Scattering theory for the Hodge-Laplacian

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Abstract

We prove using an integral criterion the existence and completeness of the wave operators $W_{\pm}(\Delta_h^{(p)}, \Delta_g^{(p)}, I_{g,h}^{(p)})$ corresponding to the Hodge-Laplacians $\Delta_\nu^{(p)}$ acting on differential p-forms, for $\nu \in \{g, h\}$, induced by two quasi-isometric Riemannian metrics g and h on a complete open smooth manifold M. In particular, this result provides a criterion for the absolutely continuous spectra $\sigma_{\mathrm{ac}}(\Delta_g^{(p)}) = \sigma_{\mathrm{ac}}(\Delta_h^{(p)})$ of $\Delta_\nu^{(p)}$ to coincide. The proof is based on gradient estimates obtained by probabilistic Bismut-type formulae for the heat semigroup defined by spectral calculus. By these localised formulae, the integral criterion only requires local curvature bounds and some upper local control on the heat kernel acting on functions, but no control on the injectivity radii. A consequence is a stability result of the absolutely continuous spectrum under a Ricci flow. As an application we concentrate on the important case of conformal perturbations.

Keywords Scattering theory, Wave operators, Bismut type derivative formulae, Hodge-Laplacian, Conformal perturbations

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Contents

1 Introduction

Let (M, g) be a non-compact geodesically complete Riemannian manifold without boundary. The Hodge-Laplacian $\Delta_g^{(p)}$ acting on differential p-forms carries important geometric and topological

information about M, of particular interest is the spectrum $\sigma(\Delta_g^{(p)})$ of $\Delta_g^{(p)}$. If M is compact, then the spectrum consists of eigenvalues with finite multiplicity. If M is non-compact, then the spectrum contains some absolutely continuous part (cf. [RS79; Wei80]). A natural questions to ask is to what extent can we control the absolutely continuous part of $\sigma(\Delta_g^{(p)})$ and under which assumptions on the geometry of (M, g)?

A systematic approach to control the absolutely continuous part of the spectrum $\sigma_{\mathrm{ac}}(\Delta_g^{(p)})$ is inspired by quantum mechanics, namely scattering theory: Assume that there is another Riemannian metric h on M such that h is quasi-isometric to g, i.e. there exists a constant $C \geqslant 1$ such that $(1/C)g \leqslant h \leqslant Cg$. We show that under suitable assumptions the wave operators

$$W_{\pm}(\Delta_h^{(p)}, \Delta_g^{(p)}, I_{g,h}^{(p)}) = \operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{it\Delta_h^{(p)}} I_{g,h}^{(p)} \mathrm{e}^{-it\Delta_g^{(p)}} \mathsf{P}_{\mathrm{ac}}(-\Delta_g^{(p)})$$

exist and are complete, where the limit is taken in the strong sense, and

$$I_{g,h}^{(p)} : \Omega_{\mathsf{L}^2}^p(M, g) \to \Omega_{\mathsf{L}^2}^p(M, h)$$

denotes a bounded identification operator between the Hilbert spaces of equivalence classes of square-integrable Borel p-forms on M corresponding to the metric g and h respectively (cf. Theorem A in the Appendix and Section 2 for details). Then as well-known, it follows in particular that

$$\sigma_{\mathrm{ac}}(\Delta_h^{(p)}) = \sigma_{\mathrm{ac}}(\Delta_g^{(p)}).$$

In [MS07], Müller & Salomonsen considered Laplacians acting on 0-forms, i.e. functions, on M. The authors assumed both metrics to have a C^{∞}-bounded geometry and a weighted integral condition involving a second order deviation of the metrics. Later in [BGM17], Bei, Güneysu & Müller could generalise the previous result for conformally equivalent metrics on differential forms under a mild first order control on the conformal factor. Hempel, Post & Weder [HPW14] improved the result of [MS07] by assuming only a zeroth order deviation of the metric g from h and a weighted integral condition involving a local lower bound of the injectivity radius and the Ricci curvatures. However, detailed control on the sectional curvature is needed to get control over the injectivity radii. In general, injectivity radii are hard to calculate.

Recently, Güneysu & Thalmaier [GT20] established a rather simple integral criterion induced by two quasi-isometric Riemannian metrics only depending on a local upper bound on the heat kernel and certain explicitly given local lower bound on the Ricci curvature using stochastic methods, namely a Bismut-type formula for the derivative of the heat semigroup [AT10].

Using a similar method, very recently Boldt & Güneysu [BG20] extended the result to a non-compact spin manifold with a fixed topological spin structure and two complete Riemannian metrics with bounded sectional curvatures. As the metrics induce Dirac operators $\mathbf{D}_g$ and $\mathbf{D}_h$, they can show existence and completeness of the wave operators corresponding to the Dirac operators $W_{\pm}(\mathbf{D}_h, \mathbf{D}_g, I_{g,h})$ and their squares $W_{\pm}(\mathbf{D}_h^2, \mathbf{D}_g^2, I_{g,h})$.

In this paper, we address the natural question: Can we extend the results to the setting of differential p-forms?

We will show that previous results can be extended to the setting of differential p-forms, for a large class of potentials assuming an integral criterion only depending on a local upper bound on the heat kernel and certain explicitly given local curvature bounds. In addition, a necessary assumption will be a bound on a weight function measuring the first order deviation of the metrics in terms of the corresponding covariant derivatives ∇^h and ∇^g which is a one-form on M with values in $\operatorname{End} TM$.

Therefore, we consider the Hodge-Laplacian Δ_g, also known as Laplace-de Rham operator, acting on the full exterior bundle $\Omega(M) = \Gamma(\bigwedge T^*M)$, i.e. the complex separable Hilbert space of differential forms on M. The Hodge-Laplacian Δ_g is related to the horizontal Laplacian $\square_g = (\nabla^g)^* \nabla^g$ by the Weitzenböck formula $\Delta_g = \square_g - \mathscr{R}_g$, where Weitzenböck curvature operator $\mathscr{R}_g \in \Gamma(\operatorname{End} \Omega(M))$ is a symmetric field of endomorphisms. In particular, when acting on 1-forms, $\mathscr{R}_g\big|_{\Omega^1(M,g)} = \operatorname{Ric}_g$ and, acting on functions, $\mathscr{R}_g\big|_{\Omega^0(M,g)} = 0$. We assume that $\mathscr{R}_g$ is in the Kato class (cf. Definition 3.3). We are now in the position to state our main result.

Main result. *Assume that g and h are two geodesically complete and quasi-isometric Riemannian metrics on M, denoted $g \sim h$, such that $\mathscr{R}_\nu$ is in the Kato class, for $\nu \in \{g, h\}$, and assume that there exists $C < \infty$ such that $\left|\delta^\nabla_{g,h}\right| \leqslant C$ and that for the both $\nu \in \{g, h\}$*

$$\int \max\left\{ \delta_{g,h}(x) \Xi_g(x,s), \delta^\nabla_{g,h}(x) + \Xi_g(x,s), \Xi_\nu(x,s) \right\} \Phi_\nu(x,s) \operatorname{vol}_\nu(\mathrm{d}x) < \infty, \qquad s > 0, \qquad (1.1)$$

where

- vol_ν *denotes the Riemannian volume measure with respect to the metric ν,*
- $\Xi_\nu(\cdot, s) : M \to (0, \infty)$ *is a function explicitly given terms of local curvature bounds (cf. (3.13) in Section 3),*
- $\Phi_\nu(\cdot, s) : M \to (0, \infty)$ *is a local upper bound on the heat kernel acting on functions on (M, ν) (cf. (2.11) in Section 2),*
- $\delta_{g,h} : M \to (0, \infty)$ *a zeroth order deviation of the metrics from each other (cf. (2.7) in Sec. 2),*
- $\delta^\nabla_{g,h} : M \to [0, \infty)$ *a first order deviation of the metrics (cf. (2.8) in Section 2).*

Then the wave operators $W_\pm(\Delta_h, \Delta_g, I_{g,h})$ exists and are complete. In particular, $\sigma_{ac}(\Delta_g) = \sigma_{ac}(\Delta_h)$.

We will see that a zeroth order deviation $\delta_{g,h}$ of the metrics from each other is induced by quasi-isometry. In comparison to the case of 0-forms, i.e. functions, it turns out, working on higher degree differential forms, also a first order deviation of the metrics $\delta^\nabla_{g,h} = \left|\nabla^h - \nabla^g\right|^2_g$ is necessary. But note that $\nabla^h - \nabla^g$ is a one-form on M with values in $\operatorname{End} TM$.

To this end, we first look at total differential forms and use probabilistic Bismut-type derivative formulae for the exterior derivative, codifferential and covariant derivative of the heat equation [DT01]. Finally everything filters through the form degree to differential p-forms. The particularly important case, in which two quasi-isometric Riemannian metrics differ by a conformal metric change, is a direct consequence of our main result.

Because our result is independent of the injectivity radii we have the following application to the Ricci flow. Let R_g be the Riemannian curvature tensor with respect to the metric g.

Corollary 6.1. *Let $S > 0$, $\lambda \in \mathbb{R}$ and assume that*

(a) the family $(g_s)_{0 \leqslant s \leqslant S} \subset \mathrm{Metr}\, M$ evolves under a Ricci-type flow

$$\partial_s g_s = \lambda \, \mathrm{Ric}_{g_s}, \qquad \text{for all } 0 \leqslant s \leqslant S$$

(b) the initial metric g_0 is geodesically complete

(c) there is some $C > 0$ such that $\left| R_{g_s} \right|_{g_s}, \left| \nabla^{g_s} R_{g_s} \right|_{g_s} \leqslant C \qquad$ for all $0 \leqslant s \leqslant S$.

We set, for all $x \in M$,

$$M_1(x) := \sup \left\{ \left| \mathrm{Ric}_{g_s}(v,v) \right| : 0 \leqslant s \leqslant S, \ v \in \mathsf{T}_x M, \ |v|_{g_s} \leqslant 1 \right\},$$

$$M_2(x) := \sup \left\{ \left| \nabla^{g_s}_v \mathrm{Ric}_{g_s}(u,w) + \nabla^{g_s}_u \mathrm{Ric}_{g_s}(v,w) + \nabla^{g_s}_w \mathrm{Ric}_{g_s}(u,v) \right| : 0 \leqslant s \leqslant S, \right.$$

$$\left. u,v,w \in \mathsf{T}_x M, \ |u|_{g_s}, |v|_{g_s}, |w|_{g_s} \leqslant 1 \right\}.$$

Let $\mathsf{B}_g(x,R)$ denote the open geodesic ball. If

$$\int \mathrm{vol}_{g_0}(\mathsf{B}_g(x,1))^{-1} \max \left\{ \sinh \left(\frac{m}{4} S \, |\lambda| \, M_1(x) \right), M_2(x) \right\} \mathrm{vol}_{g_0}(\mathrm{d}x) < \infty,$$

then $\sigma_{\mathrm{ac}}(\Delta_{g_s}) = \sigma_{\mathrm{ac}}(\Delta_{g_0})$ for all $0 \leqslant s \leqslant S$.

Thereupon, we reify our main results to the case of global curvature bounds: The curvature operator (with respect to the metric g) Q_g is uniquely determined by the equation

$$\left(Q_g(X \wedge Y), U \wedge V \right)_g = \left(R_g(X,Y)U, V \right)_g$$

for all smooth vector fields $X,Y,U,V \in \Gamma_{C^\infty}(\mathsf{T}M)$. By the Gallot–Meyer estimate [GM75], a global bound $Q_g \geqslant -K$, for some constant $K > 0$, already implies that the curvature endomorphism in the Weitzenböck formula (3.1) is globally bounded by $\mathscr{R}_g^{(p)} \geqslant -Kp(m-p)$. In this case, our main result reads as follows.

Theorem 6.7. Let $Q_\nu \geqslant -K$, for some constant $K > 0$ ($\nu \in \{g,h\}$). Let $g,h \in \mathrm{Metr}\, M$ such that $g \sim h$ and assume that there exists $C < \infty$ such that $\left| \delta^\nabla_{g,h} \right| \leqslant C$ and that for some $\nu \in \{g,h\}$

$$\int \max \left\{ \delta_{g,h}(x) \Xi_g(x,s), \delta^\nabla_{g,h}(x) + \Xi_g(x,s), \Xi_\nu(x,s) \right\} \Phi_\nu(x,s) \, \mathrm{vol}_\nu(\mathrm{d}x) < \infty, \qquad s > 0.$$

Then, the wave operators $W_\pm(\Delta_h, \Delta_g, I)$ exist and are complete. Moreover, $W_\pm(\Delta_h, \Delta_g, I)$ are partial isometries with initial space $\mathrm{ran}\, P_{\mathrm{ac}}(\Delta_g)$ and final space $\mathrm{ran}\, P_{\mathrm{ac}}(\Delta_h)$, and we have $\sigma_{\mathrm{ac}}(\Delta_g) = \sigma_{\mathrm{ac}}(\Delta_h)$.

A direct consequence and additional application is the particularly important case of conformal perturbations under local and global curvature bounds, generalising the results in [BGM17].

A final application is provided through a result by Cheeger, Fukaya and Gromov [CFG92] known as Cheeger-Gromov's thick/thin decomposition: On any complete Riemannian m-manifold (M,g) with bounded *sectional curvature* $|\kappa_g| \leqslant 1$, there exists a Riemannian metric g_ε on M such that g_ε is ε-quasi-isometric to g and has bounded covariant derivatives. Hence, in this case, the assumptions of our main result may be suitably relaxed, cf. Theorem 6.10.

Let us end the introduction with a short outline of the paper. Section 2 introduces the necessary notation and deviation maps. In Section 3, we calculate bounds for exterior derivative, codifferential and covariant derivative of the heat semigroup defined by spectral calculus using Bismut-type derivative formulae. Our main results are explained in Section 4. After this, we prove the main result in Section 5 by making use of a slight variant of the abstract Belopol'skii-Birman theorem A. We close in Section 6 with applications to the Ricci flow 6.1, the particularly important cases of conformal perturbations 6.2, specify our results for global curvature bounds 6.3 and ε-close Riemannian metrics 6.4.

