

# Bargaining in Standing Committees with an Endogenous Default\*

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## Abstract

Committee voting has mostly been investigated from the perspective of the standard Baron-Ferejohn model of bargaining over the division of a pie, in which bargaining ends as soon as the committee reaches an agreement. In standing committees, however, existing agreements can be amended. This paper studies an extension of the Baron-Ferejohn framework to a model with an evolving default that reflects this important feature of policymaking in standing committees: In each of an infinite number of periods, the ongoing default can be amended to a new policy (which is, in turn, the default for the next period). The model provides a number of quite different predictions. (i) From a positive perspective, the key distinction turns on whether the quota is less than unanimity. In that case, patient enough players waste substantial shares of the pie each period and the size principle fails in some pure strategy Markov perfect equilibria. By contrast, the unique Markov perfect equilibrium payoffs in a unanimity committee coincide with those in the corresponding Baron-Ferejohn framework. (ii) If players have heterogeneous discount factors then a large class of subgame perfect equilibria (including all Markov perfect equilibria) are inefficient.

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# 1 Introduction

Committee voting has mostly been investigated from the perspective of the standard Baron-Ferejohn model of bargaining in an ad hoc committee over the division of a single pie: players earn an exogenously fixed default payoff until the committee reaches an agreement, when negotiations end. However, many committees (such as legislatures) are dynamic in two senses: (i) their members reach a sequence of policy agreements (so the committee is *standing*), and (ii) a new pie is divided according to the same proportions as the last pie unless the last agreement is amended (so the default is endogenous). In this paper, we follow a literature initiated by Baron (1996) and Kalandrakis (2004) by studying a model which captures these dynamic aspects of policy making.<sup>1</sup> Each period begins with a default policy (i.e. a division of the pie among players) inherited from the previous period; and a player is randomly drawn to make a proposal which is then voted up or down by the committee; if voted up, the proposal is implemented and becomes the new default; if voted down, the ongoing default is implemented and remains in place until the next period. That is, the default payoff is *endogenous*, rather than exogenously fixed. A pie is available for division each period; and this process continues ad infinitum. This model naturally represents Congressional legislation on social policy and entitlements: the previously agreed law remains in place until Congress decides to amend it.<sup>2</sup>

In further contrast to Baron and Ferejohn (1989), we allow players to have different discount factors, and any concave utility functions; we consider any quota (including majority and unanimity rules); and we allow players to be selected to propose with different probabilities.<sup>3</sup> Analysis of this model of a standing committee raises various interesting (related) questions, such as: (1) When do stationary Markov perfect equilibria (SMPEs) exist and, when they do, are equilibrium payoffs unique? (2) Must each pie be divided between a minimal winning majority — as predicted by the size principle — in every SMPE? (3) Must each pie be fully divided (that is, is the division of the pie statically efficient) in every SMPE? (4) Are equilibria Pareto efficient? (5) How does the endogeneity of the evolving default affect SMPE outcomes? And (6) How do the answers to these questions depend on the quota?

The literature on standing committees (with an endogenous default) has only posed

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<sup>1</sup>We survey this literature in the next section.

<sup>2</sup>Similarly, majority Supreme Court opinions remain in force unless revisited.

<sup>3</sup>Similar extensions are allowed in Banks and Duggan (2000), Eraslan (2002) and Eraslan and McLennan (2013): all bargaining models with a fixed default.

the first two questions. Our contribution is to bypass technical issues which have stymied progress, and thereby to say much more about each of the six questions. We provide the following answers:

**(1) Equilibrium existence and multiplicity of equilibria.** We construct pure strategy SMPEs for any game with a non-unanimity quota and patient enough players, and also prove (again using constructive arguments) that unanimity games possess pure strategy SMPEs, irrespective of patience. However, we have radically different results on multiplicity for games with and without a unanimity quota. We start with the latter case. Take *any* point in the policy space at which at least a minimal winning majority have a positive share of the pie. If players are sufficiently patient then we can construct a pure strategy SMPE in which that policy is implemented in the first period and never amended (a property which we call *no-delay*). By contrast, any game with a unanimity quota has unique SMPE payoffs.

The previous literature (cf. Section 2) has focused on existence of SMPEs in bargaining games with an evolving default. Our results demonstrate that if players are patient enough or if there is a unanimity quota, then existence is not a problem. The multiplicity of equilibria limits the predictive power of the model. Nevertheless, it is interesting that play in some equilibria of standing committees with an endogenous default is consistent with some important stylized facts:

**(2) The size principle.** The size principle predicts that only minimal winning coalitions should receive a positive share of the pie. It has been central to the study of legislatures since Riker (1962), even though majorities in legislatures are typically supraminimal. The class of solutions which we construct for non-unanimity games contains SMPEs in which the pie is shared amongst more than a minimal winning coalition.

**(3) Waste.** Our results on the division of the pie again differ, depending on the quota. We show that SMPE agreements in games without a unanimity quota may waste some of the pie when all players are patient enough. Specifically, for every  $\varepsilon > 0$ , we can construct an SMPE in which a policy which wastes a proportion  $1 - \varepsilon$  of the pie is agreed to in the first period and never amended. By contrast, none of the pie is wasted in an SMPE, irrespective of players' patience, in games with a unanimity quota.

More strongly, players can waste any proportion of the pie in SMPEs of nonunanimity

games which also fail the size principle. Our model is therefore consistent with features which are common in pork barrel politics (cf. Evans (2004)).

**(4) Pareto inefficiency.** If all players share the same discount factor then Pareto efficiency only turns on whether the entire pie is distributed in every period and (with risk-averse players) on whether the policy sequence is deterministic. With heterogeneous discount factors, however, temporal patterns also matter. For instance, our no-delay SMPEs (including those without waste) support policy sequences which can be Pareto improved by operating transfers across periods. More generally, our analysis reveals that Pareto inefficiency is not limited to those SMPEs. If preferences are linear in share of the current pie (as in Baron and Ferejohn (1989)) then, in the generic case where all players have different discount factors, a subgame perfect equilibrium (SPE) can be efficient only if it relies on more complex, player-specific punishments. Dynamic equilibria — i.e. those SPEs in which behavior depends (at most) on the policies implemented in all previous periods — are all Pareto inefficient. Indeed, our characterization of Pareto efficient policy sequences reveals that some player must eventually earn the entire pie in any efficient policy sequence; and no such policy sequence can be played in any dynamic equilibrium of a nonunanimity committee. On the other hand, even if utilities are nonlinear, all dynamic equilibria of a unanimity committee are Pareto inefficient if two or more players have different discount factors. While Pareto efficient policy sequences that allocate the entire pie to the same player in every period can be supported by an SPE, irrespective of players' patience, dynamic equilibria even fail the weaker criterion of *ex post* Pareto efficiency (e.g. Merlo and Wilson (1995)) — which only requires one of the realizations of the equilibrium policy sequence to be Pareto efficient.

**(5) The effects of an endogenous default.** These results stand in sharp contrast to the properties of the Baron-Ferejohn model of an ad hoc committee, in which a single pie is divided. Eraslan (2002) shows that, in a Baron-Ferejohn model with linear utilities, heterogeneous discount factors and any quota, stationary equilibrium payoffs are unique, only minimal winning coalitions form, and none of the pie is wasted.<sup>4</sup> These properties clearly carry over to a couple of dynamic variants with exogenous defaults: in one variant, an ad hoc committee agrees once to the divisions of a sequence of pies; in another variant,

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<sup>4</sup>Ex post efficiency and the size principle hold when players have strictly concave preferences, but uniqueness and ex ante efficiency might fail (because of random proposers).

a standing committee negotiates division of a new pie once it has agreed on division of the existing pie, earning nothing each period till a winning coalition forms. Using those variants as benchmarks, our results imply that default endogeneity has profound implications for standing committees with a nonunanimity quota: Default endogeneity may cause static inefficiency (waste), allow supraminimal coalitions to form, and create a large multiplicity of equilibrium payoffs. None of these properties can hold with a unanimity quota. More strikingly, we show that there is a unique SMPE payoff vector, which coincides with the unique stationary SPE payoff vector in the equivalent Baron-Ferejohn static model (and in its standing committee variant sketched above).

As for Pareto inefficiency, the same conclusion as in our model may also apply to the standing committee model with an exogenous default sketched above. For instance, when utility functions are linear, each player's expected utility is constant across periods in a stationary equilibrium (for the same reasons as in Baron and Ferejohn's (1989) static model); so the equilibrium policy sequence must be inefficient when players have heterogeneous discount factors. By contrast, an ad hoc committee which negotiates over the sequence of pie divisions must reach a Pareto efficient agreement in any equilibrium (because any proposer is a residual claimant). The main difference from our model is that such an ad hoc committee effectively commits not to renegotiate an agreement. This suggests that efficiency may fail in our model because the committee cannot commit not to renegotiate agreements.

**(6) Effect of the quota.** Our positive results above reveal that default endogeneity only matters if the quota is less than unanimity: With a unanimity quota, there is a unique SMPE payoff and the statically efficient policy reached by an ad hoc committee is implemented immediately and never amended; with a lower quota and patient enough players, there is a multiplicity of pure strategy no-delay SMPEs, some of which are statically inefficient. As for normative results, however, default endogeneity matters even with a unanimity quota: if discount factors are heterogeneous then all dynamic equilibria are Pareto inefficient.

We relate our model and results to the literature in the next section. We present our model in Section 3, and provide results on committees with a nonunanimity and a unanimity quota respectively in Sections 4 and 5. We consider the implications of an endogenous default in Section 6. Section 7 concludes. Most of the proofs appear in the

Appendix.

## 2 Related Literature

Baron and Ferejohn (1989) has spawned an enormous literature; we refer readers to Eraslan and McLennan (2013) for a recent list of contributions, including existence and uniqueness results for any quota. The literature on bargaining in standing committees with an endogenous default was initiated by Baron (1996),<sup>5</sup> who established a dynamic median voter theorem in an environment where the policy space is one-dimensional and utilities are single-peaked. Kalandrakis (2004) was then the first to apply the endogenous-default approach to pie-division problems.<sup>6</sup> Despite its relevance, this literature has remained small, most likely for technical reasons: in equilibrium, the proposals which would be accepted may vary discontinuously with the default policy because of expectations about future play. The ensuing discontinuous transition probabilities preclude the use of conventional fixed point arguments to establish existence of even mixed strategy equilibria.

Most of this literature has focused on majority rule games, for obvious reasons; but unanimity rule games are also empirically important because they represent long-run contractual relationships. Our model allows for any quota; and our positive results reveal important differences between unanimity and nonunanimity committees.

Kalandrakis (2004) and Baron and Bowen (2013) study majority rule games with three equally patient, risk neutral players, equiprobable proposers, and a statically efficient initial default; Kalandrakis (2010) extends the model to games with five or more players whose preferences are concave. Kalandrakis (2004) and (2010) show that, for any common discount factor, these games have an SMPE in which the default immediately reaches an ergodic distribution where each proposer takes the entire pie, but players mix over extra-equilibrium proposals;<sup>7</sup> Baron and Bowen construct a no-delay SMPE in which the proposer mixes over her (single) coalition partner.<sup>8</sup> In the SMPEs which we construct, the

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<sup>5</sup>Papers which we do not survey include Gomes and Jehiel (2005, Section III.A), Bernheim et al (2006), Anesi (2010), Diermeier and Fong (2011, 2012), Zápala (2011a,b), Battaglini et al (2012), Bowen and Zahran (2012), Diermeier et al (2013), Nunnari (2014), Anesi and Seidmann (2014), and Bowen et al (forthcoming).

<sup>6</sup>Banks and Duggan (2006) consider an intermediate model, with an arbitrary (possibly statically efficient) status quo.

<sup>7</sup>In contrast, our existence results for nonunanimity committees only apply when players are sufficiently patient.

<sup>8</sup>In Kalandrakis (2004) [resp. Baron and Bowen (2013)], indifferent voters always accept [resp. re-

default reaches a single policy (immediately), and no player mixes, on or off the path.

Duggan and Kalandrakis (2012) use a fixed point argument to establish existence of pure strategy SMPEs for games with any quota in which preferences and the default are subject to stochastic shocks.<sup>9</sup> By contrast, we prove existence in unperturbed games (by and large) using constructive arguments.

Kalandrakis' (2004, 2010) equilibria violate the size principle, in the sense that a *sub-minimal* winning coalition shares the pie. Field evidence (e.g. on appropriations bills in Congress) and lab evidence both suggest that the size principle is more likely to be violated by agreements which share the pie amongst a supraminimal winning coalition.<sup>10</sup> These agreements might, in principle, be explained by social preferences; but Battaglini and Palfrey (like Kalandrakis (2010)) suggest that concavity might be a better explanation. A supramajority of players earn a positive share in (some of) our constructed equilibria, and also in the equilibria constructed by Bowen and Zahran (2012) and Richter (2014):

Bowen and Zahran require preferences to be strictly concave and the initial default to be statically efficient, and show that the size principle is violated when discount factors take intermediate values and the initial default is not too inequitable. We also allow for (but do not require) strictly concave preferences; but the size principle fails in our construction whenever all players are patient enough. Unlike Kalandrakis (2004, 2010) and Bowen and Zahran (2012), we allow the committee to choose statically inefficient policies. However, all our results about existence of SMPEs and the size principle would carry over to the case where the policy set coincides with the unit simplex. In particular, our construction of SMPEs in nonunanimity games relies on the existence of a simple solution (see Definition 1 in Subsection 4.1 below). It is readily checked that there exist simple solutions inside the unit simplex, which violate the size principle. In addition, our constructions do not depend on the initial default (which could also be in the simplex).

Richter (2014) constructs a no-delay SMPE in which the first proposer offers the egalitarian division in a model where offers may waste some of the pie.<sup>11</sup> These offers are only  

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ject]. Indifferent voters respond differently to amendments of policies on and off the equilibrium paths we construct.

<sup>9</sup>Their results apply to a class of stage games which includes pie division. Dziuda and Loeper (2010) also consider a model with preference shocks to study efficiency of SMPEs under unanimity rule. Focusing on preference polarization in environments with a finite policy space, their model does not accommodate pie-division settings.

<sup>10</sup>In Battaglini and Palfrey (2012), 45% of agreed policies were close to the centroid of the simplex.

<sup>11</sup>Baron and Ferejohn (1989) also allow for waste in their model of an ad hoc committee.

made in order to punish deviations from equilibrium play, and are therefore never observed on the path. We also allow for statically inefficient offers; but these offers are made on the equilibrium path in (some of) our constructions.

Baron (1991) argues that Congress often both wastes resources and splits the remainder among a supraminimal majority during distributive bargaining. He shows that closed and open rule models based on Baron and Ferejohn (1989) can explain statically inefficient policies (aka pork), but can only explain these violations of the size principle by appealing to a norm of universalism.<sup>12</sup> By contrast, equilibria in our model exhibit both features.

Seidmann and Winter (1998) and Okada (2000), *inter alia*, study bargaining with an endogenous default in superadditive characteristic function games.<sup>13</sup> Hyndman and Ray (2007) prove that all (including history-dependent) subgame perfect equilibria of games with binding agreements and no externalities are absorbing, and that these equilibria are asymptotically statically efficient if there is a finite number of feasible policies. They also show by example that these results do not carry over to games with externalities. Hyndman and Ray’s results are only applicable in our framework when the quota is unanimity. We exploit their first result when proving that every dynamic equilibrium of a unanimity game is no-delay; their second result also holds in our model (without requiring finiteness). Furthermore, as in Hyndman and Ray’s model with externalities, statically inefficient equilibria exist in our model with a non-unanimity quota. However, Hyndman and Ray focus on asymptotic static efficiency, and assume a common discount factor; we consider Pareto efficiency and, crucially for associated results, allow discount factors to differ.

We turn finally to the no-delay property. Policy outcomes of our no-delay SMPs can be interpreted as a special case of Acemoglu et al’s (2012) “dynamically stable states,” which are defined as political states reached in a finite number of periods (and never changed) in pure strategy SMPs of bargaining games with an endogenous default and patient players. Hence, our results characterize and prove existence of a class of dynamically stable states in voting situations where, in contrast to those studied in Acemoglu et al (2012), the set of policies is infinite and policy preferences are not acyclic. Baron and Bowen’s (2013) notion of a coalition Markov perfect equilibrium exhibits a similar no-delay property; the equilibria they construct are in mixed strategies.

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<sup>12</sup>The size principle holds in open rule games if there are enough players.

<sup>13</sup>Seidmann and Winter focus on equilibria in which the grand coalition forms after a number of steps. While we cannot exclude delay with a non-unanimity quota, our constructions all involve no-delay equilibria.



By definition, the default changes just once in a no-delay equilibrium: policy is persistent. A related literature explains why statically inefficient policies may be persistent (so the policy sequence is inefficient). However, the mechanisms in this literature rely on privately incurred adjustment costs (Coate and Morris (1999)), incomplete information (e.g. Mitchell and Moro (2006)) or the growing power of incumbent factions (Persico et al (2011)). By contrast, no-delay equilibria are inefficient in our model because relatively impatient players cannot commit to decreasing shares of the pie.

### 3 Notation and Definitions

#### 3.1 The Standing Committee Game

In each of an infinite number of discrete periods, indexed  $t = 1, 2, \dots$ , up to a unit of a divisible resource — the “pie” — can be allocated among the members of a committee  $N \equiv \{1, \dots, n\}$ ,  $n \geq 3$ . Thus, the set of feasible policies each period is

$$X \equiv \left\{ (x_1, \dots, x_n) \in [0, 1]^n : \sum_{i=1}^n x_i \leq 1 \right\} .$$

We denote the policy implemented in period  $t$ , and therefore the default at the beginning of period  $t + 1$ , by  $x^t = (x_1^t, \dots, x_n^t)$ . At the start of each period  $t$ , player  $i$  is selected with probability  $p_i \in (0, 1)$  to propose a policy in  $X$ . We say that a player who proposes the existing default *passes*. If the selected proposer passes then the default is implemented; otherwise all players simultaneously vote to accept or to reject the chosen proposal. The voting rule used in every period  $t$  is a quota  $q$  which satisfies  $n/2 < q \leq n$ . Specifically, if at least  $q$  players accept proposal  $y \in X$  then it is implemented as the committee decision in period  $t$  and becomes the default next period (i.e.  $x^t = y$ ); and if  $y$  secures less than  $q$  votes then the previous default,  $x^{t-1}$ , is implemented again and becomes the default in period  $t + 1$  (i.e.  $x^t = x^{t-1}$ ). The default in period 1 is  $x^0 = (0, \dots, 0)$ . We will refer to  $\{x^t\}_{t=1}^\infty$  such that every  $x^t$  is feasible as a *policy sequence*.

Once policy  $x^t$  has been implemented, every player  $i$  receives an instantaneous payoff of  $(1 - \delta_i) u_i(x_i^t)$ , where  $u_i$  is a strictly increasing, continuously differentiable concave utility function with  $u_i(0) = 0$ , and  $\delta_i \in [0, 1)$  is  $i$ 's discount factor. Thus, player  $i$ 's payoff from a policy sequence  $\{x^t\}_{t=1}^\infty$  is  $(1 - \delta_i) \sum_{t=1}^\infty \delta_i^{t-1} u_i(x_i^t)$ . We say that discount factors are *heterogeneous* if  $\delta_i \neq \delta_j$  for some pair of players  $i$  and  $j$ ; and that discount factors are *strictly heterogeneous* if  $\delta_i \neq \delta_j$  for every pair of players  $i$  and  $j$ .

The assumptions above define a dynamic game, which we will refer to as a *standing committee* game. Our main purpose is to analyze the equilibria of this game.

### 3.2 Equilibrium and Absorbing Points

**Equilibrium concept.** We follow the standard approach of concentrating throughout on stage-undominated subgame perfect equilibria (SPEs); i.e., SPEs in which, at any voting stage, no player uses a weakly dominated strategy. This excludes strategy combinations in which players with different preferences all vote one way, and are indifferent when  $q < n$  because they are nonpivotal — put differently, players never vote against their (dynamic) preferences. Henceforth, we leave it as understood that any reference to “equilibria” is to equilibria that satisfy this property.

For our positive analysis, we will concentrate (like the previous literature) on the stricter criterion of stationary Markov perfect equilibria (SMPEs), i.e., SPEs in which all players use strategies which only depend on the current payoff-relevant state: in proposal stages, players’ choices (of probability distributions over  $X$ ) only depend on the ongoing default; in voting stages, players’ choices (of probability distributions over {accept, reject}) only depend on the current default and the proposal just made. We will be particularly interested in pure strategy SMPEs, where every player’s choice is deterministic after every history.

Our normative results exploit a refinement of SPE (“dynamic equilibrium”) which is weaker than SMPEs. We follow Bernheim and Slavov (2009), Vartiainen (2011, 2014), and Anesi and Seidmann (2014, Section 5.2), who consider dynamic voting frameworks in which behavior in every period only depends on the list of policies implemented in all previous periods. More specifically, a *dynamic equilibrium* (or *DE*) is an SPE in which behavior in any period  $t$  only depends on  $(x^1, \dots, x^{t-1})$ , plus the current proposal in the voting stage. We adopt this weaker criterion because our aim is to establish equilibrium inefficiency — which is too easy to prove of SMPEs because Markov perfection restricts the set of policy sequences which can be supported. For instance, it precludes those sequences in which the same policy is implemented in a finite number of consecutive periods and then switches to a different policy. By contrast, we do not impose any restriction on the set of policy sequences which can be supported by allowing behavior to depend on previous periods’ policies.

**Absorbing points and no-delay strategies.** A *complete history of length  $t$*  describes all that has transpired in each period  $\tau \in \{1, 2, \dots, t\}$ : the selection of a proposer, her proposal, the associated pattern of votes (if applicable), and the implementation of period- $\tau$  policy  $x^\tau$ .<sup>14</sup> Of particular interest are “implementation histories” (to use the language of Hyndman and Ray (2007)), i.e. those which end just before the implementation of a policy. More precisely, an *implementation history of length  $t$*  is a complete history of length  $t - 1$  (or the null history if  $t = 1$ ) together with the selection of the proposer, her proposal, and (if applicable) the associated pattern of votes (but not the implementation of the policy) in period  $t$ . Hence, at such a history, players know the policy that will be implemented in period  $t$ , but they have not yet received their payoffs from that policy. An *implementation history* is an implementation history of finite length. For each  $x \in X$ , the set of implementation histories at which  $x$  is the policy about to be implemented is denoted by  $H_x$ ; that is,  $h \in H_x$  if there exists  $t \in \mathbb{N}$  such that  $h$  is an implementation history of length  $t$  and  $x^t = x$ . Let  $H \equiv \bigcup_{x \in X} H_x$  be the set of all possible implementation histories.

Every strategy profile  $\sigma$  (in conjunction with recognition probabilities) generates a transition function  $P^\sigma$  on implementation histories, where  $P^\sigma(h, H')$  is the probability (given  $\sigma$ ) that the next period’s implementation history is in  $H'$ , given that the implementation history for the current period is  $h$ . Thus, for all  $i \in N$ , all  $x \in X$  and all  $h \in H_x$ , player  $i$ ’s continuation value at  $h$  — i.e. the payoff that player  $i$  receives *from  $h$  on* — is given by

$$V_i^\sigma(h) = (1 - \delta_i) u_i(x_i) + \delta_i \int V_i^\sigma(h') P^\sigma(h, dh') .$$

We say that  $x \in X$  is an *absorbing point* of  $\sigma$  if and only if  $P^\sigma(h, H_x) = 1$  for all  $h \in H_x$ , and denote by

$$A(\sigma) \equiv \{x \in X : P^\sigma(h, H_x) = 1 \text{ for all } h \in H_x\}$$

the set of absorbing points of  $\sigma$ .

