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Essays on Networks, Information Economics, and Dynamic Games of Populism and Conflict

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Essays on Networks, Information Economics, and Dynamic Games of Populism and Conflict

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Abstract

This dissertation consists of three chapters based on one pure theory paper and two applied theory papers. The overarching concept of the thesis is the development of tools and models to study strategic interactions among agents.

Symmetric Markovian Games of Commons with Potentially Sustainable Endogenous Growth. The objective of this study to develop a tool which give an exact formula for finding an interior symmetric Markovian strategies in differential games with linear constraints and a general time-separable utility function. Differential games of common resources that are governed by linear accumulation constraints have several applications. Examples include political rent-seeking groups expropriating public infrastructure, oligopolies expropriating common resources, industries using specific common infrastructure or equipment, capital flight problems, pollution, etc. Most of the theoretical literature employs specific parametric examples of utility functions. For symmetric differential games with linear constraints and a general time-separable utility function depending only on the player's control variable, we provide an exact formula for interior symmetric Markovian strategies. This exact solution (a) serves as a guide for obtaining some new closed-form solutions and for characterizing multiple equilibria and (b) implies that if the utility function is an analytic function, then the Markovian strategies are analytic functions, too. This analyticity property facilitates the numerical computation of interior solutions of such games using polynomial projection methods and gives potential for computing modified game versions with corner solutions by employing a homotopy approach

Populism and Polarization in Social Media Without Fake News: the Vicious Circle of Biases, Beliefs and Network Homophily. The objective of this study is jointly explaining the phenomenes of polarization in social networks and downgrading of

expert(unbiased) opinion from a new angle. We build a model of network dynamics with decision-making under incomplete information in order to understand the determinants of the observed gradual downgrading of expert opinion on complicated issues and the decreasing trust in science. We suggest a search and matching mechanism behind network formation of friends, claiming that internet has made search and matching less costly and more intensive. According to our simulations, just combining the internet's ease of forming networks with (a) individual biases, such as confirmation bias or assimilation bias, and (b) people's tendency to align their actions with those of peers, can lead to populist dynamics over time through a vicious circle. Even without fake news, biases lead to more network homophily and, over time, more homophily leads to actions that put more weight on biases and less weight on expert opinion. Networks allow fundamental biases to be enhanced by peer-induced amplification factors, a finding suggesting that education should perhaps focus on mitigating fundamental biases by promoting evidence-based attitudes towards complicated social and scientific issues.

Can a social planner manipulate network dynamics and solve coordination problems? The objective of this study is to develop the mechanism of welfare-improving network evolution under incomplete information. This paper aims to build an algorithm of network dynamics with decision-making under incomplete information. Accordingly, it tries to identify if a social planner reduces the influence of individual biases, such as confirmation bias or assimilation bias on agents' actions, and solve a coordination problem. The research questions are the following: " Can the social planner increase social welfare, by manipulating the set of possible invitations and annoyances, without directly changing a network structure?", " What are the main drivers of increasing social-planner utility functions?" "How do the results change if the social planner has incomplete information or wrong priors about

the fundamental state variable?" For this research, a "Liberal Social Planner" was created; a process through which network members get suggestions depending on its utility function. The results have potential applications for the management of social media platforms by the owners of these platforms. Platforms can develop robots that can help their users be more informed and more satisfied.

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Table of contents

INTRODUCTION

In this dissertation, we focus on developing tools and models for studying strategic interaction among agents in dynamics games. This dissertation consists of one pure theory paper and two applied theory papers. The first chapter develops a general result in differential games. This is an area for studying strategic interactions when agents are forward-looking and calculate their future strategic interactions. The second and third chapters, deal with the problem of strategic interactions under incomplete information, in the context of networks. Analytical tools for linking up strategic actions with the evolution of a network are developed in the second chapter. The last chapter extends the second chapter by recommending network-manager strategies that can lead to welfare-improving network evolution under incomplete information.

The first chapter is titled "Symmetric Markovian Games of Commons with Potentially Sustainable Endogenous Growth." It is joint work with Christos Koulovatianos. It appeared as a CFS (Center for Financial Studies) Working paper, No. 638 in 2019 and is forthcoming in *Dynamic Games and Applications*, in 2020.[1] The literature using Markovian differential games with linear constraints focuses on strategic interactions among agents and the growth rate of a common property resource. Such games are used for analyzing corruption, capital flight, pollution, etc. The models developed in the past relevant literature have used specific functional forms for utility functions and parametric assumptions for finding a closed-form solution. Going beyond these examples is complicated, as it requires knowledge of dynamic programming, metric space, and functional-analysis methods for characterizing the Markovian strategies.

In order to extend the analysis of differential games to more general utility functions

[1] The paper is available at the following link: https://doi.org/10.1007/s13235-020-00349-w

without having to resort to very abstract mathematics, the research question addressed in this paper is: "Can we develop the exact formula that can serve as a guide for developing well-grounded numerical solutions to Markovian differential games, in order to generalize these parametric examples so as to take the models closer to the data?". We answer to this research question in two steps. First, we develop an exact formula for finding the main interior solution of symmetric Markovian games with linear accumulation constraints of a common resource, shedding light on when such a solution exists. Second, we characterize the general solution, which can be used as a guide for finding corner solutions using numerical methods and a homotopy approach. For achieving the first step, of finding an exact solution, we show that the Hamilton-Jacobi-Bellman equation of a player's dynamic problem can be transformed into a Lagrange d'Alembert differential equation. We show that this differential equation has an exact interior solution when the integration constant is equal to zero. In this case, the Markovian strategy is equal to the indefinite integral of the inverse function of marginal utility. For achieving the second step, of characterizing the solution, we prove the analyticity of Markovian strategy in the case that the utility function of players is analytic. Analyticity is crucial for proving the existence of approximate numerical solutions. We further demonstrate our findings by characterizing some closed-form interior solutions, which are well-known in the literature, as well as demonstrating some new examples admitting closed form solutions that are new to the literature.

The second chapter, titled "Populism and Polarization in Social Media Without Fake News: the Vicious Circle of Biases, Beliefs and Network Homophily", is joint work with Christos Koulovatianos. It appeared as a CFS Working paper, No 629 in 2019, and as a Higher School of Economics Research Paper No. WP BRP 227/EC/2020. The aim of this research to provide a partial explanation for the phenomena of polarization, populistic behavior, and of downgrading expert opinion, observed in opinion polls in the last decades.

The second chapter titled "Populism and Polarization in Social Media Without Fake News: the Vicious Circle of Biases, Beliefs and Network Homophily", is also joint work with Christos Koulovatianos. We develop a model/method using evolutionary games on networks in order to provide a partial explanation that fake news is not the sole or dominant explanation to the observed growing polarization and populistic behavior in the past few decades. We use an extended version of the information-seeking "beauty-contest" game with higher-order beliefs, which was developed by Morris and Shin (2002) and Golub and Morris (2017). Compared to the standard beauty-contest model of Morris and Shin (2002), we introduce "biased assimilation" in agents' utility functions. Biased assimilation is a structural (perhaps education-based) inclination to push facts slightly away from reality. At the same time, agents try to align their actions closer to the actions of their friends ("belongingness" in agents' preferences). We introduce a search-and-matching mechanism for creating and deleting links, and we study the evolutionary dynamics of network structure. In equilibrium the network structure is characterized by more homophily, and homophily brings peer-induced amplification to structural biases, contributing to gradually downgrading expert opinions over time, as observed in the data.

The third chapter, titled "Can a social planner manipulate network dynamics and solve coordination problems?", appeared as a Higher School of Economics Research Paper No. WP BRP 229/EC/2020. The third chapter, titled "Can a social planner manipulate network dynamics and solve coordination problems?", is a single-authored paper. This paper recommends a way to solve the polarization problem in social networks arising from the mechanism explained in the second Chapter. In this chapter we also develop an evolutionary dynamic model/method, that introduces a "Liberal Social planner" who has no bias in his preferences. This social planner manipulates network dynamics in order to make agents' actions more pragmatist, closer to fundamentals. The research questions in this chapter are: "Can a social planner increase social welfare, by manipulating the set of possible invitations sent to network members for making new friends, or by manipulating the set of annoyances among friends that are exposed, without directly changing the network structure by obliging people to make friends or cut ties with existing friends?", "What are the main drivers behind welfare increases?", "How do the results change if the social planner has incomplete information or wrong priors about the fundamentals?".

Crucially, the third chapter focuses on developing a liberal manipulation strategy by a social planner, such as a network manager of a social-media platform, who does not directly affect agents' signals or actions. The mechanism which we developed in this paper can be used in the countries, where manipulation is not allowed by law. The concept of the "Liberal Social Planner" that we suggest, influences the process through which network members acquaint themselves with other network members for making new friends. Then the social planner leaves people free to choose whom to make a friend, and this approach helps people be more pragmatist. But how does this indirect manipulation work? We find that the key mechanism behind increasing social welfare is to increase the number of indegree nodes of central agents. This happens because agents can substitute expert information with private information from central nodes and make more informed decisions. Social planners who are more confident (or even sure, even if biased) about the fundamentals (e.g., of pricing houses for buying/selling) achieve better results. These results have potential applications to the management of social media platforms by the owners of these platforms. Platforms can develop robots that can help their users in bcoming more informed and more satisfied about real-life issues, such as housing prices.

Two chapters in this dissertation is based on a joint work with Christos Koulovatianos. In each project, my co-author and I contributed equaly to each stage of the work: the formulation of the problem, the exploration of potential results, the mathematical derivations.

1. CHAPTER

Symmetric Markovian Games of Commons with Potentially Sustainable Endogenous Growth

1.1 Introduction

Markovian differential games of common property resources have far-reaching applications. A substantial literature using such games with linear constraints, focusing on the question of how strategic interactions affect the growth rate of a common-property resource includes Tornell and Velasco (1992), Lane and Tornell (1996), Tornell (1997), Tornell and Lane (1999), Sorger (2005), and Long and Sorger (2006). This literature is surveyed and explained in Long (2010, pp. 130-136).[2] The questions examined by these models are corruption, rent seeking and cross-country capital flight. Similar applications include pollution problems and oligopolies exploiting a common resource.[3] The commons problems arising may lead to slow or negative growth of capital. Instead, resolving such commons problems may guarantee sustainable growth. While these models are useful, the literature restricts itself to parametric models with closed-form solutions. There is a need to develop further results that can serve as guides for developing well-grounded numerical solutions to such models, generalizing these parametric examples and taking the models to data.[4] Here, we first contribute to

[2] An earlier survey paper in differential games is Clemhout and Wan (1994). A recent paper by Kunieda and Nishimura (2018), extends the Tornell and Velasco (1992) model by introducing uncertainty and financial constraints. This study examines how commons problems are affected by imperfect financial markets and how the possibility for sustainable growth is affected by these commons problems. Although our model is deterministic, it can contribute to extending such analyses by using more general utility functions.

[3] An early application of Markovian differential games to pollution is Dockner and Long (1993).

[4] Typically, Markovian differential-game models require metric-space or other functional-analysis methods in order to prove that solutions exist, that they are well-behaved, or that they possess certain desirable functional properties. Such approaches are necessitated by the complexity of dynamic programming problems, especially if their constraints are nonlinear. Regarding the approximation-theory difficulties posed by dynamic-programming problems and an exposition of metric-space methods see, for example, Chow and Tsitsiklis (1989). Theoretical foundations of differential games are provided by Basar and Olsder (1999) and Dockner et al. (2000).

developing such results for the case of interior solutions of symmetric Markovian games with linear accumulation constraints of a common resource. Second, we further develop a characterization of the general solution that can serve as a guide for extending numerical solutions to addressing parameterizations with corner solutions.

We achieve the first goal of the paper, the derivation of an analytical characterization, in two steps. In a first step, restricting attention to Markovian differential games of common property resources with linear accumulation constraints and interior solutions, we provide a full characterization of the interior solution for any time-separable utility function depending only on the player's control variable. We show that the Hamilton-Jacobi-Bellman equation of a player's dynamic problem can be reduced into a form of the Lagrange-d'Alembert differential equation. We show that this differential equation has an exact solution that involves an integration constant. In the case that this constant of integration is equal to zero, the Markovian strategy equals the indefinite integral of an expression involving the inverse function of marginal utility. This solution best characterizes an interior solution of the game, provided that this interior solution exists. For the case where the constant of integration is different from zero, the possibility of multiple equilibria arises.[5] We do not focus on characterizing these multiple equilibria here. Nevertheless, the differential equation that we provide can serve as a guide for either characterizing these equilibria analytically, or for obtaining them numerically.

This exact formula that we derive for the case where the integration constant is zero, allows us to achieve the second goal of this paper. We prove the analyticity of the Markovian exploitation strategies under the assumption that the utility function of players is an analytic function, subject to some weak requirements.

[5] See, for example, Tsutsui and Mino (1990) and Dockner and Long (1993), who use a similar approach for characterizing multiple Markovian equilibria, but who are restricted to linear quadratic games.

We demonstrate the usefulness of our exact solution through finding some closed-form interior solutions which, in some cases, are not listed in the literature.[6] In addition, we discuss the usefulness of the analyticity result for Markovian strategies. Analyticity can help in employing polynomial projection methods for computing interior solutions and for guiding the parameterization of games in order to guarantee interior solutions. Yet, despite that the scope of our paper is restricted to characterizing games with interior solutions, our results can be useful for pursuing interesting extensions to commons problems involving corner solutions, such as resource-depletion, exploitation quota policies, etc. Specifically, our exact interior solution can provide a starting point for homotopy approaches that lead to corner solutions after gradually changing the parameterization of the problem.

The homotopy computational approach, explained by Garcia and Zangwill (1981) and Eaves and Schmedders (1999), further adapted to dynamic games by, e.g., Borkovsky et al. (2010) and Besanko et al. (2010), starts from a well-behaved and well-characterized solution to a model for certain parameter values. By changing parameter values gradually, one can proceed to more complicated versions of the model. For the common-property applications we have in mind, some parameterizations can imply a well-behaved interior solution and some other parameterizations of the same model can imply a complicated corner solution that is difficult to compute recursively. A key contribution of our paper regarding such a homotopy approach is that it can provide ways to find a well-behaved solution that can serve as the starting point for this method.

The exact or numerical solutions of our setup can also be extended to studying stochastic games numerically. An early paper showing that stochastic Markovian games of common-property resource extraction have tractable continuity properties is Amir (1996).

[6] Such solutions can provide insights for other extensions of dynamic games of commons with piece-wise linear constraints such as Colombo and Labrecciosa (2015) or partly-linear/partly- non-linear constraints, such as Benchekroun (2008).

1.2 Statement of the problem

There are N identical (symmetric) players consuming a common resource, k. Player $i \in \{1, ..., N\}$ consumes $c_i(t)$ units of $k(t)$ at time $t \geq 0$, and the evolution of the common resource, k, is driven by,

$$\dot{k}(t) = Ak(t) - \sum_{i=1}^{N} c_i(t) \ , \tag{1.1}$$

with $A > 0$.[7] Each player $i \in \{1, ..., N\}$ is infinitely-lived and maximizes the same utility time-separable utility function,

$$U\left((c_i(t))_{t\geq 0}\right) = \int_0^{\infty} e^{-\rho t} u(c_i(t))\, dt \ , \quad \text{for all } i \in \{1, ..., N\} \ , \tag{1.2}$$

with parameters $\rho > 0$ being the rate of time preference.

Assumption 1 *Function $u : \mathbb{C} \to \mathbb{R}$, $\mathbb{C} \subseteq \mathbb{R}_+$, is twice continuously differentiable and has $u'(c) > 0$ and $u''(c) < 0$ for all $c \in \mathbb{C}$.*

We state further assumptions on the momentary utility function, u, as we proceed with our analysis in order to intuitively justify them.[8] We focus on Markovian (memoryless) strategies, $\left(\{c_i(t) = C_i(k(t))\}_{i=1}^{N}\right)_{t\geq 0}$, i.e., on consumption strategies $\{C_i(k)\}_{i=1}^{N}$ that are time-invariant.

Definition 1 *A Markov perfect Nash equilibrium (MPNE) is a set of strategies $\{C_i^*(k)\}_{i=1}^{N}$ such that the corresponding consumption paths $\left(\{c_i^*(t) = C_i^*(k(t))\}_{i=1}^{N}\right)_{t\geq 0}$*

[7] Notice that we exclude $A = 0$, which is games with non-renewable resources. We focus on games with potentially sustainable resource outcomes.

[8] For example, unlike in many papers, such as in Dockner and Sorger (1996, p. 213), an upper bound is imposed on the consumption level, c, and the resource-reproduction function is also bounded in their study. Here in some cases of sustainable growth, c can grow to infinity. In examples that we present in a later section we identify the cases where an upper bound must be placed on c and cases in which such a bound does not apply.

simultaneously solve problems $\{\mathcal{P}_i\}_{i=1}^N$*, with* $\mathcal{P}_i$ *being player* i*'s problem for all* $i \in \{1, ..., N\}$*, given by,*

$$\mathcal{P}_i \begin{cases} \max_{(c_i(t),k(t))_{t\geq 0}} \int_0^\infty e^{-\rho t} u\left(c_i\left(t\right)\right) dt \\ \text{subject to,} \\ \dot{k}\left(t\right) = Ak\left(t\right) - \sum_{j\neq i} C_j^*\left(k\left(t\right)\right) - c_i\left(t\right) \ , \\ c_i\left(t\right) \in \mathbb{C} \ , \ k\left(t\right) \in \mathbb{K} \subseteq \mathbb{R}_+ \\ \text{given } k\left(0\right) = k_0 > 0 \ , \quad \lim_{t\to\infty} e^{-\rho t} J_i'\left(k\left(t\right)\right) k\left(t\right) = 0 \end{cases}$$

with $J_i(k)$ *being the value function of problem* $\mathcal{P}_i$*, and with*

$$\mathbb{K} = \left\{k \geq 0 \ \mid C_i^*\left(k\right) \in \mathbb{C} \ , \ i \in \{1, ..., N\}\right\} \ .$$

Definition 1 is equivalent to Definition 6.6 in Basar and Olsder (1999, Definition 6.6, p. 321) for the case of $T \to \infty$ therein. The Hamilton-Jacobi-Bellman (HJB) equation of problem $\mathcal{P}_i$ is,

$$\rho J_i\left(k\right) = \max_{c_i \in \mathbb{C}} \left\{ u\left(c_i\right) + J_i'\left(k\right) \left[Ak - \sum_{j\neq i} C_j^*\left(k\right) - c_i \right] \right\} \quad \text{for all } k \in \mathbb{K} \ , \tag{1.3}$$

with first-order conditions

$$u'\left(c_i\right) = J_i'\left(k\right) \ . \tag{1.4}$$

The first concept we focus on is this of interiority of MPNE, given by Definition 2.

Definition 2 *An interior Markov perfect Nash equilibrium (IMPNE) is a set of strategies* $\{C_i^*(k)\}_{i=1}^N$ *described by Definition 1, with* $\left(\{c_i^*(t) = C_i^*(k^*(t))\}_{i=1}^N\right)_{t\geq 0}$ *and* $(k^*(t))_{t\geq 0} = \left\{ k\left(t\right) \in \mathbb{K} \ \middle| \ \dot{k}\left(t\right) = Ak\left(t\right) - \sum_{i=1}^N C_i^*\left(k\left(t\right)\right) \ , t \geq 0 \ , \ k\left(0\right) = k_0 \right\}$*, such that for all* $t \geq 0$*,* $k^*(t) \in int(\mathbb{K})$ *and* $c_i^*(t) \in int(\mathbb{C})$ *for all* $i \in \{1, ..., N\}$*, where* $int(\cdot)$ *denotes the interior of a set.*

Based on Definition 2, we make Assumption 2 below, which is an existence assumption.

Assumption 2 *Function u is such that there is a symmetric IMPNE defined as in Definition 2 is guaranteed.*

We declare that this work does not intend to provide equilibrium results. For example, two key dynamic-games textbooks, Basar and Olsder (1999) and Dockner et al. (2000), do not contain general equilibrium-existence (sufficient) conditions. The vast dynamic-games literature is aware of well-behaved (parametric) dynamic-game setups where existence is guaranteed (while much of this literature focuses mostly on linear-quadratic games). However, there are no known general existence results for Markovian dynamic games, as, e.g., in optimal-control theory. A main reason for this absence of existence results is the fact that, as equation (1.3) above shows, the structure of the optimal-control problem of a player depends on properties of the optimal strategies of other players, $\left\{C_j^*(k)\right\}_{\substack{j=1\\j\neq i}}^{N}$, a complication that makes the search for sufficient existence conditions difficult: the functional properties of the optimal strategies of other players, $\left\{C_j^*(k)\right\}_{\substack{j=1\\j\neq i}}^{N}$, are not guaranteed, as these strategies are implicit functions, characterizing the equilibrium path. Therefore, instead of proving equilibrium existence, we assume equilibrium existence and we characterize the equilibrium for the specific general class of games that we examine, using existence as a working hypothesis.

The conditions on function u guaranteeing that Assumption 2 holds must be examined for specific utility functions on a case-by-case basis. Specifically, through Assumption 2, we first assume interiority and existence. Our main goal is to use Assumption 2 as a working hypothesis, in order to provide an exact solution that characterizes and identifies parametric classes of games that have the potential for delivering such interior solutions, either analytically, or numerically (through, e.g., projection numerical methods, as discussed in Section

1.5 below, for the class of analytic utility functions, that can be used for polynomial projections). Once such classes of games are identified, parametric constraints can be found in order to restrict the class of games to those that indeed have an interior solution (see Section 1.6 below). For all parametric cases studied in Section 1.6 below, it is crucial to demonstrate that the transversality condition is met under the identified parametric constraints. In case there are non-existence problems or genericity problems in the class of candidate games, this trial- and-error procedure identifies such problems and the class of candidate games can be discarded.

Finally, the risk of characterizing an equilibrium under the existence Assumption 2 for a game where an equilibrium turns out to not exist, is not an impediment. Instead, because all known applied Markovian games are parametric, the equilibrium characterization that we provide ultimately helps in spotting the cases of gemes where our characterization results are satisfied, but an equilibrium does not exist. In brief, while there are parametric examples of Markovian games used in the literature, this paper helps in identifying more such examples.

1.3 Exploiting properties of the Hamilton-Jacobi-Bellman equation

We focus on characterizing symmetric IMPNEs, as these are given by Definition 2, having $C_i^*(k) = C_j^*(k)$ for all $i, j \in \{1, ..., N\}$. Since u is strictly concave, u' is strictly decreasing and hence invertible. Therefore, (1.4) implies,

$$c_i = (u')^{-1} (J_i'(k)) \tag{1.5}$$

is a function of k, after assuming that $J'(k)$ is a well-defined strictly monotone function. We discuss conditions guaranteeing that $J'(k)$ is a well-defined strictly monotone function below. A crucial aspect of Assumption 2, is that interior solutions do not pose problems for the monotonicity of $J'(k)$.

By the symmetry of the problem,

$$C_i^*(k) = (u')^{-1}(J_i'(k)) \text{ for all } i \in \{1, ...N\} \ . \tag{1.6}$$

Therefore, we drop subscript i, and we use (1.6) in order to substitute $C_i^*(k)$ into (1.3)[9] . Moreover, substituting $c = (u')^{-1}(J'(k))$ into (1.3) we obtain *a special case of the Lagrange-d'Alembert first-order nonlinear differential equation* (cf. Polyanin and Zaitsev (2003)),

$$J(k) = \frac{A}{\rho} k J'(k) + f(J'(k)) \ . \tag{1.7}$$

in which,

$$f(J'(k)) \equiv \frac{1}{\rho}\left[u\left((u')^{-1}(J'(k))\right) - NJ'(k) \cdot (u')^{-1}(J'(k))\right] \ .$$

Before we proceed, we introduce the problem's Lagrange multiplier, λ, as,

$$\lambda = J'(k) > 0 \ , \tag{1.8}$$

which we will use throughout the next section. Notice that $\lambda(t) = J'(k(t)) > 0$ for all $t \geq 0$, an implication of (1.4) and Assumption 1.

In order to characterize and solve a Lagrange-d'Alembert equation such as this given by (1.7), we must examine two cases separately, distinguished by the relationship between parameters A and ρ: first, the case in which $A \neq \rho$ and second the case in which $A = \rho$. In this section we focus on the more general and more interesting case, this of $A \neq \rho$.

Differentiating both sides of (1.7) with respect to k we obtain,

$$\left(1 - \frac{A}{\rho}\right)\frac{J'(k)}{J''(k)} = \frac{A}{\rho}k + f'(J'(k)) \ . \tag{1.9}$$

9 Assumption 2 and interiority allows decision rules to be monotonic, offering the property of invertibility to the decision rule, which is used for obtaining our solutions below.

Let $K(\cdot)$ be the inverse function of $J'(\cdot)$.[10] Then

$$k = K(J'(k)) \ . \tag{1.10}$$

Notice that by differentiating both sides of (1.10) we obtain $K'(J'(k)) = 1/J''(k)$. Use again $\lambda = J'(k)$, the model's Lagrange multiplier, and substitute these terms into (1.9) in order to obtain,

$$K'(\lambda) = g(\lambda) \cdot K(\lambda) + h(\lambda) \ , \tag{1.11}$$

a first-order linear differential equation in $K(\lambda)$ with variable coefficients, in which,

$$g(\lambda) \equiv \frac{\frac{A}{\rho}}{1-\frac{A}{\rho}} \frac{1}{\lambda} \quad \text{and} \quad h(\lambda) \equiv \frac{1}{1-\frac{A}{\rho}} \frac{f'(\lambda)}{\lambda} \ .$$

The solution to (1.11) is obtained through an integrating factor and is of the form,

$$K(\lambda) = \omega e^{\int g(\lambda)d\lambda} + e^{\int g(\lambda)d\lambda} \cdot \int e^{-\int g(\lambda)d\lambda} h(\lambda)\, d\lambda \ , \tag{1.12}$$

in which $\omega \in \mathbb{R}$ is an *integration constant*. The integration constant, ω, is very important for specifying the class of equilibrium solutions we focus on in this paper.

1.3.1 Characterizing the inverse of the value function of a single player in a symmetric MPNE when $A \neq \rho$

Since $K(\cdot)$ is the inverse function of $J'(\cdot)$, we can set,

$$K(\lambda) = (J')^{-1}(\lambda) \ . \tag{1.13}$$

[10]Here, the assumption that the inverse of function $J'(\cdot)$ exists, is related to Assumption 2. In order that $J'(\cdot)$ be invertible, $J'(\cdot)$ must be a function (instead, e.g., of a corresponcence), and it must be monotonic. Typically, these properties are guaranteed when an equilibrium exists. We therefore proceed, without adding additional assumptions to Assumption 2. In specific cases of parametric examples, the invertibility of $J'(\cdot)$ can be examined using a case-by-case analysis and specific parametric constraints. We remind that in the case of Markovian games, existence results for general utility functions $u(\cdot)$ are not available. For this reason, we do not attempt even to find cases or properties of functions that lead to non-existence or even to a generic set of functions that either guarantee or exclude equilibrium existence.

Therefore, by characterizing $K(\lambda)$, we characterize the inverse of the value function of a single player in a symmetric MPNE.

The integral $\int g(\lambda)\, d\lambda$ has an explicit solution, namely,

$$\int g(\lambda)\, d\lambda = \xi \ln(\lambda) \ . \tag{1.14}$$

where $\xi \equiv -\frac{A}{A-\rho}$. Notice also that the expression for $f'(\lambda)$ can be simplified. Specifically,

$$f'(\lambda) = \frac{1}{\rho}\left\{\left((u')^{-1}\right)'(\lambda)\left[u'\left((u')^{-1}\right)(\lambda) - N\lambda\right] - N(u')^{-1}(\lambda)\right\} \ ,$$

and after utilizing the identity $u'\left((u')^{-1}\right)(\lambda) = \lambda$,

$$f'(\lambda) = -\frac{1}{\rho}\left[(N-1)\lambda\left((u')^{-1}\right)'(\lambda) + N(u')^{-1}(\lambda)\right] \ . \tag{1.15}$$

Therefore, equation (1.12) can be re-written as,

$$K(\lambda) = \omega\lambda^{\xi} - \frac{1}{A}\xi\lambda^{\xi}\int \lambda^{-\xi-1}\left[(N-1)\lambda\left((u')^{-1}\right)'(\lambda) + N(u')^{-1}(\lambda)\right] d\lambda \ . \tag{1.16}$$

In order calculate the derivative of the value function of player i, $J'(k) = \lambda$, we can rewrite equation (1.16) as,

$$K(\lambda) = \omega\lambda^{\xi} + \phi(\lambda) \ , \tag{1.17}$$

and,

$$\phi(\lambda) \equiv \frac{1}{A-\rho}\lambda^{\xi}\int \lambda^{-\xi-1}\left[(N-1)\lambda\left((u')^{-1}\right)'(\lambda) + N(u')^{-1}(\lambda)\right] d\lambda \ . \tag{1.18}$$

Equation (1.18) can be simplified, after making use of the result that

$$\int \lambda^{\zeta} h'(\lambda)\, d\lambda = \lambda^{\zeta} h(\lambda) - \zeta\int \lambda^{\zeta-1} h(\lambda)\, d\lambda \ ,$$

for some $\zeta \in R$, and for a function $h(\lambda)$ that is differentiable and integrable. Specifically, we set $h(\lambda) = (u')^{-1}(\lambda)$ in (1.18) to simplify $\phi(\lambda)$ and obtain,

$$\phi(\lambda) = \frac{1}{A-\rho}\left\{(N-1)(u')^{-1}(\lambda) + \left[N + \xi(N-1)\right]\lambda^{\xi}\int \lambda^{-\xi-1}(u')^{-1}(\lambda)\, d\lambda\right\} \ . \tag{1.19}$$

1.3.2 The role of the integration constant ω when $A \neq \rho$: examining or eliminating multiple MPNEs

Based on equations (1.13) and (1.17),

$$(J')^{-1}(\lambda) = \omega\lambda^{\xi} + \phi(\lambda) \ . \tag{1.20}$$

The key to proceeding with characterizing the MPNE strategies implied by (1.20), is to either obtain the inverse of the right-hand side of equation (1.20), or to obtain $\lambda = J'(k)$ as an implicit function of (1.20). Yet, since parameter ω is an integration constant, equation (1.20) implies that there are potentially infinite MPNE strategies. This is the point made early on by Tsutsui and Mino (1990) and Dockner and Long (1993), who studied linear-quadratic games. Up to Tsutsui and Mino (1990), the literature of linear-quadratic games was focusing only on MPNEs with linear strategies, as the constant ω in equation (1.20) was considered to be only equal to 0. Specifically, if the utility function is quadratic, $\phi(\lambda)$ is an affine function of λ, which implies that after setting $\omega = 0$ to equation (1.20), the MPNE strategies, $C(k)$ are affine functions of k.[11] However, when one sets $\omega \neq 0$, non-linear strategies arise in the linear quadratic game, too, giving rise to multiple equilibria. These multiple equilibria can exist, for example, because of an "incomplete transversality condition", perhaps best explained by Tsutsui and Mino (1990, p. 153), who demonstrate the indeterminacy of stationary steady states in the linear-quadratic game they examine. Several papers deal with the characterization of such multiple equilibria in settings with different resource reproduction functions than ours, such as Rincon-Zapatero et al. (1998), Dockner and Wagener (2014), and Colombo and Labrecciosa (2015).

An important note is that equation (1.20) generalizes and extends the literature on linear-quadratic games substantially. Specifically, while the resource-reproduction function in our

[11] See Tsutsui and Mino (1990, p. 144) and Dockner and Long (1993, p. 22). We demonstrate this point in a later section of this paper, too.

paper is linear of the "Ak" type, the objective function u of the players is quite general. The key to our generalization is the differential equation (1.9) and the proposed transformation given by (1.10). For example, the key differential equation in Tsutsui and Mino (1990, eq. 4.4, p. 143) that is a transformed analogue of our equation (1.9), is restricted to linear-quadratic games only, which are just a special case of our analysis in this paper.

Equation (1.20) can help in characterizing multiple equilibria, beyond standard "guess and verify" approaches. Finding or characterizing the inverse of the right-hand side of equation (1.20), $\omega\lambda^{\xi} + \phi(\lambda)$, can be challenging, as there are two additively-separable terms involving λ. Nevertheless, characterizing the special case with integration constant $\omega = 0$ can serve as a starting point. By setting $\omega = 0$, one can focus on finding the inverse function $\phi^{-1}(\lambda)$, in order to obtain $J'(k)$. Once this mission is accomplished analytically, then multiple equilibria can be derived analytically or can be numerically computed. In general, equation (1.20) can serve as a guide for computing all equilibria, including the case of $\omega \neq 0$, numerically. In the next section we focus on deriving an explicit solution for the case $\omega = 0$. The rest of the paper focuses on characterizing this special case of $\omega = 0$.

1.4 An explicit solution for the case with integration constant $\omega = 0$

There are two cases, $A \neq \rho$ and $A = \rho$ that we will examine separately. In the case of $A \neq \rho$, we focus on characterizing interior solutions, i.e., the symmetric IMPNE (see Definition 2), for the case of setting $\omega = 0$, that provides an exact solution to the problem. In the case of $A = \rho$, the solution is implicit but straightforward to characterize.

1.4.1 Case 1: $A \neq \rho$

The Lagrange-d'Alembert first-order nonlinear differential equation given by (1.7) allows us to arrive at an exact solution for the Markovian strategies $\{C_i^*(k)\}_{i=1}^N$. This solution is given by Proposition 1.

Proposition 1 *Under Assumptions 1 and 2, with $A \neq \rho$, the symmetric interior Markov perfect Nash equilibrium, IMPNE (see Definition 2), $\{C_i^*(k)\}_{i=1}^N$, corresponding to the case of setting $\omega = 0$ in equation (1.20), is given by,*

$$C_i^*(k) = C(k) = (u')^{-1}\left(\phi^{-1}(k)\right) \ , \ \ i \in \{1, ..., N\} , \tag{1.21}$$

with $\phi(\lambda)$ given by equation (1.19), provided that function u is such that $\phi'(\lambda) \neq 0$, $\phi(\lambda)$ is invertible, and $\lim_{t\to\infty} e^{-\rho t}\phi^{-1}(k(t))\, k(t) = 0$.

Proof

Due to Assumption 2 we do not need to worry about corner solutions. Therefore, we can set $\omega = 0$ in equation (1.20) and use the initial condition, $k(0) = k_0$ in order to identify function $J'(k)$ by solving,

$$k_0 = K(\lambda_0)|_{\omega=0} = \phi(\lambda_0) \ . \tag{1.22}$$

Since problem $\mathcal{P}_i$ falls in the class of discounted dynamic-programming problems with interior solutions, any admissible value k can be treated as initial conditions and (1.22) implies that the identification of function $J'(k)$ can be obtained by the rule,

$$J'(k) = \phi^{-1}(k) \ , \tag{1.23}$$

provided that ϕ is invertible. Equation (1.23) leads to (1.21). □

Proposition 1 is one of the key results of this paper, leading to further characterizations of the solution that we provide below. Nevertheless, we cannot ignore the special case of $A = \rho$ that follows.

1.4.2 Case 2: $A = \rho$

Setting $A = \rho$, makes (1.7) to collapse into Clairaut's differential equation,[12]

$$J(k) = kJ'(k) + f(J'(k)) \ . \tag{1.24}$$

After differentiating both sides of (1.24), we arrive at,

$$0 = [k + f'(J')] \cdot J'' \ . \tag{1.25}$$

Equation (1.25) leads to an explicit characterization of the Markovian strategies $\{C_i^*(k)\}_{i=1}^N$. This characterization is given by Proposition 2.

Proposition 2 *Under Assumptions 1 and 2, with $A = \rho$, the symmetric interior Markov perfect Nash equilibrium, IMPNE (see Definition 2), $\{C_i^*(k) = C(k)\}_{i=1}^N$ with $C(k)$ being an implicit function derived from the expression,*

$$(N-1)\frac{u'(C(k))}{u''(C(k))} = \rho k - NC(k) \ , \tag{1.26}$$

provided that $\lim_{t\to\infty} e^{-\rho t} u'(C(k(t)))\, k(t) = 0$.

Proof

Equation (1.25) implies,

$$k = -f'(J'(k)) \ . \tag{1.27}$$

[12] See Clairaut (1734).

Equation (1.27) holds because $J''(k) = 0$ is ruled out. To see that $J''(k) = 0$ cannot hold, consider the contrary, namely that $J'(k) = b$ for some constant $b \in \mathbb{R}$, which can be substituted into (1.24) to obtain the general solution,

$$J(k) = bk + f(b) \ . \tag{1.28}$$

However, the solution suggested by (1.28) is not acceptable, because it violates the transversality condition. Specifically, if (1.28) were acceptable, then equation (1.5) implies that $c(t) = (u')^{-1}(b) = \xi$, a constant, for all $t \geq 0$. But then $k(t)$ should satisfy the linear equation $\dot{k}(t) = \rho k(t) - N\xi$ with solution $k(t) = N\xi/\rho + e^{\rho t}(k_0 - N\xi/\rho)$, which implies that the transversality condition is violated unless $b = 0$, since $\lim_{t\to\infty} e^{-\rho t} J'(k(t))\, k(t) = b(k_0 - N\xi/\rho)$, which would not be equal to 0 for some k_0 if $b \neq 0$. But if $b = 0$, then $u'(c) = J'(k) = 0$, a contradiction since $u'(c) > 0$ for all $c \geq 0$.

Equations (1.27) and (1.15) imply,

$$\rho k = (N-1)\, J'(k) \left((u')^{-1}\right)' (J'(k)) + N (u')^{-1} (J'(k)) \ . \tag{1.29}$$

From equation (1.5),

$$C(k) = (u')^{-1}(J'(k)) \ , \tag{1.30}$$

which implies,

$$C'(k) = \left((u')^{-1}\right)' (J'(k))\, J''(k) \ . \tag{1.31}$$

Combining (1.30) and (1.31) with (1.29), we obtain,

$$\rho k - NC(k) = (N-1)\, \frac{J'(k)}{J''(k)} C'(k) \ . \tag{1.32}$$

From equation (1.5), $J'(k) = u'(C(k))$, which implies $J''(k) = u''(C(k))\, C'(k)$. Substituting these two last expressions into (1.32) proves equation (1.26). $\square$

Proposition 2 offers the ability to compute $C(k)$ using numerical or analytical methods when applicable. In the next section we combine Propositions 1 and 2 in order to show that the analyticity of the utility function implies the analyticity of $C(k)$.

However, before we proceed, we can note that, given the requirement of interior solutions (see Assumption 2), the strategies given by both Proposition 1 and Proposition 2 either are continuous or they can be constructed so as to be continuous. To see this, consider the function $\phi(\lambda)$, given by (1.19). Given Assumption 1, $\phi(\lambda)$ is derived from derivatives and integrals of the inverse of the twice continuously differentiable and strictly increasing function u. Therefore, $\phi(\lambda)$ is also a continuous function that we assume to be strictly monotonic. Therefore, $\phi^{-1}(\lambda)$ is also a continuous function and the strategy $C(k) = (u')^{-1}\left(\phi^{-1}(\lambda)\right)$ is continuous, too.[13] In addition, (1.26) yields the strategy, $C(k)$, as an implicit function. If we assume that u is thrice continuously differentiable, then $C(k)$ will also be continuously differentiable and, hence, continuous.[14]

Typically, the case of $A = \rho$ is not particularly interesting, as $A = \rho$ does not imply interesting dynamics in most cases. However, for the class of games restricted to obeying $A = \rho$, general sufficient conditions for existence of equilibrium may be easier to find. As in this paper we focus on characterizing parametric examples of Markovian games, we leave this extension for future research.[15]

[13]See, for example, Lang (1997, Theorem 4.2, p. 60), proving that the composition of continuous functions gives a continuous function.

[14]This property of continuity of strategies differs from Dockner and Sorger (1996, Theorem 1, p. 2015), where the strategies can be disconinuous functions.

[15]Notice also that the case of $\omega \neq 0$ (or $\omega = 0$) does not apply under $A = \rho$, as the integration constant ω does not appear in the Clairaut equation given by (1.24). The integration constant ω appears only in the Lagrange-d'Alembert equation (1.7).

1.5 Analytic utility functions

Propositions 1 and 2 give the opportunity to characterize the functional properties of Markovian strategies if the utility function of players is a real analytic function. For the definition of a real analytic function see Krantz and Parks (2002, Definition 1.1.5, p. 3). Specifically, a function $f : D \rightarrow \mathbb{R}$ with $D \subseteq \mathbb{R}$, is real analytic if for all $x_0 \in D$, the value $f(x)$ can be written as a power series of the form,

$$f(x) = \sum_{i=0}^{\infty} \alpha_i (x - x_0)^i \ .$$

Examples of analytic functions include polynomial function, the exponential function, the logarithmic function, or the power function. Proposition 3 proves that if the utility function of players is real analytic, then the symmetric Markovian strategies are also real analytic.

Proposition 3 *Under Assumptions 1 and 2, if $u(c)$ is a real analytic function, then the function $C(k)$ characterizing the symmetric Markov perfect Nash equilibrium $\{C_i^*(k) = C(k)\}_{i=1}^N$ is also a real analytic function.*

Proof

The proof is straightforward through the use of known results regarding analytic functions. Specifically, in the case of $A \neq \rho$, the expression of $\phi(\lambda)$ given by equation (1.19) involves inverses, derivatives, and integrals of utility functions. In addition, the exponential function, λ^α for some α, is also analytic. By the definition of real analytic functions, it is immediate to prove that the products and sums of real analytic functions are also real analytic. That the derivative of a real analytic function is also real analytic is proved in Krantz and Parks (2002, Proposition 1.1.14, p. 9). That the inverse of a real analytic function is also real analytic, the proof is in Krantz and Parks (2002, Theorem 1.5.3, p. 22). That the indefinite integral of a real analytic function is also real analytic is proved in Krantz and Parks (2002, Proposition 2.2.3, p. 30). These Propositions and Theorems show that $\phi^{-1}(\lambda)$ is real analytic. The expression for the Markovian strategies, $C(k)$, is given by (1.6) which involves the composition of real analytic functions. That compositions of real analytic functions are also real analytic is proved in Krantz and Parks (2002, Proposition 1.4.2, p. 19). These arguments prove the proposition for the case of $A \neq \rho$.

For the case of $A = \rho$, the Markovian strategies, $C(k)$, are an implicit function of equation (1.26). The proof of the implicit function theorem for real analytic functions, stating that the implicit functions of real analytic functions are also real analytic is given by Krantz and Parks (2002, Theorem 2.3.5, p. 40). □

1.5.1 Usefulness of analyticity for interior solutions

Proposition 3 is an important result for solving the problem numerically. For example, by using polynomial approximations to value functions and Markovian strategies, Proposition 3 guarantees that the approximated functions may remain in the same space of approximating polynomials and be convergent. With the approximation error remaining bounded, a well-behaved computation is guaranteed. For computation one can use either the exact solution given by (1.19) and (1.6), or recursive methods on the Lagrange-d'Alembert differential equation given by (1.7).

1.5.2 Analyticity and extensions to corner solutions through homotopy approaches

The central assumption we have made in this paper is Assumption 2, namely that the utility function allows for an interior solution. In practice, if solving such a game requires a numerical approach, it is difficult to know in advance which combinations of parameter values of the utility function with A and ρ indeed deliver an interior solution. Yet, a trial-and-error approach can help in verifying whether the dynamics of k implied by the strategies based on Proposition 1 or Proposition 2 reconfirm that the solution is interior or not. In brief, the game can be solved using Proposition 1 or Proposition 2 under the working hypothesis that for some parameter values the problem has an interior solution. If the interiority of the solution is not reconfirmed, then parameters can be re-calibrated.[16]

In the literature of differential games modeling commons problems interesting applications involve studying the potential depletion of a common-property resource, or placing quotas on resource exploitation. Such applications involve corner solutions. Analytical results for differential games with corner solutions are more difficult to obtain. Therefore, using numerical approximations may be the only resort. Nevertheless, calibrating a Markovian differential game with corner solutions so as to achieve convergence using recursive methods may be challenging. A possibility is to employ a homotopy computational approach as in Eaves and Schmedders (1999).

The homotopy computational approach is the practice of starting from calibrated parameters of a well-behaved solution. Afterwards, changing parameter values in a gradual, step-by-step fashion, one arrives to the desired parameterization of the computational problem. This homotopy procedure is explained in detail by Garcia and Zangwill (1981). Examples of papers such as Borkovsky et al. (2010) and Besanko et al. (2010) adapt the homotopy approach to some classes of dynamic games.

For following such a homotopy procedure, our results in this paper can be proved very useful. Propositions 1 or 2 can provide well-behaved interior solutions and can guide through parameterizations that guarantee the interiority of solutions. In a next step, parameteriza-

[16]The next section, where we present several closed-form solutions, gives "hands-on" examples of how the choice of parameters affects whether a Markov-perfect Nash-equilibrium solution is interior or not.

tions leading to corner solutions of research usefulness can be pursued in a step-by-step manner.

1.6 Examples with closed-form solutions

Propositions 1 and 2 lead to immediate results in cases where the problem we study admits closed-form solutions in the special case of setting $\omega = 0$ in equation (1.20). We list these examples below, demonstrating the usefulness of Propositions 1 and 2. First, we list cases that are more or less known. These known examples use utility functions from the comprehensive class of functions guaranteeing linear aggregation in dynamic models, identified by Koulovatianos et al. (2019).[17] The common feature of these examples is that resource exploitation strategies, $C(k)$, for the case of setting $\omega = 0$ in equation (1.20), are all linear functions in k. This common feature is essential for aggregation. In addition, it helps in deriving explicit dynamics for k, which helps in identifying parametric constraints guaranteeing that the solution is interior.[18] Examining these known cases is useful, as it helps in demonstrating our solution method.[19]

At a second stage, after our method is demonstrated, we present a final example that, to the best of our knowledge, does not exist in the literature. For this particular new case the resource exploitation strategies, $C(k)$, for the case of setting $\omega = 0$ in equation (1.20), are non-linear.[20] Crucially, our suggested method is essential for identifying this closed-form

[17]By linear aggregation we mean the concept of "exact linear aggregation", as it is defined, e.g., in Gorman (1961). In our case, this class of utility functions implies that exploitation strategies are affine functions.
[18]Examining also whether steady states are unique, and examining the stability properties of these steady states, are two tasks that can be pursued within these parametric examples. As these tasks usually require more parametric constraints and as these tasks are beyond the scope of this paper, that intends to provide the solution characterization, we leave this steady-state examination for future research.
[19]Most of our examples, except the sligtly more generalized case with "Gorman preferences" and the case of constant-absolute-risk-averstion preferences, which we present below, have been thoroughly studied by Gaudet and Lohoues (2008), who go beyond the use of linear resource-reproduction functions, specifying the types of resource reproduction functions that allow for linear strategies. We thank Hassan Benchekroun for pointing this paper to us.
[20]In Tasneem, Engle-Warnick and Benchekroun (2017) there is experimental evidence that players may

solution. Therefore, we believe that this new example demonstrates the usefulness of our approach.

