

Riemannian Structures on \mathbb{Z}_2^n -Manifolds

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Abstract: Very loosely, \mathbb{Z}_2^n -manifolds are ‘manifolds’ with \mathbb{Z}_2^n -graded coordinates and their sign rule is determined by the scalar product of their \mathbb{Z}_2^n -degrees. A little more carefully, such objects can be understood within a sheaf-theoretical framework, just as supermanifolds can, but with subtle differences. In this paper, we examine the notion of a Riemannian \mathbb{Z}_2^n -manifold, i.e., a \mathbb{Z}_2^n -manifold equipped with a Riemannian metric that may carry non-zero \mathbb{Z}_2^n -degree. We show that the basic notions and tenets of Riemannian geometry directly generalize to the setting of \mathbb{Z}_2^n -geometry. For example, the Fundamental Theorem holds in this higher graded setting. We point out the similarities and differences with Riemannian supergeometry.

Keywords: \mathbb{Z}_2^n -manifolds; Riemannian structures; affine connections

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1. Introduction

One can hardly overstate the importance of (pseudo-)Riemannian geometry in physics, and in particular within metric theories of gravity theories such as general relativity. The notion of a Riemannian metric is also central in geometric mechanics, classical field theories, various non-linear sigma models, string theories, as well as global analysis, stochastic differential geometry, etc. Riemannian structures on supermanifolds are not so well studied as their classical counterparts. Relatively recent papers on the subject include [1–4]. One of the interesting aspects of the Riemannian supermanifolds is that we can have metrics that are either degree zero or degree one, referred to as even and odd Riemannian metrics, respectively. Within the context of string theory, it is now well appreciated that generalized complex geometry is the right geometrical framework to describe NS-NS fluxes. It appears that RR-fluxes require even Riemannian supergeometry as their geometric framework (see for example [5]). In contrast, odd Riemannian supergeometry remains a mathematical curiosity in the sense that applications in physics have not been so forthcoming. This is in stark contrast with odd symplectic geometry which is central to the BV-BRST formalism.

Recently, there has been a series of papers developing \mathbb{Z}_2^n -geometry ($\mathbb{Z}_2^n := \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$) and related algebraic questions, including [6–12]. Alongside this, there has been a renewed interest in the physical applications of \mathbb{Z}_2^n -gradings, see for example [13–21]. The origin of \mathbb{Z}_2^n -Lie algebras and their associative algebra counterparts can be traced back to Rittenberg & Wyler [22] and Scheunert [23], both motivated by physical considerations. It has long been appreciated that $\mathbb{Z}_2 \times \mathbb{Z}_2$ -gradings are important in parastatistics, see for example [24] and references therein. However, the rôle and importance of these higher gradings in physics are not entirely clear. That said, \mathbb{Z}_2^n -manifolds represent a natural, but challenging generalization of supermanifolds. Informally, \mathbb{Z}_2^n -manifolds are ‘manifolds’ for which we have coordinates that are \mathbb{Z}_2^n -commutative, the sign rule being determined by their

\mathbb{Z}_2^n -degree and the standard scalar product. Importantly, this means that we have formal coordinates that are not nilpotent, provided $n \geq 2$, naturally, $n = 1$ is exactly the theory of supermanifolds. We view \mathbb{Z}_2^n -manifolds, much like supermanifolds, as a ‘pit stop’ on the journey from classical differential geometry to noncommutative geometry. Higher graded-commutative versions of geometry act as a testbed for ideas in fully fledged noncommutative geometry.

In this paper, we present the notion of a Riemannian \mathbb{Z}_2^n -manifold and make an initial study of the generalization of the standard objects found in classical Riemannian geometry, i.e., Riemannian metrics, Levi–Civita connections, and the curvature tensors. The bulk of Section 2 is devoted to developing the general theory of Riemannian \mathbb{Z}_2^n -geometry. We begin with a recollection of the aspects of \mathbb{Z}_2^n -manifolds that we need for the remainder of this paper. Like Riemannian supergeometry where we have even and odd metric, on \mathbb{Z}_2^n -manifolds we can have Riemannian metrics that carry non-zero \mathbb{Z}_2^n -degree (see Definition 4). The non-degeneracy of the Riemannian metric places constraints on the dimension of the \mathbb{Z}_2^n -manifold depending on the \mathbb{Z}_2^n -degree of the metric (see Proposition 1). Most notably, we have a generalization of the fundamental theory of Riemannian geometry to the setting of \mathbb{Z}_2^n -geometry which does not depend on the \mathbb{Z}_2^n -degree of the Riemannian metric (see Theorem 1). In short, there is a natural generalization of the Levi–Civita connection in the setting of \mathbb{Z}_2^n -geometry. From there we define the Riemann curvature tensor and show that it satisfied the expected generalization of the Bianchi identities (see Definition 13, Propositions 8 and 10). The Ricci curvature tensor and Ricci scalar can, of course, also be defined (see Definitions 14 and 15). We can also classify metrics by their total degree and so we have a natural notion of even and odd Riemannian metrics on \mathbb{Z}_2^n -manifolds. For the most part, even and odd Riemannian metrics are not so different: all the notions and basic constructions are insensitive to the degree. However, for odd Riemannian metrics, the Ricci scalar identically vanishes (see Proposition 13). Moreover, we show that the connection Laplacian (acting on functions) associated with an odd Riemannian metric vanishes (see Theorem 2). We end this paper in Section 3 with a discussion of the consequences of some of the results derived in this paper in generalizing de Sitter, anti-de Sitter and Einstein manifolds to \mathbb{Z}_2^n -geometry.

We must remark that the global algebra of functions on a \mathbb{Z}_2^n -manifold is an example of an almost commutative algebra in the sense of Bongaarts & Pijls [25], which were constructed to give a very workable class of noncommutative geometries. It is known that a \mathbb{Z}_2^n -manifold is fully described by its algebra of global sections (see ([8], Corollary 3.8)). Linear connections on almost commutative algebras were given by Ciupală [26]. Moreover, the notion of a metric and the Levi–Civita connection etc. on an almost commutative algebra were given by Nagekeu [27]. Although there is some crossover with the results of Nagekeu, the key difference is that we have both global and local algebras of functions as well as coordinate expressions. Some of the results in this paper represent a generalization of the discussion of Asorey & Lavrov [28] comparing even and odd Riemannian metrics on supermanifolds. For a comparison of the Laplacians found in Poisson and Riemannian supergeometry, the reader can consult Khudaverdian & Voronov [29].

2. Riemannian \mathbb{Z}_2^n -Manifolds

2.1. The Basics of \mathbb{Z}_2^n -Geometry

The locally ringed space approach to \mathbb{Z}_2^n -manifolds was originally given by Covoilo, Grabowski and Poncin in [10,11]. Technical issues related to the functional analytic properties of the structure sheaves of \mathbb{Z}_2^n -manifolds and their products were carefully explored in [7,8]. We will draw upon these works and not present proofs of any formal statements. To set some initial notation, by \mathbb{Z}_2^n we mean the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ where the Cartesian product is over n factors. With an ordering fixed, we will often denote elements of \mathbb{Z}_2^n as γ_i for $i = 0, 1, \dots, N$, where $N = 2^n - 1$. We will use the convention of ordering the elements of \mathbb{Z}_2^n by filling in the zeros and the ones from the

left, and then placing the even elements first and then the odd elements. For example, with this choice of ordering

$$\begin{aligned}\mathbb{Z}_2^2 &= \{(0,0), (1,1), (0,1), (1,0)\}, \\ \mathbb{Z}_2^3 &= \{(0,0,0), (0,1,1), (1,0,1), (1,1,0), (0,0,1), (0,1,0), (1,0,0), (1,1,1)\}.\end{aligned}$$

Naturally, other choices of ordering are possible and have been used in the literature. Throughout this paper we set $\mathbf{q} = (q_1, q_2, \dots, q_N)$, where $N = 2^n - 1$ and $q_i \in \mathbb{N}$.

Definition 1. A locally \mathbb{Z}_2^n -ringed space, $n \in \mathbb{N} \setminus \{0\}$, is a pair $S := (|S|, \mathcal{O}_S)$ where $|S|$ is a second-countable Hausdorff space, and a \mathcal{O}_S is a sheaf of \mathbb{Z}_2^n -graded \mathbb{Z}_2^n -commutative associative unital \mathbb{R} -algebras, such that the stalks $\mathcal{O}_{S,p}$, $p \in |S|$ are local rings.

In this context, \mathbb{Z}_2^n -commutative means that any two sections $s, t \in \mathcal{O}_S(|U|)$, $|U| \subset |S|$ open, of homogeneous degree $\deg(s) \in \mathbb{Z}_2^n$ and $\deg(t) \in \mathbb{Z}_2^n$ commute up to the sign rule

$$st = (-1)^{\langle \deg(s), \deg(t) \rangle} ts,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product on \mathbb{Z}_2^n . For example, consider the $\gamma_3 = (0,1)$ and $\gamma_2 = (1,1) \in \mathbb{Z}_2^2$, then $\langle \gamma_3, \gamma_2 \rangle = 0 \times 1 + 1 \times 1 = 1$. If we write ‘symmetric’ or ‘skew-symmetric’, we will always mean in the \mathbb{Z}_2^n -graded sense unless otherwise stated.

Definition 2. A \mathbb{Z}_2^n -Grassmann algebra $\Lambda^{\mathbf{q}} := \mathbb{R}[[\xi]]$ is a formal power series generated by the \mathbb{Z}_2^n -graded variables $\{\xi^A\}$, there are q_i generators of degree $\gamma_i \in \mathbb{Z}_2^n$, ($i > 0$), subject to the relation

$$\xi^A \xi^B = (-1)^{\langle \deg(A), \deg(B) \rangle} \xi^B \xi^A,$$

where $\deg(\xi^A) =: \deg(A) \in \mathbb{Z}_2^n \setminus \{0\}$ and similar for ξ^B .

It is important to note that elements of such algebras are formal power series and not, in general, polynomials as we have non-nilpotent generators, i.e., the generators of even total degree. A \mathbb{Z}_2^n -manifold is, very loosely, a manifold whose structure sheaf is modified to include generators of a \mathbb{Z}_2^n -Grassmann algebra.

Definition 3. A (smooth) \mathbb{Z}_2^n -manifold of dimension $p|\mathbf{q}|$ is a locally \mathbb{Z}_2^n -ringed space $M := (|M|, \mathcal{O}_M)$, which is locally isomorphic to the \mathbb{Z}_2^n -ringed space $\mathbb{R}^{p|\mathbf{q}} := (\mathbb{R}^p, C_{\mathbb{R}^p}^\infty(-)[[\xi]])$. Local sections of M are formal power series in the \mathbb{Z}_2^n -graded variables ξ with smooth coefficients,

$$\mathcal{O}_M(|U|) \simeq C^\infty(|U|)[[\xi]] := \left\{ \sum_{\alpha \in \mathbb{N}^N} \xi^\alpha f_\alpha \mid f_\alpha \in C^\infty(|U|) \right\},$$

for ‘small enough’ open domains $|U| \subset |M|$. Morphisms between \mathbb{Z}_2^n -manifolds are morphisms of \mathbb{Z}_2^n -ringed spaces, i.e., pairs $(\phi, \phi^*) : (|M|, \mathcal{O}_M) \rightarrow (|N|, \mathcal{O}_N)$ consisting of a continuous map $\phi : |M| \rightarrow |N|$ and sheaf morphism $\phi^* : \mathcal{O}_N(|V|) \rightarrow \mathcal{O}_M(\phi^{-1}(|V|))$, where $|V| \subset |N|$ is open. We will refer to the global sections of the structure sheaf \mathcal{O}_M as functions on M and denote then as $C^\infty(M) := \mathcal{O}_M(|M|)$.