2 Setting & Notation

Let (M, g) be a complete smooth Riemannian manifold without boundary of dimension $m := \dim M \geqslant 2$ and $(\cdot, \cdot)_g$ its Riemannian metric. We write vol_g for the corresponding volume measure (with respect to the metric g) and denote by $\mathrm{Metr}\, M$ the set of all smooth Riemannian metrics on M. All bundles will be understood complexified, e.g. the full exterior bundle

$$\bigwedge \mathrm{T}^* M = \bigoplus_{p=0}^{m} \bigwedge^p \mathrm{T}^* M, \quad \text{with the usual convention } \bigwedge^0 \mathrm{T}^* M = \mathbb{C}.$$

Given smooth complex vector bundles $E_1 \to M$ and $E_2 \to M$ the complex linear space of smooth linear partial differential operators from E_1 to E_2 of order $\leqslant k \in \mathbb{N}_0$ is denoted by $\mathscr{D}^{(k)}(M; E_1, E_2)$, with shorthand notation $\mathscr{D}^{(k)}(M; E_1)$ if $E_1 \equiv E_2$. On a vector bundle $E \to M$ (e.g. $E = \bigwedge^p \mathrm{T}^* M$) the corresponding fibre norms are denoted by

$$|\varphi|_g := (\varphi, \varphi)_g^{1/2} \qquad \text{for any section } \varphi \in \Gamma(E),$$

where $\Gamma(E) := \Gamma_{\mathrm{C}\infty}(E)$ denotes all *smooth* sections of E and Γ_{L^2} the L^2-section of E.

In the case of $E = \bigwedge^p \mathrm{T}^* M$, we indicate the corresponding form degree by an index: For example, $\nabla^{g,(p)}$ or $(\cdot, \cdot)_g^{(p)}$ for some smooth function $f : M \to \mathbb{C}$ etc.

We denote by $\Omega_{\mathrm{L}^2}(M, g) := \Gamma_{\mathrm{L}^2}(\bigwedge \mathrm{T}^* M)$ the complex separable Hilbert space of equivalence classes α of square-integrable Borel forms on M such that

$$\|\alpha\|_g^2 := \|\alpha\|_{\Omega_{\mathrm{L}^2}(M,g)}^2 := \int_M |\alpha(x)|_g^2 \, \mathrm{vol}_g(\mathrm{d}x) < \infty,$$

with inner product

$$\langle \alpha, \beta \rangle_g := \langle \alpha, \beta \rangle_{\Omega_{\mathrm{L}^2}(M,g)} := \int_M (\alpha(x), \beta(x))_g \, \mathrm{vol}_g(\mathrm{d}x).$$

Analogously, we write $\Omega_{\mathrm{L}^2}^p(M, g)$ for the Hilbert space of Borel p-forms. In particular,

$$\Omega_{\mathrm{L}^2}(M, g) = \bigoplus_{p=0}^{m} \Omega_{\mathrm{L}^2}^p(M, g).$$

To relax notation, we set

$$\Omega(M, g) := \Omega_{\mathrm{C}\infty}(M, g) \qquad \text{and} \qquad \Omega^p(M, g) := \Omega_{\mathrm{C}\infty}^p(M, g).$$

for the set of all *smooth* forms, and *smooth p*-forms respectively, on (M, g).

Further, for some $\alpha \in \Omega^p(M)$, we denote by

$$\bullet \wedge \alpha \in \mathscr{D}^{(0)}(M; \textstyle\bigwedge^p \mathrm{T}^* M)$$

the exterior product and its formal adjoint with respect to g, the interior multiplication, by

$$\bullet \lrcorner_g \alpha := (\bullet \wedge \alpha)^{*g} \in \mathscr{D}^{(0)}(M; \textstyle\bigwedge^p \mathrm{T}^* M).$$

The interior multiplication corresponds to the contraction of $\alpha \in \Omega^p(M)$ with a vector field $X \in \Gamma(\mathrm{T}M)$, i.e.

$$X \lrcorner_g \alpha(X_1, \ldots, X_{p-1}) := \alpha(X, X_1, \ldots, X_{p-1}), \qquad \text{for all} \ \ X_1, \ldots, X_{p-1} \in \Gamma(\mathrm{T}M),$$

and is an antiderivation, to wit

$$X \lrcorner_g (\alpha \wedge \beta) = (X \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (X \lrcorner \beta) \qquad \text{for all} \ \ \alpha \in \Omega^p(M), \ \beta \in \Omega(M).$$

We denote by

$$\mathbf{d}^{(p)} \in \mathscr{D}^{(1)}(M; \textstyle\bigwedge^p \mathrm{T}^* M, \bigwedge^{p+1} \mathrm{T}^* M)$$
$$\boldsymbol{\delta}_g^{(p)} \in \mathscr{D}^{(1)}(M; \textstyle\bigwedge^p \mathrm{T}^* M, \bigwedge^{p-1} \mathrm{T}^* M)$$

the exterior derivative on p-forms and, respectively, the codifferential as the formal adjoint of $\mathbf{d}^{(p-1)}$. Then the Hodge-Laplacian can be written as the sum

$$\Delta_g^{(p)} := \boldsymbol{\delta}_g^{(p+1)} \mathbf{d}^{(p)} + \mathbf{d}^{(p-1)} \boldsymbol{\delta}_g^{(p)} \in \mathscr{D}^{(2)}(M; \textstyle\bigwedge^p \mathrm{T}^* M),$$

and its Friedrichs realisation in $\Omega_{\mathrm{L}^2}^p(M, g)$ will be again denoted by $\Delta_g^p \geqslant 0$. In particular, for $p = 0$, we recover the special case of the Laplace-Beltrami operator acting on 0-forms, i.e. functions,

$$\Delta_g^{(0)} = \boldsymbol{\delta}_g^{(1)} \mathbf{d}^{(0)} \in \mathscr{D}^{(2)}(M).$$

Furthermore, we set

$$\mathbf{d} := \bigoplus_{p=0}^m \mathbf{d}^{(p)} \in \mathscr{D}^{(1)}(M; \textstyle\bigwedge \mathrm{T}^* M)$$

$$\boldsymbol{\delta}_g := \bigoplus_{p=0}^m \boldsymbol{\delta}_g^{(p)} \in \mathscr{D}^{(1)}(M; \textstyle\bigwedge \mathrm{T}^* M)$$

and define the underlying Dirac-type operator $\mathbf{D}_g$, and the (total) Hodge-Laplacian Δ_g

$$\mathbf{D}_g := \mathbf{d} + \boldsymbol{\delta}_g \in \mathscr{D}^{(1)}(M; \textstyle\bigwedge \mathrm{T}^* M)$$
$$\Delta_g := \mathbf{D}_g^2 \in \mathscr{D}^{(2)}(M; \textstyle\bigwedge \mathrm{T}^* M)$$

where the Friedrichs realisation of Δ_g in $\Omega_{\mathrm{L}^2}^p(M, g)$ will be again denoted by $\Delta_g \geqslant 0$. In particular,

$$\Delta_g\big|_{\Omega^p(M, g)} = \Delta_g^{(p)} \in \mathscr{D}^{(2)}(M; \textstyle\bigwedge^p \mathrm{T}^* M)$$

and

$$\Delta_g = \bigoplus_{p=0}^{m} \Delta_g^{(p)} \qquad \text{as self-adjoint operators.}$$

Since g is (geodesically) complete, it follows that the operators $\mathbf{D}_g, \Delta_g, \Delta_g^{(p)}$ are essentially self-adjoint on the corresponding space of smooth compactly supported forms [Str83].

Next, recall that for the p-fold exterior product of the vector space $\bigwedge^p T^*M$, we obtain a scalar product $(\cdot, \cdot)_g$ on $\bigwedge^p T^*M$ by the bilinear extension of

$$\left(\alpha_1 \wedge \ldots \wedge \alpha_p, \beta_1 \wedge \ldots \wedge \beta_p \right)_g = \det \left(\alpha_k, \beta_l \right)_g . \tag{2.1}$$

Any $A \in \text{End}(T^*M)$ induces a linear map

$$\bigwedge^m A : \bigwedge^m T^*M \to \bigwedge^m T^*M$$

$$\alpha_1 \wedge \ldots \wedge \alpha_m \mapsto A\alpha_1 \wedge \ldots \wedge A\alpha_m.$$

As $\bigwedge^m T^*M$ is one-dimensional, the map $\bigwedge^m A$ is given by multiplication with a unique number, denoted by $\det \bigwedge^m A$,

$$\bigwedge^m A(e_1 \wedge \ldots \wedge e_m) = (\det A) \, e_1 \wedge \ldots \wedge e_m,$$

where $(e_1, \ldots, e_m)$ is a basis for $\bigwedge^m T_x^*M$.

A Riemannian metric $(u, v)_g = g(u, v)$ for $u, v \in T_xM$ gives by definition an inner product on each tangent space T_xM ($x \in M$). By Riesz' representation theorem, g provides a natural isomorphism between tangent and cotangent bundle given by $v \mapsto (v, \cdot)_g$,

$$TM \underset{\sharp^g}{\overset{\flat^g}{\rightleftarrows}} T^*M.$$

More precisely, we define the **sharp operator** $\sharp^g$ (with respect to g) by

$$\sharp^g : T^*M \to TM, \qquad \alpha(\beta) = g(\alpha^{\sharp^g}, \beta).$$

The Riemannian metric g defines a metric g on T^*M, the **cometric**, via

$$g(\alpha, \beta) := g(\alpha^{\sharp^g}, \beta^{\sharp^g}) \qquad \text{for all} \ \ \alpha, \beta \in T_x^*M, \ x \in M,$$

which extends to a metric on $\bigwedge^p T^*M$ according to (2.1).

Definition 2.1. A smooth Riemannian metric $h \in \text{Metr}\,M$ is called **quasi-isometric to** g, denoted $g \sim h$, if there exists a constant $C \geqslant 1$ such that (to be understood pointwise, as bilinear forms)

$$\frac{1}{C} g \leqslant h \leqslant C g.$$

Given $g, h \in \text{Metr}\,M$, we define a vector bundle morphism

$$A := A_{g,h} : TM \overset{\sim}{\longrightarrow} TM, \qquad h(u, v) = g(Au, v), \qquad \text{for all} \ \ x \in M, \ u, v \in T_xM. \tag{2.2}$$

Note that the vector bundle morphism $A = A_{g,h}$ induces a vector bundle morphism on the cotangent bundle via

$$A : T^*M \overset{\sim}{\longrightarrow} T^*M, \qquad \alpha \mapsto A\alpha := \alpha \circ A.$$

Lemma 2.2 (and **Definition**).   *In terms of the notations above, we have*

$$h(\alpha, \beta) = g(A^{-1}\alpha, \beta) \qquad \text{for all } \alpha, \beta \in \mathsf{T}_x^* M, \ x \in M.$$

Extending $A^{-1} = A_{g,h}^{-1}$ to a smooth vector bundle morphism by

$$\mathscr{A} := \mathscr{A}_{g,h}(x) := (\textstyle\bigwedge A_{g,h}^{-1})_x : \bigwedge \mathsf{T}^* M \overset{\sim}{\longrightarrow} \bigwedge \mathsf{T}^* M, \tag{2.3}$$

we obtain

$$g(\mathscr{A}_{g,h}(x)\alpha, \beta) = h(\alpha, \beta) \qquad \text{for } x \in M, \ \alpha, \beta \in \textstyle\bigwedge \mathsf{T}_x^* M.$$

In the following the induced metrics will be understood complexified (conjugate-linear in the first variable and linear in the second).

Remark 2.3.   (i) By the positive-definiteness of h (or g), $\mathscr{A}_{g,h}(x)$ has only positive eigenvalues ($x \in M$). By the symmetry of g and h the endomorphism $\mathscr{A}_{g,h}$ is fibrewise self-adjoint with respect to g and h. Therefore, the fibrewise operator norm $|\cdot|_g$ (or $|\cdot|_h$) induced by the metric g (or h) of $\mathscr{A}$ is equivalent to absolute value of the largest eigenvalue on the given fibre for both g and h. Thus to relax notation, we may suppress the metric and simply write $|\mathscr{A}|$.

(ii) By the very definition, $\mathscr{A}^{1/2}$ is a (pointwise) isometry from $(\bigwedge \mathsf{T}^* M, g)$ to $(\bigwedge \mathsf{T}^* M, h)$.

Proof of Lemma 2.2. We prove the Lemma in several steps.

1º We calculate the sharp operator in the new metric. For $x \in M$, let $v \in \mathsf{T}_x M$ and $\alpha \in \mathsf{T}_x^* M$. By duality,

$$g\left(\alpha^{\sharp g}, v\right) = \alpha(v) = h\left(\alpha^{\sharp h}, v\right) = g\left(A\alpha^{\sharp h}, v\right) \qquad \Longrightarrow \qquad \alpha^{\sharp g} = A\alpha^{\sharp h}, \tag{2.4}$$

for all $v \in \mathsf{T}_x M$.

2º Let $\alpha, \beta \in \mathsf{T}_x^* M$, then

$$h\left(\alpha, \beta\right) = h\left(\alpha^{\sharp h}, \beta^{\sharp h}\right)$$
$$= g\left(A\alpha^{\sharp h}, \beta^{\sharp h}\right) = g(\alpha^{\sharp h}, A\beta^{\sharp h}) = g(A^{-1}\alpha^{\sharp g}, \beta^{\sharp g}) = g(\alpha \circ A^{-1}, \beta),$$

where we used that $A^{-1}\alpha^{\sharp g} = (\alpha \circ A^{-1})^{\sharp g}$ in the last equality.