We say that  $\sigma$  is *no-delay* if and only if: (i)  $A(\sigma) \neq \emptyset$ ; and (ii) for all  $h \in H$ , there is  $x \in A(\sigma)$  such that  $h \in H_x$ . In words, a strategy profile is no-delay if the committee implements an absorbing point *at every implementation history* (including those off the equilibrium path). It is worth noting that this notion of no-delay differs from that conventionally used in models of bargaining with fixed defaults, where it is often associated with efficiency (e.g.

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<sup>14</sup>The implementation stage can be inferred from previous stages and, therefore, may appear redundant. For expositional clarity, however, it is convenient to include it in the definition of a complete history.

Austen-Smith and Banks (2005), p. 211). In contrast, no-delay equilibria can be statically inefficient in this model.

In the case of stationary Markov strategies, we will indulge in a slight abuse of notation and replace implementation histories by policies in the definitions above. For instance,  $P^\sigma(x, Y)$  will denote the probability (given stationary Markov strategy  $\sigma$ ) that the committee chooses a policy in  $Y$  in the next period given that policy  $x$  is implemented in the current period — so that  $A(\sigma) \equiv \{x \in X : P^\sigma(x, \{x\}) = 1\}$ .

### 3.3 (In)efficiency

Previous papers in the related literature have only explored the equilibrium correspondence. Given this focus, the assumption that players share a common discount factor simplifies exposition. By contrast, we will also be interested in the welfare evaluation of equilibria; and here the supposition of equal patience is problematic.

It is useful to distinguish between potential inefficiencies which arise in models with an exogenous default, and those which are peculiar to models with an endogenous default. First, the pie might not be entirely distributed in some periods: A policy  $x \in X$  is *statically inefficient* if  $\sum_{i \in N} x_i < 1$  (and is *statically efficient* otherwise). We will refer to  $1 - \sum_{i \in N} x_i$  as waste. (Recall that  $u_i$  is strictly increasing in  $x_i$ .) The uncertainty created by recognition probabilities and (possibly) mixed strategies could also result in welfare losses if utility functions are not linear. The dynamic structure of our model engenders another source of inefficiency: If players have different discount factors then inter-temporal transfers may facilitate Pareto improvements.

Our efficiency criterion captures all of these features. Formally, a (possibly stochastic) policy sequence  $\{\tilde{x}^t\}$  is *Pareto efficient* if there is no other (possibly stochastic) policy sequence  $\{\tilde{y}^t\}$  such that

$$\mathbb{E} \left[ \sum_{t=1}^{\infty} \delta_i^{t-1} u_i(\tilde{y}_i^t) \right] \geq \mathbb{E} \left[ \sum_{t=1}^{\infty} \delta_i^{t-1} u_i(\tilde{x}_i^t) \right]$$

for all  $i \in N$ , with at least one strict inequality for some  $i \in N$ . We will say that a strategy profile is *Pareto inefficient* if, from the initial default  $(0, \dots, 0)$ , it generates a policy sequence that is not Pareto efficient.

Assuming linear utilities, we can establish the following result:

**Lemma 1.** *Suppose that  $u_i(x_i) = x_i$  for all  $i \in N$ . In every Pareto efficient policy sequence, the following is true for every  $i, j \in N$  such that  $\delta_j < \delta_i$ : If player  $i$ 's expected*

share of the pie in some period  $t$  is positive then player  $j$ 's expected shares in all periods  $\tau > t$  are zero.

Thus, if an efficient policy sequence allocates a positive share of the pie to any player  $i$  with a positive probability in some period  $t$ , then all players who are less patient than  $i$  must receive a zero share (with probability 1) in all subsequent periods. This result will be very useful when we come to establish Pareto inefficiency of all dynamic equilibria in nonunanimity games. To prove it (see the Appendix), we show that the linearity of the  $u_i$ 's would otherwise permit mutually advantageous utility transfers across periods and, therefore, contradict Pareto efficiency. Observe that Lemma 1 implies that if the players all have distinct discount factors then, from some period on, some player must receive the entire pie with probability one in any Pareto efficient policy sequence. Indeed, if the most patient player receives a positive (expected) share of the pie in some period  $t$ , then all other players receive a zero share from period  $t+1$  on. If the most patient player receives a positive share with probability zero, then we can apply the same argument to the next most patient player and continue until the most patient player among those who ever receive a positive share of the pie with positive probability. As the pie cannot be wasted in an efficient sequence, such a player exists and we obtain this implication of the lemma.

Lemma 1 relies on the supposition that all utility functions are linear. Despite its relevance, characterization of the set of Pareto efficient policy sequences remains, to the best of our knowledge, an open and very delicate question.<sup>15</sup> Indeed, when players have heterogeneous discount factors, the set of payoff profiles that can be obtained through time averaging does not coincide with the set of feasible policies, even if utilities are linear.

Pareto efficiency is an *ex ante* concept in the sense that it compares the payoffs of the different policy sequences prior to their realizations. We will also consider the weaker notion of *ex post* Pareto efficiency (e.g. Merlo and Wilson (1995)), in which the payoffs of different policy sequences are compared after the realizations of those sequences. More precisely, we will say that a (possibly stochastic) policy sequence  $\{\tilde{x}^t\}$  is *ex post Pareto efficient* if at least one of the realizations  $\{x^t\}$  in its range is Pareto efficient. We will say that a strategy profile is *ex post Pareto inefficient* if it generates a policy sequence that is *ex post Pareto inefficient*: that is, each of its possible realizations can be Pareto improved. Though this definition of *ex post Pareto efficiency* is very weak, we prove in Section 5 that

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<sup>15</sup>It is readily checked that, whenever all marginal utilities satisfy  $u'_i \in [b, B]$  for some  $0 < b < B < \infty$ , some players must receive zero shares infinitely often. The proof is provided in the Supplementary Appendix.

all dynamic equilibria fail this criterion in the unanimity case.

## 4 Nonunanimity Committees

Let  $\mathcal{W}$  be the collection of winning coalitions:  $\mathcal{W} \equiv \{C \subseteq N : |C| \geq q\}$ . Throughout this section, we assume that  $q < n$ : agreement requires less than unanimous consent.

### 4.1 Simple Solutions

We will construct a class of pure strategy no-delay SMPs, in which each player  $j \in N$  is only offered two different shares of the pie — a “high” offer  $x_j > 0$  and a “low” offer  $y_j < x_j$  — after any history. In every period and for any ongoing default, each proposer  $i$  (conditional on being recognized to make an offer) implicitly selects a winning coalition  $C_i \ni i$  by making high offers to the members of  $C_i$  and low offers to the members of  $N \setminus C_i$ . If each player receives a low offer from at least one proposer, then we refer to the set of such proposals (one for each player) as a *simple solution*. Formally:

**Definition 1.** Let  $\mathcal{C} \equiv \{C_i\}_{i \in N} \subseteq \mathcal{W}$  be a class of coalitions such that, for each  $i \in N$ ,  $i \in C_i$  and  $i \notin C_j$  for some  $j \in N \setminus \{i\}$ . Let  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  be two vectors in  $[0, 1]^n$  satisfying  $x_i > y_i$  and

$$\sum_{j \in C_i} x_j + \sum_{j \notin C_i} y_j \leq 1 ,$$

for all  $i \in N$ . The simple solution induced by  $(\mathcal{C}, x, y)$  is the set of policies  $S \equiv \{x^{C_i}\}_{i \in N}$ , where

$$x_j^{C_i} \equiv \begin{cases} x_j & \text{if } j \in C_i , \\ y_j & \text{if } j \notin C_i , \end{cases} \quad \text{for all } i, j \in N .$$

A set of policies  $S \subseteq X$  is a simple solution if there exists a triplet  $(\mathcal{C}, x, y)$  (as defined above) such that  $S$  is a simple solution induced by  $(\mathcal{C}, x, y)$ .

Before we turn our attention to the construction of equilibria themselves, a few remarks are in order about simple solutions:

1. A simple solution exists if and only if  $q < n$ : if  $q < n$  then the main simple solution, in which the pie is divided equally among every minimal winning coalition, is a notable example of a simple solution (cf. Wilson (1971)); if  $q = n$  then each player must be included in the unique winning coalition  $N$  and, therefore, there is no simple solution.

2. If  $q < n$  then any policy which assigns a positive share to at least  $q$  players is part of some simple solution. To see this, take an arbitrary policy  $z \in X$  such that  $|\{i : z_i > 0\}| \geq q$ . For expositional convenience, we order the players in  $N$  in such a way that  $z_i \geq z_{i+1}$  for each  $i = 1, \dots, n-1$  (thus ensuring that  $z_i > 0$  for all  $i \leq q$ ). Consider the simple solution induced by  $(\mathcal{C}, x, y)$ , where

$$x_i = \begin{cases} z_i & \text{if } i \leq q \\ z_i + \frac{\varepsilon}{n-q} & \text{if } i > q \end{cases}, y_i = \begin{cases} z_i - \varepsilon & \text{if } i \leq q \\ z_i & \text{if } i > q \end{cases}, \varepsilon > 0 \text{ arbitrarily small,}$$

and  $C_i$  is the coalition that includes  $i$  and the next  $q-1$  players following the order  $1, 2, \dots, n-1, n, 1, 2, \dots, q-1$ . It is readily checked that  $(\mathcal{C}, x, y)$  satisfies all the conditions of Definition 1 (in particular  $y_i \geq 0$  for all  $i \in N$ ), and that  $x^{C_1} = z$ .

3. Policies which assign a positive share to fewer than a minimal winning coalition cannot be included in a simple solution. Such policies include the initial default and the vertices of the simplex.
4. The definition of the class  $\mathcal{C}$  of coalitions does not require all of them to be distinct; but it is easy to confirm that  $\mathcal{C}$  must contain at least  $n/(n-q)$  distinct coalitions.
5. The policies in a simple solution may all assign a positive share to a supraminimal coalition, and might all involve waste.

## 4.2 Preliminary Intuitions

If all players are myopic then there is a unique SPE outcome in which each proposer successfully claims the entire pie. More generally, it is easy to show that there is no absorbing SPE when players' discount factors are small. Indeed, owing to the emptiness of the core, there is always a winning coalition which can make all its (short-sighted) members strictly better off by amending any potential absorbing point to another policy in  $X$ . For future reference, we record this observation as:<sup>16</sup>

**Observation 1.** *Let  $q < n$ . If  $\delta_i = 0$  for each  $i \in N$  then each period's proposer receives the entire pie in every SPE. Furthermore, there exists  $\hat{\delta} \in (0, 1)$  such that there is no absorbing SPE whenever  $\max_{i \in N} \delta_i < \hat{\delta}$ .*

<sup>16</sup>The proof of this observation is provided in the supplementary appendix.

Nevertheless, we will show that it is possible to construct a no-delay (and therefore absorbing) SMPE when players' discount factors are sufficiently large. The following example illustrates Definition 1, and provides an intuitive presentation of some key mechanisms behind our equilibrium construction.

**Example 1.** Let  $n = 3$ ,  $q = 2$ ,  $p_i = 1/3$ ,  $\delta_i = \delta$  and  $u_i(x_i) = x_i$  for all  $i \in N$ .<sup>17</sup> Take, for example, the simple solution  $S = \{(1/3, 1/3, 1/6), (1/6, 1/3, 1/3), (1/3, 1/6, 1/3)\}$  — that is,  $C_1 = \{1, 2\}$ ,  $C_2 = \{2, 3\}$ ,  $C_3 = \{1, 3\}$  and  $x_j = 1/3$ ,  $y_j = 1/6$  for every player  $j = 1, 2, 3$ . If  $\delta \geq 12/13$  then the following strategy profile forms a pure strategy, no-delay SMPE whose set of absorbing points is  $S$ :

- Player  $i$  always offers  $1/3$  to the players in  $C_i$  and  $1/6$  to the player outside  $C_i$  if the ongoing default does not belong to  $S$ , and passes otherwise;
- Player  $i$  accepts proposal  $z$  when the ongoing default is  $w$  if and only if one of the following conditions holds: (i)  $w \in S$  and  $w_i = 1/6$ ; (ii)  $w \notin S$ ,  $z \in S$ , and  $z_i \geq (1 - \delta)w_i + (5\delta/18)$ ; or (iii)  $w, z \notin S$  and  $z_i \geq w_i$ .

A formal proof of this statement is obtained as a special case of Theorem 1. The intuition is as follows. It is readily checked that this (pure) strategy profile is no-delay and that  $S$  is the set of absorbing points: each policy  $x^{C_i}$  in  $S$  is proposed by player  $i$  with probability  $1/3$ , accepted by the two members of majority coalition  $C_i$ , and never amended.

To see why this is an SMPE, observe first that each (patient) player  $i = 1, 2, 3$  can only end up in two possible states in the long-run: a “good state” in which she receives  $1/3$  in all periods, and a “bad state” in which she receives  $1/6$  in all periods. Indeed, any ongoing default  $w$  is either an absorbing point itself or will lead immediately to some absorbing point  $x^{C_j} \in S$ , with  $x_i^{C_j} \in \{1/6, 1/3\}$ . In the former case, player  $i$ 's expected payoff is  $w_i = 1/3$  if  $i \in C_j$ , and  $w_i = 1/6$  otherwise. In the latter case (i.e. if the current period's proposer fails to amend  $w$ ),  $i$  receives  $w_i$  in the current period and  $2/3 \times 1/3 + 1/3 \times 1/6 = 5/18$  in the next period ( $i \in C_j$  with probability  $2/3$ ). Her expected payoff is therefore  $(1 - \delta)w_i + (5\delta/18)$ , which is less than  $1/3$  for all  $w_i \in [0, 1]$  (recall that  $\delta \geq 12/13$ ).

In every voting stage, players know that the next period's proposer  $i$  will successfully offer absorbing point  $x^{C_i}$  in  $S$  if the default is not already in  $S$ , and will pass otherwise.

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<sup>17</sup>These are precisely the assumptions made by Kalandrakis (2004). In contrast to that paper, however, we require the initial default to be  $(0, \dots, 0)$ , and allow for policies which do not exhaust the pie. In addition, our equilibrium construction does not allow for small discount factors.



As  $x_j^{C_i} \in \{1/3, 1/6\}$  for all  $i, j \in N$ , every player  $j$  anticipates that her shares of the pie in all future periods will either be equal to  $1/3$  or to  $1/6$ . Hence, each player  $j$ 's continuation value is bounded from above by  $u_j(1/3) = 1/3$ . This implies that it is optimal for farsighted player  $j$  to reject [resp. accept] any proposal to amend default  $x^{C_i}$  whenever  $j \in C_i$  [resp.  $j \notin C_i$ ]: changing  $x^{C_i}$  to another policy can only decrease [resp. increase] her long-run payoffs — as  $\delta$  is sufficiently close to 1, only long-run payoffs matter to her. As the  $C_i$ 's are winning coalitions, this ensures that it is impossible to amend the  $x^{C_i}$ 's once they have been implemented. If the current default is not in  $S$  then, by the same logic, it is optimal for each member of  $C_i$  to accept  $x^{C_i}$  and for any other player to reject it. These voting strategies in turn imply that there is no profitable deviation from proposal strategies. If the current default is outside  $S$  then it is optimal for proposer  $i$  to offer  $x^{C_i}$ : this proposal will be accepted by all members of winning coalition  $C_i$  and guarantees her the highest possible long-term payoff of  $1/3$ . If the default is outside  $S$  then any attempt to amend it is unsuccessful; so that passing is optimal. Thus, we obtain an SMPE.

□

This example illustrates why our results are radically different from those obtained in the standard Baron-Ferejohn model of an ad hoc committee. In particular, it explains why shares of the pie can be perpetually wasted and/or shared amongst more than a minimal winning coalition in equilibrium: Players can be locked into equilibria where any deviation to proposing a Pareto-superior policy would be rejected. Interestingly, what prevents any (minimal) winning coalition  $C$  from agreeing on a non-wasteful policy in such an equilibrium is its members' *inability to commit* not to revert to one of the statically inefficient absorbing points  $x^{C_i}$  with  $C_i \neq C$ , where some members of  $C$  lose out. The nonunanimity quota ( $q < n$ ) ensures that such coalitions  $C_i$  always exist. In the example above, farsighted player 1 rejects (off the equilibrium path) any proposal such as  $(1/2, 1/2, 0) \in X$  when the default is absorbing point  $x^{C_1} = (1/3, 1/3, 1/6)$ . Indeed, she anticipates that if such a proposal were successful in the current period then, with probability  $p_2 > 0$ , player 2 would successfully propose  $x^{C_2} = (1/6, 1/3, 1/3)$  in the next period — which would never be amended. As  $\delta$  is close to one, player 1 prefers to earn  $1/3$  with certainty in all future periods. Evidently, this commitment problem would not arise in the Baron-Ferejohn model where, once implemented, policies can never be amended.

### 4.3 Positive Results

Our first result generalizes the argument above to any nonunanimity quota, any concave utility functions, and any simple solution. We describe a pure strategy no-delay SMPE in which each policy in a simple solution is proposed by some player, and no other policy is proposed after any history as a *simple equilibrium*.

**Theorem 1.** *Suppose that  $q < n$ , and let  $S$  be a simple solution. There exists  $\bar{\delta} \in (0, 1)$  such that the following is true whenever  $\min_{i \in N} \delta_i \geq \bar{\delta}$ : There exists a pure-strategy no-delay SMPE whose set of absorbing points is  $S$ .*

The proof of this theorem, like those of all other theorems in the paper, is provided in the Appendix. Theorem 1 has several interesting implications:

**Multiplicity of SMPE payoffs.** We noted above that any policy (say,  $z$ ) which assigns a positive share to  $q$  or more players is part of a simple solution. Theorem 1 therefore implies that  $z$  is an absorbing point of an SMPE of any game with  $q < n$  and patient enough players. In that SMPE, player 1 proposes  $z$  which is accepted by all members of coalition  $C_1 = \{1, \dots, q\} \in \mathcal{W}$ , and never amended.

This argument does not apply to policies which assign a positive share to fewer than  $q$  players (including the initial default), and can therefore not be part of a simple solution. Policies which assign a zero share to some winning coalition cannot be absorbing points of an SMPE because every member of such a coalition could profitably deviate as a proposer.<sup>18</sup>

**Minimal winning coalitions.** The Baron-Ferejohn model predicts that only minimal winning coalitions share the pie in any stationary SPE. Theorem 1 immediately implies that this property, often referred to as the size principle, may fail in our model with an evolving default: As mentioned earlier, policies in a simple solution may all assign a positive share to a supraminimal coalition.

**Waste.** Another important implication of Theorem 1 is that endogeneity of the default may create substantial (static) inefficiencies in equilibrium. For any  $\varepsilon \in (0, 1)$ , let  $X_\varepsilon$  be the set of policies such that the committee “wastes” more than  $1 - \varepsilon$ :  $X_\varepsilon \equiv \{x \in X : \sum_{i \in N} x_i < \varepsilon\}$ . It is easy to find simple solutions that are subsets of  $X_\varepsilon$ . For instance, take the simple solution induced by  $(\mathcal{C}, x, y)$  where, for each  $i \in N$ ,  $x_i = \varepsilon/2q$ ,

<sup>18</sup>As Kalandrakis (2004, 2010) demonstrates, such policies could nevertheless be part of an ergodic set.

$y_i = 0$ , and  $C_i$  is the coalition that includes  $i$  and the next  $q - 1$  players following the order  $1, 2, \dots, n - 1, n, 1, 2, \dots, q - 1$ . Theorem 1 implies that any nonunanimity game with patient enough players has a pure-strategy no-delay SMPE whose absorbing points all belong to  $X_\varepsilon$ : *the committee wastes at least  $1 - \varepsilon$  in every period along the equilibrium path*. This again stands in sharp contrast to the stationary SPEs of the Baron-Ferejohn model, in which waste never occurs.

Agreements may in fact be even worse relative to the initial default than our presentation has hitherto suggested. Specifically, the proof of Theorem 1 does not rely on our supposition that  $x^0 = (0, \dots, 0)$ ; so we can construct simple equilibria in which every absorbing policy is strictly Pareto-dominated by the initial default (by appropriately selecting  $x^0$ ).<sup>19</sup>

Theorem 1 also implies that there are SMPEs in which statically inefficient policies are retained indefinitely. This property is empirically interesting: for example, Brainard and Verdier (1997) describe persistent protection as “one of the central stylized facts in trade” (p222). Theorem 1 therefore contributes to the literature on policy persistence, without requiring (as in Coate and Morris (1999) and Acemoglu and Robinson (2008)) that players can unilaterally invest in sustaining policies.

**Pork barrel politics.** We have noted that SMPE agreements may waste some of the pie and that the size principle may fail. Theorem 1 says that both properties can hold in the same equilibrium. According to Schattschneider (1935), this combination of properties characterized US trade policy before 1934. Indeed, Baron (1991) claims that legislation on distributive issues often exhibits this combination.<sup>20</sup> He also argues that models of ad hoc committees can explain pork, but not violations of the size principle. By contrast, Theorem 1 implies that equilibrium agreements in a standing committee may satisfy both properties without appealing to a norm of universalism.

We record the observations above as

**Corollary 1.** *Suppose that  $q < n$ . For each of the following statements, there exists  $\bar{\delta} \in (0, 1)$  such that this statement is true whenever  $\min_{i \in N} \delta_i \geq \bar{\delta}$ :*

<sup>19</sup>This property is stronger than a related result in Bernheim et al’s (2006) and Anesi and Seidmann’s (2014) models of bargaining with an evolving default: that the equilibrium agreement is worse than  $x^0$  for some winning coalition.

<sup>20</sup>Evans (2004) documents the failure of the size principle, and argues that Congress may often pass inefficient public good projects.

- (i) *There exist multiple pure-strategy no-delay SMPEs;*
- (ii) *Any policy which assigns a positive share to  $q$  or more players is an absorbing point in some pure-strategy no-delay SMPE;*
- (iii) *There are SMPEs which fail the size principle;*
- (iv) *For every  $\varepsilon \in (0, 1)$ , there is a pure-strategy SMPE  $\sigma$  such that  $P^\sigma(x, X_\varepsilon) = 1$  for all  $x \in X$ ;*
- (v) *There are no-delay SMPEs in which the agreement wastes some of the pie and fails the size principle.*

#### 4.4 Pareto Efficiency

Corollary 1(ii) implies that some simple equilibria are statically efficient. If players have a common discount factor and are risk neutral then these equilibria are also Pareto efficient; but wasting some of the pie is not the only possible kind of inefficiency in dynamic models when discount factors are heterogeneous.

It is possible to construct Pareto efficient SPEs. Indeed, we can construct SPEs in which some player earns the entire pie every period (irrespective of the identity of the proposer) if players are patient enough. However, these SPEs — which are trivially Pareto efficient — rely on player-specific punishments: any player who proposes another policy is “punished” with an indefinite zero allocation. Other players reject such proposals because they then always share the pie. These punishments are excluded in a DE (cf. Section 3.2). Our main result in this subsection also states that if all players have linear preferences ( $u_i(x_i) = x_i$ ) and discount factors are strictly heterogeneous then all DEs are Pareto inefficient. (Players have linear preferences in Baron and Ferejohn (1989), and much of the ensuing literature.)