Example 1 (The local model). The locally \mathbb{Z}_2^n -ringed space $\mathcal{U}^{p|\mathbf{q}} := (\mathcal{U}^p, C_{\mathcal{U}^p}^\infty(-)[[\xi]])$, where $\mathcal{U}^p \subseteq \mathbb{R}^p$ is naturally a \mathbb{Z}_2^n -manifold—we refer to such \mathbb{Z}_2^n -manifolds as \mathbb{Z}_2^n -domains of dimension $p|\mathbf{q}|$. We can employ (natural) coordinates $x^I := (x^a, \xi^A)$ on any \mathbb{Z}_2^n -domain, where x^a form a coordinate system on \mathcal{U}^p and the ξ^A are formal coordinates.

Associated with any \mathbb{Z}_2^n -graded algebra \mathcal{A} is the ideal J of \mathcal{A} generated by all homogeneous elements of \mathcal{A} that have non-zero degree. The associated J -adic topology plays an important rôle in the theory of \mathbb{Z}_2^n -manifolds. Given a morphism of \mathbb{Z}_2^n -graded algebras $f : \mathcal{A} \rightarrow \mathcal{A}'$, then $f(J_{\mathcal{A}}) \subset J_{\mathcal{A}'}$. These notions can be ‘sheafified’, i.e., for any \mathbb{Z}_2^n -manifold M , there exists an *ideal sheaf* \mathcal{J}_M , defined by $\mathcal{J}(|U|) := \langle f \in \mathcal{O}_M(|U|) \mid \deg(f) \neq 0 \rangle$. The \mathcal{J}_M -adic topology on \mathcal{O}_M can then be defined in the obvious way. Furthermore, for any \mathbb{Z}_2^n -manifold M , there exists a short exact sequence of sheaves of \mathbb{Z}_2^n -graded \mathbb{Z}_2^n -commutative associative \mathbb{R} -algebras

$$0 \longrightarrow \ker \epsilon \longrightarrow \mathcal{O}_M \xrightarrow{\epsilon} C_{|M|}^{\infty} \longrightarrow 0, \quad (1)$$

such that $\ker \epsilon = \mathcal{J}_M$. Informally, $\epsilon_{|U|} : \mathcal{O}_M(|U|) \rightarrow C^{\infty}(|U|)$ is simply “throwing away” the formal coordinates.

Much like the theory of manifolds, or indeed supermanifolds, one can global geometric concepts via the gluing of local geometric concepts, i.e., we can consider a \mathbb{Z}_2^n -manifold as being cover by \mathbb{Z}_2^n -domains together with the appropriate gluing information, i.e., admissible coordinate transformations. Fundamental here is the *chart theorem* ([30], Theorem 7.10) that allows us to write morphisms of \mathbb{Z}_2^n -manifolds in terms of the local coordinates. Specifically, suppose we have two \mathbb{Z}_2^n -domains $\mathcal{U}^{p|q}$ and $\mathcal{V}^{r|s}$. Then morphisms $\phi : \mathcal{U}^{p|q} \rightarrow \mathcal{V}^{r|s}$ correspond to *graded unital \mathbb{R} -algebra morphisms*

$$\phi^* : C^{\infty}(\mathcal{V}^r)[[\eta]] \longrightarrow C^{\infty}(\mathcal{U}^p)[[\xi]],$$

which are fully determined by their coordinate expressions. For any $M := (|M|, \mathcal{O}_M)$, we define open \mathbb{Z}_2^n -submanifolds of M as \mathbb{Z}_2^n -manifolds of the form $U := (|U|, \mathcal{O}_{M|U|})$, where $|U| \subset |M|$ is open. Due to the local structure of a \mathbb{Z}_2^n -manifold of dimension $p|q$ we know that for a ‘small enough’ $|U| \subset |M|$ there exists an isomorphism of \mathbb{Z}_2^n -manifolds

$$h : U \longrightarrow \mathcal{U}^{p|q}.$$

This isomorphism allows us to employ local coordinates. A pair (U, h) we refer to as a (*coordinate*) *chart*, and a family of charts $\{(U_i, h_i)\}_{i \in \mathcal{I}}$ we refer to as an *atlas* if the family $\{|U_i|\}_{i \in \mathcal{I}}$ forms an open cover of $|M|$. The local structure of a \mathbb{Z}_2^n -manifold guarantees that an atlas always exists. As a result of the local structure of a \mathbb{Z}_2^n -manifold any morphism $\phi : M \rightarrow N$ can be uniquely specified by a family of local morphisms between \mathbb{Z}_2^n -domains, once atlases on M and N have been fixed. Thus, morphisms of \mathbb{Z}_2^n -manifolds can be fully described using local coordinates. We will regularly exploit this and employ the standard abuses of notation as found in classical differential geometry when writing morphisms using local coordinates.

Remark 1. *There is an analogue of the Batchelor–Gawędzki theorem for (real) \mathbb{Z}_2^n -manifolds, see ([10], Theorem 3.2), i.e., any \mathbb{Z}_2^n -manifold is noncanonically isomorphic to a $\mathbb{Z}_2^n \setminus \{0\}$ -graded vector bundle over a smooth manifold. We will not use this theorem in this paper.*

2.2. Riemannian Structures

For notion in classical Riemannian geometry, we refer the reader to Eisenhart [31]. The *tangent sheaf* $\mathcal{T}M$ of a \mathbb{Z}_2^n -manifold M is defined as the sheaf of derivations of sections of the structure sheaf, i.e., $\mathcal{T}M(|U|) := \text{Der}(\mathcal{O}_M(|U|))$, for arbitrary $|U| \subset |M|$. Naturally, this is a sheaf of locally free \mathcal{O}_M -modules. Global sections of the tangent sheaf are referred to as *vector fields*. We denote the $\mathcal{O}_M(|M|)$ -module of vector fields as $\text{Vect}(M)$. The dual of the tangent sheaf is the *cotangent sheaf*, which we denote as \mathcal{T}^*M . This is also a sheaf of locally free \mathcal{O}_M -modules. Global section of the tangent sheaf we will refer to as *one-forms* and we denote the $\mathcal{O}_M(|M|)$ -module of one-forms as $\Omega^1(M)$. For more details about the tangent and cotangent sheaves, the reader should consult [30].

Definition 4. A Riemannian metric on a \mathbb{Z}_2^n -manifold M is a \mathbb{Z}_2^n -homogeneous, \mathbb{Z}_2^n -symmetric, non-degenerate, \mathcal{O}_M -linear morphisms of sheaves

$$\langle - | - \rangle_g : \mathcal{T}M \otimes_{\mathcal{O}_M} \mathcal{T}M \longrightarrow \mathcal{O}_M.$$

A \mathbb{Z}_2^n -manifold equipped with a Riemannian metric is referred to as a Riemannian \mathbb{Z}_2^n -manifold.

We will insist that the Riemannian metric is homogeneous with respect to the \mathbb{Z}_2^n -degree, and we will denote the degree of the metric as $\deg(g) \in \mathbb{Z}_2^n$. We will largely formulate Riemannian geometry globally using vector fields.

Explicitly, a Riemannian metric has the following properties:

1. $\deg(\langle X | Y \rangle_g) = \deg(X) + \deg(Y) + \deg(g)$,
2. $\langle X | Y \rangle_g = (-1)^{\langle \deg(X), \deg(Y) \rangle} \langle Y | X \rangle_g$,
3. If $\langle X | Y \rangle_g = 0$ for all $Y \in \text{Vect}(M)$, then $X = 0$,
4. $\langle fX + Y | Z \rangle_g = f \langle X | Z \rangle_g + \langle Y | Z \rangle_g$,

for arbitrary (homogeneous) $X, Y, Z \in \text{Vect}(M)$ and $f \in C^\infty(M)$. We will say that a Riemannian metric is *even* if and only if it has total degree zero. Similarly, we will say that a Riemannian metric is *odd* if and only if it has total degree one. Any Riemannian metric we consider will be either even or odd as we will only be considering homogeneous metrics.

Example 2. Pseudo-Riemannian and Riemannian manifolds are Riemannian \mathbb{Z}_2^0 -manifolds. Naturally, there are only even Riemannian metrics in the classical setting. Similarly, Riemannian supermanifolds either with an even or odd Riemannian metric are Riemannian \mathbb{Z}_2 -manifolds.

A Riemannian metric is completely specified by a symmetric rank-two covariant tensor $g = \delta x^I \delta x^J g_{IJ}(x)$, where the components are given by $g_{IJ} := \langle \partial_I | \partial_J \rangle_g$. Under changes of local coordinates $x^I \mapsto x^{I'}(x)$ the components of the metric transform as

$$g_{I'J'} = (-1)^{\langle \deg(I'), \deg(I) \rangle} \left(\frac{\partial x^I}{\partial x^{I'}} \right) \left(\frac{\partial x^J}{\partial x^{J'}} \right) g_{IJ}, \quad (2)$$

where we have used the symmetry $g_{IJ} = (-1)^{\langle \deg I, \deg(J) \rangle} g_{JI}$. The Riemannian metric can then locally be written as

$$\langle X | Y \rangle_g = (-1)^{\langle \deg(Y), \deg(I) \rangle} X^I(x) Y^J(x) g_{IJ}(x), \quad (3)$$

where (locally) $X = X^I \frac{\partial}{\partial x^I}$ and similar for $Y = Y^J \frac{\partial}{\partial x^J}$. A direct computation will show that the above local expression for the metric invariant under changes of coordinates. As customary, we will refer to g as the *metric tensor* or by minor abuse of language, we may refer to g itself as the Riemannian metric. Properties (1), (2) and (4) of the Riemannian metric are easily seen to hold for the local expression (3). Just as for Riemannian supermanifolds, the non-degeneracy condition places constraints on the number of “formal directions”. In particular, the dimension of a supermanifold must be $n|2m$ if it is to admit an even Riemannian metric, and the dimension must be $n|n$ if it is to admit an odd Riemannian metric. In the continuation of this paper, we will denote a Riemannian \mathbb{Z}_2^n -manifold by a pair (M, g) .

Proposition 1. Let (M, g) be a Riemannian \mathbb{Z}_2^n -manifold. We set $p|q = p|q_1, q_2, \dots, q_N =: q_0, q_1, q_2, \dots, q_N$. Then the non-degeneracy condition (3) requires that

- (i) $q_i = q_j$ when $\gamma_i + \gamma_j = \deg(g)$,
- (ii) If in addition to the above, $\langle \gamma_i, \gamma_j \rangle = 1$, then q_i (and so q_j) must be even integers, i.e., $q_i = 2n_i$ for some integer n_i .