3º For any $\alpha, \beta \in \Omega^p(M)$,

$$h(\alpha, \beta) = h\left(\alpha_1^{\sharp h} \wedge \dots \wedge \alpha_p^{\sharp h}, \beta_1^{\sharp h} \wedge \dots \wedge \beta_p^{\sharp h}\right)$$
$$\overset{(2.1)}{=} \det\left(\alpha_k^{\sharp h}, \beta_l^{\sharp h}\right)_h$$
$$= \det\left(A\alpha_k^{\sharp h}, \beta_l^{\sharp h}\right)_g$$
$$\overset{(2.4)}{=} \det\left(\alpha_k^{\sharp g}, A^{-1}\beta_l^{\sharp g}\right)_g$$
$$\overset{A}{\underset{\text{s.a.}}{=}} \det\left(A^{-1}\alpha_k^{\sharp g}, \beta_l^{\sharp g}\right)_g$$
$$= g\left(A^{-1}\alpha_1^{\sharp g} \wedge \dots \wedge A^{-1}\alpha_p^{\sharp g}, \beta_1^{\sharp g} \wedge \dots \wedge \beta_p^{\sharp g}\right)$$
$$= g\left(\textstyle\bigwedge^p A^{-1}\left(\alpha_1 \wedge \dots \wedge \alpha_p\right), \beta_1 \wedge \dots \wedge \beta_p\right) = g\left(\mathscr{A}\alpha, \beta\right) \qquad \blacksquare$$

Obviously, denoting by $0 < \rho_{g,h} \in C^\infty(M)$ the Radon-Nikodým density, i.e.

$$\mathrm{d\,vol}_h = \rho_{g,h} \mathrm{d\,vol}_g,$$

the following identities hold

$$\rho_{h,g} = \rho_{g,h}^{-1}, \quad \mathscr{A}_{h,g} = \mathscr{A}_{g,h}^{-1}, \quad \rho_{g,h} = (\det A)^{1/2}, \quad 0 < \inf \rho_{g,h} \leqslant \sup \rho_{g,h} < \infty. \tag{2.5}$$

The following estimates are necessary tools for the main proof noting that it is independent of the quasi-isometry of g and h. We point out that a similar argument was recently developed in [BG20].

Lemma 2.4. *Let $\mathscr{A} := \mathscr{A}_{g,h}$ be the smooth vector bundle morphism defined by* (2.3). *For any vector field $X \in \Gamma(\mathrm{T}M)$, we get*

$$\nabla_X^g \mathscr{A} = \mathscr{A}(\nabla_X^h - \nabla_X^g) + (\nabla_X^h - \nabla_X^g)^* \mathscr{A},$$

and the pointwise estimate

$$\left|\nabla_X^g \mathscr{A}\right|_g \leqslant 2\left|\mathscr{A}\right|\left|\nabla_X^h - \nabla_X^g\right|_g,$$

where $|\cdot|_g$ denotes the operator norm induced by the inner product g.

Proof. We divide the proof into two steps.

1⁰ If we differentiate the identity

$$(\alpha, \beta)_h = (\mathscr{A}\alpha, \beta)_g$$

in direction of X, we get on the one hand

$$\begin{aligned} X\,(\alpha, \beta)_h &= \left(\nabla_X^h \alpha, \beta\right)_h + \left(\alpha, \nabla_X^h \beta\right)_h \\ &= \left(\mathscr{A}\nabla_X^h \alpha, \beta\right)_g + \left(\mathscr{A}\alpha, \nabla_X^h \beta\right)_g \end{aligned}$$

and on the other hand

$$\begin{aligned} X\,(\mathscr{A}\alpha, \beta)_g &= \left(\nabla_X^g(\mathscr{A}\alpha), \beta\right)_g + \left(\mathscr{A}\alpha, \nabla_X^g \beta\right)_g \\ &= \left((\nabla_X^g \mathscr{A})\alpha, \beta\right)_g + \left(\mathscr{A}\nabla_X^g \alpha, \beta\right)_g + \left(\mathscr{A}\alpha, \nabla_X^g \beta\right)_g. \end{aligned}$$

Hence,

$$\left((\nabla_X^g \mathscr{A})\alpha, \beta\right)_g = \left(\mathscr{A}(\nabla_X^h - \nabla_X^g)\alpha, \beta\right)_g + \left(\alpha, \mathscr{A}(\nabla_X^h - \nabla_X^g)\beta\right)_g.$$

using the self-adjointness of $\mathscr{A}$.

2⁰ We estimate

$$\begin{aligned} \left|\nabla_X^g \mathscr{A}\right|_g &\leqslant \sup_{|v|,|w|\leqslant 1} \left|\left((\nabla_X^g \mathscr{A})v, w\right)_g\right| \\ &\leqslant \sup_{|v|,|w|\leqslant 1} \left(\left|\left(\mathscr{A}(\nabla_X^h - \nabla_X^g)v, w\right)_g\right| + \left|\left(v, \mathscr{A}(\nabla_X^h - \nabla_X^g)w\right)_g\right|\right) \\ &\leqslant 2\left|\mathscr{A}\right|\left|\nabla_X^h - \nabla_X^g\right|_g. \end{aligned}$$

∎

Lemma 2.5. *Let $\mathcal{A} := \mathcal{A}_{g,h}$ be the smooth vector bundle morphism defined by* (2.3). *For any vector field $X \in \Gamma(\mathsf{T}M)$, we get the pointwise estimate*

$$\left|\nabla_X^g \mathcal{A}^{1/2}\right|_g \leqslant |\mathcal{A}|\left|\mathcal{A}^{-1/2}\right|\left|\nabla_X^h - \nabla_X^g\right|_g$$

$$\left|\nabla_X^g \mathcal{A}^{-1/2}\right|_g \leqslant |\mathcal{A}|\left|\mathcal{A}^{-1/2}\right|^3 \left|\nabla_X^h - \nabla_X^g\right|_g,$$

where $|\cdot|_g$ denotes the operator norm induced by the inner product g.

Proof. We divide the proof into three steps.

1° Recall that $\mathcal{A}^{1/2}$ is an isometry, so that

$$\left(\mathcal{A}^{1/2}\alpha, \mathcal{A}^{1/2}\alpha\right)_g = (\alpha, \alpha)_h.$$

If we differentiate the identity in the direction of X, we get on the one hand

$$X\left(\mathcal{A}^{1/2}\alpha, \mathcal{A}^{1/2}\alpha\right)_g = 2\left(\nabla_X^g \mathcal{A}^{1/2}(\alpha), \mathcal{A}^{1/2}\alpha\right)_g + 2\left(\mathcal{A}^{1/2}\nabla_X^g\alpha, \mathcal{A}^{1/2}\alpha\right)_g,$$

whereas, we have on the other hand

$$X(\alpha, \alpha)_h = 2\left(\nabla_X^h\alpha, \alpha\right)_h = 2\left(\mathcal{A}\nabla_X^h\alpha, \alpha\right)_g.$$

Thus,

$$\left(\nabla_X^g \mathcal{A}^{1/2}\alpha, \mathcal{A}^{1/2}\alpha\right)_g = \left(\mathcal{A}^{1/2} \circ \left(\nabla_X^h - \nabla_X^g\right)\alpha, \mathcal{A}^{1/2}\alpha\right)_g. \tag{2.6}$$

2° Clearly, as $\mathcal{A}^{1/2}$ is self-adjoint so is $\nabla_X^g \mathcal{A}^{1/2}$. So, let λ be an eigenvalue of $\nabla_X^g \mathcal{A}^{1/2}$ with $|\lambda| = \left|\nabla_X^g \mathcal{A}^{1/2}\right|_g$ at a fixed point $x \in M$. Let $v \in \mathsf{T}_x M$ be a corresponding g-normalised eigenvector. We estimate

$$\left|\left(\mathcal{A}^{1/2}\nabla_X^g \mathcal{A}^{1/2}v, v\right)_g\right| = |\lambda|\left|\left(\mathcal{A}^{1/2}v, v\right)_g\right| = \left|\nabla_X^g \mathcal{A}^{1/2}\right|_g \left|\left(\mathcal{A}^{1/2}v, v\right)_g\right| \geqslant \left|\nabla_X^g \mathcal{A}^{1/2}\right|_g \left|\mathcal{A}^{-1}\right|^{-1/2},$$

where we used that $\mathcal{A}$, hence $\mathcal{A}^{1/2}$, has only (strictly) positive eigenvalues: More precisely, since $\mathcal{A}^{1/2}$ is self-adjoint, there are eigenvalues $\lambda_{\min}$ and $\lambda_{\max}$ of $\mathcal{A}^{1/2}$ such that

$$\lambda_{\min(\mathcal{A}^{1/2})} \leqslant \left(\mathcal{A}^{1/2}u, u\right)_g \leqslant \lambda_{\max}(\mathcal{A}^{1/2}).$$

By definition, the smooth vector bundle morphism $\mathcal{A}$ has only strictly positive eigenvalues, so does $\mathcal{A}^{1/2}$ and we have that $\left(\mathcal{A}^{1/2}u, u\right)_g > 0$. Moreover, the eigenvalues of $\mathcal{A}^{-1/2}$ are just given by $\frac{1}{\lambda(\mathcal{A}^{1/2})}$ and thus

$$\left|\mathcal{A}^{1/2}\right| \geqslant \lambda_{\min}(\mathcal{A}^{1/2}) = \frac{1}{\lambda_{\max}(\mathcal{A}^{-1/2})} = \left|\mathcal{A}^{-1/2}\right|^{-1}.$$

Hence the first estimate follows:

$$\left|\nabla_X^g \mathcal{A}^{1/2}\right|_g \leqslant \left|\mathcal{A}^{-1}\right|^{1/2} \sup_{|v|\leqslant 1}\left|\left(\mathcal{A}^{1/2}\nabla_X^g \mathcal{A}^{1/2}v, v\right)_g\right|$$

$$\overset{(2.6)}{=} \left|\mathcal{A}^{-1}\right|^{1/2} \sup_{|v|\leqslant 1}\left|\left(\mathcal{A} \circ \left(\nabla_X^h - \nabla_X^g\right)v, v\right)_g\right|$$

$$\leqslant \left|\mathcal{A}^{-1}\right|^{1/2} |\mathcal{A}|\left|\nabla_X^h - \nabla_X^g\right|_g.$$

3⁰ Covariantly differentiating the identity

$$\mathrm{id} = \mathscr{A}^{1/2} \circ \mathscr{A}^{-1/2},$$

we get

$$0 = \nabla_X^g \mathscr{A}^{1/2} \circ \mathscr{A}^{-1/2} + \mathscr{A}^{1/2} \nabla_X^g \mathscr{A}^{-1/2}.$$

Thus,

$$\nabla_X^g \mathscr{A}^{-1/2} = -\mathscr{A}^{-1/2} \nabla_X^g \mathscr{A}^{1/2} \circ \mathscr{A}^{-1/2},$$

and the second estimate follows from by part **2⁰**. ∎

We now define the zeroth order deviation of the two metrics (considered as multiplicative perturbations of each other) as

$$\delta_{g,h}(x) := 2\sinh\left(\frac{m}{4} \max_{\lambda \in \sigma(A_{g,h}(x))} |\log \lambda|\right) = \max_{\lambda \in \sigma(A_{g,h}(x))} \left|\lambda^{\frac{m}{4}} - \lambda^{-\frac{m}{4}}\right| : M \to (0, \infty), \qquad (2.7)$$

symmetric in g and h by quasi-isometry, i.e. $\delta_{g,h} = \delta_{h,g}$. We will make use of the fact [HPW14, Appendix A] that

$$\sup \delta_{g,h}(x) < \infty \quad \Longleftrightarrow \quad g \sim h.$$

The definition is becoming clearer in the Proof of the main result in Section 4. Moreover, let

$$\delta_{g,h}^{\nabla}(x) := \left|\nabla^h - \nabla^g\right|_g^2 (x) : M \to [0, \infty) \qquad (2.8)$$

be a weight function defined in terms of the corresponding covariant derivatives ∇^h and ∇^g, defined in terms of the operator norm induced by the inner product g.

Remark 2.6. Note that the difference of two connections $\nabla^h - \nabla^g$ is a one-form on M with values in $\mathrm{End}\,\mathrm{T}M$, i.e.

$$\nabla^h - \nabla^g \in \Gamma(\mathrm{T}^*M \otimes \mathrm{End}\,\mathrm{T}M).$$

Example 2.7 (Conformal metric change). Let $h := g_\psi$ be a conformal perturbation of g, i.e. we set $g_\psi := \mathrm{e}^{2\psi} g$ for some smooth function $\psi : M \to \mathbb{R}$. We take $A := \mathrm{e}^{2\psi}$, so $A^{-1} = \mathrm{e}^{-2\psi}$ and

$$\delta_{g,h}(x) = 2\sinh\frac{m}{4}|\psi(x)|.$$

Hence,

$$g \sim h \quad \Longleftrightarrow \quad \psi \quad \text{bounded}.$$

By (6.1c), for any smooth vector field $X, Y \in \Gamma_{\mathrm{C}^\infty}(\mathrm{T}M)$,

$$\left(\nabla_X^{g_\psi} - \nabla_X^g\right) Y = \mathbf{d}\psi(X)Y + \mathbf{d}\psi(Y)X - (X, Y)_g \operatorname{grad}_g \psi.$$

By norm equivalence on finite dimensional spaces, let us work with the Hilbert-Schmidt norm for the calculation. Let $(X_i)_{i=1}^m$ a smooth local g-orthonormal frame of vector fields. Then in local coordinates,

$$|\mathbf{d}\psi|_{g\otimes\mathrm{HS}}^2 = \sum_{i=1}^m \left|X_i\psi\right|^2,$$

$$\left|\nabla^{g_\psi} - \nabla^g\right|_{g\otimes\mathrm{HS}}^2 = \sum_{j,k=1}^m \left|\left(\nabla^{g_\psi} - \nabla^g\right)(X_j, X_k)\right|^2$$

$$= \sum_{j,k=1}^m \left|(X_j\psi)X_k + (X_k\psi)X_j - \delta_{jk}\sum_i (X_i\psi)X_i\right|^2$$

$$= \sum_{j<k} 2\left|(X_j\psi)X_k + (X_k\psi)X_j\right|^2 + \sum_{i=1}^m \left|X_i\psi\right|^2,$$

so that

$$|\mathbf{d}\psi|_g^2 \lesssim \left|\nabla^{g_\psi} - \nabla^g\right|_g^2 = \delta_{g,g_\psi}^\nabla.$$

Next, we give a formula how to express the codifferential $\boldsymbol{\delta}_h$ with respect the metric h in terms of the codifferential $\boldsymbol{\delta}_g$ in terms of g.