**Theorem 2.** *Let  $q < n$ .*

- (i) *If  $u_i(x_i) = x_i$  for all  $i \in N$  and  $\delta_i \neq \delta_j$  for all  $i, j \in N$  then all DEs are Pareto inefficient.*
- (ii) *There exists  $\tilde{\delta} \in (0, 1)$  such that the following is true whenever  $\min_{i \in N} \delta_i > \tilde{\delta}$ : Any (Pareto efficient) policy sequence that allocates the entire pie to the same player in every period can be supported by an SPE.*

Lemma 1 implies that, with linear utilities, any Pareto efficient policy sequence must eventually assign the pie to a single player with probability one. By definition, this is impossible in a simple equilibrium: every player earns a positive share of the pie with

positive probability. More generally, suppose that a DE  $\sigma$  prescribes that one player, say  $i$ , always earns the entire pie for sure after some history  $h$ . The proof of Theorem 2 establishes that another player could profitably deviate if selected as proposer after  $h$  by successfully proposing a more equitable division of the current pie. The nonunanimity quota guarantees that this cannot be prevented by player  $i$ . Hence, a DE cannot be Pareto efficient.

A few remarks are in order concerning Theorem 2. First, it does not assert that every DE is ex post Pareto inefficient. Indeed, such a claim would be false: We can construct a DE  $\sigma$  at which each period's realized proposer earns the entire pie when players are sufficiently patient. (This construction relies on Corollary 1(iv): any deviation from the prescribed path is punished by using a simple equilibrium in which every player's continuation value is smaller than some small  $\varepsilon > 0$  — details of the construction are provided in Section B of the Supplementary Appendix.) There are realizations of the stochastic policy sequence engendered by  $\sigma$  — i.e. those corresponding to cases where the same player is selected to propose in every period — which allocate the entire pie to the same player in every period. These realizations are Pareto efficient and, therefore,  $\sigma$  is ex post Pareto efficient.

Second, Theorem 2(i) relies on Lemma 1, whose premise requires that all utilities are linear. If some player were risk averse then DEs which generate stochastic policy sequences would obviously be Pareto inefficient; but we can construct a no-delay, deterministic DE in which the first proposer offers  $1/n$  to every player, all of whom are risk averse and have different, but large enough discount factors.<sup>21</sup> This equilibrium is inefficient because no player would earn 0 infinitely often (see footnote 15). In sum, generalization of Theorem 2 to nonlinear utilities remains an open question.

Third, in contrast to Theorem 1, the premise of Theorem 2(i) does not require that players be patient enough. It only requires strict heterogeneity. It is easy to confirm that the argument works as long as enough players have different discount factors.

Fourth, our construction of the efficient SPE extends Shaked's example (cf. Sutton (1986)) to games with an endogenous default, a nonunanimity quota, and a random protocol. It depends on the initial default assigning no share to either all or all but one player, else another player who starts with a positive share could profitably deviate by passing when selected to propose at the null history. Extending the construction to arbitrary initial defaults would require strategy profiles to support Pareto efficient policy sequences which

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<sup>21</sup>The logic behind the construction is analogous to that described in the previous paragraph: play reverts to a simple SMPE which wastes enough of the pie once the default is off the equilibrium path.

do not allocate the entire pie to the same player in every period. As we noted above, the characterization of Pareto efficient policy sequences in the infinitely repeated pie division problem is an open question.

## 5 Unanimity Committees

This section examines equilibria of standing committee games in which agreement requires unanimous consent: that is,  $q = n$ .

### 5.1 Preliminary Example

As in the previous section, we begin with a simple example that will provide some intuition for the general results that follow.

**Example 1 Continued.** Consider a variant on Example 1 (of Section 4.2) in which the default can only be changed if all three players accept a proposal: that is,  $q = n = 3$ . The other primitives of the example remain the same:  $p_i = 1/3$ ,  $\delta_i = \delta$  and  $u_i(x) = x_i$  for all  $i \in N$ . We will construct a no-delay equilibrium  $\sigma$  in which, at any default  $x \in X$ , the selected proposer (say  $i$ ) successfully offers the committee a policy  $x + s^i(x) \in \Delta_{n-1}$ . We can think of proposer  $i$  offering to share the amount of pie not distributed yet — i.e.  $1 - (x_1 + x_2 + x_3)$  — with the other players, with  $s_j^i(x)$  being the (extra) share offered by proposer  $i$  to player  $j$ .<sup>22</sup> In such a situation, proposer  $i$ 's optimal offer to player  $j$ ,  $x_j + s_j^i(x)$ , must leave the latter indifferent between accepting and rejecting. If  $j$  rejected  $i$ 's offer, she would receive her payoff from the ongoing default in the current period,  $(1 - \delta)x_j$ , and would then receive offer  $x_j + s_j^k(x)$  from each proposer  $k = 1, 2, 3$  with probability  $1/3$  in the next period. The following condition must therefore hold:

$$x_j + s_j^i(x) = (1 - \delta)x_j + \delta \left[ x_j + \frac{s_j^1(x) + s_j^2(x) + s_j^3(x)}{3} \right]$$

or, equivalently

$$s_j^i(x) = \frac{\delta}{3} [s_j^1(x) + s_j^2(x) + s_j^3(x)] \quad (1)$$

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<sup>22</sup>Hence, all proposers pass when the ongoing default is already in the unit simplex:  $s^i(x) = (0, 0, 0)$  for all  $i = 1, 2, 3$  whenever  $x \in \Delta_2$ .

for each  $i$  and  $j \neq i$ . Given the shares of the pie offered to the other committee members, proposer  $i$  receives the residual:

$$x_i + s_i^i(x) = 1 - \sum_{j \neq i} [x_j + s_j^i(x)] . \quad (2)$$

Combining (1) and (2), we obtain the policy  $x + s^i(x)$  (absorbing point) successfully offered by each player  $i$  at any default  $x \in X$ :

$$x_i + s_i^i(x) = x_i + \frac{3 - 2\delta}{3} \left( 1 - \sum_{j=1}^3 x_j \right) ,$$

$$x_j + s_j^i(x) = x_j + \frac{\delta}{3} \left( 1 - \sum_{j=1}^3 x_j \right) , \forall j \neq i .$$

In particular, each player expects to earn  $1/3$  in the game itself:  $V_i^\sigma(x^0) = 1/3$ .

□

Its simplicity notwithstanding, there are two noteworthy features of this example. First, the set of absorbing points of the no-delay SMPE  $\sigma$  coincides with the unit simplex:  $x_j + s_j^i(x) \in \Delta_2$  for all  $x \in X$  and all  $i, j \in N$ . Second, the SMPE payoffs coincide with those of the analogous Baron-Ferejohn model with a unanimity quota. As the rest of this section will demonstrate, these properties do not rely on our parametric assumptions.

## 5.2 Positive Results

Our first result generalizes some properties of Example 1 above to all DEs. These properties will be the key to proving the uniqueness of SMPE payoffs in Theorem 4(ii) and the inefficiency result of Theorem 5.

**Theorem 3.** *If  $q = n$  then every DE  $\sigma$  is no-delay with  $A(\sigma) = \Delta_{n-1}$ .*

Thus, under unanimity rule, a standing committee selects an absorbing point in the simplex immediately *at any ongoing default*. In contrast to nonunanimity committees, therefore, waste never occurs in a DE of unanimity committee games. In other words, the unanimity game has and only has no-delay, statically efficient DEs.

Unanimity voting implies that continuation values in a DE satisfy a “temporal-monotonicity” property, which provides the key to understanding the intuition behind Theorem 3. Unanimity rule gives every player the power to prevent any amendment of the default. Therefore, in equilibrium, any player’s continuation value of implementing a new policy must be

at least as great as her utility from the current default — otherwise she could profitably deviate to a strategy that prevents any amendment of the default in all future periods. This implies that, as bargaining goes on, all players' continuation values are nondecreasing. This allows us to exploit Hyndman and Ray (2007) Proposition 1, which implies (in our model) that the equilibrium default converges almost surely. The monotonicity of continuation values thus implies the no-delay property: players will not wait for several periods to get their limit payoffs. Moreover, if the limit allocation were not statistically efficient, then any proposer could profitably deviate by offering a policy that makes all players strictly better off — the monotonicity property of continuation values ensures that none of them would lose out in the future. Hence, absorbing policies must be in the unit simplex. In addition, all policies in the unit simplex must be absorbing: any amendment of a statically efficient default must make some players worse off in the current period, and temporal monotonicity of continuation values implies that the other players will never compensate them for their current losses. Hence, it is impossible to amend a policy in  $\Delta_{n-1}$ .

Our next result asserts existence of a pure strategy no-delay SMPE in which resources are never wasted. The premise of Theorem 4 differs from the premise of Theorem 1 (our analogous result for  $q < n$ ) in two important respects. First, we no longer require that players be patient enough. Indeed, unlike nonunanimity games (recall Observation 1), a no-delay equilibrium exists in unanimity games even when discount factors are small. Second, Theorem 4 asserts that the policies reached from any default (including the initial default) are statically efficient. The latter property also holds in the standard Baron-Ferejohn model with a unanimity quota (Banks and Duggan (2000, 2006)). The second part of the theorem strengthens the analogy between equilibrium play in our game and in Baron and Ferejohn (1989).

**Theorem 4.** *If  $q = n$  then: (i) a pure strategy no-delay SMPE exists; and (ii) SMPE payoffs are unique, and coincide with the stationary SPE payoffs of the Baron-Ferejohn model.*

We prove Theorem 4(i) using a construction which generalizes that employed in Example 1 above: A fixed point argument is used to show that there are proposals for each player which move the default into the simplex and make every respondent indifferent between accepting and rejecting, given that defaults in the simplex would not be amended; and that no player can profitably deviate from proposing such policies or accepting such an offer.



The proof of Theorem 4(ii) stems from Theorem 3, which establishes the no-delay property of DEs and, therefore, of SMPEs. This property implies that the strategic incentives at work in SMPEs of the standing committee game resemble those in stationary SPEs of the Baron-Ferejohn model, where the selected proposer makes a successful proposal (thus ending the game) in every subgame. This allows us to show that, for every SMPE  $\sigma$  of the standing committee game, one can construct a stationary SPE of the Baron-Ferejohn model that generates the same payoffs as  $\sigma$  in the standing committee game. Uniqueness of SMPE payoffs then follows from the following observation.<sup>23</sup>

**Observation 2.** *If  $q = n$  then the Baron-Ferejohn model has a unique stationary SPE.*

The uniqueness results of Merlo and Wilson (1995) and Eraslan (2002) could be applied to our setting under additional restrictions on the utility functions; but Observation 2 only relies on concavity and differentiability.

In the Introduction, we asked how play in standing and ad hoc committees differs. Our results in the last section entail a significant contrast across stationary equilibrium outcomes in the two games when  $q < n$ . Theorem 4(ii) implies that this contrast does not carry over to games with a unanimity quota.

### 5.3 Pareto efficiency

If  $q < n$  and players are patient enough then there is a Pareto efficient SPE in which the same player earns the entire pie each period (cf. Section 4.4). If  $q = n$  then such an SPE exists irrespective of players' patience. The construction again relies on player-specific punishments; so we turn to DEs which preclude such punishments.

Theorem 2 states that every DE of a nonunanimity game with linear preferences is Pareto inefficient if discount factors are strictly heterogeneous. Pareto efficiency then requires that some player eventually gets the entire pie: which is impossible in equilibrium; but a DE could be *ex post* Pareto efficient. In addition, Corollary 1(ii) states that there are no-delay, statically inefficient equilibria. If  $q = n$  then waste is impossible in any DE (by Theorem 3). However, we have an even stronger inefficiency result:

**Theorem 5.** *Suppose that  $q = n$ .*

*(i) If  $\delta_i \neq \delta_j$  for some  $i, j \in N$  then every DE is ex post Pareto inefficient.*

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<sup>23</sup>We are grateful to Sergiu Hart for suggesting the simple proof provided at the end of the Appendix.

(ii) Any (Pareto efficient) policy sequence that allocates the entire pie to the same player in every period can be supported by an SPE.

Note that parts (i) and (ii) respectively refer to DEs and to SPEs.

In contrast to Theorem 2(i), the premise of Theorem 5(i) does not require linear preferences, and weakens strict heterogeneity to heterogeneity. We obtain this stronger result because the DEs of a unanimity game are no-delay (Theorem 3), which is not necessarily the case in nonunanimity games. No-delay is useful for two reasons:

First, the no-delay property allows us to show that, on a DE path, all players receive a positive share of the pie each period (see Lemma 5 in the proof of Theorem 5(i)). To see why, suppose instead that on some DE path a player, say  $i$ , receives a zero share of the pie in some period. By the no-delay property (and unanimity rule), this implies that  $i$  accepts a zero share in the first period and, consequently, a total payoff of  $u_i(0)$  (i.e. her lowest possible payoff). If player  $i$  instead rejected that proposal, thus deviating from the equilibrium path, then the period-2 default would still be  $x^0 = (0, \dots, 0)$ . Lemma 5 proves that this deviation is profitable: conditional on being selected to propose in period 2, player  $i$  could propose a policy allocating a positive share of the pie to all players (including herself), which all players would prefer to accept rather than to have to wait until the next period to obtain a positive share of the pie. Thus,  $x_i^t \in (0, 1)$  for all  $i \in N$  and all  $t \in \mathbb{N}$ ; so that transfers among players and across periods are always feasible from a given DE policy sequence.

Second, the no-delay property allows us to prove inefficiency by constructing a Pareto-improving policy sequence. Consider a DE policy sequence and two players with different discount factors. By the no-delay property, their shares of the pie remain constant over time. The feasibility of transfers implies that we can make the two players better off (and leave the others indifferent) by operating a transfer from the more to the less patient player in some period  $t$ , and a transfer in the opposite direction in period  $t + 1$ .

Theorem 5(ii) extends Shaked's example (cf. Sutton (1986)) to games with an endogenous default (and a random protocol). We prove it by constructing an SPE in which any proposal to deviate from the policy sequence that allocates the entire pie to the same player is unanimously rejected. If it is not then (off the path) the equilibrium prescribes that players who rejected the proposal be "rewarded" with strictly positive future payoffs (i.e. one of them is randomly selected to receive then entire pie in all future periods), and that those who accepted it be "punished" with zero future payoffs. Therefore, independently

of her discount factor, each player strictly prefers to reject a proposal to deviate from the equilibrium path, even though she is not pivotal: this has no impact on her payoff in the current period, but yields her a strictly larger future payoff than accepting. Rejecting is thus a stage-undominated action.

The results in this subsection imply that Pareto efficiency requires player-specific punishments. By contrast, Pareto *inefficiency* requires player-specific punishments in conventional models of an ad hoc committee: see, for example, Sutton (1986).

## 6 The Effects of an Endogenous Default

The analysis in the previous sections has revealed important differences between our standing committee game and Baron and Ferejohn’s (1989) static game with an ad hoc committee. The comparison is directly relevant to committees like the Supreme Court, whose application of the stare decisis rule determines whether a decision can be amended. If the rule is strictly applied then the first decision establishes a precedent: the Court can then not revisit a case it has already decided (as in Baron-Ferejohn). By contrast, previous decisions only govern lower court rulings until amended if stare decisis is inoperative.<sup>24</sup>

In this section, we compare equilibrium outcomes in our dynamic model with dynamic variants of the Baron-Ferejohn model in which a committee decides on the policy implemented in an infinite sequence of periods. (We will return to the Supreme Court example in the next section.) We focus on two such models:

- *Ad hoc committee with commitment ability.* In this variant, the game ends once the committee has agreed to a single “policy”; but in contrast to the standard Baron-Ferejohn model, a policy specifies the way in which a sequence of pies will be divided.
- *Standing committee with an exogenous default.* In this variant, the committee negotiates over division of a single pie each period. Once an agreement is reached, players earn utility from their share of the pie, and the committee starts to negotiate division of another pie. The initial default for the new negotiations is exogenously fixed as the  $n$ -vector  $(0, \dots, 0)$ .

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<sup>24</sup>The two models naturally capture other aspects of the Court: justices bargain before voting on each case. Furthermore, life tenure stabilizes membership of the Court, with the (arguable) consequence that justices are relatively patient.

Stationary equilibria of both models clearly share a couple of properties with Baron and Ferejohn's (1989) static model: Each pie is shared by a minimal winning coalition of players; and each pie is fully shared — there is no waste. The argument for standing committees with an exogenous default corresponds to that used to derive equilibria in the conventional Baron-Ferejohn model: for stationarity precludes conditioning the current division on the history up to the current period.<sup>25</sup> Conventional arguments also entail these properties for an ad hoc committee with commitment ability, where the same coalition shares the pie in every period. However, the sequence of policies agreed by the two committees in equilibrium differ when discount factors are heterogeneous. In particular, an ad hoc committee with commitment ability must agree to a Pareto-efficient sequence of policies, as any proposer is a residual claimant of every pie.

These observations can serve as benchmarks with which to compare the results of the previous sections. Some notable differences can be observed:

(i) Substantial shares of the pie can be indefinitely wasted and the size principle may fail in *nonunanimity* standing committees with an endogenous default, whereas waste never occurs and only minimal winning coalitions form in both dynamic variants on the Baron-Ferejohn model. Thus, while these models cannot explain either statically inefficient policies or violations of the size principle, agreements in a standing committee with an endogenous default may possess both properties. Interestingly, default endogeneity does not generate waste when the quota is unanimity.

(ii) Equilibrium play in the standing committee game is Pareto inefficient when discount factors are strictly heterogeneous and preferences are linear. Theorem 2 also applies to standing committees with an exogenous default, as stationarity requires repetition of the same expected payoff. As mentioned above, however, ad hoc committees with commitment ability reach Pareto-efficient agreements in every equilibrium. The key difference from our model is that an ad hoc committee with commitment ability cannot renegotiate an agreement. Viewed in this light, our model demonstrates that equilibrium play in a standing committee with an endogenous default is inefficient with generic discount factors because players cannot commit not to renegotiate the existing agreement.

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<sup>25</sup>Indeed, all stationary SPEs are no-delay and payoffs are unique.

## 7 Concluding Remarks

This paper has identified a class of pure strategy (stationary Markov perfect) equilibria for pie-division bargaining games with an endogenous default, nonunanimity committees and patient enough players, which supplements existing constructions. This has allowed us to provide a number of predictions about decision making in standing committees, and to identify important implications of an endogenous default. The identified equilibria of the standing committee game have a no-delay property: the first policy proposal is accepted and remains in place in all future periods. In addition, our analysis has revealed that, unless committee members use history-dependent strategies based on player-specific punishments, heterogeneous discount factors cause Pareto inefficient policymaking. Differences in committee members' inherent time preferences may not be the only source of such heterogeneity. For example, if members of the committee are district representatives (like Senators) then their time preferences may be affected by institutional features (like the probability of re-election), which vary across districts.

Banks and Duggan (2000, 2006) have generalized the standard model of bargaining in ad hoc committees to include any convex set of policies as well as purely distributional policies, and established existence of a (mixed-strategy) stationary SPE. Before concluding, we discuss a similar extension of our model of bargaining in standing committees to more general policy spaces.

Our positive results for nonunanimity games relied on the existence of simple solutions. Though the definition of a simple solution needs to be extended to this more general setting, the logic behind this extension remains the same as for Definition 1. Each player  $i$  can be in two possible states: a “good state,” in which she has a high utility  $u_i$ , or a “bad state,” in which she has a low utility  $v_i$ . Each proposer  $i$  selects a policy  $x^{C_i}$  which gives all members of winning coalition  $C_i$  their high utility, and gives the other players their low utility. Put differently, each proposer  $i$  selects the coalition  $C_i$  of players who will be in a good state.

Figure 1 provides an example in the standard spatial model:  $n = 3$ ;  $q = 2$ ;  $X$  is a nonempty, compact and convex subset of  $\mathbb{R}^2$ ; and  $u_i(x) = -\|x - \hat{x}_i\|$  for all  $x \in X$  and all  $i \in N$ , where  $\hat{x}_i \in X$  stands for the ideal policy of player  $i$ . Baron and Herron (2003) use computational methods to study this setting in a finite-horizon version of our standing committee game. Given their results, Baron and Herron conjecture that proposals are always statically efficient in the infinite horizon case; and that proposals are closer to the

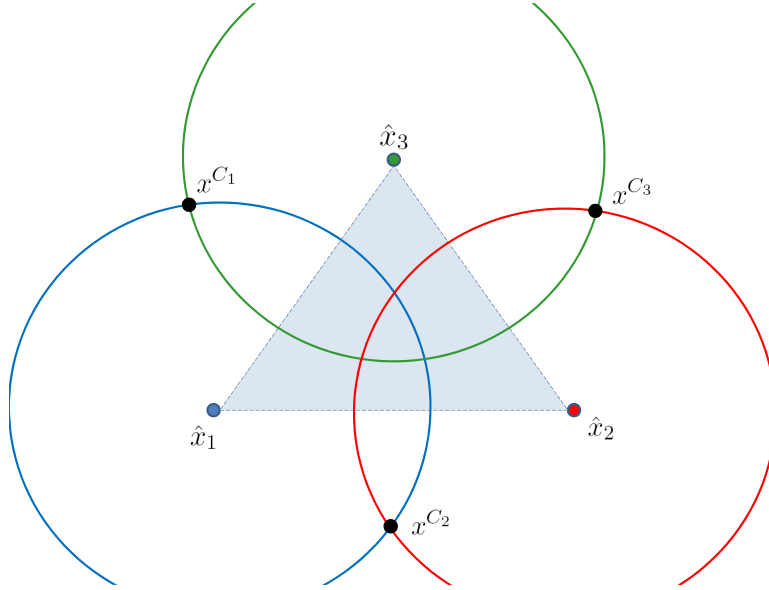


Figure 1: Simple Solution in the Spatial Model

centroid of the shaded triangle in Figure 1, the more patient are players, and the longer is the horizon. The example in Figure 1 disproves their conjecture: The set of policies  $S = \{x^{C_1}, x^{C_2}, x^{C_3}\}$  in Figure 1 constitutes a simple solution and, therefore, the set of absorbing points of some pure-strategy no-delay SMPE whenever players are sufficiently patient. (The arguments used to prove Theorem 1 still apply.) This equilibrium is both statically and Pareto inefficient: all the policies in  $S$  lie outside the static Pareto set (the grey triangle in Figure 1) and all players would be strictly better off if the expected policy  $\sum_i p_i x^{C_i}$  were agreed immediately and never amended. This is in accord with our findings for the distributive setting.

These remarks suggest that our results may be applicable to committees like the Supreme Court, whose policy space is (arguably) more naturally thought of as spatial than as divisions of a pie. The literature on precedent in constitutional law has considered how stare decisis affects the trade-off between predictability of the law and the risk of error:<sup>26</sup> stare decisis forces predictability; and the literature supposes that a divided Court would otherwise regularly overturn precedent.<sup>27</sup> We have argued above that a Court which

<sup>26</sup>Relevant papers include Schauer (1987), Stone (1988), Waldron (2012) and Kozel (forthcoming).

<sup>27</sup>The literature has typically treated the Court as a unitary body. However, Barrett (2013) considers how

operates according to strict stare decisis is equivalent to a Baron-Ferejohn ad hoc committee, whereas our model represents a Court which does not recognize precedent; and have suggested that the justices are typically patient. Our results then provide two contributions to the literature. First, our construction of no-delay SMPEs when players are patient suggests that the law may well be stable, even if precedent is not recognized.<sup>28</sup> Second, our comparison of Baron-Ferejohn with our model suggests that stare decisis may prevent the Court from reaching (statically) inefficient decisions.

Having discussed how simple solutions may exist with different policy spaces, we should also note that simple solutions need not exist (and indeed cannot exist in unanimity-rule committees). Pie-splitting problems possess a main simple solution; but this is only known to exist for strong simple symmetric games in characteristic function form with transferable utility, and remains an open question for more general simple games.<sup>29</sup> Indeed, no simple solution can exist when  $X$  is a compact interval on the real line, as the median voter cannot be excluded from any winning coalition. If players are patient enough then both ad hoc and standing committees reach policies close to the median voter's ideal policy in no-delay equilibria (cf. Baron (1996) and Banks and Duggan (2006)).