Proof. As this is a local question, it is sufficient to examine the invertibility of the metric tensor g_{IJ} in some chosen set of local coordinates. The i, j block of g is of \mathbb{Z}_2^n -degree $\gamma_i + \gamma_j + \deg(g)$. The entries of this matrix are in $\mathcal{O}_M(|U|)$ for some “small enough” $|U| \subset |M|$. It is known that any such matrix is invertible if and only if the underlying real matrix $\epsilon_{|U|}(g)$ is itself invertible (see [9] and for the super-case see [32]). Here $\epsilon : \mathcal{O}_M \rightarrow C_{|M|}^\infty$ is the canonical sheaf morphisms. Under this projection, the only non-zero blocks are those for which $\gamma_i + \gamma_j = \deg(g)$.

- (i) Each block itself must be invertible and so a square matrix. Thus, $q_i = q_j$ whenever $\gamma_i + \gamma_j = \deg(g)$.
- (ii) The symmetry of the metric tensor implies that if $\langle \gamma_i, \gamma_i \rangle = 1$, these blocks are, in classical terminology, skew-symmetric matrices. This directly implies that the q_i must be even. \square

Lemma 1. Let (M, g) be a Riemannian \mathbb{Z}_2^n -manifold. Then there is a one-to-one correspondence between elements of $\text{Vect}(M)$ and $\Omega^1(M)$, where a one-form ω corresponds to a vector field X if and only if

$$\omega(Z) = \langle Z | X \rangle_g,$$

for all $Z \in \text{Vect}(M)$.

Proof. This directly follows from the non-degeneracy of the Riemannian metric. In local coordinates we have

$$\omega(Z) = Z^I \omega_I = (-1)^{\langle \deg(X), \deg(I) \rangle} Z^I X^J g_{IJ},$$

and so $\omega_I = (-1)^{\langle \deg(X), \deg(I) \rangle} X^J g_{IJ}$. \square

Please note that $\deg(\omega) = \deg(X) + \deg(g)$ and so the above correspondence does not preserve the \mathbb{Z}_2^n -degree unless the metric is degree zero.

Proposition 2. Let (M, g) be a Riemannian \mathbb{Z}_2^n -manifold with a degree zero Riemannian metric, i.e., $\deg(g) = (0, 0, \dots, 0)$. Then the reduced manifold $|M|$ comes naturally equipped with an induced pseudo-Riemannian structure $|g|$.

Proof. Let \bar{X} and $\bar{Y} \in \text{Vect}(|M|)$ be vector fields on $|M|$. We then define the reduced pseudo-Riemannian metric as

$$\langle \bar{X} | \bar{Y} \rangle_{|g|} = \epsilon_{|M|}(\langle X | Y \rangle_g),$$

for any degree zero X and $Y \in \text{Vect}(M)$ such that $\bar{X} = X \circ \epsilon_{|M|}$ and $\bar{Y} = Y \circ \epsilon_{|M|}$ (see (1)). This construction gives a pseudo-Riemannian metric on $|M|$ as all the required properties are inherited from that of the original Riemannian metric. Specifically, with respect to some chosen local coordinates we have

$$|g|_{ab} = \epsilon_{|U|} g_{ab},$$

here $|U| \in |M|$ is chosen ‘small enough’ to allow the use of local coordinates. It is clear that as g_{IJ} is an invertible degree zero matrix, $|g|_{ab}$ is itself invertible and so non-zero. The symmetry $|g|_{ab} = |g|_{ba}$ is also evident. \square

Remark 2. Proposition 2 does not generalize to non-zero \mathbb{Z}_2^n -degree Riemannian metrics. For example, an even Riemannian metric on a supermanifold induces a (pseudo-)Riemannian on the reduced manifold via the above construction, while an odd Riemannian metric cannot induce such a structure on the reduced manifold.

Example 3. Consider the \mathbb{Z}_2^2 -domain $\mathbb{R}^{2|1,2,2}$ equipped with global coordinates

$$\left(\underbrace{x^a}_{(0,0)}, \underbrace{z}_{(1,1)}, \underbrace{\xi^b}_{(0,1)}, \underbrace{\eta^c}_{(1,0)} \right),$$

where a, b and $c = 1, 2$. This \mathbb{Z}_2^2 -manifold can be equipped with a degree $(0, 0)$ Riemannian metric

$$g = (\delta x^1)^2 + (\delta x^2)^2 + (\delta z)^2 + 2\delta \xi^1 \delta \xi^2 + 2\delta \eta^1 \delta \eta^2.$$

The reduced Riemannian metric on the reduced manifold is the standard Euclidean metric on \mathbb{R}^2 .

Example 4. Recall that the open disk is defined as $\mathbb{D} := \{(x^1, x^2) \in \mathbb{R}^2 \mid (x^1)^2 + (x^2)^2 < 1\}$. We define the \mathbb{Z}_2^2 -disk as

$$\mathbb{D}^{2|1,2,2} := (\mathbb{D}, \mathcal{C}_{\mathbb{D}}^{\infty}(-) \hat{\otimes}_{\mathbb{R}} \mathbb{R}[[z, \xi^a, \eta^b]]),$$

where a and $b = 1, 2$. The algebras here are nuclear Fréchet algebras, and as such the tensor product is completed with respect to any reasonable topology, for example the projective topology. By construction the reduced manifold is the open disk. We can equip this \mathbb{Z}_2^2 -manifold with a degree $(0, 0)$ Riemannian metric given by

$$g = 4 \frac{(\delta x^1)^2 + (\delta x^2)^2 + (\delta z)^2 + 2\delta \xi^1 \delta \xi^2 + 2\delta \eta^1 \delta \eta^2}{(1 - (x^1)^2 - (x^2)^2 - z^2 - \xi^1 \xi^2 - \eta^1 \eta^2)^2}.$$

Please note that the denominator has non-zero real part and so is invertible. The reduced Riemannian metric is the metric of the Poncaré disk model.

2.3. The Cartesian and Warped Product of Riemannian \mathbb{Z}_2^n -Manifolds

It is known that the category of \mathbb{Z}_2^n -manifolds admit all finite products, for details the reader should consult [8]. In particular, given two \mathbb{Z}_2^n -manifolds M_1 and M_2 , their Cartesian product $M_1 \times M_2$ is well-defined and is itself a \mathbb{Z}_2^n -manifold. One can construct an atlas on $M_1 \times M_2$ using atlases on M_1 and M_2 in more-or-less the same way as one can in the category of supermanifolds.

We will denote the projections as $\pi_i : M_1 \times M_2 \rightarrow M_i$ ($i = 1, 2$). The underlying continuous map $|\pi_i| : |M_1| \times |M_2| \rightarrow |M_i|$ is the standard smooth projection. Using a basis of the topology for both M_1 and M_2 consisting of small enough opens $|U_i| \subset |M_i|$, such that the corresponding open \mathbb{Z}_2^n -submanifolds are diffeomorphic to coordinate domains, we see that

$$\pi_{i,|U_i|}^* : \mathcal{O}_{M_i}(|U_i|) : \mathcal{O}_{M_1}(|U_1|) \hat{\otimes}_{\mathbb{R}} \mathcal{O}_{M_2}(|U_2|)$$

is the standard inclusion of algebras. As the algebras in question are nuclear Fréchet, the tensor product is completed with respect to any reasonable topology, say for concreteness, the projective topology (for details of the functional analytic properties of the structure sheaf of a \mathbb{Z}_2^n -manifold, the reader should consult [7]).

Proposition 3. Let (M_i, g_i) ($i = 1, 2$) be Riemannian \mathbb{Z}_2^n -manifolds whose Riemannian metric are of the same \mathbb{Z}_2^n -degree. Then,

$$(M_1 \times M_2, g := \pi_1^* g_1 + \pi_2^* g_2)$$

is itself a Riemannian \mathbb{Z}_2^n -manifold whose Riemannian metric is of degree $\deg(g) = \deg(g_i)$.

Proof. It is clear that this construction means that

$$\langle X|Y \rangle_g = \langle X|Y \rangle_{\pi_1^* g_1} + \langle X|Y \rangle_{\pi_2^* g_2},$$

for all X and $Y \in \text{Vect}(M_1 \times M_2)$. We then need to check that this has the right properties. It is clear that we have the required \mathbb{Z}_2^n -degree, that this is symmetric, and that this is linear. The non-degeneracy also directly follows as the initial structures are themselves non-degenerate. \square

By employing local coordinates x^I on M_1 and y^A on M_2 , the Riemannian metric is given locally by

$$\langle X|Y \rangle_g = (-1)^{\langle \deg(Y), \deg(I) \rangle} X^I(x, y) Y^J(x, y) g_{JI}(x) + (-1)^{\langle \deg(Y), \deg(A) \rangle} X^A(x, y) Y^B(x, y) g_{BA}(y),$$

or as a tensor

$$g = \delta x^I \delta x^J g_{IJ}(x) + \delta y^A \delta y^B g_{BA}(y).$$

This construction generalizes to a warped product.

Definition 5. Let (M_i, g_i) ($i = 1, 2$) be Riemannian \mathbb{Z}_2^n -manifolds whose Riemannian metric are of the same \mathbb{Z}_2^n -degree. Let $\mu \in C^\infty(M_1)$ be an invertible global function that is strictly positive, i.e., $\epsilon_{M_1}\mu$ is a strictly positive function on $|M_1|$. Then the warped product is defined as

$$M_1 \times_\mu M_2 := (M_1 \times M_2, g := \pi_1^* g_1 + (\pi_1^* \mu) \pi_2^* g_2).$$

Please note that if μ is the constant function of value one, then the warped product is simply the Cartesian product.

Proposition 4. The warped product $M_1 \times_\mu M_2$ is a Riemannian \mathbb{Z}_2^n -manifold.

Proof. It is clear that the definition of the warped product means that

$$\langle X|Y \rangle_g = \langle X|Y \rangle_{\pi_1^* g_1} + (\pi_1^* \mu) \langle X|Y \rangle_{\pi_2^* g_2},$$

for all X and $Y \in \text{Vect}(M_1 \times M_2)$. We then need to check that this has the right properties. It is clear that we have the required \mathbb{Z}_2^n -degree, i.e., the Riemannian metric is homogeneous. The symmetry and linearity directly from the properties of the starting pair of Riemannian metrics. Please note that $\pi_1^* \mu$, as it is invertible has a non-zero real component. Thus, the second term in the sum will not vanish unless $\langle X|Y \rangle_{\pi_2^* g_2}$ vanishes. Thus, we are back to the non-degeneracy condition of the pair of Riemannian metrics. \square

Example 5. It is easy to observe that $\mathbb{R}^{1|1,1,1} = \mathbb{R}^{1|1,0,0} \times \mathbb{R}^{0|0,1,1}$. Moreover, in global coordinates (x, z, ξ, η) the canonical degree $(1, 1)$ Riemannian metric is $g_0 = \delta x \delta z + \delta \eta \delta \xi$. Please note that this is, of course, the sum of the two Riemannian metrics on each factor, i.e., we have a Cartesian product of Riemannian \mathbb{Z}_2^2 -manifold. However, we can warp this construction. We choose, somewhat arbitrarily, $\mu(x, z) = \exp((x^2 + z^2)/k^2)$, with $k \in \mathbb{R}$ a non-zero constant. The exponential of a formal variable is understood as a power series, so explicitly

$$\mu(x, z) = \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{x^2 + z^2}{k^2}\right)^i \exp\left(\frac{x^2 + z^2}{k^2}\right).$$

The \mathbb{Z}_2^2 -manifold $\mathbb{R}^{1|1,1,1}$ can then be equipped with the warped Riemannian metric

$$g = \delta x \delta z + \delta \eta \delta \xi \exp\left(\frac{x^2 + z^2}{k^2}\right).$$

Please note that in the limit $k \rightarrow \infty$ the warping factor goes likes $\mu(x, z) \rightarrow 1$ and so we recover the canonical metric.