Lemma 2.8. *Let $\mathscr{A} := \mathscr{A}_{g,h}$ be the smooth vector bundle morphism defined by* (2.3). *Then the codifferential with respect to the metric h is given by*

$$\boldsymbol{\delta}_h\eta = \mathscr{A}^{-1}\left(\boldsymbol{\delta}_g(\mathscr{A}\eta) - \mathbf{d}\log\rho \lrcorner_g (\mathscr{A}\eta)\right) \qquad \text{for all } \eta \in \Omega_{\mathrm{C}^\infty}(M).$$

Proof. For any $\eta_1 \in \Omega_{\mathrm{C}^\infty}(M)$, $\eta_2 \in \Omega_{\mathrm{C}^\infty}(M)$, we calculate

$$\left\langle \eta_1, \mathscr{A}^{-1}\left(\boldsymbol{\delta}_g(\mathscr{A}\eta_2) - \mathbf{d}\log\rho \lrcorner_g (\mathscr{A}\eta_2)\right)\right\rangle_h = \left\langle \rho\mathscr{A}\eta_1, \mathscr{A}^{-1}\left(\boldsymbol{\delta}_g(\mathscr{A}\eta_2) - \frac{\mathbf{d}\rho}{\rho} \lrcorner_g (\mathscr{A}\eta_2)\right)\right\rangle_g$$

$$= \left\langle \eta_1, \rho\boldsymbol{\delta}_g(\mathscr{A}\eta_2) - \mathbf{d}\rho \lrcorner_g (\mathscr{A}\eta_2)\right\rangle_g$$

$$= \left\langle \mathbf{d}(\rho\eta_1), \mathscr{A}\eta_2\right\rangle_g - \left\langle \mathbf{d}\rho \wedge \eta_1, \mathscr{A}\eta_2\right\rangle_g$$

$$= \left\langle \mathbf{d}\rho \wedge \eta_1, \mathscr{A}\eta_2\right\rangle_g + \left\langle \rho\mathbf{d}\eta_1, \mathscr{A}\eta_2\right\rangle_g - \left\langle \mathbf{d}\rho \wedge \eta_1, \mathscr{A}\eta_2\right\rangle_g$$

$$= \left\langle \mathbf{d}\eta_1, \eta_2\right\rangle_h,$$

where we used that $\mathscr{A}$ is fibrewise self-adjoint. ∎

In the Proof of the main result, the gradient of the logarithm of the Radon-Nikodým density $\rho_{g,h}$ can be estimated in terms of smooth vector bundle morphism $\mathscr{A}_{g,h}$ and $\delta_{g,h}^\nabla$ which is reflected in the next Proposition 2.12. Therefore, we note two auxiliary lemmas.

Lemma 2.9. *Let A, B be two complex $m \times m$-matrices. Then,*

$$|\mathrm{tr}(AB)| \leqslant \|A\|_{\mathrm{HS}} \|B\|_{\mathrm{HS}},$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm.

Proof. Let A, B be two complex $m \times m$-matrices with singular values

$$\sigma_1(A) \geqslant \sigma_2(A) \geqslant \ldots \geqslant \sigma_m(A) \qquad \text{of } A \text{ and}$$

$$\sigma_1(B) \geqslant \sigma_2(B) \geqslant \ldots \geqslant \sigma_m(B) \qquad \text{of } B.$$

Then, with the Hilbert-Schmidt norm $\|\cdot\|_{\mathrm{HS}}$,

$$\|\sigma(A)\| = \sqrt{\sum_{i=1}^{m} \sigma_i^2(A)} = \sqrt{\mathrm{tr}(A^*A)} =: \|A\|_{\mathrm{HS}} \tag{2.9}$$

by norm equivalence on finite-dimensional vector spaces. By the well-known von Neumann trace formula [Mir75], we get

$$\begin{aligned}
|\mathrm{tr}(AB)| &\leqslant \sum_{i=1}^{m} \sigma_i(A)\sigma_i(B) \\
&= (\sigma(A), \sigma(B))_{\mathbb{R}^m} \\
&\leqslant \|\sigma(A)\| \ \|\sigma(B)\| = \|A\|_{\mathrm{HS}} \|B\|_{\mathrm{HS}},
\end{aligned}$$

where we used Cauchy-Schwarz for the first inequality and (2.9) in the last equality. ∎

Lemma 2.10 (Classical Jacobi's formula). *Let $A = A(t)$ an $m \times m$-matrix parametrised by t. If A is invertible, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \det A(t) = \det A(t) \ \mathrm{tr}\left(A(t)^{-1} \frac{\mathrm{d}}{\mathrm{d}t} A(t)\right).$$

Lemma 2.11 (Jacobi's formula). *Let $A := A_{g,h}$ be the smooth vector bundle morphism defined by (2.2) $(x \in M)$. Then, for any $X \in \Gamma(\mathrm{T}M)$,*

$$\mathbf{d}(\det A)X = \det A \ \mathrm{tr}\left(A^{-1}\nabla_X A\right).$$

Proof. For $\varepsilon > 0$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve on M such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X$. Then, computing the differential using a velocity vector,

$$\begin{aligned}
\mathbf{d}(\det A)_x X &= \frac{\mathrm{d}}{\mathrm{d}t} \det A(\gamma(t))\Big|_{t=0} = \det A(\gamma(t)) \ \mathrm{tr}\left(A(\gamma(t))^{-1} \frac{\mathrm{d}}{\mathrm{d}t} A(\gamma(t))\right)\Big|_{t=0} \\
&= \det A(\gamma(t)) \ \mathrm{tr}\left(A(\gamma(t))^{-1} \nabla_{\dot{\gamma}(t)} A(\gamma(t))\right)\Big|_{t=0} \\
&= \det A(x) \ \mathrm{tr}\left(A(x)^{-1}\nabla_X A(x)\right),
\end{aligned}$$

using Lemma 2.10 in the second step. ∎

Proposition 2.12. *Let $\mathscr{A} := \mathscr{A}_{g,h}$ be the smooth vector bundle morphism defined by (2.3) and $\rho = (\det A)^{1/2}$ the Radon-Nikodým density defined by (2.5). Then we can estimate as follows:*

$$|\mathbf{d}\log\rho|_g \leqslant C(m)\left|\mathscr{A}^{-1}\right| |\nabla^g\mathscr{A}|_g.$$

Proof. We first recall, that for $x \in M$,

$$\dim \bigwedge^p \mathrm{T}_x^* M = \binom{m}{p} \qquad \Longrightarrow \qquad \dim \bigwedge \mathrm{T}_x^* M = 2^m < \infty,$$

so dim $\bigwedge T_x^* M$ is finite-dimensional ($x \in M$). Recall that the Radon-Nikodým density ρ is given by $\rho = (\det A)^{1/2}$. By Jacobi's formula, Lemma 2.11 above,

$$\mathbf{d}(\log \rho)_x X = \frac{1}{2} \operatorname{tr}\left(A(x)^{-1} \nabla_X A(x)\right).$$

Hence, using Lemma 2.9,

$$\begin{aligned}
\left| \mathbf{d}(\log \rho)_x X \right| &\leqslant \frac{1}{2} \left\| A(x)^{-1} \right\|_{\mathrm{HS}} \left\| \nabla_X A(x) \right\|_{g \otimes \mathrm{HS}} \\
&\leqslant C(m) \left| A(x)^{-1} \right| \left| \nabla_X A(x) \right|_g \\
&\leqslant C(m) \left| \mathscr{A}(x)^{-1} \right| \left| \nabla_X \mathscr{A}(x) \right|_g
\end{aligned}$$

by norm equivalence on finite-dimensional spaces. $\blacksquare$

Next, we define the bounded identification operator

$$\begin{aligned}
I = I_{g,h} &: \Omega_{\mathsf{L}^2}(M, g) \to \Omega_{\mathsf{L}^2}(M, h) \\
I_{g,h}\alpha(x) &\mapsto \mathscr{A}_{g,h}^{-1/2}(x)\alpha(x),
\end{aligned}$$

well-defined by $g \sim h$.

Lemma 2.13. *The adjoint I^* of the bounded identification operator I is given by*

$$\begin{aligned}
I^* = I_{g,h}^* &: \Omega_{\mathsf{L}^2}(M, h) \to \Omega_{\mathsf{L}^2}(M, g) \\
I_{g,h}^*\alpha(x) &\mapsto \rho_{g,h}(x)\mathscr{A}_{g,h}^{1/2}(x)\alpha(x).
\end{aligned} \tag{2.10}$$

Proof. For compactly supported $\alpha \in \Omega(M, h)$ and $\beta \in \Omega(M, g)$, we get

$$\begin{aligned}
\left\langle I_{g,h}^*\alpha, \beta \right\rangle_{\Omega_{\mathsf{L}^2}(M,g)} &= \left\langle \alpha, I_{g,h}\beta \right\rangle_{\Omega_{\mathsf{L}^2}(M,h)} \\
&= \int_M \left(\alpha, \mathscr{A}_{g,h}^{-1/2}\beta \right)_h \mathrm{d}\operatorname{vol}_h \\
&= \int_M \left(\mathscr{A}_{g,h}^{1/2}\alpha, \mathscr{A}_{g,h}^{1/2}\mathscr{A}_{g,h}^{-1/2}\beta \right)_g \rho_{g,h}\mathrm{d}\operatorname{vol}_g \\
&= \int_M \left(\rho_{g,h}\mathscr{A}_{g,h}^{1/2}\alpha, \beta \right)_g \mathrm{d}\operatorname{vol}_g = \left\langle \rho_{g,h}I_{g,h}^{-1}\alpha, \beta \right\rangle_{\Omega_{\mathsf{L}^2}(M,g)}. \quad \blacksquare
\end{aligned}$$

For every $g \in \operatorname{Metr} M$,

$$(P_s^g)_{s>0} := \left(\mathrm{e}^{-\frac{s}{2}\Delta_g} \right)_{s>0} \subset \mathscr{L}(\Omega_{\mathsf{L}^2}(M, g))$$

is the heat semigroup defined by the spectral theorem, choosing $f : \mathbb{R} \to \mathbb{R}$, $f(\lambda) := \mathrm{e}^{-t/2\lambda}$. Let us denote by

$$(0, \infty) \times M \times M \ni (s, x, y) \mapsto p_s^g(x, y) := \mathrm{e}^{-\frac{s}{2}\Delta_g}(x, y) \in \operatorname{Hom}\left(\bigwedge T_y^* M, \bigwedge T_x^* M \right)$$

the corresponding jointly smooth integral kernel of P_s^g. The smooth representative of $P_s\alpha(x)$ is given by

$$P_s^g\alpha(x) = \int_M \mathrm{e}^{-s/2\Delta_g}(x, y)\alpha(y) \operatorname{vol}_g(\mathrm{d}y).$$

It is well-known [Gün17, Theorem II.1.] that we have

$$\int_M \left| p_s^g(x,y) \right|_g^2 \mathrm{vol}_g(\mathrm{d}y) < \infty, \qquad \text{for all } s > 0, \ x \in M.$$

The form degree p will be indicated again by round brackets $p_s^{g,(p)}$. Finally, we set

$$\Phi_g(x,s) := \sup_M p_s^{g,(0)}(x,\cdot), \tag{2.11}$$

indicating the minimal heat kernel $p_s^{g,(0)}$ acting on 0-forms, i.e. functions. Then it is well-known that, for all $(x,s) \in M \times (0,\infty)$, it follows that $\Phi_g(x,s) < \infty$. One can even show [Gün17] that

$$\sup_K \Phi_g(\cdot, s) < \infty \qquad \text{for all } s > 0, \ K \subset M \text{ compact.}$$

3 Bismut-type Formulae & Gradient Estimates

In this section, we will *omit the metric g in the notation if unambiguous*. We first assume the form α to be real-valued.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqslant 0}, \mathbb{P})$ be a filtered probability space satisfying the usual hypotheses, i.e. a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with a filtration that is right-continuous

$$\mathscr{F}_t = \mathscr{F}_{t+} := \bigcap_{s > t} \mathscr{F}_s \quad \text{for all } t \geqslant 0,$$

and complete, i.e. $\mathscr{F}_0$ contains all subsets of $\mathbb{P}$-null sets.

Let $X(x)$ be a Brownian motion on M starting at $x \in M$ and $\zeta(x)$ its maximal lifetime, i.e. an M-valued stochastic process $\left(X_t = X_t(x) \right)_{0 \leqslant t < \zeta}$ such that $X_0 = x \in M$ and

$$N_t^f := f(X_t) - f(X_0) - \frac{1}{2} \int_0^t \Delta_M f(X_s) \, \mathrm{d}s \quad \text{on } \{t < \zeta(x)\}$$

is a real local martingale for all $f \in C^\infty(M)$ and Δ_M denoting the Laplace-Beltrami operator on M. In other words, the process $X(x)$ is a diffusion process in M starting at $x \in M$ which is generated by $\frac{1}{2}\Delta_M$.