1.6.1 Gorman preferences

Let's consider preferences as in Gorman (1961), given by,

$$u(c) = \frac{(c+\chi)^{1-\theta}}{1-\theta} \ , \tag{1.33}$$

in which $\theta > 0$ and $\chi \in \mathbb{R}$. Based on (1.18), the corresponding function ϕ is

$$\phi(\lambda) = \frac{1-(1-\theta)N}{\rho-(1-\theta)A}\lambda^{-\frac{1}{\theta}} - N\frac{\chi}{A} \ . \tag{1.34}$$

Therefore, if parameters ρ, A, θ, and N are such that,

$$\frac{1-(1-\theta)N}{\rho-(1-\theta)A} > 0 \ , \tag{1.35}$$

then $\phi' < 0$, implying that ϕ^{-1} exists. In particular equation (1.23) gives,

$$J'(k) = \phi^{-1}(k) = \left[\frac{\rho-(1-\theta)A}{1-(1-\theta)N}\right]^{-\theta} \left(k + N\frac{\chi}{A}\right)^{-\theta} \ , \tag{1.36}$$

which implies,

$$J(k) = \left[\frac{\rho-(1-\theta)A}{1-(1-\theta)N}\right]^{-\theta} \frac{\left(k+N\frac{\chi}{A}\right)^{1-\theta}}{1-\theta} \ .$$

Moreover, (1.6) gives,

$$c^* = C(k) = \frac{\rho-(1-\theta)A}{1-(1-\theta)N}k - \frac{\chi}{A} \cdot \frac{A-N\rho}{1-(1-\theta)N} \ . \tag{1.37}$$

Substituting (1.37) into the constraint $\dot{k} = Ak - NC(k)$, and solving the resulting linear differential equation, we obtain the explicit dynamics of $k(t)$, namely,

$$k(t) = -\frac{N\chi}{A} + e^{\frac{A-N\rho}{1-(1-\theta)N}t}\left[k(0) + \frac{N\chi}{A}\right] \ . \tag{1.38}$$

choose both linear and nonlinear strategies. The theoretical model employed in Tasneem, Engle-Warnick and Benchekroun (2017) allows for multiple equilibria, providing a clear distinction between linear and nonlinear equilibria. The evidence that non-linear strategies may be chosen by players, supports the usefulness of our new example. We are indebted to Hassan Benchekroun for making this point to us.

Equation (1.38) implies that once $k(0) > -N\chi/A$, $k(t) > -N\chi/A$ for all $t \geq 0$, making $J(k(t))$ be well-defined for all $t \geq 0$. Therefore,

$$k(0) > -\frac{N\chi}{A} \ , \tag{1.39}$$

is one of the necessary parametric constraints of this problem.[21] In order to examine the parametric constraints for guaranteeing that the transversality condition is met, we use (1.38) and (1.36) to obtain,

$$J'(k(t)) = \left[\frac{\rho - (1-\theta)A}{1-(1-\theta)N}\right]^{-\theta} \left[k(0) + N\frac{\chi}{A}\right]^{-\theta} e^{-\theta\frac{A-N\rho}{1-(1-\theta)N}t} \ . \tag{1.40}$$

Combining (1.40) and (1.38) shows that $\lim_{t\to\infty} e^{-\rho t} J_i'(k(t))\, k(t) = 0$ is equivalent to,

$$\lim_{t\to\infty}\left\{-\frac{N\chi}{A}e^{-\left[\rho+\frac{\theta(A-N\rho)}{1-(1-\theta)N}\right]t} + \left[k(0) + \frac{N\chi}{A}\right] e^{-\left[\rho-\frac{(1-\theta)(A-N\rho)}{1-(1-\theta)N}\right]t}\right\} = 0 \ . \tag{1.41}$$

After some algebra, the requirement implied by (1.41) that $\rho+\theta(A-N\rho)/[1-(1-\theta)N] > 0$, is simplified to,

$$\frac{\rho(1-N) + \theta A}{1-(1-\theta)N} > 0 \ . \tag{1.42}$$

The condition $\rho - (1-\theta)(A-N\rho)/[1-(1-\theta)N] > 0$, which is the other requirement implied by (1.41), is equivalent to (1.35). This equivalence can be verified after some algebra. Therefore, conditions (1.35), (1.39) and (1.42) are the three parametric requirements that guarantee a well-behaved solution.[22] An important observation from (1.38) is that these three conditions do not rule out the possibility of sustainable perpetual growth, i.e. $k(t) \to \infty$. Obviously from (1.38), sustainable growth occurs if $(A-N\rho)/[1-(1-\theta)N] > 0$.

A crucial observation is that equation (1.37) holds also for the case where $A = \rho$. Specifically, after setting $A = \rho$ in equation (1.37), the resulting strategy $C(k)|_{A=\rho}$ satisfies the

[21] Apparently, combining (1.39), (1.38), and (1.37) is necessary in order to identify parametric restrictions guaranteeing that $C(k(t)) > 0$ for all $t \geq 0$, consistently with an interior solution.

[22] Importantly, these three parametric requirements are related to the invertibility of function $J'(\cdot)$, as discussed above.

condition given by (1.26). We are not aware of any paper in the literature solving this problem for $\chi \neq 0$. Therefore, equation (1.37) is a novelty of this paper.

Finally, notice that by adding the constant $-1/(1-\theta)$ to the utility function given by (1.33), a modification that does not affect optimization, we can consider the case where $\theta = 1$, which leads to having logarithmic utility, since, after using L'Hôpital's rule,

$$\lim_{\theta \to 1} \frac{(c+\chi)^{1-\theta} - 1}{1-\theta} = \ln(c+\chi) \ .$$

1.6.1.1.Special case: CRRA preferences ($\chi = 0$) A familiar example in the literature (see Lane and Tornell, 1996) is this of CRRA preferences,

$$u(c) = \frac{c^{1-\theta}}{1-\theta} \ ,$$

which is the case of setting $\chi = 0$ in (1.33). Setting $\chi = 0$ in (1.37) gives

$$c^* = C(k) = \frac{\rho - (1-\theta)A}{1-(1-\theta)N} k \ ,$$

which coincides with the solution in Lane and Tornell (1996, eq. 17, p. 221).

1.6.2 Constant absolute risk aversion (CARA) preferences

Let a utility function representing CARA preferences, namely,

$$u(c) = -e^{-\frac{1}{\beta}c} \ , \tag{1.43}$$

in which $\beta > 0$. Based on (1.18), the corresponding function ϕ is

$$\phi(\lambda) = -\frac{\beta N}{A}\left[\ln(\beta) + \frac{N + \xi(N-1)}{\xi N} + \ln(\lambda)\right] \ . \tag{1.44}$$

Equation (1.44) implies that $\phi'(\lambda) < 0$ for all $\lambda > 0$. Therefore, ϕ^{-1} exists, with equation (1.23) implying,

$$J'(k) = \phi^{-1}(k) = \frac{e^{\frac{A-\rho}{A} - \frac{N-1}{N}}}{\beta N} e^{-\frac{A}{\beta N}k} \ , \tag{1.45}$$

leading to,

$$J(k) = -\frac{e^{\frac{A-\rho}{A}-\frac{N-1}{N}}}{A} e^{-\frac{A}{\beta N}k} .$$

In turn, (1.6) gives,

$$c^* = C(k) = \frac{A}{N}k - \frac{\beta(A-\rho)}{A} + \beta\frac{N-1}{N} . \tag{1.46}$$

After inserting (1.46) into the constraint $\dot{k} = Ak - NC(k)$, we obtain $\dot{k}(t) = \beta(A-\rho N)/A$, which has the obvious solution,

$$k(t) = k(0) + \beta\frac{A-\rho N}{A}t . \tag{1.47}$$

Equations (1.47) and (1.46) reveal the parametric constraint that guarantee an interior solution. Specifically, equations (1.47) shows that,

$$A - \rho N \geq 0 , \tag{1.48}$$

guarantees $k(t) \geq k(0) > 0$ for all $t \geq 0$. Equation (1.46) reveals that placing a constraint on the initial conditions $k(0)$, namely,

$$k(0) \geq \frac{\beta(A-\rho)}{A} - \beta\frac{N-1}{N} , \tag{1.49}$$

guarantees $c(t) \geq c(0) > 0$ for all $t \geq 0$ if (1.48) also holds. In summary, inequalities (1.48) and (1.49) guarantee the interiority of the solution given by (1.46).

Combining (1.47) with (1.45) leads to a tractable expression for the transversality condition. Specifically,

$$\lim_{t\to\infty} e^{-\rho t} J_i'(k(t))\, k(t) = \frac{e^{\frac{A-\rho}{A}-\frac{N-1}{N}}}{A} e^{-\frac{A}{\beta N}k(0)} \lim_{t\to\infty}\left[k(0)\, e^{-\frac{A}{N}t} + \beta\frac{A-\rho N}{A}\frac{t}{e^{\frac{A}{N}t}}\right] . \tag{1.50}$$

Equation (1.50) reveals that, in the case of CARA preferences, no parametric restrictions on A, ρ, β, N, and $k(0)$ beyond inequalities (1.48) and (1.49) are needed in order to ensure that the transversality condition is met. Importantly, equation (1.47) reveals that sustainable growth is possible. Finally, exactly as in the case of Gorman preferences, equation (1.46) holds also for the case where $A = \rho$, with the strategy $C(k)|_{A=\rho}$ satisfying condition (1.26).

1.6.3 Quadratic preferences

Let the utility function be,

$$u(c) = -\frac{1}{2}(\chi - c)^2 \ , \tag{1.51}$$

with $0 \leq c < \chi$. It is broadly known that linear quadratic differential games have linear strategies as solutions.[23] Combining (1.51) with (1.18), the corresponding function ϕ is

$$\phi(\lambda) = -\frac{2N-1}{2A-\rho}\lambda + N\frac{\chi}{A} \ .$$

Therefore, ϕ^{-1} exists and equation (1.23) implies,

$$J'(k) = \phi^{-1}(k) = \frac{2A-\rho}{2N-1}\left(N\frac{\chi}{A} - k\right) \ , \tag{1.52}$$

leading to,

$$J(k) = -\frac{1}{2}\frac{2A-\rho}{2N-1}\left(N\frac{\chi}{A} - k\right)^2 \ .$$

Equation Moreover, (1.6) combined with (1.52) give the formula of the Markovian strategy,

$$c^* = C(k) = \frac{2A-\rho}{2N-1}k + \frac{\chi}{A}\frac{\rho N - A}{2N-1} \ . \tag{1.53}$$

Important in this example is to restrict parameters so that $2A - \rho > 0$, and to use (1.53) in order to select N, A, ρ, χ, $k(0)$ so that the dynamics of k imply $k(t) < N\chi/A$ for all $t \geq 0$, guaranteeing that $J'(k(t)) > 0$ for all $t \geq 0$, and that the solution is interior.

[23] A study explaining that non-linear strategies can also exist is Tsutsui and Mino (1990). Nevertheless, focusing on interior solutions is important on whether such non-linear strategies can exist or not in linear quadratic games.

These conditions can be found after we substitute the strategies given by (1.53) into $\dot{k}(t) = Ak(t) - NC(k(t))$, and after solving for the explicit dynamics of $k(t)$. Specifically, these dynamics are given by,

$$k(t) = N\frac{\chi}{A} + e^{-\frac{A-\rho N}{2N-1}t}\left[k(0) - N\frac{\chi}{A}\right] . \tag{1.54}$$

Equation (1.54) tells us that while $k(0) < N\chi/A$, the parametric constraint needed for guaranteeing that $k(t) < N\chi/A$ for all $t \geq 0$ is,

$$A > \rho N . \tag{1.55}$$

Notice that (1.55) implies $A > \rho/2$, which is a necessary condition guaranteeing $J'(k(t)) > 0$ for all $t \geq 0$. In addition, using (1.54) and (1.52), we can verify that the transversality condition holds if (1.55) holds, since $\lim_{t\to\infty} e^{-\rho t} J_i'(k(t))\, k(t) = 0$ is equivalent to,

$$\lim_{t\to\infty} \frac{N\chi}{A} e^{-\left[\rho + \frac{A-\rho N}{2N-1}\right]t} - \left[N\frac{\chi}{A} - k(0)\right] e^{-\left[\rho + 2\frac{A-\rho N}{2N-1}\right]t} = 0 .$$

As we saw above, similar parametric constraints are needed in the case of Gorman preferences in order to guarantee that $J'(k(t)) > 0$ for all $t \geq 0$ and that the transversality condition is met.[24] Nevertheless, these parametric constraints in the case of Gorman preferences do not rule out the possibility of sustainable growth. On the contrary, in the case of quadratic preferences, because the utility function has a bliss point, the growth of $k(t)$ must be bounded above. This feature of linear quadratic games motivates the main purpose of this paper, which is to discover further solutions for Markovian games of commons with linear constraints.

[24]Specifically, in the case of Gorman preferences, the parametric restrictions on N, A, ρ, χ, θ, $k(0)$, given by conditions (1.35), (1.39) and (1.42), are needed in order to guarantee that $k(t) > -N\chi/A$ for all $t \geq 0$.

1.6.4 New Example with non-linear exploitation strategy: demonstrating our method

Here we consider a utility function that does not fall in the class of preferences that lead to aggregation (see Koulovatianos et al., 2019, p. 172, Theorem 1). Specifically, the resource exploitation strategies, $C(k)$, are not linear in k. The utility function is,

$$u(c) = c - \kappa c^{\frac{3}{2}} , \tag{1.56}$$

where $\kappa > 0$. Notice that for $u'(c) = 1 - 3/2\kappa c^{1/2} > 0$ we need to place an upper bound on c. The requirement $u'(c) > 0$ holds if and only if,

$$0 \leq c < \frac{4}{9\kappa^2} \equiv \bar{c} . \tag{1.57}$$

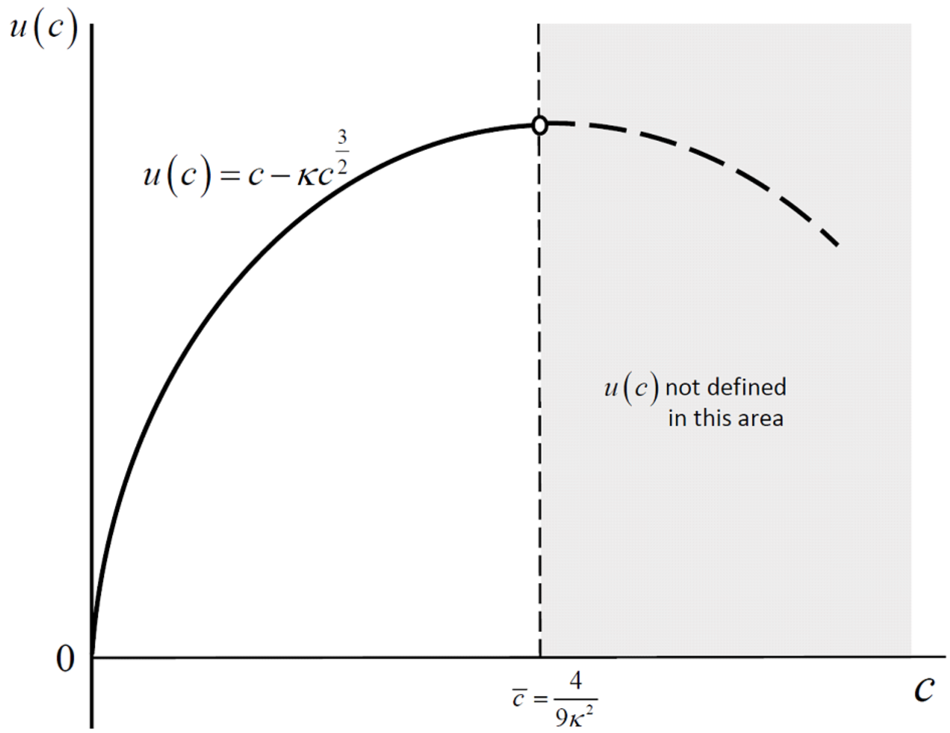

Figure 1.1 The utility function $u(c) = c - \kappa c^{3/2}$ is not defined in the shaded area.

Beyond the value of $\bar{c} = 4/(9\kappa^2)$ for c, $u(c)$ becomes downward-sloping, as shown by Figure 1.1. As marginal utility becomes negative for $c > \bar{c}$, one can view $\bar{c}$ as the bliss point of this utility function. Think, for example, that c is the consumption of a renewable resource such as fish, and that $\bar{c}$ is the maximum instantaneous flow of a country's fish consumption, as the residents of a country become instantaneously satiated by consuming fish. As for the linear technology of the fish reproduction stock, k, driven by Ak, think of Ak as a production function of fish through sustainable fish-farming.[25]

Within the range of values for c given by (1.57), the inverse function of $u'(c)$ is given by,

$$(u')^{-1}(\lambda) = \frac{4}{9\kappa^2}(1-\lambda)^2 \ . \tag{1.58}$$

After introducing (1.58) into (1.18) and after some algebra, the corresponding function ϕ is,[26]

$$\phi(\lambda) = \eta(\lambda-\theta)^2 + \psi \ , \tag{1.59}$$

where,

$$\psi = \frac{4}{9\kappa^2}\left[\frac{N}{A} - \left(\frac{2N-1}{2A-\rho}\right)^2 \frac{3A-2\rho}{3N-2}\right] , \tag{1.60}$$

$$\theta = \frac{3A-2\rho}{2A-\rho}\frac{2N-1}{3N-2} , \tag{1.61}$$

and

$$\eta = \frac{4}{9\kappa^2}\frac{3N-2}{3A-2\rho} \ . \tag{1.62}$$

Inverting $\phi(\lambda)$ seems straightforward, but one must pay attention to one feature. Specifically, the procedure for inverting $\phi(\lambda)$ is setting $k = \phi(\lambda)$, and then using equation (1.59) in order to solve for variable λ. During this function inversion process, a step is given by,

$$|\lambda-\theta| = \frac{1}{\eta^{\frac{1}{2}}}(k-\psi)^{\frac{1}{2}} \ . \tag{1.63}$$

[25] Fish reproduction is the application in Sorger (2005), who also uses a linear, constant-reproduction rate, Ak. Alternative interpretations would include exogenously supplied infrastructure by governments to users, such as public roads, assuming that users have an upper capacity of usage, $\bar{c}$.

[26] For the derivation of $\phi(\lambda)$ see the Appendix.

Notice a first parametric constraint implied by (1.63), that

$$k(t) \geq \max\{\psi, 0\}, \text{ for all } t \geq 0 \ . \tag{1.64}$$

A second parametric constraint implied by (1.63) comes from the requirement that $\eta^{1/2}$ exists and that it is different from 0, i.e., that $\eta > 0$. Since $3N - 2 > 0$ for all $N \in \{1, 2, ...\}$, $\eta > 0$ if and only if,

$$A > \frac{2}{3}\rho \ . \tag{1.65}$$

Condition (1.65) implies that,[27]

$$0 < \theta < \frac{3}{2} \tag{1.66}$$

which means,

$$(1-\theta)^2 < 1 \ . \tag{1.67}$$

We will examine conditions that guarantee (1.64) below. There are two possibilities for the left-hand side of equation (1.63). The first is to have $\lambda - \theta \geq 0$, which leads to, a value function, $J(k)$ that is strictly convex and which implies dynamics that violate the property that the solution is interior.[28] The second possibility, of $\lambda - \theta \leq 0$, is admissible, but it still remains to identify parameter restrictions guaranteeing that u is such that the solution is interior.

Let,

$$\lambda - \theta \leq 0 \ , \tag{1.68}$$

hold. Combining (1.68) with (1.63) implies,

$$\lambda = J'(k) = \phi^{-1}(k) = \theta - \frac{1}{\eta^{\frac{1}{2}}}(k - \psi)^{\frac{1}{2}} \ , \ \text{if } \lambda \leq \theta \ . \tag{1.69}$$

27 For a proof of this result, see the Appendix.

28 See the Appendix for details on this point.

Equation (1.69) gives the variable part of the formula for $J(k)$, namely,

$$J(k) = \theta k - \frac{2}{3\eta^{\frac{1}{2}}}(k-\psi)^{\frac{3}{2}} \ , \tag{1.70}$$

where a constant of integration can be added, specified by the HJB equation of the problem of each player. Importantly, this value function is concave, since (1.69) implies,

$$J''(k) = -\frac{1}{2\eta^{\frac{1}{2}}}(k-\psi)^{-\frac{1}{2}} < 0 \ . \tag{1.71}$$

Based on (1.21), (1.58) together with (1.69) reveal the formula of the optimal symmetric strategy, which is given by,

$$C(k) = \frac{4}{9\kappa^2}\left[\frac{1}{\eta^{\frac{1}{2}}}(k-\psi)^{\frac{1}{2}} + 1 - \theta\right]^2 \ . \tag{1.72}$$

Given (1.57), it should be that $0 \leq C(k) < 4/(9\kappa^2) = \bar{c}$, an inequality that leads to,

$$k \in \left[\underline{k}, \bar{k}\right) \ , \ \text{with} \ \ \bar{k} = \eta\theta^2 + \psi = \frac{4N}{9\kappa^2 A} \ , \ \text{and} \ \ \underline{k} = \max\left\{0, \eta(1-\theta)^2 + \psi\right\} \ . \tag{1.73}$$

with,

$$\eta(1-\theta)^2 + \psi = \bar{k} - (2\theta - 1)\eta \ . \tag{1.74}$$

To ensure that the interval $\left[\underline{k}, \bar{k}\right)$ is non-empty, the formulas given by (1.73) and (1.74) indicate that $2\theta - 1 > 0$, as (1.62) together with condition (1.65) imply that $\eta > 0$. After some algebra we can prove that $2\theta - 1 > 0$ is equivalent to having $A > (5N-2)\rho/[2(3N-1)]$. Therefore, a sufficient condition to guarantee that $\left[\underline{k}, \bar{k}\right)$ is non-empty, is given by,

$$A > \frac{5}{6}\rho \ , \tag{1.75}$$

and notice that the parametric constraint given by (1.75), implies the parametric constraint given by (1.65).[29]

[29] To see why (1.75) implies $A > (5N-2)\rho/[2(3N-1)]$ for all $N \geq 1$, define $H(N) = (5N-2)\rho/[2(3N-1)]$. Notice that $H(1) = 3\rho/4$, with $H'(N) = \rho/\left[2(3N-1)^2\right] > 0$, and with $\lim_{N\to\infty} H(N) = 5\rho/6$.

To examine the dynamics of this game, we first examine the monotonicity of the symmetric strategy $C(k)$. Based on the first-order condition given by (1.5),

$$u'(C(k)) = J'(k) \ , \tag{1.76}$$

differentiating both sides of (1.76) implies,

$$J''(k) = u''(C(k))\, C'(k) \ . \tag{1.77}$$

Because $u''(c) < 0$ for all c complying with (1.57), and because of (1.71),

$$J''(k) < 0 \ \text{ combined with (1.77) imply that } C'(k) > 0 \ . \tag{1.78}$$

After some algebra, we can verify that $C'(k) > 0$ is equivalent to having $\eta^{-1/2}(k-\psi)^{1/2} + 1 - \theta > 0$.

Introducing strategies $C(k)$ into (1.1) implies,

$$\dot{k} = Ak - NC(k) \ . \tag{1.79}$$

Differentiating (1.79) with respect to k we obtain,

$$\frac{\partial \dot{k}}{\partial k} = A - NC'(k) \ . \tag{1.80}$$

The monotonicity of $C(k)$ given by (1.78), together with (1.80), jointly imply that it is possible to have stable dynamics toward a 0-growth steady state of k. Such a 0-growth steady state of k can either lie within the domain of admissible interior strategies of the game, the interval $\left[\underline{k}, \bar{k}\right)$, or it can be the supremum of the interval $\left[\underline{k}, \bar{k}\right)$, which is $\bar{k}$. We explore parametric conditions that allow for this possibility.

After expanding the quadratic term in (1.72), the law of motion (1.79) becomes,

$$\dot{k} = Ak - \frac{4N}{9\kappa^2\eta}(k-\psi) - \frac{8N(1-\theta)}{9\kappa^2\eta^{\frac{1}{2}}}(k-\psi)^{\frac{1}{2}} - \frac{4N(1-\theta)^2}{9\kappa^2} \ . \tag{1.81}$$

Given the nature of (1.81), it is useful to introduce a function,

$$z(k) \equiv (k-\psi)^{\frac{1}{2}} \ . \tag{1.82}$$

In the Appendix we show that (1.81) can be re-written as,

$$\dot{k} = A\left(1-\frac{\bar{k}}{\eta}\right)z(k)^2 - 2A\frac{\bar{k}(1-\theta)}{\eta^{\frac{1}{2}}}z(k) + A\left[\bar{k}(2-\theta)-\eta\theta\right]\theta \tag{1.83}$$

The right-hand side of (1.83) can be seen as a quadratic polynomial in terms of the function $z(k)$, with discriminant, Δ, given by,

$$\Delta = 4A^2\left\{\frac{\left[\bar{k}(1-\theta)\right]^2}{\eta} - \left(1-\frac{\bar{k}}{\eta}\right)\left[\bar{k}(2-\theta)-\eta\theta\right]\theta\right\} \ . \tag{1.84}$$

The discriminant given by (1.84) can inform us on whether real roots of the quadratic polynomial exist. In order to achieve this goal, we use the formulas given by (1.62) and (1.61). After some algebra, we can show that,[30]

$$A \gtreqless \rho N \Leftrightarrow \theta \gtreqless 1 \Leftrightarrow \bar{k} \gtreqless \eta \Leftrightarrow \bar{k}(2-\theta)-\eta\theta \gtreqless 0 \ , \text{ for all } N \geq 2 \text{ and all } A > \frac{5}{6}\rho \ . \tag{1.85}$$

The equivalence given by (1.85) serves as a guide, indicating that we must focus on the relationship between A and ρN, keeping in mind that condition (1.75) should always hold. A first implication of (1.85) is that,

$$\left(1-\frac{\bar{k}}{\eta}\right)\left[\bar{k}(2-\theta)-\eta\theta\right] \leq 0 \ , \text{ for all } A > \frac{5}{6}\rho, \text{ with equality iff } A = \rho N \ . \tag{1.86}$$

Based on (1.66), $\theta > 0$, and (1.86) implies that,

$$\Delta \geq 0 \ , \text{ for all } A > \frac{5}{6}\rho, \text{ with equality iff } A = \rho N \ . \tag{1.87}$$

Therefore, as long as $A > 5\rho/6$ and $A \neq \rho N$, there are always two real roots to the quadratic polynomial in $z(k)$ given by the right-hand side of (1.83). Let's call these two real roots r_1

[30] See the Appendix for a proof of this statement.

and r_2. The first root, r_1, is easy to identify using (1.73) and (1.72) and introducing them into (1.79), namely,

$$\dot{k}\Big|_{k=\bar{k}} = A\bar{k} - NC\left(\bar{k}\right) = 0 \ . \tag{1.88}$$

The result in (1.88) implies that,

$$r_1 = z\left(\bar{k}\right) = \left(\bar{k} - \psi\right)^{\frac{1}{2}} = \eta^{\frac{1}{2}}\theta > 0 \ . \tag{1.89}$$

The right-hand side of (1.83) implies,

$$r_1 r_2 = \frac{\left[\bar{k}\left(2-\theta\right) - \eta\theta\right]\theta}{1 - \frac{\bar{k}}{\eta}} \ , \quad \text{and} \quad r_1 + r_2 = 2\frac{\bar{k}\left(1-\theta\right)}{\eta^{\frac{1}{2}}\left(1 - \frac{\bar{k}}{\eta}\right)} \ . \tag{1.90}$$

It is verifiable that all three equations in (1.89) and (1.90) comply with,

$$r_2 = \frac{\bar{k}\left(2-\theta\right) - \eta\theta}{\eta^{\frac{1}{2}}\left(1 - \frac{\bar{k}}{\eta}\right)} < 0 \ . \tag{1.91}$$

To see why r_2 is strictly negative, observe from (1.90) and (1.86) that for all $A > 5\rho/6$ with $A \neq \rho N$, $r_1 r_2 < 0$. Since $r_1 > 0$, it follows that $r_2 < 0$.

In brief, (1.83) can be re-written as,

$$\dot{k} = A\left(1 - \frac{\bar{k}}{\eta}\right)\left[z\left(k\right) - \eta^{\frac{1}{2}}\theta\right]\left[z\left(k\right) - \frac{\bar{k}\left(2-\theta\right) - \eta\theta}{\eta^{\frac{1}{2}}\left(1 - \frac{\bar{k}}{\eta}\right)}\right] \ . \tag{1.92}$$

Since $z\left(k\right) \geq 0$ and since, according to (1.91), $r_2 < 0$, the last term of (1.92), $z\left(k\right) - r_2 > 0$. In addition, since $z'\left(k\right) = 1/2\left(k - \psi\right)^{-1/2} > 0$, and since $k < \bar{k}$, $z\left(k\right) < z\left(\bar{k}\right) = \eta^{1/2}\theta$, i.e., the penultimate term of (1.92), $z\left(k\right) - r_1 < 0$. Given these two observations, (1.92) and (1.85) imply,

$$\dot{k} \gtreqless 0 \Leftrightarrow A \gtreqless \rho N \ , \ \text{ for all } N \geq 2 \text{ and all } A > \frac{5}{6}\rho \ . \tag{1.93}$$

Therefore, (1.93) implies that the parametric constraint

$$A > \rho N \ , \tag{1.94}$$

guarantees an interior solution where $k \to \bar{k}$ asymptotically, as $t \to \infty$, as is depicted by Figure 1.2.[31]

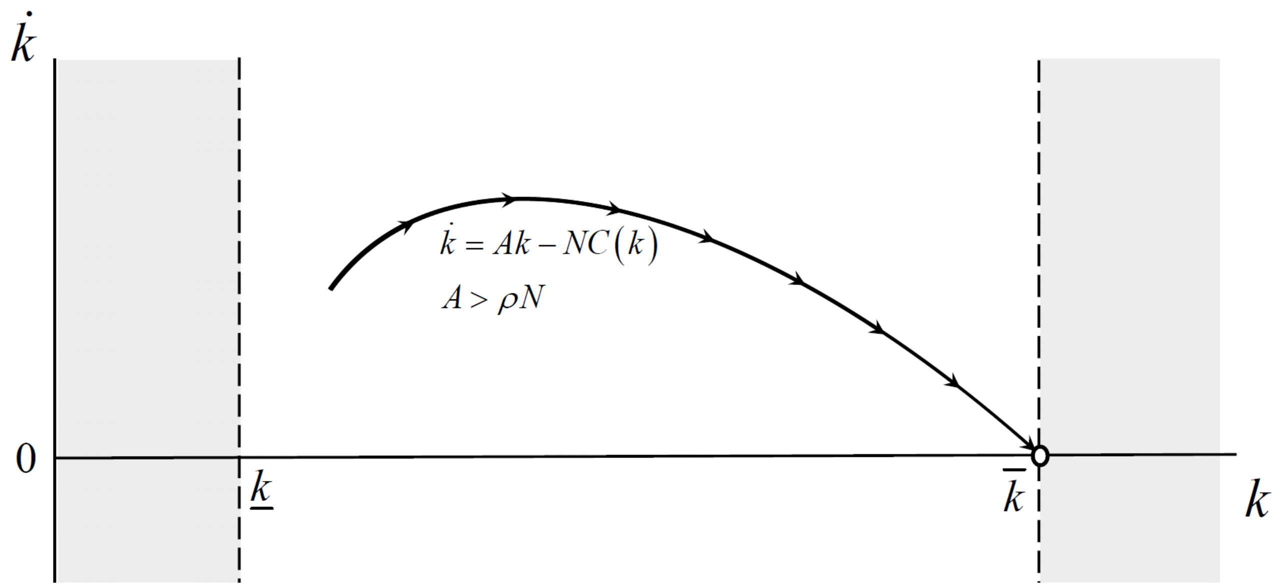

Figure 1.2 Dynamics of k toward the supremum of the $[\underline{k}, \bar{k})$ interval, $\bar{k}$. The resource, k, converges asymptotically to $\bar{k}$ from any $k(0) \in [\underline{k}, \bar{k})$, guaranteeing an interior solution.

This asymptotic-convergence property guarantees that the transversality condition holds as well. Specifically, from (1.69) we can see that,

$$\lim_{t\to\infty} e^{-\rho t} J'(k(t))\, k(t) = \left[\theta - \frac{1}{\eta^{\frac{1}{2}}}\left(\bar{k} - \psi\right)^{\frac{1}{2}}\right] \bar{k} \lim_{t\to\infty} e^{-\rho t} = 0 \ .$$

We conclude proving the uniqueness of the optimal response to symmetric strategies played by other players for the case of setting $\omega = 0$ in equation (1.20) for this novel example presented in this section, by demonstrating that $C(k)$ given by (1.72) is the unique

[31] Trivially, the case where the initial condition is above $\bar{k}$ does not qualify as an interior solution.

maximizer of $\mathcal{P}_i$, for all $i \in \{1, ..., N\}$ (see Definition 1 and set $C(k) = C^*(k)$). First, notice that, under the parametric constraint (1.94),

$$C''(k) = \frac{-2(1-\theta)}{9\kappa^2\eta^{\frac{1}{2}}}(k-\psi)^{-\frac{3}{2}} > 0 \; , \tag{1.95}$$

which is an implication of (1.85). For a symmetric equilibrium in $C(k)$, a player's Hamiltonian is,

$$\mathcal{H} = u(c) + \lambda\left[Ak - (N-1)C(k) - c\right] \; .$$

The Hessian matrix of this Hamiltonian with respect to variables (c, k) is,

$$H_{\mathcal{H}} = \begin{bmatrix} u''(c) & 0 \\ 0 & -\lambda(N-1)C''(k) \end{bmatrix} \; .$$

Given that $u''(c) < 0$, (1.95) implies that $H_{\mathcal{H}}$ is negative definite. Therefore, given Mangasarian's theorem (see, for example, Sydsaeter et al., 2008, p. 330, Theorem 9.7.1), the optimal strategy, $C(k)$, that solves the individual problem of player $i \in \{1, ..., N\}$, is a unique maximizer in response to the symmetric strategies, $C(k)$, played by the other $N-1$ players.

An alternative interpretation and application of this game is the case where N monopolists, co-exploit a common-property resource each supplying to a different market at zero cost, facing a constant interest rate equal to ρ.[32] The HJB equation of player $i \in \{1, ..., N\}$ in such a setup is,

$$\rho J_i(k) = \max_{q_i \geq 0}\left\{p(q_i)\,q_i + J_i'(k)\left[Ak - \sum_{j \neq i} Q_j^*(k) - q_i\right]\right\} \; ,$$

[32] Think, for example, of a railroad that is provided exogenously by a government, with railway companies utilizing this railroad infrastructure in a rivalrous and non-excludable, manner, at no cost. This infrastructure, k, can depreciate with utilization, i.e., by the number of passengers of each company, q_i, according to an endogenous depreciation function that is linear in q_i, say, $\delta(q_i) = \psi k_i$, and parameter A in the law of motion of k is normalized so as to set $\psi = 1$. A discrete-time version of this setup is given, for example, in Koulovatianos and Mirman (2007, p. 203).

in which q_i is the extracted and supplied quantity of the common resource by monopolist i and $Q_j^*(k)$ is the optimal Markovian strategy of oligopolist $j \neq i$. The first-order conditions of this problem are,

$$p'(q_i)\,q_i + p(q_i) = J_i'(k) \ . \tag{1.96}$$

Finding the inverse of the derivative of the revenue function in (1.96), where $R(q_i) = p(q_i)\,q_i$ is the revenue function and $R'(q_i) = p'(q_i)\,q_i + p(q_i)$, can be challenging to do analytically, unless we express the inverse demand function in the form,

$$p(q_i) = r(q_i)\,q_i^{-1} \ .$$

In this case, the revenue function, $R(q_i) = p(q_i)\,q_i = r(q_i)$, and (1.96) becomes $r'(q_i) = J'(k)$, exactly as in (1.4), making this game identical to the one examined in this paper, after setting $r(q) = u(q)$. An advantage of this interpretation is that much of the literature focuses on linear-demand functions, making the revenue function quadratic, while the method we propose can lead to other functional forms for the inverse-demand function.[33] In the example examined here, the inverse-demand function would be,

$$p(q_i) = 1 - \kappa q_i^{\frac{1}{2}} \ .$$

1.7 Conclusion

We have provided a thorough and comprehensive characterization of the interior solution to the class of symmetric Markovian differential games of commons problems with linear constraints. For a broad class of time separable utility functions that depend only on the player's control variable and that allow for interior solutions, we have provided an exact interior solution to the problem when the coefficient of the linear resource reproduction function differs from the rate of time preference ($A \neq \rho$). The solution to the special case

[33] We thank an anonymous referee for pointing this interpretation to us.

where the rate of time preference equals the coefficient of the linear resource reproduction function is given as an implicit function of a simple expression. In the more interesting and more common case of $A \neq \rho$, our analytical approach involves a differential equation with an explicit solution involving an integration constant. When we give this constant the value of zero, we obtain an analytical for the Markovian strategies. This particular case with the zero-integration constant, quickly leads to the verification of closed-form solutions. Moreover, our solution gives an immediate result regarding analytic functions. If the utility function is analytic, then the resulting Markovian strategies are also analytic functions. This analyticity property can facilitate the numerical computation of such games using, e.g. polynomial approximations for value functions and Markovian strategies. Additionally, the analyticity property can be useful in numerically approximating commons problems with corner solutions. For the cases where the integration constant is not equal to zero, multiple Markovian strategies can arise. This case can be intractable analytically, but it can help in either characterizing these multiple equilibria, or in numerically computing them. An interesting extension of our findings is to study conditions for the sustainability of cooperation as Jørgensen, Martin-Herran and Zaccour (2005) have done for linear-quadratic games. Finally, a future extension could be to characterize this class of games for an infinite horizon.

1.8 Appendix

Derivation of function $\phi(\lambda)$ in equation (1.59)

Substituting (1.58) into (1.19) leads to,

$$\phi(\lambda) = \frac{1}{A-\rho}\left\{\frac{4(N-1)}{9\kappa^2}(1-\lambda)^2 + [N+\xi(N-1)]\frac{4\lambda^{\xi}}{9\kappa^2}\int \lambda^{-\xi-1}(1-\lambda)^2 d\lambda\right\} . \quad \text{(A.1.1)}$$

To calculate the integral in (A.1.1) we expand the quadratic form, namely,

$$\int \lambda^{-\xi-1}(1-\lambda)^2 d\lambda = \int \left(\lambda^{-\xi-1} - 2\lambda^{-\xi} + \lambda^{-\xi+1}\right) d\lambda \ ,$$

which leads to,

$$\int \lambda^{-\xi-1}(1-\lambda)^2 d\lambda = \frac{1}{-\xi+2}\lambda^{-\xi}\left(\lambda^2 - 2\frac{-\xi+2}{-\xi+1}\lambda + \frac{-\xi+2}{-\xi}\right) . \quad \text{(A.1.2)}$$

Combining (A.1.2) with (A.1.1) gives,

$$\phi(\lambda) = \alpha\cdot\left(\lambda^2 - 2\lambda + 1\right) + \beta\cdot\left(\lambda^2 - 2\frac{-\xi+2}{-\xi+1}\lambda + \frac{-\xi+2}{-\xi}\right) , \quad \text{(A.1.3)}$$

where

$$\alpha \equiv \frac{4(N-1)}{9\kappa^2(A-\rho)} \quad \text{and} \quad \beta \equiv \frac{4[N+\xi(N-1)]}{9\kappa^2(A-\rho)(-\xi+2)} . \quad \text{(A.1.4)}$$

Collecting terms in (A.1.3) leads to,

$$\phi(\lambda) = (\alpha+\beta)\left[\lambda^2 - 2\frac{\alpha+\zeta\beta}{\alpha+\beta}\lambda + \left(\frac{\alpha+\zeta\beta}{\alpha+\beta}\right)^2\right] + \alpha + \beta\frac{-\xi+2}{-\xi} - (\alpha+\beta)\left(\frac{\alpha+\zeta\beta}{\alpha+\beta}\right)^2 ,$$

or,

$$\phi(\lambda) = (\alpha+\beta)\left(\lambda - \frac{\alpha+\zeta\beta}{\alpha+\beta}\right)^2 + \alpha + \beta\frac{-\xi+2}{-\xi} - (\alpha+\beta)\left(\frac{\alpha+\zeta\beta}{\alpha+\beta}\right)^2 , \quad \text{(A.1.5)}$$

where,

$$\zeta \equiv \frac{-\xi+2}{-\xi+1} . \quad \text{(A.1.6)}$$

Substituting the expressions for α, β, and ζ given by (A.1.4) and (A.1.6) into (A.1.5), gives equation (1.59), together with the expressions given by (1.60), (1.61) and (1.62).

□

Proof of inequality (1.66)

Fix any value of ρ and observe that

$$\theta = F(A)\, G(N) \ , \tag{A.1.7}$$

where,

$$F(A) = \frac{3A - 2\rho}{2A - \rho} \ , \tag{A.1.8}$$

and,

$$G(N) = \frac{2N - 1}{3N - 2} \ . \tag{A.1.9}$$

Notice that, according to (1.65), $A > 2/3\rho > 1/2\rho$, and therefore, $F(A) > 0$. Moreover,

$$F'(A) = \frac{\rho}{(2A - \rho)^2} > 0 \ , \tag{A.1.10}$$

and

$$\lim_{A \downarrow \frac{2}{3}\rho} F(A) = 0 \ , \quad \text{and} \ \lim_{A \to \infty} F(A) = \frac{3}{2} \ . \tag{A.1.11}$$

Combining (A.1.11) with (A.1.10) gives,

$$0 < F(A) < \frac{3}{2} \quad \text{for all } A \text{, given any } \rho \text{ complying with (1.65)} \tag{A.1.12}$$

Similarly, notice that,

$$G'(N) = \frac{-1}{(3N - 2)^2} < 0 \ , \tag{A.1.13}$$

while,

$$G(1) = 1 \ \text{ and } \ \lim_{N \to \infty} G(N) = \frac{2}{3} \ . \tag{A.1.14}$$

Therefore, (A.1.7), (A.1.13) and (A.1.14) imply,

$$\frac{2}{3} < G(N) \leq 1 \text{ , for all } N \in \{1, 2, ...\} \ . \tag{A.1.15}$$

Combining, (A.1.7), (A.1.13) and (A.1.15) proves inequality (1.66). □

Why the case of $\lambda - \theta \geq 0$ in equation (1.63) is not admissible

Substituting $\lambda - \theta \geq 0$ into (1.63) gives,

$$\lambda = \phi^{-1}(k) = \frac{1}{\eta^{\frac{1}{2}}}(k - \psi)^{\frac{1}{2}} + \theta \ , \ \text{ if } \lambda \geq \theta \ . \tag{A.1.16}$$

Recall that $\lambda = J'(k)$. Differentiating the right-hand side of (A.1.16) we can see that $J''(k) = 1/\left(2\eta^{1/2}\right)(k - \psi)^{-1/2} > 0$ for all $k \geq \max\{0, \psi\}$. Yet, $J''(k) > 0$ is not a property of the value function that complies with the transversality condition. To see this, consider the first-order condition given by (1.5), which implies,

$$u'(C(k)) = J'(k) \ . \tag{A.1.17}$$

Differentiating both sides of (A.1.17) implies,

$$J''(k) = u''(C(k))\, C'(k) \ . \tag{A.1.18}$$

Because $u''(c) < 0$ for all c complying with (1.57),

$$J''(k) > 0 \ \text{ combined with (A.1.18) imply that } C'(k) < 0 \ . \tag{A.1.19}$$

Yet, remember that the budget constraint given by (1.1) implies,

$$\dot{k}(t) = Ak(t) - NC(k(t)) \ . \tag{A.1.20}$$

Combining (A.1.20) with $C'(k) < 0$ means that the right-hand side of equation (A.1.20) is upward-sloping in $k(t)$. Based on (1.21), we can combine (A.1.16) with (1.58) to obtain the explicit formula for $C(k)$, namely,

$$C(k) = \frac{4}{9\kappa^2}\left[1 - \theta - \frac{1}{\eta^{\frac{1}{2}}}(k - \psi)^{\frac{1}{2}}\right]^2 . \tag{A.1.21}$$

Using (A.1.21) we derive the first and second derivatives of the strategies $C(k)$, i.e.,

$$C'(k) = \frac{-4}{9\kappa^2\eta^{\frac{1}{2}}}\left[(1 - \theta)(k - \psi)^{-\frac{1}{2}} - \frac{1}{\eta^{\frac{1}{2}}}\right] . \tag{A.1.22}$$

Notice that (A.1.22) combined with (A.1.19) implies that

$$C'(k) < 0 \text{ holds if } k < \psi + \eta(1 - \theta)^2 . \tag{A.1.23}$$

In addition, (A.1.22) implies,

$$C''(k) = \frac{2}{9\kappa^2\eta^{\frac{1}{2}}}(1 - \theta)(k - \psi)^{-\frac{3}{2}} > 0 . \tag{A.1.24}$$

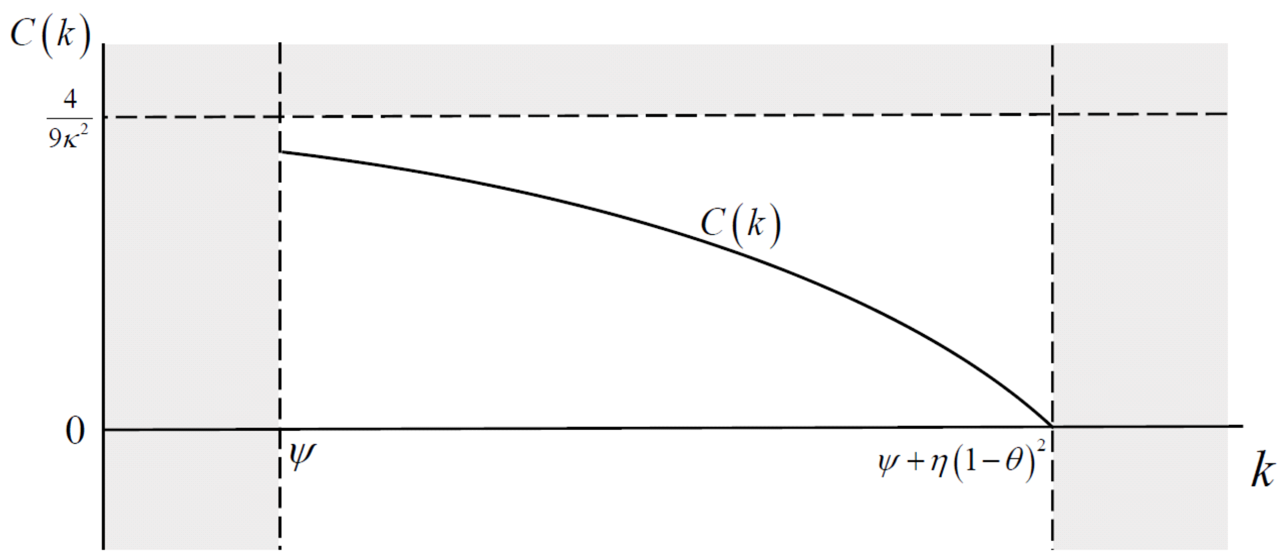

Figure A.1.1 Properties of the decision rule if $\lambda - \theta \geq 0$

All properties of $C(k)$ described by (1.57), (A.1.21), (A.1.22), (A.1.23), and (A.1.24) are depicted by Figure A.1.1, where the shaded areas indicate value regions where the strategies $C(k)$ are not defined. Without loss of generality, Figure A.1.1 depicts a case where $\psi > 0$. The case of $\psi \leq 0$ would simply depict a picture with $C(k)$ exhibiting the same properties for $k \in \left[0, \psi + \eta(1-\theta)^2\right]$.