2.4. Isometries and Killing Vector Fields

The notions of isometries and infinitesimal isometries of a Riemannian \mathbb{Z}_2^n -manifold are the same as the classical notions.

Definition 6. Let (M, g) and (N, h) be Riemannian \mathbb{Z}_2^n -manifolds. Then a diffeomorphism $\phi : M \rightarrow N$ is said to be an isometry if and only if

$$\langle X|Y \rangle_g = \langle \phi_* X | \phi_* Y \rangle_h,$$

for all X and $Y \in \text{Vect}(M)$.

Observation 1. As we are dealing with diffeomorphisms it is clear that $\dim(M) = \dim(N)$. Moreover, as $\deg(X) = \deg(\phi_* X)$ and similar for Y , it must be the case that $\deg(g) = \deg(h)$. Thus, it is impossible to have isometries between Riemannian \mathbb{Z}_2^n -manifolds that have Riemannian metrics of different \mathbb{Z}_2^n -degrees.

Moving on to infinitesimal symmetries, any homogeneous vector field $X \in \text{Vect}(M)$ defines an infinitesimal local diffeomorphism, which in coordinates is given by

$$\phi_\lambda^* x^I = x^I + \lambda X^I(x),$$

where $\deg(\lambda) = \deg(X)$ is a formal parameter taken to \mathbb{Z}_2^n -commute with the coordinates. This induces the map

$$\delta x^I \mapsto \delta x^I + \lambda \delta x^J \frac{\partial X^I}{\partial x^J}.$$

It is then a direct calculation to show that under such a local diffeomorphism the metric changes as

$$\begin{aligned} g \mapsto g + \lambda \delta x^J \delta x^I & \left((-1)^{\langle \deg(X), \deg(I) \rangle} \frac{\partial X^K}{\partial x^J} g_{KI} + (-1)^{\langle \deg(X) + \deg(J), \deg(I) \rangle} \frac{\partial X^K}{\partial x^I} g_{KJ} \right. \\ & \left. + (-1)^{\langle \deg(X), \deg(I) + \deg(J) \rangle} X^K \frac{\partial g_{IJ}}{\partial x^K} \right) + \mathcal{O}(\lambda^2). \end{aligned}$$

The Lie derivative is then defined as the $\mathcal{O}(\lambda)$ term of the above, i.e.,

$$(L_X g)_{IJ} = (-1)^{\langle \deg(X), \deg(I) \rangle} \frac{\partial X^K}{\partial x^J} g_{KI} + (-1)^{\langle \deg(X) + \deg(J), \deg(I) \rangle} \frac{\partial X^K}{\partial x^I} g_{KJ} + (-1)^{\langle \deg(X), \deg(I) + \deg(J) \rangle} X^K \frac{\partial g_{IJ}}{\partial x^K}.$$

Here, L_X is the *Lie derivative along X*, defined on the sheaf of tensors fields on M analogously to the case of standard supermanifolds. In particular, L_X is the derivation of degree $(\deg(X), 0)$ on the sheaf of covariant tensor fields (which is a sheaf of $\mathbb{Z}_2^n \times \mathbb{N}$ -graded (tensor) algebras) such that $L_X(f) = X(f)$ and $L_X(df) = d(X(f))$ for $f \in \mathcal{O}_M$:

$$L_X(\alpha \otimes \beta) = L_X(\alpha) \otimes \beta + (-1)^{\langle \deg(X), \deg(\alpha) \rangle} \alpha \otimes L_X(\beta).$$

A Killing vector field is then a vector field whose Lie derivative annihilates the metric. More formally we have the following definition.

Definition 7. A vector field X on a Riemannian \mathbb{Z}_2^n -manifold (M, g) is said to be a Killing vector field if and only if

$$L_X g = 0.$$

Just as in the classical and super-case, we have the following result.

Proposition 5. Let (M, g) be a Riemannian \mathbb{Z}_2^n -manifold. The space of Killing vector fields forms a \mathbb{Z}_2^n -Lie algebra under the standard \mathbb{Z}_2^n -graded Lie bracket of vector fields.

Proof. This follows directly from $L_{[X,Y]} = [L_X, L_Y]$ for all vector fields X and $Y \in \text{Vect}(M)$. Indeed, this is clearly true for the action on 1-forms. Therefore, if $g = \alpha \otimes \beta$ for 1-forms α and β , then

$$\begin{aligned} L_X \circ L_Y(g) &= L_X \circ L_Y(\alpha \otimes \beta) = L_X \circ L_Y(\alpha) \otimes \beta + (-1)^{\langle \deg(X), \deg(\alpha) + \deg(Y) \rangle} L_Y(\alpha) \otimes L_X(\beta) \\ &\quad - (-1)^{\langle \deg(Y), \deg(\alpha) \rangle} L_X(\alpha) \otimes L_Y(\beta) + (-1)^{\langle \deg(Y) + \deg(X), \deg(\alpha) \rangle} (\alpha) \otimes L_X \circ L_Y(\beta) \end{aligned}$$

and consequently

$$\begin{aligned} [L_X, L_Y](g) &= (L_X \circ L_Y - (-1)^{\langle \deg(X), \deg(Y) \rangle} L_Y \circ L_X) (\alpha \otimes \beta) \\ &= [L_X, L_Y](\alpha) \otimes \beta + (-1)^{\langle \deg(Y) + \deg(X), \deg(\alpha) \rangle} \alpha \otimes [L_X, L_Y](\beta) \\ &= L_{[X, Y]}(\alpha) \otimes \beta + (-1)^{\langle \deg(Y) + \deg(X), \deg(\alpha) \rangle} \alpha \otimes L_{[X, Y]}(\beta) = L_{[X, Y]}(g). \quad \square \end{aligned}$$

Example 6. Consider a \mathbb{Z}_2^n -manifold of the form $M = N \times \mathbb{R}^2$. Such a \mathbb{Z}_2^n -manifold is a pp-wave type \mathbb{Z}_2^n -manifold if it is a Riemannian \mathbb{Z}_2^n -manifold with a degree zero Riemannian metric, such that in any coordinate system (x^I, s, t) , where (s, t) are canonical coordinates on \mathbb{R}^2 , the Riemannian metric is of the form

$$g = \delta x^I \delta x^J g_{IJ}(x) + 2 \delta s \delta t + \delta s \delta s H(x, s),$$

where $H : M \times \mathbb{R} \rightarrow \mathbb{R}_*$ is a smooth map of \mathbb{Z}_2^n -manifolds, and $g_N = \delta x^I \delta x^J g_{IJ}(x)$ is a degree zero Riemannian metric on N . It is easy to observe that there is a null Killing vector field $\frac{\partial}{\partial t}$, i.e., $\langle \partial_t | \partial_t \rangle_g = 0$ and $L_{\partial_t} g = 0$. Such \mathbb{Z}_2^n -manifolds are generalizations of pp-wave type spacetimes as defined by Flores & Sánchez [33].

2.5. The Inverse Metric

The inverse metric is locally defined via

$$g^{IK} g_{KJ} = g_{JK} g^{KI} = \delta_J^I.$$

Please note that the inverse metric will have the same \mathbb{Z}_2^n -degree as the metric. The above relation can be used to deduce the symmetry property of the inverse metric.

Proposition 6. The inverse metric has the following symmetry

$$(-1)^{\langle \deg(K), \deg(K) \rangle} g^{IK} = (-1)^{\langle \deg(I), \deg(K) \rangle + \langle \deg(g), \deg(g) \rangle} ((-1)^{\langle \deg(I), \deg(I) \rangle} g^{KI}).$$

Proof.

$$\begin{aligned} g^{IK} g_{KJ} &= (-1)^{\langle \deg(I) + \deg(K) + \deg(g), \deg(K) + \deg(J) + \deg(g) \rangle} g_{KJ} g^{IK} \\ &= (-1)^{\langle \deg(I) + \deg(K) + \deg(g), \deg(K) + \deg(J) + \deg(g) \rangle + \langle \deg(K), \deg(J) \rangle + \phi} g_{JK} g^{KI} \end{aligned}$$

where ϕ represents the symmetry of the inverse metric, which is to be determined. Expanding the scalar product shows that it must be the case that $\phi + \langle \deg(I), \deg(J) \rangle + \langle \deg(I), \deg(g) \rangle + \langle \deg(K), \deg(K) \rangle + \langle \deg(K), \deg(J) \rangle + \langle \deg(g), \deg(J) \rangle + \langle \deg(g), \deg(g) \rangle = 0$ when $I = J$. This means that $\phi = \langle \deg(J), \deg(J) \rangle + \langle \deg(K), \deg(K) \rangle + \langle \deg(K), \deg(J) \rangle + \langle \deg(g), \deg(g) \rangle$, i.e.,

$$g^{IK} = (-1)^{\langle \deg(J), \deg(J) \rangle + \langle \deg(K), \deg(K) \rangle + \langle \deg(K), \deg(J) \rangle + \langle \deg(g), \deg(g) \rangle} g^{KL}.$$

This directly implies the symmetry property stated in the proposition. \square

Observation 2. If the Riemannian metric is even, then its corresponding inverse metric defines a symmetric contravariant rank-2 tensor. On the other hand, if the Riemannian metric is odd, then its corresponding inverse metric defines a skew-symmetric contravariant rank-2 tensor. This is in complete agreement with the case of even and odd Riemannian metrics on supermanifolds.

The inverse metric can be used to raise indices of covariant tensors and in particular, it is essential in defining the trace of rank-two covariant tensors. Specifically, starting with a rank-two

covariant tensor ω_{IJ} , where we do not assume any symmetry, we can use Lemma 1 and the inverse metric to define

$$X_J^K := (-1)^{\langle \deg(I), \deg(\omega) + \deg(g) + \deg(J) \rangle} \omega_{IJ} g^{IK}.$$

The trace of ω_{IJ} is then defined as the \mathbb{Z}_2^n -graded trace (see [9], Theorem 1.) of X_J^K .

Definition 8. Let (M, g) be a Riemannian \mathbb{Z}_2^n -manifold and let ω_{IJ} be (the components of) a rank-two covariant tensor. Then the trace of ω is defined as

$$\mathrm{tr}_g(\omega) := (-1)^{\langle \deg(\omega) + \deg(g), \deg(I) + \deg(J) \rangle} (-1)^{\langle \deg(I), \deg(J) \rangle} \omega_{IJ} g^{IJ} (-1)^{\langle \deg(J), \deg(J) \rangle} \in C^\infty(M).$$

Please note that $\deg(\mathrm{tr}_g(\omega)) = \deg(\omega) + \deg(g)$. The formula for the trace of a rank-two tensor is identical to the classical formula up to the obligatory sign factors. We have chosen the second index in our definition, but this is sufficient as any covariant tensor can be written as the sum of its symmetrization and skew-symmetrization.