Further, let $/\!/$ be the stochastic parallel transport along $X(x)$ and B the stochastic anti-development of X to $\mathsf{T}_x M$, which is a standard Brownian motion on $\mathsf{T}_x M \cong \mathbb{R}^m$. For every $r > 0$ let

$$\tau := \tau(x,r) := \inf \left\{ t \geqslant 0 : X_t(x) \notin \mathsf{B}(x,r) \right\} : \Omega \to [0, \infty]$$

be the first exit time of X from the open ball $\mathsf{B}(x,r)$.

Let $E, \widetilde{E}$ be two Riemannian vector bundles over M, endowed with a metric connection ∇^E and $\nabla^{\widetilde{E}}$ respectively. For a given multiplication map $m \in \Gamma(\mathrm{Hom}(\mathsf{T}^*M \otimes E, \widetilde{E})) \cong \Gamma(\mathsf{T}M \otimes E^* \otimes \widetilde{E})$, we consider the Dirac-type operator

$$\mathsf{D}_m := m\nabla^E : \Gamma(E) \to \Gamma(\widetilde{E})$$

which is understood as the composition

$$\Gamma(E) \xrightarrow{\nabla^E} \Gamma(\mathsf{T}^*M \otimes E) \xrightarrow{m} \Gamma(\widetilde{E}).$$

A multiplication map m is said to be compatible with ∇ provided $\nabla m = 0$, i.e.

$$\nabla_v^{\widetilde{E}}(m_U \alpha) = m_{\nabla_v^{\mathsf{T}M} U} \alpha + m_U(\nabla_v^E \alpha) \qquad \text{for all } U \in \Gamma(\mathsf{T}M),\ \alpha \in \Gamma(E),\ v \in \mathsf{T}M.$$

The horizontal Laplacian $\square$ is the second order differential operator given by the following decomposition

$$\square := -\nabla^*\nabla\ :\ \Gamma(E) \xrightarrow{\nabla^E} \Gamma(\mathsf{T}^*M \otimes E) \xrightarrow{\nabla^{\mathsf{T}^*M \otimes E}} \Gamma(\mathsf{T}^*M \otimes \mathsf{T}^*M \otimes E) \xrightarrow{\mathrm{tr}} \Gamma(E).$$

Next, we consider two second order differential operators

$$\mathsf{L} := \square - \mathscr{R} \quad \text{on } \Gamma(E) \quad \text{and} \quad \widetilde{\mathsf{L}} := \square - \widetilde{\mathscr{R}} \quad \text{on } \Gamma(\widetilde{E}),$$

obeying the commutation relation

$$\varrho := \widetilde{\mathsf{L}}\mathsf{D}_m - \mathsf{D}_m\mathsf{L}.$$

In geometrically natural situations, we have $\varrho = 0$ or $\rho \in \Gamma(\mathrm{Hom}(E,\widetilde{E}))$ is of zeroth order, and $\mathscr{R}$ and $\widetilde{\mathscr{R}}$ are zeroth order operators, provided m is compatible with the Levi-Civita connection. Under those assumptions, Driver and Thalmaier [DT01] can proof derivative formulae for the heat semigroup in the general setting of vector bundles using martingale methods. For a detailed discussion we refer the reader to [DT01]. Let us note two important examples.

Example 3.1. The exterior bundle of total forms $E = \bigwedge \mathsf{T}^*M \to M$ with its natural connection

$$\nabla := \nabla^{\bigwedge \mathsf{T}^*M} := \bigoplus_{p=0}^m \nabla^{\bigwedge^p \mathsf{T}^*M}$$

and Clifford action $c\ :\ \mathsf{T}M \to \mathrm{End}(\bigwedge \mathsf{T}^*M)$, $c(\alpha)\beta := \alpha \wedge \beta - \alpha^\sharp \lrcorner \beta$. The Hodge-Laplacian Δ is related to the horizontal Laplacian $\square$ by the Weitzenböck formula

$$\Delta = \square - \mathscr{R}, \tag{3.1}$$

where Weitzenböck curvature operator $\mathscr{R} \in \Gamma(\mathrm{End}\,\Omega_{C^\infty}(M))$ is a symmetric field of endomorphisms. Acting on p-forms, the field of endomorphisms is specified again by an index

$$\mathscr{R}^{(p)} := \mathscr{R}\big|_{\Omega_{C^\infty}^p(M,g)}.$$

In particular, note that $\mathscr{R}^{(1)} = \mathrm{Ric}$ and $\mathscr{R}^{(0)} = 0$. Moreover, it can be written explicitly (cf. e.g. [DT01, Lemma A.7]), for any orthonormal basis $(e_k)_{1 \leqslant k \leqslant m}$,

$$\mathscr{R}^{(p)} = -\sum_{k,l=1}^m \mathsf{R}(e_l, e_k)(e_l^\flat \wedge \bullet)(e_k \lrcorner \bullet),$$

where $\mathsf{R}(e_l, e_k)$ is the curvature tensor acting on p-forms (cf. [DT01, Lemma A.9]). Then

$$\mathsf{D}_g \equiv \mathsf{D}_c = \mathsf{d} + \boldsymbol{\delta}, \qquad \mathsf{L} = \widetilde{\mathsf{L}} = \Delta, \qquad m = c \quad \text{and} \quad \varrho = 0.$$

In particular, for $E := \bigwedge^p \mathsf{T}^*M$ and $\widetilde{E} := \bigwedge^{p+1} \mathsf{T}^*M$, then $\quad \varrho = 0 \quad$ with

$$\mathsf{D}_m = \mathsf{d}\big|_{\Omega^p}, \qquad \mathsf{L} = -\Delta^{(p)}, \quad \widetilde{\mathsf{L}} = -\Delta^{(p+1)}, \qquad \mathscr{R} = \mathscr{R}^{(p)}, \quad \widetilde{\mathscr{R}} = \mathscr{R}^{(p+1)}, \qquad m(\alpha \otimes \beta) = \alpha \wedge \beta.$$

If instead $\widetilde{E} := \bigwedge^{p-1} \mathsf{T}^*M$, then again $\quad \varrho = 0 \quad$ but with

$$\mathsf{D}_m = \boldsymbol{\delta}\big|_{\Omega^p}, \qquad \mathsf{L} = -\Delta^{(p)}, \quad \widetilde{\mathsf{L}} = -\Delta^{(p-1)}, \qquad \mathscr{R} = \mathscr{R}^{(p)}, \quad \widetilde{\mathscr{R}} = \mathscr{R}^{(p-1)}, \qquad m(\alpha \otimes \beta) = -(\alpha^\sharp \lrcorner \beta).$$

Example 3.2 ([DT01], cf. Proposition 2.15).   Let $\widetilde{E} = \mathsf{T}^*M \otimes E$ and $m = \mathrm{id}_{\widetilde{E}}$. For $\mathbf{D}_m = \nabla$ and given $\mathscr{R} \in \mathsf{End}\, E$, we set

$$\widetilde{\mathscr{R}} = \mathrm{Ric}^{\mathrm{tr}} \otimes \mathbf{1}_E - 2\mathsf{R}^E \cdot + \mathbf{1}_{\mathsf{T}^{\circ *}M} \otimes \mathscr{R} \in \mathsf{End}\, \widetilde{E}, \tag{3.2}$$

$$\varrho = \nabla \cdot \mathsf{R}^E + \nabla^{\mathsf{End}\, E}\mathscr{R} \in \Gamma(\mathsf{Hom}(E, \widetilde{E})), \tag{3.3}$$

where R^E denotes the Riemannian curvature tensor to ∇ on E and

$$\mathrm{Ric}^{\mathrm{tr}} \in \Gamma(\mathsf{End}\, \mathsf{T}^*M)$$

denotes the transpose of the Ricci curvature tensor $\mathrm{Ric} \in \Gamma(\mathsf{End}\, \mathsf{T}M)$ on M, and for any $\eta \in \widetilde{E}_x$, $v \in \mathsf{T}_xM$, $\alpha \in E_x$, (e_i) an orthonormal frame for T_xM,

$$\left(\mathsf{R}^E \cdot \eta\right)(v) := \sum_{i=1}^m \mathsf{R}^E(v, e_i)\eta(e_i),$$

$$\left(\nabla \cdot \mathsf{R}^E \alpha\right)(v) := \sum_{i=1}^m \nabla_{e_i}\mathsf{R}^E(e_i, v)\alpha,$$

$$(\nabla\mathscr{R}\alpha)(v) := (\nabla_v\mathscr{R})\alpha.$$

Choosing $m = \mathrm{id}$, for $\mathbf{L} := \square - \mathscr{R}$, $\widetilde{\mathbf{L}} = \square - \widetilde{\mathscr{R}}$, it follows that $\varrho \in \Gamma(\mathsf{Hom}(E, \widetilde{E}))$.

Next, we introduce the stochastic representation of the semigroup. To this end, recall that $/\!/_t^E : E_x \to E_{X_t}$ is the parallel transport along our diffusion $X_t = X_t(x)$ started at $x \in M$. We define the linear operators on E_x and $\widetilde{E}_x$, respectively,

$$\mathscr{R}_{/\!/_t} := (/\!/_t^E)^{-1}\mathscr{R}\, /\!/_t^E \qquad \text{and} \qquad \widetilde{\mathscr{R}}_{/\!/_t} := (/\!/_t^{\widetilde{E}})^{-1}\widetilde{\mathscr{R}}\, /\!/_t^{\widetilde{E}}$$

along the paths of $X(x)$ in the following way: Via the stochastic parallel transport we get to a random point on the tangent space at X_t and apply the curvature (in case $p = 0$ just Ric and $E = \mathsf{T}M$) considered as a linear transformation. Then we parallel transport back to where the diffusion started. Thus, we get a linear mapping $E_x \to E_x$ which now depends on random, i.e.

$$
\begin{array}{ccc}
E_x & \overset{\mathscr{R}_{/\!/_t^E}}{\dashrightarrow} & E_x \\
{\scriptstyle /\!/_t^E}\downarrow & & \uparrow{\scriptstyle /\!/_t^{E,-1}} \\
E_{X_t} & \underset{\mathscr{R}_{X_t}}{\longrightarrow} & E_{X_t}
\end{array}
$$

Let Q_t be the $\mathsf{End}(E_x)$-valued, and $\widetilde{Q}_t$ the $\mathsf{End}(\widetilde{E}_x)$-valued respectively, pathwise solutions to the ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_t = -\frac{1}{2}\mathscr{R}_{/\!/_t}Q_t, \qquad Q_0 = \mathrm{id}_{E_x},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{Q}_t = -\frac{1}{2}\widetilde{\mathscr{R}}_{/\!/_t}\widetilde{Q}_t, \qquad \widetilde{Q}_0 = \mathrm{id}_{\widetilde{E}_x}. \tag{3.4}$$

The composition $Q \circ /\!/^{E,-1}$ maps from a random point X_t back to the starting point x and is called the (inverse) **damped parallel transport** along the paths of $X(x)$.

Given a potential $w : M \to \mathbb{C}$, the Feynman-Kac semigroup

$$P_s^w f(x) := \mathbb{E}\left(e^{-\frac{1}{2}\int_0^s w(X_r(x))\mathrm{d}r} f(X_s(x)) \mathbb{1}_{\{s<\zeta(x)\}} \right)$$

acts on (bounded) measurable functions f on M. Further, let $\underline{\mathscr{R}}^E = \sigma_{\min}(\mathscr{R}^E)$, i.e.

$$\underline{\mathscr{R}}^E(x) := \min\left\{ (\mathscr{R}_x v, v), v \in E_x, |v| = 1 \right\}.$$

By uniform continuity, $\underline{\mathscr{R}}^E$ is a continuous function on M. By Gronwall's inequality,

$$\left| \mathcal{Q}_s \right|_{\mathrm{op}} \leqslant \exp\left(-\frac{1}{2}\int_0^s \underline{\mathscr{R}}^E(X_r(x))\mathrm{d}r \right).$$

We have the following probabilistic representation of the semigroup, for all $s > 0$ and every $x \in M$,

$$P_s^g \alpha(x) = \int_M p_s^g(x, y)\alpha(y) \operatorname{vol}_g(\mathrm{d}y) = \mathbb{E}^x\left(\mathcal{Q}_s /\!/_s^{E,-1} \alpha(X_s) \mathbb{1}_{\{s<\zeta\}} \right) \qquad \text{for all } \ \alpha \in \Gamma_{\mathsf{L}^2}(E),$$

provided the scalar semigroup $P_t^{\underline{\mathscr{R}}^E} |\alpha|(x) < \infty$. In particular, we get semigroup domination

$$\left| P_s^g \alpha(x) \right| \leqslant P_s^{\underline{\mathscr{R}}^E} |\alpha|(x).$$

To ensure the existence, we therefore always assume that $\underline{\mathscr{R}}^E \in \mathsf{K}(M)$ is in the Kato class.

Definition 3.3 (Kato class). Let $w : M \to \mathbb{C}$ be a measurable function. Then w is said to be in the **contractive Dynkin class** $\mathsf{D}(M)$ (also **extended Kato class**), if there is a $t > 0$ with

$$\sup_{x \in M} \int_0^t \mathbb{E}^x\left(\mathbb{1}_{\{s<\zeta\}} \left| w(X_s) \right| \right) \mathrm{d}s < 1,$$

and w is in the **Kato class** $\mathsf{K}(M)$, if

$$\lim_{t \searrow 0} \sup_{x \in M} \int_0^t \mathbb{E}^x\left(\mathbb{1}_{\{s<\zeta\}} \left| w(X_s) \right| \right) \mathrm{d}s = 0.$$

The function w is said to be in the **local Dynkin class** $\mathsf{D}_{\mathrm{loc}}(M)$ or **local Kato class** $\mathsf{K}_{\mathrm{loc}}(M)$, if $\mathbb{1}_K w \in \mathsf{D}(M)$ or $\mathbb{1}_K w \in \mathsf{K}(M)$, respectively, for all compact $K \subset M$.