Introducing strategies $C(k)$ into (1.1) we obtain,

$$\dot{k} = Ak - NC(k) \ . \tag{A.1.25}$$

Differentiating (A.1.25) with respect to k we obtain,

$$\frac{\partial \dot{k}}{\partial k} = A - NC'(k) > 0 \ , \ \text{ for all } k \in \left[\max\{0, \psi\}, \psi + \eta(1-\theta)^2\right] \ . \tag{A.1.26}$$

Equation (A.1.26) is a consequence of equation (A.1.23). The key message of (A.1.26) it implies dynamics of k. These unstable dynamics of k imply a violation of the feature that the solution is interior. In the absence of an interior solution, Proposition 1 does not apply and, therefore, the closed form solution of the strategies, $C(k)$, given by (A.1.21), is invalid.

Figures A.1.2 and A.1.3 depict (A.1.25) and the dynamics of k, based on all parametric cases. Specifically, we distinguish cases of parametric values of A such that $\psi > 0$ and otherwise. Based on equation (1.60), after some algebra, and making use of the parametric constraint given by (1.65), we can show that,

$$\psi > 0 \Leftrightarrow \left(A - \frac{3N-2}{4N-3}\rho\right)(A - \rho N) > 0 \Leftrightarrow A \in \left(\frac{2}{3}\rho \ , \ \frac{3N-2}{4N-3}\rho\right) \cup (\rho N \ , \ \infty) \ , \tag{A.1.27}$$

$$\psi = 0 \Leftrightarrow A = \rho N \ \text{ or } A = \frac{3N-2}{4N-3}\rho \ , \tag{A.1.28}$$

and

$$\psi < 0 \Leftrightarrow A \in \left(\frac{3N-2}{4N-3}\rho \ , \ \rho N\right) \ . \tag{A.1.29}$$

A common feature between Figures A.1.2 and A.1.3 is that when $k = \psi + \eta(1-\theta)^2$, which is the upper bound of k for which $C(k)$ is admissible in this case of $\lambda - \theta \geq 0$, $\dot{k} > 0$. To see this, insert $k = \psi + \eta(1-\theta)^2$ into (A.1.25) to obtain,

$$\dot{k}\Big|_{k=\psi+\eta(1-\theta)^2} = A\left[\psi + \eta(1-\theta)^2\right] > 0\ . \tag{A.1.30}$$

Inequality (A.1.30) justifies why in both Figures A.1.2 and A.1.3 the curve depicting the law of motion for k is above the 0-line.

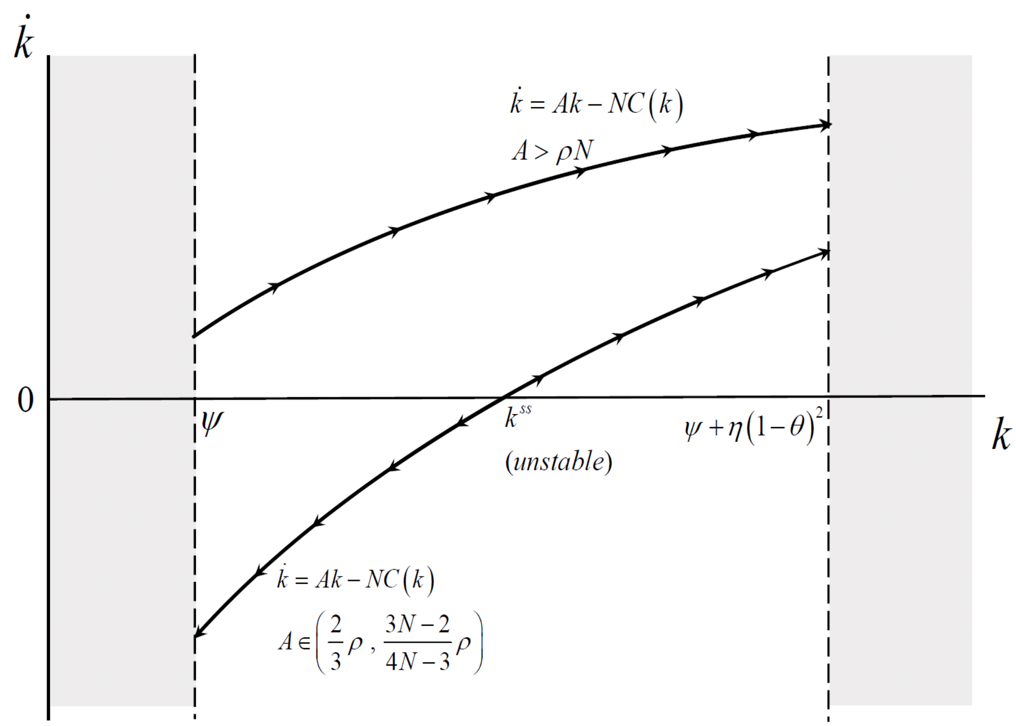

Figure A.1.2 Resource dynamics in the case where A is such that $\psi > 0$.

To understand why there are two curves depicting (A.1.25) in Figure A.1.2, which focuses on parameter values implying $\psi > 0$, consider the equivalence given by (A.1.27) and focus on the specific value of k, $k = \psi$. By inserting $k = \psi$ into (A.1.25),

$$\dot{k}\Big|_{k=\psi} = A\psi - \frac{4}{9\kappa^2}(1-\theta)^2 > 0 \Leftrightarrow (N-1)(A-\rho N) > 0\ . \tag{A.1.31}$$

In the trivial case of $N = 1$, $\dot{k}\Big|_{k=\psi} = 0$. Yet, this does not correspond to an interior solution with free initial conditions. We therefore focus on cases with $N \geq 2$. When $N \geq 2$, the equivalence given by (A.1.31) implies that,

$$\dot{k}\Big|_{k=\psi} \gtreqless 0 \Leftrightarrow A \gtreqless \rho N \ . \tag{A.1.32}$$

Given that,

$$\frac{3N-2}{4N-3} \in \left(\frac{3}{4}, 1\right) \quad \text{for all } N \in \{2, 3, ...\} \, ,$$

the two curves depicting (A.1.25) in Figure A.1.2 are justified. The equivalence implied by (A.1.27) implies that, in the case where $A \in (2/3\rho \ , \ \ (3N-2)\,/\,(4N-3)\,\rho)$, there is a value k^{ss} for which $\dot{k}\Big|_{k=k^{ss}} = 0$. Yet, this unstable 0-growth value does not correspond to an interior solution with free initial conditions for the problem.

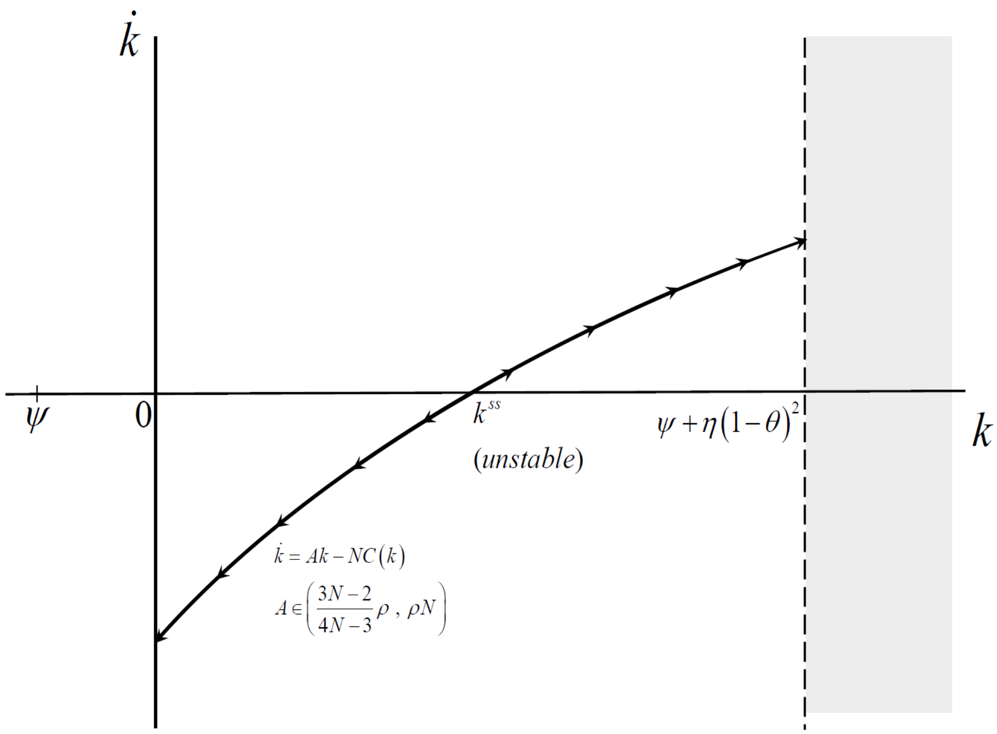

Figure A.1.3 Resource dynamics in the case where A is such that $\psi < 0$.

Figure A.3 focuses on the case implied by (A.1.29). Because of (A.1.32), in Figure A.1.3.

we have once more a value k^{ss} for which $\dot{k}\Big|_{k=k^{ss}} = 0$. Again, this unstable 0-growth value does not correspond to an interior solution with free initial conditions for the problem. The same problem arises for the two specific values of A given by (A.1.28), for which $\psi = 0$.

In summary, the case of $\lambda - \theta \geq 0$ does not correspond to an interior solution and it should, therefore, be discarded. □

Proof of equivalence (1.85)

To prove that $\theta \gtreqless 1 \Leftrightarrow A \gtreqless \rho N$, use (1.61) to obtain,

$$\theta \gtreqless 1 \Leftrightarrow \frac{3A - 2\rho}{2A - \rho}\frac{2N - 1}{3N - 2} \gtreqless 1 \, . \tag{A.1.33}$$

Based on the parametric constraint given by (1.75) numerators and denominators in the fractions appearing in (A.1.33) are strictly positive. This feature leads to verifying that

$$\frac{3A - 2\rho}{2A - \rho}\frac{2N - 1}{3N - 2} \gtreqless 1 \Leftrightarrow A \gtreqless \rho N \ ,$$

which confirms the first part of (1.85), that $\theta \gtreqless 1 \Leftrightarrow A \gtreqless \rho N$.

For proving the second part of (1.85), that $\bar{k} \gtreqless \eta \Leftrightarrow A \gtreqless \rho N$, observe that (1.62) and (1.73) imply,

$$\frac{\bar{k}}{\eta} = \frac{(3A - 2\rho)\, N}{3AN - 2A}. \tag{A.1.34}$$

Using the parametric constraint given by (1.75), which also implies $\eta > 0$, together we can show that

$$\frac{\bar{k}}{\eta} \gtreqless 1 \Leftrightarrow A \gtreqless \rho N \ ,$$

which proves the second part of (1.85), that $\bar{k} \gtreqless \eta \Leftrightarrow A \gtreqless \rho N$.

Finally, for proving that $\bar{k}\,(2 - \theta) - \eta\theta \gtreqless 0 \Leftrightarrow A \gtreqless \rho N$, use (1.61) and (1.62) to see that,

$$\bar{k}\,(2 - \theta) - \eta\theta \gtreqless 0 \Leftrightarrow \frac{N}{A}\left(2 - \frac{6AN - 3A - 4\rho N + 2\rho}{6AN - 4A - 3\rho N + 2\rho}\right) \gtreqless \frac{2N - 1}{2A - \rho} \Leftrightarrow$$

$$\Leftrightarrow \frac{6AN - 5A - 2\rho N + 2\rho}{2A - \rho} \gtreqless \frac{A}{N} \frac{2N - 1}{2A - \rho} \Leftrightarrow$$

$$\Leftrightarrow 2\left(3A - \rho\right) N^2 - \left(5A - 2\rho\right) N \gtreqless A\left(6N^2 - 7N + 2\right) \Leftrightarrow$$

$$\Leftrightarrow \left(N - 1\right)\left(A - \rho N\right) \gtreqless 0 \ ,$$

confirming that $\bar{k}\left(2 - \theta\right) - \eta\theta \gtreqless 0 \Leftrightarrow A \gtreqless \rho N$ for all $N \geq 2$. □

2. CHAPTER

Populism and Polarization in Social Media Without Fake News: the Vicious Circle of Biases, Beliefs and Network Homophily

2.1 Introduction

A crucial feature of populism is a separatist tendency in society, a tendency for having social groups with strong within-group ties and similar within-group biases. Such groups often define their identity by strongly differentiating themselves from other groups with different beliefs and biases.[34] People within such groups tend to downgrade expert opinion on highly technical matters even outside politics, e.g., medical facts about immunizations, scientific findings in physics and biology that may be oppose traditional religious views, etc.[35] In the past two decades, there is evidence that populism rose over time.[36] Together with this rise, there is a growing tendency for downgrading expert opinion, celebrating the term "post-truth" era in politics and society.[37] Social media and internet-based networks are the focus of recent research on understanding the causes of this uprising downgrading of expert opinion. Much of research related to networks has focused on measuring the spread of fake news through social media, studying also the effectiveness of combating fake news through

[34]Although there is no generally accepted definition of populism in the academia, a common element among suggested definitions of populism in politics, is a tendency of citizens to split between groups of "pure people" versus supporters of the "corrupt elite" (see, for example, Mudde, 2004, and Stanley, 2008). A more general way of describing both this separatist tendency among different groups and the tendency of persons to connect with persons of similar features is the concept of homophily, explained in the survey paper by McPherson et al. (2001).

[35]Gauchat (2012) provides evidence that measures of trust in science tend to differ among groups with different political views, reporting a decline in the trust in science by conservatives in the US from 1974 to 2010. Hamilton et al. (2015) provide consistent evidence on lower trust to science by conservatives regarding vaccine issues and climate change, using a survey in 2014.

[36]For evidence on the rise in populism in the past decades see Rodrik (2018), and Guiso et al. (2018).

[37]See the review article of Lewandowsky et al. (2017).

internet websites that debunk information.[38] While we think that this strong focus on fake news is crucial, in this study we take one step beyond the role of fake news in order to understand why expert opinion is downgraded over time, why populism and polarization rise over time, and how these two processes are interrelated.

We build a simulated model of network dynamics and limited information. We remove the possibility of fake news from the model and demonstrate that, given the search and matching facility that social media offer for connecting with new online friends, two social elements alone, are sufficient for producing, (a) networks that gradually exhibit more homophily and polarization over time, and, (b) a gradual downgrading of expert opinion on issues for which knowledge is limited. The two social elements that are sufficient for producing these dynamics are, (i) individual biases, such as biased assimilation and confirmation bias, and (ii) the tendency that people have for socially aligning their actions with actions of network friends.[39]

For distinguishing the two eras of social networks, the pre-social-media era and the post-social-media era, the key is to introduce a search-and-matching mechanism that can bring together new friends.[40] Compared to traditional social networks without internet, internet-based social media are distinguished by the speed and intensity of the search-and-matching possibilities they offer. Due to this difference in speed of search and matching, in traditional social networks without internet, the evolution of some social processes, such as populism

[38] See the online platform "Hoaxy" for detecting fake news (Shao et al. 2016 and 2018) and a related discussion in Ciampaglia et al. (2018) on debunking fake news, reviewing preliminary results on this new area of research.

[39] Evidence on the role that biases play in promoting attitude polarization was provided by Lord et al. (1979), contributing to the literature on biased asssimilation and confirmation bias. Confirmation bias, as is explained by Nickerson (1998), together with biased assimilation, are the closest concepts of bias we employ in our model. For the coordination motive among people in a sociaty to align their actions to these of their friends and peers, see the famous "beauty contest" example proposed by Keynes (1936) and the formulation in Morris and Shin (2002) and Golub and Morris (2018).

[40] Search and matching models have been used, for example, in monetary economics (see Kiyotaki and Wright, 1993) and in modeling unemployment (see Mortensen and Pissarides, 1994).

and polarization, might be too slow, requiring a lifetime to evolve, so the overall process might be stalled in society. On the contrary, internet-based social media can speed up search and matching of new internet friends (or can even make matches among distantly located people possible), speeding up the evolution of some social processes as well. The search-and-matching framework we suggest, and the uncomplicated simulated evolutionary dynamics it produces, are two key contributions of this paper.

Our search-and-matching process involves features of coordination games with incomplete information. In these games, players need to form beliefs about a fundamental value. In our framework, there is a public noisy signal that captures the role of expert opinion on this fundamental value. In addition, players have access to private signals and also try to coordinate with network friends. In order to take actions (e.g., immunizations, political votes, etc.) related to this fundamental value (e.g., the risk of a disease, the risk of a fiscal crisis, etc.), players form beliefs on what other players believe, i.e., they form higher-order beliefs. In this environment, fundamental (structural) biases of players, more related to their education level or culture, such as confirmation biases, cause a preference for choosing internet social media friends with similar biases, the network feature known as homophily.

Our main result is that, even in the absence of producing and re-producing fake news, fundamental biases combined with the need for aligning actions to those of friends, lead to a network evolution characterized by homophily, high network density and closeness centrality among friends of similar biases. These network features are mapped to actions of players, strengthening populist characteristics that lead to polarization over time: players gradually put more weight on their biases and less weight on expert opinions. *Networks make fundamental biases be enhanced by peer-induced amplification factors and these biases lead to more network features that promote these biases, a vicious circle of populist trends.*

The crucial distinction between fundamental (structural) biases and peer-induced amplification of biases in decision making provides three main insights that we demonstrate through simulation experiments and through some analytical characterizations. First, the tendency of people to connect with those who have similar fundamental biases is endogenous, depending on the existing network structure. Specifically, as the existing network exhibits more homophily, and as subnetworks of connected persons with similar bias also exhibit more density and closeness centrality, the tendency to match with new persons of similar biases becomes stronger. Second, we analytically show that, in decision-making, there is a tradeoff between peer-induced amplification of biases and importance of expert opinion. Whenever the role of biases increases in decision-making, the role of expert opinion becomes downgraded. This tradeoff is clear in our model because, as our model has no fake news, the weight that individual decisions place on noisy private signals is constant, independently of the network structure. Third, the size of fundamental biases, measured in relation to the standard deviations of private and public signals, affects both the intensity of the long-term homophily outcome and the speed of transition to this outcome. Specifically, weak fundamental biases lead to weaker homophily outcomes. These dynamics occur in a framework where agents have myopia regarding the evolution of the network, despite that they make sophisticated decisions based on the existing structure of the networks, using all available information. We conjecture that a more sophisticated model, with foresight and rational expectations about the network evolution, would strengthen this relationship between fundamental biases and network dynamics.[41]

Our findings give a clear message. Combating fake news through network debunkers is not a complete treatment against populist trends. For preventing populism, it may also

[41]Such an extension is beyond the scope of this paper, as it demands the development of new analytical tools in dynamic games with foresight, where whole networks are the state variables affecting each individual forward-looking decision.

be crucial to focus on removing the structural feature of individual biases, e.g., through providing better education to younger individuals and through promoting an evidence-based mentality to society.

2.1.1 Related literature

Our paper contributes to two literature strands. The first strand is the growing literature on the determinants of homophily in networks and on how homophily affects a number of economic and social decisions, including the speed of learning. Examples of this literature are Jackson (2008), Currarini et al. (2009), Kossinets and Watts (2009), Golub and Jackson (2012a,b), Bramoulle et al. (2012), Jackson and Lopez-Pintado (2013), Centola (2013), Lobel and Sadler (2015), Currarini and Mengel (2016), and Halberstam and Knight (2016).

The second strand is the literature interested in fake news, despite that we do not study fake-news extensions in this paper. In this paper we model biased assimilation as a structural feature, showing that, over time, due to the dynamics of network peers and due to interactions with network peers, actions tend to be more and more biased, while expert opinion is gradually downgraded. Nevertheless, studying the interplay between our suggested mechanism of biases in this paper and fake-news mechanisms suggested in the growing fake-news literature on networks, should be a topic of future research. We think that establishing our model's mechanics is a stepping stone for such a synthesis. Papers in this fake-news strand of literature include Mullainathan and Shleifer (2005), Baron (2006), Gentzkow and Shapiro (2006), Besley and Prat (2006), Bernhardt et al. (2008), Gentzkow et al. (2015) and Allcott and Gentzkow (2017).

Notably, a paper sharing similar concepts to ours is Dandekar et al. (2013), which builds on the model of DeGroot (1974), exploring how biased assimilation leads to homophily. A crucial difference from Dandekar et al. (2013), is that we place emphasis on how expert signals might be ignored due to biases and progressing homophily. We follow a different approach. We use a dynamic variant of frameworks suggested by Morris and Shin (2002) and Golub and Morris (2018), introducing a search-and-matching mechanism. We build a tractable algorithm suggesting an efficient way for calculating higher-order beliefs, offering different insights and results.

Acemoglu et al. (2013) develop a political approach to populism, sharing one common feature with us, the role of biases. Nevertheless, Acemoglu et al. (2013) focus on modeling the political process in a representative democracy, while we focus on the social dynamics of how incomplete information and network externalities lead to a gradual downgrading of expert biases.

A recent paper that offers empirical evidence that friendship networks make political opinions more tightly related is Algan et al. (2019). Another paper offering theory and evidence on information transmission through gossips is Banerjee et al. (2019). Other recent papers of related focus to ours include Candogan (2019), Candogan and Drakopoulos (2019), Myatt and Wallace (2019), and Egorov and Sonin (2019). These papers focus on the signaling mechanisms and their relationship to the network structure. A more directly related paper, focusing on the role that social media play in transmitting biased information that enhances polarization is Campbell, Leister, and Zenou (2019). The key difference of our paper is our focus on studying the role that people's fundamental preference biases play in the evolution of simulated network dynamics, even when biased or fake news are absent.

Finally, an evolutionary model that has a similar flavor to the network dynamics we suggest is the Schelling (1969, 1971) model. Two key differences in our framework is that we focus on network dynamics and that we propose a seach-and-matching mechanism of network formation.

2.2 Model

There is a network of $N < \infty$ persons. In period $t \in \{0, 1, ...\}$ the network is represented by an adjacency matrix $\mathbf{M}_t$. Matrix $\mathbf{M}_t$ is a symmetric $N \times N$ matrix with entries in $\{0, 1\}$, where $M_t^{ij} = M_t^{ji} = 1$ denotes that two individuals (nodes), are connected. The symmetry of $\mathbf{M}_t$ implies that we restrict attention to undirected networks, where each node represents an individual. In addition, we do not consider self-loops, meaning that all diagonal elements of $\mathbf{M}_t$ are equal to 0.

Let function $d_i(\mathbf{M}_t) \equiv \sum_{j=1}^{N} M_t^{ij}$ calculate the degree of node i, i.e., the sum of other individuals i is connected to. Given this degree, we define an associated $N \times N$ matrix $\boldsymbol{\gamma}_t$, defined by the function,

$$\boldsymbol{\gamma}_t = \Gamma(\mathbf{M}_t) \quad \text{with} \quad \gamma_t^{ij} \equiv \frac{M_t^{ij}}{d_i(\mathbf{M}_t)} \, . \tag{2.1}$$

Observe that, as in Golub and Morris (2018), matrix $\boldsymbol{\gamma}_t$ is a row-stochastic matrix, where γ_t^{ij} is the weight that i assigns to j, with agents putting equal weights to all of their friends.

The objective of each network member involves two tasks in each period. The first task is to understand the value of a fundamental quantity for which information is limited. This fundamental quantity can be the outcome of a vote on a political issue, a scientific finding about, e.g., a medical issue such as a vaccine for an epidemic, a price outcome, e.g. a house-price index, etc. The second task of each individual is to coordinate actions with peers, especially with those connected to them. This is the (Keynes, 1936) "beauty contest" motive, of trying to guess the actions of peers. In our framework, apart from this "beauty-contest" motive, agents will be trying to be more socially accepted by coordinating actions with their network peers who are connected with them.

In our model, we divide agents into two types, A and B, distinguished by differences in fundamental biases. As in Morris and Shin (2002), the action a_i of the agent gives higher utility if it is, (i) closer to an underlying state, θ_t, +/- some bias, which depends on the agent's type, and (ii) closer to the "beauty contest" term, which leads to an externality: each agent tries to second-guess the decisions of their friends. Specifically, the payoff function of a type-A agent i is given by,

$$u_i^A(a_t, \theta_t) = -(1-r)\left[a_{i,t} - (\theta_t + b)\right]^2 - r\sum_{j=1}^{N}\gamma_t^{ij}\left(a_{j,t} - a_{i,t}\right)^2 \ , \tag{2.2}$$

while the payoff function of a type-B agent i is,

$$u_i^B(a_t, \theta_t) = -(1-r)\left[a_{i,t} - (\theta_t - b)\right]^2 - r\sum_{j=1}^{N}\gamma_t^{ij}\left(a_{j,t} - a_{i,t}\right)^2 \ , \tag{2.3}$$

where $r \in (0,1)$, $b > 0$, and $a_t = [a_{1,t}, ..., a_{N,t}]$. According to (2.2) and (2.3), the feature distinguishing agent types is the bias: type A agents prefer that their action be closer to $(\theta_t + b)$, while type-B agents prefer being closer to $(\theta_t - b)$. Agents of each type have a preference to taking actions shifted away from the true value of θ_t. Intuitively, this bias in preferred actions reflects political, religious, and other similar biases, falling in the categories of biased assimilation and confirmation bias (see Lord et al., 1979, and Nickerson, 1998).[42] Assume that there is a total number of N_A type-A players and a total number of N_B type-B players, with $N_A + N_B = N$.

Parameter r captures the relative importance of the "beauty-contest" externality. In our setup, there is a key difference in the specification of the "beauty-contest" externality, compared to the standard "beauty-contest" concept used, e.g. in Morris and Shin (2002). In our setup the "beauty-contest" concept externality refers only to network "friends", i.e., to people who are connected with player i in period t. Therefore, while r and b are constant parameters of the utility function over time, the network externality can potentially differ over time, implicitly affecting the relative importance of the bias parameter, b, as well.

[42]The assumption of bias symmetry is made for simplicity.

2.2.1 Signals and Information Structure

The key assumption we make is that, in each period $t \in \{0, 1, ...\}$, there is a new task carrying a new fundamental value, θ_t, that is unknown and needs to be learned through signals available in period t. Therefore, the time horizon available for learning about parameter θ is one period only. Despite that the fundamental value to be learned is new in every period, we assume, for simplicity, that the stochastic structure underlying the signals that guide learning of θ_t, is the same in every period.

Specifically, in a similar fashion to Morris and Shin (2002), the information set available to player $i \in \{1, ..., N\}$ in each period is $\mathcal{I}_{i,t} = (y_t, x_{i,t})$, where y_t is a public signal with,

$$y_t = \theta_t + \eta_t \ , \quad \text{with} \ \ \eta_t \sim N\left(0, \sigma_\eta^2\right) \quad , \quad t = 0, ..., \tag{2.4}$$

and $x_{i,t}$ is a private signal to agent i only, with,

$$x_{i,t} = \theta_t + \varepsilon_{i,t} \ , \quad \text{with} \ \ \varepsilon_{i,t} \sim N\left(0, \sigma_\varepsilon^2\right) \quad , \quad t = 0, ..., \tag{2.5}$$

and the precisions of the public and the private signals are $\alpha = 1/\sigma_\eta^2$ and $\beta = 1/\sigma_\varepsilon^2$. Importantly, η_t is independent from $\varepsilon_{i,t}$ for all $i \in \{1, ..., N\}$, and $\varepsilon_{i,t}$ is independent from $\varepsilon_{j,t}$ for all $i \neq j$.

Since our goal is to produce an algorithm for running network simulations, the data-generating process of $\mathcal{I}_{i,t} = (y_t, x_{i,t})$ in every period needs a "true" parameter, θ_t^*, unknown to players in the model, to be used by a modeler. From a modeler's perspective, θ_t^* can vary (randomly) over time or it can be constant over time. Trying different sequences $\{\theta_t^*\}_{t=0}^T$ in simulated paths does not change the optimal strategic rules of players, since players do not know θ_t^* in each period and since the learning horizon is only one period for each t. Yet, even with the same strategic rules, the progression and noisiness of θ_t^* will affect the samples of signals $\{\mathcal{I}_{i,t} = (y_t, x_{i,t})\}_{i=1}^N$ and it will affect the simulated paths of actions, as these actions depend on $(y_t, x_{i,t})$. We return to this point when we discuss the strategies and simulation results below.

2.2.2 Belief sophistication, evolutionary myopia and taking optimal actions

The evolving state variable of the problem is the network structure, summarized by the $N \times N$ matrix $\boldsymbol{\gamma}_t$. Because the nodes of matrix $\boldsymbol{\gamma}_t$ enter the utility functions of individuals given by (2.2) and (2.3), each individual needs to be aware of the agents with whom they are connected. However, because of the direct interaction of player i with other players, in order to make an optimal decision, player i needs to second-guess the beliefs of other agents. In order to second-guess beliefs of other players, player i needs to be aware of all nodes in matrix $\boldsymbol{\gamma}_t$. We assume this level of sophistication in order to introduce and analyze the element of higher-order beliefs: each individual i must understand what other individuals believe about θ_t, and also i must understand what other individuals believe that i believes about θ_t. This *belief sophistication*, that the structure of $\boldsymbol{\gamma}_t$ is understood, and that higher-order beliefs are calculated, is a reasonable assumption, as each individual develops a sufficient understanding of the connectedness among players in social media in a given period t, which influences decisions.

Nevertheless, we assume away that individuals have foresight about the evolution of $\boldsymbol{\gamma}_t$ over time. Every individual only evaluates a myopic, narrow-sighted local evolution of its peer connections, at the stage of evaluating the random invitations for friendship or annoyances received in each period, that we explain below in the section explaining the period-by-period search and matching mechanism. We call this nearsightedness of the local evolution of $\boldsymbol{\gamma}_t$ for one period only, *evolutionary myopia*.

Decision-making on *taking optimal actions* involves maximizing the expected utility given by (2.2) and (2.3). Specifically, the objective function is the conditional expectation $E(u_i^A(a_t,\theta_t) \mid \mathcal{I}_{i,t})$ for type-A players and $E(u_i^B(a_t,\theta_t) \mid \mathcal{I}_{i,t})$ for type-B players. Denoting optimal actions by $a_{i,t}^{A*}$ and $a_{i,t}^{B*}$, first-order conditions give,

$$a_{i,t}^{A*} = (1-r)E(\theta \mid \mathcal{I}_{i,t}) + (1-r)b + r\sum_{j=1}^{N} \gamma_t^{ij} E(a_j \mid \mathcal{I}_{i,t}) \ , \quad i = 1,...,N_A \ , \tag{2.6}$$

and

$$a_{i,t}^{B*} = (1-r)E(\theta \mid \mathcal{I}_{i,t}) - (1-r)b + r\sum_{j=1}^{N} \gamma_t^{ij} E(a_j \mid \mathcal{I}_{i,t}) \ , \quad i = 1,...,N_B \ . \tag{2.7}$$

Based on the stochastic structure given by (2.4) and (2.5), Bayesian learning implies,[43]

$$E(\theta_t \mid \mathcal{I}_{i,t}) = \frac{\alpha y_t + \beta x_{i,t}}{\alpha+\beta} \ . \tag{2.8}$$

In addition, since the objective functions of all players are quadratic, it is reasonable to focus on linear strategies of the form,

$$a_{j,t}^{A*} = \omega_y^j y + \omega_b^j b + \left(1-\omega_y^j-\omega_b^j\right) x_j \ , \quad j = 1,...,N_A \ , \tag{2.9}$$

and

$$a_j^{B*} = w_y^j y + w_b^j (-b) + \left(1-w_y^j-w_b^j\right) x_j \ , \quad j = 1,...,N_B \ . \tag{2.10}$$

Notice that the linear-weights normalization, $\omega_y^j+\omega_b^j+\omega_x^j = 1$ and $w_y^j+w_b^j+w_x^j = 1$, is possible because the objective functions are ordinal utility functions. Substituting equations (2.8), (2.9) and (2.10) into (2.6) and (2.7) gives a linear system of $2N$ equations (the transformed equations (2.6) and (2.7) and equations (2.9) and (2.10)), in $2N$ unknowns, the coefficients $\left(\{\omega_y^j\}_{j=1}^{N_A}, \{w_y^j\}_{j=1}^{N_B}, \{\omega_b^j\}_{j=1}^{N_A}, \{w_b^j\}_{j=1}^{N_B}\right)$.

[43]See Morris and Shin (2002, p. 1526) and the Appendix of this paper.

Solving this linear problem through matrix inversion, leads to the fixed-point strategies of the form,[44]

$$a_{i,t}^{A*} = a_i^A \left(y_t, x_{i,t} \mid \boldsymbol{\gamma}_t \right) = \omega_y^i \left(\boldsymbol{\gamma}_t \right) y + \omega_b^i \left(\boldsymbol{\gamma}_t \right) b + \left[1 - \omega_y^i \left(\boldsymbol{\gamma}_t \right) - \omega_b^i \left(\boldsymbol{\gamma}_t \right) \right] x_i \ , \quad i = 1, ..., N_A \ , \tag{2.11}$$

and

$$a_i^{B*} = a_i^B \left(y_t, x_{i,t} \mid \boldsymbol{\gamma}_t \right) = w_y^i \left(\boldsymbol{\gamma}_t \right) y + w_b^i \left(\boldsymbol{\gamma}_t \right) (-b) + \left[1 - w_y^i \left(\boldsymbol{\gamma}_t \right) - w_b^i \left(\boldsymbol{\gamma}_t \right) \right] x_i \ , \quad i = 1, ..., N_B \ . \tag{2.12}$$

Substituting these strategies in the objective function of each player gives the value functions (indirect utility functions),

$$V_i^A \left(\boldsymbol{\gamma}_t \right) = E \left(u_i^A \left(a_{i,t}^{A*}, \theta_t \right) \mid \mathcal{I}_{i,t} \right) \ , \quad i = 1, ..., N_A \ , \tag{2.13}$$

and

$$V_i^B \left(\boldsymbol{\gamma}_t \right) = E \left(u_i^B \left(a_{i,t}^{B*}, \theta_t \right) \mid \mathcal{I}_{i,t} \right) \ , \quad i = 1, ..., N_B \ . \tag{2.14}$$

In the Appendix, we explain how the derivation of value functions $V_i^A \left(\boldsymbol{\gamma}_t \right)$ and $V_i^B \left(\boldsymbol{\gamma}_t \right)$ is achieved through matrix algebra.

Returning to the remark about the "true" parameter, θ_t^*, used by a modeler for simulating this model, in each period t, the strategy coefficients, $\omega_y^i \left(\boldsymbol{\gamma}_t \right)$, $\omega_b^i \left(\boldsymbol{\gamma}_t \right)$, $w_y^i \left(\boldsymbol{\gamma}_t \right)$, and $w_b^i \left(\boldsymbol{\gamma}_t \right)$ in equations (2.11) and (2.12) are not affected by the pattern of sequences $\{\theta_t^*\}_{t=0}^T$ in simulated paths. Yet, since different sequences $\{\theta_t^*\}_{t=0}^T$ give different average patterns of signals $(y_t, x_{i,t})$, simulated actions, $a_{i,t}^{A*}$ and $a_{i,t}^{B*}$ given by equations (2.11) and (2.12) will follow different patterns, depending on each sequences $\{\theta_t^*\}_{t=0}^T$. Accordingly, the value functions,

[44] We give details on how this problem is solved in Section 3, which focuses on characterizing the equilibrium, in order to convey the intuition of how the network structure, $\boldsymbol{\gamma}_t$, influences optimal strategies. At this stage, the statement made by equations (2.11) and (2.12) is that actions do depend on network structure, $\boldsymbol{\gamma}_t$, and so do indirect utility functions.

$V_i^A(\boldsymbol{\gamma}_t)$ and $V_i^B(\boldsymbol{\gamma}_t)$, given by (2.13) and (2.14) will follow different patterns as well. As we will see below, these value functions drive the dynamics of the network, $\boldsymbol{\gamma}_t$.

2.2.3 Myopic search and matching equilibrium: the evolution of the network

The evolving state variable of the problem is the network structure $\boldsymbol{\gamma}_t$. We assume that in each period each player, i, randomly (a) *sends one invitation to one non-friend* (individuals in the i-th row of $\boldsymbol{\gamma}_t$ with $\gamma_t^{ij} = 0$), and (b) *causes one annoyance to one friend* (individuals in the i-th row of $\boldsymbol{\gamma}_t$ with $\gamma_t^{ij} = 1$). After these invitations have been sent and annoyances have been caused, players who receive these invitations and experience these annoyances are prompted to make decisions on selecting new friends and on excluding old friends from their social network. Below we explain the details of the algorithm that governs these decisions, leading to the evolution of network $\boldsymbol{\gamma}_t$.

2.2.3.1. Sending invitations *The invitation that player i sends* to a non-friend in period t, is drawn from a uniform distribution, by counting the total number of 0's in the i-th row of $\boldsymbol{\gamma}_t$. This random invitation is a spontaneous social attempt to make friends, reaching out to agents of both types.

2.2.3.2. Causing annoyances Similarly, *the annoyance that player i causes* to a friend in period t, is also drawn from a uniform distribution, by counting the total number of 1's in the i-th row of $\boldsymbol{\gamma}_t$. Again this random annoyance is a spontaneous social event of hostility, provoking examination by the friend of i who has experienced the annoyance.

2.2.3.3. First stage of decision-making: examining received invitations and experienced annoyances Player i's decision of making new friends and of excluding old friends in period t, is based on examining only period t's received invitations or experienced annoyances. If player i has received no invitations and has experienced no annoyances, then player i has no decision to make. If player i has received/experienced a total number of m *invitations and annoyances altogether*, then i has 2^m cases to examine. These cases consist of $\{0, 1\}$ choices. Choice "0" stands for either rejecting a received friendship invitation or excluding an old friend based on a caused annoyance. On the contrary, choice "1" stands for either accepting a received friendship invitation or keeping an old friend despite a caused annoyance.

Once all the 2^m potential decision outcomes of i's received invitations and annoyances are created, they are introduced in the i-th row of matrix $\mathbf{M}_t$, which satisfies $\boldsymbol{\gamma}_t = \Gamma(\mathbf{M}_t)$. Therefore, the algorithm creates 2^m versions of the original matrix $\mathbf{M}_t$. Specifically, denote by $\mathbf{m}_{i,t}$ the $2^m \times m$ matrix with each row being an $1 \times m$ vector representing each $\{0, 1\}$ constellation of alternative friendships acceptances/exclusions of all m invitations of player i in period t. For each $k \in \{1, 2, ..., 2^m\}$, we place the elements of the k-th row of $\mathbf{m}_{i,t}$, in the corresponding ordered positions of invitations/annoyances received by player i, in the i-th row and the i-th column of matrix $\mathbf{M}_t$. This leads to the transformed symmetric matrix $\mathbf{M}_{i,t,k}$.[45] Using the transformation $\boldsymbol{\gamma}_{i,t,k} = \Gamma(\mathbf{M}_{i,t,k})$, we use the mapping $V_i^A(\boldsymbol{\gamma}_{i,t,k})$ or $V_i^B(\boldsymbol{\gamma}_{i,t,k})$, depending on whether player i is type A or type B. We store all values $\{V_i^A(\boldsymbol{\gamma}_{i,t,k})\}_{k=1}^{2^m}$ or $\{V_i^B(\boldsymbol{\gamma}_{i,t,k})\}_{k=1}^{2^m}$, in a $2^m \times 1$ vector $\mathbf{v}_{i,t}$. The maximizing element $k_{i,t}^*$ of vector $\mathbf{v}_{i,t}$ governs the optimal decision of player i on which invitations to accept/reject, or on which old friends to exclude, if any, based on caused annoyances. Therefore, the i-th row of $\mathbf{M}_t$ is replaced by the i-th row of matrix $\mathbf{M}_{i,t,k^*}$. At this stage, when this procedure is completed for all $i \in \{1, ..., N\}$, matrix $\mathbf{M}_t$ is transformed into an interim matrix $\hat{\mathbf{M}}_t$.

[45] The symmetry of matrix $\mathbf{M}_{i,t,k}$ guarantees that each scenario of accepting/rejecting potential or actual friends based on received invitations/annoyances is respected by both counterparts: player i who received the invitations/annoyances and any other player who sent the invitations/annoyances.

Notice that interim matrix $\hat{\mathbf{M}}_t$ is not symmetric. Matrix $\hat{\mathbf{M}}_t$ is further transformed at the second stage of decision-making.

2.2.3.4. Second stage of decision-making: treating simultaneous invitations and simultaneous annoyances At the first stage, each agent i examined all *received* invitations and annoyances he/she *experienced*, and made optimal decisions. At the second stage, outcomes of invitations that player j *sent* and annoyances that player j *caused* are aligned with the decisions of any other agent i who has received these specific invitations/annoyances.

Let's start with an invitation that player j sent to player i. We use a *convention*: no matter if the inclusion of the invited person, i, increases agent j's utility or not, in case j's invitation is accepted, agent j will add i as a friend. Therefore, player j must update his/her row of matrix $\hat{\mathbf{M}}_t$, for this accepted invitation he/she sent to i. In order to achieve this goal, we isolate such cases where the invitation has not been updated, using the indicator function,

$$\bar{\mathbb{I}}_t^{ij} = \begin{cases} 1 & , \quad \text{if } \left[m_t^{ij} - \hat{m}_t^{ij} = -1\right] \,\&\, \left[\hat{m}_t^{ij} - \hat{m}_t^{ji} \neq 0\right] \\ 0 & , \quad \text{else} \end{cases}$$

where m_t^{ij} and $\hat{m}_t^{ij}$ are elements of matrices $\mathbf{M}_t$ and $\hat{\mathbf{M}}_t$. Denote by $\bar{\mathbf{M}}_t$ the $N \times N$ matrix comprised solely by the indicator function $\bar{\mathbb{I}}_t^{ij}$. We transform the original matrix, $\hat{\mathbf{M}}_t$, into a new one, denoted by $\tilde{\mathbf{M}}_t$, with element $\tilde{m}_t^{ij}$ given by,

$$\tilde{m}_t^{ij} = \begin{cases} 1 & , \quad \text{if } \bar{m}_t^{ij} + \bar{m}_t^{ji} = 1 \\ \hat{m}_t^{ij} & , \quad \text{else} \end{cases}$$

where $\bar{m}_t^{ij}$ is an element of matrix $\bar{\mathbf{M}}_t$.

This transformation of matrix $\hat{\mathbf{M}}_t$ into matrix $\widetilde{\mathbf{M}}_t$ registers any invitation sent from j to i , that i had accepted, but player j had not registered in the j-th row of matrix $\hat{\mathbf{M}}_t$. Importantly, the transformation of matrix $\hat{\mathbf{M}}_t$ into matrix $\widetilde{\mathbf{M}}_t$ takes care of cases where both j had sent an invitation to i, and i had sent an invitation to j, but only one of the two accepted the invitation, while the other player rejected it. In this case of at least one acceptance in mutual invitations, matrix $\widetilde{\mathbf{M}}_t$ sets $\tilde{m}_t^{ij} = \tilde{m}_t^{ji} = 1$, following the convention that random invitations sent which are ultimately accepted, must be respected by both players.

We proceed with an annoyance that player j caused to player i. Again we follow a similar *convention* to the case of invitations: no matter if the exclusion of the annoyed person, i, increases j's utility or not, in case i excludes j from his/her network of friends, j must respect this decision and update his/her row of matrix $\widetilde{\mathbf{M}}_t$ accordingly. In order to isolate such cases where the outcome of the annoyance has not been updated, we use the indicator function

$$\underline{\mathbb{I}}_t^{ij} = \begin{cases} 1 & , \quad \text{if } \left[m_t^{ij} - \tilde{m}_t^{ij} = 1\right] \,\&\, \left[\tilde{m}_t^{ij} - \tilde{m}_t^{ji} \neq 0\right] \\ 0 & , \quad \text{else} \end{cases}$$

where m_t^{ij} and $\tilde{m}_t^{ij}$ are elements of matrices $\mathbf{M}_t$ and $\widetilde{\mathbf{M}}_t$. Denote by $\underline{\mathbf{M}}_t$ the $N \times N$ matrix comprised solely by the indicator function $\underline{\mathbb{I}}_t^{ij}$. We transform matrix $\widetilde{\mathbf{M}}_t$, into a new one, the final update of the network matrix that carries through to period $t+1$. Therefore we denote this matrix by $\mathbf{M}_{t+1}$, with element $\tilde{m}_{t+1}^{ij}$ given by,

$$m_{t+1}^{ij} = \begin{cases} 1 & , \quad \text{if } \underline{m}_t^{ij} + \underline{m}_t^{ji} = 1 \\ \tilde{m}_t^{ij} & , \quad \text{else} \end{cases}$$

where $\underline{m}_t^{ij}$ is an element of matrix $\underline{\mathbf{M}}_t$. Notice that the updated matrix, $\mathbf{M}_{t+1}$ is symmetric and that in the case of two mutually caused annoyances between any players i and j where only one of the two rejected the other, matrix $\mathbf{M}_{t+1}$ sets $m_{t+1}^{ij} = m_{t+1}^{ji} = 0$.

Finally, the updated network, $\boldsymbol{\gamma}_{t+1}$ is obtained via the transformation,

$$\boldsymbol{\gamma}_{t+1} = \Gamma\left(\mathbf{M}_{t+1}\right) \ .$$

Our model resembles the Golub and Morris (2018) general framework, which introduces limited information and higher-order learning in networks. Yet, there are numerous differences. First, in our model there are different types of persons, each having their own biases and prejudices on taking actions shifted away from the true fundamental value θ. Second, players in the Golub and Morris (2018) framework do not receive public signals, but only private signals. In our model, the presence of public signals is crucial, as public signals represent expert opinions about fundamentals. Third our model is dynamic, introducing a search-and-matching mechanism that influences these dynamics. In the next section we focus on characterizing these dynamics.