Lemma 2. Let (M, g) be a Riemannian manifold. If g is an odd Riemannian metric and ω_{IJ} is symmetric, i.e., $\omega_{IJ} = (-1)^{\langle \deg(I), \deg(J) \rangle} \omega_{JI}$, then $\mathrm{tr}_g(\omega) = 0$. Similarly, if g is an even Riemannian metric and ω_{IJ} is skew-symmetric, i.e., $\omega_{IJ} = -(-1)^{\langle \deg(I), \deg(J) \rangle} \omega_{JI}$, then $\mathrm{tr}_g(\omega) = 0$.

Proof. Using Proposition 6 we observe that in both the stipulated cases we contract tensors that are symmetric and skew-symmetric in their indices. Thus, $\mathrm{tr}_g(\omega) = -\mathrm{tr}_g(\omega)$ and so $\mathrm{tr}_g(\omega) = 0$. \square

Example 7. If g is an odd Riemannian metric, then it is clear from Proposition 6 and Lemma 2 that $\mathrm{tr}_g(g) = 0$.

2.6. Levi-Civita Connections and the Fundamental Theorem of Riemannian \mathbb{Z}_2^n -Geometry

The notion of an affine connection on a \mathbb{Z}_2^n -manifold is more-or-less the same as that of an affine connection on a supermanifold, or indeed a manifold. In particular, we generalize Koszul's notion of a connection.

Definition 9. An affine connection on a \mathbb{Z}_2^n -manifold is a \mathbb{Z}_2^n -degree preserving map

$$\begin{aligned} \nabla : \mathrm{Vect}(M) \times \mathrm{Vect}(M) &\longrightarrow \mathrm{Vect}(M) \\ (X, Y) &\mapsto \nabla_X Y, \end{aligned}$$

that satisfies the following:

- Bi-linearity

$$\begin{aligned} \nabla_X(Y + Z) &= \nabla_X Y + \nabla_X Z, \\ \nabla_{X+Y} Z &= \nabla_X Z + \nabla_Y Z, \end{aligned}$$

- $C^\infty(M)$ -linearity in the first argument

$$\nabla_{fX} Y = f \nabla_X Y,$$

- The Leibniz rule

$$\nabla_X f Y = X(f) Y + (-1)^{\langle \deg(X), \deg(f) \rangle} f \nabla_X Y,$$

for all (homogeneous) $X, Y, Z \in \mathrm{Vect}(M)$ and $f \in C^\infty(M)$.

We define the *Christoffel symbols* of any affine connection by (locally) setting

$$\begin{aligned}\nabla_X Y &= X^I \frac{\partial Y^J}{\partial x^I} \frac{\partial}{\partial x^J} + (-1)^{\langle \deg(I), \deg(Y) + \deg(J) \rangle} X^I Y^J \nabla_{\frac{\partial}{\partial x^I}} \left(\frac{\partial}{\partial x^J} \right) \\ &= X^I \frac{\partial Y^J}{\partial x^I} \frac{\partial}{\partial x^J} + (-1)^{\langle \deg(I), \deg(Y) + \deg(J) \rangle} X^I Y^J \Gamma_{JI}^K \frac{\partial}{\partial x^K}.\end{aligned}$$

That is, we make the standard definition

$$\nabla_{\frac{\partial}{\partial x^I}} \left(\frac{\partial}{\partial x^J} \right) := \Gamma_{JI}^K \frac{\partial}{\partial x^K}.$$

Definition 10. The torsion tensor of an affine connection $T_\nabla : \text{Vect}(M) \otimes_{C^\infty(M)} \text{Vect}(M) \rightarrow \text{Vect}(M)$ is defined as

$$T_\nabla(X, Y) := \nabla_X Y - (-1)^{\langle \deg(X), \deg(Y) \rangle} \nabla_Y X - [X, Y],$$

for any (homogeneous) $X, Y \in \text{Vect}(M)$. An affine connection is said to be symmetric if the torsion vanishes.

It is clear that the torsion is skew-symmetric, i.e.,

$$T_\nabla(X, Y) = -(-1)^{\langle \deg(X), \deg(Y) \rangle} T_\nabla(Y, X).$$

Furthermore, a direct calculation shows that

$$T_\nabla(fX, Y) = f T_\nabla(X, Y) = (-1)^{\langle \deg(f), \deg(X) \rangle} T_\nabla(X, fY),$$

i.e., we do indeed have a tensor. The components of the torsion tensor with respect local coordinates are

$$T_{IJ}^K = \Gamma_{JI}^K - (-1)^{\langle \deg(I), \deg(J) \rangle} \Gamma_{IJ}^K.$$

Definition 11. An affine connection on a Riemannian \mathbb{Z}_2^n -manifold (M, g) is said to be metric compatible if and only if

$$X\langle Y|Z \rangle_g = \langle \nabla_X Y|Z \rangle_g + (-1)^{\langle \deg(X), \deg(Y) \rangle} \langle Y|\nabla_X Z \rangle_g,$$

for any $X, Y, Z \in \text{Vect}(M)$.

Lemma 3. Let (M, g) be a Riemannian \mathbb{Z}_2^n -manifold. Then an affine connection ∇ that is symmetric (i.e., is torsionless) and metric compatible satisfies the Koszul formula

$$\begin{aligned}2\langle \nabla_X Y|Z \rangle_g &= X\langle Y|Z \rangle_g + \langle [X, Y]|Z \rangle_g \\ &\quad + (-1)^{\langle \deg(X), \deg(Y) + \deg(Z) \rangle} (Y\langle Z|X \rangle_g - \langle [Y, Z]|X \rangle_g) \\ &\quad - (-1)^{\langle \deg(Z), \deg(X) + \deg(Y) \rangle} (Z\langle X|Y \rangle_g - \langle [Z, X]|Y \rangle_g),\end{aligned}\tag{4}$$

for all homogeneous X, Y and $Z \in \text{Vect}(M)$.

Proof. The Koszul formula (4) can be constructed in the same way as the classical case upon being careful with the needed sign factors. From the metric compatibility (Definition 11) we have

$$\begin{aligned}X\langle Y|Z \rangle_g &= \langle \nabla_X Y|Z \rangle_g + (-1)^{\langle \deg(X), \deg(Y) \rangle} \langle Y|\nabla_X Z \rangle_g, \\ Y\langle Z|X \rangle_g &= \langle \nabla_Y Z|X \rangle_g + (-1)^{\langle \deg(Y), \deg(Z) \rangle} \langle Z|\nabla_Y X \rangle_g, \\ Z\langle X|Y \rangle_g &= \langle \nabla_Z X|Y \rangle_g + (-1)^{\langle \deg(Z), \deg(X) \rangle} \langle X|\nabla_Z Y \rangle_g.\end{aligned}$$

We then sum these three with appropriate sign factors to get (after a little rearranging)

$$\begin{aligned} & X\langle Y|Z\rangle_g + (-1)^{\langle \deg(X), \deg(Y) + \deg(Z) \rangle} Y\langle Z|X\rangle_g - (-1)^{\langle \deg(Z), \deg(X) + \deg(Y) \rangle} Z\langle X|Y\rangle_g \\ &= \langle \nabla_X Y|Z\rangle_g + (-1)^{\langle \deg(X), \deg(Y) \rangle} \langle \nabla_Y X|Z\rangle_g \\ &+ (-1)^{\langle \deg(X), \deg(Y) \rangle} \langle Y|\nabla_X Z\rangle_g - (-1)^{\langle \deg(X), \deg(Z) \rangle} \langle \nabla_Z X\rangle_g \\ &+ \langle X|\nabla_Y Z\rangle_g - (-1)^{\langle \deg(Z), \deg(Y) \rangle} \langle \nabla_Z Y\rangle_g. \end{aligned}$$

Next we use the symmetric property (Definition 10) the RHS of the above becomes

$$\begin{aligned} &= 2\langle \nabla_X Y|Z\rangle_g - \langle [X, Y]|Z\rangle_g + (-1)^{\langle \deg(X), \deg(Y) \rangle} \langle Y|[X, Z]\rangle_g + \langle X|[Y, Z]\rangle_g \\ &= 2\langle \nabla_X Y|Z\rangle_g - \langle [X, Y]|Z\rangle_g - (-1)^{\langle \deg(Z), \deg(X) + \deg(Y) \rangle} \langle [Z, X]|Y\rangle_g + (-1)^{\langle \deg(X), \deg(Y) + \deg(Z) \rangle} \langle [Y, Z]|X\rangle_g. \end{aligned}$$

Rearranging this produces the Koszul formula. \square

We are now in a position to state and prove the generalization of the Fundamental Theorem of Riemannian geometry to \mathbb{Z}_2^n -geometry. That is, we have a canonical affine connection associated with the Riemannian structure.

Theorem 1 (The Fundamental Theorem). *There is a unique symmetric (torsionless) and metric compatible affine connection on a Riemannian \mathbb{Z}_2^n -manifold (M, g) .*

Proof. The classical proof using the Koszul formula generalizes to our setting. Lemma 3 shows that if an affine connection is metric compatible and symmetric then it necessarily satisfies the Koszul formula (4). We need to argue that the Koszul formula defines a unique metric compatible and symmetric connection. With this in mind, let us fix X and $Y \in \text{Vect}(M)$ (both homogeneous) and then consider the RHS of (4) as a map $\omega(Z) : \text{Vect}(M) \rightarrow C^\infty(M)$. A direct calculation shown that $\omega(fZ) = (-1)^{\langle \deg(f), \deg(X) + \deg(Y) \rangle} f\omega(Z)$. Thus, ω is a tensor, and more specifically a one-form. We can then use Lemma 1. We define the corresponding vector field as $2\nabla_X Y$ so that $\omega(Z) = \langle 2\nabla_X Y, Z\rangle_g$ for all vector fields Z . In this way we obtain a map $\nabla : \text{Vect}(M) \times \text{Vect}(M) \rightarrow \text{Vect}(M)$. We need to show that this map does indeed define an affine connection (see Definition 9).

- Bi-linearity is clear as the RHS of the Koszul formula (4) is bi-linear.
- The $C^\infty(M)$ -linearity in the first argument follows from a direct calculation:

$$\begin{aligned} \langle 2\nabla_{fX} Y|Z\rangle_g &= fX\langle Y|Z\rangle_g + \langle [fX, Y]|Z\rangle_g \\ &+ (-1)^{\langle \deg(X) + \deg(f), \deg(Y) + \deg(Z) \rangle} (Y\langle Z|fX\rangle_g - \langle [Y, Z]|fX\rangle_g) \\ &- (-1)^{\langle \deg(Z), \deg(X) + \deg(f) + \deg(Z) \rangle} (Z\langle fX|Y\rangle_g - \langle [Z, fX]|Y\rangle_g) \\ &= f(X\langle Y|Z\rangle_g + \langle [X, Y]|Z\rangle_g) \\ &+ (-1)^{\langle \deg(X), \deg(Y) + \deg(Z) \rangle} (Y\langle Z|X\rangle_g - \langle [Y, Z]|X\rangle_g) \\ &- (-1)^{\langle \deg(Z), \deg(X) + \deg(Y) \rangle} (Z\langle X|Y\rangle_g - \langle [Z, X]|Y\rangle_g) \\ &- (-1)^{\langle \deg(X), \deg(f), \deg(Y) \rangle} Y(f)\langle X|Z\rangle_g + (-1)^{\langle \deg(X), \deg(f), \deg(Y) + \langle \deg(X), \deg(Z) \rangle \rangle} Y(f)\langle Z|X\rangle_g \\ &- (-1)^{\langle \deg(Z), \deg(X) + \deg(f) + \deg(Y) \rangle} Z(f)\langle X|Y\rangle_g + (-1)^{\langle \deg(Z), \deg(X) + \deg(f) + \deg(Y) \rangle} Z(f)\langle Y|X\rangle_g \\ &= \langle 2f\nabla_X Y|Z\rangle_g. \end{aligned}$$

This is precisely the $C^\infty(M)$ -linearity property.