Remark 3.4. (i) The Kato class $\mathsf{K}(M)$ plays an important rôle in the study of Schrödinger operators and their associated semigroups, cf. [Sim82]. The contractive Dynkin class $\mathsf{D}(M)$ appears in [Voi86] to study properties of semigroups associated to Schrödinger operators. In the case of a non-compact manifold it is well-known that there are many technical difficulties with the behaviour of the potentials at ∞. The Kato class defines a sufficiently rich class of potentials for which we can still expect the Feynman-Kac formula to make sense pointwise not only vol-a.e. x, but *for all* $x \in M$.

(ii) In particular (cf. [Gün17, Remark VI.2.]), in the Euclidean space $\mathbb{R}^m$, we get $\mathsf{L}^q(\mathbb{R}^m) \subset \mathsf{K}(\mathbb{R}^m)$, for $m \geqslant 2$ and $q > \frac{m}{2}$. Then it is well-known, that the Coulomb potential $\frac{1}{|x|}$ is in $\mathsf{K}(\mathbb{R}^3)$.

(iii) Clearly, all four classes depend on the Riemannian structure of M and we have

$$\mathsf{K}(M) \subset \mathsf{D}(M) \quad \text{and} \quad \mathsf{K}_{\mathrm{loc}}(M) \subset \mathsf{D}_{\mathrm{loc}}(M).$$

In view of those implications and since it is more common to work with Kato classes, we note that in what follows all assumptions may be relaxed from $\mathsf{K}(M)$ to $\mathsf{D}(M)$.

Lemma 3.5 ([Gün17, Lemma VI.8.]). *For any $w \in \mathsf{K}(M)$ and $\gamma > 1$, there is $c_\gamma = c_\gamma(w) > 0$, such that*

$$\sup_{x \in M} \mathbb{E}^x \left(\mathbb{1}_{\{t < \zeta\}} e^{\int_0^t |w(X_s)| \mathrm{d}s} \right) \leqslant \gamma e^{t c_\gamma} < \infty, \qquad \text{for all } t \geqslant 0.$$

Remark 3.6. The previous Lemma 3.5 can be elaborated in the case of potentials in the Dynkin class (cf. [Gün17, Lemma VI.8.]), namely: For any $w \in \mathsf{D}(M)$ there are $c_k = c_k(w) > 0$, for $k \in \{1, 2\}$, such that

$$\sup_{x \in M} \mathbb{E}^x \left(\mathbb{1}_{\{t < \zeta\}} e^{\int_0^t |w(X_s)| \mathrm{d}s} \right) \leqslant c_1 e^{t c_2} < \infty, \qquad \text{for all } t \geqslant 0.$$

Following the ideas of [DT01], we will write out the Bismut-type formulae derived for the special cases in Example 3.1 and Example 3.2.

From now on, let $E := \bigwedge \mathsf{T}^* M$ and, to shorten notation, we set $\underline{\mathscr{R}} := \underline{\mathscr{R}}^{\bigwedge \mathsf{T}^* M}$.

By [DT01, Theorem 6.1]], we get immediately the following Bismut-type derivative formulae.

Theorem 3.7. *Let $\alpha \in \Omega_{\mathsf{L}^2}(M)$ and $\underline{\mathscr{R}} \in \mathsf{K}(M)$. Then for any $v \in \bigwedge \mathsf{T}_x M$, we have the following Bismut-type formulae:*

$$\langle (\mathbf{d} P_s \alpha)_x, v \rangle = -\mathbb{E} \left\langle /\!\!/_s^{-1} \alpha(X_s(x)) \mathbb{1}_{\{s < \zeta(x)\}}, \mathscr{Q}_s \int_0^s \mathscr{Q}_r^{-1} (\mathrm{d}B_r \lrcorner \mathscr{Q}_r \dot{\ell}_r) \right\rangle,$$

$$\langle (\boldsymbol{\delta} P_s \alpha)_x, v \rangle = -\mathbb{E} \left\langle /\!\!/_s^{-1} \alpha(X_s(x)) \mathbb{1}_{\{s < \zeta(x)\}}, \mathscr{Q}_s \int_0^s \mathscr{Q}_r^{-1} (\mathrm{d}B_r \wedge \mathscr{Q}_r \dot{\ell}_r) \right\rangle,$$

where

- $\tau(x, r) < \zeta(x)$ *is the first exit time of X from the open ball $\mathsf{B}(x, r)$,*

- $\mathrm{d}B := /\!\!/^{-1} \circ \mathrm{d}X(x)$ *is a Brownian motion in $\mathsf{T}_x M$, i.e. the associated anti-development of the Brownian motion $X(x)$,*

- $(\ell_r)_{r \in [0,s]}$ *is any adapted process in $\bigwedge \mathsf{T}_x M$ with absolutely continuous paths such that for some arbitrary small $\varepsilon > 0$*

$$\mathbb{E} \left(\int_0^{(s-\varepsilon) \wedge \tau(x)} \left| \dot{\ell}_r \right|^2 \mathrm{d}r \right)^{1/2} < \infty \qquad \text{and} \qquad \ell_0 = v, \quad \ell_r = 0 \qquad \text{for all } r \geqslant (s - \varepsilon) \wedge \tau(x).$$

If, in addition, $\alpha \in \Omega_{\mathsf{L}^2}(M)$ is bounded on this neighbourhood, we may take $\varepsilon = 0$.

Note the remarkable facts in the formula above that curvature only enters locally around a point x and there is no derivative on the righthand side. Moreover, the process ℓ may be chosen

nicely, as long as it starts in v and is zero when the process X exits from the open ball $B(x, r)$ (some relatively compact neighbourhood of x would be sufficient).

Next, we prove a covariant Bismut-type formula in the setting of Example 3.2. In the proof, we briefly outline the strategy and refer the reader to Driver and Thalmaier [DT01] studying the theory in full generality, namely on vector bundles, for all the details. Their idea is to define a suitable martingale, say N_s, and stay on the *local* martingale level as long as possible. Using that a *true* martingale has constant expectation, one then shows that N_s is indeed a martingale and takes expectations at times $s = 0$ and $s = t \wedge \tau$. Note that this method solely involves the geometry and applies *especially* in the case of non-compact manifolds.

Theorem 3.8. *Suppose* $\alpha \in \Omega_{L^2}(M)$, $\underline{\mathscr{R}} \in K(M)$ *and* $\xi \in \widetilde{E}_x^* = T_x M \otimes \bigwedge T_x M$. *Let* $(\ell_r)_{r \in [0,s]}$ *a bounded adapted process with absolutely continuous paths in* $\bigwedge T_x M$ *such that* $\mathbb{E}\left(\int_0^{(s-\varepsilon) \wedge \tau(x,r)} |\dot{\ell}_r|^2 \, dr\right)^{1/2} < \infty$ *and* $\ell_0 = \xi$, $\ell_r = 0$ *for all* $r \geq (s - \varepsilon) \wedge \tau(x, r)$ *and some arbitrary small* $\varepsilon > 0$. *Then,*

$$\left\langle \nabla e^{\Delta s/2} \alpha(x), \xi \right\rangle = -\mathbb{E}^x \left\langle \mathcal{Q}_s /\!\!/_s^{-1} \alpha(X_s) \mathbb{1}_{\{s < \zeta\}}, U_{s \wedge \tau}^\ell \right\rangle,$$

where

$$U_s^\ell := \int_0^s \mathcal{Q}_s^{-1}(dB_s \lrcorner \widetilde{\mathcal{Q}}_s \dot{\ell}_s) + \frac{1}{2} \int_0^s (\mathcal{Q}_s)^{-1} \varrho_{/\!\!/_s} \widetilde{\mathcal{Q}}_s \ell_s \, ds,$$

$\varrho_{/\!\!/_s} := /\!\!/_s^{-1} \varrho /\!\!/_s$ *with* ϱ *given by* (3.2), $\mathcal{Q}$ *and* $\widetilde{\mathcal{Q}}$ *are defined by* (3.4) *and* $\widetilde{\mathscr{R}}$ *by* (3.3).

Proof. According to [DT01, Proposition 3.2 & Theorem 3.7], the process

$$N_r^\ell := \left\langle \widetilde{\mathcal{Q}}_r /\!\!/_r^{-1} \nabla e^{\Delta(s-r)/2} \alpha(X_r(x)), \ell_r \right\rangle - \left\langle \mathcal{Q}_r /\!\!/_r^{-1} e^{\Delta(s-r)/2} \alpha(X_r(x)), U_r^\ell \right\rangle$$

is a local martingale. The existence of the scalar semigroup is provided by the assumption $\underline{\mathscr{R}} \in K(M)$. By the Burkholder-Davis-Gundy inequality, N_s^ℓ is a true martingale and evaluating at the times $s = 0$ and $s = t \wedge \tau$ and taking expectations, we get by the martingale property $\mathbb{E} N_0^\ell = \mathbb{E} N_{s \wedge \tau}^\ell$,

$$\left\langle \nabla e^{\Delta s/2} \alpha(x), \xi \right\rangle = -\mathbb{E} \left\langle \mathcal{Q}_{s \wedge \tau} /\!\!/_{s \wedge \tau}^{-1} e^{\Delta(s - s \wedge \tau)/2} \alpha(X_{s \wedge \tau}(x)), U_{s \wedge \tau}^\ell \right\rangle.$$

Note that, by the strong Markov property,

$$\mathcal{Q}_{s \wedge \tau} /\!\!/_{s \wedge \tau}^{-1} \left(P_{s - s \wedge \tau} \alpha \right)(X_{s \wedge \tau}(x)) = \mathbb{E}^{\mathscr{F}_{s \wedge \tau}} \left(\mathcal{Q}_s /\!\!/_s^{-1} \alpha(X_s(x)) \mathbb{1}_{\{s < \zeta(x)\}} \right),$$

which is by definition a bounded $\mathscr{F}_{s \wedge \tau}$-measurable random variable. Hence,

$$\left\langle \nabla e^{\Delta s/2} \alpha(x), \xi \right\rangle = -\mathbb{E} \left\langle \mathcal{Q}_s /\!\!/_s^{-1} \alpha(X_s(x)) \mathbb{1}_{\{s < \zeta(x)\}}, U_{s \wedge \tau}^\ell \right\rangle. \qquad \blacksquare$$

From now on, set $D := B(x, R)$ to be a ball with small radius, say $R = 1$, and we define

$$\overline{K}(x) := \max \left\{ \langle \mathscr{R}(v), v \rangle : v \in \bigwedge T_y M, |v| = 1, y \in B(x, R) \right\}, \tag{3.5}$$

$$\underline{K}(x) := \min \left\{ \langle \mathscr{R}(v), v \rangle : v \in \bigwedge T_y M, |v| = 1, y \in B(x, R) \right\}. \tag{3.6}$$

Theorem 3.9. *Let $\alpha \in \Omega_{\mathsf{L}^2}(M)$ and $\underline{\mathscr{R}} \in \mathsf{K}(M)$. Then, for all $s > 0$,*

$$\left|(\mathbf{d}P_s\alpha)_x\right|^2 \leqslant \Psi(x,s)\Phi(x,s)\left\|\alpha\right\|^2_{\Omega_{\mathsf{L}^2}(M)},$$

$$\left|(\boldsymbol{\delta}P_s\alpha)_x\right|^2 \leqslant \Psi(x,s)\Phi(x,s)\left\|\alpha\right\|^2_{\Omega_{\mathsf{L}^2}(M)}, \tag{3.7}$$

where

$$\Psi(x,s) := \frac{1}{s}\exp\left[Cs + \left(\pi\sqrt{(m-1)\underline{K}(x)^-} + \pi^2(m+5) + (\overline{K}(x) + \underline{K}(x))^-\right)\frac{s}{2}\right]. \tag{3.8}$$

Proof. By Gronwall's inequality, we have

$$\left|\mathcal{Q}_s\right|_{\mathrm{op}} \leqslant \exp\left(-\frac{1}{2}\int_0^s \underline{\mathscr{R}}(X_r(x))\mathrm{d}r\right) \qquad \text{for all } s \geqslant 0.$$

and hence

$$\left|\mathcal{Q}_s\right|_{\mathrm{op}} \leqslant \mathrm{e}^{\underline{K}(x)s/2}, \quad \left|\mathcal{Q}_s^{-1}\right|_{\mathrm{op}} \leqslant \mathrm{e}^{\overline{K}(x)s/2} \qquad \mathbb{P}\text{-a.s. on } \{s \leqslant \tau(x,R)\}. \tag{3.9}$$

Let $q \in [2,\infty)$. By a proper choice of the Cameron-Martin space valued process ℓ_s, it is well-known how to estimate the second factor (cf. [TW98, Proof of Corollary 5.1], [TW11, Remark 3.2], [CTT18]),

$$|\ell| \leqslant |v|, \qquad \left[\mathbb{E}\left(\int_0^{s\wedge\tau(x,R)}\left|\dot{\ell}_r\right|^2\mathrm{d}r\right)^q\right]^{1/q} \leqslant s^{-1/2}\mathrm{e}^{C(m,q,R,\underline{K}^-)s/2}|v|, \tag{3.10}$$

where

$$C(m,q,R,\underline{K}^-) := \frac{\pi}{2R}\sqrt{(m-1)\underline{K}^-} + \frac{\pi^2}{4R^2}(m+q+3).$$

By the Burkholder-Davis-Gundy inequality, we get

$$\mathbb{E}\left|\int_0^s \mathcal{Q}_r^{-1}(\mathrm{d}B_r \rightharpoonup \mathcal{Q}_r\dot{\ell}_r)\right|^q \leqslant c_3(q)\mathrm{e}^{q(\overline{K}+\underline{K})^-s/2}\mathbb{E}\left(\int_0^s\left|\dot{\ell}_r\right|^2\mathrm{d}r\right)^{q/2}. \tag{3.11}$$