2.3 Equilibrium Characterization

2.3.1 Why the network structure affects strategies: higher order beliefs

Our analysis in this section focuses on how the dynamics of the network, $\boldsymbol{\gamma}_t$, affect the evolution of these optimal weights on biases and expert opinion, and how these biases further affect the evolution of the network, $\boldsymbol{\gamma}_t$.

To see why the structure of the network affects the strategy of each player, first consider equations (2.6) and (2.7). Players do not only try to coordinate with others, due to the "beauty-contest" term, but also try to form the correct beliefs about other player's expectations about the state variable θ.

The last term of the optimal action in equations (2.6) and (2.7) is given by (we simplify the expression of the conditional expectation),

$$\sum_{j=1}^{N}\gamma^{ij}E(a_j) = \begin{bmatrix}1\\ .\\ 1\\ 1\\ .\\ 1\end{bmatrix}^T \begin{bmatrix}\gamma_t^{11} & . & \gamma_t^{1k} & \gamma_t^{1k+1} & . & \gamma_t^{1N}\\ . & . & & & & .\\ \gamma_t^{k1} & & . & & & \gamma_t^{kN}\\ \gamma_t^{k+11} & & & . & & \gamma_t^{k+1N}\\ . & & & & . & .\\ \gamma_t^{N1} & . & \gamma_t^{Nk} & \gamma_t^{Nk+1} & . & \gamma_t^{NN}\end{bmatrix} \begin{bmatrix}\omega_x^1 E(\theta)+\omega_b^1 b+(1-\omega_x^1-\omega_b^1)y\\ \cdots\\ \omega_x^k E(\theta)+\omega_b^k b+(1-\omega_x^k-\omega_b^k)y\\ w_x^{k+1}E(\theta)+w_b^{k+1}(-b)+(1-w_x^{k+1}-w_b^{k+1})y\\ \cdots\\ w_x^N E(\theta)+w_b^N(-b)+(1-w_x^N-w_x^N)y\end{bmatrix}$$

Type A's optimal action is given by,

$$a_i^{A*} = (1-r)\left(E(\theta)+b\right) + \\ +r \begin{bmatrix} 1 \\ . \\ 1 \\ 1 \\ . \\ 1 \end{bmatrix}^T \begin{bmatrix} \gamma_t^{11} & . & \gamma_t^{1k} & \gamma_t^{1k+1} & . & \gamma_t^{1N} \\ . & . & . & . & . & . \\ \gamma_t^{k1} & . & . & . & . & \gamma_t^{kN} \\ \gamma_t^{k+11} & . & . & . & . & \gamma_t^{k+1N} \\ . & . & . & . & . & . \\ \gamma_t^{N1} & . & \gamma_t^{Nk} & \gamma_t^{Nk+1} & . & \gamma_t^{NN} \end{bmatrix} \begin{bmatrix} \omega_x^1 E(\theta) + \omega_b^1 b + (1-\omega_x^1 - \omega_b^1) y \\ ... \\ \omega_x^k E(\theta) + \omega_b^k b + (1-\omega_x^k - \omega_b^k) y \\ w_x^{k+1} E(\theta) + w_b^{k+1}(-b) + (1-w_x^{k+1} - w_b^{k+1}) y \\ ... \\ w_x^N E(\theta) + w_b^N(-b) + (1-w_x^N - w_x^N) y \end{bmatrix}$$

while type B's optimal action is given by,

$$a_i^{B*} = (1-r)\left(E(\theta)-b\right) + \\ +r \begin{bmatrix} 1 \\ . \\ 1 \\ 1 \\ . \\ 1 \end{bmatrix}^T \begin{bmatrix} \gamma_t^{11} & . & \gamma_t^{1k} & \gamma_t^{1k+1} & . & \gamma_t^{1N} \\ . & . & . & . & . & . \\ \gamma_t^{k1} & . & . & . & . & \gamma_t^{kN} \\ \gamma_t^{k+11} & . & . & . & . & \gamma_t^{k+1N} \\ . & . & . & . & . & . \\ \gamma_t^{N1} & . & \gamma_t^{Nk} & \gamma_t^{Nk+1} & . & \gamma_t^{NN} \end{bmatrix} \begin{bmatrix} \omega_x^1 E(\theta) + \omega_b^1 b + (1-\omega_x^1 - \omega_b^1) y \\ ... \\ \omega_x^k E(\theta) + \omega_b^k b + (1-\omega_x^k - \omega_b^k) y \\ w_x^{k+1} E(\theta) + w_b^{k+1}(-b) + (1-w_x^{k+1} - w_b^{k+1}) y \\ ... \\ w_x^N E(\theta) + w_b^N(-b) + (1-w_x^N - w_x^N) y \end{bmatrix}$$

Therefore, we have N equations and N unknowns. Using linear algebra, we find the optimal weights using,

$$\begin{bmatrix} 1 & . & . & . & . & -r\chi\gamma_t^{1N} & 0 & . & . & . & . & 0 \\ . & 1 & . & . & . & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . & . & . \\ -r\chi\gamma_t^{N1} & . & . & . & . & 1 & 0 & . & . & . & . & 0 \\ 0 & . & . & . & . & 0 & 1 & . & -r\gamma_t^{1k} & r\gamma_t^{1k+1} & . & r\gamma_t^{1N} \\ . & . & . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & r\gamma_t^{k+11} & . & . & 1 & -r\gamma_t^{k+1k+2} & . \\ . & . & . & . & . & . & . & . & . & . & 1 & . \\ 0 & . & . & . & . & 0 & r\gamma_t^{N1} & . & r\gamma_t^{Nk} & -r\gamma_t^{Nk+1} & . & 1 \end{bmatrix} \begin{bmatrix} \omega_x^1 \\ . \\ \omega_x^k \\ w_x^{k+1} \\ . \\ w^N \\ \omega_b^1 \\ . \\ \omega_b^k \\ w_b^{k+1} \\ . \\ w_b^N \end{bmatrix} = \begin{bmatrix} (1-r)\chi \\ (1-r)\chi \\ (1-r)\chi \\ (1-r)\chi \\ (1-r)\chi \\ (1-r)\chi \\ (1-r) \\ (1-r) \\ (1-r) \\ (1-r) \\ (1-r) \\ (1-r) \end{bmatrix} \tag{2.15}$$

where $\chi = \frac{\beta}{\alpha+\beta}$. The matrix of equation (2.15) consists of two blocks. The first block helps us in finding the weight of the private signal for each agent, while the second block enables us to find the weights on the bias, b. It is obvious that each agent needs to use the information of the whole network matrix, $\boldsymbol{\gamma}_t$. Therefore, equation (2.15) demonstrates the dependence of all strategy coefficients, $\omega_y^i(\boldsymbol{\gamma}_t)$, $\omega_b^i(\boldsymbol{\gamma}_t)$, $w_y^i(\boldsymbol{\gamma}_t)$, and $w_b^i(\boldsymbol{\gamma}_t)$ in equations (2.11) and (2.12) on $\boldsymbol{\gamma}_t$.

Given any network matrix $\boldsymbol{\gamma}_t$, for calculating the expected utility that gives us the value functions, $V_i^A(\boldsymbol{\gamma}_t)$ and $V_i^B(\boldsymbol{\gamma}_t)$, we use matrix algebra as well. This calculation is more involved, so it appears in the Appendix.

2.3.2 The tradeoff between biases and expert opinion

It can be proved that, for all $\boldsymbol{\gamma}_t$, the optimal weight on the private signal is given by,[46]

$$\omega_x^i(\boldsymbol{\gamma}_t) = 1 - \omega_y^i(\boldsymbol{\gamma}_t) - \omega_b^i(\boldsymbol{\gamma}_t) = \frac{(1-r)\beta}{(1-r)\beta+\alpha}, \quad i = 1, ..., N_A \ , \tag{2.16}$$

and

$$w_x^i(\boldsymbol{\gamma}_t) = 1 - w_y^i(\boldsymbol{\gamma}_t) - w_b^i(\boldsymbol{\gamma}_t) = \frac{(1-r)\beta}{(1-r)\beta+\alpha}, \quad i = 1, ..., N_B \ . \tag{2.17}$$

[46] A formal proof can be provided by the authors upon request.

An immediate implication of equations (2.16) and (2.17) is that, for all $\boldsymbol{\gamma}_t$,

$$\omega_y^i(\boldsymbol{\gamma}_t) + \omega_b^i(\boldsymbol{\gamma}_t) = w_y^j(\boldsymbol{\gamma}_t) + w_b^j(\boldsymbol{\gamma}_t) = \frac{\alpha}{(1-r)\beta + \alpha}, \quad i = 1, ..., N_A \ , \quad j = 1, ..., N_B \ . \tag{2.18}$$

Equation (2.18) says that whenever the weight on the bias, $\omega_b^i(\boldsymbol{\gamma}_t)$ increases, the weight and the attention to the public signal, the expert opinion, $\omega_y^i(\boldsymbol{\gamma}_t)$, has to decrease. This relationship captures the tradeoff between paying attention to biases versus paying attention to expert opinion. In our model, this tradeoff is explicitly defined by equation (2.18), which provides intuition for the main results in the simulations below.

2.4 Simulation Experiments

In our benchmark calibration we use a weight on the "beauty-contest" term of $r = 0.65$. The noisiness of private signals, σ_ε, is higher than the noisiness of expert signals, σ_η. We set $\sigma_\varepsilon = 0.32$, which implies $\beta = 10$, and $\sigma_\eta = 0.18$, which implies $\alpha = 30$. We set a small bias value, $b = 0.02$, which is about 9 times smaller than one standard deviation of the noisiness of the expert signal. In addition, we split the network into two groups of equal size. We set $N = 100$ and we let $N_A = N_B = 50$. Finally, we set $\theta_t^* = 0$ for all t.

To start examining the properties of our benchmark calibration, Figure 2.1 shows a sample of a totally random initial network in period 0, $\boldsymbol{\gamma}_0$, that we call the "original network". In the original network $\boldsymbol{\gamma}_0$, agents of all types are mixed and connected. The probability of randomly appearing 0's in the original network matrix $\boldsymbol{\gamma}_0$ is set to $p = 0.7$. As time passes, already 20 periods ahead, one can see that the two group types start becoming split (type-A agents are numbered from 1 to 50). As time moves even further ahead, homophily increases.

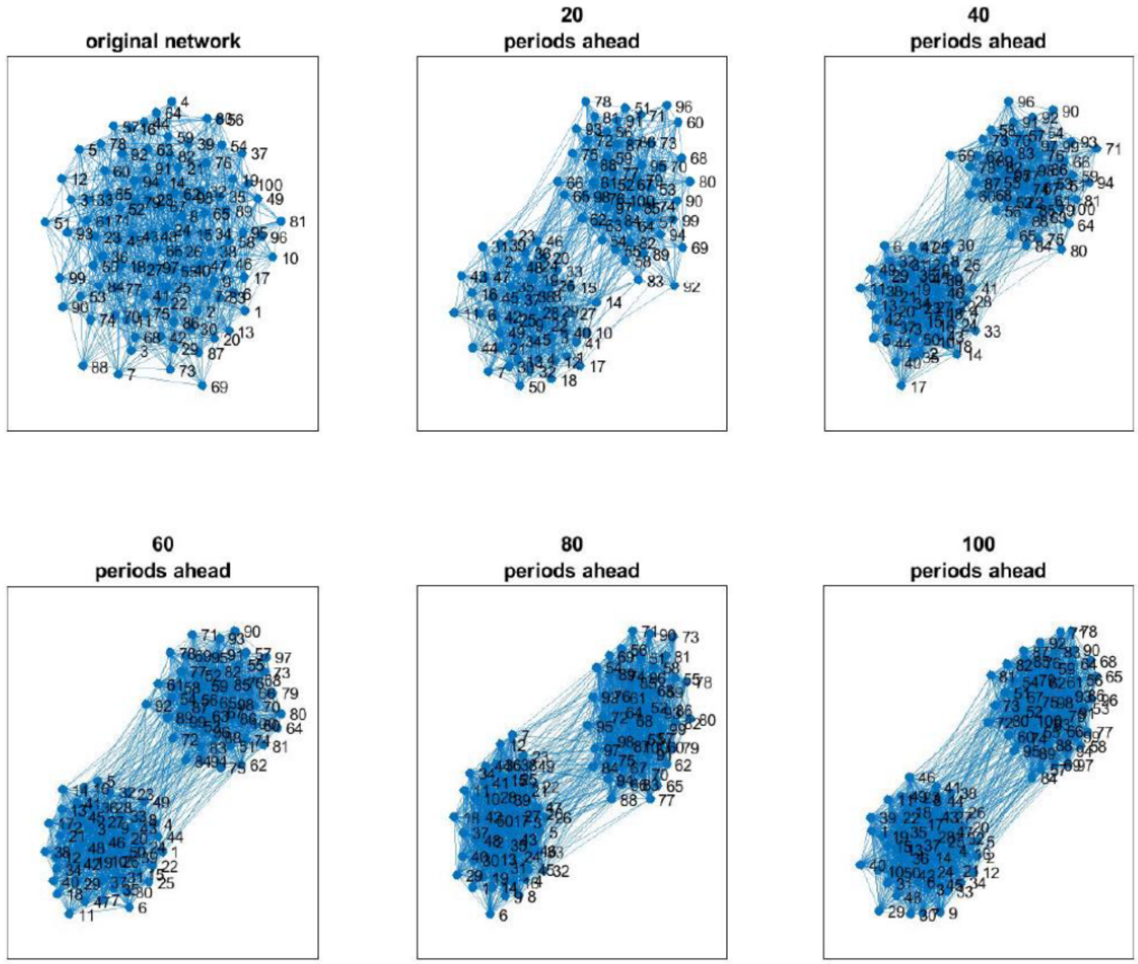

Figure 2.1 Sample network dynamics for the benchmark calibration.

In order to have a more concrete view of the benchmark model, we calculate 200 Monte-Carlo simulation trials, with time horizon equal to 100[47] . Figure 2.2.a depicts the dynamics of the network $\boldsymbol{\gamma}_t$. We use three metrics to describe the evolution of $\boldsymbol{\gamma}_t$: (i) the subnetwork inbreeding homophily index recommended by Currarrini et al. (2009, p. 1008), (ii) the subnetwork density index and (iii) the subnetwork closeness centrality.

[47]In Appendix 2.6.C we demonstrate the stability of results for 600 periods.

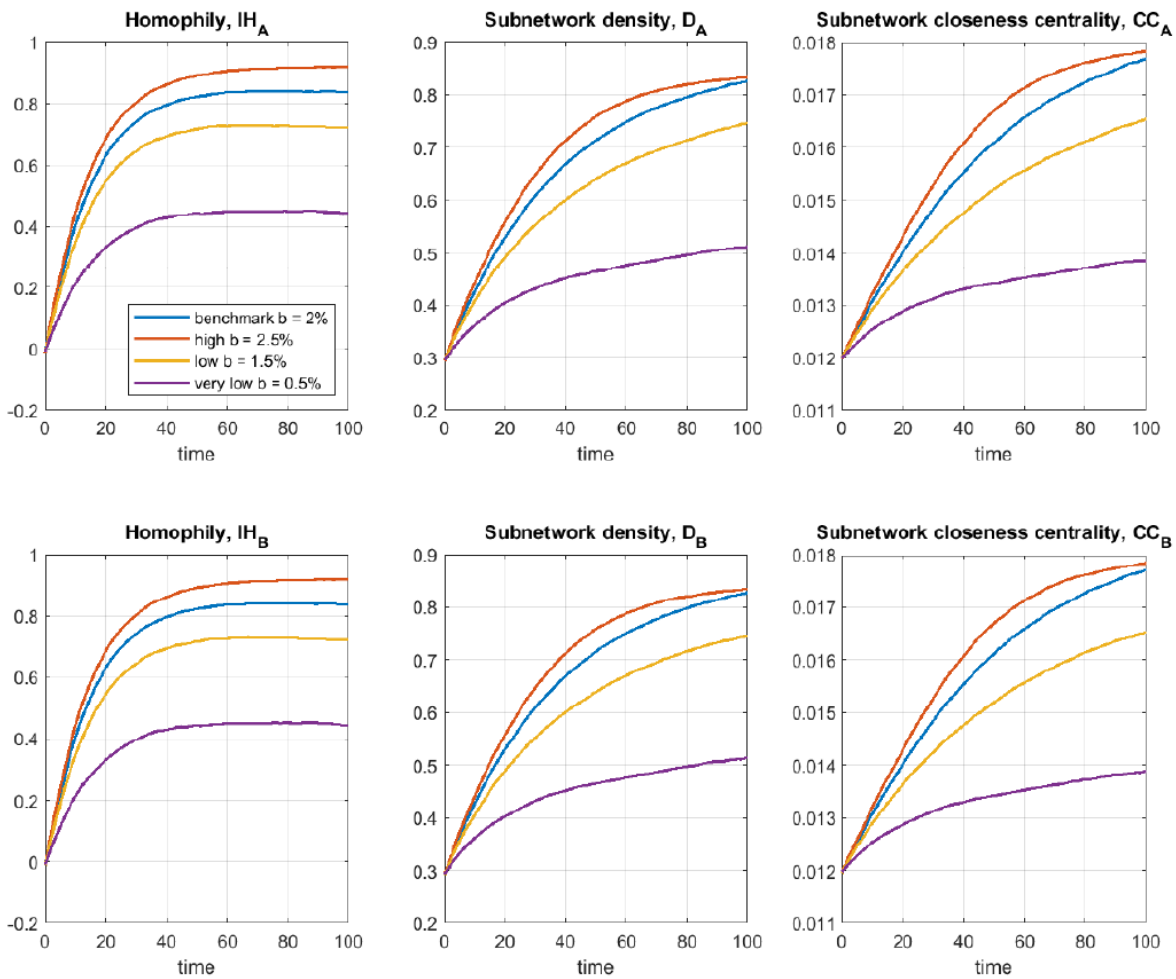

Figure 2.2.a Evolution of network $\boldsymbol{\gamma}_t$ with sensitivity analysis on the bias parameter, b. Benchmark calibration ($b = 2\%$) is compared to alternative cases with high and low b values.

The inbreeding homophily index depicted in the two panels on the left of Figure 2.2.a, is given by the formula,

$$IH_k = \frac{H_k - W_k}{1 - W_k} \ , \quad k \in \{A, B\} \ ,$$

where,

$$H_k = \frac{s_k}{s_k + d_k} \ ,$$

with s_k being the average number of friendships that agents of type k have with other type-k

agents, while d_k is the average number of friendships that type-k agents have with non-type-k agents. In addition $W_k \equiv N_k/N$. Density is defined as $D_k = s_k/N_k$, and the closeness centrality index is calculated in the standard way (see Jackson, 2008, Ch. 2).

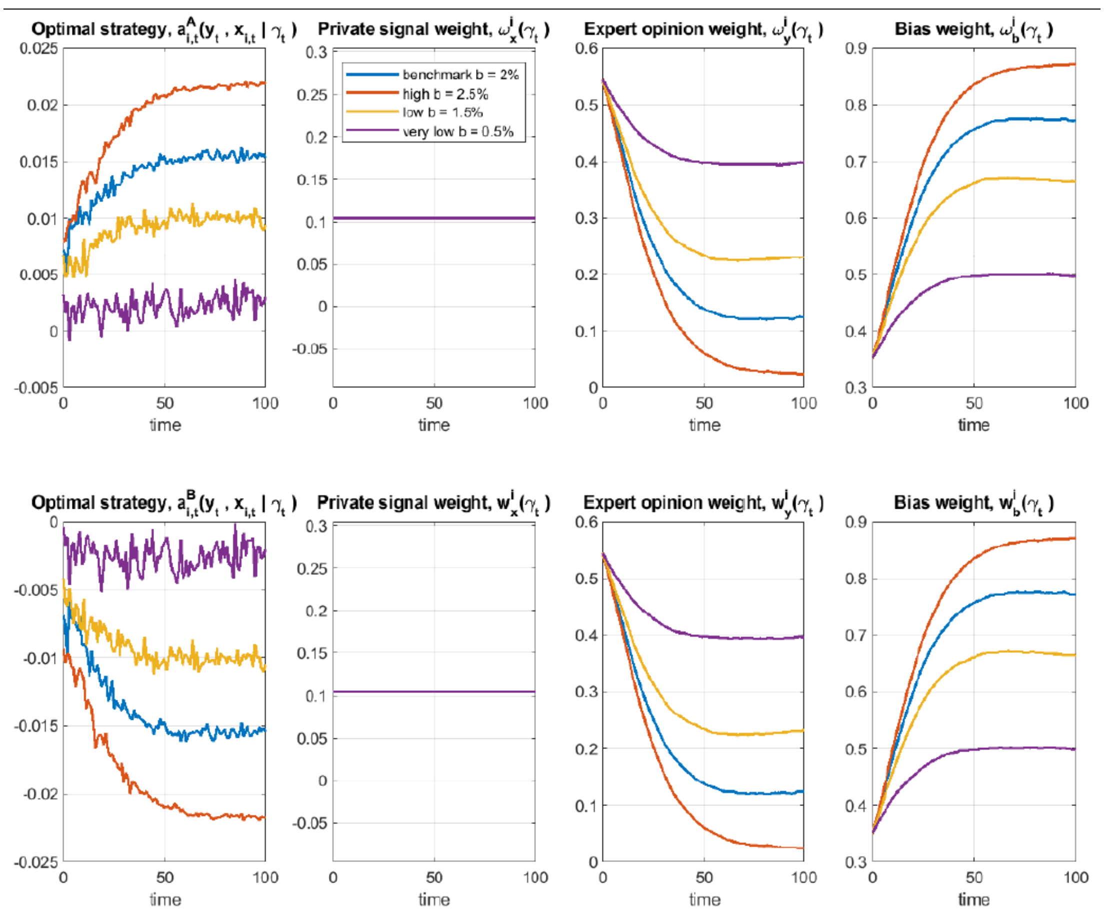

Figure 2.2.b Optimal actions and weights over time with sensitivity analysis on the bias parameter, b. Benchmark calibration ($b = 2\%$) is compared to alternative cases with high and low b values. The strategies of type A (top panels) and type B (bottom panels) are,

$$a^{A*}_{i,t} = a^A_i(y_t, x_{i,t} \mid \boldsymbol{\gamma}_t) = \omega^i_x(\boldsymbol{\gamma}_t)\, x_i + \omega^i_y(\boldsymbol{\gamma}_t)\, y + \omega^i_b(\boldsymbol{\gamma}_t)\, b, \text{ and}$$

$$a^{B*}_i = a^B_i(y_t, x_{i,t} \mid \boldsymbol{\gamma}_t) = w^i_x(\boldsymbol{\gamma}_t)\, x_i + w^i_y(\boldsymbol{\gamma}_t)\, y + w^i_b(\boldsymbol{\gamma}_t)\,(-b).$$

As we can see in Figure 2.2.a, as time passes, homophily increases and the within-group ties become stronger, because the density index, D_k, and the closeness centrality index,

CCA_k, of the subnetwork of friends of each of the two groups ($k \in \{A, B\}$) increase over time. Notably, for higher values of b, these dynamics of $\boldsymbol{\gamma}_t$ are accelerated, leading to a more segregated network faster, with more intense homophily, subnetwork density, and closeness centrality than the network depicted by the bottom right panel of Figure 1. Lower values of b seem to decelerate this segregation process. When the value of b is sufficiently low ($b = 0.5\%$, one fourth of the benchmark value $b = 2\%$), the homophily and density dynamics seem to slow down substantially.

The network dynamics of matrix $\boldsymbol{\gamma}_t$, depicted by Figure 2.2.a, are reflected in the optimal actions of players. Figure 2.2.b plots the optimal actions and action weights. Consistently with equations (2.11) and (2.12), and consistently with the characterization provided by equations (2.16), (2.17) and (2.18), over time the weight on the private signals remains constant, while the weights on bias increase and the weights on expert opinion decrease. Thus, the model provides not only homophily dynamics, but also a gradual downgrading of the expert opinion and an increase in biases. Notice that, despite the 200 Monte-Carlo simulation trials, there is still some unsuppressed noise of actions in the left top and left bottom panels of Figure 2.2.b. This unsuppressed noise is due to the fact that the noise levels of expert opinions and private signals are substantially high ($\sigma_\eta = 18\%$ and $\sigma_\varepsilon = 32\%$). Having in mind expert opinions about complicated public-policy issues (strategies to reduce unemployment, to increase growth, to strengthen international trade, to reduce fiscal debt, etc.), we assume that experts might disagree. Other sources of signals (internet bloggers, peers, etc.), exhibit even more disagreement. That agents in the model are aware of the values of σ_η and σ_ε, means that agents are aware of these kinds of disagreement.

2.4.1 The role of fundamental biases

To understand the role of fundamental biases captured by parameter b in the model, we must focus on the bias factors $\omega_b^i(\boldsymbol{\gamma}_t)$ and $w_b^i(\boldsymbol{\gamma}_t)$ in strategies (2.11) and (2.12). Parameter b is the fundamental bias parameter, while the optimal-strategy factors $\omega_b^i(\boldsymbol{\gamma}_t)$ and $w_b^i(\boldsymbol{\gamma}_t)$ are the peer-induced bias amplification factors. These peer-induced bias amplification factors enhance biases in actions, $a_i^{A*} = a_i^A(y_t, x_{i,t} \mid \boldsymbol{\gamma}_t)$ and $a_i^{B*} = a_i^B(y_t, x_{i,t} \mid \boldsymbol{\gamma}_t)$. In turn, the high peer-induced bias in these actions changes the value functions, $V_i^A(\boldsymbol{\gamma}_t)$ and $V_i^B(\boldsymbol{\gamma}_t)$, that players use in order to decide who to make friend and who to kick out of their personal network of peers. Therefore, given a level of fundamental biases captured by parameter b, the model produces additional peer-induced bias, captured by optimal strategy factors $\omega_b^i(\boldsymbol{\gamma}_t)$ and $w_b^i(\boldsymbol{\gamma}_t)$, which further enhances the homophily/segregation dynamics of network $\boldsymbol{\gamma}_t$. These segregation dynamics of $\boldsymbol{\gamma}_t$ lead to more peer-induced bias that accelerates the future segregation dynamics of $\boldsymbol{\gamma}_t$ even more. This acceleration is the the vicious circle of biases, beliefs and network homophily.

Since the model has no fake news, it emphasizes the role of the parameter, b. Specifically, in Figure 2.b we can see that a very small value of $b = 0.5\%$, gives agent actions that are not particularly polarized and without strong polarization dynamics. This finding is a theoretical argument indicating that one strategy for coping with populism might be to develop strategies for reducing b through educational reforms that may focus on mitigating fundamental biases by promoting evidence-based attitudes towards complicated social and scientific issues.

2.4.2 The role of asymmetry in the size of different groups

Here we study how different subgroup sizes influence the dynamics of the network, the actions, the dynamics of peer-induced biases and the dynamics of the downgrading of expert opinion. Using the same calibrating parameters as in the benchmark case, we make type-A agents a larger group with $N_A = 65$, and type-B agents a smaller group with $N_B = 35$.

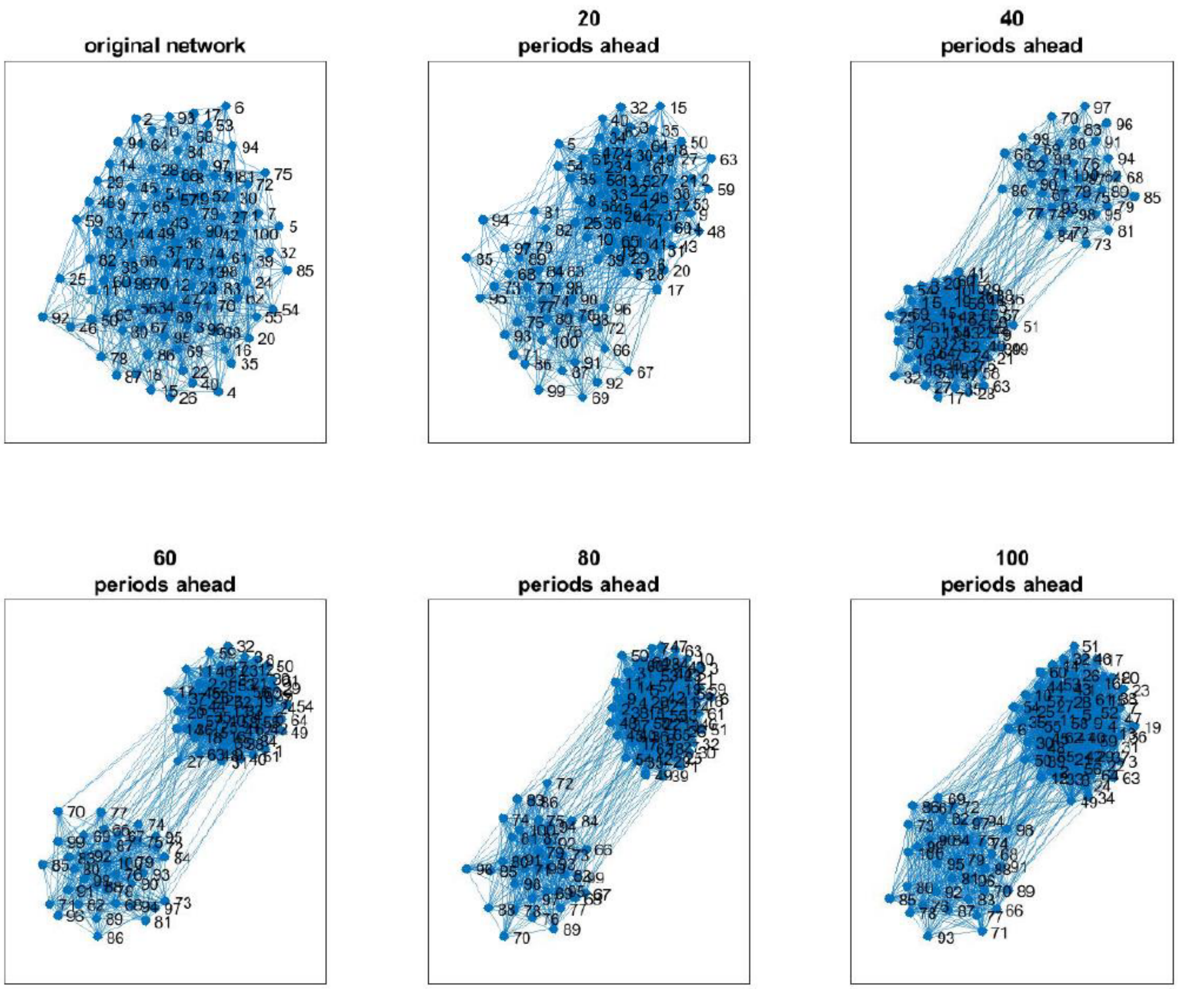

Figure 2.3 Sample network dynamics for the calibration with $N_A = 65$ and $N_B = 35$.

Figure 2.3 presents the sample dynamics of such a network. Just 20 periods ahead, the homophily dynamics are at work. Yet, the density of the small, type-B subnetwork seems to be increasing at a lower pace compared to the density of the larger, type-A subnetwork.

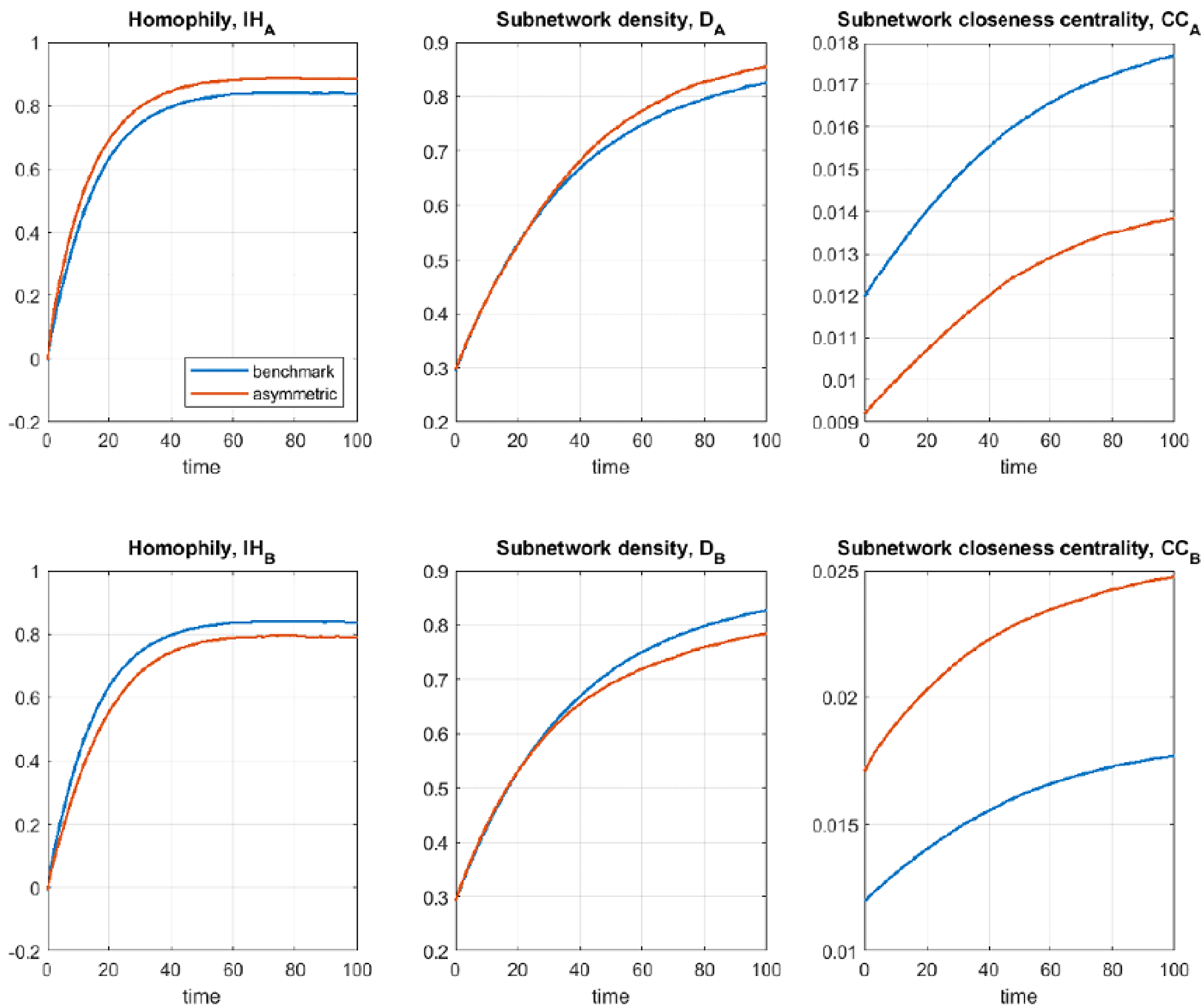

Figure 2.4.a Evolution of network γ_t varying the sizes of subgroups A and B. Benchmark calibration ($N_A = N_B = 50$) is compared to a case of asymmetric groups with $N_A = 65$ and $N_B = 35$.

To see if the sample dynamics depicted by Figure 2.3 are robust, we run a Monte-Carlo simulation of 200 Monte-Carlo trials. In Figure 2.4.a we compare the dynamics of this asymmetric network with $N_A = 65$ and $N_B = 35$ to the dynamics of the benchmark network with groups of the same size ($N_A = N_B = 50$). The intuition visually conveyed by Figure 2.3 concerning the evolution of the density between the two groups is confirmed by the Monte-Carlo averaging: the larger group, type A, exhibits higher subnetwork density and more homophily, too. Yet, the closeness centrality measure evolves in the opposite way: the smaller group, type B, exhibits higher closeness centrality than the larger group.

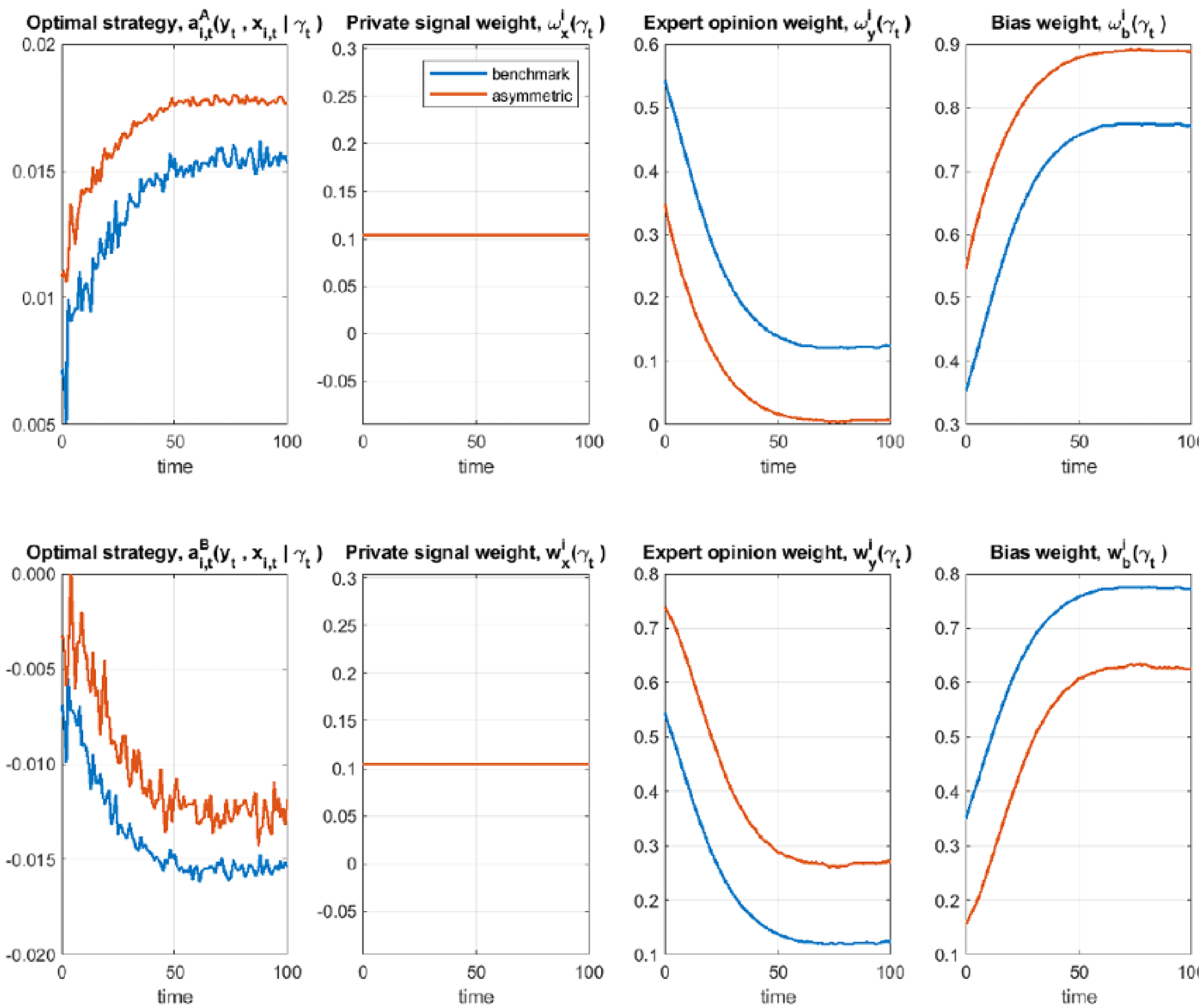

Figure 2.4.b Optimal actions and weights over time varying the size of subgroups A and B. Benchmark calibration ($N_A = N_B = 50$) is compared to a case of asymmetric groups with $N_A = 65$ and $N_B = 35$.

Figure 2.4.b investigates the effects of the network dynamics depicted by Figure 2.4.a on actions and peer-induced biases. Small groups need a bigger fundamental bias, b, in order to exhibit higher peer-induced bias. Otherwise, if the fundamental bias of a small group is the same as the fundamental bias of a large group, the interactions of the small group with the larger group lead to smaller peer-induced biases $w_b^i(\boldsymbol{\gamma}_t)$, and a more moderate decline in the downgrading of expert opinion. In brief, we find a moderate tendency of smaller groups to assimilate with the larger network and a moderate tendency of larger groups to exhibit higher homophily, subnetwork density, peer-induced bias and peer-induced neglect for expert opinion. Future work trying to understand the fanaticism of small groups might focus on studying the role that fundamental biases and fake news play within the subnetwork of such smaller groups.

2.4.3 Stability of results and different speed of meeting agents

In this subsection we present the stability of results using more periods. In addition we show that stability depends on different group sizes and on how it connects to the speed of meeting new people.

The analysis of stability of results uses more periods. Therefore, Figure 2.5.a presents the network dynamics after 1000 periods using the same calibration parameters as the benchmark. As we can see, the results are stable, there is polarization in network structure and increasing subnetwork closeness centrality.

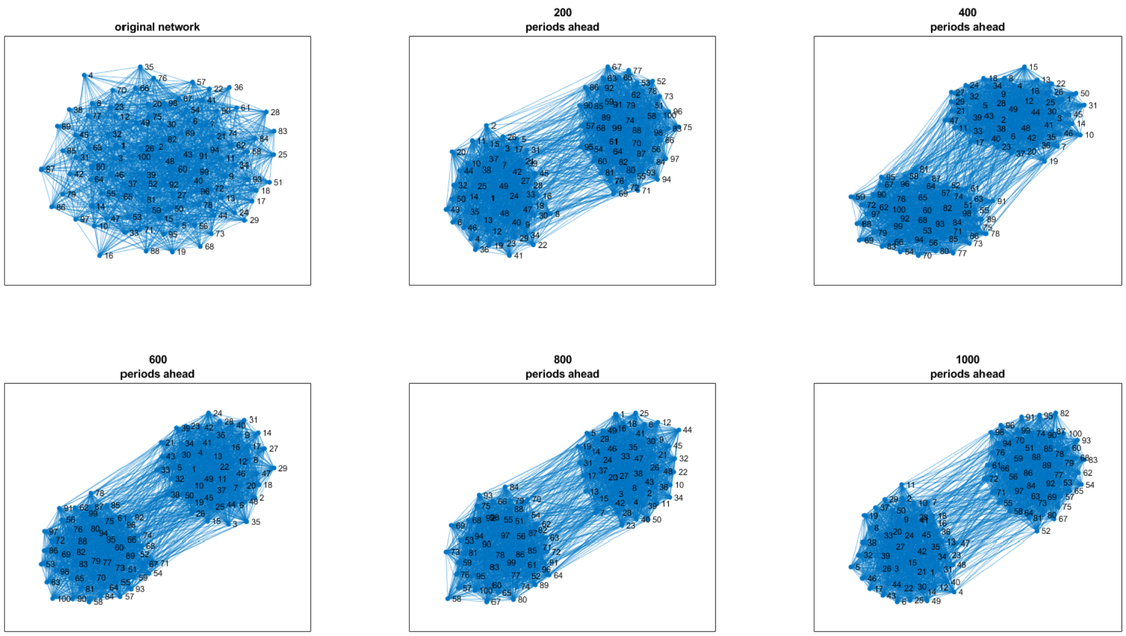

Figure 2.5.a Sample network dynamics for the benchmark calibration 1000 periods.

After 1000 periods there are still connections between two groups. There are three different components that lead to completely disjoint subnetworks: (1) the original network structure, which is randomly generated in period 1, (2) group size, (3) the Search and Matching process. The first and the third components are random processes and we cannot distinguish them in any formal way. In Figure 2.5.b we will demonstrate that the completely disjoint subnetworks depends on group size. In Figure 2.5.b we can see that all members of group B, except agent 96, break all their connections. As the time passes, there are no more B type agents with whom they can create a new link. Furthermore, as they experience the annoyances from their friends' side, they decide to break connections after computing their value function. In Figure 2.5.c we demonstrate that A-type agents can come up without any connection. In this case the first and third components play a very important role. But as we mention in the beginning, they are random processes, so we cannot distinguish which component will produce larger effect using formal methods.

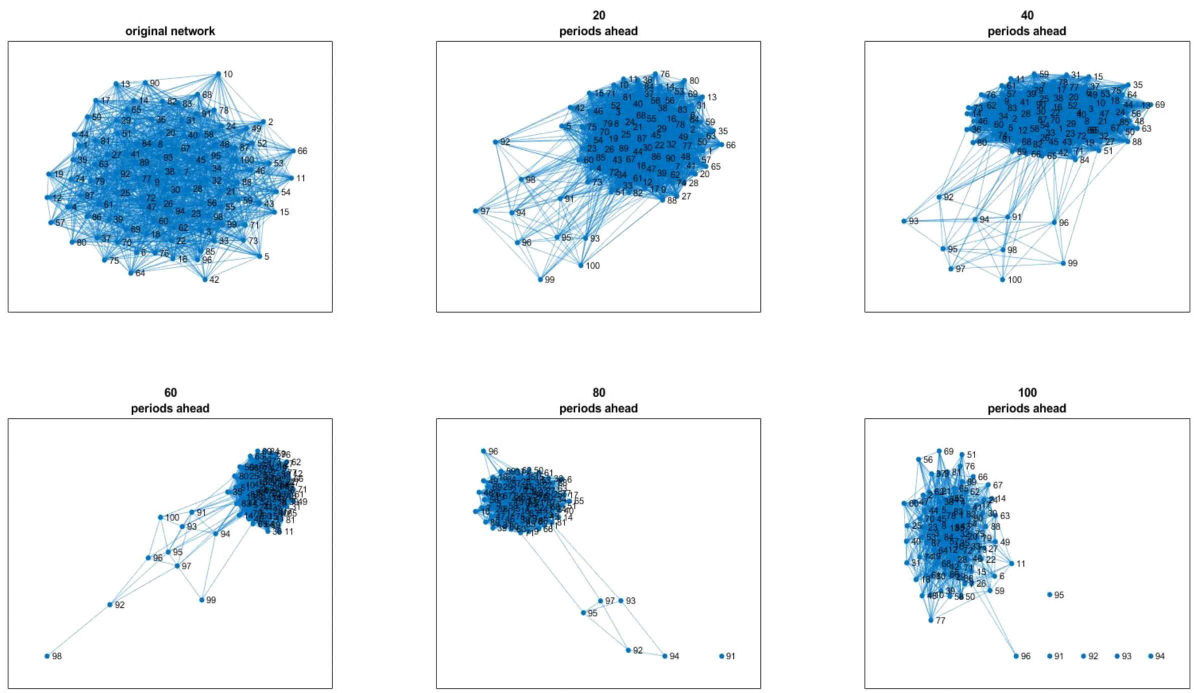

Figure 2.5.b Sample network dynamics for the calibration with $N_A = 95$ and $N_B = 5$.

