Decomposition schemes for symmetric n-ary bands

Jimmy Devillet

University of Luxembourg
Luxembourg

in collaboraton with Pierre Mathonet

n-ary semigroups

$n \geq 2$

(X, F) n-ary groupoid : X nonempty set and $F\colon X^n \to X$

Definition. (Dörnte, 1928)
$F\colon X^n \to X$ is *associative* if

$$\begin{aligned} & F(x_1, \ldots, x_{i-1}, F(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{2n-1}) \\ = \quad & F(x_1, \ldots, x_i, F(x_{i+1}, \ldots, x_{i+n}), x_{i+n+1}, \ldots, x_{2n-1}) \end{aligned}$$

for all $x_1, \ldots, x_{2n-1} \in X$ and all $1 \leq i \leq n-1$.

$\Longrightarrow$ (X, F) is an *n-ary semigroup*

$F\colon X^n \to X$ is

- *idempotent* if $F(x, \ldots, x) = x \; \forall \, x \in X$
- *symmetric* if F is invariant under the action of permutations

$\Longrightarrow$ *symmetric n-ary band*

$n = 2$: semilattices

$n \geq 3$: ?

Example of symmetric ternary band

Consider $X = \{1,2,3\}$ and define $F\colon X^3 \to X$ by its level sets given (up to permutations) by

- $F^{-1}(\{1\}) = \{(1,1,1)\}$,
- $F^{-1}(\{2\}) = \{(1,1,2),(1,2,2),(1,3,3),(2,2,2),(2,3,3)\}$,
- $F^{-1}(\{3\}) = \{(1,1,3),(1,2,3),(2,2,3),(3,3,3)\}$.

This operation is clearly idempotent and symmetric.

Is it associative? Yes! $\to$ tedious computations.

$\forall x,y,z \in X$, $F(x,y,z) = G(x,G(y,z))$ for the associative $G\colon X^2 \to X$ defined by

G	1	2	3
1	1	2	3
2	2	2	3
3	3	3	2

How to prove the existence of such a G? How to construct it?

Example of symmetric ternary band

Consider $X = \{1,2,3\}$ and define $F\colon X^3 \to X$ by its level sets given (up to permutations) by

- $F^{-1}(\{1\}) = \{(1,1,1)\}$,
- $F^{-1}(\{2\}) = \{(1,1,2),(1,2,2),(1,3,3),(2,2,2),(2,3,3)\}$,
- $F^{-1}(\{3\}) = \{(1,1,3),(1,2,3),(2,2,3),(3,3,3)\}$.

This operation is clearly idempotent and symmetric.

Is it associative? Yes! $\to$ tedious computations.

$\forall x,y,z \in X$, $F(x,y,z) = G(x,G(y,z))$ for the associative $G\colon X^2 \to X$ defined by

G	1	2	3
1	1	2	3
2	2	2	3
3	3	3	2

How to prove the existence of such a G? How to construct it?

Example of symmetric ternary band

Consider $X = \{1, 2, 3\}$ and define $F \colon X^3 \to X$ by its level sets given (up to permutations) by

- $F^{-1}(\{1\}) = \{(1, 1, 1)\}$,
- $F^{-1}(\{2\}) = \{(1, 1, 2), (1, 2, 2), (1, 3, 3), (2, 2, 2), (2, 3, 3)\}$,
- $F^{-1}(\{3\}) = \{(1, 1, 3), (1, 2, 3), (2, 2, 3), (3, 3, 3)\}$.

This operation is clearly idempotent and symmetric.

Is it associative? Yes! $\to$ tedious computations.

$\forall x, y, z \in X,\ F(x, y, z) = G(x, G(y, z))$ for the associative $G \colon X^2 \to X$ defined by

G	1	2	3
1	1	2	3
2	2	2	3
3	3	3	2

How to prove the existence of such a G? How to construct it?

n-ary extensions

(X, G) semigroup

Define a sequence $(G^m)_{m \geq 1}$ of $(m+1)$-ary operation inductively by the following rules $G^1 = G$ and

$$G^m(x_1, \ldots, x_{m+1}) = G^{m-1}(x_1, \ldots, x_{m-1}, G(x_m, x_{m+1})), \qquad m \geq 2.$$

Setting $F = G^{n-1}$, the pair (X, F) is the *n-ary extension* of (X, G).