By Lemma 3.5, for any $\gamma > 1$, there is a constant $c_\gamma = c_\gamma(\underline{\mathscr{R}})$ such that

$$\sup_{x\in M}\mathbb{E}^x\left(\mathbb{1}_{\{s<\zeta\}}\mathrm{e}^{\int_0^s|\underline{\mathscr{R}}(X_s)|\mathrm{d}s}\right) \leqslant \gamma\mathrm{e}^{sc_\gamma} < \infty. \tag{3.12}$$

Now, we can estimate as follows. Let $|v| \leqslant 1$, then using Hölder and Cauchy-Schwarz inequality,

$$\left|(\mathbf{d}P_s\alpha)_x\right| \leqslant \left[\mathbb{E}^x\left|\alpha(X_s)\mathbb{1}_{\{s<\zeta\}}\right|^p\right]^{1/p}\left[\mathbb{E}^x\left|\mathcal{Q}_s\mathbb{1}_{\{s<\zeta\}}\int_0^s\mathcal{Q}_r^{-1}(\mathrm{d}B_r \rightharpoonup \mathcal{Q}_r\dot{\ell}_r)\right|^q\right]^{1/q}$$

$$\leqslant \left[\mathbb{E}^x\left|\alpha(X_s)\mathbb{1}_{\{s<\zeta\}}\right|^p\right]^{1/p}\left[\mathbb{E}^x\left(\left|\mathcal{Q}_s\right|^{2q}\mathbb{1}_{\{s<\zeta\}}\right)\right]^{1/(2q)}\left[\mathbb{E}^x\left(\int_0^s\mathcal{Q}_r^{-1}(\mathrm{d}B_r \rightharpoonup \mathcal{Q}_r\dot{\ell}_r)\right)^{2q}\right]^{1/(2q)}$$

$$\overset{(3.12)}{\underset{(3.11)}{\leqslant}} \left[\mathbb{E}^x\left(|\alpha|^p(X_s)\mathbb{1}_{\{s<\zeta\}}\right)\right]^{1/p}\gamma\mathrm{e}^{sc_\gamma}c_3(q)^{1/q}\mathrm{e}^{(\overline{K}+\underline{K})^-s/2}\left[\mathbb{E}\left(\int_0^s\left|\dot{\ell}_r\right|^2\mathrm{d}r\right)^{2q}\right]^{1/(2q)}$$

$$\overset{(3.10)}{\leqslant} \mathrm{e}^{C_1(\gamma,c_\gamma,c_3(q)^{1/q})s + (\overline{K}+\underline{K})^-s/2}\mathrm{e}^{C(m,2q,R,\underline{K}^-)s/2}s^{-1/2}\left[\int_M p_s^{g,(0)}(x,y)|\alpha(y)|^p\mathrm{vol}_g(\mathrm{d}y)\right]^{1/p}$$

$$\leqslant \sqrt{\Psi(x,s)}\sqrt[p]{\Phi(x,s)}\left\|\alpha\right\|_{\Omega_{\mathsf{L}^p}(M)},$$

where $\Psi(x,s)$ is given by (3.13). In particular, for $p = 2 = q$ the result follows.

By an analogous calculation, we obtain the estimate (3.7). Since complexifications are norm preserving, the proof is complete. ∎

Using similar techniques as in the Proof of the previous theorem, we can show the following estimate.

Theorem 3.10. *Let $\alpha \in \Omega_{\mathsf{L}^2}(M)$ and $\underline{\mathscr{R}} \in \mathsf{K}(M)$. Then, for all $\xi \in \mathsf{T}_x M \otimes \bigwedge \mathsf{T}_x M$ and $s > 0$,*

$$\left| \langle \nabla P_s \alpha, \xi \rangle \right|^2 \leqslant |\xi|^2 \, \Xi(x,s) \Phi(x,s) \, \|\alpha\|_{\Omega_{\mathsf{L}^2}(M)}^2 \,,$$

where

$$\Xi(x,s) := e^{Cs + (\overline{K}+\underline{K})^- s/2} \left[e^{C(m,R,\underline{K}^-)s/2} s^{-1} + \max_{y \in B(x,R)} |\nabla R(y)| \, s \right]$$

$$= \Psi(x,s) + s^2 \Psi(x,s) \max_{y \in B(x,R)} |\nabla R(y)| \,. \tag{3.13}$$

Proof. Set

$$U_s^\ell = \underbrace{\int_0^s \mathcal{Q}_s^{-1}(\mathrm{d} B_s \lrcorner \widetilde{\mathcal{Q}}_s \ell_s)}_{=: \; \ell_s^{(1)}} + \frac{1}{2} \underbrace{\int_0^s \mathcal{Q}_s^{-1} \varrho_{/\!/_s} \widetilde{\mathcal{Q}}_s \ell_s \, \mathrm{d} s}_{=: \; \ell_s^{(2)}} = \ell_s^{(1)} + \frac{1}{2}\ell_s^{(2)} \,.$$

As in the previous Proof of Theorem 3.9, we find

$$|\ell| \leqslant |\xi| \,, \qquad \left[\mathbb{E} \left(\int_0^{s \wedge \tau(x,R)} |\dot{\ell}_r|^2 \, \mathrm{d} r \right)^q \right]^{1/q} \leqslant s^{-1/2} e^{C(m,q,R,\underline{K}^-)s/2} \, |\xi| \,.$$

Again using Gronwall's inequality (3.9) and

$$\left| \varrho(X_s(x)) \right| \leqslant \max_{y \in B(x,R)} |\varrho(y)| \leqslant \max_{y \in B(x,R)} |\nabla R(y)| \qquad \mathbb{P} - \text{a.s. on } \{s \leqslant \tau(x,R)\} \,,$$

we have

$$\mathbb{E} \left| \int_0^s \mathcal{Q}_s^{-1} \varrho_{/\!/_s} \widetilde{\mathcal{Q}}_s \ell_s \, \mathrm{d}s \right| \leqslant e^{(\overline{K}_1 + \underline{K}_2)^- s/2} \max_{y \in B(x,R)} |\nabla R(y)| \, s \, |\xi| \,.$$

By Lemma 3.5, for any $\gamma > 1$, there is a constant $c_\gamma = c_\gamma(\underline{\mathscr{R}})$ such that

$$\sup_{x \in M} \mathbb{E}^x \left(\mathbb{1}_{\{t < \zeta\}} e^{\int_0^s |\underline{\mathscr{R}}(X_s)| \mathrm{d}s} \right) \leqslant \gamma e^{sc_\gamma} < \infty \,.$$

As in the proof of Theorem 3.9, a similar calculation shows, using Hölder's inequality and the elementary inequality $(a+b)^c \leqslant 2^{c-1}(a^c + b^c)$,

$$\left| \langle \nabla P_s \alpha(x), \xi \rangle \right| = \left| \mathbb{E}^x \left\langle /\!/_s^{-1} \alpha(X_s) \mathbb{1}_{\{s<\zeta\}}, \mathcal{Q}_s U_{s\wedge\tau}^\ell \right\rangle \right|$$

$$\leqslant 2 \left| \mathbb{E}^x \left\langle /\!/_s^{-1} \alpha(X_s) \mathbb{1}_{\{s<\zeta\}}, \mathcal{Q}_s \ell_{s\wedge\tau}^{(1)} \right\rangle \right| + \left| \mathbb{E}^x \left\langle /\!/_s^{-1} \alpha(X_s) \mathbb{1}_{\{s<\zeta\}}, \mathcal{Q}_s \ell_{s\wedge\tau}^{(2)} \right\rangle \right|$$

$$\leqslant \left[\mathbb{E}^x \left| \alpha(X_s) \mathbb{1}_{\{s<\zeta\}} \right|^p \right]^{1/p} \left(2 \left[\mathbb{E}^x \left(\mathcal{Q}_s \mathbb{1}_{\{s<\zeta\}} \ell_{s\wedge\tau}^{(1)} \right)^q \right]^{1/q} + \left[\mathbb{E}^x \left(\mathcal{Q}_s \mathbb{1}_{\{s<\zeta\}} \ell_{s\wedge\tau}^{(2)} \right)^q \right]^{1/q} \right)$$

$$\leqslant |\xi| \left[\mathbb{E}^x \left(|\alpha|^p (X_s) \mathbb{1}_{\{s<\zeta\}} \right) \right]^{1/p} \gamma e^{sc_\gamma} c_3(q)^{1/q} e^{(\overline{K}+\underline{K})^- s/2} \left(2 \left[\mathbb{E} \left(\int_0^s |\dot{\ell}_r|^2 \, \mathrm{d}r \right)^{2q} \right]^{1/(2q)} + \right.$$

$$\left. + s \max_{y \in B(x,R)} |\nabla R(y)| \right)$$

$$\leqslant |\xi| \, e^{C_1(\gamma, c_\gamma, c_3(q)^{1/q})s \, + \, (\overline{K}+\underline{K})^- s/2} \Big[e^{C(m,2q,R,\underline{K}^-)s/2} s^{-1} + \Big(\max_{y \in B(x,R)} |\nabla R(y)| \Big) s \Big] \times$$

$$\times \Big[\int_M p_s^{g,(0)}(x,y) \, |\alpha(y)|^p \, \mathrm{vol}_g(\mathrm{d}y) \Big]^{1/p}$$

$$\leqslant |\xi| \, \sqrt{\Xi(x,s)} \, \sqrt[p]{\Phi(x,s)} \, \|\alpha\|_{\Omega_{\mathsf{L}^p}(M)} \, .$$

where $\Xi(x,s)$ is given by (3.13).

In particular, for $p = 2 = q$ the result follows. Since complexification is norm preserving, the proof is complete. ∎

In the remaining part of this section, we will make use of Theorem 3.10 and derive similar estimates for the exterior derivative and codifferential transformed by the smooth vector bundle morphism $\mathscr{A}_{g,h}$ defined in (2.3). The key observation will be Proposition 3.15 showing how to estimate the transformed codifferential $\boldsymbol{\delta}_g$ (with respect the metric g) applied to the semigroup in terms of covariant derivative ∇^g of g applied to the semigroup. In addition, using Lemma 2.8, a direct consequence is an analogous result (cf. Corollary 3.18) in terms of the new metric, i.e. for the transformed codifferential $\boldsymbol{\delta}_h$ (with respect the metric h) applied to the semigroup.

To this end, we will make use of the well-known metric descriptions of the exterior derivative $\mathbf{d}$ and the codifferential $\boldsymbol{\delta}_g$ (with respect the metric g), namely:

Lemma 3.11 ([Jos17, Lemma 4.3.4]). *Let $e_1, \ldots, e_m$ (where $m = \dim M$) be a local orthonormal frame for $\mathsf{T}_x M$ ($x \in M$) and $\varepsilon^1, \ldots, \varepsilon^m$ be its dual coframe, i.e. $\varepsilon^j(e_i) = \delta_i^j$. Then*

$$\mathbf{d} = \sum_{i=1}^m \varepsilon^i \wedge \nabla_{e_i}^g \qquad and \qquad \boldsymbol{\delta}_g = -\sum_{i=1}^m e_i \lrcorner \nabla_{e_i}^g. \tag{3.14}$$

Let $\mathscr{A} := \mathscr{A}_{g,h}$ be the smooth vector bundle morphism defined by (2.3).

Proposition 3.12. *Let $\alpha \in \Omega_{\mathsf{L}^2}(M)$ and $\underline{\mathscr{R}} \in \mathsf{K}(M)$. Then, for any orthonormal frame $(e_i)_{i=1}^m$ for $\mathsf{T}_x M$ ($x \in M$) and dual coframe $(\varepsilon^i)_{i=1}^m$, we decompose*

$$\mathbf{d}(\mathscr{A}^{1/2} P_s \alpha) = \sum_{i=1}^m \Big(\varepsilon^i \wedge \nabla_{e_i}^g P_s \alpha(\mathscr{A}^{1/2}) + \varepsilon^i \wedge P_s \alpha \left(\nabla_{e_i}^g \mathscr{A}^{1/2} \right) \Big).$$

Proof. From (3.14), we get for any $\eta \in \Omega(M)$

$$\mathbf{d}(\mathscr{A}^{1/2} \eta) = \mathbf{d}(\eta \circ \mathscr{A}^{1/2}) = \sum_{i=1}^m \varepsilon^i \wedge \nabla_{e_i}^g \eta \circ \mathscr{A}^{1/2} + \sum_{i=1}^m \varepsilon^i \wedge \eta \circ \nabla_{e_i}^g \mathscr{A}^{1/2},$$

where $(e_i)_{i=1}^m$ denotes an orthonormal frame for $\mathsf{T}_x M$ ($x \in M$) and $(\varepsilon^i)_{i=1}^m$ the dual coframe. In particular, for $\eta = P_s \alpha$ the claims follows. ∎

Corollary 3.13. *Let $\alpha \in \Omega_{\mathsf{L}^2}(M)$ and $\underline{\mathscr{R}} \in \mathsf{K}(M)$. Then,*

$$\left| \mathscr{A}^{-1/2} \mathbf{d}(\mathscr{A}^{1/2} P_s \alpha(x)) \right|^2 \lesssim \left(\delta_{g,h}^\nabla(x) + \Xi(x,s) \right) \Phi(x,s) \, \|\alpha\|_{\Omega_{\mathsf{L}^2}(M)}^2.$$

Proof. By Theorem 3.10, we have

$$\left| \nabla^g_{e_i} P_s \alpha (\mathscr{A}^{1/2}) \right|^2 \leqslant C(m) \left| \mathscr{A}(x) \right| \Xi(x,s) \Phi(x,s) \left\| \alpha \right\|^2_{\Omega_{\mathsf{L}^2}(M)}$$

and, combined with Lemma 3.5 and Lemma 2.4,

$$\left| P_s \alpha \left(\nabla^g_{e_i} \mathscr{A}^{1/2} \right) \right|^2 \leqslant C(m, \gamma, c_\gamma, s) \left| \nabla^g \mathscr{A}^{1/2}(x) \right|^2_g \Phi(x,s) \left\| \alpha \right\|^2_{\Omega_{\mathsf{L}^2}(M)}$$

$$\lesssim \left| \mathscr{A}^{1/2}(x) \right|^2 \delta^\nabla_{g,h}(x) \Phi(x,s) \left\| \alpha \right\|^2_{\Omega_{\mathsf{L}^2}(M)}.$$

Thus the claim follows:

$$\left| \mathscr{A}^{-1/2} \mathbf{d}(\mathscr{A}^{1/2} P_s \alpha(x)) \right|^2 \lesssim \left| \mathscr{A}^{-1/2}(x) \right|^2 \left| \mathscr{A}^{1/2}(x) \right|^2 \left(\delta^\nabla_{g,h}(x) + \Xi(x,s) \right) \Phi(x,s) \left\| \alpha \right\|^2_{\Omega_{\mathsf{L}^2}(M)}. \qquad \blacksquare$$

Remark 3.14. By a similar calculation the estimate also holds if we interchange the rôles of $\mathscr{A}^{1/2}$ and $\mathscr{A}^{-1/2}$ in the previous Corollary 3.13.