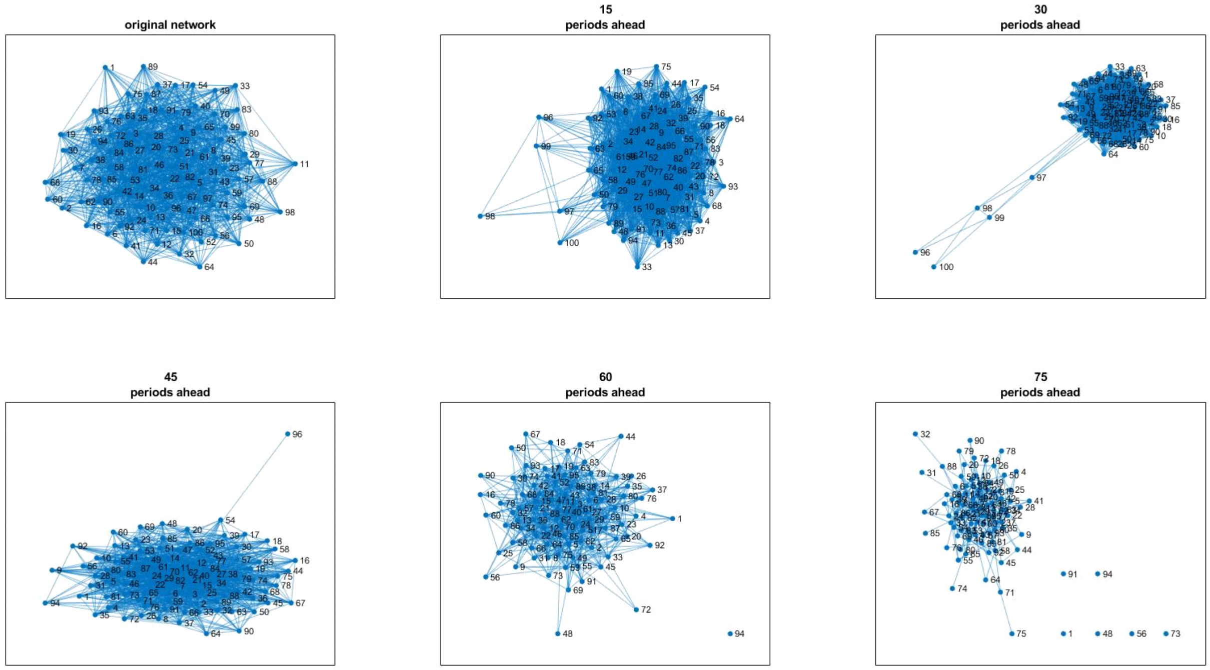

Figure 2.5.c Sample network dynamics for the calibration with $N_A = 95$ and $N_B = 5$.

Stability of results depends on group size: In this subsection we estimate the monotonicity of network characteristics with respect to network asymmetry. We produce the following group asymmetry, so $(N_A; N_B) = (50; 50), (55; 45), ..., (90; 10), (95; 5)$. Figure 2.6.a and Figure 2.6.b show, that compare with main results, monotonicity of increasing dynamics does not increase, when the size of group B is not small enough. The original matrix in the first period, together with random search and matching process influences the monotonicity of the following groups (85;15), (90;10), (95; 5).

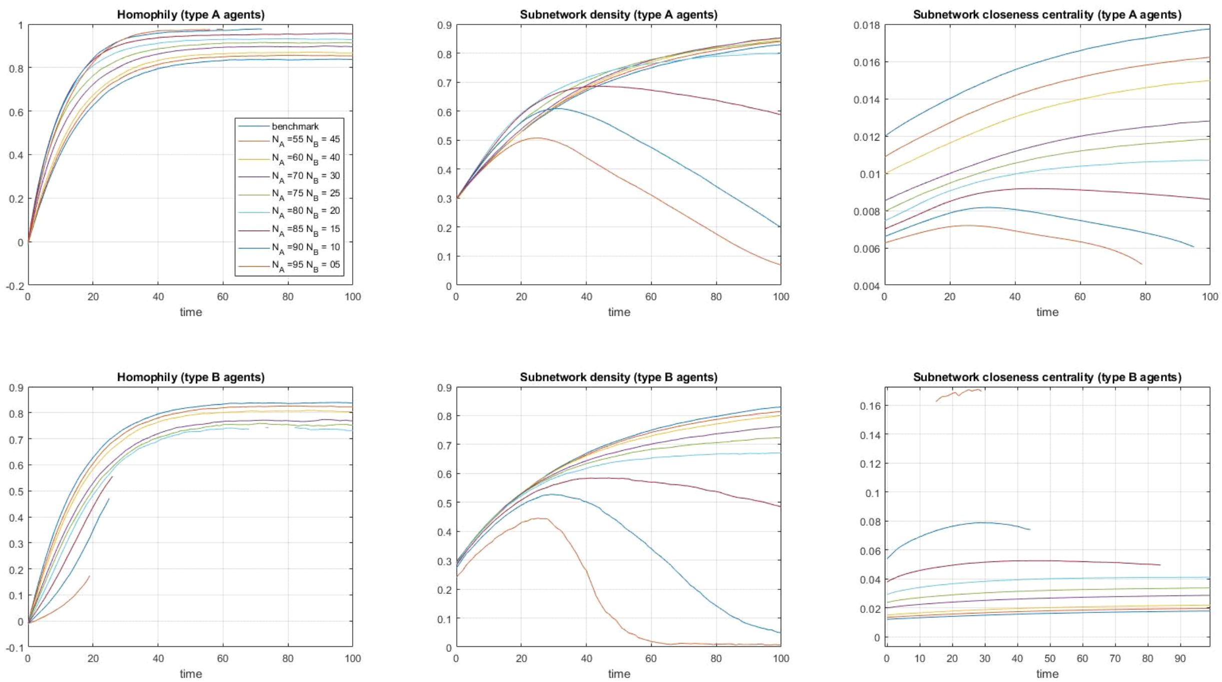

Figure 2.6.a Monotonicity of homophily, subnetwork density and subnetwork closeness centrality measures with respect to network asymmetry. Evolution of network $\boldsymbol{\gamma}_t$ varying the sizes of subgroups A and B.

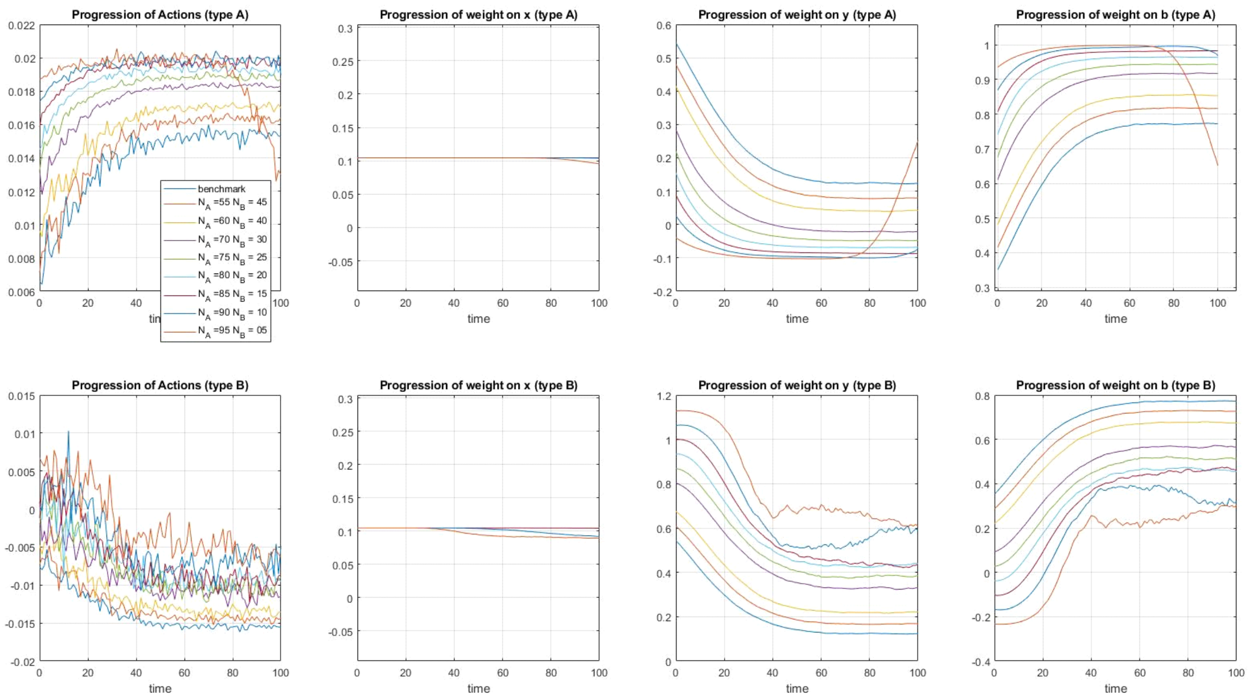

Figure 2.6.b Monotonicity on actions with respect to network asymmetry. Evolution of network $\boldsymbol{\gamma}_t$ varying the sizes of subgroups A and B.

If B-type agents get disconnected from all other A-type agents(either by random matching or by choice), then , as in Figure 2.6.a, we would never see increasing dynamics on subnetwork density and closeness centrality equal to 0.

In Figure 2.6.b we can see that the weight for the bias parameter b is negative, which means that B-type agents are mimicing themselves as A-type agents. As time passes, they meet more B-type agents, so the weight on bias b increases. As a consequence, they do not have incentives to demonstrate themselves as A-type agents any more.

Figure 2.7.a presents Homophily, Subnetwork density and Subnetwork closeness centrality dynamics, for 600 periods. In this figure, we also plot the speed of meeting new friends and, consequently, get the chance to break the old links. As it can be seen, for the benchmark

case the homophily increases during the first 150 periods. After 150 periods, it slightly decreases and stays stable. The main explanation of this non-linear dynamics of homophily is the group size: the bigger the group, the higher the chances that all agents with the same type meet each other and create a link. However, as time passes, there are no other agents of the same type that are not already connected. On the other hand, friends can make an annoyance, driving agents of the same type to break the link. This mechanism produces the decreasing in homophily that can be seen on the graphs. Besides the benchmark case, we add to the plots cases where probability of meeting new people or breaking links with old friends are set to 1/2 or 1/5.

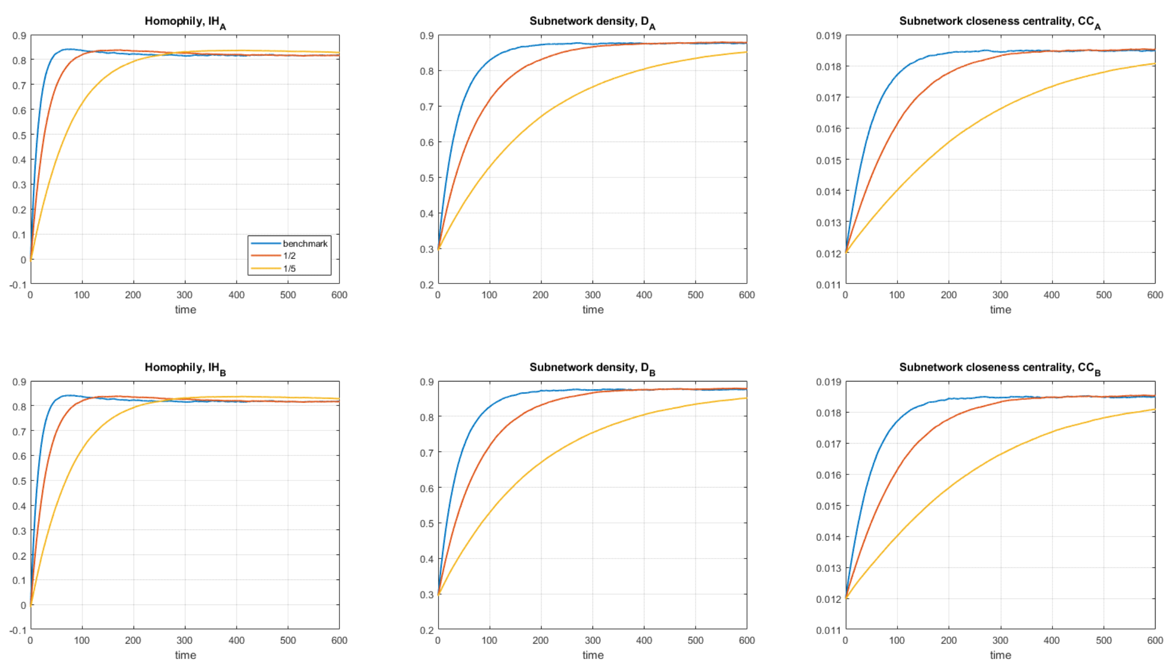

Figure 2.7.a Evolution of network with sensitivity analysis on bias parameter b_i , 600 periods, with the speed of meeting new people using the same benchmark calibration

In the main body of paper, we show the vicious cycle between homophily and biases. The decrease in the homophily after 150-200 periods lowers the weight on bias (Figure 2.7.b). Note that the decrease in weight begins on period 150 and lasts till period 300 for benchmark case, after which it gets stable.

Figure 2.7.b shows another important result. When the probability of meeting new friends is 1/5, we need more periods to get the same level of bias weight. This dynamics is an explanation to the phenomena described on Zeynep Tufekci's talk "How the Internet has made social change easy to organize, hard to win", where she compares the protest using new social media platforms (Facebook, Twitter) versus Montgomery bus boycott in 1955 in Alabama. Despite it usually takes 5 minutes to organize protest in the former cases, whereas for the latest it takes more than one year to gather strangers, protest duration and rent achievements are larger in the second case.

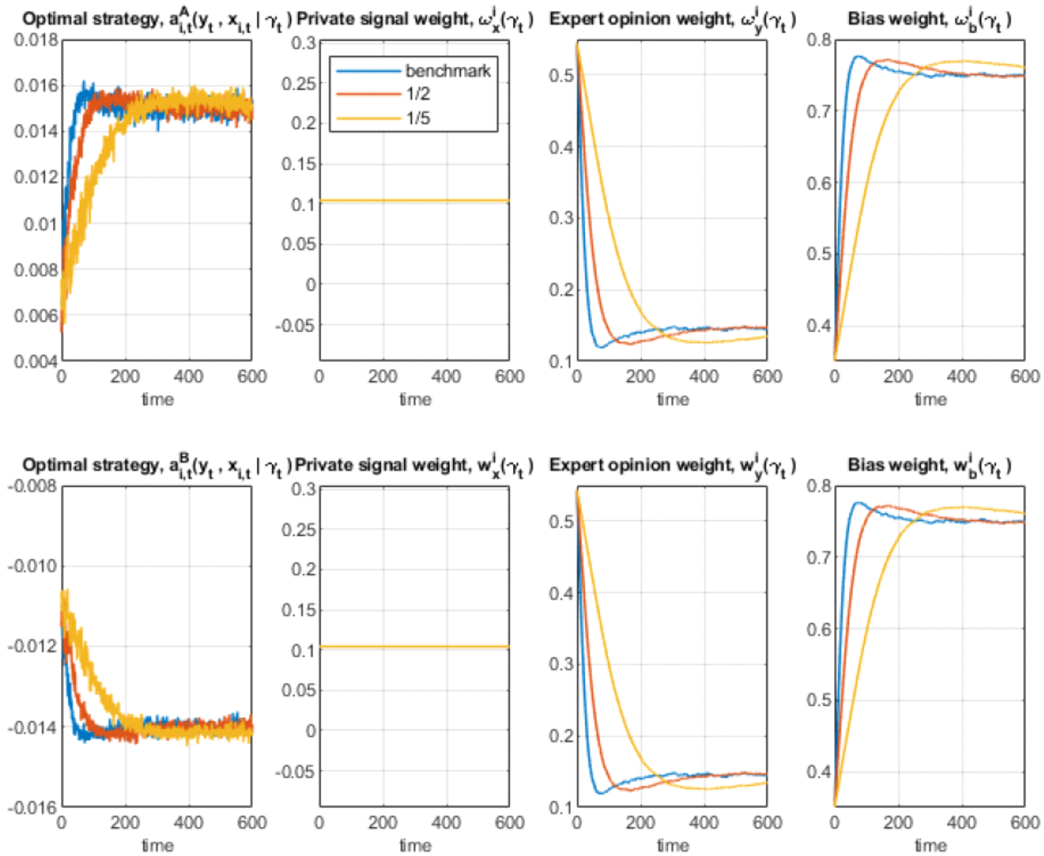

Figure 2.7.b Optimal actions and weights over 600 periods with sensitivity analysis on the speed of meeting new friends and breaking the links with old friends.The strategy of type A is demonstrated in the top panel and type B in the bottom panels.

The fact that the frequency of invitation/annoyances does not seem to change the simulated steady stated of action, homophily, network density and centrality, points at one new direction for future extensions. This new direction is to introduce a random birth/death process of internet users, that can capture the concept of average time horizons that network users have to meet new people and to revise their friends. Again, this is a direction for future research.

2.5 Conclusion

Populism has risen substantially in the past few decades. Among other factors explaining this rise, much research has focused on internet social media as one of the core culprits. Internet and social media have decreased the cost of forming new networks and of exchanging information. Populists tend to spend much energy on networking and on spreading information that is not fact-based or expert-reviewed. Naturally, much of current research has focused on fake news.

There is an obvious implicit and legitimate motivation behind the development of this fake-news literature: it is hoped that by understanding the determinants of fake news and by developing ways of combating fake news, problems of populism, of neglecting expert opinion, of fanaticism, etc. may be mitigated. While we do not object this view, we have argued that combating fake news may not be sufficient for combating the rising populist tendency of neglecting expert opinion. Just combining the internet's ease of forming networks with two fundamental features of most people, fundamental biases in attitudes towards a number of life aspects, and people's fundamental preference for being liked by their peers, can lead to populist dynamics over time through a vicious circle. Even without fake news, biases lead to more homophily and, over time, more homophily leads to actions that put more weight on biases and less weight on expert opinion.

Certainly, it is impossible to reverse the technological improvements behind the development of the internet and online social media. Yet, a message of our findings is that, in addition to the fake-news research initiative, societies might need to invest more intensely in ways of mitigating fundamental biases from people. This might be possible to be achieved through educational reforms and educational approaches that train citizens in developing a fact-based attitude towards knowledge and new information, trust for science and respect for expert views. Understanding the determinants of biases and ways of making people aware of biases may be a new focus of future research that aims at mitigating populism in society.

To the best of our knowledge, our paper is the first study to propose a search and matching mechanism of network friends in an environment of incomplete information, higher order beliefs and evolutionary dynamics. An appealing feature of our model is that it rationalizes decisions under incomplete information. We have tackled a demanding fixed-point problem of calculating higher-order beliefs, and have simplified the computation of value functions that are crucial for the search-and-matching decisions, using linear algebra. Yet, our model is still demanding in terms of the required computational power, even in cases with $N = 1000$. Future research might focus on simulating networks with millions of network members and many different groups, distinguished by identifiable biases. For this research agenda, the search-and-matching mechanism may be simplified, perhaps by finding some quasi-solutions to the calculation of value functions, in order to avoid sacrificing the key mechanism of rationalizing friendship choices.

Finally, future work can focus on evolving networks where the number of network participants, N, changes over time. This extension can be rather straightforward, provided that the "birth-and-death" process of internet and social media users relies on empirical observations. Such extensions are among the numerous directions one can take in future research.

2.6 Appendix

2.6.A Calculating key expectations

Agent i's information set consists of her private signal x_i, and public signal y. Since all signals are random variables, centered around θ, to predict the state of the world conditional on its information set, agent i should consider the following probability density function:

$$p(\theta \mid \mathcal{I}_i) = p(\theta \mid \ (y, x_i)) \propto p(y, x_i \mid \theta)p(\theta) \propto \exp\left\{-\frac{1}{2}\left[\alpha(\theta - y)^2 + \beta(\theta - x_i)^2\right]\right\}$$

in the case of the flat (absolutely non-informative: $p(\theta) \propto 1$) prior of θ. The expression in the exponential function can be transformed in the following way:

$$\begin{aligned} L &= \alpha(\theta - y)^2 + \beta(\theta - x_i)^2 = \alpha\left[\theta^2 - 2\theta y\right] + \beta(\theta^2 - 2\theta x_i) + C = \\ &= \theta^2(\alpha + \beta) - 2\theta(\alpha y + \beta x_i) + C \end{aligned}$$

where C is a constant. Such transformations are frequently used in the Bayesian statistics literature (see, for instance, Koop et.al., 2007). Therefore, we find that,

$$\theta|_{(y,x_i)} \sim N\left(\frac{\alpha y + \beta x_i}{\alpha + \beta}\ ,\ \frac{1}{\alpha + \beta}\right)\ , \tag{A.1}$$

which implies,

$$E(\theta \mid \mathcal{I}_i) = E(\theta \mid \ (y, x_i)) = \frac{\alpha y + \beta x_i}{\alpha + \beta}\ .$$

Next step is to calculate $E(\theta^2 \mid \mathcal{I}_i)$. Observe that any normally distributed variable, $x \sim N(\mu, \sigma^2)$, can be written as a linear transformation of a standard normal, i.e., $x = \mu + \sigma z$, with $z \sim N(0, 1)$. Therefore, equation (A.1) implies,

$$\theta = \frac{\alpha y + \beta x_i}{\alpha + \beta} + \frac{1}{\sqrt{(\alpha + \beta)}} z\ ,$$

where $z \sim N(0, 1)$. Let,

$$\sigma = \frac{1}{\sqrt{(\alpha + \beta)}}$$

and

$$\mu = \frac{\alpha y + \beta x_i}{\alpha + \beta}\ ,$$

implying that $\theta = \sigma z + \mu$. Therefore, $\theta^2 = \sigma^2 z^2 + \mu^2 + 2\sigma\mu z$, implying,

$$E(\theta^2 \mid \mathcal{I}_i) = \sigma^2 E(z^2 \mid \mathcal{I}_i) + \mu^2 + 2\sigma\mu E(z \mid \mathcal{I}_i) \ . \tag{A.2}$$

Since $z \sim N(0,1)$, $z^2 \sim \chi^2(1)$, and $E(z^2 \mid \mathcal{I}_i) = 1$. Therefore, equation (A.2) implies,

$$E(\theta^2 \mid \mathcal{I}_i) = \left(\frac{\alpha y + \beta x_i}{\alpha + \beta}\right)^2 + \frac{1}{\alpha + \beta} \ .$$

2.6.B Calculating the value functions

The value functions are equal to $E(u_i(a^*, \theta))$. Using (2.11) and (2.12), we find the optimal action a_i^* of each agent, and then we put the optimal action a_i^* into the expected utility function and find the expected utility from everyone's side. The calculations are summarized by,

$$\begin{bmatrix} E(u_i(a,\theta)) \\ . \\ E(u_k(a,\theta)) \\ E(u_{k+1}(a,\theta)) \\ . \\ E(u_N(a,\theta)) \end{bmatrix} = \begin{bmatrix} -(1-r)\left(\frac{1}{\alpha+\beta} + \left(\frac{\alpha y + \beta x_1}{\alpha+\beta}\right)^2\right) \\ . \\ -(1-r)\left(\frac{1}{\alpha+\beta} + \left(\frac{\alpha y + \beta x_k}{\alpha+\beta}\right)^2\right) \\ -(1-r)\left(\frac{1}{\alpha+\beta} + \left(\frac{\alpha y + \beta x_{k+1}}{\alpha+\beta}\right)^2\right) \\ . \\ -(1-r)\left(\frac{1}{\alpha+\beta} + \left(\frac{\alpha y + \beta x_N}{\alpha+\beta}\right)^2\right) \end{bmatrix} - r \begin{bmatrix} \frac{1}{\alpha+\beta} + \left(\frac{\alpha y + \beta x_1}{\alpha+\beta}\right)^2 \\ . \\ \frac{1}{\alpha+\beta} + \left(\frac{\alpha y + \beta x_k}{\alpha+\beta}\right)^2 \\ \frac{1}{\alpha+\beta} + \left(\frac{\alpha y + \beta x_{k+1}}{\alpha+\beta}\right)^2 \\ . \\ \frac{1}{\alpha+\beta} + \left(\frac{\alpha y + \beta x_N}{\alpha+\beta}\right)^2 \end{bmatrix} .*$$

$$.* \left(\begin{bmatrix} \gamma_t^{11} & . & \gamma_t^{1k} & \gamma_t^{1k+1} & . & \gamma_t^{1N} \\ . & . & . & . & . & . \\ \gamma_t^{k1} & . & . & . & . & \gamma_t^{kN} \\ \gamma_t^{k+11} & . & . & . & . & \gamma_t^{k+1N} \\ . & . & . & . & . & . \\ \gamma_t^{N1} & . & \gamma_t^{Nk} & \gamma_t^{Nk+1} & . & \gamma_t^{NN} \end{bmatrix} \begin{bmatrix} (\omega_x^1)^2 \\ . \\ (\omega_x^k)^2 \\ (w_x^{k+1})^2 \\ . \\ (w_x^N)^2 \end{bmatrix} \right) - \begin{bmatrix} a_1^2 \\ . \\ a_k^2 \\ a_{k+1}^2 \\ . \\ a_N^2 \end{bmatrix} -$$

$$- \begin{bmatrix} \frac{r}{\alpha} \\ \frac{r}{\alpha} \\ \frac{r}{\alpha} \\ \frac{r}{\alpha} \\ \frac{r}{\alpha} \\ \frac{r}{\alpha} \end{bmatrix} .* \left(\begin{bmatrix} \gamma_t^{11} & . & \gamma_t^{1k} & \gamma_t^{1k+1} & . & \gamma_t^{1N} \\ . & . & . & . & . & . \\ \gamma_t^{k1} & . & . & . & . & \gamma_t^{kN} \\ \gamma_t^{k+11} & . & . & . & . & \gamma_t^{k+1N} \\ . & . & . & . & . & . \\ \gamma_t^{N1} & . & \gamma_t^{Nk} & \gamma_t^{Nk+1} & . & \gamma_t^{NN} \end{bmatrix} \begin{bmatrix} (\omega_x^1)^2 \\ . \\ (\omega_x^k)^2 \\ (w_x^{k+1})^2 \\ . \\ (w_x^N)^2 \end{bmatrix} \right) +$$

$$+b^2 \left(\begin{bmatrix} -(1-r) \\ . \\ -(1-r) \\ -(1-r) \\ . \\ -(1-r) \end{bmatrix} - r \begin{bmatrix} \gamma_t^{11} & . & \gamma_t^{1k} & \gamma_t^{1k+1} & . & \gamma_t^{1N} \\ . & . & . & . & . & . \\ \gamma_t^{k1} & . & . & . & . & \gamma_t^{kN} \\ \gamma_t^{k+11} & . & . & . & . & \gamma_t^{k+1N} \\ . & . & . & . & . & . \\ \gamma_t^{N1} & . & \gamma_t^{Nk} & \gamma_t^{Nk+1} & . & \gamma_t^{NN} \end{bmatrix} \begin{bmatrix} \left(\omega_b^1\right)^2 \\ . \\ \left(\omega_b^k\right)^2 \\ \left(w_b^{k+1}\right)^2 \\ . \\ \left(w_b^N\right)^2 \end{bmatrix} \right) +$$

$$+2b \begin{bmatrix} \frac{\alpha y+\beta x_1}{\alpha+\beta} \\ . \\ \frac{\alpha y+\beta x_k}{\alpha+\beta} \\ \frac{\alpha y+\beta x_{k+1}}{\alpha+\beta} \\ . \\ \frac{\alpha y+\beta x_N}{\alpha+\beta} \end{bmatrix} .* \left(\begin{bmatrix} -(1-r) \\ . \\ -(1-r) \\ -(1-r) \\ . \\ -(1-r) \end{bmatrix} - \begin{bmatrix} \gamma_t^{11} & . & \gamma_t^{1k} & -\gamma_t^{1k+1} & . & -\gamma_t^{1N} \\ . & . & . & . & . & . \\ \gamma_t^{k1} & . & . & . & . & -\gamma_t^{kN} \\ \gamma_t^{k+11} & . & . & . & . & -\gamma_t^{k+1N} \\ . & . & . & . & . & . \\ \gamma_t^{N1} & . & \gamma_t^{Nk} & -\gamma_t^{Nk+1} & . & -\gamma_t^{NN} \end{bmatrix} \begin{bmatrix} \omega_b^1\omega_x^1 \\ . \\ \omega_b^k\omega_x^k \\ w_b^{k+1}w_x^{k+1} \\ . \\ w_b^N w_x^N \end{bmatrix} \right) +$$

$$+2 \begin{bmatrix} a_1 \\ . \\ a_k \\ a_{k+1} \\ . \\ a_N \end{bmatrix} .* \begin{bmatrix} \frac{\alpha y+\beta x_1}{\alpha+\beta} \\ . \\ \frac{\alpha y+\beta x_k}{\alpha+\beta} \\ \frac{\alpha y+\beta x_{k+1}}{\alpha+\beta} \\ . \\ \frac{\alpha y+\beta x_N}{\alpha+\beta} \end{bmatrix} .* \left(\begin{bmatrix} (1-r) \\ . \\ (1-r) \\ (1-r) \\ . \\ (1-r) \end{bmatrix} + r \begin{bmatrix} \gamma_t^{11} & . & \gamma_t^{1k} & \gamma_t^{1k+1} & . & \gamma_t^{1N} \\ . & . & . & . & . & . \\ \gamma_t^{k1} & . & . & . & . & \gamma_t^{kN} \\ \gamma_t^{k+11} & . & . & . & . & \gamma_t^{k+1N} \\ . & . & . & . & . & . \\ \gamma_t^{N1} & . & \gamma_t^{Nk} & \gamma_t^{Nk+1} & . & \gamma_t^{NN} \end{bmatrix} \left[\begin{bmatrix} \omega_x^1 \\ . \\ \omega_x^k \\ w_x^{k+1} \\ . \\ w_x^N \end{bmatrix} \right] \right) +$$

$$+2b \begin{bmatrix} a_1 \\ . \\ a_k \\ a_{k+1} \\ . \\ a_N \end{bmatrix} .* \left(\begin{bmatrix} (1-r) \\ . \\ (1-r) \\ -(1-r) \\ . \\ -(1-r) \end{bmatrix} + r \begin{bmatrix} \gamma_t^{11} & . & \gamma_t^{1k} & -\gamma_t^{1k+1} & . & -\gamma_t^{1N} \\ . & . & . & . & . & . \\ \gamma_t^{k1} & . & . & . & . & -\gamma_t^{kN} \\ \gamma_t^{k+11} & . & . & . & . & -\gamma_t^{k+1N} \\ . & . & . & . & . & . \\ \gamma_t^{N1} & . & \gamma_t^{Nk} & -\gamma_t^{Nk+1} & . & -\gamma_t^{NN} \end{bmatrix} \begin{bmatrix} \omega_b^1 \\ . \\ \omega_b^k \\ w_b^{k+1} \\ . \\ w_b^N \end{bmatrix} \right) -$$

$$-ry^2 \begin{bmatrix} \gamma_t^{11} & . & \gamma_t^{1k} & \gamma_t^{1k+1} & . & \gamma_t^{1N} \\ . & . & . & . & . & . \\ \gamma_t^{k1} & . & . & . & . & \gamma_t^{kN} \\ \gamma_t^{k+11} & . & . & . & . & \gamma_t^{k+1N} \\ . & . & . & . & . & . \\ \gamma_t^{N1} & . & \gamma_t^{Nk} & \gamma_t^{Nk+1} & . & \gamma_t^{NN} \end{bmatrix} \begin{bmatrix} (1-\omega_x^1-\omega_b^1)^2 \\ . \\ (1-\omega_x^k-\omega_b^k)^2 \\ (1-w_x^{k+1}-w_b^{k+1})^2 \\ . \\ (1-w_x^N-w_b^N)^2 \end{bmatrix} -$$

$$-2ry \begin{bmatrix} \frac{\alpha y+\beta x_1}{\alpha+\beta} \\ . \\ \frac{\alpha y+\beta x_k}{\alpha+\beta} \\ \frac{\alpha y+\beta x_{k+1}}{\alpha+\beta} \\ . \\ \frac{\alpha y+\beta x_N}{\alpha+\beta} \end{bmatrix} .* \left(\begin{bmatrix} \gamma_t^{11} & . & \gamma_t^{1k} & \gamma_t^{1k+1} & . & \gamma_t^{1N} \\ . & . & . & . & . & . \\ \gamma_t^{k1} & . & . & . & . & \gamma_t^{kN} \\ \gamma_t^{k+11} & . & . & . & . & \gamma_t^{k+1N} \\ . & . & . & . & . & . \\ \gamma_t^{N1} & . & \gamma_t^{Nk} & \gamma_t^{Nk+1} & . & \gamma_t^{NN} \end{bmatrix} \begin{bmatrix} (1-\omega_x^1-\omega_b^1)\omega_x^1 \\ . \\ (1-\omega_x^k-\omega_b^k)\omega_x^k \\ (1-w_x^{k+1}-w_b^{k+1})w_x^{k+1} \\ . \\ (1-w_x^N-w_b^N)w_x^N \end{bmatrix} \right) -$$

$$
-2ryb \begin{bmatrix} \gamma_t^{11} & . & \gamma_t^{1k} & -\gamma_t^{1k+1} & . & -\gamma_t^{1N} \\ . & . & . & . & . & . \\ \gamma_t^{k1} & . & . & . & . & -\gamma_t^{kN} \\ \gamma_t^{k+11} & . & . & . & . & -\gamma_t^{k+1N} \\ . & . & . & . & . & . \\ \gamma_t^{N1} & . & \gamma_t^{Nk} & -\gamma_t^{Nk+1} & . & -\gamma_t^{NN} \end{bmatrix} \begin{bmatrix} (1-\omega_x^1-\omega_b^1)\omega_b^1 \\ . \\ (1-\omega_x^k-\omega_b^k)\omega_b^k \\ (1-w_x^{k+1}-w_b^{k+1})w_b^{k+1} \\ . \\ (1-w_x^N-w_b^N)w_b^N \end{bmatrix} +
$$

$$
+2ry \begin{bmatrix} a_1 \\ . \\ a_k \\ a_{k+1} \\ . \\ a_N \end{bmatrix} .* \left(\begin{bmatrix} \gamma_t^{11} & . & \gamma_t^{1k} & \gamma_t^{1k+1} & . & \gamma_t^{1N} \\ . & . & . & . & . & . \\ \gamma_t^{k1} & . & . & . & . & \gamma_t^{kN} \\ \gamma_t^{k+11} & . & . & . & . & \gamma_t^{k+1N} \\ . & . & . & . & . & . \\ \gamma_t^{N1} & . & \gamma_t^{Nk} & \gamma_t^{Nk+1} & . & \gamma_t^{NN} \end{bmatrix} \begin{bmatrix} (1-\omega_x^1-\omega_b^1) \\ . \\ (1-\omega_x^k-\omega_b^k) \\ (1-w_x^{k+1}-w_b^{k+1}) \\ . \\ (1-w_x^N-w_b^N) \end{bmatrix} \right)
$$

where the operation ".$*$" denotes element-by-element multiplication.

3. CHAPTER

Can a social planner manipulate network dynamics and solve coordination problems?

3.1 Introduction

Coordination problems in case of uncertainty about some fundamental parameter are everywhere.[48] Examples of such fundamental parameters include the outcome of a vote on a political issue, scientific findings of, e.g., a medical issue such as a vaccine for an epidemic, a price outcome, e.g. a stock-price in financial markets, etc. Decision-makers, solving coordination problems with others, try to take into account other agents' beliefs about some parameter.The main focus is on a network model where everyone has a fundamental parameter governing assimilation or confirmation bias in their preferences (for example, left or right-wing political views, religious view vs scientific findings). To simplify the analysis, these fundamental biases are considered as constant, and that all network agents know these constant parameters of biased assimilation. However,the main focus is on developing a mechanism that endogenizes the weight people put on these biases while making decisions, and how this weight is affected by efforts of people to align their actions with the actions of others. More importantly, the question of whether a social planner can influence this endogenous coordination, leading to better social outcomes is studied.

Much of the literature on network theory develops coordination games where agents try to align their actions with these of their neighbors.

Therefore, the network structure is crucial for their action-alignment efforts and for their ability to elicit information about unknown parameters.[49] In this paper, we use a

[48]Such a coordination motive is well known as *convention* in economics literature developed by Shin and Williamson (1996), and Young (1996).

[49]

utility function similar to this in Morris and Shin (2002), where agents care not only for their friends' actions but also take into account other agents' actions. Key feature of the Morris and Shin (2002) model is that it can offer an analysis of the relationship between public/private information on unknown parameters and social welfare.[50]

In order to capture the alignment mechanism, in this model, the "beauty-contest" structure of Morris and Shin (2002) is kept. The model is developed not only for understanding the dynamics of endogenous biased assimilation regarding political issues, such as putting efforts into organizing protests and voting behavior, but also for understanding herd behavior in financial markets that appears during speculative attacks. For understanding the dynamics of such political and market phenomena, agents need not only second-guess the actions of their network friends, but also the actions of network non-friends too. Hakobyan and Koulovatianos (2020) develop a search-and-matching algorithm of network dynamics, focusing on an explanation of how expert opinions have been downgraded over time, and how network agents have been taking more polarized actions while forming more polarized subnetworks. In this paper, we address how a social planner can solve such polarized and populistic behavior in order to bring agents' actions closer to the true values of unknown parameters.

A simulation model of network dynamics with incomplete information is built, where the social planner tries to manipulate the sample of possible invitations and annoyances in order bring agent actions closer to the model's fundamentals. For example, if we consider herd behavior in financial markets, we can notice that agents trying to follow other agents' actions, can play another equilibrium strategy, where they just move away from market

See, for example Golub and Morris (2017), Myatt and Wallace (2019), Ballester et al. (2006) and Denti (2017), among others.

[50]The idea of Morris and Shin (2002) has been applied to models studying how political issues influence social welfare, and to the study of financial markets and business cycles. See, for example Angeletos and Pavan (2007), Myatt and Wallace (2012, 2015, 2019).

fundamentals, leading the market to exhibiting price bubbles. In financial markets, bubbles occur during times of aggressive speculative attacks. Morris and Shin (2002) show that increasing the precision of public signal can harm social welfare if agents have private signals. In this paper, as it was demonstrated, such results can vanish if agents are connected in a particular network structure. If the social planner manipulates the network dynamics in a particular way, then social welfare can increase.

In the model of this paper, two types of social planners are analyzed: (i) a social planner with perfect information, and (ii) a social planner with incomplete information. The two different scenarios are considered: (a) the case where, similarly to network agents, the social planner receives private and public signals, and calculates expectations, and (b) the case where the social planner has wrong expectations. In the latter scenario, the social planner's expectations about the state variable are equal to a constant, which is different from the optimal state variable.

It is demonstrated that social planners can improve welfare, not by directly influencing/changing the network structure, or by providing to agents fake news in order to manipulate the agents' actions. In my model social planners give opportunities to non-connected agents, to be introduced to each other and to meet. The decision of the evolution of the network structure rests entirely upon the agents. Therefore, in this paper, the focus is to understand how different "agent sampling" in a search-and-matching environment influences social welfare. One of the key and novel features in this paper is that agents are heterogeneous, making the setup more realistic.

The remainder of the paper is organized as follows. Section 2 presents the setup of the model, the utility function of agents, the network structure, the signals and the information structure. Section 3 focuses on presenting the linear equilibrium and the fixed-point strategies under evolutionary myopia. Section 4 demonstrates the network formation process , while Section 5 shows some simulation experiments. Section 6 concludes.

3.1.1 Related Literature

This paper links four strands of literature. (1)The first strand uses quadratic-utility functions trying to understand how agents play coordination games under information asymmetry. The closest paper in the literature using such a benchmark without network structure is Morris and Shin (2002). Compared to Morris and Shin (2002), this paper has three differences: we introduce (i) a network structure, (ii) assimilation and confirmation bias in the utility functions, and (iii) evolutionary dynamics of network structure with endogenous weights on signals. The literature combining global and coordination games in the fashion of Morris and Shin (2002), with network structure are Golub and Morris (2017), Dewan and Myatt (2012), Myatt and Wallace (2012), Bonfiglioli and Gancia (2013), Llosa and Venkateswaran (2012) and Pavan (2014). Almost all papers with quadratic utility use symmetric agents in their model. The closest paper using asymmetric agents is Myatt and Wallace (2019), which uses two types of asymmetry: (a) asymmetry in conformity (coordination motive), and (b) different weights for friends (with whom agents coordinate). Myatt and Wallace (2018) use only coordination asymmetry. Compared to Myatt and Wallace (2018, 2019), in this paper, agents have these two types of asymmetry. However, in this paper agents try to coordinate their actions with people they are not connected too, and agents have asymmetry in assimilation bias. The difference between this paper and Myatt and Wallace (2018, 2019) is also that this paper has an evolutionary dynamic of network structure. In addition, there is a difference in the information structure. In this paper, agents share information using their network connections, while Myatt and Wallace (2018, 2019) do not consider such types of information transmission. They describe "accuracy" of the information source, and agents decide how much attention be put in which signal, paying the cost for a signal. In Leister (2017), agents are asymmetric in general, but they get one private signal. In this paper, we develop mostly the idea which was used in social media platforms in the sense that information is cheap, and agents share information more cheaply. The only requirement in my model for cheap information transmission is being friends with those transmitting information in the network. This paper is also an extended version of Hakobyan and Koulovatianos (2020), trying to give an answer which arose in their model, which is how to deal with the reinforcement of populism due to the evolving and gradually strengthened polarization and homophily.

(2) The literature focusing on understanding games on networks, coordination on networks, key players, homophily and degree centrality. Examples of this literature are Jackson (2008), Currarini et al. (2009), Kossinets and Watts (2009), Golub and Jackson(2012a,b), Bramoulle et al. (2012), Jackson and Lopez-Pintado (2013), Centola (2013), Lobel and Sadler (2015), Currarini and Mengel (2016), and Halberstam and Knight (2016). In this model, we show that the network structure, specifically indegree and outdegree centrality, are tightly linked with social welfare. We demonstrate that a social planner can manipulate indegree and outdegree links indirectly, and thus increase social welfare.

(3) The literature on strategic disclosure or information manipulation and fake news. My model is different from standard sender/receiver games such as Crawford and Sobel (1982), Kartik(2009), and Edmond (2013). In this paper, we demonstrate the advantages that new social media platforms give to agents. Crawford and Sobel (1982) characterize two types of equilibrium in sender/receiver games with conflict of interest: (i) Separating, which is not a part of a Nash equilibrium and, (ii) Babbling equilibrium, which is a part of an equilibrium,

but in this case, there is no information transmission. In my paper, agents have a conflict of interest, so directly sending them perfect information would not give any results. For this reason, we develop a mechanism that tries to solve the coordination problem without directly sending information. The literature developed on information manipulation, such as Edmond (2013), Edmond and Lu (2017) use biased signals, trying to manipulate the agents' behavior. In our model, the social planner does not use any biased or unbiased signals. Therefore, differently from the literature on information manipulation, our manipulation of network dynamic is not direct. In many countries, there is a law against information manipulation.[51] In this case, even if a social planner has good intentions, whishing to bring agents' actions closer to fundamentals, this is infeasible, according to the law. Our model gives a solution for such problems by creating wrong priors, because it influences agents' decisions indirectly. While there is no direct link to the fake news literature, this paper can solve the consequences of polarization in networks, bringing the agents' actions closer to the model's fundamentals. Therefore, researchers who are interested in hte fake news literature can find our framework useful for developing further research.

(4) The literature on social policy. Researchers who are interested in social-planner and social-welfare maximization can find this framework useful for developing further work. The most relevant reference in this literature is Dyckman (1966), Cavallo (2008), and Bernheim (1989).

3.2 Model

There is a directed network of $N < \infty$ agents. We denote this network by $\mathcal{G} := \{V, E\}$, where V is the set of agents/nodes, and E is the set of edges in this network. In period t the network is represented by an adjacency matrix $\mathbf{M}_t$ with entries in $\{0, 1\}$. The graph to

[51] One country with such a law is France https://www.gouvernement.fr/en/against-information-manipulation

be used is directed and unweighted, i.e., $M_t^{ij} \neq M_t^{ji}$. Edges between agent i and j represent the private information transmission,where the link between i to j means that agent i gets agent j private signal.[52] The diagonal elements of matrix $\mathbf{M}_t$ are 0, which means that there is no self-loops in this model.

In each period, agents make two decisions, (i) to guess a fundamental variable $\theta_t \in \mathbb{R}$ by using all available information (the state variable θ_t is unknown regarding to political issues, or to a scientific finding, such as a vaccine or global warming), and (ii) to align their actions closer to other agents' actions (the known "beauty contest" motive, defined by Keynes, 1936). The fundamental varible θ_t is i.i.d., so agents need to guess new fundamentals in each period $t \in \{0, 1, ...\}$, and there is no learning in the model.

We divide agents into two groups: (a) agent i's neighbors/friends ($j \in \{1, 2, ..., N_i\}$), and (b) non-neighbors ($k \in \{1, 2, ..., N_{-i}\}$). In addition, there are two *types* of agents denoted by "+" and "−", depending on the direction of their structural biases b_i, i.e., whether the bias is above or below the value of θ.[53] Specifically, the payoff function for agent i with positive bias "+" is given by,

$$u_i^+ \left(a_t, \theta_t\right) = -\left(1 - r_i\right)\left(a_{i,t} - \left(\theta_t + b_i\right)\right)^2 - r_i \left[\frac{q_i}{\#N_i} \sum_{j \in N_i} \left(a_{j,t} - a_{i,t}\right)^2 + \frac{\left(1 - q_i\ \right)}{\#N_{-i}} \sum_{k \in N_{-i}} \left(a_{k,t} - a_{i,t}\right)^2\right], \tag{3.1}$$

while the payoff function of a agent i with negative bias "−" looks like

[52] For example, social network structure as a Twitter. Agent i can follow agent j, gets his private signal, but if agent j is not following back to agent i, he cann't get agent i's private signal.

[53] Intuitively, this structural bias in preferred actions reflects political, religious, and other similar biases, falling in the categories of biased assimilation and confirmation bias (see Lord et al., 1979, and Nickerson, 1998).

$$u_i^-\left(a_t,\theta_t\right)=-\left(1-r_i\right)\left(a_{i,t}-\left(\theta_t-b_i\right)\right)^2-r_i\left[\frac{q_i}{\#N_i}\sum_{j\in N_i}\left(a_{j,t}-a_{i,t}\right)^2+\frac{\left(1-q_i\ \right)}{\#N_{-i}}\sum_{k\in N_{-i}}\left(a_{k,t}-a_{i,t}\right)^2\right], \tag{3.2}$$

where $a_{i,t}=[a_{1,t},...,a_{N,t}]$. In equation (3.1) and (3.2) parameter $r_i\in(0,1)$captures the second-guessing motive. Each agent tries to have their action closer to $\theta_t\pm b_i$, where $b_i\in(0,1)$ is the individual bias.