G is a *binary reduction* of F

$\rightarrow$ (X, F) is an n-ary semigroup

Not every n-ary semigroup is obtained like this

Example.

$(\mathbb{R}, F)$ where $F\colon \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$F(x, y, z) = x - y + z, \qquad x, y, z \in \mathbb{R}$$

Easy examples of symmetric n-ary bands

$e \in X$ is said to be a *neutral element for F* if

$$F(x, e, \ldots, e) = F(e, x, e, \ldots, e) = \cdots = F(e, \ldots, e, x) = x, \qquad x \in X$$

A group $(X, *)$ with neutral element e has *bounded exponent* if $\exists\, m \geq 1$ such that

$$\underbrace{x * \cdots * x}_{m \text{ times}} \quad = \quad e, \qquad x \in X$$

Examples

- $(X, \curlywedge)$ semillatice $\Rightarrow (X, \curlywedge^{n-1})$ symmetric n-ary band
- $(X, *)$ Abelian group whose exponent divides $n-1$
  $\Rightarrow (X, *^{n-1})$ symmetric n-ary band

$$*^{n-1}(x, \ldots, x) \quad = \quad \underbrace{x * \cdots * x}_{= e} * \, x \quad = \quad x$$

n-ary semilattices of n-ary semigroups

Extend the concept of semilattices of semigroups (Clifford, Yamada,...) to n-ary semigroups

$(Y, \curlywedge^{n-1})$ n-ary semilattice

$\{(X_\alpha, F_\alpha)\colon \alpha \in Y\}$ set of n-ary semigroups such that

$$X_\alpha \cap X_\beta = \varnothing, \qquad \alpha \neq \beta$$

Definition
(X, F) is an *n-ary semilattice $(Y, \curlywedge^{n-1})$ of n-ary semigroups (X_α, F_α)* if

- $X = \bigcup_{\alpha \in Y} X_\alpha$
- $F|_{X_\alpha^n} = F_\alpha \ \forall \alpha \in Y$
- $\forall (x_1, \ldots, x_n) \in X_{\alpha_1} \times \cdots \times X_{\alpha_n}$

$$F(x_1, \ldots, x_n) \in X_{\alpha_1 \curlywedge \cdots \curlywedge \alpha_n}$$

We write $(X, F) = ((Y, \curlywedge^{n-1}), (X_\alpha, F_\alpha))$

Not an n-ary semigroup in general!

Strong n-ary semilattices of n-ary semigroups

Extend the concept of strong semilattices of semigroups (Kimura,...) to n-ary semigroups

Definition
Let $(X,F) = ((Y, \curlywedge^{n-1}), (X_\alpha, F_\alpha))$. Suppose that $\forall\, \alpha \succeq \beta \in Y\ \exists\, \varphi_{\alpha,\beta}\colon X_\alpha \to X_\beta$ homomorphism such that

(a) $\varphi_{\alpha,\alpha}$ is the identity on X_α

(b) $\forall\, \alpha \succeq \beta \succeq \gamma \in Y$ we have $\varphi_{\beta,\gamma} \circ \varphi_{\alpha,\beta} = \varphi_{\alpha,\gamma}$

(c) $\forall\, (x_1, \ldots, x_n) \in X_{\alpha_1} \times \cdots \times X_{\alpha_n}$ we have

$$F(x_1, \ldots, x_n) = F_{\alpha_1 \curlywedge \cdots \curlywedge \alpha_n}(\varphi_{\alpha_1, \alpha_1 \curlywedge \cdots \curlywedge \alpha_n}(x_1), \ldots, \varphi_{\alpha_n, \alpha_1 \curlywedge \cdots \curlywedge \alpha_n}(x_n))$$

(X,F) is a *strong n-ary semilattice of n-ary semigroups*

We write $(X,F) = ((Y, \curlywedge^{n-1}), (X_\alpha, F_\alpha), \varphi_{\alpha,\beta})$

Strong n-ary semilattices of n-ary semigroups

Proposition

Every strong n-ary semilattice of n-ary semigroups is an n-ary semigroup

Description of symmetric n-ary bands

Theorem

The following assertions are equivalent.