We now turn to unanimity games. We showed in Section 5 that, in distributive settings, the same equilibrium payoffs can be obtained in our model as in models of standing committees with an exogenous default. The proof of this result (i.e. Theorem 4(ii)) proceeds in two steps: the first step shows that any SMPE payoff of the endogenous-default model is a stationary SPE payoff of the fixed-default model; the second uses the uniqueness result of Observation 2 to show that these payoffs must coincide. Inspection of the first step reveals that it does not rely on the restriction to pie-division problems. Hence, when  $q = n$ , the equilibrium payoffs of the extended standing committee game are also stationary SPE payoffs of the related ad hoc committee game. However, we do not know whether the two sets of payoffs coincide. In particular, Observation 2 relies on pie division and can therefore not be directly applied to spatial settings.

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stare decisis affects play by individual justices with different constitutional viewpoints (or, equivalently, preferences). She focuses on bargaining once a precedent has been set; whereas we consider how stare decisis determines which precedent would be set.

<sup>28</sup>This prediction is surely plausible: the Court rarely overturns precedent in areas (like constitutional law) where stare decisis has less force. See Gerhardt (2008) Ch. 1 for a discussion of the evidence.

<sup>29</sup>We refer the reader to Ordeshook (1986, Chapter 9) for an in-depth discussion.

## Appendix: Proofs of the Main Results

**Lemma 1.** *Suppose that  $u_i(x_i) = x_i$  for all  $i \in N$ . In any Pareto efficient policy sequence, the following is true for any  $i, j \in N$  such that  $\delta_j < \delta_i$ : If player  $i$ 's expected share of the pie in some period  $t$  is positive then player  $j$ 's expected shares in all periods  $\tau > t$  are zero.*

**Proof:** Take any Pareto efficient policy sequence and let  $\mathbf{u}_l^t$  denote player  $l$ 's expected period- $t$  utility in this sequence.

Take any two players  $i, j \in N$  with  $\delta_j < \delta_i$ . Suppose that, contrary to the statement of the lemma,  $\mathbf{u}_i^t > 0$  and that  $\mathbf{u}_j^\tau > 0$  for some  $\tau > t$ . This implies that there is a feasible marginal utility transfer  $d\mathbf{u}_j^t > 0$  from player  $i$  to player  $j$  in period  $t$ , and a feasible marginal utility transfer  $d\mathbf{u}_j^\tau > 0$  from player  $j$  to player  $i$  in period  $\tau$ . In particular, consider transfers that would leave player  $j$  indifferent; that is:  $d\mathbf{u}_j^t - \delta_j^{\tau-t} d\mathbf{u}_j^\tau = 0$ . The resulting change in player  $i$ 's payoff would therefore be equal to

$$-d\mathbf{u}_j^t + \delta_i^{\tau-t} d\mathbf{u}_j^\tau = -\delta_j^{\tau-t} d\mathbf{u}_j^\tau + \delta_i^{\tau-t} d\mathbf{u}_j^\tau = \left( \delta_i^{\tau-t} - \delta_j^{\tau-t} \right) d\mathbf{u}_j^\tau > 0$$

(where the inequality follows from  $\delta_i > \delta_j$  and  $d\mathbf{u}_j^\tau > 0$ ). Thus, our initial supposition that  $\mathbf{u}_i^t > 0$  and that  $\mathbf{u}_j^\tau > 0$  for some  $j < i$  and some  $\tau > t$  implies that the vector of payoffs generated by the policy sequence is Pareto dominated (one can make player  $i$  strictly better-off without making the other players worse off). This is impossible since the policy sequence is by supposition Pareto efficient. □

**Theorem 1.** *Suppose that  $q < n$ , and let  $S$  be a simple solution. There exists  $\bar{\delta} \in (0, 1)$  such that the following is true whenever  $\min_{i \in N} \delta_i \geq \bar{\delta}$ : There exists a pure-strategy no-delay SMPE whose set of absorbing points is  $S$ .*

**Proof:** Let  $\{C_1, \dots, C_n\} \subseteq \mathcal{W}$  and  $(x_1, \dots, x_n), (y_1, \dots, y_n) \in [0, 1]^n$  satisfy the conditions in Definition 1, and let  $S \equiv \{x^{C_i}\}$  be the simple solution induced by  $(\{C_i\}_{i \in N}, x, y)$ . To establish Theorem 1, we proceed in three steps: first, we define threshold  $\bar{\delta} \in (0, 1)$ ; second, we construct a stationary Markov pure-strategy profile  $\sigma$ ; and third, we prove that  $\sigma$  is a no-delay, stage-undominated SPE such that  $A(\sigma) = S$ . In this last step, Claims 1 and 2 determine the continuation-value functions induced by  $\sigma$  (the  $V_i^\sigma$ 's), and show



that  $\sigma$  is a no-delay strategy profile with absorbing set  $S$ . Using the continuation-value functions, Claim 3 shows that, in every voting stage, no player uses a weakly dominated strategy. Finally, Claim 4 proves that there is no profitable one-shot deviation from  $\sigma$  in the proposal stage of any period. By the one-shot deviation principle, Claims 3 and 4 establish that  $\sigma$  is a stage-undominated SPE, thus completing the proof of the theorem.

**Step 1: Definition of  $\bar{\delta}$ .** Let  $p^{\min}$  be the minimal probability of recognition among the members of the committee:  $p^{\min} \equiv \min_{i \in N} p_i$ . For each player  $i \in N$ , define the threshold  $\bar{\delta}_i$  as

$$\bar{\delta}_i \equiv \max \left\{ \frac{u_i(1) - u_i(x_i)}{u_i(1) - p^{\min} u_i(y_i) - (1 - p^{\min}) u_i(x_i)}, \frac{u_i(y_i) - u_i(0)}{p^{\min} u_i(x_i) + (1 - p^{\min}) u_i(y_i) - u_i(0)} \right\} \in (0, 1) .$$

The threshold  $\bar{\delta}$  is defined as  $\bar{\delta} \equiv \max_{i \in N} \bar{\delta}_i$ .

We henceforth assume that  $\min_{i \in N} \delta_i \geq \bar{\delta}$ .

**Step 2: Construction of stationary Markov strategy profile  $\sigma = (\sigma_1, \dots, \sigma_n)$ .** For each  $i \in N$ , define the function  $\phi^i : X \rightarrow S$  as follows: (1) if  $w \in S$  then  $\phi^i(w) = w$ ; (2) if  $w \notin S$  then  $\phi^i(w) \equiv (\phi_1^i(w), \dots, \phi_n^i(w))$  where  $\phi_j^i(w) = x_j^{C_i}$  for all  $j \in N$ .

Equipped with functions  $(\phi^i)_{i \in N}$ , we are now in a position to define  $\sigma$ . For each  $i \in N$ ,  $\sigma_i$  prescribes the following behavior to player  $i$ :

(a) In the proposal stage of any period  $t$  with ongoing default  $w$ ,  $i$ 's proposal (conditional on  $i$  being selected to make a proposal) is  $\phi^i(w)$ ,<sup>30</sup>

(b) In the voting stage of any period  $t$  with ongoing default  $w$ , player  $i$  accepts proposal  $z \in X \setminus \{w\}$  if and only if: either (a)  $w \in S$  and  $w_i = y_i$ ; or (b)  $w \notin S$  and

$$(1 - \delta_i) u_i(z_i) + \delta_i \sum_{j \in N} p_j u_i(\phi_i^j(z)) \geq (1 - \delta_i) u_i(w_i) + \delta_i \sum_{j \in N} p_j u_i(\phi_i^j(w)) .$$

Observe that  $\sigma$  is a pure strategy stationary Markov strategy profile.

**Step 3: Proof that  $\sigma$  is a no-delay, stage-undominated SPE such that  $A(\sigma) = S$ .**

We proceed in a number of steps:

*Claim 1: The collection of functions  $(\phi^i)_{i \in N}$  satisfies the following inequality for all  $i \in N$  and  $w \notin S$ :*

$$(1 - \delta_i) u_i(w_i) + \delta_i \sum_{j \in N} p_j u_i(\phi_i^j(w)) \leq u_i(x_i) .$$

<sup>30</sup>Recall that proposing the default  $w$  is interpreted as passing.

*Proof:* Consider any player  $i \in N$  and any policy  $w \notin S$ . By definition of the  $\phi^j$ 's, we have

$$\begin{aligned} (1 - \delta_i) u_i(w_i) + \delta_i \sum_{j \in N} p_j u_i(\phi_i^j(w)) &= (1 - \delta_i) u_i(w_i) + \delta_i \left[ u_i(x_i) \sum_{j: i \in C_j} p_j + u_i(y_i) \sum_{j: i \notin C_j} p_j \right] \\ &\leq (1 - \delta_i) u_i(1) + \delta_i [(1 - p^{\min}) u_i(x_i) + p^{\min} u_i(y_i)] \\ &\leq u_i(x_i) \end{aligned}$$

where the last inequality follows from  $\delta_i \geq \bar{\delta} \geq \bar{\delta}_i$ .

*Claim 2:* (a) According to  $\sigma$ , in every period that starts with default  $w \notin S$ , each proposer  $j$  (if selected to make a proposal) successfully offers  $\phi^j(w) \in S$ , which is never amended;  $\sigma$  is no-delay with  $A(\sigma) = S$ .

(b) For all  $w \in X$  and  $i \in N$ ,

$$V_i^\sigma(w) = (1 - \delta_i) u_i(w_i) + \delta_i \sum_{j \in N} p_j u_i(\phi_i^j(w)). \quad (3)$$

*Proof:* (a) Consider a period that starts with default  $w \notin S$ . Each player  $j \in N$  is selected to make a proposal with probability  $p_j$ . From the definition of proposal strategies, she proposes  $z = \phi^j(w)$ . As the range of  $\phi^j$  is equal to  $S$ , this implies that  $z \in S$ . From part (1) in the definition of  $\phi^i$ , we thus have  $\phi^i(z) = z$  for all  $i \in N$ : proposal strategies prescribe all proposers to pass when the default is  $z$ . Hence, proposer  $j$ 's offer,  $z$ , would be implemented in all future periods if it were voted up. Given that the default  $w$  does not belong to  $S$ , proposal  $z$  is voted up if there is a winning coalition of players  $i$  for which

$$(1 - \delta_i) u_i(z_i) + \delta_i \sum_{j \in N} p_j u_i(\phi_i^j(z)) \geq (1 - \delta_i) u_i(w_i) + \delta_i \sum_{j \in N} p_j u_i(\phi_i^j(w))$$

(see part (b) in the definition of voting strategies). To see that this is the case, consider the winning coalition  $C_j$ : By definition of  $\phi^j$  and  $x^{C_j}$ , we have  $\phi_i^j(z) = z_i = \phi_i^j(w) = x_i^{C_j} = x_i$  for each  $i \in C_j$  (where the first equality follows from  $z \in S$  and the third from  $w \notin S$ ). We therefore have  $u_i(z_i) = u_i(\phi_i^j(z)) = u_i(x_i)$ ; so that

$$\begin{aligned} (1 - \delta_i) u_i(z_i) + \delta_i \sum_{j \in N} p_j u_i(\phi_i^j(z)) &= (1 - \delta_i) u_i(x_i) + \delta_i \sum_{j \in N} p_j u_i(x_i) \\ &= u_i(x_i) \geq (1 - \delta_i) u_i(w_i) + \delta_i \sum_{j \in N} p_j u_i(\phi_i^j(w)) \end{aligned}$$

(where the inequality is obtained from Claim 1). Thus, each player  $i$  in  $C_j \in \mathcal{W}$  votes to accept  $z$ , which is then implemented (and never amended since  $z \in S$ ).

This also shows that  $P^\sigma(w, S) = 1$  for all  $w \notin S$ . As  $\sigma$  prescribes all proposers to pass at a default  $w$  in  $S$ , we also have  $P^\sigma(w, S) = 1$  for all  $w \in S$ . This proves part (a) of the claim.

(b) First suppose that  $w \notin S$  is implemented in the current period. Every player  $i \in N$  receives  $(1 - \delta_i)u_i(w_i)$  in the current period and, from the discussion above, her continuation value from the next period on will be  $\sum_{j \in N} p_j u_i(\phi_i^j(w))$ . This proves that equality (3) holds when  $w \notin S$ .

Now suppose that  $w \in S$  is implemented. From the definition of proposal strategies, all proposers pass in future periods — i.e.  $w_i = \phi_i^j(w)$  for all  $i, j \in N$  — so that  $i$ 's continuation value is  $u_i(w_i)$ . This implies that

$$V_i^\sigma(w) = u_i(w_i) = (1 - \delta_i)u_i(w_i) + \delta_i \sum_{j \in N} p_j u_i(\phi_i^j(w)) ,$$

fulfilling (3).

*Claim 3: Given default  $w$  and proposal  $z$ , each voter  $i \in N$  accepts only if  $V_i^\sigma(z) \geq V_i^\sigma(w)$ , and rejects only if  $V_i^\sigma(w) \geq V_i^\sigma(z)$ .*

*Proof:* If  $w \notin S$  then this claim is an immediate consequence of Claim 2 and the definition of voting strategies (part (b)).

Suppose that  $w \in S$  — so  $V_i^\sigma(w) = u_i(w_i)$ . We must prove that part (a) in the definition of voting strategies prescribes  $i$  to accept only if  $V_i^\sigma(z) \geq V_i^\sigma(w)$ , and to reject only if  $V_i^\sigma(w) \geq V_i^\sigma(z)$ . To do so, we distinguish between two different cases:

- Case 1:  $z \in S$ . In this case,  $V_i^\sigma(z) = u_i(z_i) \in \{u_i(x_i), u_i(y_i)\}$ . According to  $\sigma$ , if  $i$  accepts then  $w_i = y_i$ . Hence,  $V_i^\sigma(w) = u_i(y_i) = \min\{u_i(x_i), u_i(y_i)\} \leq V_i^\sigma(z)$ . If  $i$  rejects then  $w_i = x_i$  and  $V_i^\sigma(w) = u_i(x_i) = \max\{u_i(x_i), u_i(y_i)\} \geq V_i^\sigma(z)$ .

- Case 2:  $z \notin S$ . According to  $\sigma$ , if  $i$  accepts then  $w_i = y_i$ . As  $\delta_i \geq \bar{\delta} \geq \bar{\delta}_i$ ,

$$\begin{aligned} V_i^\sigma(w) = u_i(y_i) &\leq (1 - \delta_i)u_i(0) + \delta_i [p^{\min} u_i(x_i) + (1 - p^{\min}) u_i(y_i)] \\ &\leq (1 - \delta_i)u_i(z_i) + \delta_i \sum_{j \in N} p_j u_i(\phi_i^j(w)) = V_i^\sigma(z) . \end{aligned}$$

If  $i$  rejects then  $w_i = x_i$  (see part (a) in the definition of voting strategies); so that  $V_i^\sigma(w) = u_i(w_i) = u_i(x_i)$ . Moreover, Claim 2 implies that

$$V_i^\sigma(z) = (1 - \delta_i)u_i(z_i) + \delta_i \sum_{j \in N} p_j u_i(\phi_i^j(z)) .$$

As  $z \notin S$ , Claim 1 then implies that

$$V_i^\sigma(w) = u_i(x_i) \geq (1 - \delta_i) u_i(z_i) + \delta_i \sum_{j \in N} p_j u_i(\phi_i^j(z)) = V_i^\sigma(z),$$

thus completing the proof of Claim 3.

*Claim 4: There is no profitable one-shot deviation from  $\sigma$  in the proposal stage of any period.*

*Proof:* Suppose, first, that the current default  $w$  is an element of  $S$ . Passing is evidently an optimal action for the selected proposer, for part (a) in the definition of voting strategies implies that members of some winning coalition — i.e. those voters  $j$  who receive  $w_j = x_j$  — would reject any proposal in  $X$ .

Now suppose that  $w \notin S$ . If proposer  $i$  followed the prescription of  $\sigma_i$  then her proposal  $\phi^i(w)$  would be accepted (Claim 2) and her payoff would be  $u_i(x_i)$  (which is the highest payoff she can obtain by making a successful proposal in  $S$ ). Therefore, if she is to profitably deviate, then she must either make an unsuccessful proposal — thus obtaining payoff  $V_i^\sigma(w)$  — or successfully propose some  $z \notin S$  — thus obtaining  $V_i^\sigma(z)$ . As  $w$  and  $z$  do not belong to  $S$ , Claims 1 and 2 imply that  $\max\{V_i^\sigma(w), V_i^\sigma(z)\} \leq u_i(x)$ . This proves that no proposer has a profitable one-shot deviation from  $\sigma$ .

Combining Claims 1-4, we obtain Theorem 1.

□

**Theorem 2.** *Let  $q < n$ .*

(i) *If  $u_i(x_i) = x_i$  for all  $i \in N$  and  $\delta_i \neq \delta_j$  for all  $i, j \in N$  then all DEs are Pareto inefficient.*

(ii) *There exists  $\tilde{\delta} \in (0, 1)$  such that the following is true whenever  $\min_{i \in N} \delta_i > \tilde{\delta}$ : Any (Pareto efficient) policy sequence that allocates the entire pie to the same player in every period can be supported by an SPE.*

**Proof:**

(i) **Pareto inefficiency of DEs in the linear-utility case**

We assume without loss of generality that  $\delta_i < \delta_{i+1}$  for each  $i = 1, \dots, n - 1$ . Now suppose, contrary to the statement of Theorem 2(i), that a Pareto efficient DE  $\sigma$  exists.

Let  $\mathbf{u}_i^t$  denote player  $i$ 's expected period- $t$  payoff in this equilibrium. To obtain the desired contradiction, we first need to establish the following result:

*Claim :* For every  $i \in N$  and every  $t \in \mathbb{N}$ , there exists  $\tau > t$  such that  $\mathbf{u}_i^\tau < 1$ .

*Proof.* Suppose that, contrary to the claim, there exist  $i \in N$  and  $t \in \mathbb{N}$  such that  $\mathbf{u}_i^\tau = 1$  for all  $\tau > t$  — so that  $\mathbf{u}_i^\tau = 0$  for all  $l \neq i$  and all  $\tau > t$ . Now consider a potential deviation by player  $j \neq i$  in some period  $\tau$  following some “policy history”  $(x^1, \dots, x^{\tau-1})$ : She proposes  $(1/n, \dots, 1/n)$  whenever selected as proposer. This proposal must be rejected by at least one member of  $N \setminus \{i\}$  — say  $k$  — and, following the new policy history  $(x^1, \dots, x^{\tau-1}, x^\tau) = (x^1, \dots, x^{\tau-1}, x^{\tau-1})$ , player  $j$  must receive 0 in all future periods; otherwise the deviation would be profitable and, consequently,  $\sigma$  would not be an SPE. As player  $k$  never uses a weakly dominated strategy (recall from Section 3.2 that equilibria are required to be stage-undominated), she behaves as if pivotal in voting stages. Hence, her payoff from rejecting  $j$ 's proposal (and inducing policy history  $(x^1, \dots, x^{\tau-1}, x^{\tau-1})$ ) must be at least as great as her payoff from accepting it. As the latter payoff is greater than or equal to  $(1 - \delta_k)/n > 0$ , player  $k$ 's expected payoff following policy history  $(x^1, \dots, x^{\tau-1}, x^{\tau-1})$  must be strictly positive. But this in turn implies that if  $k$  is recognized to make a proposal at policy history  $(x^1, \dots, x^{\tau-1})$  (which occurs with probability  $p_k > 0$ ), then she can profitably deviate from  $\sigma$  by passing (or, equivalently, proposing the current default  $x^{\tau-1}$ ). This would indeed yield policy history  $(x^1, \dots, x^{\tau-1}, x^{\tau-1})$  and, from the previous discussion, yield her a positive payoff (recall that  $k \neq i$ ). This contradicts  $\sigma$  being a DE.  $\square$

By supposition,  $\sigma$  generates a Pareto efficient policy sequence. From Lemma 1, this implies that if  $\mathbf{u}_n^t > 0$  in some period  $t$  then  $\mathbf{u}_j^\tau = 0$  for all  $j \neq n$  and all  $\tau > t$ . As  $u_i(x_i) = x_i$  for each  $i \in N$  by assumption, this in turn implies that  $\mathbf{u}_n^\tau = 1$  for all  $\tau > t$  which, from the claim above, is impossible. Consequently,  $\mathbf{u}_n^t = 0$  for every period  $t$ . We can now proceed inductively by applying the same argument to each player  $i < n$  until we reach the conclusion that  $\mathbf{u}_i^t = 0$  for all  $i$  and all  $t$  — which is evidently impossible in a DE.

**(ii) Construction of a Pareto efficient DE**

Let  $x^* \equiv (1, 0, \dots, 0)$  and, for each  $i \in N$ , let  $x^{-i} \in X$  be the policy defined as

$$x_j^{-i} \equiv \begin{cases} 0 & \text{if } j = i, \\ 1/(n-1) & \text{if } j \neq i. \end{cases}$$

To prove the result we construct a strategy profile  $\sigma$  that prescribes proposers to successfully offer policies in  $\{x^*\} \cup \{x^{-i} : i \in N\}$ , which are then never amended. On the path,  $\sigma$  induces the constant policy sequence  $\{x^*\}$  which is Pareto efficient, irrespective of players' payoff functions and discount factors. (An analogous construction can be used to support *any* policy sequence in which the same player receives the entire pie each period.)

Before proceeding, we begin with an informal description of the construction. On the path,  $\sigma$  prescribes all proposers to successfully offer  $x^*$  in the first period, and then to pass in all future periods. Any proposal to deviate from this path would be unsuccessful and the proposer — say  $i$  — would be “punished” with the perpetual implementation of  $x^{-i}$ . If a winning coalition  $C$  accepted such a proposal, then one of its members — say  $j$  — would be “punished” with the perpetual implementation of  $x^{-j}$ . Coalitions that fail to implement the prescribed punishments face similar punishments. For instance, farsighted player 2 does not offer players  $l > 1$  to deviate from policy sequence  $\{x^*\}$ , because she anticipates that those players would coalesce with 1 to implement policy  $x^{-2}$  indefinitely. By the same logic, farsighted players  $l > 1$  find it optimal to coalesce with player 1, because if they do not then they will face themselves similar punishments with positive probability.

The formal construction below proceeds in four steps. In Step (a), we define threshold  $\tilde{\delta}$ . Step (b) partitions the set of histories of the game into subsets  $\tilde{H}(C)$ , where “ $\tilde{h} \in \tilde{H}(C)$ ” is interpreted as “some member  $j$  of coalition  $C$  must be ‘punished’ (with the implementation of  $x^{-j}$ ) at history  $\tilde{h}$ .” Step (c) provides a formal definition of  $\sigma$ . In step (d), we check that, when  $\min_{i \in N} \delta_i > \tilde{\delta}$ , there is no history at which a player has a profitable one-shot deviation and no player uses weakly dominated voting strategies. By the one-shot deviation principle, this proves that  $\sigma$  is an undominated SPE.