The second part of the utility function is normalized for controlling network effects on calculations. Parameter $q_i\in(0,1)$ differentiates the weight that agent i puts on friends versus non-friends. Parameters, r_i, q_i and b_i are iid across individuals, generalized in period 0, and are common knowledge to all agents.

3.2.1 Signals and Information Structure

Agents face uncertainty about state variable, θ_t, in each period $t\in\{0,1,...\}$. Every period generates a new task, which agents try to best-guess. Agents get public and private signals in each period, with the learning duration being confined to one period. The information set available to player $i\in\{1,...,N\}$ in each period is $\mathcal{I}_{i,t}=\left(y_t,x_{i,t},\sum_j x_{j,t}\right)$, where y_t is a public signal with,

$$y_t=\theta_t+\eta_t\ ,\quad \text{with}\ \ \eta_t\sim N\left(0,\sigma_\eta^2\right)\quad ,\ \ t=0,1..,\tag{3.3}$$

and $x_{i,t}$ is a private signal to agent i only with,

$$x_{i,t}=\theta_t+\varepsilon_{i,t}\ ,\quad \text{with}\ \ \varepsilon_{i,t}\sim N\left(0,\sigma_\varepsilon^2\right)\quad ,\quad t=0,1,.,\tag{3.4}$$

and the precision of the public and the private signals are $\alpha=1/\sigma_\eta^2$ and $\beta=1/\sigma_\varepsilon^2$. Importantly, η_t, $\varepsilon_{i,t}$ are i.i.d. over time. η_t is independent from $\varepsilon_{i,t}$ for all $i\in\{1,...,N\}$, and $\varepsilon_{i,t}$ is independent from $\varepsilon_{j,t}$ for all $i\neq j$. Contrary to the Morris and Shin (2002), we assume that if agents are connected in network then they can see the private information of their neighbors.

The main goal is to understand network evolution. For this reason, we develop an algorithm and run simulations for understanding network evolution. The algorithm needs that a modeler generate the information set $\mathcal{I}_{i,t} = (y_t, x_{i,t}, \{x_{j,t}\}_{j \in N_i})$ for every period and the modeler needs a "true" parameter, θ_t^*, unknown to agents in the model. Using different values for $\{\theta_t^*\}_{t=0}^T$ does not change the optimal strategy chosen by agents, as learning is only one period, and the modeler can choose the same θ_t^* for every period.

3.3 Linear Equilibrium, fixed-point strategies and evolutionary myopia.

The model focuses on an incomplete-information benchmark, where the evolving state variable is the network structure, $\mathbf{M}_t$. Each agent i needs to second-guess the actions of all other agents, which means that each player needs to second-guess the beliefs of other players. The information asymmetry among agents is low compared with Golub and Morris(2017) and Hakobyan and Koulovatianos (2020). If agent i is connected with agent j, this means that the information set which is available to agent i intersects with the information set of agent j: $\mathcal{I}_{i,t} \cap \mathcal{I}_{j,t} = \{x_{i,t}, x_{j,t}, \{s_l\}_{l \in \Omega ij}\}$, where $\{s_l\}_{l \in \Omega ij}$–represents agents i's and agent j's common-friend signals. At the same time, agent i tries to second-guess non-friend ($k \in N_{-i}$) beliefs about the state variable θ_t. Agent i understands that the intersection of her information set with non-friends can be non-empty, because of common friends $\mathcal{I}_{i,t} \cap \mathcal{I}_{k,t} = \{s_e\}_{e \in \Omega_{ik}}$, where s_e represents agents i's and agent k's common-friend signals.

The structure of $\mathbf{M}_t$ is common knowledge for all agents. This common knowledge is one of the key assumptions in this paper.

Nevertheless, there is limited foresight about the network structure's evolution. In period

t, agents only perceive a myopic, narrow-sighted local evolution of their peer connections. This happens at the stage of evaluating the modeler's sampling of invitations for friendship or annoyances received in each period, that we explain below in the section Network Formation Process. We call this nearsightedness of the local evolution of $\mathbf{M}_t$ for one period only, i.e evolutionary myopia[54] .

At first, let's find the fixed-point strategies(the myopic best reply function) for every period, which depend on higher-order belief. The myopic best-reply function is similar to Myatt and Wallace (2019) and Golub and Morris (2017)[55] the only exception is that in our model agents also care about their assimilation bias, and non-friends' action . We denote the optimal action taken by players by a_i^{+*} and a_i^{-*}. Each agent makes their decision based on the information set $\mathcal{I}_{i,t}$ available to her. We will skip the notation of the information set and denote agent i's mathematical expectation by $\mathbb{E}_i(\bullet)$ instead of $\mathbb{E}(\bullet|I_i)$.

Agent i maximizes the expected utility function (3.1)/(3.2) by her own action. First-order conditions imply the following solution for agent i's action:

$$a_{i,t}^{+*} = (1 - r_i) E(\theta_t + b_i) + r_i \left[\frac{q_i}{\#N_i} \sum_{j \in N_i} E(a_{j,t}) + \frac{(1 - q_i\ )}{\#N_{-i}} \sum_{k \in N_{-i}} E(a_{k,t}) \right], \tag{3.5}$$

$$a_{i,t}^{-*} = (1 - r_i) E(\theta_t - b_i) + r_i \left[\frac{q_i}{\#N_i} \sum_{j \in N_i} E(a_{j,t}) + \frac{(1 - q_i\ )}{\#N_{-i}} \sum_{k \in N_{-i}} E(a_{k,t}) \right]. \tag{3.6}$$

[54]The evolutionary myopia is a reasonable assumption in directed networks.For example, Twitter is one of the directed network structure, and each agent made a decision which links to create not taking account of other agents action. At the end of the period, where all agents made their decision. The network structure $\mathbf{M}_t$ becomes common knowledge for everyone. If we consider undirected network structure, such as Facebook, creating a link needs to be accepted from both sides, so agent i need to understand if creating a link is valuable for agent k, or not.

[55]For more reference about myopic best-response functions and average based updating of information, see Calvó-Armengol et. al (2009), Bramoullé et al., DeGroot (1974) , Young (1996), Fudenberg et al. (1998) and others. For referense best-response function and bayesian learning, see Acemoglu et al. (2011), Mueller-Frank, M. (2013).

Therefore, an agent's optimal decision depends on the expectation of the state variable, $E_i(\theta_t)$, the expectation of the actions of friends, $E_i\left(\sum\limits_{j\in N_i} a_{j,t}\right)$, and of non-friends, $E_i\left(\sum\limits_{k\in N_{-i}} a_{k,t}\right)$. Moreover, based on both private and public information, the expectation of θ_t is given by (probability density function is defined for the case of the flat (absolutely non-informative: $p(\theta) \propto 1$) prior of θ, see Appendix for proof),

$$E_i(\theta_t) = \frac{\alpha y + \beta_i x_i + \sum_j \beta_j x_j}{\alpha + \beta_i + \sum_j \beta_j} \tag{3.7}$$

The linear equilibrium for each agent's action in period t is defined as a weighted sum of all signals in agent i's information set and of the weight on the structural bias, b_i. We call each weight associated with the j-th signal in this sum a 'j-th signal weight'. The presence of network signal transmission results in non-equal weights between private signals. Our educated guess is that the weight of signal j in agent i's action w_{ij} depends on agent i's and agent j's network degrees and on the precisions of signals β.

Given a network structure, We group agents into clusters depending on their closeness centrality measure and precision. Normalized closeness centrality CC_i for the node i is defined as the inverse of the average of the lengths of the shortest paths between the node i and all other nodes in the graph $\mathcal{G}$:

$$CC_{i,t} = \left(\frac{\sum_{j\in V\setminus\{i\}} dist(i,j)}{N-1}\right)^{-1}, \tag{3.8}$$

where $dist(i,j)$ is the number of the edges in the shortest path between nodes i and j in the network $\mathcal{G}$.

Definition 1 (Clusters) *We shall say that there exists cluster $C_{q,t}$ of agents $i \in V$, $q \in \{1,2,...,Q_i\}$(q is the number of clusters in the network) if and only if these conditions are satisfied: (i) all members of cluster $C_{q,t}$ are characterized by the same closeness centrality;(ii) all members of cluster $C_{q,t}$ are characterized by the same private signal precision; (iii) any agent who does not belong to cluster $C_{q,t}$ has closeness centrality different from the closeness* *centrality of this cluster's members, or different precision of private signal:*

$$C_{q,t} = \{i \in C_{q,t} : \forall j \in C_{q,t} : CC_{i,t} = CC_{j,t};\ \forall k \notin C_{q,t} : CC_{i,t} \neq CC_{k,t}\} \tag{3.9}$$

An agent who has different closeness centrality and different precision from others will belong to their own cluster. In this case, we simply separate this agent from the set of agents with different centrality measures.

The following proposition defines the linear equilibrium characteristics.

Proposition 2 (Linear equilibrium characteristics) *Given a network $\mathcal{G}$, there exists a linear equilibrium in period t, in which agent i's action can be represented in the following way:*

$$a^{+}_{i,t} = \sum_q \omega_{i,t,q}\left(\mathbf{M}_t\right) x_{i,t,q} + w_{i,t}\left(\mathbf{M}_t\right) b_i + \left[1 - \sum_q \omega_{i,t,q}\left(\mathbf{M}_t\right) - w_{i,t}\left(\mathbf{M}_t\right)\right] y_t \tag{3.10}$$

$$a^{-}_{i,t} = \sum_q \omega_{i,t,q}\left(\mathbf{M}_t\right) x_{i,t,q} + w_{i,t}\left(\mathbf{M}_t\right)\left(-b_i\right) + \left[1 - \sum_q \omega_{i,t,q}\left(\mathbf{M}_t\right) - w_{i,t}\left(\mathbf{M}_t\right)\right] y_t \tag{3.11}$$

where $0 \leq \sum_{q \in NC_i} \omega_{iq} \leq 1$ and $\omega_{iq} \geq 0\ \ \forall i = 1..\mathcal{N}$, where $q = 1..Q_i$, all clusters which appear in agent i's networks.Formulation of the weight for private signals is depicted in the figure below.

$$\begin{aligned} \overline{x}_{i,q,t} &= \frac{\sum_{s \in INC_{i,q,t}} x_s}{|INC_{iq}|}, \\ NC_i &= \{INC_{i,q,t}\}_{q : \exists j \in \{C_q\} \cap \{N_i\}} \\ INC_{i,q,t} &= \{s : x_s \in \{N_i\} \cap \{C_q\}\} \end{aligned}$$

Here $INC_{i,q,t}$ — individual neighbour cluster, or the set of agents that are simultaneously in cluster q and in agent i's neighbors set. The notation $s \in INC_{i,q,t}$ stands for the index number of agent s from this set. $NC_{i,t}$— neighbour cluster, or the set of individual neighbour clusters that are not empty for agent i. The notation $q \in NC_{i,t}$ stands for cluster q from the set $INC_{i,t}$.

For demonstrating how the definition and proposition of fixed-point strategies works let us look at the following example.

Example 3 *Let us consider a simple example with* 6 *agents, who are connected in the network, depicted in Figure 2 .*

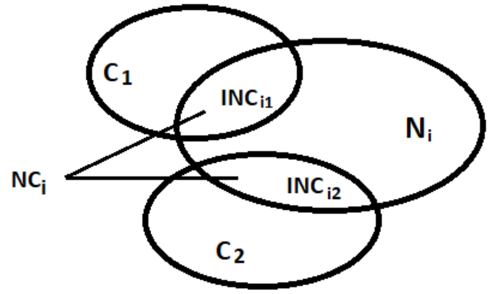

Figure 1: Figure 3.1 Illustration of cluster definition

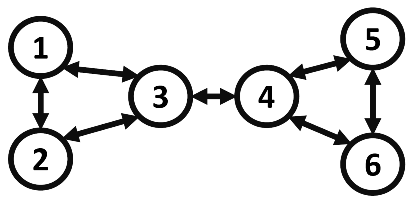

Figure 3.2 Example with 6 agents

In this network there are two different clusters. The first one consists of agents $C_1 = \{3, 4\}$ *who are characterized by closeness centrality* $CC_3 = CC_4 = 5/7$*. The second cluster consists of agents* $C_2 = \{1, 2, 5, 6\}$ *with closeness centrality* $CC_1 = CC_2 = CC_5 = CC_6 = 1/2$*. We demonstrate the difference between clusters using different weight simbols on private signals* υ *for cluster* C_1 *and* ω *for cluster* C_2*.*

$$a_{1,1} = \omega_1 \frac{(x_1 + x_2)}{2} + \omega_2 x_3 + w_1 b_1 + (1 - \omega_1 - \omega_2 - w_1)y$$

$$a_{2,1} = \omega_1 \frac{(x_1 + x_2)}{2} + \omega_2 x_3 + w_2 b_2 + (1 - \omega_1 - \omega_2 - w_2)y$$

$$a_{3,1} = \upsilon_1 \frac{(x_1 + x_2)}{2} + \upsilon_2 \frac{(x_3 + x_4)}{2} + w_3 b_3 + (1 - \upsilon_2 - \upsilon_1 - w_3)y$$

$$a_{4,1} = \upsilon_1 \frac{(x_5 + x_6)}{2} + \upsilon_2 \frac{(x_3 + x_4)}{2} + w_4 b_4 + (1 - \upsilon_2 - \upsilon_1 - w_4)y$$

$$a_{5,1} = \omega_1 \frac{(x_5 + x_6)}{2} + \omega_2 x_4 + w_5 b_5 + (1 - \omega_1 - \omega_2 - w_5)y$$

$$a_{6,1} = \omega_1 \frac{(x_5 + x_6)}{2} + \omega_2 x_4 + w_6 b_6 + (1 - \omega_1 - \omega_2 - w_6)y$$

Hence, in this example, the weights are the same within each cluster and differ between clusters. In this example, we consider that $\beta_1 = \beta_3 = \beta_2 = \beta_4$ *and* $b_1 = b_2 = b_3 = b_4$*. This* *example demonstrates what happens in the first period*[56] *. In Appendix, we generalize the solution of finding linear equilibrium weights for $Q_i = \mathcal{N}$ different clusters.*

The linear equilibrium action should be optimal for all agents in the network. It means that the action, characterized by (3.10) and agent i's optimal actions (2.2) should be the same. This gives us a system of linear equations. This system provides us with a solution for equilibrium weights as a function of the parameters. Once the weights are found, optimal actions should be calculated using the following algorithm.

Algorithm 4 (Finding optimal actions) (i)*Substitute all other agents' actions (3.10) into the term $\sum\limits_{j \in N_i} E_i(a_{j,t})$, $\sum\limits_{k \in N_{-i}} E_i(a_{k,t})$. If the agent has no information about the signal j, then $E_i x_j = E_i \theta$;*

(ii)*Rearrange the terms to get a coefficient preceding the $E_i(\theta_t)$ term. Then substitute the mathematical expectation as a function of agent i's signals (3.7).*

(iii)*Rearrange the terms to get a coefficient preceding each agent i's signal. These coefficients should be equal to the corresponding weights in the linear equilibrium (3.10). The solution to the resulting system of linear equations is the vector of equilibrium weights.*

(iv) *Find the optimal action.*

Substituting the optimal action strategies in the objective function of each player gives the value functions (indirect utility functions),

$$V_i^+(\mathbf{M}_t) = E\left(u_i^+\left(a_{i,t}^{+*}, \theta_t\right) \mid \mathcal{I}_{i,t}\right), \quad i = 1, ..., N_+ , \tag{3.12}$$

and

$$V_i^-(\mathbf{M}_t) = E\left(u_i^-\left(a_{i,t}^{-*}, \theta_t\right) \mid \mathcal{I}_{i,t}\right), \quad i = 1, ..., N_- . \tag{3.13}$$

The value function will influence the evolution of the network structure. We demonstrate this influence in the next section.

[56] *Notice that we normalize the weights. Normalizing is possible because the objective functions are ordinal utility functions.*

3.4 Network formation process

The state variable in each period is the network structure. The network dynamics are governed by two main processes: *(1)* The Social Planner uses different sampling processes in order to create possible invitations(a link which can be created) to non-friends for each agent and annoyances(a link which can be broken) from friends of each agent. *(2)* Subject to the sampling process of possible invitations/annoyances chosen by the social planner, agent i uses his own value-function criterion in order to decide upon whom to add from the sample of non-friends ($k \in N_{-i}$) and whom to exclude among friends $j \in N_i$.[57]

In process *(1)* above, the Social Planner selects different processes in order to manipulate network dynamics. Given, however, that process *(2)* gives freedom to people to choose their network friends, it is a liberal social-planner manipulation[58] .

3.4.1 First stage of decision making: Sampling process

Our main goal is to examine how a social planner who cares only about bringing optimal actions of agents closer to fundamentals, can solve coordination problems among agents by varying the sampling process of invitations and annoyances sent to non-friends and friends of each agent. We will call this selection of sampling processes by the Social Planner "Social planner manipulation of network dynamics". We will compare this sampling process to two other sampling algorithms, the uniformly random sampling and the biased sampling. The next subsections will provide a more detailed explanation of these algorithms.

3.4.1.1. Social planner manipulation of network dynamics In the role of a modeler, we introduce a social planner, who doesn't care about individuals' biases. The

[57]We describe this process below in subsection "second stage decision making process: creating/deleting a link".

[58]In the role of a social-planner can be social-media platforms owners/government.

social planner cares how to bring agents' actions closer to the state variable θ_t. The utility function of a social planner is given by,

$$W = -\frac{\sum_{i \in N} (a_{i,t} - \theta_t)^2}{N} \quad \text{for all } i = 1, 2, ..., N \tag{3.14}$$

The utility function of social planner looks like as Morris and Shin (2002) utility, but contrary to their case, in my model agents optimal actions contain bias. The social planner understands that optimal actions of agents are influenced by their biases, and he tries to minimize the effect of these biases by manipulating the set of possible invitations and annoyances[59] . In Hakobyan and Koulovatianos (2020) authors show that biases, such as assimilation bias or confirmation bias, can increase polarization and populistic behavior, as time passes. In this paper, we show that the social planner can solve the problem of polarization in network dynamics, using his power to manipulate the sampling process of possible invitations and annoyances. We consider two social-planner types: *(a) a social planner with perfect information* about the state variable θ_t in each period, and *(b) a social planner with incomplete information* about θ_t. For the second case, we will assume that, like common agents, the social planner gets public and private information or has some prior about θ_t, which is constant, but generally $\theta_t^{sp} \neq \theta_t^*$.

The social planner directly manipulates network dynamics. In the first period, he takes the adjacency matrix $\mathbf{M}_t$ and calculates the social welfare W. After fixing the welfare level, he calculates all possible changes in the network structure that are driven by agents' decisions, and suggests a vector of possible invitations together with a vector of possible annoyances, which increase his utility function.

[59] As a future extension, we will employ a utilitarian social welfare function and compare it to the welfare functions appearing here. Such a comparison can shed more light on which bias/externality parameters are crucial for increasing social welfare.

Algorithm 5 (Social-planner manipulation of network dynamics) *(1) The social planner takes the adjancency matrix $\mathbf{M}_t$, calculates social welfare W and fixes it. (2) The social planner takes each row of the adjancency matrix, $\mathbf{M}_t$, and calculates social welfare taking into account all possible changes of the adjancency matrix driven by potential agent link connections. (3) The social planner chooses one agent from a sample of k with whom agent i could create a link, making social welfare to increase as a result of establishing this link. (4) The social planner chooses some agent j with whom agent i can delete a link, making social welfare to increase as a result of deleting this link.*

A matrix of possible invitations and annoyances (PIA matrix) is created. The size of the PIA matrix is $N \times 2$, where the first column shows possible invitations and the second column shows the possible annoyances. After the creation the PIA matrix, the game proceeds to the second stage of decision making, where agents decide which link to create and which link to delete, depending on their value functions.

3.4.1.2. ***Uniformly Random Sampling*** In each period the social planner randomly creates a vector of possible invitations and annoyances, using the following algorithm.

Algorithm 6 *(1) The social planner takes the adjancency matrix $\mathbf{M}_t$ and randomly chooses one agent from a set of k (individuals in the i-th row of $\mathbf{M}_t = 0$) with whom agent i can create a link, and saves the index of agent k in the sample of possible invitations. (2) The social planner randomly takes, from a set of j (individuals in the i-th row of $\mathbf{M}_t = 1$), one agent with whom agent i can delete a link.*

3.4.1.3. ***Biased Sampling*** There is a vast literature examining whether the network structure is random or not.[60] Here the social planner uses a biased sampling algorithm. In this model agents get invitations from friends of friends. Such algorithms are used in real-world social-network platforms.[61]

[60]See, for example, Jackson et al. (2007), Snijders, et al. (2010), Bhattacharya et al. (2017), Golub and Livne (2011), among others.

[61]This biased sample is very common in such networks like Facebook or Vkontakte. For example Facebook suggests a potential friends list (from a group of non-friends who have common characteristics to these of agent i). These characteristics can include friends of friends, or sometimes people who belong to the same groups of interests (in my model this corresponds to agents with the same fundamental bias, $\pm b$).

Algorithm 7 *(1) The social planner calculates the number of common links agent i has with each of his non-friends, suggesting the agent k, with whom agent i has the most common friends (e.g., meeting friends of friends). (2) For determining the set of annoyances, the social planner uses the opposite. He calculates, among friends, the agents with whom agent i has the fewest common friends, creating a set of agents who cause an annoyance to agent i.*[62]

3.4.2 Second stage of decision making: creating/deleting the links

Agent i makes the decision of creating a new link or of deleting an old friend, conditional on the set of invitations and annoyances that has been created by the social planner. Player i receives one invitation and experiences one annoyance, i.e., he examines 2^2 cases. These cases consist of $\{0, 1\}$ choices. Choice "0" stands for either not creating a new link or excluding an old friend based on a caused annoyance. On the contrary, choice "1" stands for either creating a new link or keeping an old friend, despite a caused annoyance. The algorithm creates a 2^2 versions of the original matrix $\mathbf{M}_t$, with each agent calculating his value function for all possible cases, choosing the $\mathbf{M}_t$ version that gives him the maximum value-function level. The generalized version of calculating the value function for N different clusters is introduced in the Appendix (7.4). This paper is focused on directed graphs, so the game evolves as $\mathbf{M}_{t+1} = \widetilde{\mathbf{M}_t}$, where $\widetilde{\mathbf{M}_t}$ is the updated version of the adjancency matrix. Notice that $\widetilde{\mathbf{M}_t}$ is not symmetric.[63]

3.5 Simulation experiments

[62] *Suggestions of breaking links is not crucial in this model. In social media platforms as Facebook, Vkontakte agents can ignore friends messages, or unfollow friends news, but platform will continue show them as friends. In our model we consider such behavior as breaking a link, because the link between agents show the information transmission process.*

[63] An interesting extension is to add one more step for capturing a feedback effect, where agents i and j can update information only if both of them decide to add each other to their subnetwork of friends.

Due to the complexity of equilibrium conditions, and the network formation process, we perform 100 Monte-Carlo simulation experiments in order to analyze the network dynamics, provide answers to the following questions:

- Is the manipulating power of the social planner, capable of solving the coordination problem among agents, bringing their actions closer to fundamental variables, and thus increasing social welfare?

- Can the social planner increase social welfare, just by manipulating the set of possible invitations and annoyances, without directly changing a network structure?

- What are the main drivers of increasing social-planner utility functions?

- How do the results change if the social planner has incomplete information or wrong priors about the fundamental variable?

For simulation experiments we use the following parameters: the "beauty-contest" parameter $r = 0.7$ (the results are similiar with $r \sim U[0,1]$) for all agents, the weight on the action of friends is set to $q \sim U[0,1]$.[64] The precision of the public signal is $\alpha = 1/\sigma_\eta = 30$, while the precision of the private signals is randomly distributed following a uniform distribution, parametrized as, $\beta_i \sim U[10,45]$. We differentiate agents into two groups: agents with positive biases and agents with negative biases, $b_i \sim U[0,1]$. There are $N = 20$ agents, and we split them into two groups $N_1^+ = 10$ and $N_2^- = 10$.

For performing comparative dynamics among different algorithms and for studyung their influence on social welfare, we fix the adjancency matrix in period 0. We randomly generate

[64] Svensson, (2006) argues that the main Morris and Shin (2002) result is present only if the second-guessing motive is relatively high $r \in (0.5, 1)$.

a non-symmetric original matrix $\mathbf{M}_t$ in period $t = 0$ and fix it in order to understand how different sampling processes implemented by the social planner influence network dynamics.

We perform comparative dynamics in two main cases: *(1) A social planner with perfect knowledge* of the true state variable θ_t, and *(2) a social planner with incomplete information/knowledge* about state variable θ_t.

3.5.1 *Social planner with perfect knowledge* about state variable θ_t

In this subsection we assume that the social planner has perfect knowledge about the fundamental variable $\{\theta_t^*\}_{t=0}^T$ in every period. As mentioned in Section 3, there is no learning between periods. In my simulations, we choose the same fundamental variable, $\theta_t^* = 0$, in every period, and, in this Section, we assume that the social planner knows that $\theta_t^* = 0$, in every period. The welfare function of the social planner, given by (3.14) can be transformed into a matrix form, after doing some algebra. The social planner's welfare function in a matrix form is,

$$W = -\ \mathbf{1}^T \left(\bar{\omega}_{b\circ}^2 \circ \bar{b}_\circ^2 + \bar{\omega}_{b\circ}^2 \circ E^*(\theta^2) - 2 * \bar{\omega}_{b\circ}^2 \circ \bar{b} \circ E^*(\theta) + W_{\text{private}\circ}^2 \left(\frac{\mathbf{1}_\circ}{\overline{\beta}} \right) + \frac{W_y^2}{\alpha} \right) \quad (3.15)$$

where $\mathbf{1}$ is a vector of ones, the size of this vector is $N \times 1$.[65] $\bar{\omega}_b$ is the weight for bias which was describe by equation (A.1.3)[66] .$\overline{\beta}$ is a vector of private signal precision (size of this vector is $N \times 1$). $E^*(\theta^2)$ is described in Appendix. Symbol $\circ$ represents element-by-element multiplication.

In the case of perfect knowledge of the state variable ($\theta_t^* = 0$), following equation (3.15), one can see, that the social planner's welfare depends on the weight that every agent puts

[65]Notice that the expression in the brackets is an $N \times 1$ matrix, and multiplying it with the transponse of vector ones ($\mathbf{1}^T$) will give us a scalar.
[66]Please notice, that the optimal-action weight on the bias depends on network structure.

to the bias, and on weights in the Kronecker products of private and public signals with the precision of signals. Therefore, the social planner manipulates network dynamics in order to decrease these components in his utility function.

For understanding the network evolution of well-known social platforms, and compare them with social planner manipulation, we run simulation experiments. We begin with our comparative dynamics, with a benchmark parameters, and examine how network evolution depends on social-planner's manipulation. Figure 3.3a depicts these network dynamics. As it can be seen in Figure 3.3a, in the last periods the key agents share their information with others more actively, and the numbers of indegree links increase. We prove analyticaly and by using simulation example the importance of indegree and outdegree links below.

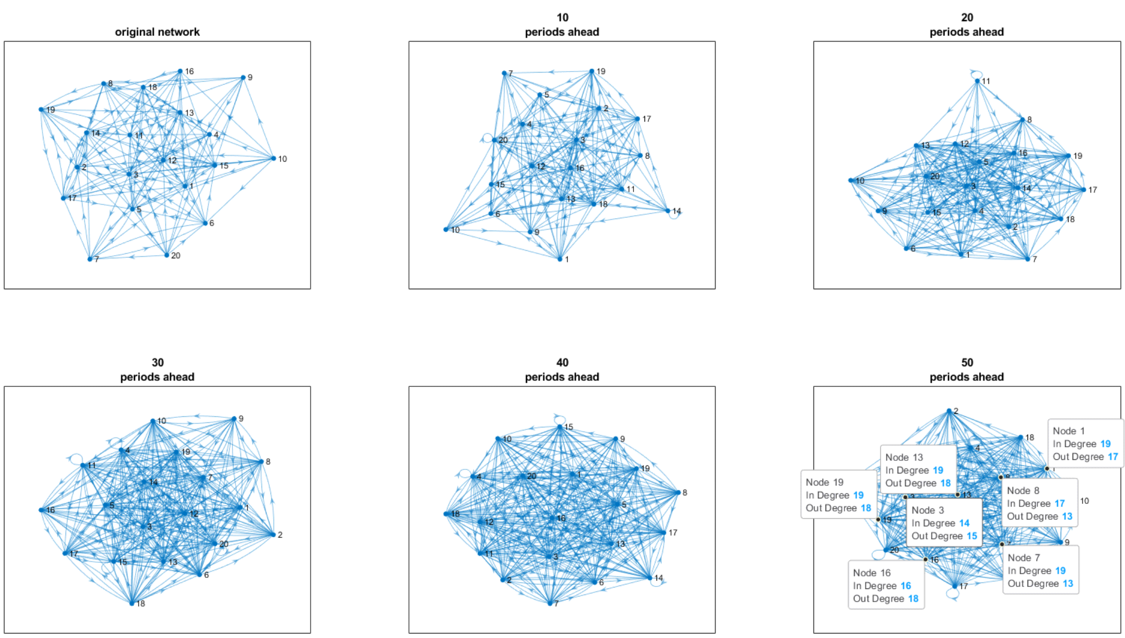

Figure 3.3a Social planner manipulation of network dynamics

The number of nodes directed towards agent i (indegree nodes) shows the number of agents who receive agent i 's signal. As we can see in the last periods, the graph looks like a combination of a star network and a ring network. Comparing the three cases of network evolution that we can see, there are similarities in the node degrees between the network dynamics also in the cases of social planner manipulation (Figure 3.3a) and the biased sampling by the social planner (Figure 3.3b). On the contrary, social planner manipulation (Figure 3.3a) and random sampling (Figure 3.3c) there are no similarities: as time passes the outdegree links of key players increase, instead of the indegree links increasing. Therefore, a combination of indegree and outdegree links seem influence social welfare the most.[67] Below we take a closer look on how social welfare depends on the evolution of networks in these cases.

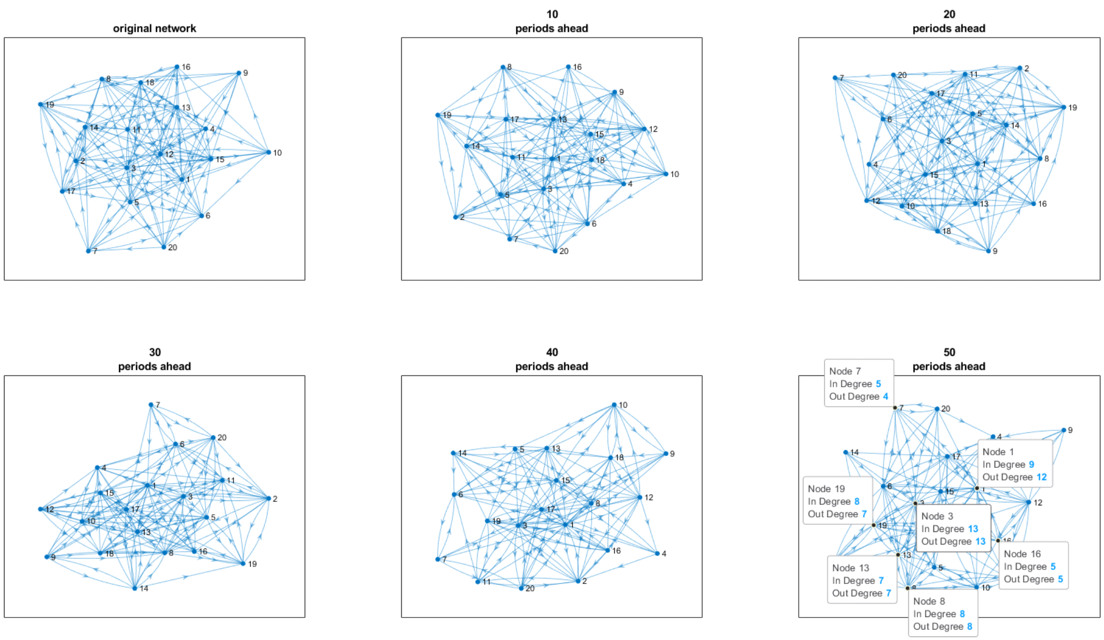

Figure 3.3b Evolution of network dynamics—bias sampling

[67]For these particular network dynamics look at the social welfare dynamics in the Appendix. In main body we present results of 100 Monte-Carlo simulations.

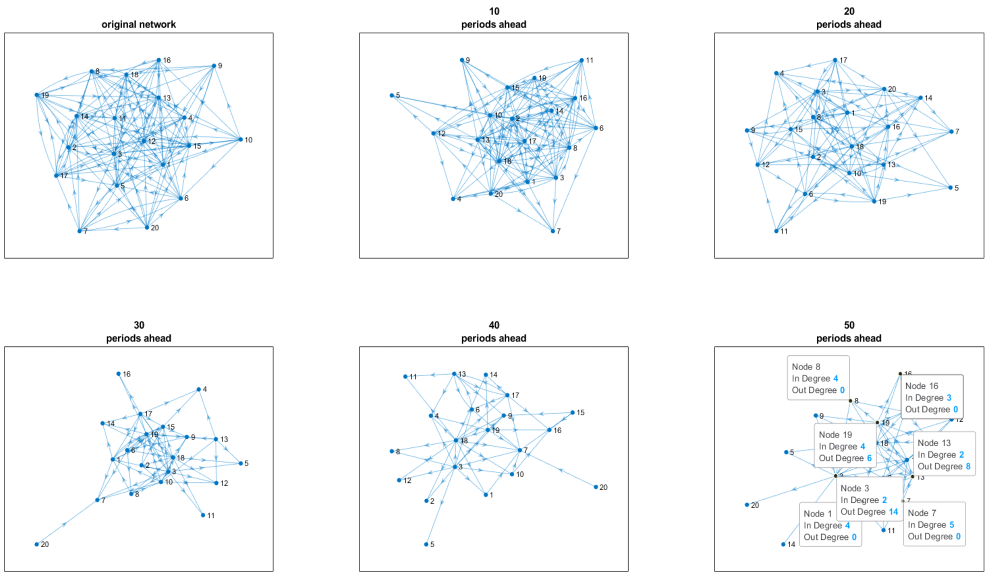

Figure 3.3c Network evolution with randomly invitations and annoyances

3.5.1.1Comparative dynamics and social welfare As mentioned above, the original adjancency matrix $\mathbf{M}_0$ is the same for all sampling algorithms. But from $t = 1$ the set of possible invitations and annoyances differ, depending on the different algorithms. Figure 3.4 illustrates how welfare dynamics differ across cases. The time horizon is set to $T = 50$. As we can see in the benchmark case, where $r = 0.7$, $q = U[0, 1]$ for all agents the difference in friends of friends between the social planner manipulation case and that case of biased sampling is not big. Yet, the difference between random sampling and the two other sampling process is substantially big.

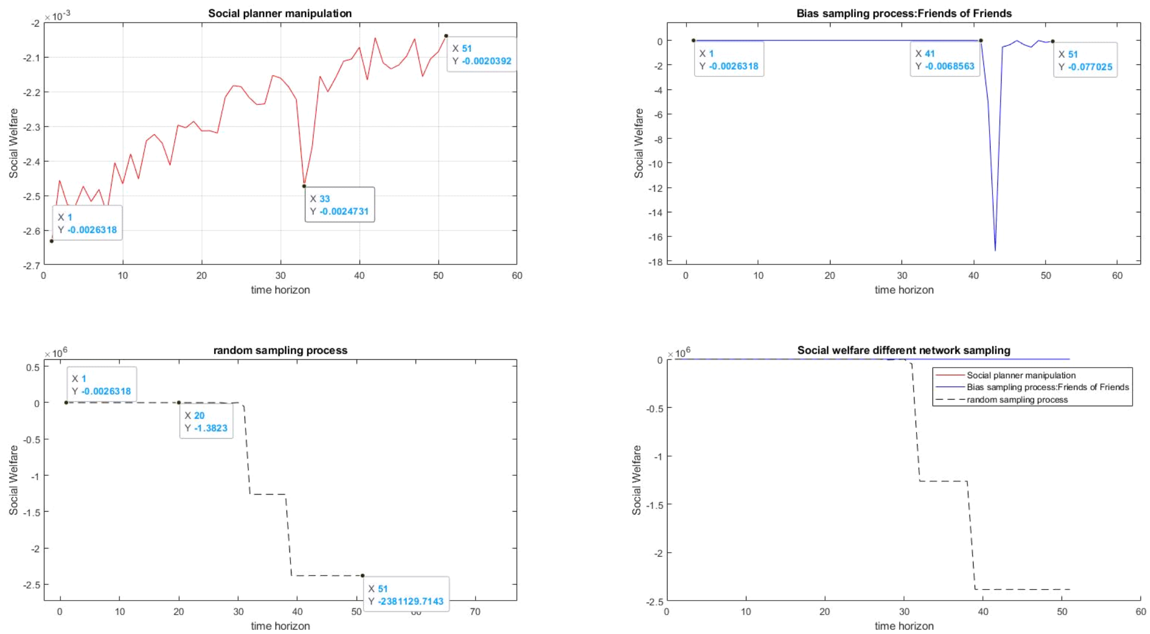

Figure 3.4 Comperative statics: Social Welfare: 100 Monte-Carlo Simulations.

The key explanation of this distance lies on how many indegree and outdegree links agents have. As we can see from Figure 4 in some periods there is a sudden drop in social welfare. There can be several reason of such drop, *(1) "tragedy of the commons" problem.* The algorithm of social planner maximize changes in adjancency matrix by row, which influence on the second stage decision making process. One of the solutions of such drops can be changing the social planner manipulation strategy, by adding a step in the algorithm, where the social planner tries to compare pairwise stability of changing rows, and decide which row to change and which row to keep as it is in period t.[68] *(2) "Not enough Monte-Carlo simulations."* One of the solution of such drops increase the numbers of Monte-Carlo simulations. In 100 Monte-Carlo simulations only one simulation demonstrate such drop in

[68] We will add this step in a future version of the draft.

social welfare, and effect of this simulations influence on other 99 Monte-Carlo simulations. Increasing the number of Monte-Carlo simulations can smooth such drops.[69]

The key result of Figure 4 is increasing curve for social planner manipulation strategy, and one of explanation is connection between node degree and social welfare. We analyze the connection between node degree and social welfare looking at a one-period (static) game.

3.5.1.2. Analyzing the connection between node degree and social welfare In order to keep the analysis simple, we examine the static game, with the following network topology: We consider a central agent with a ring network, demonstrated in Figure 3.5.

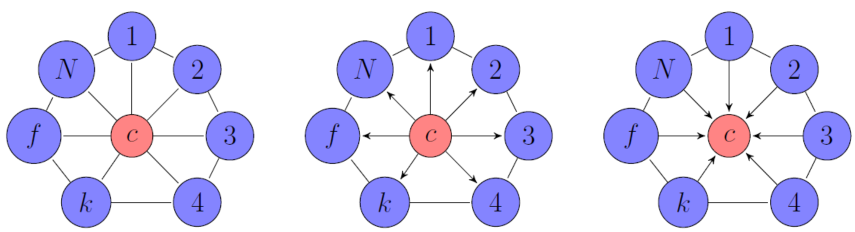

Figure 3.5 Star network with combination of ring network.

The first graph shows the undirected graphs, so everyone can see the signal of each other. The second graph, shows that the central agent receives signals frm other agents, not sharing his information ($C_{\text{Outdegree}} > C_{\text{Indegree}}$). The third graphs shows that the central agent shares his private signal, but doesn't get signals from others ($C_{\text{Outdegree}} < C_{\text{Indegree}}$).[70] For the sake of simplification, we consider a graph with two clusters only (see the definition of clusters in Section 3). Let's assume that the precision of the private signal is the same for all agents $\beta_i = \beta$, and in this static game there is no bias $b_i = 0$. This simplification will give us the

[69] In future work we will further extend the results for 500-1000 Monte-Carlo simulations and robustness checks of results.
[70] An analytical solution can be found in Appendix .

opportunity to demonstrate that the Morris and Shin (2002) results do not go through in particular network structures.

Lemma 8 *Let $\mathcal{G}$ is described as in Figure 5. Suppose $\beta_i = \beta$, $r \in (0.65, 1)$,$b_i = 0$. The optimal weights w_s^*,w_r^*,w_c^* is described by formula (A.1.5) ,(A.1.8) and (A.1.6) . Using this optimal weights, the social welfare is described by the equation (A.1.9) ,(A.1.10) and (A.1.11). And the following holds:*

1. For every $\beta = \overline{\beta}$ and $\alpha < \overline{\beta}$, the $\frac{\partial W_c}{\partial \alpha} > 0$ if agent i put weight to central agents signal and $\beta = \overline{\beta}$ and $\alpha > \overline{\beta}$, the $\frac{\partial W_c}{\partial \alpha} > 0$ if agent i put weight to public signal.

2. For every $\beta = \overline{\beta}$ and $\alpha < \overline{\beta}$, the $\frac{\partial W_s}{\partial \alpha} > 0$ if agent i put weight to central agents signal and $\beta = \overline{\beta}$ and $\alpha > \overline{\beta}$, the $\frac{\partial W_s}{\partial \alpha} > 0$ if agent i put weight to public signal.

3. For every $\beta = \overline{\beta}$ and $\alpha \lesseqqgtr \overline{\beta}$, the $\frac{\partial W_r}{\partial \alpha} < 0$.

In Figure 6a. we will illustrate the effect of the public and private signals precision on the welfare. More formal proof is available on online Appendix.

We call these networks "central agent", "central receiver" and "central sender" respectively. The effect of the public and private signals precision on the welfare in these network structures is illustrated in Figure 3.6a.

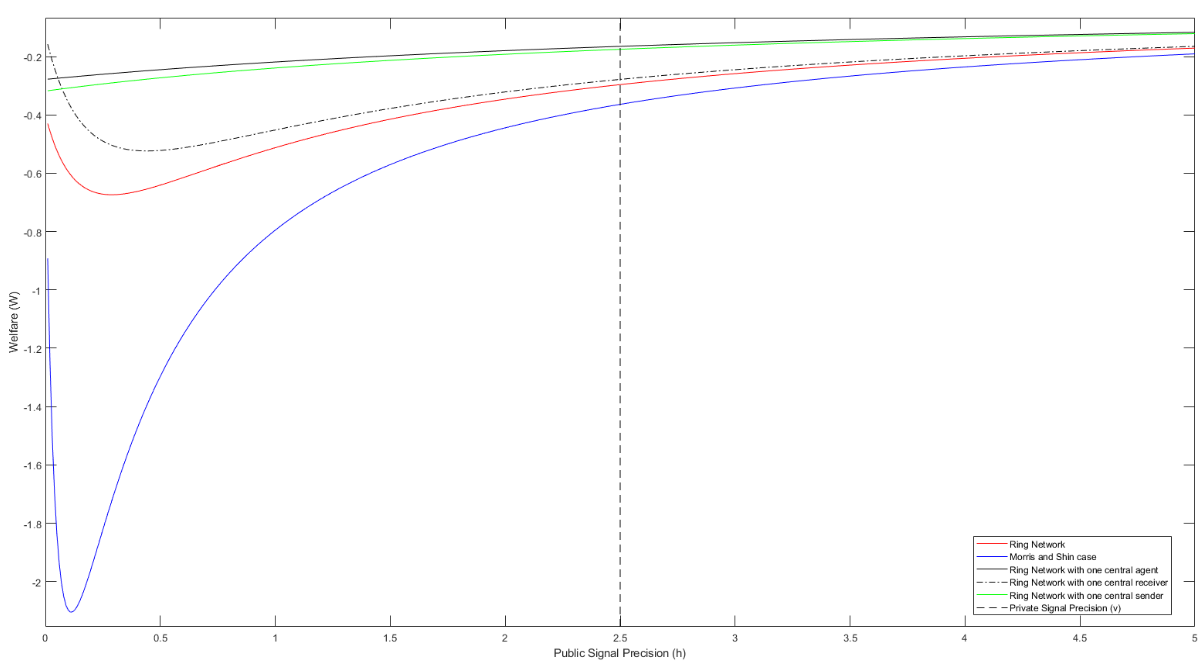

Figure 3.6a Social Welfare, comparing information sharing in central agent contest.

It can be seen that in the case of one central receiver the behavior of welfare is similar to this in the ring network. However, for each value of α, the welfare for the former network is higher than the welfare for the latter one. It can be caused by the presence of the agent, who is better informed compared to other agents. Since this central agent does not send any signal, her effect on the other agents' actions is minor (only through the presence of the unobserved signal in the transparency term $(E(\sum a_k))$.

However, in the case of the central-sender type of network, the Morris and Shin (2002) result vanishes. Even for high values of r, the welfare function is monotonically increasing with respect to α. Therefore, crucial result in this paper is that each agent's private signal sent in network can be viewed as a substitute to the public signal sent by the authority. Here we considered the extreme case: each agent in the network receives the same signal from the central agent. It means that agents are able to choose between two signals, that has the same characteristics as long as their precision parameters are the same. The effect of switching between signals is illustrated by Figure 3.6b.

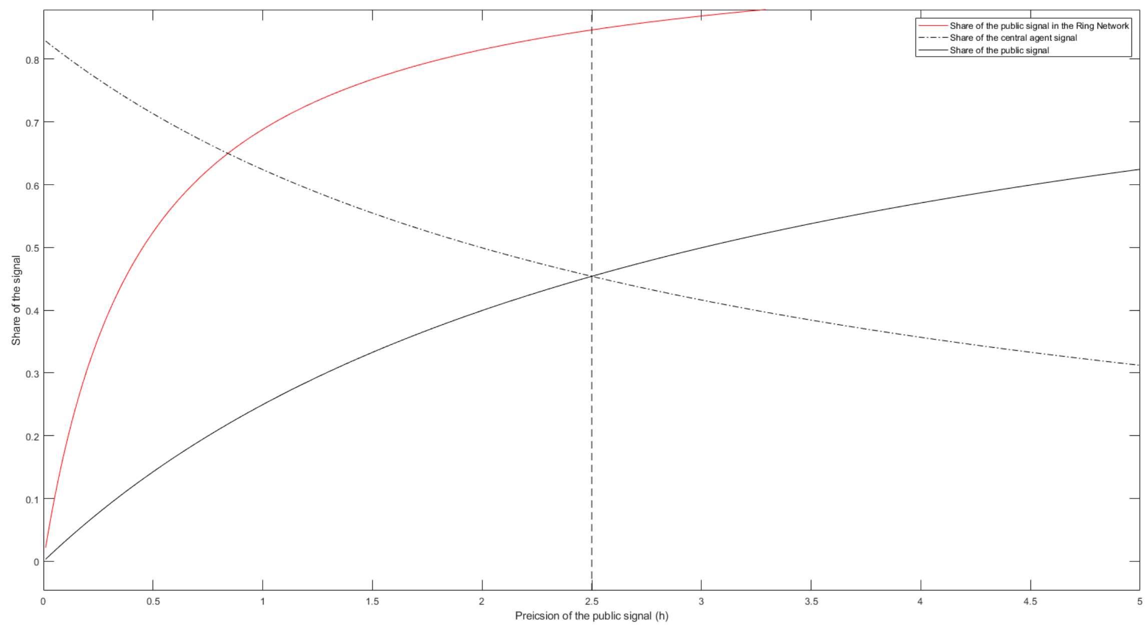

Figure 3.6b Share of signals in non-central agents actions.