- The Leibniz rule follows from a direct calculation:

$$\begin{aligned}
\langle 2\nabla_X fY \rangle_g &= X\langle fY|Z \rangle_g + \langle [X, fY]|Z \rangle_g \\
&\quad + (-1)^{\langle \deg(X), \deg(Y) + \deg(f) + \deg(Z) \rangle} (fY\langle Z|X \rangle_g - \langle [fY, Z]|X \rangle_g) \\
&\quad + (-1)^{\langle \deg(Z), \deg(Y) + \deg(f) + \deg(X) \rangle} (Z\langle X|fY \rangle_g - \langle [Z, X]|fY \rangle_g) \\
&= X(f)\langle Y|Z \rangle_g + (-1)^{\langle \deg(X), \deg(f) \rangle} fX\langle Y|Z \rangle_g \\
&\quad + X(f)\langle Y|Z \rangle_g + (-1)^{\langle \deg(X), \deg(f) \rangle} f\langle [X, Y]|Z \rangle_g \\
&\quad + (-1)^{\langle \deg(X), \deg(f) \rangle + \langle \deg(X), \deg(Y) + \deg(Z) \rangle} f(Y\langle Z|X \rangle_g - \langle [Y, Z]|X \rangle_g) \\
&\quad + (-1)^{\langle \deg(X), \deg(f) \rangle + \langle \deg(X), \deg(Y) \rangle + \langle \deg(Z), \deg(X) + \deg(Y) + \deg(f) \rangle} Z(f)\langle Y|X \rangle_g \\
&\quad - (-1)^{\langle \deg(X), \deg(f) \rangle + \langle \deg(Z), \deg(X) + \deg(Y) \rangle} f(Z\langle X|Y \rangle_g - \langle [Z, X]|Y \rangle_g) \\
&\quad - (-1)^{\langle \deg(X), \deg(f) \rangle + \langle \deg(X) + \deg(Y) + \deg(f) \rangle} Z(f)\langle X|Y \rangle_g \\
&= (-1)^{\langle (X), (f) \rangle} ((fX\langle Y|Z \rangle_g + \langle f[X, Y]|Z \rangle_g) \\
&\quad + (-1)^{\langle \deg(X), \deg(Y) + \deg(Z) \rangle} (fY\langle Z|X \rangle_g + \langle f[Y, Z]|X \rangle_g) \\
&\quad - (-1)^{\langle \deg(Z), \deg(X) + \deg(Y) \rangle} (fZ\langle X|Y \rangle_g + \langle f[Z, X]|Y \rangle_g)) + \langle 2X(f)Y|Z \rangle_g \\
&= \langle 2X(f)Y \rangle_g + (-1)^{\langle \deg(X), \deg(f) \rangle} f\langle 2\nabla_X Y|Z \rangle_g.
\end{aligned}$$

This is precisely the Leibniz rule.

Thus, the Koszul formula defines a connection ∇ . We can then use the Lemma 3 to conclude that this connection is indeed metric compatible and symmetric. \square

Definition 12. The symmetric, metric compatible affine connection defined by Theorem 1 is referred to as the Levi-Civita connection.

Observation 3. The Fundamental Theorem holds on any Riemannian \mathbb{Z}_2^n -manifold irrespectively of the \mathbb{Z}_2^n -degree of the Riemannian metric, i.e., any Riemannian \mathbb{Z}_2^n -manifold canonically comes equipped with the Levi-Civita connection. This encompasses the well-known result for Riemannian supermanifolds.

Proposition 7. In local coordinates x^I the Christoffel symbols Γ_{JI}^L of the Levi-Civita connection on a Riemannian \mathbb{Z}_2^n -manifold (M, g) are given by

$$\Gamma_{JI}^L = \frac{1}{2} \left(\frac{\partial g_{JK}}{\partial x^I} + (-1)^{\langle \deg(I), \deg(J) \rangle} \frac{\partial g_{IK}}{\partial x^J} - (-1)^{\langle \deg(K), \deg(I) + \deg(J) \rangle} \frac{\partial g_{IJ}}{\partial x^K} \right) g^{KL}.$$

Proof. It follows from direct computation using the Koszul formula, see Lemma 3 and Theorem 1. In particular, it is easy to see that

$$2\langle \nabla_I \partial_J | \partial_K \rangle_g = 2\Gamma_{JI}^L g_{LK} = \left(\frac{\partial g_{JK}}{\partial x^I} + (-1)^{\langle \deg(I), \deg(J) \rangle} \frac{\partial g_{IK}}{\partial x^J} - (-1)^{\langle \deg(K), \deg(I) + \deg(J) \rangle} \frac{\partial g_{IJ}}{\partial x^K} \right).$$

Multiplying on the right of the above by the inverse metric yields the result. \square

2.7. The Riemann Curvature Tensor

We now turn attention to the generalization of the Riemann curvature tensor and its basic properties.

Definition 13. The Riemannian curvature tensor of an affine connection

$$R_\nabla : \text{Vect}(M) \otimes_{C^\infty(M)} \text{Vect}(M) \otimes_{C^\infty(M)} \text{Vect}(M) \rightarrow \text{Vect}(M)$$

is defined as

$$R_{\nabla}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z,$$

for all X, Y and $Z \in \text{Vect}(M)$.

Directly from the definition it is clear that

$$R_{\nabla}(X, Y)Z = -(-1)^{\langle \deg(X), \deg(Y) \rangle} R_{\nabla}(Y, X)Z, \quad (5)$$

for all X, Y and $Z \in \text{Vect}(M)$.

Proposition 8. *The Riemannian curvature tensor associated with the Levi–Civita connection satisfies the \mathbb{Z}_2^n -graded first Bianchi identity*

$$(-1)^{\langle \deg(X), \deg(Z) \rangle} R_{\nabla}(X, Y)Z + (-1)^{\langle \deg(Y), \deg(X) \rangle} R_{\nabla}(Y, Z)X + (-1)^{\langle \deg(Z), \deg(Y) \rangle} R_{\nabla}(Z, X)Y = 0,$$

for any and all X, Y and $Z \in \text{Vect}(M)$.

Proof. The proof follows in almost identically the same way as the classical proof upon a little care with the sign factors and as such we just sketch the proof here. It is more convenient to re-write the LHS of the first Bianchi identity as

$$Bi(X, Y, Z) := R_{\nabla}(X, Y)Z + (-1)^{\langle \deg(Y), \deg(X) + \deg(Z) \rangle} R_{\nabla}(Y, Z)X + (-1)^{\langle \deg(Z), \deg(Y) + \deg(X) \rangle} R_{\nabla}(Z, X)Y.$$

Then using the definition of the Riemann curvature tensor (Definition 13) we obtain

$$\begin{aligned} Bi(X, Y, Z) &= \nabla_X(\nabla_Y Z - (-1)^{\langle \deg(Z), \deg(Y) \rangle} \nabla_Z Y) - \nabla_{[X, Y]}Z \\ &+ (-1)^{\langle \deg(X), \deg(Y) + \deg(Z) \rangle} \nabla_Y(\nabla_Z X - (-1)^{\langle \deg(Z), \deg(X) \rangle} \nabla_X Z) - (-1)^{\langle \deg(X), \deg(Y) + \deg(Z) \rangle} \nabla_{[X, Y]}Z \\ &+ (-1)^{\langle \deg(Z), \deg(X) + \deg(Y) \rangle} \nabla_Z(\nabla_X Y - (-1)^{\langle \deg(X), \deg(Y) \rangle} \nabla_Y X) - (-1)^{\langle \deg(Z), \deg(X) + \deg(Y) \rangle} \nabla_{[Z, X]}Y, \end{aligned}$$

then using the fact that the Levi–Civita connection is torsionless (Definition 10),

$$\begin{aligned} &= \nabla_X[Y, Z] - \nabla_{[X, Y]}Z \\ &+ (-1)^{\langle \deg(X), \deg(Y) + \deg(Z) \rangle} \nabla_Y[Z, X] - (-1)^{\langle \deg(X), \deg(Y) + \deg(Z) \rangle} \nabla_{[Y, Z]}X \\ &+ (-1)^{\langle \deg(Z), \deg(X) + \deg(Y) \rangle} \nabla_Z[X, Y] - (-1)^{\langle \deg(Z), \deg(X) + \deg(Y) \rangle} \nabla_{[Z, X]}Y \end{aligned}$$

once again using the fact that the Levi–Civita connection is torsionless,

$$= [X, [Y, Z]] - [[X, Y], Z] - (-1)^{\langle \deg(X), \deg(Y) \rangle} [Y, [X, Z]].$$

This is nothing but the \mathbb{Z}_2^n -graded Jacobi identity for the standard Lie bracket of vector fields (here written in Loday–Leibniz form). Thus, $Bi(X, Y, Z) = 0$ and so the \mathbb{Z}_2^n -graded first Bianchi identity holds. \square

Remark 3. *Just as in the classical case, the proof of the first Bianchi identity only relies on the affine connection being symmetric, i.e., the torsion is zero. Thus, the first Bianchi identity holds for any symmetric affine connection and not just the Levi–Civita connection.*

Proposition 9. *The Riemann curvature tensor associated with the Levi–Civita connection satisfies*

$$\langle R_{\nabla}(X, Y)Z | W \rangle_g = -(-1)^{\langle \deg(Z), \deg(W) \rangle} \langle R_{\nabla}(X, Y)W | Z \rangle_g,$$

for all X, Y, Z and $W \in \text{Vect}(M)$.