**(a) Definition of  $\tilde{\delta}$ .** Let  $p^{\min} \equiv \min_{i \in N} p_i$ . For each  $j \in N$ , define

$$W_j^1(\delta_j) \equiv \max \left\{ (1 - p^{\min}) u_j \left( \frac{1}{1-n} \right), \frac{p_j (1 - \delta_j) u_j(1)}{1 - p_j \delta_j} \right\}$$

$$W_j^2(\delta_j) \equiv \min_{i \in N} \frac{(1 - p_i) u_j \left( \frac{1}{n-1} \right)}{1 - p_i \delta_j}.$$

Observe that  $W_j^1(\delta_j) \rightarrow (1 - p^{\min}) u_j \left( \frac{1}{1-n} \right)$  as  $\delta_j \rightarrow 1$ , and that  $W_j^2(\delta_j) \rightarrow u_j \left( \frac{1}{1-n} \right)$  as  $\delta_j \rightarrow 1$ . As  $1 - p^{\min} < 1$ , there exists  $\tilde{\delta}_j \in (0, 1)$  such that

$$(1 - \delta_j) u_j(1) + \delta_j W_j^1(\delta_j) < \delta_j W_j^2(\delta_j) \quad (4)$$

whenever  $\delta_j > \tilde{\delta}_j$ .

Define  $\tilde{\delta} \equiv \max_{j \in N} \tilde{\delta}_j$ , and assume henceforth that  $\min_{j \in N} \delta_j > \tilde{\delta}$ ; so that inequality (4) holds for all  $j \in N$ .

**(b) Histories.** In our construction, we only need to refer to histories at which a proposer is about to be selected. Accordingly, we will abuse terminology by referring to such paths as “histories.” A typical period- $t$  history is denoted by  $\tilde{h}^t$ ,<sup>31</sup> and we use  $\tilde{h}^t = \left( \tilde{h}^{t-1}, \tilde{h}^1 \right)$  to denote the concatenation of a period- $(t-1)$  history with a one-period history  $\tilde{h}^1$  — more precisely,  $\tilde{h}^1$  describes everything that happened in period  $t$  (proposer selection, proposal, pattern of votes, and implementation of a policy).

As explained above, we want to identify every history with the coalition of players to punish — or, equivalently, with the policies in  $\{x^{-i} : i \in N\}$  to implement indefinitely — at that history. To this end, we will partition the set of histories into a collection  $\{\tilde{H}(C) : C \subseteq N\}$  where, for each  $C \subseteq N$ ,  $\tilde{H}(C)$  can be thought of as the set of histories at which a member of  $C$  should be “punished” — in the sense that a policy in  $X(C) \equiv \{x^{-i} : i \in C\}$  should be indefinitely implemented. We define the elements of the partition as follows. Let  $X(\emptyset) \equiv \{x^*\}$ .

- (i)  $\tilde{H}(\emptyset)$  contains the null history, and all histories at which  $x^*$  has been proposed and (if there was a vote) unanimously accepted in all previous periods;
- (ii) And for any other history  $\tilde{h}^t = \left( \tilde{h}^{t-1}, \tilde{h}^1 \right)$  with  $\tilde{h}^{t-1} \in \tilde{H}(C)$  for some  $C \subseteq N$ :
  - (iia) If some  $y = x^{-i} \in X(C)$ , with  $C \neq \emptyset$ , is proposed and (if there is a vote) unanimously accepted in  $\tilde{h}^1$  then  $\tilde{h}^t \in \tilde{H}(\{i\})$ ;
  - (iib) If some  $y \in X(C)$  is proposed and rejected by the members of some (nonempty)  $C' \subseteq N$  in  $\tilde{h}^1$  then  $\tilde{h}^t \in \tilde{H}(C')$ ;
  - (iic) If player  $k$  proposes some  $y \notin X(C)$  which (if a vote takes place) is unanimously rejected in  $\tilde{h}^1$  then  $\tilde{h}^t \in \tilde{H}(\{k\})$ ;

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<sup>31</sup>We use a tilde to distinguish these histories from the implementation histories used in the rest of the paper.

- (**iiid**) If player  $k$  proposes some  $y \notin X(C)$  which is accepted by the members of some (nonempty)  $C' \subseteq N$  in  $\tilde{h}^1$  then  $\tilde{h}^t \in \tilde{H}(C')$ .

These conditions can be informally interpreted as follows: (**i**) At the start of the game and until policy  $x^*$  is amended, nobody should be punished; (**iiia**) If, in the previous period, a member of  $C$  was supposed to be punished, the proposer offered to punish some  $i \in C$  and the offer was unanimously accepted, then player  $i$  should be punished at the new history; (**iiib**) If, in the previous period, a member of  $C$  was supposed to be punished, the proposer offered to punish some  $i \in C$  and the offer was rejected by the members of some coalition  $C'$ , then some member of  $C'$  should be punished at the new history; (**iiic**) If, in the previous period, a member of  $C$  was supposed to be punished, proposer  $k$  did not offer to punish any  $i \in C$  and her offer was unanimously rejected, then player  $k$  should be punished at the new history; and (**iiid**) If, in the previous period, a member of  $C$  was supposed to be punished, the proposer did not offer to punish any  $i \in C$  and the offer was accepted by the members of some coalition  $C'$ , then some member of  $C'$  should be punished at the new history. Thus, any possible history belongs to an element of  $\{\tilde{H}(C) : C \subseteq N\}$ .

(**c**) **Definition of  $\sigma$ .** For each  $i \in N$ , we define the linear orders  $\triangleright_1, \dots, \triangleright_n$  on  $\{x^*\} \cup X(N)$  as:

- $x^* \triangleright_1 x^{-n} \triangleright_1 x^{-(n-1)} \triangleright_1 \dots \triangleright_1 x^{-1}$ ;
- $x^* \triangleright_i x^{-(i-1)} \triangleright_i \dots \triangleright_i x^{-1} \triangleright_i x^{-n} \triangleright_i \dots \triangleright_i x^{-i}$  for all  $1 < i < n$ ; and
- $x^* \triangleright_n x^{-(n-1)} \triangleright_n \dots \triangleright_n x^1 \triangleright_n x^{-n}$ .

Suppose that a history in  $\tilde{H}(C)$  — where  $C \subseteq N$  may be empty — has occurred, and let  $d \in X$  be the current default.  $\sigma$  prescribes the following behavior to each player  $j \in N$  after such a history:

*In proposal stages:* If  $C \neq \{j\}$  or  $d_j = 0$  then player  $j$  proposes the  $\triangleright_j$ -maximum in  $X(C)$ ; otherwise, she passes.

*In a voting stage with proposal  $y$  (irrespective of the proposer):* If  $y \in X(C)$  then player  $j$  accepts  $y$ ; if  $y \notin X(C)$  then player  $j$  rejects  $y$ .

According to  $\sigma$ , the following happens on the path. The null history belongs to  $\tilde{H}(\emptyset)$  and  $X(\emptyset) = \{x^*\}$ . As the default is  $d = (0, \dots, 0)$ ,  $\sigma$  prescribes all proposers to offer  $x^*$  which is unanimously accepted. From (**i**) in the definition of proposer histories, therefore, the following happens in every period  $t > 1$ : the ongoing default is  $x^*$  and  $X(C) = \{x^*\}$ , so that  $\sigma$  prescribes all proposers to offer  $x^*$  (i.e. pass). Hence,  $\sigma$  sustains the Pareto



efficient policy sequence in which  $x^*$  is implemented in every period.

Before we proceed to show that  $\sigma$  is an SPE in which no player uses weakly dominated strategies in voting stages, we establish two useful claims.

*Claim 1: Let  $C \subseteq N$ . At any history  $\tilde{h} \in \tilde{H}(C)$  ending with default  $d \in X$ , player  $j$ 's continuation value  $\tilde{V}_j^C(d)$  (engendered by  $\sigma$ ) is as follows:*

(i) If  $C = \emptyset$  then:  $\tilde{V}_j^C(d) = u_j(1)$  for  $j = 1$ , and  $\tilde{V}_j^C(d) = 0$  for  $j \neq 1$ ;

(ii) If  $C = \{i\}$  and  $d_i = 0$  for some  $i \in N$  then:  $\tilde{V}_j^C(d) = 0$  for  $j = i$ , and  $\tilde{V}_j^C(d) = u_j(1/(n-1))$  for  $j \neq i$ ;

(iii) If  $C = \{i\}$  and  $d_i > 0$  for some  $i \in N$  then:

$$\tilde{V}_j^C(d) = \begin{cases} \frac{p_i(1-\delta_i)u_i(d_i)}{1-p_i\delta_i} & \text{if } j = i, \\ \frac{p_i(1-\delta_j)u_j(d_j)+(1-p_i)u_j(\frac{1}{n-1})}{1-p_i\delta_j} & \text{otherwise;} \end{cases}$$

(iv) If  $\emptyset \neq C \neq \{i\}$  for all  $i \in N$  then:

$$\tilde{V}_j^C(d) = (1-p_j^-) u_j \left( \frac{1}{n-1} \right),$$

where  $p_j^- \equiv \sum_{l \in N^j} p_l$  and  $N^j \equiv \{l \in N : x^{-j} \triangleright_l x^{-i} \text{ for all } i \in C\}$ .

*Proof:* (i) At the null history (which belongs to  $\tilde{H}(\emptyset)$ ),  $\sigma$  prescribes all proposers to successfully offer  $x^*$ . In addition, if  $x^*$  was implemented in the first period and all selected proposers passed in subsequent periods then, from the definition of  $\sigma$  and (i) in the definition of histories, all future proposers will also pass. Thus, from any history in  $\tilde{H}(\emptyset)$ , policy  $x^*$  is indefinitely implemented; so that continuation values are given by  $\tilde{V}_j^\emptyset(d) = u_j(1)$  if  $j = 1$ , and  $\tilde{V}_j^\emptyset(d) = u_j(0) = 0$  for  $j \neq 1$ .

(ii) Suppose that  $C = \{i\}$  and  $d_i = 0$  for some  $i \in N$ . In this case,  $\sigma$  prescribes all proposers (including player  $i$ ) to offer  $x^{-i}$ , which is unanimously accepted. From (iia) in the definition of histories, the next period's history will also belong to  $\tilde{H}(\{i\})$ , ending with default  $x^{-i}$ . As  $x_i^{-i} = 0$ ,  $\sigma$  prescribes all proposers to offer  $x^{-i}$  (i.e. to pass) at that history. Applying the same argument to all future period, we obtain that policy  $x^{-i}$  is indefinitely implemented; so that continuation values are given by  $\tilde{V}_j^C(d) = u_j(x_i^{-i}) = u_j(0) = 0$  if  $j = i$ , and  $\tilde{V}_j^C(d) = u_j(x_j^{-i}) = u_j(1/(n-1))$  if  $j \neq i$ .

(iii) Suppose that  $C = \{i\}$  and  $d_i > 0$  for some  $i \in N$ . If any player  $l \neq i$  is selected to propose at  $\tilde{h}$  (which happens with probability  $(1-p_i)$ ) then she proposes  $x^{-i}$ , which (from

$\sigma$  and **(iia)** in the definition of histories) is unanimously accepted and never amended. If player  $i$  is selected to propose at  $\tilde{h}$  (which happens with probability  $p_i$ ) then she passes. From  $\sigma$  and **(iic)** in the definition of histories, the next period's history  $\tilde{h}'$  is still in  $\tilde{H}(\{i\})$  and the default remains  $d$  (with  $d_i > 0$ ). Hence,  $\sigma$  prescribes the same behavior at  $\tilde{h}$  and  $\tilde{h}'$ . This implies that players' continuation values from inducing histories  $\tilde{h}$  and  $\tilde{h}'$  are the same. Therefore, player  $i$ 's continuation value  $\tilde{V}_i^{\{i\}}(d)$  satisfies

$$\tilde{V}_i^{\{i\}}(d) = (1 - p_i) u_i(x_i^{-i}) + p_i \left[ (1 - \delta_i) u_i(d_i) + \delta_i \tilde{V}_i^{\{i\}}(d) \right]$$

or, equivalently,

$$\tilde{V}_i^{\{i\}}(d) = \frac{p_i(1 - \delta_i) u_i(d_i)}{1 - p_i \delta_i}$$

(recall that  $u_i(x_i^{-i}) = u_i(0) = 0$ ). Similarly, if  $j \neq i$  then player  $j$ 's continuation value  $\tilde{V}_j^{\{i\}}(d)$  satisfies

$$\tilde{V}_j^{\{i\}}(d) = (1 - p_i) u_i(x_j^{-i}) + p_i \left[ (1 - \delta_j) u_j(d_j) + \delta_j \tilde{V}_j^{\{i\}}(d) \right]$$

or, equivalently,

$$\tilde{V}_j^{\{i\}}(d) = \frac{p_i(1 - \delta_j) u_j(d_j) + (1 - p_i) u_j\left(\frac{1}{n-1}\right)}{1 - p_i \delta_j}$$

(recall that  $u_j(x_j^{-i}) = u_j(1/(n-1))$ ).

(iv) If  $C$  includes more than one player then  $\sigma$  prescribes each proposer  $l \in N$  to offer the  $\triangleright_l$ -maximum in  $X(C)$ . Thus, all proposers in  $N^j \equiv \{l \in N : x^{-j} \triangleright_l x^{-i} \text{ for all } i \in C\}$  offer  $x^{-j}$ , which (from  $\sigma$  and **(iia)** in the definition of histories) is unanimously accepted and never amended. Player  $j$ 's payoff is then  $u_j(x_j^{-j}) = u_j(0) = 0$  with probability  $p_j^-$ . With probability  $(1 - p_j^-)$ , the selected proposer successfully offers a policy  $x^{-i}$  with  $i \neq j$  (which by the same logic as above is never amended). Player  $j$  then receives  $u_j(x_j^{-i}) = u_j(1/(n-1))$ ; and her continuation value at history  $\tilde{h}$  is

$$\tilde{V}_j^C(d) = (1 - p_j^-) u_j\left(\frac{1}{n-1}\right).$$

◇

*Claim 2: For every player  $j \in N$  and every coalition  $C \ni j$ , we have*

$$(1 - \delta_j) u_j(1) + \delta_j \max_{d \in X} \tilde{V}_j^C(d) < \delta_j \min_{d \in X} \tilde{V}_j^{\{i\}}(d) \leq u_j\left(\frac{1}{n-1}\right), \text{ for all } i \neq j.$$

*Proof:* Fix  $j \in N$  and  $C \ni j$ , and take an arbitrary player  $i \neq j$ . From Claim 1(ii)-(iv),

$$\max_{d \in X} \tilde{V}_j^C(d) = \begin{cases} 0 & \text{if } C = \{j\} \text{ \& } d_j = 0, \\ \frac{p_j(1-\delta_j)u_j(1)}{1-p_j\delta_j} & \text{if } C = \{j\} \text{ \& } d_j > 0, \\ \left(1 - p_j^-\right) u_j\left(\frac{1}{n-1}\right) & \text{if } C \neq \{j\}, \end{cases}$$

and

$$\min_{d \in X} \tilde{V}_j^{\{i\}}(d) = \begin{cases} u_j\left(\frac{1}{n-1}\right) & \text{if } d_i = 0, \\ \frac{p_i(1-\delta_j)u_j(0) + (1-p_i)u_j\left(\frac{1}{n-1}\right)}{1-p_i\delta_j} = \frac{(1-p_i)u_j\left(\frac{1}{n-1}\right)}{1-p_i\delta_j} & \text{if } d_i > 0. \end{cases}$$

By construction, there is at least one proposer  $k \in N$  such that  $x^{-j}$  is the  $\triangleright_k$ -maximum in  $X(C)$ . Hence,  $p_j^- \geq p_k \geq p^{\min}$  or, equivalently,  $1 - p_j^- \leq 1 - p^{\min}$ . This implies that

$$\begin{aligned} (1 - \delta_j) u_j(1) + \delta_j \max_{d \in X} \tilde{V}_j^C(d) &\leq (1 - \delta_j) u_j(1) + \delta_j W_j^1(\delta_j) \\ &< \delta_j W_j^2(\delta_j) \leq \delta_j \frac{(1 - p_i) u_j\left(\frac{1}{n-1}\right)}{1 - p_i \delta_j} \\ &\leq \delta_j \min_{d \in X} \tilde{V}_j^{\{i\}}(d) \leq u_j\left(\frac{1}{n-1}\right), \end{aligned}$$

where the second inequality follows from  $\delta_j \geq \min_{i \in N} \delta_i > \tilde{\delta}$  (recall inequality (4)), and the last two inequalities from  $1 - p_i < 1 - p_i \delta_j$ .

◇

**(d)  $\sigma$  is an SPE:**

**(i) Voting strategies.** Suppose that a history in  $\tilde{H}(C)$ ,  $C \subseteq N$ , with current default  $d$  has occurred and that the selected proposer — say  $k$  — has offered  $y \neq d$ . Consider player  $j$ 's voting behavior in such a situation.

- *Case 1:  $C$  is empty and  $y = x^*$ .* Observe first that, by construction, the default is  $d = (0, \dots, 0)$  — if  $C = \emptyset$  and  $d = x^*$  then  $y = x^*$  implies that the proposer passes: there is no vote. From Claim 1(i), player  $j$ 's payoff if she does not deviate from  $\sigma$  is  $\tilde{V}_j^\emptyset(d) = u_j(1)$  if  $j = 1$ , and  $\tilde{V}_j^\emptyset(d) = 0$  if  $j \neq 1$ . Now consider a deviation by player 1. If she rejects  $x^*$  then the next period's history will be in  $\tilde{H}(\{1\})$  (see case **(iib)** in the definition of histories). As all the other players vote to accept  $x^*$  and  $q < n$ , her payoff from deviating is therefore

$$(1 - \delta_1) u_1(1) + \delta_1 \tilde{V}_1^{\{1\}}(x^*) = (1 - \delta_1) u_1(1) < u_1(1),$$

where the equality follows from Claim 1(ii) (and  $d_1 = 0$ ). This proves that the deviation is not profitable for player 1. In addition, as the inequality above is strict, she is strictly better off accepting  $x^*$  when all the other players also accept it. This proves that accepting  $x^*$  is not weakly dominated in this voting stage.

Now consider a deviation by player  $j \neq 1$ . By rejecting  $x^*$  (while all the other players accept it), she induces a history in  $\tilde{H}(\{j\})$  (see **(iib)** in the definition of histories). As  $d_j = 0$ , her payoff from deviation is therefore  $u_j(0)$  (Claim 1(ii)). This proves that the deviation is not profitable. To see that accepting  $x^*$  is not a weakly dominated strategy in this voting game, consider an action profile (in this stage) in which player 1 [resp. each player  $i \notin \{1, j\}$ ] rejects [resp. accepts]  $x^*$ . If  $j$  votes to accept  $x^*$  then the next history will be in  $\tilde{H}(\{1\})$  (see **(iib)** in the definition of histories); if she votes to reject  $x^*$  then the next history will be in  $\tilde{H}(\{1, j\})$ . In the former case, whether she is pivotal or not, her payoff is

$$(1 - \delta_j) u_j(0) + \delta_j \tilde{V}_j^{\{1\}}(x^*) = \delta_j \min_{d' \in X} \tilde{V}_j^{\{1\}}(d')$$

(Claim 1(iii)); in the latter case, whether she is pivotal or not, her payoff is lower than  $(1 - \delta_j) u_j(1) + \delta_j \max_{d' \in X} \tilde{V}_j^{\{1, j\}}(d')$  (Claim 1(iv)). It follows from Claim 2 that she is strictly better off accepting  $x^*$ . This proves that accepting  $x^*$  is not weakly dominated in this voting stage.

• *Case 2:  $C$  is nonempty and  $y \in X(C)$ .* Let  $i$  be the player in  $C$  such that  $y = x^{-i}$ . In this case,  $\sigma$  prescribes all players to accept proposal  $x^{-i}$ . Therefore, the next period's history will be in  $\tilde{H}(\{i\})$  (see case **(iia)** in the definition of histories). From Claim 1(ii), player  $j$ 's payoff is then

$$(1 - \delta_j) u_j(x_j^{-i}) + \delta_j \tilde{V}_j^{\{i\}}(x^{-i}) = u_j\left(\frac{1}{n-1}\right)$$

if  $j \neq i$  and

$$(1 - \delta_i) u_i(x_i^{-i}) + \delta_i \tilde{V}_i^{\{i\}}(x^{-i}) = u_i(0)$$

if  $j = i$ . Now suppose that player  $j \neq i$  deviates by rejecting proposal  $x^{-i}$ . As  $q < n$ , policy  $x^{-i}$  is still implemented in the current period. Moreover, the next period's history will be in  $\tilde{H}(\{j\})$  (see case **(iib)** in the definition of histories). From Claim 1(iii), her payoff from deviating is

$$(1 - \delta_j) u_j(x_j^{-i}) + \delta_j \tilde{V}_j^{\{j\}}(x^{-i}) < (1 - \delta_j) u_j(1) + \delta_j \max_{d' \in X} \tilde{V}_j^{\{j\}}(d') < u_j\left(\frac{1}{n-1}\right),$$

where the second inequality follows from Claim 2. Hence, she is strict better off accepting  $x^{-i}$  when all the other players also accept it. This proves that  $i$  cannot profitably deviate from accepting  $x^{-i}$ , which is not a weakly dominated strategy in this voting stage.

Now consider a deviation by player  $i$ : If she rejects  $x^{-i}$ , then  $x^{-i}$  is still implemented in the current period (she is not pivotal) and the next period's history will still be in  $\tilde{H}(\{i\})$  (see case **(iib)** in the definition of histories). Therefore, her payoff remains the same and the deviation is not profitable. To see that accepting  $x^{-i}$  is not a weakly dominated strategy in the stage game, consider an action profile (in the voting stage) in which all the other players but one — say  $l \neq i$  — vote to accept  $x^{-i}$ . If player  $i$  votes to accept  $x^{-i}$  then it is implemented ( $q < n$ ) and the next period's history will be in  $\tilde{H}(\{l\})$  (see case **(iib)**). From Claim 1(iii), her payoff is then

$$(1 - \delta_i) u_i(x_i^{-i}) + \delta_i \tilde{V}_i^{\{l\}}(x^{-i}) = \delta_i \tilde{V}_i^{\{l\}}(x^{-i}) = \delta_i \min_{d' \in X} \tilde{V}_i^{\{l\}}(d') .$$

If she votes to reject  $x^{-i}$  then, from case **(iib)** in the definition of histories, the next period's history will be in  $\tilde{H}(\{i, l\})$ . As  $\max\{u_i(0), u_i(d_i)\} \leq u_i(1)$ , it follows from Claim 1(iv) that (whether she is pivotal or not) her payoff must be lower than

$$(1 - \delta_i) u_i(1) + \delta_i \max_{d' \in X} \tilde{V}_i^{\{i, l\}}(d') < \delta_i \min_{d' \in X} \tilde{V}_i^{\{l\}}(d') ,$$

where the inequality follows from Claim 2. This proves that she is strictly better off accepting  $x^{-i}$  which, therefore, is not weakly dominated in this voting stage.

- *Case 3:  $y$  is not in  $X(C)$ .* Recall that the proposer is player  $k$ . In this case,  $\sigma$  prescribes all players to reject  $y$ . From case **(iic)** in the definition of histories, the next period's history will be in  $\tilde{H}(\{k\})$ . Hence, player  $j$ 's payoff is given by

$$(1 - \delta_j) u_j(d_j) + \delta_j \tilde{V}_j^{\{k\}}(d)$$

(Claim 1(ii)-(iii)). If player  $j \neq k$  deviates by accepting  $y$ , then default  $d$  is still implemented in the current period and the next period's history will be in  $\tilde{H}(\{j\})$  (see **(iib)** in the definition of histories). Her payoff is then equal to

$$\begin{aligned} (1 - \delta_j) u_j(d_j) + \delta_j \tilde{V}_j^{\{j\}}(d) &\leq (1 - \delta_j) u_j(1) + \delta_j \max_{d' \in X} \tilde{V}_j^{\{j\}}(d') \\ &< \delta_j \min_{d' \in X} \tilde{V}_j^{\{k\}}(d') \\ &\leq (1 - \delta_j) u_j(d_j) + \delta_j \tilde{V}_j^{\{k\}}(d) , \end{aligned}$$

where the second inequality follows from Claim 2 (recall that  $u_j(0) = 0 \leq u_j(d_j)$  for all  $d \in X$ ). Player  $j$  is thus *strictly* better off rejecting  $y$  which, consequently, is not a weakly dominated action in this voting stage.