(i) (X, F) is a symmetric n-ary band

(ii) (X, F) is a strong n-ary semilattice of n-ary extensions of Abelian groups whose exponents divide $n-1$

Sketch of the proof

(ii) $\Rightarrow$ (i): Check the axioms

(i) $\Rightarrow$ (ii): Follows from the following steps

(1) $\forall\, x \in X$, define $\ell_x\colon X \to X$ by $\ell_x(y) = F(x, \ldots, x, y)\ \forall\, y \in X$

(2) The binary relation $\sim$ on X defined by

$$x \sim y \quad \Leftrightarrow \quad \ell_x = \ell_y, \qquad x, y \in X,$$

is a congruence on (X, F) such that $(X/{\sim}, \tilde{F})$ is an n-ary semilattice

(3) $\forall\, x \in X$, $([x]_\sim, F|_{[x]^n_\sim})$ is the n-ary extension of an Abelian group whose exponent divides $n-1$

(4) $\forall\, [x]_\sim \succeq_{\tilde{F}} [y]_\sim \in X/{\sim}$, the map $\ell_y|_{[x]_\sim}\colon ([x]_\sim, F|_{[x]^n_\sim}) \to ([y]_\sim, F|_{[y]^n_\sim})$ is a homomorphism

(5) Check that $(X, F) = ((X/{\sim}, \tilde{F}), ([x]_\sim, F|_{[x]^n_\sim}), \ell_y|_{[x]_\sim})$

Reducibility of symmetric n-ary bands

Theorem

$(X, F) = ((Y, \curlywedge^{n-1}), (X_\alpha, F_\alpha), \varphi_{\alpha,\beta})$ is the n-ary extension of a semigroup (X, G) if and only if $\exists\, e\colon Y \to X$ such that

(a) $\forall\, \alpha \in Y$, $e(\alpha) = e_\alpha \in X_\alpha$

(b) $\forall\, \alpha \succeq \beta \in Y$, we have $\varphi_{\alpha,\beta}(e_\alpha) = e_\beta$

Moreover, $(X, G) = ((Y, \curlywedge), (X_\alpha, G_\alpha), \varphi_{\alpha,\beta})$ where $G_\alpha^{n-1} = F_\alpha$ $\forall \alpha \in Y$

Sketch of the proof

$\Leftarrow$: Computations

$\Rightarrow$: Follows from the following steps

(1) G is surjective and symmetric

(2) The congruence $\sim$ on (X, F) is a congruence on (X, G) such that $(X/\sim, \tilde{G})$ is a semilattice

(3) $\forall\, x \in X$, $([x]_\sim, G|_{[x]_\sim^2})$ is an Abelian group whose exponent divides $n-1$ and $G|_{[x]_\sim^2}^{n-1} = F|_{[x]_\sim^n}$

(4) Step (3) $\Rightarrow$ $G|_{[x]_\sim^2}$ has a neutral element $e_{[x]_\sim} \in [x]_\sim$

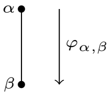
(5) $\forall\, [x]_\sim \succeq_{\tilde{G}} [y]_\sim \in X/\sim$, the map $\ell_y|_{[x]_\sim} : ([x]_\sim, G|_{[x]_\sim^2}) \to ([y]_\sim, G|_{[y]_\sim^2})$ is a group homomorphism

(6) Take $e\colon X/\sim \to X$ defined by $e([x]_\sim) = e_{[x]_\sim}$

Example of symmetric ternary band

Consider $X = \{1, 2, 3\}$ and define $F\colon X^3 \to X$ by its level sets given (up to permutations) by

- $F^{-1}(\{1\}) = \{(1, 1, 1)\}$,
- $F^{-1}(\{2\}) = \{(1, 1, 2), (1, 2, 2), (1, 3, 3), (2, 2, 2), (2, 3, 3)\}$,
- $F^{-1}(\{3\}) = \{(1, 1, 3), (1, 2, 3), (2, 2, 3), (3, 3, 3)\}$.