Proof. The proposition can be established via direct calculation along the same lines as the classical case. First, directly from the definition of the Riemann curvature tensor and using the metric compatibility condition we observe that

$$\begin{aligned} \langle R_{\nabla}(X, Y)Z|W \rangle_g &= \langle \nabla_X \nabla_Y Z|W \rangle_g - (-1)^{\langle \deg(X), \deg(Y) \rangle} \langle \nabla_Y \nabla_X Z|W \rangle_g - \langle \nabla_{[X, Y]} Z|W \rangle_g \\ &= X \langle \nabla_Y Z|W \rangle_g - (-1)^{\langle \deg(X), \deg(Y) + \deg(Z) \rangle} \langle \nabla_Y Z|\nabla_X W \rangle_g \\ &\quad - (-1)^{\langle \deg(X), \deg(Y) \rangle} Y \langle \nabla_X Z|W \rangle_g + (-1)^{\langle \deg(Y), \deg(Z) \rangle} \langle \nabla_X Z|\nabla_Y W \rangle_g - \langle \nabla_{[X, Y]} Z|W \rangle_g. \end{aligned}$$

Then, after a little algebra

$$\begin{aligned} &\langle R_{\nabla}(X, Y)Z|W \rangle_g + (-1)^{\langle \deg(Z), \deg(W) \rangle} \langle R_{\nabla}(X, Y)W|Z \rangle_g \\ &= X \langle \nabla_Y Z|W \rangle_g + (-1)^{\langle \deg(Z), \deg(Y) \rangle} X \langle Z|\nabla_Y W \rangle_g \\ &\quad - (-1)^{\langle \deg(X), \deg(Y) \rangle} Y \langle \nabla_X Z|W \rangle_g - (-1)^{\langle \deg(X), \deg(Y) + \deg(Z) \rangle} Y \langle Z|\nabla_X W \rangle_g \\ &\quad - \langle \nabla_{[X, Y]} Z|W \rangle_g - (-1)^{\langle \deg(Z), \deg(X) + \deg(Y) \rangle} \langle Z|\nabla_{[X, Y]} W \rangle_g, \end{aligned}$$

then using metric compatibility again

$$= (XY - (-1)^{\langle \deg(X), \deg(Y) \rangle} YX - [X, Y]) \langle Z|W \rangle_g = 0. \quad \square$$

The covariant derivative of the Riemann curvature tensor is defined via the product rule for the covariant derivative, i.e.,

$$\begin{aligned} (\nabla_Z R_{\nabla})(X, Y)W &= \nabla_Z (R_{\nabla}(X, Y)W) - R_{\nabla}(\nabla_Z X, Y)W \\ &\quad - (-1)^{\langle \deg(Z), \deg(X) \rangle} R_{\nabla}(X, \nabla_Z Y)W - (-1)^{\langle \deg(Z), \deg(X) + \deg(Y) \rangle} R_{\nabla}(X, Y)\nabla_Z W \end{aligned}$$

for all X, Y, Z (homogeneous) and $W \in \text{Vect}(M)$.

Proposition 10. The Riemann curvature tensor associated with the Levi–Civita connection satisfies the \mathbb{Z}_2^n -graded second Bianchi identity

$$(-1)^{\langle \deg(Y), \deg(Z) \rangle} (\nabla_Z R_{\nabla})(X, Y) + (-1)^{\langle \deg(X), \deg(Y) \rangle} (\nabla_Y R_{\nabla})(Z, X) + (-1)^{\langle \deg(Z), \deg(X) \rangle} (\nabla_X R_{\nabla})(Y, Z) = 0.$$

Proof. The proposition follows from a direct computation along the same lines as the classical calculation. We will only outline the proof here. We denote the LHS of the second Bianchi identity as $Bii(X, Y, Z)W$, for arbitrary X, Y, Z (homogeneous) and $W \in \text{Vect}(M)$. Using the fact that the Levi–Civita connection is torsionless, we can, after a little algebra, write

$$Bii(X, Y, Z)W = \sum_{cyclic} (-1)^{\langle \deg(X), \deg(Z) \rangle} ((\nabla_X R_{\nabla})(Y, Z)W + R_{\nabla}(X, Y)\nabla_Z W - R_{\nabla}([X, Y], Z)W).$$

Then using the definition of the Riemannian curvature tensor, we obtain an expression involving one, two and three ∇ 's. However, all but the terms involving a single ∇ cancel in the cyclic sum. One then obtains

$$Bii(X, Y, Z)W = \sum_{cyclic} (-1)^{\langle \deg(X), \deg(Z) \rangle} \nabla_{[[X, Y], Z]} W,$$

and then we recognize the Jacobi identity for the Lie bracket of vector fields and so $Bii(X, Y, Z) = 0$. \square

Proposition 11. In local coordinate x^I the components of the Riemann curvature tensor, which are defined by $R_{\nabla}(\partial_I, \partial_J)\partial_K := R_{IJK}^L \partial_L$, of the Levi–Civita connection on a Riemannian \mathbb{Z}_2^n -manifold (M, g) are given by

$$\begin{aligned} R_{IJK}^L &= \frac{\partial \Gamma_{KJ}^L}{\partial x^I} - (-1)^{\langle \deg(I), \deg(J) \rangle} \frac{\partial \Gamma_{KI}^L}{\partial x^J} \\ &\quad + (-1)^{\langle \deg(I), \deg(J) + \deg(K) + \deg(M) \rangle} \Gamma_{KJ}^M \Gamma_{MI}^L \\ &\quad - (-1)^{\langle \deg(J), \deg(K) + \deg(M) \rangle} \Gamma_{KI}^M \Gamma_{MJ}^L. \end{aligned}$$

Proof. This follows from direct computation. \square

2.8. The Ricci Tensor and Ricci Scalar

The Ricci curvature tensor and Ricci scalar can also be generalized to the setting of Riemannian \mathbb{Z}_2^n -manifolds.

Definition 14. The Ricci curvature tensor of an affine connection is the symmetric rank-2 covariant tensor defined as

$$\text{Ric}_{\nabla}(X, Y) = \text{tr} \left(Z \mapsto \frac{1}{2} (R_{\nabla}(Z, X)Y + (-1)^{\langle \deg(X), \deg(Y) \rangle} R_{\nabla}(Z, Y)X) \right),$$

for all X and $Y \in \text{Vect}(M)$. Here tr is the \mathbb{Z}_2^n -graded trace (see [9], Theorem 1.).

Remark 4. One can show that the tensor $\text{tr}(Z \mapsto R_{\nabla}(X, Y)Z)$ vanishes if the Riemannian metric is even, but not necessarily so if the Riemannian metric is odd. Combining this with the first Bianchi identity (see Proposition 8) one can prove that $\text{tr}(Z \mapsto R_{\nabla}(Z, X)Y)$ is not symmetric if the Riemannian metric is odd. Thus, we define the Ricci curvature tensor as the symmetrization of a trace to obtain a tensor with a well-defined symmetry for both even and odd Riemannian metrics.

Defining $R_{IJ} := \text{Ric}_{\nabla}(\partial_I, \partial_J)$, it is clear that the symmetry equates to $R_{IJ} = (-1)^{\langle \deg(I), \deg(J) \rangle} R_{JI}$.

Definition 15. Let (M, g) be a Riemannian \mathbb{Z}_2^n -manifold. The Ricci scalar (associated with the Levi–Civita connection) is defined as

$$S_{\nabla} := \text{tr}_g(\text{Ric}_{\nabla}) \in C^{\infty}(M),$$

where tr_g is the metric trace (see Definition 8).

Please note that $\deg(S_{\nabla}) = \deg(g)$, and that this has some direct consequences.

Proposition 12. Let (M, g) be a Riemannian \mathbb{Z}_2^n -manifold, then the Ricci scalar S_{∇} cannot be a non-zero constant if the metric g has non-zero \mathbb{Z}_2^n -degree.

Proof. It is sufficient to consider this question locally and so we can employ local coordinates. If the metric is of non-zero \mathbb{Z}_2^n -degree, then a non-zero Ricci scalar must locally be at least linear in one or more of the formal coordinates. Thus, it cannot be constant, i.e., locally the derivative of the Ricci scalar with respect to at least one of the formal coordinates is non-vanishing. The only possible constant of non-zero \mathbb{Z}_2^n -degree is zero. \square

Observation 4. The only possible Riemannian \mathbb{Z}_2^n -manifolds with constant, but non-zero, Ricci scalar are those whose Riemannian metric is degree zero.

Proposition 13. Let (M, g) be a Riemannian \mathbb{Z}_2^n -manifold with an odd metric. Then the Ricci scalar vanishes identically.

Proof. In any set of local coordinates, we see that

$$S_{\nabla} = (-1)^{\langle \deg(g), \deg(I) + \deg(J) \rangle} R_{JI} g^{IJ} (-1)^{\langle \deg(I), \deg(J) \rangle}.$$

Using the fact that the Ricci tensor is symmetric in its indices, while the inverse metric is skew-symmetric for an odd Riemannian metric (see Proposition 6), the contraction of the two will vanish. \square

2.9. The Covariant Divergence and the Connection Laplacian

The notions of the gradient of a function, the covariant divergence of a vector field and the connection Laplacian all generalize to the setting of \mathbb{Z}_2^n -manifolds. We restrict attention to the covariant divergence and the connection Laplacian (with respect to the Levi–Civita connection) to avoid subtleties of introducing Berezin volumes on \mathbb{Z}_2^n -manifolds (see [12]). The reader should note that unless the Riemannian metric is of degree zero, there is no canonical Berezin volume. Thus, in order not to introduce extra structure, one should consider the connection Laplacian and not the Laplace–Beltrami operator.

Definition 16. Let $f \in C^\infty(M)$ be an arbitrary function on a Riemannian \mathbb{Z}_2^n -manifold (M, g) . The gradient of f is the unique vector field $\text{grad}_g f$ such that

$$X(f) = (-1)^{\langle \deg(f), \deg(g) \rangle} \langle X | \text{grad}_g f \rangle_g,$$

for all $X \in \text{Vect}(M)$.

In local coordinates we have

$$\text{grad}_g f = (-1)^{\langle \deg(f), \deg(g) \rangle + \langle \deg(f) + \deg(g), \deg(I) \rangle} \frac{\partial f}{\partial x^I} g^{II} \frac{\partial}{\partial x^I}.$$

Please note that $\deg(\text{grad}_g f) = \deg(f) + \deg(g)$. It is easy to observe that

$$\text{grad}_g (ff') = (-1)^{\langle \deg(f), \deg(g) \rangle} f \text{grad}_g f' + (-1)^{\langle \deg(f'), \deg(f) + \deg(g) \rangle} f' \text{grad}_g f, \quad (6)$$

for homogeneous f and $f' \in C^\infty(M)$. Moreover, it is clear from the definition of the gradient that if $f = \mathbb{1}$, i.e., the constant function of value one, then its gradient is zero.

Definition 17. Let (M, g) be a Riemannian \mathbb{Z}_2^n -manifold and let ∇ be the associated Levi–Civita connection. The covariant divergence is the map

$$\text{Div}_{\nabla} : \text{Vect}(M) \rightarrow C^\infty(M),$$

given by

$$\text{Div}_{\nabla} X := \text{tr} (Z \mapsto \nabla_Z X),$$

for any arbitrary $X \in \text{Vect}(M)$.

Locally, the above definition amounts to

$$\text{Div}_{\nabla} X = (-1)^{\langle \deg(I), \deg(I) + \deg(X) \rangle} (\nabla_I X)^I = (-1)^{\langle \deg(I), \deg(I) + \deg(X) \rangle} \frac{\partial X^I}{\partial x^I} + (-1)^{\langle \deg(I), \deg(I) + \deg(J) \rangle} X^J \Gamma_{II}^I.$$

The covariant divergence satisfies the expected properties (it is straightforward to check these)

$$\operatorname{Div}_{\nabla}(X + cY) = \operatorname{Div}_{\nabla}X + c \operatorname{Div}_{\nabla}Y, \text{ and} \quad (7)$$

$$\operatorname{Div}_{\nabla}(fX) = (-1)^{\langle \deg(f), \deg(X) + \deg(g) \rangle} \langle X | \operatorname{grad}_g f \rangle_g + f \operatorname{Div}_{\nabla}X, \quad (8)$$

with X and $Y \in \operatorname{Vect}(M)$ and $c \in \mathbb{R}$, and $f \in C^\infty(M)$. The connection Laplacian is then defined as “div of the grad”. More carefully we have the following definition.

Definition 18. Let (M, g) be a Riemannian \mathbb{Z}_2^n -manifold and let ∇ be the associated Levi–Civita connection. The connection Laplacian (acting on functions) is the differential operator of \mathbb{Z}_2^n -degree $\deg(g)$ defined as

$$\Delta_g f = \operatorname{Div}_{\nabla}(\operatorname{grad}_g f),$$

for any and all $f \in C^\infty(M)$.