Now consider a deviation by player  $k$ . She earns the same payoff after either accepting or rejecting her own proposal  $y$  because she is not pivotal in the current period, and the next period's history is still in  $\tilde{H}(\{k\})$  (see **(iid)** in the definition of histories). This proves that she cannot profitably deviate from rejecting  $y$ . To see that rejecting  $y$  is not weakly dominated in the stage game, consider an extra-equilibrium action profile in which all the other players but one — say  $l \neq k$  — reject  $y$ . If player  $k$  rejects  $y$  then default  $d$  is implemented (as  $q < n$ ) and the next period's history will be in  $\tilde{H}(\{l\})$  (see **(iid)** in the definition of histories). Her payoff is therefore equal to

$$(1 - \delta_k) u_k(d_k) + \delta_k \tilde{V}_k^{\{l\}}(d) \geq \delta_k \min_{d' \in X} \tilde{V}_k^{\{l\}}(d') .$$

If she accepts  $y$  then the next period's history will be in  $\tilde{H}(\{k, l\})$  (see case **(iid)**). As  $\max\{u_k(y_k), u_k(d_k)\} \leq u_k(1)$ , it follows from Claim 1(iv) that (whether she is pivotal or not) her payoff cannot exceed

$$(1 - \delta_k) u_k(1) + \delta_k \max_{d' \in X} \tilde{V}_k^{\{k, l\}}(d') < \delta_k \min_{d' \in X} \tilde{V}_k^{\{l\}}(d')$$

(where the inequality follows from Claim 2). This implies that player  $k$  is strictly better off rejecting  $y$ , which is therefore not weakly dominated.

**(ii) Proposal strategies.** Take an arbitrary history in  $\tilde{H}(C)$ ,  $C \subseteq N$ , and let  $d \in X$  be the current default. Let the selected proposer be player  $k \in N$ . If she proposes some policy  $y \in X(C)$  then, from the definition of  $\sigma$ ,  $y$  is unanimously accepted and never amended. Her payoff is therefore  $u_k(y_k)$ . If she proposes a policy outside  $X(C)$  then, by definition of  $\sigma$ , her proposal is unanimously rejected. Default  $d$  is implemented in the current period and, from **(iic)** in the definition of histories, the next period's history will belong to  $\tilde{H}(\{k\})$ . (The same applies if she passes when default  $d$  is not in  $X(C)$ .) Her payoff is therefore

$$\tilde{v}_k(d) \equiv (1 - \delta_k) u_k(d_k) + \delta_k \tilde{V}_k^{\{k\}}(d) .$$

We now prove that  $\sigma$  prescribes optimal behavior at any such history. To this end, suppose first that  $C = \emptyset$ ; so that the default  $d$  is either  $(0, \dots, 0)$  (at the null history) or  $x^*$ . According to  $\sigma$ , proposer  $k$  should offer the only element in  $X(C)$ ,  $x^*$ , which by

definition of  $\sigma$  would then be accepted and never amended.<sup>32</sup> If  $k = 1$  then there is evidently no profitable deviation from  $\sigma_k$  at this history: she obtains her maximal payoff of  $u_1(1)$ . If  $k \neq 1$  then a profitable deviation must yield a payoff that strictly exceeds  $u_k(x_k^*) = u_k(0) = 0$ . As explained above, any proposal that differs from  $x^*$  would be unanimously rejected, and player  $k$ 's payoff would be

$$\tilde{v}_k(d) = (1 - \delta_k) u_k(0) + \delta_k \tilde{V}_k^{\{k\}}(d) = 0$$

(since  $d_k = 0$ ), making the deviation unprofitable.

Now suppose that  $C$  is nonempty and that  $C \neq \{k\}$ . According to  $\sigma$ , proposer  $k$  should offer the  $\triangleright_k$ -maximum element  $y$  in  $X(C)$ , which would then be accepted and never amended. As  $C \neq \{k\}$ ,  $y_k = 1/(n-1)$  and she obtains a payoff of  $u_k(1/(n-1))$ . By construction, she could not earn a larger payoff by proposing another policy in  $X(C)$  (which would also be unanimously accepted). If she deviated by proposing a policy outside  $X(C)$  then she would receive a payoff of

$$\tilde{v}_k(d) \leq (1 - \delta_k) u_k(1) + \delta_k \max_{d' \in X} \tilde{V}_k^{\{k\}}(d') < u_k\left(\frac{1}{n-1}\right)$$

(where the second inequality follows from Claim 2). Hence, the deviation would not be profitable.

Next, suppose that  $C = \{k\}$  and  $d_k = 0$ . According to  $\sigma$ , proposer  $k$  should (successfully) offer  $y = x^{-k}$  — i.e. the  $\triangleright_k$ -maximum in  $X(C) = \{x^{-k}\}$  — thus obtaining a payoff of  $u_k(y_k) = u_k(0) = 0$ . If she deviates from  $\sigma$  by passing or (unsuccessfully) proposing another policy then, from Claim 1(ii), her payoff will be

$$\tilde{v}_k(d) = (1 - \delta_k) u_k(0) + \delta_k \tilde{V}_k^{\{k\}}(d) = 0 .$$

This proves that she cannot profitably deviate.

Finally, suppose that  $C = \{k\}$  and  $d_k > 0$  — so that  $d \notin X(C)$ . In this case,  $\sigma$  prescribes proposer  $k$  to pass, thereby obtaining  $\tilde{v}_k(d) > u_i(0) = 0$ . From the discussion above, proposing any other policy outside  $X(C)$  would yield the same payoff  $\tilde{v}_k(d)$ . If she deviates by proposing policy  $x^{-k}$  — i.e. the only policy in  $X(C)$  — then, by definition of  $\sigma$ , her offer will be accepted and never be amended. Hence, she gets  $u_i(x_k^{-k}) = u_k(0) = 0 < \tilde{v}_k(d)$ . As a result,  $k$  does not have a profitable deviation.

By the one-shot deviation principle,  $\sigma$  is an SPE. □

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<sup>32</sup>Observe that, when  $d = x^*$ , the proposer passes, precluding a vote.

**Theorem 3.** *If  $q = n$  then every DE  $\sigma$  is no-delay with  $A(\sigma) = \Delta_{n-1}$ .*

**Proof:** We prove Theorem 3 in four steps. The first step provides three preliminary lemmata: Lemmata 2 and 4 state useful results on continuation values and their relations to instantaneous utilities, and Lemma 3 shows that the set of absorbing points of any DE is nonempty and contained in the unit simplex. Using these results, Step 2 shows that, in every DE, the committee implements a policy in the unit simplex in every period, and Step 3 that the set of absorbing points of any DE coincides with the simplex. Step 4 concludes: the previous three steps jointly prove the theorem.

**Step 1: Preliminary results.** Recall that we use  $h^t$  to denote a typical “implementation history” — i.e. those just before the implementation of a new policy — and  $V_i^\sigma(h^t)$  to denote player  $i$ ’s continuation value at this history. For each  $x \in X$ , let  $H_x$  be the set of implementation histories just prior to the implementation of  $x$ . The proof of Theorem 3 hinges on the following lemmata. (Two of these lemmata not only apply to DEs, but to SPEs more generally.)

**Lemma 2.** *Suppose that  $q = n$ , and let  $\sigma$  be an SPE. For every  $i \in N$  and every history  $h^t \in H_x$ , we have*

- (i)  $V_i^\sigma(h^{t+1}) \geq u_i(x_i)$  for every equilibrium realization of  $h^{t+1}$  conditional on  $h^t$ ;
- (ii)  $V_i^\sigma(h^t) \geq u_i(x_i)$ ; and
- (iii)  $\mathbb{E}[V_i^\sigma(h^{t+1}) | h^t] \geq V_i^\sigma(h^t)$ .

*Proof:* (i) This is an immediate consequence of the unanimity rule. If  $V_i^\sigma(h^{t+1}) < u_i(x_i)$  for some realization of  $h^{t+1}$ , then player  $i$  could profitably deviate from  $\sigma$  by rejecting (and therefore preventing) any amendment of  $x$  leading to  $h^{t+1}$ .

(ii) Part (i) implies that  $V_i^\sigma(h^{t+1}) \geq u_i(x_i)$  for  $P^\sigma(h^t, \cdot)$ -almost all  $h^{t+1}$ . Hence  $V_i^\sigma(h^t)$ , which is a time average, must also exceed  $u_i(x_i)$ : By monotonicity of integrals, we have

$$\begin{aligned} V_i^\sigma(h^t) &= (1 - \delta_i) u_i(x_i) + \delta_i \int V_i^\sigma(h^{t+1}) P^\sigma(h^t, dh^{t+1}) \\ &\geq (1 - \delta_i) u_i(x_i) + \delta_i \int u_i(x_i) P^\sigma(h^t, dh^{t+1}) = u_i(x_i) . \end{aligned}$$

(iii) Suppose that  $\mathbb{E}[V_i^\sigma(h^{t+1}) | h^t] < V_i^\sigma(h^t)$ . From part (ii), this implies that

$$V_i^\sigma(h^t) = (1 - \delta_i) u_i(x_i) + \delta_i \mathbb{E}[V_i^\sigma(h^{t+1}) | h^t] < V_i^\sigma(h^t) ,$$

which is obviously impossible.



◇

**Lemma 3.** *Suppose that  $q = n$ . If  $\sigma$  is a DE then  $\emptyset \neq A(\sigma) \subseteq \Delta_{n-1}$ .*

*Proof:* It is easy to see that  $A(\sigma) \neq \emptyset$  (for instance, take policy  $(1, 0, \dots, 0) \in \Delta_{n-1}$ ).

Let  $\sigma$  be a DE and suppose, contrary to the statement of the result, that there exists  $x \in A(\sigma) \setminus \Delta_{n-1}$ . As  $x \notin \Delta_{n-1}$ , there is  $y \in \Delta_{n-1}$  such that  $u_i(y_i) > u_i(x_i)$  for each  $i \in N$ . By definition of  $A(\sigma)$ , there must be a sequence of (consecutive) implementation histories  $\{h^m\}$  such that, for all  $m$ ,  $h^m \in H_x$  and  $h^{m+1}$  is induced from  $h^m$  by  $\sigma$ . (In words,  $x$  is indefinitely implemented from  $h^1$  on according to  $\sigma$ .) Thus, the following is true for each  $i \in N$  and each  $h^m$ :

$$V_i^\sigma(h^m) = u_i(x_i) < u_i(y_i) \leq V_i^\sigma(h') \quad , \quad (5)$$

for any history  $h' \in H_y$  — the second inequality follows from Lemma 2(ii).

Let the first element of  $\{h^m\}$ ,  $h^1$ , be a period- $(\tau - 1)$  implementation history for some  $\tau \in \mathbb{N}$ . Consider the period- $\tau$  proposal stage that follows history  $h^1$ , and let the sequence of policies implemented prior to this stage be denoted by  $(x^1, x^2, \dots, x^{\tau-1})$  (so that  $x^{\tau-1} = x$ ). Now  $x$  is absorbing at  $h^1$ . Consequently, if proposal  $y \neq x$  were made at this stage of period  $\tau$ , then it would be rejected by at least one player  $j$  — otherwise any proposer  $i$  could profitably deviate from  $\sigma$  by inducing a history  $h' \in H_y$  (inequality (5) above). As player  $j$  never uses a weakly dominated strategy (recall from Section 3.2 that equilibria are required to be stage-undominated), she behaves as if pivotal in voting stages. Hence, her payoff from rejecting proposal  $y$  (and inducing policy history  $(x^1, \dots, x^{\tau-1}, x)$ ) must be at least as great as her payoff from accepting it. As  $\sigma$  is a DE (so that behavior only depends on policy histories rather than entire histories), her payoff from rejecting  $y$  is equal to  $V_j^\sigma(h^2)$ ; her payoff from accepting it is equal to  $V_j^\sigma(h')$  for some  $h' \in H_y$ . Hence, we have  $V_j^\sigma(h^2) \geq V_j^\sigma(h')$ , which contradicts (5).

◇

At this point, we need some notation. Any strategy profile  $\sigma = (\sigma_i)_{i \in N}$  induces a stochastic process  $\{\tilde{x}^t\}$  on the policy space, where the random variable  $\tilde{x}^t$  stands for the policy implemented in period  $t$ . For any period- $t$  implementation history  $h^t$  and any  $m \in \mathbb{N}$ , we can define a random variable  $\tilde{x}^m(h^t)$ , which describes the policy implemented in period  $t + m$  conditional on  $h^t$ . Thus,  $\mathbb{E}[\tilde{x}_i^m(h^t)] = \mathbb{E}[\tilde{x}_i^{t+m}|h^t]$ , where  $\mathbb{E}[\cdot]$  is the expectation operator with respect to the stochastic process engendered by  $\sigma$ .

**Lemma 4.** *If  $\sigma$  is an SPE then the following statements are true for all  $x \in X$  and all  $h^t \in H_x$ :*

- (i)  $(\tilde{x}^m(h^t))$  converges almost surely to a limit  $\tilde{x}(h^t)$ ;
- (ii) For every  $i \in N$ , we have

$$u_i(x_i) \leq \mathbb{E}[V_i^\sigma(h^{t+1}) | h^t] \leq \mathbb{E}[u_i(\tilde{x}_i(h^t))] . \quad (6)$$

*Proof:* Take an arbitrary  $x \in X$  and an arbitrary  $h^t \in H_x$ .

(i) By Proposition 1 in Hyndman and Ray (2007), the stochastic sequence  $(u_i(\tilde{x}_i^m(h^t)))_{i \in N}$  converges almost surely to a limit.<sup>33</sup> As the  $u_i$ 's are strictly increasing functions, the stochastic sequence of policies  $(\tilde{x}^m(h^t))$  converges along any sample path for which  $(u_i(\tilde{x}_i^m(h^t)))_{i \in N}$  converges. Hence,  $(\tilde{x}^m(h^t))$  converges almost surely to a limit  $\tilde{x}(h^t)$ .

(ii) The first inequality in (6) is an immediate implication of Lemma 2(ii)-(iii).

To complete the proof of the lemma, therefore, it remains to establish that

$$\mathbb{E}[V_i^\sigma(h^{t+1}) | h^t] \leq \mathbb{E}[u_i(\tilde{x}_i(h^t))]$$

for all  $i \in N$ . To do so, observe first that Lemma 2(iii) (applied recursively) implies that  $\mathbb{E}[V_i^\sigma(h^{t+m}) | h^t] \leq \mathbb{E}[V_i^\sigma(h^{t+m+1}) | h^t]$  and, therefore, that  $\mathbb{E}[V_i^\sigma(h^{t+1}) | h^t] \leq \mathbb{E}[V_i^\sigma(h^{t+m}) | h^t]$  for all  $m \in \mathbb{N}$ . Now suppose that, contrary to our assertion,  $\mathbb{E}[V_i^\sigma(h^{t+1}) | h^t] - \mathbb{E}[u_i(\tilde{x}_i(h^t))] = \varepsilon > 0$ . By definition,

$$\mathbb{E}[V_i^\sigma(h^{t+m}) | h^t] = (1 - \delta_i) \mathbb{E}\left[\sum_{\tau=0}^{\infty} \delta_i^\tau u_i(\tilde{x}^{m+\tau}(h^t))\right] = (1 - \delta_i) \sum_{\tau=0}^{\infty} \delta_i^\tau \mathbb{E}[u_i(\tilde{x}^{m+\tau}(h^t))] .$$

As  $(\tilde{x}^m(h^t))$  converges almost surely to a limit  $\tilde{x}(h^t)$ , Lebesgue's Dominated Convergence Theorem implies that  $\mathbb{E}[u_i(\tilde{x}_i^m(h^t))] \rightarrow \mathbb{E}[u_i(\tilde{x}_i(h^t))]$ . This in turn implies that there exists  $M \geq 1$  such that  $\mathbb{E}[u_i(\tilde{x}_i^m(h^t))] \leq \mathbb{E}[u_i(\tilde{x}_i(h^t))] + \frac{\varepsilon}{2}$  and, consequently,

$$\mathbb{E}[V_i^\sigma(h^{t+m}) | h^t] \leq \mathbb{E}[u_i(\tilde{x}_i(h^t))] + \frac{\varepsilon}{2} < \mathbb{E}[V_i^\sigma(h^{t+1}) | h^t] \quad (7)$$

for all  $m > M$ . This contradicts our initial observation that  $\mathbb{E}[V_i^\sigma(h^{t+1}) | h^t] \leq \mathbb{E}[V_i^\sigma(h^{t+m}) | h^t]$  for all  $m \in \mathbb{N}$ .

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<sup>33</sup>Note that Hyndman and Ray's result applies to a more general class of coalitional games — in which unanimous voting is only a special case.

**Step 2: If  $\sigma$  is a DE then, in every period, the committee implements a policy in  $\Delta_{n-1}$  according to  $\sigma$ .** For any  $w \in X$  and any  $h \in H_w$ , define policy  $y(h)$  as  $y(h) \equiv \mathbb{E}[\tilde{x}(h)]$ . As the  $u_i$ 's are concave, Jensen's inequality implies that

$$\mathbb{E}[u_i(\tilde{x}_i(h))] \leq u_i(y_i(h)) \quad (8)$$

for all  $i \in N$ .

Now suppose, contrary to the statement of the lemma, that there is some implementation history  $h^t$  such that the committee implements  $x \notin \Delta_{n-1}$  — so that  $h^t \in H_x$ . Using (8) and Lemma 4(ii), we obtain

$$u_i(x_i) \leq \mathbb{E}[V_i^\sigma(h^{t+1}) | h^t] \leq \mathbb{E}[u_i(\tilde{x}_i(h^t))] \leq u_i(y_i(h^t)) \quad , \quad (9)$$

which implies that  $y_i(h^t) \geq x_i$  for all  $i \in N$ . Consequently, there must be some  $y \in \Delta_{n-1}$  such that  $y_i \geq y_i(h^t)$  and  $y_i > x_i$  for all  $i \in N$ . We therefore have

$$V_i^\sigma(h^t) = (1 - \delta_i) u_i(x_i) + \delta_i \mathbb{E}[V_i^\sigma(h^{t+1}) | h^t] < u_i(y_i) \leq V_i^\sigma(h') \quad (10)$$

for all  $i \in N$  and all  $h' \in H_y$  (where the last inequality follows from Lemma 2(ii)). This in turn implies that any proposer whose proposal induces implementation history  $h^t$  (and therefore policy history  $(x^1, \dots, x^{t-1}, x)$ ) can profitably deviate by inducing  $h' \in H_y$  (and therefore policy history  $(x^1, \dots, x^{t-1}, y)$ ) instead. Proposal  $y$  is unanimously accepted for the following reason. Once  $y$  has been offered, voters implicitly have to choose between policy sequences  $(x^1, \dots, x^{t-1}, y)$  and  $(x^1, \dots, x^{t-1}, x^{t-1})$  — as  $\sigma$  is a DE, each player's continuation value  $V_i^\sigma(h)$  at any history  $h$  only depends on the history of policies in  $h$ . If  $x^{t-1} = x$  (so that  $x$  is the default at the start of period  $t$ ), then  $(x^1, \dots, x^{t-1}, x^{t-1})$  is the policy sequence in  $h^t$  and (10) ensures that all players are strictly better off accepting  $y$ . If  $x^{t-1} \neq x$ , then  $x$  is in the acceptance set after policy sequence  $(x^1, \dots, x^{t-1})$  and, by (10), so is  $y$ . We therefore have a contradiction with  $\sigma$  being a DE.

**Step 3: If  $\sigma$  is a DE then  $A(\sigma) = \Delta_{n-1}$ .** We already know from Lemma 3 that  $A(\sigma) \subseteq \Delta_{n-1}$ . To complete the proof of this step, we must show that every point in the unit simplex is absorbing.

Let  $x' \in \Delta_{n-1}$  and suppose, contrary to the statement above, that there is a history at which  $x'$  is amended to some  $x \neq x'$  with positive probability. From Step 2, we have  $x \in \Delta_{n-1}$ . By (9), we have  $u_i(x_i) \leq u_i(y_i(h^t))$  for all  $i \in N$  and all  $h^t \in H_x$ . But, as  $x \in \Delta_{n-1}$ , this implies that  $x = y(h^t)$ , and therefore, that  $u_i(x_i) = u_i(y_i(h^t))$  for all

$i \in N$  and all  $h^t \in H_x$ . Using (9) again, this implies that  $u_i(x_i) = \mathbb{E}[V_i^\sigma(h^{t+1}) | h^t]$  for every  $i \in N$  and all  $h^t \in H_x$ . Hence, at implementation history  $h^t \in H_x$  where the current default  $x'$  is about to be amended to  $x$ ,

$$u_i(x_i) = (1 - \delta_i) u_i(x_i) + \delta_i \mathbb{E}[V_i^\sigma(h^{t+1}) | h^t] = V_i^\sigma(h^t) \geq u_i(x'_i)$$

for all  $i \in N$  (where the inequality follows from Lemma 2(i)). As  $x'$  is by assumption an element of the simplex, the inequality above implies that  $x = x'$ , thus yielding the desired contradiction.

**Step 4: Every DE  $\sigma$  is a no-delay DE with  $A(\sigma) = \Delta_{n-1}$ .** Combining Steps 2 and 3, we obtain that every DE is no-delay and that its absorbing set coincides with the unit simplex.

□

For future reference (proof of Theorem 4(ii)), observe that before a policy in  $\Delta_{n-1}$  is implemented no proposer randomizes over proposals. Indeed, the no-delay property implies that, at any default outside  $\Delta_{n-1}$ , each proposer makes an offer that is accepted by all players. By sequential rationality, the proposer must give the other players the minimum shares that they are willing to accept.

**Theorem 4.** *If  $q = n$  then: (i) a pure strategy no-delay SMPE exists; and (ii) SMPE payoffs are unique and coincide with the stationary SPE payoffs of the Baron-Ferejohn model.*

**Part (i)**

To prove Theorem 4(i), we will construct an equilibrium  $\sigma$  in which, at any default  $x \in X$ , the selected proposer — say  $i$  — offers the committee a policy  $x + s^i(x) \in \Delta_{n-1}$ , which is accepted by all players and then never amended. We can think of proposer  $i$  offering to share the amount of money not distributed yet — i.e.  $1 - \sum_{j \in N} x_j$  — with the other players, with  $s_j^i(x)$  being the share offered by proposer  $i$  to player  $j$ . The first step of the proof is to define these transfers and to use them to construct the stationary Markov pure-strategy profile  $\sigma$ . Step 2 then establishes that  $\sigma$  is a no-delay stage-undominated SMPE. To do so, we first establish that the set of absorbing points of  $\sigma$  coincides with the

simplex, and determine the continuation-value function induced by  $\sigma$  (Claim 1). Using these results, we then show that players never use weakly dominated strategies in voting stages (Claim 2), and that there are no profitable one-shot deviations from  $\sigma$  in proposal stages (Claim 3). By the one-shot deviation principle, this establishes Theorem 4(i).