$\alpha \curlywedge \beta = \beta$

$X = \{1, 2, 3\}$, $X_\alpha = \{1\}$, $X_\beta = \{2, 3\}$

(X_α, F_α) ternary extension of the trivial group on $\{1\}$

(X_β, F_β) isomorphic to the ternary extension of $(\mathbb{Z}_2, +)$

$\varphi_{\alpha,\beta}\colon X_\alpha \to X_\beta$ defined by $\varphi_{\alpha,\beta}(1) = 2$

$\varphi_{\alpha,\alpha} = \mathrm{id}|_{X_\alpha}\colon X_\alpha \to X_\alpha$

$\varphi_{\beta,\beta} = \mathrm{id}|_{X_\beta}\colon X_\beta \to X_\beta$

Example of a symmetric ternary band

$F\colon X^3 \to X$ defined by

- $F|_{\{1\}^3} = F_\alpha$, $F|_{\{2,3\}^3} = F_\beta$
- $F(1,1,2) = F_\beta(\varphi_{\alpha,\beta}(1), \varphi_{\alpha,\beta}(1), \varphi_{\beta,\beta}(2)) = F_\beta(2,2,2) = 2 = F(1,2,1) = F(2,1,1)$
- $F(1,1,3) = F_\beta(\varphi_{\alpha,\beta}(1), \varphi_{\alpha,\beta}(1), \varphi_{\beta,\beta}(3)) = F_\beta(2,2,3) = 3 = F(1,3,1) = F(3,1,1)$
- $F(1,2,2) = F(2,1,2) = F(2,2,1) = 2$
- $F(1,3,3) = F(3,1,3) = F(3,3,1) = 2$
- $F(1,2,3) = F(1,3,2) = F(2,1,3) = F(3,1,2) = F(2,3,1) = F(3,2,1) = 2$

$\to$ (X, F) is a strong ternary semilattice of ternary extensions of Abelian groups whose exponents divide 2

Reducible to a semigroup (X, G) (take $e\colon \{\alpha, \beta\} \to X$ defined by $e(\alpha) = 1$ and $e(\beta) = 2$) where G is defined by

G	1	2	3
1	1	2	3
2	2	2	3
3	3	3	2

Example of a symmetric ternary band

$F\colon X^3 \to X$ defined by

- $F|_{\{1\}^3} = F_\alpha$, $F|_{\{2,3\}^3} = F_\beta$
- $F(1,1,2) = F_\beta(\varphi_{\alpha,\beta}(1), \varphi_{\alpha,\beta}(1), \varphi_{\beta,\beta}(2)) = F_\beta(2,2,2) = 2 = F(1,2,1) = F(2,1,1)$
- $F(1,1,3) = F_\beta(\varphi_{\alpha,\beta}(1), \varphi_{\alpha,\beta}(1), \varphi_{\beta,\beta}(3)) = F_\beta(2,2,3) = 3 = F(1,3,1) = F(3,1,1)$
- $F(1,2,2) = F(2,1,2) = F(2,2,1) = 2$
- $F(1,3,3) = F(3,1,3) = F(3,3,1) = 2$
- $F(1,2,3) = F(1,3,2) = F(2,1,3) = F(3,1,2) = F(2,3,1) = F(3,2,1) = 2$

$\to$ (X, F) is a strong ternary semilattice of ternary extensions of Abelian groups whose exponents divide 2

Reducible to a semigroup (X, G) (take $e\colon \{\alpha, \beta\} \to X$ defined by $e(\alpha) = 1$ and $e(\beta) = 2$) where G is defined by

G	1	2	3
1	1	2	3
2	2	2	3
3	3	3	2

References

N. L. Ackerman.
A characterization of quasitrivial n-semigroups.
To appear in *Algebra Universalis*.

M. Couceiro and J. Devillet.
Every quasitrivial n-ary semigroup is reducible to a semigroup.
Algebra Universalis, 80(4), 2019.

M. Couceiro, J. Devillet, J.-L. Marichal, and P. Mathonet.
Reducibility of n-ary semigroups: from quasitriviality towards idempotency.
arXiv:1909.10412.

J. Devillet and P. Mathonet.
On the structure of symmetric n-ary bands.
arXiv:2004.12423.

W. A. Dudek and V. V. Mukhin.
On n-ary semigroups with adjoint neutral element.
Quasigroups and Related Systems, 14:163–168, 2006.