Boy or Girl?

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We investigate various assertions concerning the probability of some children being boys or girls. We suppose that being born a boy (respectively, a girl) happens exactly in one case out of two. If you prefer a different and childfree setting, you can translate the problems into considerations about a fair coin, and tossing either heads or tails.
Contrary to our intuition, flipping a fair coin has no memory about previous results, namely if I have obtained ten times heads, then on the eleventh toss I still have 50% probability of obtaining heads. The fact that obtaining eleven heads in a row is very unlikely does not change the fact that obtaining eleven heads or obtaining ten heads and then one tails are equally probable unlikely events. So if I tell you:

> My oldest child is a daughter: is my second child a boy or a girl?

then you have no reason to prefer one answer over the other. Going back to our coin example, if my first ten children are daughters, then there is still 50% probability that my eleventh child is also a daughter.

If I have two children, then the chance that I have two daughters is 25% (1 case out of 4). This is the same probability as throwing twice heads with two tosses of a coin. But suppose that I tell you:

> I have two children. I have at least a daughter. How likely it is that I have two girls?

Since you exclude the case in which I have two boys, the 1 favorable case out of 4 becomes 1 favorable case out of 3, so the probability is 1/3. This leads to the following assertion, which may look paradoxical (because the female presence is guaranteed):

> I have two children. I have at least a daughter. Then it is more likely than not that I have a boy too.

Let's reformulate the above probabilities with frequencies. So let us consider all families with two children, and let us suppose that the boys and girls are evenly distributed: one fourth of all families has two girls, one fourth of all families has two boys, one half of all families has one boy and one girl.

> From all families with two children, at least one of whom is a girl, a family is chosen at random. What is the probability that there are two girls?

Here the problem is identical to the previous one with the parent stating to have at least one daughter, so the probability is 1/3. Now consider the following:

> From all families with two children, one child is selected at random, and the sex of that child is specified to be a girl. What is the probability that in that family there are two girls?

In this case the probability is 1/2. This is because we are singling out one child, the chosen one. And this is analogous to claiming that the older child is a daughter.

Now consider the birth date of the children, and suppose that there is equal probability for a child to be born on Monday, on Tuesday, and so on. Also suppose that on each day of the week half of the born children are girls and half are boys. Now consider the following problem:

> Mrs. Smith has two children, and one of them is a girl, who was born on a Tuesday. What is the probability that the second child is a girl?

At a first glance the information on the day of the week is irrelevant. But it is relevant, as we will see. Recall that being born on a Tuesday is an event which has probability 1/7.
Let us call $\mathrm{P(GirlGirl \mid GirlTue)}$ the probability of having two girls among two children, given that one child is a girl born on Tuesday. Let us call $\mathrm{P(GirlTue \mid GirlGirl)}$ the probability that at least one girl is born on a Tuesday, given that both children are girls. By Bayes' Theorem we have

$$\mathrm{P(GirlGirl \mid GirlTue)} = \frac{\mathrm{P(GirlTue \mid GirlGirl)} \times \mathrm{P(GirlGirl)}}{\mathrm{P(GirlTue)}}$$

where $\mathrm{P(GirlGirl)} = 1/4$ is the probability of having two girls and $\mathrm{P(GirlTue)}$ is the probability of having at least one child which is a girl born on Tuesday. We have, by considering the complementary probabillity $\mathrm{P(GirlTue \mid GirlGirl)} = \frac{13}{49}$ (because having twice one day out of the six days which are not Tuesday has probability $\frac{6}{7} \cdot \frac{6}{7}$).
Let us now compute $\mathrm{P(GirlTue)}$. If we have two girls (which happens with probability 1/4), then the relative probability is $\frac{13}{49}$ (see above) while if we have exactly one girl (which happens with probability 1/2), then the relative probability is $\frac{1}{7}$. So we get

$$\mathrm{P(GirlTue)} = \frac{1}{4} \cdot \frac{13}{49} + \frac{1}{2} \cdot \frac{1}{7} = \frac{15}{64}$$

and finally we can compute

$$\mathrm{P(GirlGirl \mid GirlTue)} = \frac{13}{27} = 0,\overline{481}\,.$$

References

[1] Alex Bellos, *So you think you have got problems?*, Guardian Faber, Faber & Faber Ltd, 2019.

[2] Wikipedia contributors. *Boy or Girl paradox.* Wikipedia, The Free Encyclopedia, `https://en.wikipedia.org/wiki/Boy_or_Girl_paradox`, retrieved December 31, 2020.

Exercises for the reader

1. Mrs. Smith is the mother of two. We meet her walking along the street with a young girl whom she proudly introduces as her daughter. What is the probability that Mrs. Smith's other child is also a girl?

2. Mrs. Smith is the mother of two. She is the mother of one girl which is born on a leap year. None of the children is born on February 29. What is the probability that the other child is a girl?

Solutions to the exercises

1. We should suppose that it is equally probable that the walking companion is either one of the two children, and that there are no preferences of girls over boys or conversely in this respect. Then this problem is equivalent to randomly selecting one of the children and noticing that it is a daughter. In the problem with the chosen child, we had seen that the probability of having two girls was 1/2.

2. Suppose that all birthdays are equally probable, having excluded the possibility of February 29. Then the probability of being born in a leap year is 1/4. Replace $1/7$ with $1/4$ and adapt the reasoning. We similarly have

$$\mathrm{P}(\text{GirlGirl} \mid \text{GirlYear}) = \frac{\mathrm{P}(\text{GirlYear} \mid \text{GirlGirl}) \times \mathrm{P}(\text{GirlGirl})}{\mathrm{P}(\text{GirlYear})}\,.$$

We get $\mathrm{P}(\text{GirlYear} \mid \text{GirlGirl}) = \frac{7}{16}$ (because having twice one year which is not a leap year has probability $\frac{3}{4} \cdot \frac{3}{4}$). We also have

$$\mathrm{P}(\text{GirlYear}) = \frac{1}{4} \cdot \frac{7}{16} + \frac{1}{2} \cdot \frac{1}{4} = \frac{15}{64}$$

and finally we can compute

$$\mathrm{P}(\text{GirlGirl} \mid \text{GirlYear}) = \frac{7}{15} = 0,4\overline{6}\,.$$