leq b \leq p-2$ and k is a positive integer such that $1 \leq k \leq p+1$. Assume that $(N(\rho_F), k) \neq (1, 2)$. Then we have the following four cases:*
(i). If $\varepsilon_2 \neq 1$ and $k-1 \equiv b-a (mod\, p-1)$, then ρ_F comes from $g = \theta^a(\overline{E_k^{\varepsilon_1,\varepsilon_2}})$, i.e., $\rho_F \cong \rho_g$.
(ii). If $\varepsilon = \varepsilon' = 1, k = 2, b-a \equiv 1 (mod\ p-1)$ and $p \neq 2, 3$, then ρ_F comes from $g = \theta^a(\overline{E_{p^2+1}^{1,1}})$.
(iii). If $\varepsilon_1 = \varepsilon_2 = 1, k-1 \equiv b-a (mod\, p-1)$, $k \neq 2$ and $p \nmid Den\left(\frac{B_k}{2k}\right)$, then ρ_F comes from $g = \theta^a(\overline{E_k^{1,1}})$.
(iv). If $\varepsilon_1 \neq 1, \varepsilon_2 = 1, k-1 \equiv b-a (mod\ p-1)$ and $p \nmid Den\left(\frac{B_k^{\varepsilon_1}}{2k}\right)$, then ρ_F comes from $g = \theta^a(\overline{E_k^{\varepsilon_1,1}})$.

Proof. In the case when $a = p-1$ we have the following cases. (i). We have that ρ_F comes from $E_k^{\varepsilon,\varepsilon',t}$ for some positive integer t. Then since $(N(\rho), k) \neq (1, 2)$ taking $t = 1$ gives a normalised eigenform $E_k^{\varepsilon,\varepsilon'}$ with an optimal level. Here $a_0(E_k^{\varepsilon_1,\varepsilon_2}) = 0$, so we can take modulo p reduction and apply the theta operator to get a normalised cuspidal Katz eigenform $g = \theta^{p-1}(\overline{E_k^{\varepsilon_1,\varepsilon_2}})$ such that $\rho_F \cong \rho_g$.
(ii). Let $\rho_F \cong \overline{\rho}_{E_2^{1,1,t}}$ for some positive integer t. Then for prime $p \neq 2, 3$ we have $\rho_{\theta^{p-1}\left(\overline{E_2^{1,1,t}}\right)} \cong \rho_{\overline{E_{p^2+1}^{1,1}}}$. Then set $g = \theta^{p-1}\left(\overline{E_{p^2+1}^{1,1}}\right)$.
(iii). Here the assumption $p \nmid \text{Den}\left(\frac{B_k^{\varepsilon_1}}{2k}\right)$ implies that the modulo p reduction is well defined. Similarly (iv) holds.
On the other hand, when $a \neq p-1$, by applying the above method to the twist $\theta^{p-(1+a)}F$ one can get modular form g'. Then use the relation $\rho_{\theta^{p-1}F} \cong \rho_F \cong \rho_{\theta^a g'}$. ■

Let us assume that we are in the same notation and under the same assumptions as in Lemma 61. Then as an immediate consequence of Lemma 48 we have

Theorem 62. *Let $F \in S_{k'}(\Gamma_1(M), \overline{\mathbb{F}}_p)_{Katz}$ be the above normalised Katz eigenform which we consider. Suppose that $a_p(F) = 0$ and g is the modular form associated to F as in Lemma 61. Then up to a suitable power multiple of the Hasse invariant, F is in the level old space of g.*

As an application of Corollary 52 we can compute all possible coefficients of any of the above type cuspidal Katz eigenform which has reducible mod p Galois representation in terms of the associated normalised Katz eigenform of optimal level.

Let $F \in S_k(\Gamma_1(M), \overline{\mathbb{F}}_p)_{\mathrm{Katz}}$ be a normalised Katz eigenform. Suppose that $a_p(F) \neq 0$ and ρ_F is reducible. Then by [[18], Theorem 4.12] we can assume that $2 \leq k \leq p+1$ is an optimal weight. Then $\rho_F \cong \varepsilon' \oplus \varepsilon\chi_p^{k-1}$ and suppose that $(N(\rho), k) \neq (1, 2)$ and $p \nmid 2k \cdot \mathrm{Den}(B_k^{\varepsilon_1})$ when $\varepsilon_2 = 1$. Then by Lemma 48 we have $\theta F \in \mathcal{L}_M^{\mathrm{old}}(\theta(\overline{E_k^{\varepsilon_1,\varepsilon_2}}))$.

Chapter 3

Numerical Examples

In this section, we will give numerical examples which illustrates Theorem 3. In our last example, we determine all the possible coefficients of a Katz eigenform which admits a reducible Galois representation. The modular forms we consider are taken from the L-functions and Modular Forms Database (LMFDB).