Figure 6b shows that, the higher the public signal precision, the higher its share in the non-central agent's action. Moreover, share of all other signals is approximately equal to zero. This means that non-central agents simple switch between two signals: the public one and the central agent's one.

Since they have freedom of choice, agents can choose the most precise signal available. This fact leads to the absence of the Morris and Shin (2002) result (the noisy signal is simply ignored). Therefore, the central agent's and central sender's strategy can solve the problem raised in the paper by Morris and Shin (2002).

Therefore, coming back to our social-planner manipulation of networks dynamics agents with high precision of private signals end up with high indegree links, and agents which have low precision of private signal end up with high outdegree centrality and this effect increases social welfare.

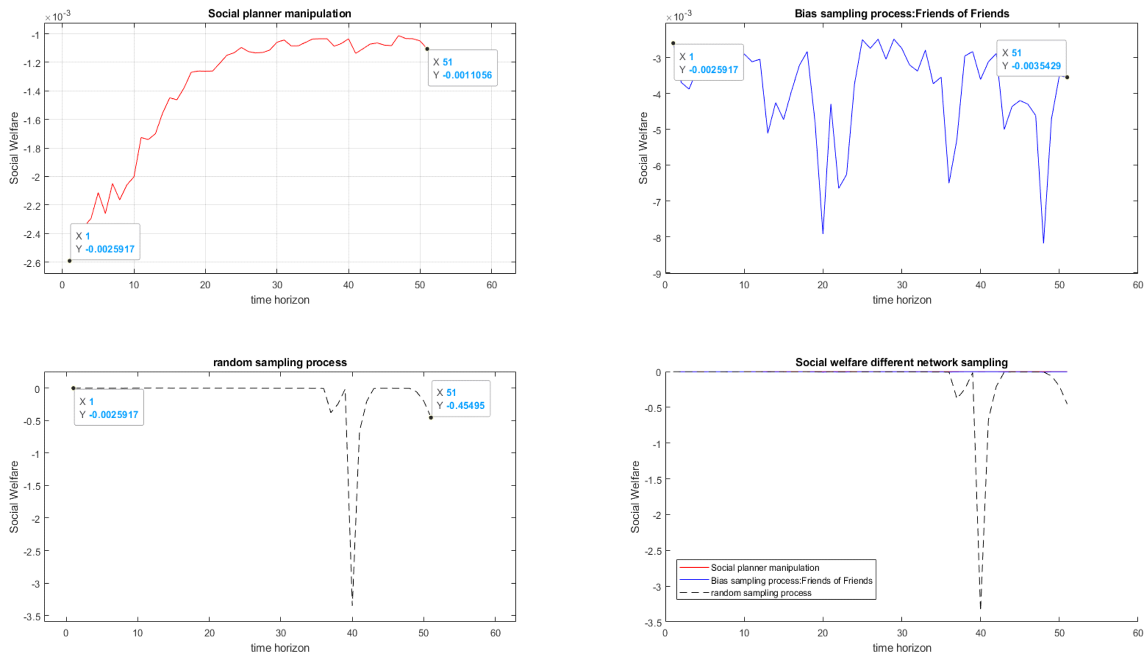

Figure 3.7 Social welfare dynamics for the case described in Figure 3a,3b,3c

The connections between node-degree, private and public signal precision can be demonstrated in the Table 3.1. Please note that the weight of signal j in agent i's action w_{ij} depends on agent i's and agent j's the precisions of signals and degrees in the network. So the Table 1 answer partially to the question.

Agent_N	Beta	Gam_indegree	Gam_outdegree	SPindegree	SPoutdegree	bindegree	boutdegree	r_indegree	r_outdegree
1	28	11	8	19	18	12	12	4	1
2	21	10	9	17	16	10	7	5	1
3	39	12	10	15	16	14	15	3	16
4	39	12	3	19	10	6	6	4	1
5	30	13	5	20	17	7	9	3	1
6	19	6	8	20	18	11	9	4	6
7	34	6	7	20	15	6	5	6	1
8	18	11	6	17	13	8	8	6	1
9	26	6	5	14	16	4	5	5	1
10	23	4	9	17	16	7	6	5	5
11	29	10	6	17	18	8	7	3	1
12	45	11	12	17	16	9	9	3	1
13	37	9	10	20	18	9	9	3	8
14	45	7	9	13	17	4	5	2	1
15	18	8	10	10	17	7	9	2	7
16	29	6	8	17	18	5	4	4	1
17	11	6	11	4	18	8	8	3	6
18	37	8	10	14	18	6	7	4	9
19	31	6	9	20	18	9	9	5	7
20	40	3	10	19	16	6	7	2	1

Table 3.1. Node degree data, precision of private signal. The precision of public signal is $\alpha = 30$.

In Table 1. Gam_indegree and Gam_outdegree describe the original matrix $\mathbf{M}_0$. SPindegree and SPoutdegree colums describe the node degree after 50 periods in the case of social planner manipulation of network dynamics (Figure 3.3a). bindegree and bourdegree columns show the results in Fugure 3.3b, and r_indegree and r_outdegree columns are the illustration of Figure 3.3c.

3.5.2 *Social planner with incomplete information* about state variable θ_t

In the real world it is diffucult to find a social planner who exactly knows all fundamental variables, in every period. Therefore, we examine the case of an imperfectly informed social planner. In this section we examine two social-planner types: *(1) A social planner with incomplete information,* and *(2) a social planner with wrong expectations* $E(\theta_t) = \widetilde{\theta}_t \neq \theta_t^*$. The second case can be useful if we consider public policy issues in the autocratic regimes.

Where agents/citizens have precise signals about the fundamental variables, but the regime has wrong expectations and tries to manipulate agents actions closer to their expectation.

3.5.2.1. Social planner with incomplete information In this case the social planner has incomplete information about the state variable θ_t, and receives private and public signals like all other agents in the model. Therefore the social planner calculates $E\left(\theta_t|\mathcal{I}_{sp}\right)$, with $\mathcal{I}_{sp}$ denoting the information set of the social planner, consisting of a private and a public signal. The social planner's public signal is the same public signal as that of all other agents. For the benchmark case we consider that the social planner's private signal has higher precision, than the precision of the public signal and precision of the privete signals of all other agents. Under imperfect information, $E\left(\theta_t|\mathcal{I}_{sp}\right) \neq 0$, and social welfare also depends on the expectations of the social planner.

$$W = -\ \mathbf{1}^T\ \left(\bar{\omega}^2_{b\circ} \circ \bar{b}^2_{\circ} + \bar{\omega}^2_{b\circ} \circ E\left(\theta^2_t|\mathcal{I}_{sp}\right) - \bar{\omega}^2_{b\circ} \circ \bar{b} \circ E\left(\theta_t|\mathcal{I}_{sp}\right) + W^2_{\text{private}\circ}\left(\frac{\mathbf{1}_{\circ}}{\overline{\beta}}\right) + \frac{W^2_y}{\alpha}\right)$$

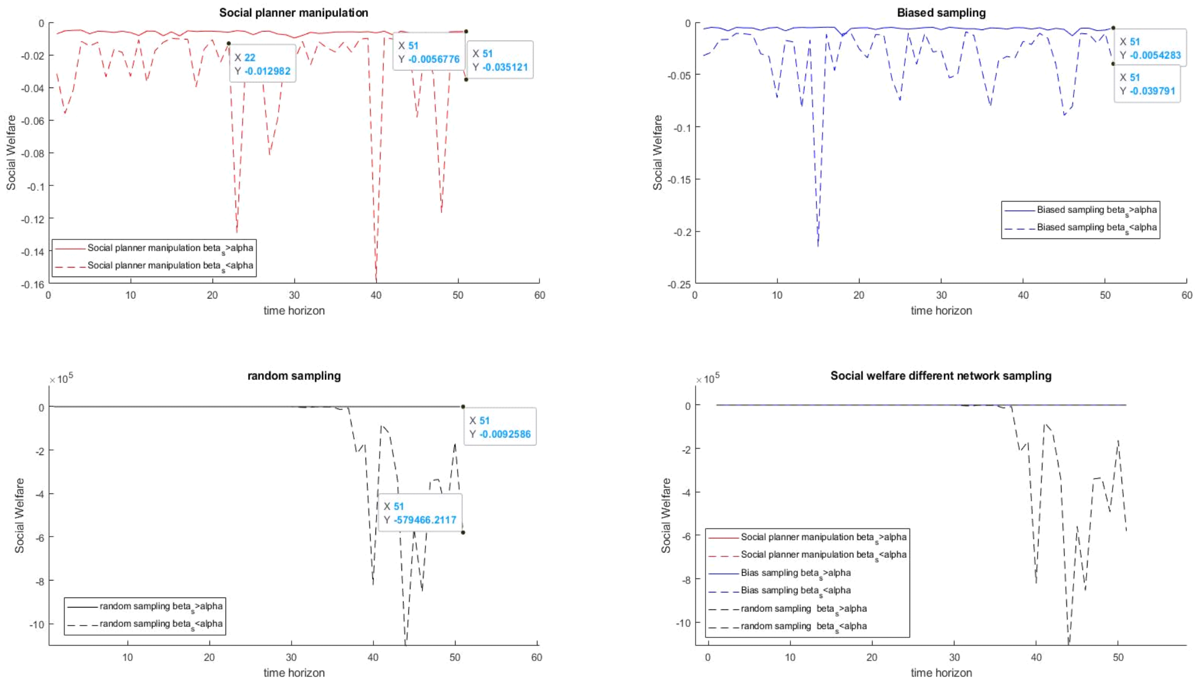

Figure 3.8a Social welfare: Incomplete information benchmark

As we can see from Figure 3.8a, this setting of imperfect information leads to worse results than in the case of biased sampling. This happens because the social planner cannot make the indegree links of key players to increase. A demonstration through simulated evolution dynamics that the indegree links are higher in the biased sampling setting than in the examined case here, can be found in Appendix . The key conclusion that can be reached in the incomplete information benchmark is that following any social planner manipulation of network dynamics can be useful only in short horizons. Therefore, we decrease the time periods and we examine the short-horizon case.

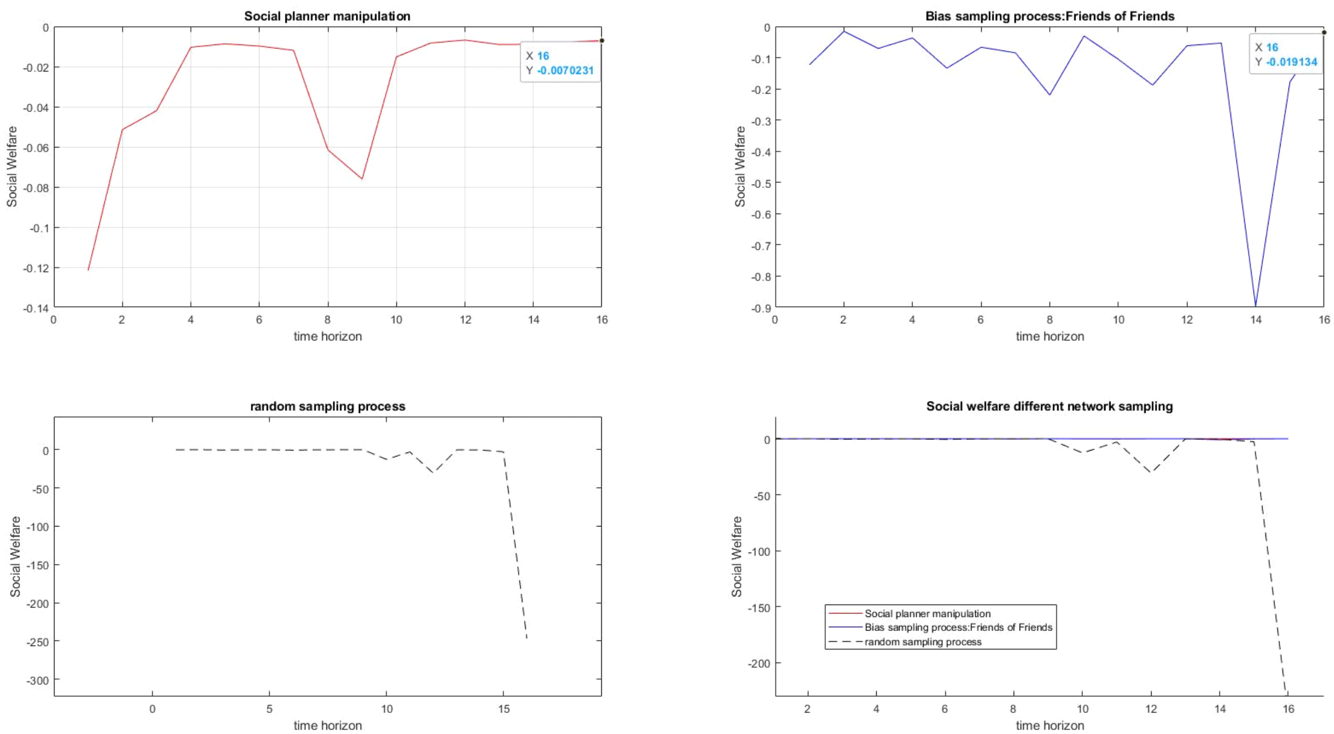

Figure 3.8b Social welfare: short-time periods

As it can be seen, in the incomplete information case, social-planner manipulation works better for short horizons. This means that the social planner does not need to manipulate the sampling process in all periods.

3.5.2.2. Social planner with wrong expectations In this case of imperfect knowledge on the side of the social planner, I examine that the planner has wrong expectations about the state variable, i.e., $E(\theta_t) = \widetilde{\theta}_t \neq \theta_t^*$. Under wrong expectations, the social planner has some prior beliefs about the state variable of the type $\theta_t = \widetilde{\theta}_t = const$. For example, if the true value is θ_t^*, the social planner believes that $\theta_t^{*,sp} = \widetilde{\theta}_t \neq \theta_t^*$. Can we see increasing social welfare in this setup in every period?

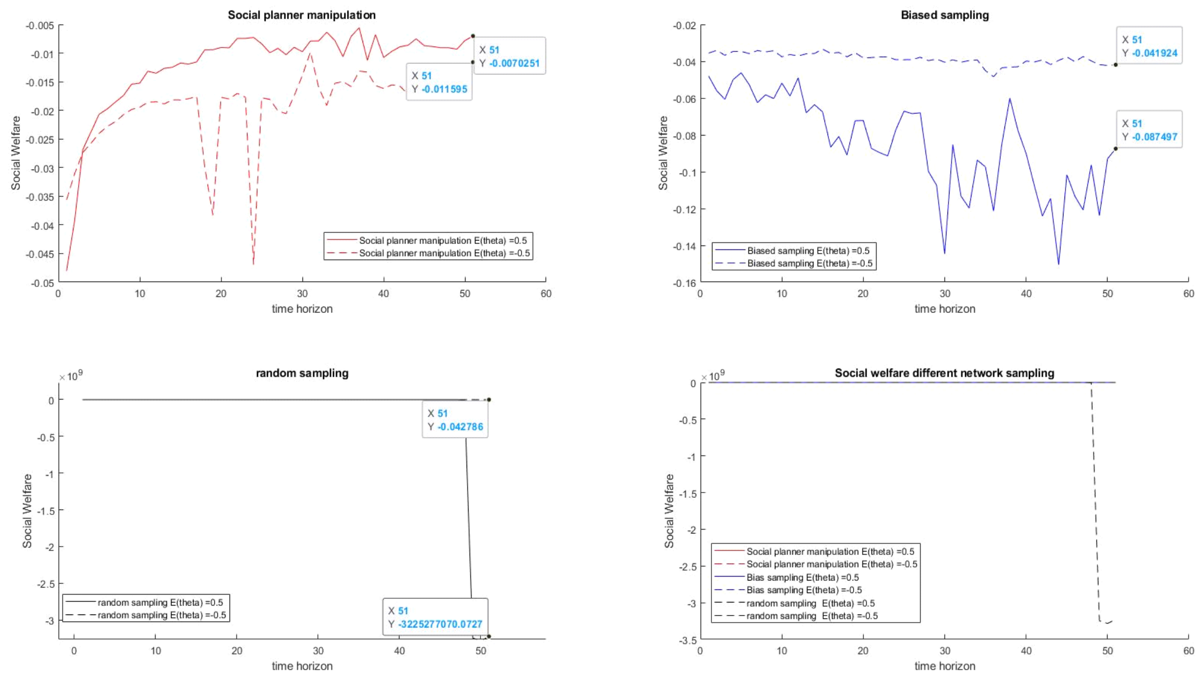

Figure 3.9 Social welfare. Wrong expectations of social planner.

As we can see, under the incomplete-information benchmark, it is better if the social planner has wrong constant expectations compared to receiving private and public signals. This happens because the noise term and the precision of signals that influence expectations about the state variable, $E(\theta_t)$, change every period. Therefore, it is better if $E(\theta_t)$ is equal to wrong constant in social-planner's mind, compared to the case that the social planner gets more precise signals then all other agents.

3.6 Conclusion

Polarization has increased during the last decades. There is a large literature in political economics, trying to understand how to mitigate the problems caused by polarization in networks, by spotting the key players who distort information, trying to bring agents' action closer to fundamental variables.

We examined whether a social planner, such as the manager of a social-media internet platform, can manipulate network dynamics so as to bring agents' actions closer to pragmatic viewpoints, thus increasing social welfare. Specifically, we examined if social planners can influence network dynamics by recommending people as network friends to online-platform users, and by pointing annoying behaviors by existing social-media friends. Importanty, we let network users to decide alone whom to make a new friend and whom to abolish as network friend. We also examined how my analysis changes if the social planner also has incomplete information or wrong priors concerning fundamental variables.

We built a dynamic network formation model, where each agent has strong incentives to coordinate their action with the actions of other agents and also to align their actions with payoff-relevant fundamentals. Using simulations, we demonstrated that if the social planner is perfectly informed about fundamentals, then his policy will be to suggest agents create more indegree links, if their private signal precision is high compare with public signal, and agents, who have low private signal precision, create more outdegree links. This strategy is crucial for increasing social welfare. Using a static game example, we provided a formal proof of why social welfare goes up when nodes with indegree centrality increase.

One of the main explanations of this result is the following: if central agents share their information with others, their signals have the same characteristics as public information, and agents can decide on switching from one signal to another if the precision of the public signal increases. This characteristic of my analysis improves a feature of the standard Morris and Shin (2002) model who find that increasing the precision of public information can decrease social welfare. Specifically, in our model, we demonstrated that social planners try to organize the network structure as a combination of star networks. It is this structure that can increase social welfare.

Interesting is also the case where the social planner has imperfect imperfect knowledge about fundamentals. We demonstrated, that in both cases, where the social planner has noisy information, and where the social planner is sure about the wrong fundamental value, social-planner manipulation can increase social welfare. But the most interesting result is that with a social planner being sure about the wrong expectation, welfare improvement is higher than having noisy information about fundamentals, even if the public and private signals the social planner receives have lower noise than those of the agents. One of the key explanations can be that if the social planner fixes an expectation of the fundamental value, then it can be easier for him to organize the network dynamics. In the case of noisy signals, the changing signals of agents combined with the noisy signals of the social planner, bring some mess to the social planner's strategy.

To the best of my knowledge, this is the first paper where social planner tries to manipulate network in a dynamic setting, not by directly influencing agents' action, but by just trying to introduce agents to each other in a way that social welfare will increase. An appealing feature of the examined model is that it rationalizes decisions under incomplete information. Agents in the model make decisions to create new links or to delete some of the old links, depending on their value functions. The calculation of value functions is challenging, because of the complexity of the model. Our model can be still demanding even with a case of $N = 50$, but it can offer new venues for improvement.

Finally, future work can focus on extending the biased-sampling setup, focusing on some real-world challenges that social-network platforms suggest. We can use this method of creating and deleting links using the biases b_i, and endogenizing the strength of peer-induced assimilation bias. Future works also need to be focused on whether social media platforms need to be regulated or not. Recently social media platforms owners such as Facebook and Twitter blocked president Trump's video. They claim the following: "This video includes false claims that a group of people is immune from COVID-19 which is a violation of our policies around harmful COVID misinformation." After this incident, Twitter announced that they plan to share the state for top profiles. Twitter shares the following information in its blog: "We believe that people have a right to know when a media account is directly or indirectly affiliated with a state actor". Therefore, social platform owners care about agents' decisions, and future extensions are needed in order to understand which mechanism is more efficient.

3.7 Appendix

3.7.A Expectation of the state of the world

Agent i's information set consists of her own private signal $x_{i,t}$, the set of her neighbors' private signals $\{x_{j,t}\}_{j \in N_i}$ and public signal y_t. Since all signals are random variables, centered at the θ, to predict the state of the world conditional on it's information set, agent i should consider the following probability density function[71] :

$$p\left(\theta_t|\mathcal{I}_{i,t}\right) = p\left(\theta_t|x_{i,t},\{x_{j,t}\}_{j\in N_i},y_t\right) \propto p(x_i,\{x_{j,t}\}_{j\in N_i}y|\theta)p(\theta) \propto$$
$$\propto \exp\left[-\frac{1}{2}\left(\beta_i(\theta_t-x_{i,t})^2+\sum_{j\in N_i}\beta_{j,t}(\theta_t-x_{j,t})^2+\alpha(\theta_t-y_t)^2\right)\right]$$

Using standard derivations frequently used in the Bayesian statistics literature (see, for instance, Koop (2007)). So we get the following result:

$$\theta|_{(x_{i,t},\{x_{j,t}\}_{j\in N_i},y_t)} \sim \mathbf{N}\left(\frac{\alpha y+\beta_i x_i+\sum_j \beta_j x_j}{\alpha+\beta_i+\sum_j \beta_j},\frac{1}{\alpha+\beta_i+\sum_j \beta_j}\right)$$

For calculating value functions, we need to calculate $E(\theta^2|\mathcal{I}_i)$. Following the Hakobyan and Koulovatianos (2020) we will find the following:

$$E(\theta^2|\mathcal{I}_i) = \left(\frac{\alpha y+\beta_i x_i+\sum_j \beta_j x_j}{\alpha+\beta_i+\sum_j \beta_j}\right)^2 + \frac{1}{\alpha+\beta_i+\sum_j \beta_j} \tag{A.1.1}$$

3.7.B More detail examples: 6 Agents case

Let's consider the network structure which include 6 agents. The graph $\mathcal{G}$ is unweighted and undirected, so $M_t^{ij} = M_t^{ji}$ as demonstrated in the Figure 3.2.

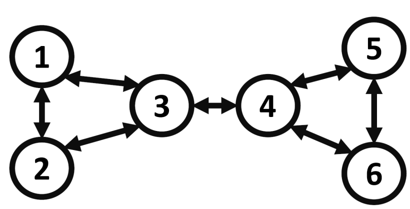

Figure 3.2 Example with 6 agents

[71]With absolutely non-informative prior where $p(\theta) \propto 1$.

In the Section 3.1 we define the information set as $\mathcal{I}_{i,t} = \left(y_t, x_{i,t}, \sum_j x_{j,t} \right)$. Let's begin from the first period $t = 1$, the information set will be the following $\mathcal{I}_{1,1} = (y_1,\, x_{1,1},\, x_{2,1},\, x_{3,1})$; $\mathcal{I}_{2,1} = (y_1,\, x_{1,1},\, x_{2,1},\, x_{3,1})$; $\mathcal{I}_{3,1} = (y_1,\, x_{1,1},\, x_{2,1},\, x_{3,1},\, x_{4,1})$; $\mathcal{I}_{4,1} = (y_1,\, x_{3,1},\, x_{4,1},\, x_{5,1},\, x_{6,1})$; $\mathcal{I}_{5,1} = (y_1,\, x_{4,1},\, x_{5,1},\, x_{6,1})$ and $\mathcal{I}_{6,1} = (y_1,\, x_{4,1},\, x_{5,1},\, x_{6,1})$. In Section 4 I introduce the algorithm of finding the equilibrium. The linear strategy which I define in equation *(3.10)* looks like the follows:

$$
\begin{aligned}
a_{1,1} &= \omega_{11}^1 x_{1,1} + \omega_{12}^1 x_{2,1} + \omega_{13}^1 x_{3,1} + w_{b_1}^1 b_1 + \left(1 - \left(\omega_{11}^1 + \omega_{12}^1 + \omega_{13}^1 + w_{b_1}^1\right)\right) y_1 \\
a_{2,1} &= \omega_{21}^1 x_{1,1} + \omega_{22}^1 x_{2,1} + \omega_{23}^1 x_{3,1} + w_{b_2}^1 b_2 + \left(1 - \left(\omega_{21}^1 + \omega_{22}^1 + \omega_{23}^1 + w_{b_2}^1\right)\right) y_1 \\
a_{3,1} &= \omega_{31}^1 x_{1,1} + \omega_{32}^1 x_{2,1} + \omega_{33}^1 x_{3,1} + \omega_{34}^1 x_{4,1} + w_{b_3}^1 b_3 + \left(1 - \left(\omega_{31}^1 + \omega_{32}^1 + \omega_{33}^1 + \omega_{34}^1 + w_{b_3}^1\right)\right) y_1 \\
a_{4,1} &= \omega_{43}^1 x_{3,1} + \omega_{44}^1 x_{4,1} + \omega_{45}^1 x_{5,1} + \omega_{46}^1 x_{6,1} + w_{b_4}^1 b_4 + \left(1 - \left(\omega_{43}^1 + \omega_{44}^1 + \omega_{45}^1 + \omega_{46}^1 + w_{b_4}^1\right)\right) y_1 \\
a_{5,1} &= \omega_{54}^1 x_{4,1} + \omega_{55}^1 x_{5,1} + \omega_{56}^1 x_{6,1} + w_{b_5}^1 b_5 + \left(1 - \left(\omega_{54}^1 x + \omega_{55}^1 + \omega_{56}^1 + w_{b_5}^1\right)\right) y_1 \\
a_{6,1} &= \omega_{64}^1 x_{4,1} + \omega_{65}^1 x_{5,1} + \omega_{66}^1 x_{6,1} + w_{b_6}^1 b_6 + \left(1 - \left(\omega_{64}^1 x + \omega_{65}^1 + \omega_{66}^1 + w_{b_6}^1\right)\right) y_1
\end{aligned}
$$

We normalized the weights, so $\omega_{11} + \omega_{12} + \omega_{13} + w_{b_1} + w_{y_1} = 1$ and we will solve system of linear equations for $\omega_{11}; \omega_{12}; \omega_{13}; w_{b_1}$. Weight which agent 1 put on public signal we will find in the following way $w_{y_1} = 1 - (\omega_{11} + \omega_{12} + \omega_{13} + w_{b_1})$. If the agent has no information about the signal j, then $E_i x_j = E_i \theta$. Let's consider the optimal action from 1st agent side, which looks like the following equation:

$$
\begin{aligned}
a_1 = {} & (1 - r_1) E_1(\theta_1) + (1 - r_1) b_1 + r_1 \frac{q_1}{\# N_1} \left[\omega_{21} x_1 + \omega_{22} x_2 + \omega_{23} x_3 + w_{b_2} b_2 + (1 - \omega_{21} - \omega_{22} - \omega_{23} - w_{b_2}) y\right] \\
& + r_1 \frac{q_1}{\# N_1} \left[\omega_{31} x_1 + \omega_{32} x_2 + \omega_{33} x_3 + \omega_{34} E_1(\theta_1) + w_{b_3} b_3 + (1 - \omega_{31} - \omega_{32} - \omega_{33} - \omega_{34} - w_{b_3}) y\right] + \\
& + r_1 \frac{(1 - q_1)}{\# N_{-1}} \left[\omega_{43} x_3 + \omega_{44} E_1(\theta_1) + \omega_{45} E_1(\theta_1) + \omega_{46} E_1(\theta_1) + w_{b_4} b_4 + (1 - \omega_{43} - \omega_{44} - \omega_{45} - \omega_{46} - w_{b_4}) y\right] + \\
& + r_1 \frac{(1 - q_1)}{\# N_{-1}} \left[\omega_{54} E_1(\theta_1) + \omega_{55} E_1(\theta_1) + \omega_{56} E_1(\theta_1) + w_{b_5} b_5 + (1 - \omega_{54} - \omega_{55} - \omega_{56} - w_{b_5}) y\right] + \\
& + r_1 \frac{(1 - q_1)}{\# N_{-1}} \left[\omega_{64} E_1(\theta_1) + \omega_{65} E_1(\theta_1) + \omega_{66} E_1(\theta_1) + w_{b_6} b_6 + (1 - \omega_{64} - \omega_{65} - \omega_{66} - w_{b_6}) y\right]
\end{aligned}
$$

In the second step, we need to rearrange the terms, and get the coefficient preceding the $E_1(\theta_1)$.

$$\begin{aligned} a_1 &= E_1(\theta_1)\left[(1-r_1)+r_1\left[\frac{q_1}{\#N_1}\omega_{34}+\frac{(1-q_1)}{\#N_{-1}}\left[\omega_{44}+\omega_{45}+\omega_{46}+\omega_{54}+\omega_{55}+\omega_{56}+\omega_{64}+\omega_{65}+\omega_{66}\right]\right]\right]+ \\ & (1-r_1)b_1+r_1\frac{q_1}{\#N_1}\left[\omega_{21}+\omega_{31}\right]x_1+r_1\frac{q_1}{\#N_1}\left[\omega_{22}+\omega_{32}\right]x_2+r_1\left[\frac{q_1}{\#N_1}\left[\omega_{23}+\omega_{33}\right]+\frac{(1-q_1)}{\#N_{-1}}\omega_{43}\right]x_3+ \\ & +r_1\frac{q_1}{\#N_1}\left(w_{b_2}b_2+w_{b_3}b_3\right)+r_1\frac{(1-q_1)}{\#N_{-1}}\left(w_{b_4}b_4+w_{b_5}b_5+w_{b_6}b_6\right)+ \\ & +r_1\left(\frac{q_1}{\#N_1}\left[w_{2,y}+w_{3,y}\right]+\frac{(1-q_1)}{\#N_{-1}}\left[w_{4,y}+w_{5,y}+w_{6,y}\right]\right)y \end{aligned}$$

Let's use the algebra from Appendix A1 to calculate the $E_i(\theta_1)$.

$$\begin{aligned} E_1(\theta_1) &= E_2(\theta_1)=\frac{\beta_1x_1+\beta_2x_2+\beta_3x_3+\alpha y_1}{\beta_1+\beta_2+\beta_3+\alpha}; \quad E_3(\theta_1)=\frac{\beta_1x_1+\beta_2x_2+\beta_3x_3+\beta_4x_4+\alpha y_1}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha}; \\ E_4(\theta_1) &= \frac{\beta_3x_3+\beta_4x_4+\beta_5x_5+\beta_6x_6+\alpha y_1}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha}; \quad E_4(\theta_1)=E_5(\theta_1)=\frac{\beta_4x_4+\beta_5x_5+\beta_6x_6+\alpha y_1}{\beta_4+\beta_5+\beta_6+\alpha}; \end{aligned}$$

Now we can find the weights which agent 1 put in y, x_1, x_2, x_3 and b_1.

Weight for x_1

$$\begin{aligned} \omega_{11} &= \frac{\beta_1}{\beta_1+\beta_2+\beta_3+\alpha}\left[(1-r_1)+r_1\left[\frac{q_1}{\#N_1}\omega_{34}+\frac{(1-q_1)}{\#N_{-1}}\left[\omega_{44}+\omega_{45}+\omega_{46}+\omega_{54}+\omega_{55}+\omega_{56}+\omega_{64}+\omega_{65}+\omega_{66}\right]\right]\right]+ \\ & +r_1\frac{q_1}{\#N_1}\left[\omega_{21}+\omega_{31}\right] \end{aligned}$$

Weight for x_2

$$\begin{aligned} \omega_{12} &= \frac{\beta_2}{\beta_1+\beta_2+\beta_3+\alpha}\left[(1-r_1)+r_1\left[\frac{q_1}{\#N_1}\omega_{34}+\frac{(1-q_1)}{\#N_{-1}}\left[\omega_{44}+\omega_{45}+\omega_{46}+\omega_{54}+\omega_{55}+\omega_{56}+\omega_{64}+\omega_{65}+\omega_{66}\right]\right]\right]+ \\ & +r_1\frac{q_1}{\#N_1}\left[\omega_{22}+\omega_{32}\right] \end{aligned}$$

Weight for x_3

$$\begin{aligned} \omega_{13} &= \frac{\beta_3}{\beta_1+\beta_2+\beta_3+\alpha}\left[(1-r_1)+r_1\left[\frac{q_1}{\#N_1}\omega_{34}+\frac{(1-q_1)}{\#N_{-1}}\left[\omega_{44}+\omega_{45}+\omega_{46}+\omega_{54}+\omega_{55}+\omega_{56}+\omega_{64}+\omega_{65}+\omega_{66}\right]\right]\right]+ \\ & +r_1\left[\frac{q_1}{\#N_1}\left[\omega_{23}+\omega_{33}\right]+\frac{(1-q_1)}{\#N_{-1}}\omega_{43}\right] \end{aligned}$$

Weight for b_1

$$
\begin{aligned}
w_{b_1} b_1 &= (1-r_1)\, b_1 + r_1 \frac{q_1}{\#N_1} \left(w_{b_2} b_2 + w_{b_3} b_3\right) + r_1 \frac{(1-q_1)}{\#N_{-1}} \left(w_{b_4} b_4 + w_{b_5} b_5 + w_{b_6} b_6\right) \Longleftrightarrow \\
&\Longleftrightarrow \quad w_{b_1} = (1-r_1) + \frac{r_1}{b_1} \frac{q_1}{\#N_1} \left(w_{b_2} b_2 + w_{b_3} b_3\right) + \frac{r_1}{b_1} \frac{(1-q_1)}{\#N_{-1}} \left(w_{b_4} b_4 + w_{b_5} b_5 + w_{b_6} b_6\right)
\end{aligned}
$$

We can do the same Algebra from 2nd agent side.

$$
\begin{aligned}
\omega_{21} &= \frac{\beta_1}{\beta_1+\beta_2+\beta_3+\alpha} \left[(1-r_2) + r_2 \left[\frac{q_2}{\#N_2}\omega_{34} + \frac{(1-q_2)}{\#N_{-2}} \left[\omega_{44}+\omega_{45}+\omega_{46}+\omega_{54}+\omega_{55}+\omega_{56}+\omega_{64}+\omega_{65}+\omega_{66}\right]\right]\right] + \\
&+ r_2 \frac{q_2}{\#N_2} \left[\omega_{11}+\omega_{31}\right]
\end{aligned}
$$

$$
\begin{aligned}
\omega_{22} &= \frac{\beta_2}{\beta_1+\beta_2+\beta_3+\alpha} \left[(1-r_2) + r_2 \left[\frac{q_2}{\#N_2}\omega_{34} + \frac{(1-q_2)}{\#N_{-2}} \left[\omega_{44}+\omega_{45}+\omega_{46}+\omega_{54}+\omega_{55}+\omega_{56}+\omega_{64}+\omega_{65}+\omega_{66}\right]\right]\right] + \\
&+ r_2 \frac{q_2}{\#N_2} \left[\omega_{12}+\omega_{32}\right]
\end{aligned}
$$

$$
\begin{aligned}
\omega_{23} &= \frac{\beta_3}{\beta_1+\beta_2+\beta_3+\alpha} \left[(1-r_2) + r_2 \left[\frac{q_2}{\#N_2}\omega_{34} + \frac{(1-q_2)}{\#N_{-2}} \left[\omega_{44}+\omega_{45}+\omega_{46}+\omega_{54}+\omega_{55}+\omega_{56}+\omega_{64}+\omega_{65}+\omega_{66}\right]\right]\right] + \\
&+ r_2 \left[\frac{q_2}{\#N_2} \left[\omega_{13}+\omega_{33}\right] + \frac{(1-q_2)}{\#N_{-2}}\omega_{43}\right]
\end{aligned}
$$

$$
w_{b_2} = (1-r_2) + \frac{r_2}{b_2} \frac{q_2}{\#N_2} \left(w_{b_1} b_1 + w_{b_3} b_3\right) + \frac{r_2}{b_2} \frac{(1-q_2)}{\#N_{-2}} \left(w_{b_4} b_4 + w_{b_5} b_5 + w_{b_6} b_6\right)
$$

Using the same strategy we will find the optimal action for 3rd agent.

$$
\begin{aligned}
a_3 &= E(\theta) \left[(1-r_3) + r_3 \left[\frac{q_3}{\#N_3} \left[\omega_{45}+\omega_{46}\right] + \frac{(1-q_3)}{\#N_{-3}} \left[\omega_{55}+\omega_{56}+\omega_{65}+\omega_{66}\right]\right]\right] + (1-r_3)\, b_3 + \\
&+ r_3 \frac{q_3}{\#N_3} \left[\omega_{11}+\omega_{21}\right] x_1 + r_3 \frac{q_3}{\#N_3} \left[\omega_{12}+\omega_{22}\right] x_2 + r_3 \frac{q_3}{\#N_3} \left[\omega_{13}+\omega_{23}+\omega_{43}\right] x_3 + \\
&+ r_3 \left[\frac{q_3}{\#N_3}\omega_{44} + \frac{(1-q_3)}{\#N_{-3}} \left[\omega_{54}+\omega_{64}\right]\right] x_4 + r_3 \frac{q_3}{\#N_3} \left(w_{b_1} b_1 + w_{b_2} b_2 + w_{b_4} b_4\right) + \\
&+ r_3 \frac{(1-q_3)}{\#N_{-3}} \left(w_{b_5} b_5 + w_{b_6} b_6\right) + r_3 \left(\frac{q_3}{\#N_3} \left[w_{1,y}+w_{2,y}+w_{4,y}\right] + \frac{(1-q_3)}{\#N_{-3}} \left[w_{5,y}+w_{6,y}\right]\right) y
\end{aligned}
$$

$$\omega_{31} = \frac{\beta_1}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha}\left[(1-r_3)+r_3\left[\frac{q_3}{\#N_3}[\omega_{45}+\omega_{46}]+\frac{(1-q_3)}{\#N_{-3}}[\omega_{55}+\omega_{56}+\omega_{65}+\omega_{66}]\right]\right]+ +r_3\frac{q_3}{\#N_3}[\omega_{11}+\omega_{21}]$$

$$\omega_{32} = \frac{\beta_2}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha}\left[(1-r_3)+r_3\left[\frac{q_3}{\#N_3}[\omega_{45}+\omega_{46}]+\frac{(1-q_3)}{\#N_{-3}}[\omega_{55}+\omega_{56}+\omega_{65}+\omega_{66}]\right]\right]+ +r_3\frac{q_3}{\#N_3}[\omega_{12}+\omega_{22}]$$

$$\omega_{33} = \frac{\beta_3}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha}\left[(1-r_3)+r_3\left[\frac{q_3}{\#N_3}[\omega_{45}+\omega_{46}]+\frac{(1-q_3)}{\#N_{-3}}[\omega_{55}+\omega_{56}+\omega_{65}+\omega_{66}]\right]\right]+ +r_3\frac{q_3}{\#N_3}[\omega_{13}+\omega_{23}+\omega_{43}]$$

$$\omega_{34} = \frac{\beta_4}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha}\left[(1-r_3)+r_3\left[\frac{q_3}{\#N_3}[\omega_{45}+\omega_{46}]+\frac{(1-q_3)}{\#N_{-3}}[\omega_{55}+\omega_{56}+\omega_{65}+\omega_{66}]\right]\right]+ +r_3\left[\frac{q_3}{\#N_3}\omega_{44}+\frac{(1-q_3)}{\#N_{-3}}[\omega_{54}+\omega_{64}]\right]$$

$$w_{b_3} = (1-r_3)+\frac{r_3}{b_3}\frac{q_3}{\#N_3}(w_{b_1}b_1+w_{b_2}b_2+w_{b_4}b_4)+\frac{r_3}{b_3}\frac{(1-q_3)}{\#N_{-3}}(w_{b_5}b_5+w_{b_6}b_6)$$

Consider the optimal strategy from agent 4 side.

$$\begin{aligned} a_4 &= E(\theta)\left[(1-r_4)+r_4\left[\frac{q_4}{\#N_4}[\omega_{31}+\omega_{32}]+\frac{(1-q_4)}{\#N_{-4}}[\omega_{11}+\omega_{12}+\omega_{21}+\omega_{22}]\right]\right]+(1-r_4)b_4+ \\ &+r_4\frac{q_4}{\#N_4}[\omega_{55}+\omega_{65}]x_5+r_4\frac{q_4}{\#N_4}[\omega_{56}+\omega_{66}]x_6+r_4\frac{q_4}{\#N_4}[\omega_{34}+\omega_{54}+\omega_{64}]x_4+ \\ &+r_4\left[\frac{q_4}{\#N_4}\omega_{33}+\frac{(1-q_4)}{\#N_{-4}}[\omega_{13}+\omega_{23}]\right]x_3+r_4\frac{q_4}{\#N_4}(w_{b_3}b_3+w_{b_5}b_5+w_{b_6}b_6)+ \\ &+r_4\frac{(1-q_4)}{\#N_{-4}}(w_{b_1}b_1+w_{b_2}b_2)+r_4\left(\frac{q_4}{\#N_4}[w_{3,y}+w_{5,y}+w_{6,y}]+\frac{(1-q_4)}{\#N_{-4}}[w_{1,y}+w_{2,y}]\right)y \end{aligned}$$

$$\omega_{43} = \frac{\beta_3}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_4)+r_4\left[\frac{q_4}{\#N_4}\left[\omega_{31}+\omega_{32}\right]+\frac{(1-q_4)}{\#N_{-4}}\left[\omega_{11}+\omega_{12}+\omega_{21}+\omega_{22}\right]\right]\right]+ \\ +r_4\left[\frac{q_4}{\#N_4}\omega_{33}+\frac{(1-q_4)}{\#N_{-4}}\left[\omega_{13}+\omega_{23}\right]\right]$$

$$\omega_{44} = \frac{\beta_4}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_4)+r_4\left[\frac{q_4}{\#N_4}\left[\omega_{31}+\omega_{32}\right]+\frac{(1-q_4)}{\#N_{-4}}\left[\omega_{11}+\omega_{12}+\omega_{21}+\omega_{22}\right]\right]\right]+ \\ +r_4\frac{q_4}{\#N_4}\left[\omega_{34}+\omega_{54}+\omega_{64}\right]$$

$$\omega_{45} = \frac{\beta_5}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_4)+r_4\left[\frac{q_4}{\#N_4}\left[\omega_{31}+\omega_{32}\right]+\frac{(1-q_4)}{\#N_{-4}}\left[\omega_{11}+\omega_{12}+\omega_{21}+\omega_{22}\right]\right]\right]+ \\ +r_4\frac{q_4}{\#N_4}\left[\omega_{55}+\omega_{65}\right]$$

$$\omega_{46} = \frac{\beta_6}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_4)+r_4\left[\frac{q_4}{\#N_4}\left[\omega_{31}+\omega_{32}\right]+\frac{(1-q_4)}{\#N_{-4}}\left[\omega_{11}+\omega_{12}+\omega_{21}+\omega_{22}\right]\right]\right]+ \\ +r_4\frac{q_4}{\#N_4}\left[\omega_{56}+\omega_{66}\right]$$

$$w_{b_4} = (1-r_4)+\frac{r_4}{b_4}\frac{q_4}{\#N_4}\left(w_{b_3}b_3+w_{b_5}b_5+w_{b_6}b_6\right)+\frac{r_4}{b_4}\frac{(1-q_4)}{\#N_{-4}}\left(w_{b_1}b_1+w_{b_2}b_2\right)$$

The strategy from the 5th agent side

$$a_5 = E(\theta)\left[(1-r_5)+r_5\left[\frac{q_5}{\#N_5}\omega_{43}+\frac{(1-q_5)}{\#N_{-5}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right]+ \\ (1-r_5)\,b_5+r_5\frac{q_5}{\#N_5}\left[\omega_{45}+\omega_{65}\right]x_5+r_5\frac{q_5}{\#N_5}\left[\omega_{46}+\omega_{66}\right]x_6+r_5\left[\frac{q_5}{\#N_5}\left[\omega_{44}+\omega_{64}\right]+\frac{(1-q_5)}{\#N_{-5}}\omega_{34}\right]x_4+ \\ +r_5\frac{q_5}{\#N_5}\left(w_{b_4}b_4+w_{b_6}b_6\right)+r_5\frac{(1-q_5)}{\#N_{-5}}\left(w_{b_1}b_1+w_{b_2}b_2+w_{b_3}b_3\right)+ \\ +r_5\left(\frac{q_5}{\#N_5}\left[w_{4,y}+w_{6,y}\right]+\frac{(1-q_5)}{\#N_{-5}}\left[w_{1,y}+w_{2,y}+w_{3,y}\right]\right)y$$

$$\begin{aligned}\omega_{54} &= \frac{\beta_4}{\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_5)+r_5\left[\frac{q_5}{\#N_5}\omega_{43}+\frac{(1-q_5)}{\#N_{-5}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right]+\\ &+r_5\left[\frac{q_5}{\#N_5}\left[\omega_{44}+\omega_{64}\right]+\frac{(1-q_5)}{\#N_{-5}}\omega_{34}\right]\end{aligned}$$

$$\begin{aligned}\omega_{55} &= \frac{\beta_5}{\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_5)+r_5\left[\frac{q_5}{\#N_5}\omega_{43}+\frac{(1-q_5)}{\#N_{-5}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right]+\\ &+r_5\frac{q_5}{\#N_5}\left[\omega_{45}+\omega_{65}\right]\end{aligned}$$

$$\begin{aligned}\omega_{56} &= \frac{\beta_6}{\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_5)+r_5\left[\frac{q_5}{\#N_5}\omega_{43}+\frac{(1-q_5)}{\#N_{-5}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right]+\\ &+r_5\frac{q_5}{\#N_5}\left[\omega_{46}+\omega_{66}\right]\end{aligned}$$

$$w_{b_5}=(1-r_5)+\frac{r_5}{b_5}\frac{q_5}{\#N_5}\left(w_{b_4}b_4+w_{b_6}b_6\right)+\frac{r_5}{b_5}\frac{(1-q_5)}{\#N_{-5}}\left(w_{b_1}b_1+w_{b_2}b_2+w_{b_3}b_3\right)$$

From the 6th agent side

$$\begin{aligned}a_6 &= E(\theta)\left[(1-r_6)+r_6\left[\frac{q_6}{\#N_6}\omega_{43}+\frac{(1-q_6)}{\#N_{-6}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right]+\\ &(1-r_6)\,b_6+r_6\frac{q_6}{\#N_6}\left[\omega_{45}+\omega_{55}\right]x_5+r_6\frac{q_6}{\#N_6}\left[\omega_{46}+\omega_{56}\right]x_6+r_6\left[\frac{q6}{\#N_6}\left[\omega_{44}+\omega_{54}\right]+\frac{(1-q_6)}{\#N_{-6}}\omega_{34}\right]x_4+\\ &+r_6\frac{q_6}{\#N_6}\left(w_{b_4}b_4+w_{b_5}b_5\right)+r_6\frac{(1-q_6)}{\#N_{-6}}\left(w_{b_1}b_1+w_{b_2}b_2+w_{b_3}b_3\right)+\\ &+r_6\left(\frac{q_6}{\#N_6}\left[w_{4,y}+w_{5,y}\right]+\frac{(1-q_6)}{\#N_{-6}}\left[w_{1,y}+w_{2,y}+w_{3,y}\right]\right)y\end{aligned}$$

$$\begin{aligned}\omega_{64} &= \frac{\beta_4}{\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_6)+r_6\left[\frac{q_6}{\#N_6}\omega_{43}+\frac{(1-q_6)}{\#N_{-6}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right]+\\ &+r_6\left[\frac{q6}{\#N_6}\left[\omega_{44}+\omega_{54}\right]+\frac{(1-q_6)}{\#N_{-6}}\omega_{34}\right]\end{aligned}$$

$$\omega_{65} = \frac{\beta_5}{\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_6)+r_6\left[\frac{q_6}{\#N_6}\omega_{43}+\frac{(1-q_6)}{\#N_{-6}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right]+ +r_6\frac{q_6}{\#N_6}\left[\omega_{45}+\omega_{55}\right]$$

$$\omega_{66} = \frac{\beta_6}{\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_6)+r_6\left[\frac{q_6}{\#N_6}\omega_{43}+\frac{(1-q_6)}{\#N_{-6}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right]+ +r_6\frac{q_6}{\#N_6}\left[\omega_{46}+\omega_{56}\right]$$

$$w_{b_6} = (1-r_6)+\frac{r_6}{b_6}\frac{q_6}{\#N_6}\left(w_{b_4}b_4+w_{b_5}b_5\right)+\frac{r_6}{b_6}\frac{(1-q_6)}{\#N_{-6}}\left(w_{b_1}b_1+w_{b_2}b_2+w_{b_3}b_3\right)$$

3.7.C Generalizing The Solution in Matrix Form

The matrix $\mathcal{A}$ represents the adjacency matrix plus identity matrix (shows all connections including self-loops) $\mathcal{A} = \mathbf{M}_t + eye(N)$. Let's demonstrate the matrix $\mathcal{A}$ for the example which I introduce in Figure 2. Matrix $\mathcal{A}$ looks like,

$$\mathcal{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix};$$

The expression which is multiplied by $E_i(\theta_t)$ includes the following 2 parts: $(1 - r_i)$ and the weights of the signals, which agent i doesn't observe.[72]

The weight of this signals can be not only the signals of agents who are not in agent i's network, but the same time the weight which agent i's neighbors put to their friends, which are not in agent i's friends.[73]

For finding the weight before $E_i(\theta_t)$ we will introduce the matrix B_i. As we need to find the signals which are not in agent i's network, we need to exclude the agent i's friends from the adjacency matrix.I take the row of matrix $\mathcal{A}$, build a new matrix $(N \times N)$ and repeated that row N times, for all agents $N = \{1, 2...6\}$. For example for 6 agents case, the matrix B looks like the following.

$$B_1 = B_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}; B_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix};$$

$$B_4 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}; B_5 = B_6 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix};$$

The next step will be introducing the matrix C_i, which will show all connections excluding agent i's connections.. So $C_i = A - B_i$. If there is negative elements in the matrix $C_i(C_{ij} < 0)$, we will replace to 0. For algoritm we will use $C_i(C_i < 0) = 0$. Which will find

[72]As we show in example (**??**), from the first agent side its look like the following $E_1(\theta_1)[(1 - r_1) + r_1 \frac{q_1}{\#N_1}\omega_{34} + \frac{(1-q_1)}{\#N_{-1}}[\omega_{44} + \omega_{45} + \omega_{46} + \omega_{54} + \omega_{55} + \omega_{56} + \omega_{64} + \omega_{65} + \omega_{66}]$.