Directly from the definition of the gradient, one can see that the connection Laplacian has no zeroth-order term, i.e., $\Delta_g \mathbb{1} = 0$. By construction, it is obvious that the connection Laplacian is at most second-order. The connection Laplacian on a Riemannian \mathbb{Z}_2^n -manifold is exactly what one expects given the classical connection Laplacian on a Riemannian manifold.

Proposition 14. Let (M, g) be Riemannian \mathbb{Z}_2^n -manifold with an even Riemannian metric. Furthermore, let Δ_g be the connection Laplacian associated with the Levi–Civita connection. Then the violation of the Leibniz rule over the product f' for Δ_g is given by $(-1)^{\langle \deg(f'), \deg(g) \rangle} 2 \langle \operatorname{grad}_g f | \operatorname{grad}_g f' \rangle_g$, where f and f' are homogeneous but otherwise arbitrary functions.

Proof. We prove the proposition by constructing the ‘anomaly’ to the Leibniz rule for the connection Laplacian. Let (M, g) be a Riemannian \mathbb{Z}_2^n -manifold where the Riemannian metric is of arbitrary \mathbb{Z}_2^n -degree (we will not insist that it is even at this stage). Using (6) and (8) we obtain

$$\begin{aligned} \Delta_g(ff') &= \operatorname{Div}_{\nabla} \left((-1)^{\langle \deg(f), \deg(g) \rangle} f \operatorname{grad}_g f' + (-1)^{\langle \deg(f'), \deg(f) + \deg(g) \rangle} f' \operatorname{grad}_g f \right) \\ &= (-1)^{\langle \deg(f), \deg(f') + \deg(g) \rangle} \langle \operatorname{grad}_g f' | \operatorname{grad}_g f \rangle_g + (-1)^{\langle \deg(f), \deg(g) \rangle} f(\Delta_g f') \\ &\quad + (-1)^{\langle \deg(f'), \deg(g) \rangle} \langle \operatorname{grad}_g f | \operatorname{grad}_g f' \rangle_g + (-1)^{\langle \deg(f'), \deg(f) + \deg(g) \rangle} f'(\Delta_g f) \\ &= (\Delta_g f)f' + (-1)^{\langle \deg(g), \deg(f) \rangle} f(\Delta_g f') \\ &\quad + (-1)^{\langle \deg(f'), \deg(g) \rangle} \left(1 + (-1)^{\langle \deg(g), \deg(g) \rangle} \right) \langle \operatorname{grad}_g f | \operatorname{grad}_g f' \rangle_g, \end{aligned}$$

with f and $f' \in C^\infty(M)$. Then, if the Riemannian metric is even the ‘anomaly’ is precisely as claimed in the proposition. \square

The proof of Proposition 14 implies that for odd Riemannian metrics the connection Laplacian is a vector field. However, as we shall show, the connection Laplacian associated with an odd Riemannian metric vanishes. Before we can prove this we need a lemma.

Lemma 4. Let (M, g) be a Riemannian \mathbb{Z}_2^n -manifold and let Γ_{JI}^K be the associated Christoffel symbols in some chosen coordinate system. Then

$$\begin{aligned} (-1)^{\langle \deg(I), \deg(I) + \deg(L) \rangle} 2g^{IL}\Gamma_{JI}^I &= (-1)^{\langle \deg(I), \deg(I) \rangle} g^{JI} \left(\frac{\partial g_{IM}}{\partial x^L} \right) g^{MI} \\ &\quad + (-1)^{\langle \deg(I), \deg(I) + \deg(L) \rangle} (1 - (-1)^{\langle \deg(g), \deg(g) \rangle}) g^{JL} \left(\frac{\partial g_{LM}}{\partial x^I} \right) g^{MI}. \quad (9) \end{aligned}$$

In particular, if the Riemannian metric is odd, then

$$g^{JL}\Gamma_{JI}^I = g^{JL} \left(\frac{\partial g_{LM}}{\partial x^I} \right) g^{MI}. \quad (10)$$

Proof. From Proposition 7 we observe that

$$\begin{aligned} (-1)^{\langle \deg(I), \deg(I) + \deg(L) \rangle} 2\Gamma_{LI}^I &= (-1)^{\langle \deg(I), \deg(I) + \deg(L) \rangle} \left(\frac{\partial g_{LM}}{\partial x^I} \right) g^{MI} \\ &+ (-1)^{\langle \deg(I), \deg(I) \rangle} \left(\frac{\partial g_{IM}}{\partial x^L} \right) g^{MI} \\ &- (-1)^{\langle \deg(M), \deg(I) + \deg(L) \rangle + \langle \deg(I), \deg(I) + \deg(L) \rangle} \left(\frac{\partial g_{IL}}{\partial x^M} \right) g^{IM}. \end{aligned}$$

Relabeling the contracted indices of the last term by $I \rightarrow M$ and $M \rightarrow I$, and then using the symmetry of the Riemannian metric and its inverse (see Proposition 6) we see that

$$\begin{aligned} &(-1)^{\langle \deg(M), \deg(I) + \deg(L) \rangle + \langle \deg(I), \deg(I) + \deg(L) \rangle} \left(\frac{\partial g_{IL}}{\partial x^M} \right) g^{IM} \\ &= (-1)^{\langle \deg(I), \deg(I) + \deg(L) \rangle + \langle \deg(g), \deg(g) \rangle} \left(\frac{\partial g_{LM}}{\partial x^I} \right) g^{IM}. \end{aligned}$$

This establishes the validity of (9). If the Riemannian metric is odd, then the first term of (9) vanishes as this involves the contraction of a symmetric and a skew-symmetric object, while the second term remains. This establishes (10). \square

Theorem 2. The connection Laplacian on a Riemannian \mathbb{Z}_2^n -manifold (M, g) with an odd Riemannian metric identically vanishes, i.e., it is a zero map.

Proof. A direct computation from the definitions and using

$$\frac{\partial g^{KI}}{\partial x^I} = -(-1)^{\langle \deg(I), \deg(K) + \deg(J) + \deg(g) \rangle} g^{KJ} \left(\frac{\partial g_{JL}}{\partial x^I} \right) g^{LI},$$

one can quickly show that in local coordinates

$$\begin{aligned} \Delta_g f &= (-1)^{\langle \deg(J), \deg(J) \rangle} g^{JI} \frac{\partial^2 f}{\partial x^I \partial x^J} \\ &- (-1)^{\langle \deg(I), \deg(I) + \deg(L) \rangle + \langle \deg(J), \deg(J) \rangle} g^{JL} \left(\frac{\partial g_{LM}}{\partial x^I} \right) g^{MI} \frac{\partial f}{\partial x^I} \\ &+ (-1)^{\langle \deg(I), \deg(I) + \deg(L) \rangle + \langle \deg(J), \deg(J) \rangle} g^{JL} \Gamma_{LI}^I \frac{\partial f}{\partial x^J}. \end{aligned}$$

It is clear from the proof of Proposition 14, or from the fact that the contraction of a symmetric object and a skew-symmetric object, that the second-order term vanishes for odd Riemannian metrics. Then using Lemma 4, and specifically (10), shows that $\Delta_g f = 0$ for all $f \in C^\infty(M)$ provided the Riemannian metric is odd. \square

Remark 5. The reader should be reminded of modular vector fields and the modular class of a Poisson manifold (see [34]). Via analogy, Theorem 2 is not surprising given that the modular class of a symplectic manifold vanishes, or in other words, Hamiltonian vector fields annihilate the Liouville volume. However, there is no direct generalization of the modular class for supermanifolds or \mathbb{Z}_2^n -manifolds equipped with odd Riemannian metrics.

Observation 5. *One cannot develop a theory of harmonic functions etc. on Riemannian \mathbb{Z}_2^n -manifolds with odd Riemannian metrics using the canonical connection Laplacian.*

Just for completeness, the connection Laplacian on Riemannian \mathbb{Z}_2^n -manifold with an even Riemannian metric is locally given by

$$\Delta_g f = (-1)^{\langle \deg(I), \deg(J) \rangle} g^{IJ} \left(\frac{\partial^2 f}{\partial x^I \partial x^J} - \Gamma_{IJ}^K \frac{\partial f}{\partial x^K} \right),$$

which is, up to the sign factors, in agreement with the classical connection Laplacian on a Riemannian manifold (see for example ([35], Section 2.6, p. 30). Deriving this local expression follows using the same steps as in the classical case (so Lemma 4 and Proposition 7 together with the symmetry of the metric and its inverse) and so details are left to the reader.

3. Discussion

In this paper, we have defined a Riemannian \mathbb{Z}_2^n -manifold and explored their basic properties, such as the Fundamental Theorem, i.e., a canonical symmetric metric compatible affine connection, the Levi-Civita connection, always exists irrespective of the degree of the Riemannian metric. We have deduced the \mathbb{Z}_2^n -graded generalizations of several objects of interest in Riemannian geometry such as the torsion and various curvature tensors associated with the Levi-Civita connection. In particular, it was shown that Riemannian \mathbb{Z}_2^n -manifolds with an odd metric, i.e., the total degree is 1, necessarily have vanishing scalar curvature. Thus, the Ricci scalar is not a very interesting curvature invariant in this case. A little more generally, the scalar curvature cannot be a non-zero constant unless the Riemannian metric is of \mathbb{Z}_2^n -degree zero. Thus, studying Riemannian \mathbb{Z}_2^n -manifolds with non-zero constant Ricci scalar is limited not just to the even case, but to the degree zero case. Recall that de Sitter (dS) and anti-de Sitter (AdS) are Lorentzian Riemannian manifolds with constant positive and constant negative scalar curvature, respectively. Both these manifolds are of significance in cosmology and quantum field theory (the AdS/CFT correspondence, for example). We have a higher graded “no-go theorem”.

Statement 1. *There is no direct analogue of dS or AdS space in \mathbb{Z}_2^n -geometry where the Riemannian metric is of non-zero \mathbb{Z}_2^n -degree.*

Four-dimensional Riemannian Einstein manifolds are important in the study of gravitational instantons and Lorentzian Einstein manifolds play a rôle in string theory, quantum gravity and can serve as the target space of various sigma models. A natural question here is if we have generalizations of Einstein manifolds where we replace the manifold with a Riemannian \mathbb{Z}_2^n -manifold with an arbitrary (but homogeneous) degree metric. The obvious thing to try is to set

$$\text{Ric}_\nabla = \kappa g,$$

where Ric_∇ is the Ricci curvature tensor associated with the Levi-Civita connection. It is clear that we require $\deg(\kappa) = \deg(g)$. Thus, $\kappa \in C^\infty(M)$ cannot be a non-zero constant if $\deg(g) \neq 0$. We have another higher graded “no-go theorem”.

Statement 2. *There is no direct analogue of an Einstein manifold in \mathbb{Z}_2^n -geometry where the Riemannian metric is of non-zero \mathbb{Z}_2^n -degree.*

In conclusion, the basic notions, concepts, and mathematical statements found in classical Riemannian geometry, with a little care, generalize to the setting of \mathbb{Z}_2^n -geometry. The challenge is to now find interesting examples and possible applications of Riemannian \mathbb{Z}_2^n -geometry in physics and classical differential geometry.

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