3.1 Example 1

First let us consider the elliptic curve $E : y^2 + xy = x^3 - 1$, which is of level 431. We can get from the database that E corresponds to a weight 2 mod 2 Katz eigenform

$$g(q) = q + q^2 + q^3 + q^4 + q^5 + q^6 + q^8 + q^{10} + q^{11} + q^{12} + q^{15} + q^{16} + q^{19} + q^{20} + \mathcal{O}(q^{22})$$

labeled on the database by Newform orbit 431.2.a.a. We can also observe that ρ_g comes also from a Katz newform $f \in S_1(\Gamma_1(431), \overline{\mathbb{F}}_2)_{\text{Katz}}$ which is given by

$$f(q) = q + q^3 + q^4 + q^5 + q^{11} + q^{12} + q^{15} + q^{16} + q^{19} + q^{20} + \mathcal{O}(q^{23})$$

[LMFDB, Newform orbit 431.1.b.a].

Then from Corollary 51 we have the relation

$$g(q) = Af(q) + \alpha \text{Frob} f(q), \text{ for some } \alpha \in \overline{\mathbb{F}}_2. \tag{3.1}$$

Computationally we can find $\alpha = a_2(g) - a_2(Af) = 1$. Plugging $\alpha = 1$ yields the identity we want

$$\begin{aligned} &Af(q) + \alpha \text{Frob} f(q) \\ &\qquad = q + q^3 + q^4 + q^5 + q^{11} + q^{12} + q^{15} + q^{16} + q^{19} + q^{20} + \mathcal{O}(q^{23}) + \\ &\qquad q^2 + q^6 + q^8 + q^{10} + q^{22} + q^{24} + q^{30} + q^{32} + q^{38} + q^{40} + \mathcal{O}(q^{46}) \\ &= q + q^2 + q^3 + q^4 + q^5 + q^6 + q^8 + q^{10} + q^{11} + q^{12} + q^{15} + q^{16} + q^{19} + q^{20} + q^{22} + \mathcal{O}(q^{23}) = g(q). \end{aligned}$$

3.2 Example 2

Now let us have an example which has degeneracy both in the level and in the weight. Let us start from a classical newform f in weight 2 and level 89 with trivial character, labeled on LMFDB by Newform orbit 89.2.a.a,

$$f(q) = q - q^2 - q^3 - q^4 - q^5 + q^6 - 4q^7 + 3q^8 - 2q^9 + q^{10} - 2q^{11} + q^{12} + \\ 2q^{13} + 4q^{14} + q^{15} - q^{16} + 3q^{17} + 2q^{18} - 5q^{19} + q^{20} + 4q^{21} + \mathcal{O}(q^{22}).$$

When we reduce it mod 5 we get

$$\overline{f}(q) = q + 4q^2 + 4q^3 + 4q^4 + 4q^5 + q^6 + q^7 + 3q^8 + 3q^9 + q^{10} + 3q^{11} + q^{12} + \\ 2q^{13} + 4q^{14} + q^{15} + 4q^{16} + 3q^{17} + 2q^{18} + q^{20} + 4q^{21} + \mathcal{O}(q^{22}).$$

We have the situation that $a_3(f) \equiv (3+1) (\text{mod } 5)$, so f satisfies the level raising condition at the prime 3 mod 5. This means the mod 5 representation $\overline{\rho}_f$ arises from a newform of level $3 \cdot 89 = 267$ and weight 2 modulo 5. To include degeneracy in the weight we go to weight 10 and level 267 and search for mod 5 eigensystems which correspond to the mod 5 eigenform $\overline{f}$. By using Magma we obtain two systems with coefficients that possibly differing at primes 3, 5 and 89. Indexed by consecutive primes, the eigenvalues of the normalised eigenforms, i.e., the coefficients at the prime indices of the forms, are:

$$4, 1, 4, 1, 3, 2, 3, 0, 2, 0, 1, 3, 0, 3, 3, 2, 4, 1, 2, 0, 2, 4, 2, 4, 4, 0, 1, 0, 1, 2, 3, 1, 3, 2, 4, \\ 2, 3, 0, 3, 4, 4, 2, 0, 1, 3, 2$$

and

$$4, 1, 0, 1, 3, 2, 3, 0, 2, 0, 1, 3, 0, 3, 3, 2, 4, 1, 2, 0, 2, 4, 2, 4, 4, 0, 1, 0, 1, 2, 3, 1, 3, 2, 4, \\ 2, 3, 0, 3, 4, 4, 2, 0, 1, 3, 2.$$