[73]For example, if we look from the first agent side the weight ω_{34} is the weight which agent 3 puts to his friend 4, which is not in agent's i's network.

all negative elements and change it to 0.

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \text{ so after replacing negative values } \quad C_1 = C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix};$$

$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}; C_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; C_5 = C_6 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

Agent i differ friends from non-friends by putting different weights to them. As you can see in the equation (2.2) the weight which agent i put to his friend is $\frac{q_i}{\#N_i}$, non-friends weight is equal to $\frac{(1-q_i)}{\#N_{-i}}$. We will take every row from matrix $\mathcal{A}$. Let's call it as $\mathcal{L}_i = \mathcal{A}(:,i)^T$, and replace the "1's" to $\frac{q_i}{\#N_i}$, and "0's" to $\frac{(1-q_i)}{\#N_{-i}}$. So in this way we will find the matrix which shows the weights from each agent side, we will call it as $\mathcal{D}_i = \mathcal{L}_i \circ \mathcal{A}$.[74]

$$\mathcal{D}_1 = \begin{bmatrix} \frac{q_1}{\#N_1} & \frac{q_1}{\#N_1} & \frac{q_1}{\#N_1} & 0 & 0 & 0 \\ \frac{q_1}{\#N_1} & \frac{q_1}{\#N_1} & \frac{q_1}{\#N_1} & 0 & 0 & 0 \\ \frac{q_1}{\#N_1} & \frac{q_1}{\#N_1} & \frac{q_1}{\#N_1} & \frac{q_1}{\#N_1} & 0 & 0 \\ 0 & 0 & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} \\ 0 & 0 & 0 & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} \\ 0 & 0 & 0 & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} \end{bmatrix}; \mathcal{D}_2 = \begin{bmatrix} \frac{q_2}{\#N_2} & \frac{q_2}{\#N_2} & \frac{q_2}{\#N_2} & 0 & 0 & 0 \\ \frac{q_2}{\#N_2} & \frac{q_2}{\#N_2} & \frac{q_2}{\#N_2} & 0 & 0 & 0 \\ \frac{q_2}{\#N_2} & \frac{q_2}{\#N_2} & \frac{q_2}{\#N_2} & \frac{q_2}{\#N_2} & 0 & 0 \\ 0 & 0 & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} \\ 0 & 0 & 0 & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} \\ 0 & 0 & 0 & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} \end{bmatrix};$$

$$\mathcal{D}_3 = \begin{bmatrix} \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & 0 & 0 & 0 \\ \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & 0 & 0 & 0 \\ \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & 0 & 0 \\ 0 & 0 & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} \\ 0 & 0 & 0 & \frac{(1-q_3)}{\#N_{-3}} & \frac{(1-q_3)}{\#N_{-3}} & \frac{(1-q_3)}{\#N_{-3}} \\ 0 & 0 & 0 & \frac{(1-q_3)}{\#N_{-3}} & \frac{(1-q_3)}{\#N_{-3}} & \frac{(1-q_3)}{\#N_{-3}} \end{bmatrix}; \quad \mathcal{D}_4 = \begin{bmatrix} \frac{(1-q_4)}{\#N_{-4}} & \frac{(1-q_4)}{\#N_{-4}} & \frac{(1-q_4)}{\#N_{-4}} & 0 & 0 & 0 \\ \frac{(1-q_4)}{\#N_{-4}} & \frac{(1-q_4)}{\#N_{-4}} & \frac{(1-q_4)}{\#N_{-4}} & 0 & 0 & 0 \\ \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & 0 & 0 \\ 0 & 0 & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} \\ 0 & 0 & 0 & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} \\ 0 & 0 & 0 & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} \end{bmatrix};$$

$$\mathcal{D}_5 = \begin{bmatrix} \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & 0 & 0 & 0 \\ \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & 0 & 0 & 0 \\ \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & 0 & 0 \\ 0 & 0 & \frac{q_5}{\#N_5} & \frac{q_5}{\#N_5} & \frac{q_5}{\#N_5} & \frac{q_5}{\#N_5} \\ 0 & 0 & 0 & \frac{q_5}{\#N_5} & \frac{q_5}{\#N_5} & \frac{q_5}{\#N_5} \\ 0 & 0 & 0 & \frac{q_5}{\#N_5} & \frac{q_5}{\#N_5} & \frac{q_5}{\#N_5} \end{bmatrix}; \mathcal{D}_6 = \begin{bmatrix} \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & 0 & 0 & 0 \\ \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & 0 & 0 & 0 \\ \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & 0 & 0 \\ 0 & 0 & \frac{q_6}{\#N_6} & \frac{q_6}{\#N_6} & \frac{q_6}{\#N_6} & \frac{q_6}{\#N_6} \\ 0 & 0 & 0 & \frac{q_6}{\#N_6} & \frac{q_6}{\#N_6} & \frac{q_6}{\#N_6} \\ 0 & 0 & 0 & \frac{q_6}{\#N_6} & \frac{q_6}{\#N_6} & \frac{q_6}{\#N_6} \end{bmatrix};$$

[74] Please note that when we will consider network formation process, the only matrix which which will change and influence to decision send an invitation or cause an annoyances, is matrix $\mathcal{D}_i$.

After describing matrix C_i and $\mathcal{D}_i$, in our paper we need a Hadamard product of this two matrices $\mathcal{E}_i = D_i \circ C_i$.

$$\mathcal{E}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{q_1}{\#N_1} & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} \\ 0 & 0 & 0 & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} \\ 0 & 0 & 0 & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} \end{bmatrix}; \quad \mathcal{E}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{q_2}{\#N_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} \\ 0 & 0 & 0 & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} \\ 0 & 0 & 0 & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} \end{bmatrix};$$

$$\mathcal{E}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} \\ 0 & 0 & 0 & 0 & \frac{(1-q_3)}{\#N_{-3}} & \frac{(1-q_3)}{\#N_{-3}} \\ 0 & 0 & 0 & 0 & \frac{(1-q_3)}{\#N_{-3}} & \frac{(1-q_3)}{\#N_{-3}} \end{bmatrix}; \quad \mathcal{E}_4 = \begin{bmatrix} \frac{(1-q_4)}{\#N_{-4}} & \frac{(1-q_4)}{\#N_{-4}} & 0 & 0 & 0 & 0 \\ \frac{(1-q_4)}{\#N_{-4}} & \frac{(1-q_4)}{\#N_{-4}} & 0 & 0 & 0 & 0 \\ \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\mathcal{E}_5 = \begin{bmatrix} \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & 0 & 0 & 0 \\ \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & 0 & 0 & 0 \\ \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & 0 & 0 & 0 \\ 0 & 0 & \frac{q_5}{\#N_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad \mathcal{E}_6 = \begin{bmatrix} \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & 0 & 0 & 0 \\ \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & 0 & 0 & 0 \\ \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & 0 & 0 & 0 \\ 0 & 0 & \frac{q_6}{\#N_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

So we solve the first part of optimal action, now we need to get how agent i's signal depends on signals which he gets. Let's introduce the matrix F_i, For every agent i I take the i-th row from the matrix $\mathcal{A}$ and build a new matrix F_i where other rows are 0s.

$$F_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad F_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; F_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$F_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad F_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad F_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix};$$

For finding the last part of optimal action[75] , we will Introduce the matrix $G_i = \mathcal{A} - C_i - F_i$. The matrix G_i shows the signals which agent i knows which other people gets.

[75]For example, if we look from the first agent side we need to find the following part. $r_1 \frac{q_1}{\#N_1} [\omega_{21} + \omega_{31}] x_1 + r_1 \frac{q_1}{\#N_1} [\omega_{22} + \omega_{32}] x_2 +$
$r_1 \left[\frac{q_1}{\#N_1} [\omega_{23} + \omega_{33}] + \frac{(1-q_1)}{\#N_{-1}} \omega_{43} \right] x_3$. So we need to get the weights before agent private signal.

$$G_1 = \begin{bmatrix} 0&0&0&0&0&0\\ 1&1&1&0&0&0\\ 1&1&1&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}; \quad G_2 = \begin{bmatrix} 1&1&1&0&0&0\\ 0&0&0&0&0&0\\ 1&1&1&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}; G_3 = \begin{bmatrix} 1&1&1&0&0&0\\ 1&1&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&1&0&0\\ 0&0&0&1&0&0\\ 0&0&0&1&0&0 \end{bmatrix};$$

$$G_4 = \begin{bmatrix} 0&0&1&0&0&0\\ 0&0&1&0&0&0\\ 0&0&1&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&1&1\\ 0&0&0&1&1&1 \end{bmatrix}; \quad G_5 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&1&1&1\\ 0&0&0&0&0&0\\ 0&0&0&1&1&1 \end{bmatrix}; \quad G_6 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&1&1&1\\ 0&0&0&1&1&1\\ 0&0&0&0&0&0 \end{bmatrix};$$

The private information which agent i gets, can be part of linear strategy not only his friends, but from neighbors of his friends, so we need to multiply by elements the matrix G_i and D_i. The matrix $H_i = D_i \circ G_i$.

$$H_1 = \begin{bmatrix} 0&0&0&0&0&0\\ \frac{q_1}{\#N_1}&\frac{q_1}{\#N_1}&\frac{q_1}{\#N_1}&0&0&0\\ \frac{q_1}{\#N_1}&\frac{q_1}{\#N_1}&\frac{q_1}{\#N_1}&0&0&0\\ 0&0&\frac{(1-q_1)}{\#N_{-1}}&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}; \quad H_2 = \begin{bmatrix} \frac{q_2}{\#N_2}&\frac{q_2}{\#N_2}&\frac{q_2}{\#N_2}&0&0&0\\ 0&0&0&0&0&0\\ \frac{q_2}{\#N_2}&\frac{q_2}{\#N_2}&\frac{q_2}{\#N_2}&0&0&0\\ 0&0&\frac{(1-q_2)}{\#N_{-2}}&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix};$$

$$H_3 = \begin{bmatrix} \frac{q_3}{\#N_3}&\frac{q_3}{\#N_3}&\frac{q_3}{\#N_3}&0&0&0\\ \frac{q_3}{\#N_3}&\frac{q_3}{\#N_3}&\frac{q_3}{\#N_3}&0&0&0\\ 0&0&0&0&0&0\\ 0&0&\frac{q_3}{\#N_3}&\frac{q_3}{\#N_3}&0&0\\ 0&0&0&\frac{(1-q_3)}{\#N_{-3}}&0&0\\ 0&0&0&\frac{(1-q_3)}{\#N_{-3}}&0&0 \end{bmatrix}; \quad H_4 = \begin{bmatrix} 0&0&\frac{(1-q_4)}{\#N_{-4}}&0&0&0\\ 0&0&\frac{(1-q_4)}{\#N_{-4}}&0&0&0\\ 0&0&\frac{q_4}{\#N_4}&\frac{q_4}{\#N_4}&0&0\\ 0&0&0&0&0&0\\ 0&0&0&\frac{q_4}{\#N_4}&\frac{q_4}{\#N_4}&\frac{q_4}{\#N_4}\\ 0&0&0&\frac{q_4}{\#N_4}&\frac{q_4}{\#N_4}&\frac{q_4}{\#N_4} \end{bmatrix};$$

$$H_5 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&\frac{(1-q_5)}{\#N_{-5}}&0&0\\ 0&0&0&\frac{q_5}{\#N_5}&\frac{q_5}{\#N_5}&\frac{q_5}{\#N_5}\\ 0&0&0&0&0&0\\ 0&0&0&\frac{q_5}{\#N_5}&\frac{q_5}{\#N_5}&\frac{q_5}{\#N_5} \end{bmatrix}; \quad H_6 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&\frac{(1-q_6)}{\#N_{-6}}&0&0\\ 0&0&0&\frac{q_6}{\#N_6}&\frac{q_6}{\#N_6}&\frac{q_6}{\#N_6}\\ 0&0&0&\frac{q_6}{\#N_6}&\frac{q_6}{\#N_6}&\frac{q_6}{\#N_6}\\ 0&0&0&0&0&0 \end{bmatrix};$$

Before getting the weights, lets analyze the expectation side. As we explain in the previous section the $E_t(\theta_t)$ depends on the private and public signals which every agents have.

$$E_1(\theta_1) = E_2(\theta_1) = \frac{\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \alpha y_1}{\beta_1 + \beta_2 + \beta_3 + \alpha}; \quad E_3(\theta_1) = \frac{\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \alpha y_1}{\beta_1 + \beta_2 + \beta_3 + \beta_4 + \alpha};$$

$$E_4(\theta_1) = \frac{\beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6 + \alpha y_1}{\beta_3 + \beta_4 + \beta_5 + \beta_6 + \alpha}; \quad E_4(\theta_1) = E_5(\theta_1) = \frac{\beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6 + \alpha y_1}{\beta_4 + \beta_5 + \beta_6 + \alpha};$$

I will write the algorithm for finding the optimal weights before private signals and biases. The optimal weight of public signal can be find using the optimal weight of private signals and biases. Now let consider the precision side. Let introduce β which is a vector of all private signal precision. We will use command *repmat* which will repeat this vector. For example in our example $N = 6$; we will $\beta = repmat(\beta, N, 1)$, which repeat the row β six time.

$$\beta = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \end{bmatrix}$$

For getting the precision which is known from all agents side, let's multiply every element of the matrix β with matrix A. $\beta_{new} = \beta \circ \mathcal{A}$.

$$\beta_{new} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & 0 & 0 \\ 0 & 0 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ 0 & 0 & 0 & \beta_4 & \beta_5 & \beta_6 \\ 0 & 0 & 0 & \beta_4 & \beta_5 & \beta_6 \end{bmatrix};$$

Now we need to sum the row of β_{new} and plus α, which is precision of public signal. The precision of public signal is common knowledge.

$$norm = \text{sum}\left(\beta_{new}\ , 2\right) + \alpha \circ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Longleftrightarrow norm = \begin{bmatrix} \beta_1 + \beta_2 + \beta_3 + \alpha \\ \beta_1 + \beta_2 + \beta_3 + \alpha \\ \beta_1 + \beta_2 + \beta_3 + \beta_4 + \alpha \\ \beta_3 + \beta_4 + \beta_5 + \beta_6 + \alpha \\ \beta_4 + \beta_5 + \beta_6 + \alpha \\ \beta_4 + \beta_5 + \beta_6 + \alpha \end{bmatrix};$$

Using the command $repmat(norm, 1, N)$ gives us the matrix R_{norm} which repeat the column of the matrix $norm$ N times.

$$R_{norm} = \begin{bmatrix} \beta_1+\beta_2+\beta_3+\alpha & . & . & . & . & \beta_1+\beta_2+\beta_3+\alpha \\ \beta_1+\beta_2+\beta_3+\alpha & . & . & . & . & \beta_1+\beta_2+\beta_3+\alpha \\ \beta_1+\beta_2+\beta_3+\beta_4+\alpha & . & . & . & . & \beta_1+\beta_2+\beta_3+\beta_4+\alpha \\ \beta_3+\beta_4+\beta_5+\beta_6+\alpha & . & . & . & . & \beta_3+\beta_4+\beta_5+\beta_6+\alpha \\ \beta_4+\beta_5+\beta_6+\alpha & . & . & . & . & \beta_4+\beta_5+\beta_6+\alpha \\ \beta_4+\beta_5+\beta_6+\alpha & . & . & . & . & \beta_4+\beta_5+\beta_6+\alpha \end{bmatrix};$$

After defining the R_{norm} , we can find a new precision matrix $P = \beta_{new} \circ \div R_{norm}$.

$$P = \begin{bmatrix} \frac{\beta_1}{\beta_1+\beta_2+\beta_3+\alpha} & \frac{\beta_2}{\beta_1+\beta_2+\beta_3+\alpha} & \frac{\beta_3}{\beta_1+\beta_2+\beta_3+\alpha} & 0 & 0 & 0 \\ \frac{\beta_1}{\beta_1+\beta_2+\beta_3+\alpha} & \frac{\beta_2}{\beta_1+\beta_2+\beta_3+\alpha} & \frac{\beta_3}{\beta_1+\beta_2+\beta_3+\alpha} & 0 & 0 & 0 \\ \frac{\beta_1}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha} & \frac{\beta_2}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha} & \frac{\beta_3}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha} & \frac{\beta_4}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha} & 0 & 0 \\ 0 & 0 & \frac{\beta_3}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha} & \frac{\beta_4}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha} & \frac{\beta_5}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha} & \frac{\beta_6}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha} \\ 0 & 0 & 0 & \frac{\beta_4}{\beta_4+\beta_5+\beta_6+\alpha} & \frac{\beta_5}{\beta_4+\beta_5+\beta_6+\alpha} & \frac{\beta_6}{\beta_4+\beta_5+\beta_6+\alpha} \\ 0 & 0 & 0 & \frac{\beta_4}{\beta_4+\beta_5+\beta_6+\alpha} & \frac{\beta_5}{\beta_4+\beta_5+\beta_6+\alpha} & \frac{\beta_6}{\beta_4+\beta_5+\beta_6+\alpha} \end{bmatrix};$$

So after defining all matrices which we need to find the optimal weights, we will describe a large matrix $\mathcal{Z}$ which will combine $\mathcal{Z}_1$,$\mathcal{Z}_2$,$\mathcal{Z}_3$. At first, Let rearrange and put together the expression before $E_i(\theta)$.So let's introduce each elements of $\mathcal{Z}$.

For introducing $\mathcal{Z}_1$ at first i will introduce N blocks of matrices.

$$\mathcal{Z}_1 = \begin{bmatrix} \mathcal{Z}_1^1 \\ \mathcal{Z}_1^2 \\ \mathcal{Z}_1^{\cdot} \\ \mathcal{Z}_1^{\cdot} \\ \mathcal{Z}_1^{N-1} \\ \mathcal{Z}_1^N \end{bmatrix} = \begin{bmatrix} repmat\left(reshape\left(E_1^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(E_2^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(E_{\cdot}^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(E_{\cdot}^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(E_{N-1}^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(E_N^T, 1, N*N\right), N, 1\right) \end{bmatrix}$$

So in matrix $\mathcal{Z}$ the $\mathcal{Z}_1$ represent a block of constant multiply to $E_i\left(\theta_t\right)$. We need to multiply by elements this matrix with precision matrix. So we need to rearrange the precision matrix and the matrix which represent the the weight on conformity(r_i). The new precision matrix looks like $P_{new} = reshape\left(P^T, N*N, 1\right)$. The matrix $R_{new} = reshape\left(repmat\left(R^T, 1, N\right)^T, N*N, 1\right)$, where $R = \begin{bmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 \end{bmatrix}$.

Now we will introduce the block $\mathcal{Z}_2$, which consists of N blocks matrices.

$$\mathcal{Z}_2 = \begin{bmatrix} \mathcal{Z}_2^1 \\ \mathcal{Z}_2^2 \\ \mathcal{Z}_2^{.} \\ \mathcal{Z}_2^{.} \\ \mathcal{Z}_2^{N-1} \\ \mathcal{Z}_2^N \end{bmatrix} = \begin{bmatrix} repmat\left(reshape\left(H_1^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(H_2^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(H_.^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(H_.^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(H_{N-1}^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(H_N^T, 1, N*N\right), N, 1\right) \end{bmatrix}$$

The third block $\mathcal{Z}_3$ is the identity matrix $\mathcal{Z}_3 = repmat(eye(N), N, N)$, which represent the private signal set, the first row represent x_1, the second row represent the x_2 and so on.

$$\mathcal{Z}_3 = repmat(\text{eye}(N), N, N), \text{where eye}(N) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ . & 1 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 1 & . \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So let introduce Matrix $\mathcal{X}$,which is in $(N*N)*(N*N)$ matrix.

$$\mathcal{X} = \boldsymbol{eye}\,(N*N) - R.{*}P_{new}.{*}\mathcal{Z}_1 - R.{*}\mathcal{Z}_2.{*}\mathcal{Z}_3$$

So the matrix of weights looks like the following

$$\mathcal{X} * \begin{bmatrix} \omega_{1,1} \\ . \\ \omega_{1,N} \\ \omega_{2,1} \\ . \\ \omega_{2,N} \\ . \\ \omega_{N-1,1} \\ \omega_{N-1,N-1} \\ \omega_{N,1} \\ . \\ \omega_{N,N} \end{bmatrix} = (\mathcal{I} - R)\,.{*}P_{new}$$

Therefore, the optimal weights matrix will be,

$$\mathcal{W}_x^* = (\mathcal{X}_x)^{-1}\,(\mathbf{1} - R)\,P_{new} \tag{A.1.2}$$

Now let's find the optimal weight for bias. At first I will introduce the row-vector of

$b = \begin{bmatrix} b_1 & b_2 & . & . & . & b_N \end{bmatrix}$. We will use a command $b_{new} = repmat(\beta, N, 1)$.

$$b^*_{new} = b_{new} \circ \mathcal{A} \circ \begin{bmatrix} 0 & 1 & . & . & . & 1 \\ 1 & 0 & . & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & . & . & . & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b_2 & b_3 & 0 & 0 & 0 \\ . & 0 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & b_{N-2} & 0 & b_N \\ 0 & 0 & 0 & b_{N-2} & b_{N-1} & 0 \end{bmatrix}$$

$$b_{+-} = \begin{bmatrix} -1 & . & -1_{N_{1,m}} & 1_{N_{1,k}} & . & 1_{N_{1,\text{end}}} \\ . & . & . & . & . & . \\ -1_{N_{m,1}} & . & -1_{N_{m,m}} & 1_{N_{m,k}} & . & 1_{N_{m,end}} \\ 1_{N_{k,1}} & . & 1_{N_{k,m}} & -1_{N_{k,k}} & . & -1_{N_{k,end}} \\ . & . & . & . & . & . \\ 1_{N_{N,1}} & . & 1_{N_{N,m}} & -1_{N_{\mathrm{N,k}}} & . & -1_{N_{N,N}} \end{bmatrix}$$

$$\mathcal{X}_b = \begin{bmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 1 & 0 \\ 0 & . & . & . & . & 1 \end{bmatrix} + \begin{bmatrix} \frac{r_1}{b_1} \frac{q_1}{\#N_1} \\ \frac{r_2}{b_2} \frac{q_2}{\#N_2} \\ \frac{r_.}{b_.} \frac{q_.}{\#N_.} \\ \frac{r_.}{b_.} \frac{q_.}{\#N_.} \\ \frac{r_{N-1}}{b_{N-1}} \frac{q_{N-1}}{\#N_{N-1}} \\ \frac{r_N}{b_N} \frac{q_N}{\#N_N} \end{bmatrix} \circ (b^*_{new} \circ b_{+-}) + \begin{bmatrix} \frac{r_1}{b_1} \frac{(1-q_1)}{\#N_{-1}} \\ \frac{r_2}{b_2} \frac{(1-q_2)}{\#N_{-2}} \\ \frac{r_.}{b_.} \frac{(1-q_.)}{\#N_{-.}} \\ \frac{r_.}{b_.} \frac{(1-q_.)}{\#N_{-.}} \\ \frac{r_{N-1}}{b_{N-1}} \frac{(1-q_{N-1})}{\#N_{-(N-1)}} \\ \frac{r_N}{b_N} \frac{(1-q_N)}{\#N_{-N}} \end{bmatrix} \circ (b^*_{new} \circ b_{+-})$$

The optimal weights $\mathcal{W}_b$

$$\mathcal{W}^*_b = (\mathcal{X}_b)^{-1} * \begin{bmatrix} (1 - r_1) \\ (1 - r_.) \\ (1 - r_.) \\ (1 - r_.) \\ (1 - r_N) \end{bmatrix} \tag{A.1.3}$$

The expressions (A.1.2) and (A.1.3) will give us the optimal weights for private signal and bias, and we can find the optimal weight for public signal.

$$\mathcal{W}_y = \mathbf{1} - \mathcal{W}_x - \mathcal{W}_b \tag{A.1.4}$$

3.7.D Calculating the value functions

The value functions are equal to $E(u_i(a^*, \theta))$. Using equation (3.10), we find the optimal action a_i^* of each agent, and then we put the optimal action a_i^* into the expected utility function and find the expected utility from everyone's side.

We introduce some matrices that can in decreasing the size of equations.

$$\bar{x}\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_N \end{bmatrix}; \bar{b} = \left(\begin{bmatrix} b_1 \\ . \\ . \\ . \\ b_N \end{bmatrix} \circ \begin{bmatrix} 1 \\ 1 \\ 1_m \\ -1_k \\ -1_{\text{end}} \end{bmatrix} \right); \bar{\omega}_y = \begin{bmatrix} \omega_{1,y} \\ \omega_{2,y} \\ . \\ . \\ \omega_{N,y} \end{bmatrix}; \bar{\omega}_b = \begin{bmatrix} \omega_{1,b} \\ \omega_{2,b} \\ . \\ . \\ \omega_{N,b} \end{bmatrix}; \bar{\bar{\beta}} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ . \\ . \\ \beta_N \end{bmatrix};$$

$$\begin{bmatrix} E_1(\theta) \\ E_2(\theta) \\ . \\ . \\ E_N(\theta) \end{bmatrix} = \left(\frac{\left[\mathcal{A} \circ \begin{bmatrix} X_1 \\ . \\ . \\ . \\ X_N \end{bmatrix} .* \begin{bmatrix} \beta_1 \\ . \\ . \\ . \\ \beta_N \end{bmatrix} + (\alpha * y) \circ \begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix} \right] \circ\div}{\mathcal{A} \circ \begin{bmatrix} \beta_1 \\ . \\ . \\ . \\ \beta_N \end{bmatrix} + \alpha \circ \begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}} \right)$$

$$E^*(\theta^2) = \begin{bmatrix} E_1(\theta^2) \\ E_2(\theta^2) \\ . \\ . \\ E_N(\theta^2) \end{bmatrix} = \left(\frac{\left[\mathcal{A} \circ \begin{bmatrix} X_1 \\ . \\ . \\ . \\ X_N \end{bmatrix} .* \begin{bmatrix} \beta_1 \\ . \\ . \\ . \\ \beta_N \end{bmatrix} + (\alpha * y) \circ \begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix} \right] \circ\div}{\mathcal{A} \circ \begin{bmatrix} \beta_1 \\ . \\ . \\ . \\ \beta_N \end{bmatrix} + \alpha \circ \begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}} \right)^2 + \left(\frac{\begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix} \circ\div}{\mathcal{A} \circ \begin{bmatrix} \beta_1 \\ . \\ . \\ . \\ \beta_N \end{bmatrix} + \alpha \circ \begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}} \right)$$

where matrix $\mathcal{A}$ represents the adjacency matrix plus identity matrix $\mathcal{A} = \boldsymbol{M}_t + eye(N)$. The operation "$\circ$" denotes element-by-element multiplication, $\circ\div$ denotes element-by-element division, and $\overset{2}{\circ}$ each element in the matrix are squared.

The calculations are summarized by,

$$\begin{bmatrix} E(u_i(a,\theta)) \\ . \\ . \\ . \\ . \\ E(u_N(a,\theta)) \end{bmatrix} = - \begin{bmatrix} a_1^2 \\ . \\ . \\ . \\ . \\ a_N^2 \end{bmatrix} - \begin{bmatrix} (1-r_1) \\ . \\ . \\ . \\ . \\ (1-r_N) \end{bmatrix} \circ \begin{bmatrix} E_1(\theta^2) \\ E_2(\theta^2) \\ . \\ . \\ E_N(\theta^2) \end{bmatrix} - 2 \circ \bar{b} \circ \begin{bmatrix} (1-r_1) \\ . \\ . \\ . \\ (1-r_N) \end{bmatrix} \circ \begin{bmatrix} E_1(\theta) \\ E_2(\theta) \\ . \\ . \\ E_N(\theta) \end{bmatrix} -$$

$$- \begin{bmatrix} (1-r_1) \\ . \\ . \\ . \\ (1-r_N) \end{bmatrix} \circ \begin{bmatrix} b_1^2 \\ . \\ . \\ . \\ b_N^2 \end{bmatrix} + 2 \begin{bmatrix} (1-r_1) \\ . \\ . \\ . \\ (1-r_N) \end{bmatrix} \circ \begin{bmatrix} a_1 \\ . \\ . \\ . \\ a_N \end{bmatrix} \circ \begin{bmatrix} E_1(\theta) \\ E_2(\theta) \\ . \\ . \\ E_N(\theta) \end{bmatrix} + 2 \begin{bmatrix} (1-r_1) \\ . \\ . \\ . \\ (1-r_N) \end{bmatrix} \circ \bar{b} \circ \begin{bmatrix} a_1 \\ . \\ . \\ . \\ a_N \end{bmatrix} -$$

$$- \begin{bmatrix} r_1 \\ . \\ . \\ r_N \end{bmatrix} \circ \begin{bmatrix} (diag(\mathcal{D}_1))^T * ((\mathcal{G}_1 \circ \mathcal{W}) * \bar{x})_\circ^2 \\ . \\ . \\ (diag(\mathcal{D}_N))^T * ((\mathcal{G}_1 \circ \mathcal{W}) * \bar{x})_\circ^2 \end{bmatrix} -$$

$$- \begin{bmatrix} r_1 \\ . \\ . \\ . \\ r_N \end{bmatrix} \circ \left[\begin{bmatrix} diag(D_1)^T \\ . \\ . \\ . \\ diag(D_N) \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & . & . & . & 1 \\ 1 & 0 & . & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & . & . & . & 1 & 0 \end{bmatrix} \right] * \left[\bar{\omega}_b \circ \bar{b} + y \circ \bar{\omega}_{1,y} \right]_\circ^2 -$$

$$-2 \begin{bmatrix} r_1 \\ . \\ . \\ . \\ r_N \end{bmatrix} \circ \begin{bmatrix} diag(D_1)^T * \left(((\mathcal{G}_1 \circ \mathcal{W}) * \bar{x}) \circ \left[\bar{\omega}_b \circ \bar{b} + y \circ \bar{\omega}_{1,y} \right] \right) \\ . \\ . \\ . \\ diag(D_N)^T * \left(((\mathcal{G}_N \circ \mathcal{W}) * \bar{x}) \circ \left[\bar{\omega}_b \circ \bar{b} + y \circ \bar{\omega}_{1,y} \right] \right) \end{bmatrix} -$$

$$- \begin{bmatrix} r_1 \\ . \\ . \\ . \\ r_N \end{bmatrix} \circ \begin{bmatrix} diag(D_1)^T * \left(\left(\left(\begin{bmatrix} 1 & . & . & . & 1 \end{bmatrix} * (\mathcal{C}_1 \circ \mathcal{W})^T \right)^T \right)_.^2 \right) \\ . \\ . \\ . \\ diag(D_N)^T * \left(\left(\left(\begin{bmatrix} 1 & . & . & . & 1 \end{bmatrix} * (\mathcal{C}_N \circ \mathcal{W})^T \right)^T \right)_.^2 \right) \end{bmatrix} \circ \begin{bmatrix} E_1(\theta^2) \\ E_2(\theta^2) \\ . \\ . \\ E_N(\theta^2) \end{bmatrix} -$$

$$- \begin{bmatrix} r_1 \\ . \\ . \\ . \\ r_N \end{bmatrix} \circ \begin{bmatrix} diag(\mathcal{D}_1)^T * \left((\mathcal{C}_1 \circ \mathcal{W})_\circ^2 * \begin{bmatrix} \frac{1}{\beta_1} \\ . \\ . \\ \frac{1}{\beta_N} \end{bmatrix} \right) \\ . \\ . \\ diag(\mathcal{D}_N)^T * \left((\mathcal{C}_N \circ \mathcal{W})_\circ^2 * \begin{bmatrix} \frac{1}{\beta_1} \\ . \\ . \\ \frac{1}{\beta_N} \end{bmatrix} \right) \end{bmatrix} -$$

$$-2\circ\begin{bmatrix}r_1\\ .\\ .\\ .\\ r_N\end{bmatrix}\circ\begin{bmatrix}diag(\mathcal{D}_1)^T*(((G_1\circ\mathcal{W})*\bar{x})\circ((\mathcal{C}_1\circ\mathcal{W})*\mathbf{1}))\\ .\\ .\\ diag(\mathcal{D}_N)^T*(((\mathcal{G}_N\circ\mathcal{W})*\bar{x})\circ((\mathcal{C}_N\circ\mathcal{W})*\mathbf{1}))\end{bmatrix}\circ\begin{bmatrix}E_1(\theta)\\ E_2(\theta)\\ .\\ .\\ E_N(\theta)\end{bmatrix}-$$

$$-2\circ\begin{bmatrix}r_1\\ .\\ .\\ .\\ r_N\end{bmatrix}\circ\begin{bmatrix}diag(D_1)^T*\left(\left(\bar{\omega}_b\circ\bar{b}+\bar{\omega}_y*y\right)\circ(\mathcal{C}_1\circ\mathcal{W}*\mathbf{1})\right)\\ .\\ .\\ .\\ diag(D_N)^T*\left(\left(\bar{\omega}_b\circ\bar{b}+\bar{\omega}_y*y\right)\circ(\mathcal{C}_N\circ\mathcal{W}*\mathbf{1})\right)\end{bmatrix}\circ\begin{bmatrix}E_1(\theta)\\ E_2(\theta)\\ .\\ .\\ E_N(\theta)\end{bmatrix}-$$

$$+2\circ\circ\begin{bmatrix}r_1\\ .\\ .\\ r_N\end{bmatrix}\circ\begin{bmatrix}a_1\\ .\\ .\\ a_2\end{bmatrix}\circ\begin{bmatrix}diag(D_1)^T*\left((\mathcal{G}_1\circ\mathcal{W})*\bar{x}+(\mathcal{C}_1\circ\mathcal{W})*E(\theta)+\left(\bar{\omega}_b\circ\bar{b}+\bar{\omega}_y\circ\bar{y}\right)\right)\\ .\\ .\\ diag(D_N)^T*\left((\mathcal{G}_N\circ\mathcal{W})*\bar{x}+(\mathcal{C}_N\circ\mathcal{W})*E(\theta)+\left(\bar{\omega}_b\circ\bar{b}+\bar{\omega}_y\circ\bar{y}\right)\right)\end{bmatrix}-$$

$$-2\circ\begin{bmatrix}r_1\\ .\\ .\\ r_N\end{bmatrix}\circ\begin{bmatrix}diag(D_1)^T*\sum_{i=1}^{N}(\mathcal{M}(:,i)\circ\mathcal{M}*\mathbf{1})\\ .\\ .\\ diag(D_N)^T*\sum_{i=1}^{N}(\mathcal{M}(:,i)\circ\mathcal{M}*\mathbf{1})\end{bmatrix}\circ E(\theta)^2_\circ$$

where $\mathcal{M}=\mathcal{C}_{1,new}\circ\mathcal{W}_{\text{private}}$ and $\mathcal{M}=\mathcal{M}|_{\{(M_1(:,i-1,i-2,\ldots i-N)=0\}}$.So when we take the first row from the matrix $\mathcal{M}$, we replace 0's in their place. The next step, when we take the second row, we will keep the matrix which we get before(with the first row equal to 0's) and we change and put the second row $=0$.

3.7.E Proof of the static-game result

As we can see in Figures 3a, 3b, and 3c, the final period graphs looks like combination of star networks with ring networks. Therefore, for simplification, we compare "central agent", "central sender" and "central receiver" cases. Below we describe the optimal private signal weight.

Star network joint with ring network. The central agent sends and gets the private signals of others

$$a_i=w_1\frac{x_i+x_{i-1}+x_{i+1}}{3}+w_2x_c+(1-w_1-w_2)y$$

$$a_c = \omega_1 x_c + \omega_2 \sum_{i=1}^{N-1} \frac{x_i}{N-1} + (1-\omega_1-\omega_2)y$$

$$w_c^* = \begin{bmatrix} \omega_1 \\ \omega_2 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -r \\ 0 & \frac{1}{N-1} & -\frac{r}{N-1} & 0 \\ 0 & -\frac{\phi_i}{4}\frac{(1-r)(N-4)}{(N-1)} - \frac{r}{(N-1)^2} & \frac{1}{3} - \frac{\phi_i}{4}\frac{r(N-4)}{(N-1)} - \frac{2}{3}\frac{r}{N-1} & 0 \\ \frac{r}{N-1} & -\frac{\phi_i}{4}\frac{r(N-4)}{(N-1)^2} & -\frac{\phi_i}{4}\frac{r(N-4)}{(N-1)} & 1 - \frac{r(N-2)}{(N-1)} \end{bmatrix}^{-1} \begin{bmatrix} (1-r)\frac{\phi_c}{N} \\ (1-r)\frac{\phi_c}{N} \\ (1-r)\frac{\phi_i}{4} \\ (1-r)\frac{\phi_i}{4} \end{bmatrix} \tag{A.1.5}$$

One centralized agent who gets signals from others without showing his own signal We consider network structure where central agents gets signals from other agents in the networks, but didn't share his own signal.

Linear Strategy for central agents will look like the following way.

$$a_c = \omega_1 x_c + \omega_2 \frac{\sum x_i}{N-1} + (1-\omega_1-\omega_2)y$$

Linear strategy for other agents will look like the following way.

$$a_i = w_1 \frac{x_i + x_{i+1} + x_{i-1}}{3} + (1-\xi)y$$

$$w_r^* = \begin{bmatrix} \omega_1 \\ \omega_2 \\ w_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -r \\ -\frac{\phi_i}{3}\frac{r}{N-1} & -\frac{\phi_i}{3}\frac{r(N-4)}{(N-1)^2} - \frac{r}{(N-1)^2} & \frac{1}{3} - \frac{\phi_i}{3}\frac{r(N-4)}{(N-1)} - \frac{2}{3}\frac{r}{N-1} \end{bmatrix}^{-1} \begin{bmatrix} (1-r)\frac{\phi_c}{N} \\ (1-r)\frac{\phi_c(N-1)}{N} \\ (1-r)\frac{\phi_i}{3} \end{bmatrix} \tag{A.1.6}$$

Optimal weights can be provide by request. We use this matrices to analyze the social welfare.

Central agent sends information without getting any signal Let's look at the linear strategy of agents.

$$a_i = w_1 \frac{x_i + x_{i+1} + x_{i-1}}{3} + w_2 x_c + (1 - \omega_1 - \omega_2) y$$

$$a_c = \omega_c x_c + (1 - \xi) y$$

$$\omega = \frac{v}{v+h}((1-r) + r\omega_1) + r\omega_2 \tag{A.1.7}$$

So we can find the optimal weights ω_1, w_1 and w_2.

$$w_s^* = \begin{bmatrix} \omega_1 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & -r\frac{\beta}{\alpha+\beta} & -r \\ 0 & \frac{1}{3} - \frac{\phi_i}{3}\frac{r(N-4)}{(N-1)} - \frac{2}{3}\frac{r}{N-1} & 0 \\ -\frac{r}{N-1} & -\frac{\phi_i}{4}\frac{r(N-4)}{(N-1)} & 1 - \frac{r(N-2)}{(N-1)} \end{bmatrix}^{-1} \begin{bmatrix} (1-r)\frac{\beta}{\alpha+\beta} \\ (1-r)\frac{\phi_i}{4} \\ (1-r)\frac{\phi_i}{4} \end{bmatrix} \tag{A.1.8}$$

The optimal weights are available in online Appendix, using these optimal weights we can calculate the welfare W_c^*,W_r^* and W_s^*

The optimal welfare for central agent case is described by the following equations. Please notice, that in this example precision is equal for every agents and agents doesn't distinguish weights between friends and non-friends.

$$W_c^* = \frac{1}{N}\left(\frac{\omega_1^{2*}}{\beta} + \frac{\omega_2^{2*}}{(N-1)\,\beta} + \frac{(1-\omega_1^*-\omega_2^*)^2}{\alpha}\right) + \frac{N-1}{N}\left(\frac{w_1^{2*}}{3\beta} + \frac{w_2^{2*}}{\beta} + \frac{(1-w_1^*-w_2^*)^2}{\alpha}\right) \tag{A.1.9}$$

The welfare W_r^* from central reciever side describes by the following equations.

$$W_r^* = \frac{1}{N}\left(\frac{\omega_1^{2*}}{\beta} + \frac{\omega_2^{2*}}{(N-1)\,\beta} + \frac{(1-\omega_1^*-\omega_2^*)^2}{\alpha}\right) + \frac{N-1}{N}\left(\frac{w_1^{2*}}{3\beta} + \frac{(1-w_1^*)^2}{\alpha}\right) \tag{A.1.10}$$

The welfare W_r^* from central sender side describes by the following equations.

$$W_s^* = \frac{1}{N}\left(\frac{\omega_1^{2*}}{\beta} + \frac{(1-\omega_1^*)^2}{\alpha}\right) + \frac{N-1}{N}\left(\frac{w_1^{2*}}{3\beta} + \frac{w_2^{2*}}{\beta} + \frac{(1-w_1^*-w_2^*)^2}{\alpha}\right) \tag{A.1.11}$$

4. Conclusion

In the first chapter, “Symmetric Markovian Games of Commons with Potentially Sustainable Endogenous Growth”, we develop an exact formula for finding the exact interior solution for Markovian differential games with linear accumulation constraints of a common resource. Second, we characterize the general solution, which can be used as a guide for finding corner solutions numerically, using a homotopy approach.

In the second chapter, “Populism and Polarization in Social Media Without Fake News: the Vicious Circle of Biases, Beliefs and Network Homophily”, the cheap way of making internet friends increases the speed of finding friends with similar biases, which increases homophily. In turn, homophily affects the weight that each agent places on their bias, while taking action, and this leads to more homophily. This vicious circle of biases, beliefs, and homophily, increases the peer-induced weight of their pre-existing structural biases that agents put on their actions. Crucailly, , agents gradually ignore expert opinions (unbiased signal) more and more, which matches the trend measured by opinion polls in the past few decades.

In the third chapter, “Can a social planner manipulate network dynamics and solve coordination problems?”, I introduce a “Liberal Social Planner” and find that, indeed, the social planner can indirectly manipulate network dynamics in order to bring agents’ actions closer to fundamentals. I find that the key mechanism behind increasing social welfare is to increase the number of indegree nodes of central agents. This happens because agents can substitute expert information with private information from central nodes and make more informed decisions. Social planners who are more confident (or even sure, even if biased) about the fundamentals (e.g., of pricing houses for buying/selling) achieve better results. These results have potential applications to the management of social media platforms by the owners of these platforms. Platforms can develop robots that can help their users in bcoming more informed and more satisfied about real-life issues, such as housing prices, etc.

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