**Step 1: Construction of stationary Markov pure-strategy profile  $\sigma$ .** For each  $x \in X$ , let

$$T_x \equiv \left\{ s \in [0, 1]^n : \sum_{j \in N} x_j + s_j = 1 \right\} .$$

Thus, any element of the  $n$ -fold product of  $T_x, T_x^n$ , can be thought of as a vector of shares of the budgetary surplus  $s = (s^i)_{i \in N}$ , where  $s^i \in T_x$  stands for the shares offered by proposer  $i$ . Next, let  $\phi(x)(\cdot) = (\phi^1(x)(\cdot), \dots, \phi^n(x)(\cdot))$  be a self-map on  $T_x^n$  defined as follows: for all  $i \in N$  and all  $s = (s^k)_{k \in N} \in T_x^n$ ,

$$\phi_j^i(x)(s) \equiv u_j^{-1} \left( (1 - \delta_j) u_j(x_j) + \delta_j \sum_{k \in N} p_k u_j(x_j + s_j^k) \right) - x_j, \quad \forall j \neq i,$$

$$\phi_i^i(x)(s) \equiv 1 - x_i - \sum_{j \neq i} [x_j + \phi_j^i(x)(s)] .$$

As all the  $u_i$ 's are by assumption continuous,  $\phi(x)(\cdot)$  is a continuous function from  $T_x^n$  (which is convex and compact in  $\mathbb{R}^{n^2}$ ) into itself. Brouwer's Fixed Point Theorem then implies that there is  $s(x) = (s_j^i(x))_{i, j \in N} \in T_x^n$  such that  $\phi(x)(s(x)) = s(x)$ ; that is

$$u_j(x_j + s_j^i(x)) = (1 - \delta_j) u_j(x_j) + \delta_j \sum_{k \in N} p_k u_j(x_j + s_j^k(x)), \quad \forall j \neq i, \quad (11)$$

$$x_i + s_i^i(x) = 1 - \sum_{j \neq i} [x_j + s_j^i(x)], \quad (12)$$

for all  $i \in N$ . Observe that, by construction,  $x + s^i(x) \in \Delta_{n-1}$  for all  $i \in N$  and all  $x \in X$ . Moreover, if  $x \in \Delta_{n-1}$  then  $T_x = \{(0, \dots, 0)\}$  and, therefore,  $s^i(x) = (0, \dots, 0)$  for every  $i \in N$ .

We are now in a position to define the strategy profile  $\sigma = (\sigma_1, \dots, \sigma_n)$ :

- In the proposal stage of any period  $t$  with ongoing default  $x^{t-1} = x$ ,  $i$ 's proposal (conditional on  $i$  being selected as proposer) is  $x + s^i(x)$ ;
- In the voting stage of any period  $t$  with ongoing default  $x^{t-1} = x$ , following any proposal  $y \in X \setminus \{x\}$ , player  $i$  accepts if and only if

$$(1 - \delta_i) u_i(y) + \delta_i \sum_{j \in N} p_j u_i(y_i + s_i^j(y)) \geq (1 - \delta_i) u_i(x) + \delta_i \sum_{j \in N} p_j u_i(x_i + s_i^j(x)) .$$

Observe that  $\sigma$  is a pure strategy stationary Markov strategy combination. To complete the proof of Theorem 4(i), it therefore remains to show that  $\sigma$  is a no-delay, stage-undominated SPE.

**Step 2: Proof that  $\sigma$  is a no-delay, stage-undominated SPE.** We proceed in several steps.

*Claim 1:  $\sigma$  is no-delay with  $A(\sigma) = \Delta_{n-1}$  and, for all  $i \in N$  and all  $x \in X$ :*

$$V_i^\sigma(x) = (1 - \delta_i) u_i(x_i) + \delta_i \sum_{j \in N} p_j u_i(x_i + s_i^j(x)) .$$

*Proof:* If  $x \in \Delta_{n-1}$  then  $\sigma$  prescribes all proposers to pass in all periods. This implies that  $x \in A(\sigma)$  — thus establishing that  $\Delta_{n-1} \subseteq A(\sigma)$  — and, for each  $i \in N$ ,

$$V_i^\sigma(x) = u_i(x_i) = (1 - \delta_i) u_i(x_i) + \delta_i \sum_{j \in N} p_j u_i(x_i + s_i^j(x))$$

(since  $x \in \Delta_{n-1}$  implies that  $s_i^j(x) = 0$  for all  $i, j \in N$ ).

If  $x \notin \Delta_{n-1}$  then, in the next period,  $\sigma$  prescribes each proposer  $j$  to propose policy  $x + s^j(x)$ . As  $x + s^j(x) \in \Delta_{n-1}$ , we have  $s_i^k(x + s^j(x)) = 0$  for all  $i, k \in N$ , so that

$$(1 - \delta_i) u_i(x_i + s_i^j(x)) + \delta_i \sum_{k \in N} p_k u_i(x_i + s_i^j(x) + s_i^k(x + s^j(x))) = u_i(x_i + s_i^j(x)) , \quad (13)$$

for all  $i \in N$ . From the definition of voting strategies, therefore, player  $i$  accepts if and only if

$$u_i(x_i + s_i^j(x)) \geq (1 - \delta_i) u_i(x_i) + \delta_i \sum_{k \in N} p_k u_i(x_i + s_i^k(x)) ,$$

which by equation (11) holds for all  $i \neq j$ . To prove that  $j$ 's proposal is voted up, we therefore need to confirm that she accepts her own proposal. By concavity of the  $u_i$ 's, equation (11) implies that

$$\begin{aligned} u_i(x_i + s_i^j(x)) &= (1 - \delta_i) u_i(x_i) + \delta_i \sum_{k \in N} p_k u_i(x_i + s_i^k(x)) \\ &\leq u_i\left((1 - \delta_i) x_i + \delta_i \sum_{k \in N} p_k (x_i + s_i^k(x))\right) = u_i\left(x_i + \delta_i \sum_{k \in N} p_k s_i^k(x)\right) , \end{aligned}$$

for all  $i \neq j$ , which in turn implies that  $s_i^j(x) \leq \sum_{k \in N} p_k s_i^k(x)$  for all  $i \neq j$  (recall that  $\delta_i \in (0, 1)$ ). Using this inequality and the concavity of  $u_j$ , we obtain

$$\begin{aligned}
& (1 - \delta_j) u_j(x_j) + \delta_j \sum_{k \in N} p_k u_j(x_j + s_j^k(x)) \leq u_j\left(x_j + \delta_j \sum_{k \in N} p_k s_j^k(x)\right) \\
& = u_j\left(x_j + \delta_j \sum_{k \in N} p_k \left[1 - \sum_{l \in N} x_l - \sum_{l \neq j} s_l^k(x)\right]\right) = u_j\left(x_j + \delta_j \left[\sum_{i \in N} s_i^j(x) - \sum_{i \neq j} \sum_{k \in N} p_k s_i^k(x)\right]\right) \\
& \leq u_j\left(x_j + \delta_j \left[\sum_{i \in N} s_i^j(x) - \sum_{i \neq j} s_i^j(x)\right]\right) = u_j(x_j + \delta_j s_j^j(x)) \leq u_j(x_j + s_j^j(x)) \\
& = (1 - \delta_j) u_j(x_j + s_j^j(x)) + \delta_j \sum_{k \in N} p_k u_j(x_j + s_j^j(x) + s_j^k(x + s_j^j(x))) , \tag{14}
\end{aligned}$$

where the last equality follows from (13). Thus,  $\sigma_j$  prescribes player  $j$  to accept as well, and  $x_j + s_j^j(x)$  is therefore voted up. This proves that policies outside the simplex cannot be absorbing points of  $\sigma$  — i.e.  $(X \setminus \Delta_{n-1}) \cap A(\sigma) = \emptyset$  — and, therefore, that  $A(\sigma) = \Delta_{n-1}$ . This also proves that  $P^\sigma(x, A(\sigma)) = P^\sigma(x, \Delta_{n-1}) = 1$  for all  $x \in X$ ; that is,  $\sigma$  is no-delay.

Moreover, as  $x_j + s_j^j(x) \in \Delta_{n-1}$ ,  $\sigma$  prescribes all proposers to pass in all future periods. This implies that, for all  $i \in N$  and  $x \notin X$ ,

$$V_i^\sigma(x) = (1 - \delta_i) u_i(x_i) + \delta_i \sum_{j \in N} p_j u_i(x_i + s_i^j(x)) ,$$

thus completing the proof of the claim.

For future reference (see Claim 3 below), observe that (14) implies that  $V_i^\sigma(x + s^i(x)) \geq V_i^\sigma(x)$  for any player  $i \in N$ .

*Claim 2: Given default  $x$  and proposal  $y$ , each voter  $i \in N$  accepts if and only if  $V_i^\sigma(y) \geq V_i^\sigma(x)$ , and rejects only if  $V_i^\sigma(x) \geq V_i^\sigma(y)$ .*

*Proof:* This is an immediate consequence of Claim 1 and the definition of voting strategies.

*Claim 3: There is no profitable one-shot deviation from  $\sigma$  in the proposal stage of any period.*

*Proof:* Let  $x^{t-1} = x$ , and suppose that player  $i$  is recognized to make a proposal in period  $t$ . If she plays according to  $\sigma_i$  then she proposes  $x + s^i(x)$  (or, equivalently, passes when  $x \in \Delta_{n-1}$ ). As  $\sigma$  is no-delay (Claim 1), this offer is accepted and player  $i$ 's payoff is  $u_i(x_i + s_i^i(x))$ .

In the proof of Claim 1, we showed that  $V_i^\sigma(x) \leq V_i^\sigma(x + s^i(x))$ . Hence, player  $i$  cannot profitably deviate by passing or by making a proposal that is voted down.

Now consider a deviation to a proposal  $y \neq x + s^i(x)$ , which is accepted. According to the definition of voting strategies,  $y$  must satisfy

$$(1 - \delta_j) u_j(y_j) + \delta_j \sum_{k \in N} p_k u_j(y_j + s_j^k(y)) \geq (1 - \delta_j) u_j(x_j) + \delta_j \sum_{k \in N} p_k u_j(x_j + s_j^k(x)) \quad (15)$$

for all  $j \in N$ . We distinguish between two different cases:

- Case 1:  $y \in \Delta_{n-1}$ . In this case, inequality (15) becomes

$$u_j(y_j) \geq (1 - \delta_j) u_j(x_j) + \delta_j \sum_{k \in N} p_k u_j(x_j + s_j^k(x)) = u_j(x_j + s_j^i(x))$$

for all  $j \neq i$  (the equality is obtained from (11)). As  $u_j$  is increasing, this implies that  $y_j \geq x_j + s_j^i(x)$  for all  $j \neq i$  and, consequently,

$$x_i + s_i^i(x) = 1 - \sum_{j \neq i} (x_j + s_j^i(x)) \geq 1 - \sum_{j \neq i} y_j = y_i .$$

This in turn implies that  $V_i^\sigma(x_i + s_i^i(x)) = u_i(x_i + s_i^i(x)) \geq u_i(y_i) = V_i^\sigma(y)$ . Hence, proposing  $y \in \Delta_{n-1}$  is not a profitable (one-shot) deviation for player  $i$ .

- Case 2:  $y \notin \Delta_{n-1}$ . In this case, equations (11) and (15) (as well as the concavity of the  $u_j$ 's) imply that

$$\begin{aligned} u_j \left( y_j + \delta_j \sum_{k \in N} p_k s_j^k(y) \right) &\geq (1 - \delta_j) u_j(y_j) + \delta_j \sum_{k \in N} p_k u_j(y_j + s_j^k(y)) \\ &\geq (1 - \delta_j) u_j(x_j) + \delta_j \sum_{k \in N} p_k u_j(x_j + s_j^k(x)) \\ &= u_j(x_j + s_j^i(x)) , \end{aligned}$$

so that  $y_j + \sum_{k \in N} p_k s_j^k(y) \geq x_j + s_j^i(x)$  for all  $j \neq i$  (recall that  $\delta_j \in (0, 1)$  and  $s_j^k(y) \geq 0$  for all  $j, k \in N$ ). Consequently,

$$\begin{aligned} x_i + s_i^i(x) &= 1 - \sum_{j \neq i} [x_j + s_j^i(x)] \\ &\geq 1 - \sum_{j \neq i} \left[ y_j + \sum_{k \in N} p_k s_j^k(y) \right] = 1 - \sum_{j \neq i} y_j - \sum_{k \in N} \left[ p_k \sum_{j \neq i} s_j^k(y) \right] . \quad (16) \end{aligned}$$



Moreover, by equation (12),

$$\sum_{j \neq i} s_j^k(y) = 1 - \sum_{l \in N} y_l - s_i^k(y) . \quad (17)$$

Combining (16) and (17), we obtain

$$x_i + s_i^i(x) \geq 1 - \sum_{j \neq i} y_j - \sum_{k \in N} \left[ p_k \left( 1 - \sum_{l \in N} y_l - s_i^k(y) \right) \right] = y_i + \sum_{k \in N} p_k s_i^k(y) .$$

Hence:

$$\begin{aligned} V_i^\sigma(x_i + s_i^i(x)) &= u_i(x_i + s_i^i(x)) \geq u_i \left( y_i + \sum_{k \in N} p_k s_i^k(y) \right) \geq u_i \left( (1 - \delta_i) y_i + \delta_i \sum_{k \in N} p_k [y_i + s_i^k(x)] \right) \\ &\geq (1 - \delta_i) u_i(y_i) + \delta_i \sum_{k \in N} p_k u_i(y_i + s_i^k(x)) = V_i^\sigma(y) . \end{aligned}$$

This shows that proposing  $y \notin \Delta_{n-1}$  is not a profitable deviation for player  $i$ , and completes the proof of Claim 3.

Combining Claims 1-3, we obtain Theorem 4(i).

### Part (ii)

Denote our game with an evolving default by  $\Gamma^e$ , and the game with a constant default of  $x^0 = (0, \dots, 0)$  by  $\Gamma^c$ . To prove the second part of the theorem, we first show that, for every SMPE  $\sigma$  of  $\Gamma^e$ , we can construct a stationary strategy profile  $\sigma^c$  in  $\Gamma^c$  that generates the same payoffs as  $\sigma$  in  $\Gamma^e$ . We then show that  $\sigma^c$  is a stationary SPE of  $\Gamma^c$ . Uniqueness of SMPE payoffs in  $\Gamma^e$  then follows from uniqueness of stationary SPE in  $\Gamma^c$  (Observation 2).

Let  $\sigma = (\sigma_i)_{i \in N}$  be an SMPE of  $\Gamma^e$ , and let  $\pi^i(x) \in X$  be the proposal made by player  $i$  when the ongoing default is some  $x$  outside  $\Delta_{n-1}$ . (Recall that proposers do not randomize at such a default: cf. the paragraph immediately after the proof of Theorem 3.) Hence, player  $i$ 's expected payoff as evaluated after rejection of a proposal in the first period is given by:

$$V_i^\sigma(x^0) = (1 - \delta_i) u_i(x^0) + \delta_i \sum_{j \in N} p_j u_i(\pi_i^j(x^0))$$

(recall that, by Theorem 3,  $\sigma$  must be no-delay).

Now define the stationary strategy profile  $\sigma^c = (\sigma_i^c)_{i \in N}$  in game  $\Gamma^c$  as follows. At the proposal stage of every period  $t$ , each player  $i \in N$  makes proposal  $\pi^i(x^0)$ . At the

voting stage of each period, player  $i$  accepts the proposal just made, say  $y$ , if and only if  $u_i(y) \geq V_i^\sigma(x^0)$ .

As  $\sigma$  is no-delay, proposal  $\pi^i(x^0)$ ,  $i \in N$ , must be accepted with probability 1 at default  $x^0$  in  $\Gamma^e$ . By sequential rationality and unanimity rule, this implies that  $V_j^\sigma(\pi^i(x^0)) = u_j(\pi_j^i(x^0)) \geq V_j^\sigma(x^0)$  for all  $j \in N$ , which in turn implies that proposal  $\pi^i(x^0)$  is also accepted with probability 1 in any period of  $\Gamma^c$ . Two immediate consequences of this observation are that: (i) player  $i$ 's expected payoff as evaluated after rejection of a proposal in the first period of  $\Gamma^c$  is  $V_i^\sigma(x^0)$ ; and (ii) player  $i \in N$  has no profitable deviation from the voting behavior prescribed by  $\sigma_i^c$ .

To complete the proof of the result, therefore, it remains to show that no player  $i \in N$  can profitably deviate from  $\sigma^c$  in a proposal stage of  $\Gamma^c$ . Consider the proposal of an arbitrary player  $i$  when the default is  $x^0$ . As  $\sigma$  is an SMPE of  $\Gamma^e$ , player  $i$  cannot profitably deviate by (successfully) proposing a policy  $y \in X \setminus \{\pi^i(x^0)\}$  or by making an unsuccessful proposal at default  $x^0$ . Hence,

$$V_i^\sigma(\pi^i(x^0)) = u_i(\pi^i(x^0)) \geq \max \left\{ V_i^\sigma(x^0), (1 - \delta_i) u_i(y) + \delta_i \sum_{j \in N} p_j u_i(\pi_j^i(y)) \right\} \geq u_i(y)$$

where the second inequality follows from Lemma 2(ii). Now consider a deviation from  $\pi^i(x^0)$  in  $\Gamma^c$ . If  $i$  proposed some policy  $y$  then her expected payoff would be  $u_i(y)$  if her proposal were successful, and  $V_i^\sigma(x^0)$  otherwise. Hence, the inequality above implies that  $i$  cannot improve upon proposing  $\pi^i(x^0)$  and, therefore, cannot profitably deviate from  $\sigma_i^c$  in proposal stages.

The theorem then follows from Observation 2, which says that Baron and Ferejohn's (1989) model has unique stationary SPE payoffs when  $q = n$ .

□

**Theorem 5.** *Suppose that  $q = n$ .*

- (i) *If  $\delta_i \neq \delta_j$ , for some  $i, j \in N$ , then every DE is ex post Pareto inefficient.*
- (ii) *Any (Pareto efficient) policy sequence that allocates the entire pie to the same player in every period can be supported by an SPE.*

**Proof:**

(i) **Ex post Pareto inefficiency of DEs**

We begin with a lemma which shows that, in a DE, the pie is never entirely allocated to a single player; so that transfers among players are always feasible. This will then allow us to prove that any realization of a DE policy sequence can be Pareto-improved using transfers across periods.

**Lemma 5.** *Let  $q = n$ . If  $\{\tilde{x}^t\}$  is the stochastic sequence of policies on some DE path then, for every realization  $\{x^t\}$  of  $\{\tilde{x}^t\}$ , we have  $x_i^t \in (0, 1)$  for all  $i \in N$  and all  $t \in \mathbb{N}$ .*

*Proof:* Let  $\{x^t\}$  be an arbitrary realization of the sequence  $\{\tilde{x}^t\}$  engendered by some DE  $\sigma$ . Suppose that, contrary to the statement above,  $x_j^\tau = 0$  for some  $j \in N$  and some  $\tau \in \mathbb{N}$ . Theorem 3 then implies that, in the first period, player  $j$  accepted a proposal  $x$  such that  $x_j = x_j^t = 0$  for all  $t \in \mathbb{N}$ . Player  $j$ 's payoff under  $\sigma$  is therefore  $u_j(0) = 0$ .

To prove the lemma, we will now show that  $j$  could profitably deviate — i.e. obtain a payoff strictly greater than  $u_j(0)$  — by rejecting  $x$ . To this end, suppose that  $x$  is rejected in period 1 and that  $j$  is selected to propose at the start of period 2. Define  $h_0 \in H_{x^0}$  as the implementation history that would be induced by a rejection of  $j$ 's proposal. We know from Theorem 3 that, for each  $i \in N$ ,

$$V_i^\sigma(h_0) = (1 - \delta_i) u_i(0) + \delta_i \sum_{l \in N} p_l u_i(x_i^l) ,$$

where  $x^l$  denotes player  $l$ 's successful proposal in period 3. As  $x^l \in \Delta_{n-1}$  for all  $l \in N$ ,  $W \equiv \{i \in N : u_i(0) < \sum_{l \in N} p_l u_i(x_i^l)\}$  is nonempty; so that, for each  $i \in W$ ,

$$V_i^\sigma(h_0) < \sum_{l \in N} p_l u_i(x_i^l) \leq u_i \left( \sum_{l \in N} p_l x_i^l \right) ,$$

where the second inequality follows from Jensen's inequality. By continuity of the  $u_i$ 's, therefore, there exists a sufficiently small  $\varepsilon > 0$  such that

$$V_i^\sigma(h_0) < u_i \left( \sum_{l \in N} p_l x_i^l - \varepsilon \right) , \forall i \in W .$$

By definition of  $W$ ,  $V_i(h_0) = u_i(0)$  for every  $i \in N \setminus W$ .

Now define policy  $y = (y_i)_{i \in N} \in X$  as follows:

$$y_i \equiv \sum_{l \in N} p_l x_i^l - \varepsilon , \text{ for all } i \in W , \text{ and } y_i \equiv \frac{|W|}{n - |W|} \varepsilon > 0 \text{ for all } i \in N \setminus W .$$

It is readily checked that  $u_i(y_i) > V_i^\sigma(h_0)$  and then, by Lemma 2(ii),

$$V_i^\sigma(h_0) = (1 - \delta_i) u_i(0) + \delta_i \sum_{l \in N} p_l u_i(x_i^l) < u_i(y_i) \leq V_i^\sigma(h)$$

for all  $h \in H_y$  and all  $i \in N$ . By the same argument as in Step 2 in the proof of Theorem 3, this inequality implies that  $j$  could successfully propose  $y$  in period 2 and thus get a payoff of  $V_j^\sigma(h) > V_j^\sigma(h_0) \geq u_j(0)$ . This in turn implies that her equilibrium proposal under  $\sigma$  (conditional on being recognized to make a proposal in period 2) must yield a payoff at least as great as  $V_j^\sigma(h) > u_j(0)$ . As  $p_j > 0$  and  $q = n$  (and  $u_j(0)$  is obviously the minimum payoff she can get), she must therefore reject any period-1 proposal  $x$  such that  $x_j = 0$  in equilibrium.

◇

Suppose that there are  $i, j \in N$  such that  $\delta_i > \delta_j$ . Now suppose that, contrary to the Theorem, there exists an ex post Pareto efficient DE  $\sigma$ . This implies that some realization of the policy sequence engendered by  $\sigma$  is Pareto efficient. Take one of these realizations, say  $\{x^t\}$ . From Theorem 3, there exists a policy  $\bar{x} \in \Delta_{n-1}$  such that  $x^t = \bar{x}$  for all  $t \in \mathbb{N}$ . To obtain the desired contradiction, therefore, it suffices to show that the indefinite implementation of  $\bar{x}$  can be Pareto improved.

Lemma 5 implies that  $\bar{x}_i$  and  $\bar{x}_j$  are both in  $(0, 1)$ . Consequently, there is a feasible marginal transfer  $dx_j^1$  from player  $i$  to player  $j$  in period 1, and a marginal transfer  $dx_j^2$  from  $j$  to  $i$  in period 2, such that player 1's discounted payoff remains unchanged. If we suppose by contradiction that the repeated implementation of policy  $\bar{x}$  is Pareto efficient then the changes in players  $i$  and  $j$ 's payoffs must satisfy:

$$-u'_i(\bar{x}_i) dx_j^1 + \delta_i u'_i(\bar{x}_i) dx_j^2 = 0, \text{ and } u'_j(\bar{x}_j) dx_j^1 - \delta_j u'_j(\bar{x}_j) dx_j^2 \leq 0,$$

where  $u'_i(\bar{x}_i) > 0$  and  $u'_j(\bar{x}_j) > 0$  — recall that by assumption all players' (instantaneous) payoff functions are strictly increasing. Combining these two conditions, we obtain  $\delta_i = dx_j^1/dx_j^2 \leq \delta_j$ , which contradicts our initial assumption that  $\delta_i > \delta_j$ .