The first eigensystem gives a mod 5 Katz eigenform

$$\overline{g}(q) = q + 4q^2 + q^3 + 4q^4 + 4q^5 + 4q^6 + q^7 + 3q^8 + q^9 + q^{10} + 3q^{11} + 4q^{12} + \\ 2q^{13} + 4q^{14} + 4q^{15} + 4q^{16} + 3q^{17} + 4q^{18} + q^{20} + q^{21} + 2q^{22} + 2q^{23} + \mathcal{O}(q^{24}).$$

Then by Theorem 3 we have the following equation

$$\overline{g}(q) = \beta_1 A^2 B_1^{89,267}\overline{f}(q) + \beta_2 \text{Frob} B_1^{89,267}\overline{f}(q) + \beta_3 A^2 B_3^{89,267}\overline{f}(q) + \beta_4 \text{Frob} B_3^{89,267}\overline{f}(q),$$

for some $\beta_i, \in \overline{\mathbb{F}}_5$ $(i = 1,2,3,4)$. It is easy to see that $\beta_1 = 1, \beta_2 = 0, \beta_3 = 2$ and $\beta_4 = 0$. Then we have

$$\begin{aligned}
&A^2 B_1^{89,267}\overline{f}(q) + 2A^2 B_3^{89,267}\overline{f}(q) \\
&\quad = q + 4q^2 + 4q^3 + 4q^4 + 4q^5 + q^6 + q^7 + 3q^8 + 3q^9 + q^{10} + 3q^{11} + q^{12} + \\
&\qquad 2q^{13} + 4q^{14} + q^{15} + 4q^{16} + 3q^{17} + 2q^{18} + q^{20} + 4q^{21} + \mathcal{O}(q^{22}) \\
&\qquad + 2q^3 + 3q^6 + 3q^9 + 3q^{12} + 3q^{15} + 2q^{18} + 2q^{21} + q^{24} + q^{27} + \mathcal{O}(q^{30}) \\
&\quad = q + 4q^2 + q^3 + 4q^4 + 4q^5 + 4q^6 + q^7 + 3q^8 + q^9 + q^{10} + 3q^{11} + 4q^{12} + \\
&\qquad 2q^{13} + 4q^{14} + 4q^{15} + 4q^{16} + 3q^{17} + 4q^{18} + q^{20} + q^{21} + \mathcal{O}(q^{22}) = \overline{g}(q).
\end{aligned}$$

The second eigensystem corresponds to a mod 5 Katz eigenform

$$\begin{aligned}
\overline{g}'(q) &= q + 4q^2 + q^3 + 4q^4 + 4q^6 + q^7 + 3q^8 + q^9 + 3q^{11} + 4q^{12} + \\
&\qquad 2q^{13} + 4q^{14} + 4q^{16} + 3q^{17} + 4q^{18} + q^{21} + 2q^{22} + 2q^{23} + \mathcal{O}(q^{24}).
\end{aligned}$$

Then by Theorem 3 we have the following equation

$$\overline{g}'(q) = \beta_1 A^2 B_1^{89,267}\overline{f}(q) + \beta_2 \mathrm{Frob} B_1^{89,267}\overline{f}(q) + \beta_3 A^2 B_3^{89,267}\overline{f}(q) + \beta_4 \mathrm{Frob} B_3^{89,267}\overline{f}(q),$$

for some $\beta_i, \in \overline{\mathbb{F}}_5$ $(i = 1,2,3,4)$. By an easy calculation on the q-expansions we have $\beta_1 = 1, \beta_2 = 1, \beta_3 = 2$ and $\beta_4 = 2$. Then we can check the compatibility of the coefficients

$$\begin{aligned}
&A^2 B_1^{89,267}\overline{f}(q) + \mathrm{Frob}\overline{f}(q) + 2A^2 B_3^{89,267}\overline{f}(q) + 2\mathrm{Frob} B_3^{89,267}\overline{f}(q) \\
&\quad = q + 4q^2 + 4q^3 + 4q^4 + 4q^5 + q^6 + q^7 + 3q^8 + 3q^9 + q^{10} + 3q^{11} + q^{12} + \\
&\qquad 2q^{13} + 4q^{14} + q^{15} + 4q^{16} + 3q^{17} + 2q^{18} + q^{20} + 4q^{21} + \mathcal{O}(q^{22}) \\
&\qquad + q^5 + 4q^{10} + 4q^{15} + 4q^{20} + 4q^{25} + q^{30} + q^{35} + \mathcal{O}(q^{40}) \\
&\qquad + 2q^3 + 3q^6 + 3q^9 + 3q^{12} + 3q^{15} + 2q^{18} + 2q^{21} + q^{24} + q^{27} + \mathcal{O}(q^{30}) \\
&\qquad + 2q^{15} + 3q^{30} + 3q^{45} + \mathcal{O}(q^{60}) \\
&\quad = q + 4q^2 + q^3 + 4q^4 + 4q^6 + q^7 + 3q^8 + q^9 + 3q^{11} + 4q^{12} + \\
&\qquad 2q^{13} + 4q^{14} + 4q^{16} + 3q^{17} + 4q^{18} + q^{21} + \mathcal{O}(q^{22}) = \overline{g}'(q).
\end{aligned}$$

3.3 Example 3

Let us consider one example of a Katz eigenform which has a reducible representation. We can take $k = 2$ and $p = 5$ and choose a prime N satisfying the condition $N^2 \equiv$

1(mod 5) and that the largest prime factor of $N-1$ is greater than $N^{1/4}$, say $N = 11$. Then by Theorem 1 of [3] and Mazur's Theorem [[27], Proposition 5.12] the reducible representation $\mathbf{1} \oplus \chi_5$ arises from a newform of level 11 and weight 2, labeled on the LMFDB by Newform orbit 11.2.a.a and given by

$$f(q) = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} - 2q^{12} + 4q^{13} + 4q^{14} - q^{15} - 4q^{16} - 2q^{17} + 4q^{18} + \mathcal{O}(q^{20}).$$

One can check that $a_l \equiv (1 + l) (\text{mod } 5)$ for primes $l \nmid 5 \cdot 11$. We have a modular form $g \in S_2(\Gamma_1(66), \mathbb{C})$ [LMFDB, Newform orbit 66.2.a.c] given by

$$g(q) = q + q^2 + q^3 + q^4 - 4q^5 + q^6 - 2q^7 + q^8 + q^9 - 4q^{10} + q^{11} + q^{12} + 4q^{13} - 2q^{14} - 4q^{15} + q^{16} - 2q^{17} + q^{18} + \mathcal{O}(q^{20})$$

such that the mod 5 reduction of the representations ρ_f and ρ_g are the same.

Here using Corollary 48 we can compare $\theta^{p-1} f$ and $\theta^{p-1} g$, which leads to the the congruence on q-expansions

$$a_n(g(q)) \equiv a_n\Big(\sum_{d|66/11} \alpha_d B_d^{11,66} f(q)\Big) \text{mod } p$$

for all n such that $5 \nmid n$. We can see that $\alpha_2 = a_2(g) - a_2(f) = 3, \alpha_3 = a_3(g) - a_3(f) = 2$ and $\alpha_6 = (a_2(g) - a_2(f))(a_3(g) - a_3(f)) = 6$.
To verify the relation we first have

$$\overline{g}(q) = q + q^2 + q^3 + q^4 + q^5 + q^6 + 3q^7 + q^8 + q^9 + q^{10} + q^{11} + q^{12} + 4q^{13} + 3q^{14} + q^{15} + q^{16} + 3q^{17} + q^{18} + \mathcal{O}(q^{20})$$

and

$$\overline{f}(q) = q + 3q^2 + 4q^3 + 2q^4 + q^5 + 2q^6 + 3q^7 + 3q^9 + 3q^{10} + q^{11} + 3q^{12} + 4q^{13} + 4q^{14} + 4q^{15} + q^{16} + 3q^{17} + 4q^{18} + \mathcal{O}(q^{20}).$$

Then we can compute

$$
\begin{aligned}
&B_1^{11,66}\overline{f}(q) + \overline{\alpha}_2 B_2^{11,66}\overline{f}(q) + \overline{\alpha}_3 B_3^{11,66}\overline{f}(q) + \overline{\alpha}_6 B_6^{11,66}\overline{f}(q) \\
&\quad = q + 3q^2 + 4q^3 + 2q^4 + q^5 + 2q^6 + 3q^7 + 3q^9 + 3q^{10} + q^{11} + 3q^{12} + 4q^{13} + 4q^{14} \\
&\qquad + 4q^{15} + q^{16} + 3q^{17} + 4q^{18} + \mathcal{O}(q^{20}) \\
&\qquad + 3q^2 + 4q^4 + 2q^6 + q^8 + 3q^{10} + q^{12} + 4q^{14} + 4q^{18} + \mathcal{O}(q^{20}) \\
&\qquad + 2q^3 + q^6 + 3q^9 + 4q^{12} + 2q^{15} + 4q^{18} + \mathcal{O}(q^{21}) \\
&\qquad + q^6 + 3q^{12} + 4q^{18} + \mathcal{O}(q^{24}) \\
&\equiv q + q^2 + q^3 + q^4 + q^5 + q^6 + 3q^7 + q^8 + q^9 + q^{10} + q^{11} + q^{12} + 4q^{13} + 3q^{14} + q^{15} + q^{16} + 3q^{17} \\
&\qquad + q^{18} + \mathcal{O}(q^{20}) = \overline{g}(q).
\end{aligned}
$$

Thus as described above we can determine all coefficients of g except the coefficients indexed by $5n, a_{5n}(g)$ for all positive integers n from the q-expansion of f.

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