## (ii) Construction of a Pareto efficient SPE

For every  $d = (d_1, \dots, d_n) \in X$  and  $i \in N$ , let  $x^i(d)$  be the policy in  $X$  that allocates  $d_j$  to each player  $j \neq i$  and the residual to player  $i$ ; that is, for each  $j \in N$ :

$$x_j^i(d) \equiv \begin{cases} 1 - d_{-i} & \text{if } j = i, \\ d_j & \text{if } j \neq i, \end{cases}$$

where  $d_{-i} \equiv 1 - \sum_{j \neq i} d_j$ .

To prove the result, we construct a strategy profile  $\sigma$  that has the following absorbing policies:  $x^i(d)$  for all  $d \in X$  and all  $i \in N$ . On the path,  $\sigma$  induces the constant policy sequence  $\{x^1(x^0)\} = \{(1, 0, \dots, 0)\}$  which is Pareto efficient, irrespective of players' payoff functions and discount factors. (An analogous construction can be used to support any constant policy sequence of the form  $\{x^i(x^0)\}$  for some  $i \in N$ .) The construction below proceeds in three steps. Step (a) partitions the set of histories of the game into subsets  $\tilde{H}(C)$ , where " $\tilde{h} \in \tilde{H}(C)$ " is interpreted as "some member  $i$  of coalition  $C$  must be 'rewarded' (with the implementation of  $x^i(d)$ ) at history  $\tilde{h}$ ." Step (b) provides a formal definition of  $\sigma$ . In step (c), we check that there is no history at which a player has a profitable one-shot deviation, and that none of the players use a dominated voting strategy. By the one-shot deviation principle, this proves that  $\sigma$  is a (stage-undominated) SPE.

**(a) Histories.** In our construction, we only need to refer to histories at which a proposer is about to be selected. Accordingly, we will abuse terminology by referring to such paths as "histories." A typical period- $t$  history is denoted by  $\tilde{h}^t$ ,<sup>34</sup> and we use  $\tilde{h}^t = (\tilde{h}^{t-1}, \tilde{h}^1)$  to denote the concatenation of a period- $(t-1)$  history with a one-period history  $\tilde{h}^1$  — more precisely,  $\tilde{h}^1$  describes everything that happened in period  $t$  (proposer selection, proposal, pattern of votes, and implementation of a policy).

As explained above, we want to identify every history with the players to reward at that history. To this end, we will partition the set of histories into a collection  $\{\tilde{H}(C) : \emptyset \neq C \subseteq N\}$  where, for each nonempty coalition  $C \subseteq N$ ,  $\tilde{H}(C)$  can be thought of as the set of histories at which a member of  $C$  should be "rewarded" — in the sense that a policy  $x^i(d)$ , for some  $i \in C$ , should be indefinitely implemented when the current default is  $d$ . We define the elements of the partition as follows.

- (i) The null history and all the histories at which  $(1, 0, \dots, 0)$  has been proposed and (if there was a vote) unanimously accepted in all previous periods are contained in  $\tilde{H}(\{1\})$ ;
- (ii) And for any other history  $\tilde{h}^t = (\tilde{h}^{t-1}, \tilde{h}^1)$  where  $\tilde{h}^{t-1}$  belongs to  $\tilde{H}(C)$ , for some nonempty  $C \subseteq N$ , and ends with the implementation of some  $d \in X$ :
  - (iia) If some  $x^i(d)$ , where  $i \in C$ , is proposed and (if there is a vote) unanimously accepted in  $\tilde{h}^1$ , then  $\tilde{h}^t \in \tilde{H}(\{i\})$ ;

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<sup>34</sup>As in the proof of Theorem 2, we use a tilde to distinguish these histories from implementation histories.

- (iib) If some  $x^i(d)$ , where  $i \in C$ , is proposed and rejected in  $\tilde{h}^1$ , then  $\tilde{h}^t \in \tilde{H}(\{i\})$ ;
- (iic) If player  $k$  proposes some  $y \neq x^i(d)$  for all  $i \in C$ , which (if a vote takes place) is unanimously accepted in  $\tilde{h}^1$  then  $\tilde{h}^t \in \tilde{H}(\{k\})$ ;
- (iid) If player  $k$  proposes some  $y \neq x^i(d)$  for all  $i \in C$ , which is rejected by the members of some (nonempty)  $C' \neq \{k\}$  in  $\tilde{h}^1$  then  $\tilde{h}^t \in \tilde{H}(C' \setminus \{k\})$ ;
- (iie) If player  $k$  proposes some  $y \neq x^i(d)$  for all  $i \in C$ , which is rejected by player  $k$  alone in  $\tilde{h}^1$  then  $\tilde{h}^t \in \tilde{H}(\{k\})$ .

These conditions can be informally interpreted as follows: **(i)** At the start of the game and until some player attempts to amend policy  $(1, 0, \dots, 0)$ , player 1 should be rewarded; **(iia)** If a member of  $C$  was supposed to be rewarded in the last period, the proposer offered to reward some  $i \in C$ , and the offer was unanimously accepted, then player  $i$  should be rewarded at the new history; **(iib)** If a member of  $C$  was supposed to be rewarded in the last period, the proposer offered to reward some  $i \in C$  and the offer was rejected, then player  $i$  should be rewarded at the new history; **(iic)** If a member of  $C$  was supposed to be rewarded in the last period, the proposer  $k$  did not offer to reward any  $i \in C$  and her offer was unanimously accepted, then player  $k$  should be rewarded at the new history; **(iid)** If a member of  $C$  was supposed to be rewarded in the last period, the proposer  $k$  did not offer to reward any  $i \in C$  and the offer was rejected by the members of some coalition  $C' \neq \{k\}$  (which may include player  $k$ ) then some member of  $C' \setminus \{k\}$  should be rewarded at the new history; and **(iie)** If a member of  $C$  was supposed to be rewarded in the last period, the proposer  $k$  did not offer to reward any  $i \in C$  and the offer was rejected by player  $k$  alone, then player  $k$  should be rewarded at the new history.

**(b) Definition of  $\sigma$ .** For each  $i \in N$ , we define the linear order  $\triangleright_i$  on  $N$  as:

- $1 \triangleright_1 2 \triangleright_1 \dots \triangleright_1 n$ ;
- $i \triangleright_i i+1 \triangleright_i \dots \triangleright_i n \triangleright_i 1 \triangleright_i \dots \triangleright_i i-1$  for all  $1 < i < n$ ; and
- $n \triangleright_n 1 \triangleright_n 2 \triangleright_n \dots \triangleright_n n-1$ .

Suppose that a history in  $\tilde{H}(C)$ ,  $\emptyset \neq C \subseteq N$ , ending with default  $d \in X$  has occurred. Strategy profile  $\sigma$  prescribes the following behavior after such a history:

*In proposal stages:* Player  $j$  proposes  $x^i(d)$  where  $i$  is the  $\triangleright_j$ -maximum in  $C$ .

*In a voting stage with proposal  $y$  by player  $k$ :* If  $y \in \{x^i(d) : i \in C\}$  then  $\sigma$  prescribes every player  $j$  to accept  $y$ ; if  $y \notin \{x^i(d) : i \in C\}$  then:

- when  $d \notin \Delta_{n-1}$ ,  $\sigma$  prescribes player  $j \neq k$  to reject  $y$ , and player  $k$  to accept it if and only if

$$(1 - \delta_k) u_k(y_k) + \delta_k u_k(1 - y_{-k}) > (1 - \delta_k) u_k(d_k) + \delta_k u_k(1 - d_{-k}) ;$$

- when  $d \in \Delta_{n-1}$ ,  $\sigma$  prescribes every player  $j \neq k$  to accept  $y$  if and only if  $u_j(y_j) > u_j(d_j)$ , and player  $k$  to accept it if and only if

$$(1 - \delta_k) u_k(y_k) + \delta_k u_k(1 - y_{-k}) > u_k(d_k) .$$

According to  $\sigma$ , the following happens on the path. The null history belongs to  $\tilde{H}(\{1\})$ . Therefore, in period 1, all proposers offer  $(1, 0, \dots, 0)$  which is unanimously accepted. From **(i)** in the definition of proposer histories, therefore, we have  $d = (1, 0, \dots, 0)$  and  $\tilde{h}^t \in \tilde{H}(\{1\})$  in every period  $t \geq 1$ . This in turn implies that all proposers offer  $(1, 0, \dots, 0)$  (i.e. pass) in every period  $t > 1$ . Hence,  $\sigma$  sustains the efficient policy sequence in which  $(1, 0, \dots, 0)$  is implemented in every period.

**(c)  $\sigma$  an SPE: (i) Voting strategies.** Suppose that a history in  $\tilde{H}(C)$ ,  $\emptyset \neq C \subseteq N$ , ending with default  $d \in X$  has occurred and that the selected proposer – say  $k$  – has offered  $y \neq d$ . Consider player  $j$ 's voting behavior in such a situation.

- *Case 1:  $y = x^i(d)$  for some  $i \in C$ .* If player  $j$  plays in accordance with  $\sigma_j$ , then she accepts proposal  $x^i(d)$ . As the other players do the same according to  $\sigma$ , policy  $x^i(d)$  will be implemented and never be amended. Player  $j$ 's payoff is therefore

$$u_j(x_j^i(d)) = \begin{cases} u_j(d_j) & \text{if } j \neq i , \\ u_j(1 - d_{-j}) & \text{if } j = i . \end{cases}$$

Now suppose that player  $j$  deviated from  $\sigma$  by rejecting proposal  $x^i(d)$  in the current period. The current default  $d$  would be implemented in the period and then, from **(iib)** in the definition of histories and the definition of  $\sigma$ ,  $x^i(d)$  would be successfully proposed in the next period and never amended. As  $d_j \leq 1 - d_{-j}$ , this implies that  $u_j(x_j^i(d))$  is the maximum payoff that player  $j$  can earn at such a history. This shows that, *irrespective of the other players' actions* in this voting stage, player  $j$  cannot profitably deviate from  $\sigma_j$ , which is therefore not weakly dominated.

- *Case 2:  $d \notin \Delta_{n-1}$  and  $y \neq x^i(d)$  for all  $i \in C$ .* Suppose first that  $j \neq k$  — i.e. player  $j$  is not the proposer. Strategy  $\sigma$  prescribes  $j$  and all  $l \neq k$  to reject  $y$ . If  $j$  plays according

to  $\sigma$  then  $y$  will be rejected and she will receive  $(1 - \delta_j) u_j(d_j)$  in the current period. From **(iid)** in the definition of histories, this history at the start of the next period will then be in  $\tilde{H}(N \setminus \{k\})$ ; so that, by definition of  $\sigma$ , a nonempty set of proposers (including herself) will successfully offer  $x^j(d)$  while the others will successfully propose policies in  $\{x^l(d) : l \in N \setminus \{j, k\}\}$ . In the former case she will receive  $u_j(1 - d_{-j}) > u_j(d_j)$  (recall that  $d \notin \Delta_{n-1}$ ) in all future periods; in the latter she will receive  $u_j(d_j)$  in all future periods.

If she deviated from  $\sigma$ , then the default  $d$  would still be implemented in the current period (players  $l \neq k$  would still reject  $y$ ) and some policy in  $\{x^l(d) : l \in N \setminus \{i, k\}\}$  would be implemented in all future periods. Hence, player  $j$ 's payoff would be  $u_j(d_j)$ , which is strictly less than her payoff from rejecting  $y$  (because  $j$  proposes, and therefore obtains  $u_j(1 - d_{-j}) > u_j(d_j)$ , with positive probability next period). This proves that player  $j$  cannot profitably deviate from rejecting  $y$  and that this is not a weakly dominated strategy in the stage game.

Now suppose that  $j = k$  — i.e. player  $j$  is the proposer. As the other players reject her proposal (according to  $\sigma$ ), she receives  $(1 - \delta_k) u_k(d_k)$  in the current period. From **(iid)** in the definition of histories, the next period's history belongs to  $\tilde{H}(C)$  with  $C$  not including  $k$ ; so that, *irrespective of  $k$ 's move*, a policy  $x^l(d)$  with  $l \neq k$  will be implemented in the next period and never amended. This implies that player  $k$ 's discounted sum of payoffs from the next period on is  $u_k(d_k)$  irrespective of her choice in the voting stage. Hence, under  $\sigma$ , player  $j = k$  cannot profitably deviate from  $\sigma$  because her payoff will be  $u_k(d_k)$  whether she accepts the proposal or not. This is true as long as at least one of the other players rejects  $y$ . To prove that  $\sigma_k$  is a weakly undominated strategy in the stage-game, it therefore remains to show that she could not improve on the choice prescribed by  $\sigma_k$  if all the other players accepted  $y$ . In this case, accepting  $y$  would lead to a history in  $\tilde{H}(\{k\})$  (**(iic)** in the definition of histories): in the next period,  $x^k(d)$  would be implemented (and never amended) with probability 1. Her payoff would then be  $(1 - \delta_k) u_k(y_k) + \delta_k u_k(1 - y_{-k})$ . Rejecting  $y$  would also induce a history in  $\tilde{H}(\{k\})$  (**(iie)** in the definition of histories), so that  $x^k(d)$  would be implemented with probability 1 in the next period. Her total payoff would therefore be  $(1 - \delta_k) u_k(d_k) + \delta_k u_k(1 - d_{-k})$ . From the definition of the proposer's voting strategy, this implies that her voting behavior is weakly undominated in the stage-game.

- *Case 3:  $d \in \Delta_{n-1}$  and  $y \neq x^i(d)$  for all  $i \in C$ .* As  $d$  is in the simplex,  $x_j^i(d) = d_j$  for



all  $i, j \in N$ . From the definition of  $\sigma$ , any policy in the simplex is absorbing. Therefore, rejection of  $y$  leads to the indefinite implementation of  $d$ , yielding a payoff of  $u_j(d_j)$  for player  $j$  irrespective of who voted  $y$  down. This implies that, whenever any other player rejects  $y$ , player  $j$  is indifferent between accepting and rejecting; so that her strategy is a best response.

Therefore, to show that  $j$  cannot profitably deviate from  $\sigma$  and that  $\sigma$  prescribes her a weakly undominated action in this voting stage, it suffices to show that she cannot improve on playing according to  $\sigma$  when all the other players accept  $y$ . In this case, if  $j$  accepts  $y$  then she receives  $(1 - \delta_j)u_j(y_j)$  in the current period. From **(iic)** in the definition of histories, the next period's history is in  $\tilde{H}(\{k\})$ , so that player  $j$  will receive  $u_j(x_j^k(y)) = u_j(y_j)$  in all future periods if  $j \neq k$ , and  $u_j(x_k^k(y)) = u_j(1 - y_{-k})$  in all future periods if  $j = k$ . Her total payoff from accepting  $y$  is therefore  $u_j(y_j)$  if  $j \neq k$ , and  $(1 - \delta_j)u_j(y_j) + \delta_j u_j(1 - y_{-j})$  if  $j = k$ . As explained in the previous paragraph, her payoff will be  $u_j(d_j)$  if, instead, she rejects  $y$ . By definition of  $\sigma$ , therefore,  $j$ 's choice (as prescribed by  $\sigma$ ) is a best response and weakly undominated in the stage-game.

**(ii) Proposal strategies.** Take an arbitrary history in  $\tilde{H}(C)$ ,  $\emptyset \neq C \subseteq N$ , and let  $d \in X$  be the current default. If proposer  $k$  offers some  $x^i(d)$  with  $i \in C$  (as prescribed by  $\sigma$ ), then from the definition of voting strategies,  $x^i(d)$  is unanimously accepted (and never amended). Her payoff is therefore

$$u_k(x_k^i(d)) = \begin{cases} u_k(1 - d_{-k}) & \text{if } k = i, \\ u_k(d_k) & \text{if } k \neq i. \end{cases}$$

Suppose first that  $d \notin \Delta_{n-1}$ . If  $k$  deviated from  $\sigma$  by proposing some  $y \neq x^i(d)$  for all  $i \in C$ , then her proposal would be rejected by all other players (so that  $d$  would be implemented in the current period). From **(iic)** in the definition of histories, the next period's history would be in  $\tilde{H}(N \setminus \{k\})$ . By definition of  $\sigma$ , this implies that some policy  $x^l(d)$ , with  $l \neq k$ , would be implemented indefinitely. As  $x_k^l(d) = d_k$ , this implies that her total payoff from deviating would be  $u_i(d_k) < u_k(1 - d_{-k}) \leq u_k(x_k^i(d))$ : the deviation would not be profitable.

Now suppose that  $d \in \Delta_{n-1}$  — so that  $x_k^i(d) = d_k$  for all  $i \in N$ . If player  $k$  makes a proposal that is rejected by at least one of the other players, then the same argument as in the previous paragraph shows that such a deviation cannot be profitable. If player  $k$  makes a proposal that only she rejects, then she receives  $(1 - \delta_k)u_k(d_k)$  in the current period.

As the next period's history will be in  $\tilde{H}(\{k\})$  ((**ii**) in the definition of histories), she will then receive  $u_k(x_k^k(d)) = u_k(d_k)$ . Hence, her payoff from the deviation is the same as that from not deviating; i.e.  $u_k(d_k)$ . Finally, by definition of voting strategies when  $d \in \Delta_{n-1}$ , player  $k$  would have to offer  $y_j > d_j$  to each  $j \neq k$  to make a successful proposal. As  $d$  is in the simplex, this implies that  $y_k \leq 1 - y_{-k} < 1 - d_{-k} = d_k$ . Hence, player  $k$ 's payoff from making a successful proposal  $(1 - \delta_k)u_k(y_k) + \delta_k u_k(1 - y_{-k})$  would be strictly less than  $u_k(d_k) = u_k(x_k^k(d))$ . As a result,  $i$  does not have a profitable deviation.

By the one-shot deviation principle,  $\sigma$  is an SPE. □

**Observation 2.** *If  $q = n$  then the Baron-Ferejohn model has a unique stationary SPE.*

**Proof:** We prove Observation 2 in two steps. Step 1 derives several properties of stationary SPE behavior, which we will use in Step 2 to establish the uniqueness of a stationary SPE.

*Step 1: Properties of stationary SPEs.*

Let  $\sigma$  be any stationary SPE and, for each  $i \in N$ , let  $W_i^\sigma$  be player  $i$ 's continuation value from moving to period  $t + 1$  conditional on  $\sigma$  and on any period- $t$  proposal being rejected. Because  $\sigma$  is a stationary SPE, the very first proposer makes a successful proposal. Let  $x^i = (x_1^i, \dots, x_n^i) \in X$  be player  $i$ 's proposal when she is selected to propose. As  $W_j^\sigma$  is the most that player  $j$  can expect to receive from continuing the bargaining process beyond the current period, routine arguments imply that  $j$  votes to accept proposal  $x^i$  if and only if  $u_j(x_j^i) \geq \delta_j W_j^\sigma$  (recall that  $u_j(0) = 0$ ). Since  $u_i$  is strictly increasing in  $x_i^i$ , this in turn implies that  $u_j(x_j^i) = \delta_j W_j^\sigma$  for all  $j \neq i$ . Therefore, each player  $j \in N$  receives the same share of the pie  $\bar{x}_j^\sigma \equiv u_j^{-1}(\delta_j W_j^\sigma)$  from all proposers  $i \neq j$ . (Players do not randomize in equilibrium: each proposer allocates just enough to the other players to induce them to accept the proposal and allocates the residual to herself.)

By definition of  $W_j^\sigma$ , we can rewrite  $u_j(\bar{x}_j^\sigma) = \delta_j W_j^\sigma$  as

$$u_j(\bar{x}_j^\sigma) = \delta_j \left[ p_j u_j(x_j^j) + (1 - p_j) u_j(\bar{x}_j^\sigma) \right]$$

or, equivalently, as

$$u_j(\bar{x}_j^\sigma) = \frac{\delta_j p_j}{1 - \delta_j(1 - p_j)} u_j(x_j^j) \equiv \lambda_j u_j(x_j^j) . \quad (18)$$

As  $\lambda_j \in (0, 1)$ , we have  $\bar{x}_j^\sigma < x_j^j$  for all  $j \in N$ . This implies that

$$D^\sigma \equiv 1 - \sum_{i \in N} \bar{x}_i^\sigma = 1 - \sum_{i \neq j} \bar{x}_i^\sigma - \bar{x}_j^\sigma = x_j^j - \bar{x}_j^\sigma > 0 .$$

Substituting into (18), we obtain

$$u_j(\bar{x}_j^\sigma) = \lambda_j u_j(D^\sigma + \bar{x}_j^\sigma) . \quad (19)$$

Now take an arbitrary  $i \in N$  and define  $H_i : [0, 1]^2 \rightarrow \mathbb{R}$  as

$$H_i(s, f) \equiv u_i(f) - \lambda_i u_i(s + f) , \text{ for all } s, f \in [0, 1] .$$

It is readily checked that  $H_i(0, 0) = 0$  and that, for all  $(s, f) \neq (0, 0)$ ,  $H_i$  is strictly decreasing in  $s$ , strictly increasing in  $f$  and continuous in its arguments. Let  $S_i \equiv \max \{s \in [0, 1] : H_i(s, 1) \geq 0\}$ . The properties of  $H_i$  ensure that we can (explicitly) define  $f_i : [0, S_i] \rightarrow [0, 1]$  as the unique solution to  $H_i(s, f_i(s)) \equiv 0$  for all  $s \in [0, S_i]$ . By the implicit function theorem, the derivative of  $f_i$  satisfies

$$f_i'(s) = \frac{\lambda_i u_i'(s + f_i(s))}{u_i'(f_i(s)) - \lambda_i u_i'(s + f_i(s))} > 0$$

for all  $s$  (where the inequality follows from  $\lambda_i \in (0, 1)$ ,  $u_i' > 0$  and concavity of the  $u_i$ 's).

By definition of the  $H_i$ 's and  $f_i$ 's, we know from (19) that  $\bar{x}_i^\sigma$  and  $D^\sigma$  must satisfy

$$\bar{x}_i^\sigma = f_i(D^\sigma) \text{ for all } i \in N , \quad (20)$$

in any stationary SPE  $\sigma$ .

*Step 2: Uniqueness of stationary SPE.*

To complete the proof of the Observation, it suffices to show that for any two stationary SPEs of the Baron-Ferejohn model,  $\sigma_1$  and  $\sigma_2$ , we have  $\bar{x}_j^{\sigma_1} = \bar{x}_j^{\sigma_2}$  for all  $j \in N$  — so that  $\sigma_1 = \sigma_2$ . Suppose instead that  $\bar{x}_i^{\sigma_1} < \bar{x}_i^{\sigma_2}$  for some  $i \in N$ . As  $f_i$  is strictly increasing, (20) implies that  $D^{\sigma_1} < D^{\sigma_2}$ . This in turn implies that  $\bar{x}_j^{\sigma_1} < \bar{x}_j^{\sigma_2}$  for all  $j \in N$  and, consequently, that  $\sum_{j \in N} \bar{x}_j^{\sigma_1} < \sum_{j \in N} \bar{x}_j^{\sigma_2}$ . We then have

$$D^{\sigma_1} \equiv 1 - \sum_{j \in N} \bar{x}_j^{\sigma_1} > 1 - \sum_{j \in N} \bar{x}_j^{\sigma_2} \equiv D^{\sigma_2} :$$

a contradiction.

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