Proposition 3.15. *Let $\alpha \in \Omega_{\mathsf{L}^2}(M)$ and $\underline{\mathscr{R}} \in \mathsf{K}(M)$. Then, for any orthonormal basis $(e_i)_i$ for $\mathsf{T}_x M$ ($x \in M$), we decompose*

$$\boldsymbol{\delta}_g(\mathscr{A}^{1/2} P_s \alpha) = -\sum_{i=1}^m e_i \lrcorner \left(\nabla^g_{e_i} P_s \alpha \right) (\mathscr{A}^{1/2}) - \sum_{i=1}^m e_i \lrcorner P_s \alpha \left(\nabla^g_{e_i} \mathscr{A}^{1/2} \right).$$

Proof. Let $\eta \in \Omega(M)$ be arbitrary and $(e_i)_{i=1}^m$ an orthonormal basis for $\mathsf{T}_x M$ ($x \in M$). By definition of the codifferential, (3.14) in Lemma 3.11 above, we get

$$\boldsymbol{\delta}_g(\mathscr{A}^{1/2} \eta) = -\sum_{i=1}^m e_i \lrcorner \nabla^g_{e_i}(\mathscr{A}^{1/2} \eta)$$

$$= -\sum_{i=1}^m \left(e_i \lrcorner (\mathscr{A}^{1/2} \nabla^g_{e_i} \eta) + e_i \lrcorner ((\nabla^g_{e_i} \mathscr{A}^{1/2}) \eta) \right).$$

In particular, if we set $\eta = P_s \alpha(x)$, then the following equalities hold:

$$\sum_{i=1}^m e_i \lrcorner \left(\mathscr{A}^{1/2} \nabla^g_{e_i} P_s \alpha \right) = \sum_{i=1}^m e_i \lrcorner \left(\nabla^g_{e_i} P_s \alpha \right) (\mathscr{A}^{1/2})$$

and

$$\sum_{i=1}^m e_i \lrcorner ((\nabla^g_{e_i} \mathscr{A}^{1/2}) P_s \alpha) = \sum_{i=1}^m e_i \lrcorner P_s \alpha \left(\nabla^g_{e_i} \mathscr{A}^{1/2} \right). \qquad \blacksquare$$

Corollary 3.16. *Let $\alpha \in \Omega_{\mathsf{L}^2}(M)$ and $\underline{\mathscr{R}} \in \mathsf{K}(M)$. Then,*

$$\left| \mathscr{A}^{-1/2} \boldsymbol{\delta}_g(\mathscr{A}^{1/2} P_s \alpha(x)) \right|^2 \lesssim \left(\delta^\nabla_{g,h}(x) + \Xi(x,s) \right) \Phi(x,s) \left\| \alpha \right\|^2_{\Omega_{\mathsf{L}^2}(M)}.$$

Proof. By Theorem 3.10 we have

$$\left| \sum_{i=1}^m e_i \lrcorner \left(\nabla^g_{e_i} P_s \alpha \right) (\mathscr{A}^{1/2}) \right|^2 \leqslant C(m) \left| \mathscr{A}^{1/2}(x) \right|^2 \Xi(x,s) \Phi(x,s) \left\| \alpha \right\|^2_{\Omega_{\mathsf{L}^2}(M)}$$

and, combined with Lemma 3.5 and Lemma 2.4,

$$\left| e_i \lrcorner P_s \alpha \left(\nabla^g_{e_i} \mathcal{A}^{1/2} \right) \right|^2 \leqslant C(m, \gamma, c_\gamma, s) \left| \nabla^g \mathcal{A}^{1/2}(x) \right|^2_g \Phi(x, s) \|\alpha\|^2_{\Omega_{L^2}(M)}$$

$$\lesssim \left| \mathcal{A}^{1/2}(x) \right|^2 \delta^\nabla_{g,h}(x) \Phi(x, s) \|\alpha\|^2_{\Omega_{L^2}(M)} .$$

Thus the claims follows:

$$\left| \mathcal{A}^{-1/2} \boldsymbol{\delta}_g (\mathcal{A}^{1/2} P_s \alpha(x)) \right|^2 \lesssim \left| \mathcal{A}^{-1/2}(x) \right|^2 \left| \mathcal{A}^{1/2}(x) \right|^2 \left(\delta^\nabla_{g,h}(x) + \Xi(x, s) \right) \Phi(x, s) \|\alpha\|^2_{\Omega_{L^2}(M)} . \qquad \blacksquare$$

Remark 3.17. By a similar calculation the estimate also holds if we interchange the rôles of $\mathcal{A}^{1/2}$ and $\mathcal{A}^{-1/2}$ in the previous Corollary 3.16.

Corollary 3.18. *Let $\alpha \in \Omega_{L^2}(M)$ and $\underline{\mathcal{R}} \in \mathsf{K}(M)$. Then,*

$$\left| \mathcal{A}^{1/2} \boldsymbol{\delta}_h (\mathcal{A}^{-1/2} P_s \alpha(x)) \right|^2 \lesssim \left(\delta^\nabla_{g,h}(x) + \Xi(x, s) \right) \Phi(x, s) \|\alpha\|^2_{\Omega_{L^2}(M)} .$$

Proof. By Lemma 2.8, recall that

$$\boldsymbol{\delta}_h(\eta) = \mathcal{A}^{-1} \left(\boldsymbol{\delta}_g (\mathcal{A}\eta) - \mathbf{d}\log\rho \lrcorner_g (\mathcal{A}\eta) \right) \qquad \text{for all } \eta \in \Omega_{C^\infty}(M),$$

so that

$$(\mathcal{A}^{1/2} \boldsymbol{\delta}_h \mathcal{A}^{-1/2})(\eta) = \mathcal{A}^{-1/2} \left(\boldsymbol{\delta}_g (\mathcal{A}^{1/2}\alpha) - \mathbf{d}\log\rho \lrcorner_g (\mathcal{A}^{1/2}\eta) \right) .$$

Using Proposition 2.12 combined with Lemma 2.5, in addition

$$\mathcal{A}^{-1/2} \left| \mathbf{d}\log\rho \lrcorner (\mathcal{A}^{1/2}\eta) \right|_g \leqslant C(m) \left| \mathcal{A}^{-1/2} \right| \left| \mathcal{A}^{-1} \right| \left| \nabla^g \mathcal{A} \right|_g \left| \mathcal{A}^{1/2} \right| |\eta|_g$$

$$\leqslant C(m) \left| \mathcal{A}^{-1/2} \right| \left| \mathcal{A}^{-1} \right| |\mathcal{A}| \left| \nabla^h_X - \nabla^g_X \right|_g \left| \mathcal{A}^{1/2} \right| |\eta|_g$$

$$\leqslant C(m) \left| \nabla^h_X - \nabla^g_X \right|_g |\eta|_g .$$

Finally, combined with Corollary 3.16,

$$\left| \mathcal{A}^{1/2} \boldsymbol{\delta}_h (\mathcal{A}^{-1/2} P_s \alpha(x)) \right|^2 \lesssim \left(\delta^\nabla_{g,h}(x) + \Xi(x, s) \right) \Phi(x, s) \|\alpha\|^2_{\Omega_{L^2}(M)}$$

which proves the claim. $\blacksquare$

4 Main results

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. By $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ we denote the space of all bounded operators $P : \mathcal{H}_1 \to \mathcal{H}_2$. For any $q \in [1, \infty)$, the operator P is a **Schatten operator of class** q if $\operatorname{tr} \left| P^* P \right|^q < \infty$. For $q = 1$ the operator P is trace class and for $q = 2$ the operator P is in the Hilbert-Schmidt class. Given two metric vector bundles $E, \widetilde{E}$ over M and a bounded operator

$$P \in \mathcal{L}(\Gamma_{L^2}(M, E), \Gamma_{L^2}(M, \widetilde{E}))$$

such that there exists a corresponding jointly smooth integral kernel $p(x, y)$ of P

$$M \times M \ni (x, y) \mapsto p_s(x, y) \in \operatorname{Hom}\left(E_y, \widetilde{E}_x \right),$$

the uniquely determined map such that we have

$$P\alpha(x) = \int_M p(x, y)\alpha(y) \operatorname{vol}_\bullet(\mathrm{d}y).$$

Then P is Hilbert-Schmidt, if

$$\iint_{M\times M} |p(x, y)|^2 \operatorname{vol}_\bullet(\mathrm{d}x)\operatorname{vol}_\bullet(\mathrm{d}y) < \infty.$$

In particular, it is well-known [RS79] that Hilbert-Schmidt operators and the product of a bounded operator with some Hilbert-Schmidt operator are compact, and that the product of two Hilbert-Schmidt class operators is trace class.

Since we assume M to be geodesically complete, we can restrict ourselves to smooth compactly supported differential forms. Using the common abuse of notation, the unique self-adjoint realisations of the exterior derivative $\mathbf{d}$, the codifferential $\boldsymbol{\delta}_g$ and Hodge-Laplacian Δ_g will be denoted by the same symbol.

In addition, we define the operators

$$(\hat{P}_s^g)_{s>0} := (\mathbf{d}\, P_s^g)_{s>0} \subset \mathscr{L}(\Omega_{\mathsf{L}^2}(M, g), \Omega_{\mathsf{L}^2}(M, g)),$$
$$(\check{P}_s^g)_{s>0} := (\boldsymbol{\delta}_g P_s^g)_{s>0} \subset \mathscr{L}(\Omega_{\mathsf{L}^2}(M, g), \Omega_{\mathsf{L}^2}(M, g)),$$

$$(\hat{P}_s^{g,h})_{s>0} := (I_{g,h}^{-1}\mathbf{d}\, I_{g,h} P_s^g)_{s>0} \subset \mathscr{L}(\Omega_{\mathsf{L}^2}(M, g), \Omega_{\mathsf{L}^2}(M, g)),$$
$$(\check{P}_s^{g,h})_{s>0} := (I_{g,h}\boldsymbol{\delta}_g I_{g,h}^{-1} P_s^h)_{s>0} \subset \mathscr{L}(\Omega_{\mathsf{L}^2}(M, g), \Omega_{\mathsf{L}^2}(M, g)).$$

Let $\hat{p}_s^g(x, y)$, $\check{p}_s^g(x, y)$, $\hat{p}_s^{g,h}(x, y)$ and $\check{p}_s^{g,h}(x, y)$ be the corresponding jointly smooth integral kernel of $\hat{P}_s^g$, $\check{P}_s^g$, $\hat{P}_s^{g,h}$ and $\check{P}_s^{g,h}$, respectively. For example, recall that this implies [Gün17, Theorem II.1.] that

$$(0, \infty) \times M \times M \ni (s, x, y) \mapsto \hat{p}_s(x, y) \in \operatorname{Hom}\left(\textstyle\bigwedge \mathsf{T}_y^* M, \bigwedge \mathsf{T}_x^* M\right)$$

is the uniquely determined map such that we have

$$\hat{P}_s^g \alpha(x) = \int_M \hat{p}_s^g(x, y)\alpha(y) \operatorname{vol}_g(\mathrm{d}y) \qquad \text{for all } s > 0, \ \alpha \in \Omega_{\mathsf{L}^2}(M, g), \ x \in M.$$

By Riesz' representation theorem, the next result follows from Theorem 3.9 for the exterior derivative and the codifferential.

Theorem 4.1. *For every $g \in \operatorname{Metr} M$, $(s, x) \in (0, \infty) \times M$, we have*

$$\int \left|\hat{p}_s^g(x, y)\right|_g^2 \operatorname{vol}_g(\mathrm{d}y) \leqslant \Psi_g(x, s)\Phi_g(x, s), \tag{4.1}$$

$$\int \left|\check{p}_s^g(x, y)\right|_g^2 \operatorname{vol}_g(\mathrm{d}y) \leqslant \Psi_g(x, s)\Phi_g(x, s). \tag{4.2}$$

By Riesz' representation theorem, the next result follows from Corollaries 3.13, 3.16 and 3.18 for the transformed exterior derivative and for the transformed codifferential with respect to g, and h, respectively.

Theorem 4.2. *For every $g \in \mathrm{Metr}\, M$, $(s,x) \in (0,\infty) \times M$, we have*

$$\int \left| \hat{p}_s^{g,h}(x,y) \right|_g^2 \mathrm{vol}_g(\mathrm{d}y) \lesssim \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s) \right) \Phi_g(x,s), \tag{4.3}$$

$$\int \left| \check{p}_s^{g,h}(x,y) \right|_g^2 \mathrm{vol}_g(\mathrm{d}y) \lesssim \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s) \right) \Phi_g(x,s), \tag{4.4}$$

$$\int \left| \check{p}_s^{h,g}(x,y) \right|_g^2 \mathrm{vol}_g(\mathrm{d}y) \lesssim \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s) \right) \Phi_g(x,s). \tag{4.5}$$

We can now state and prove the main result on the existence and completeness of the wave operators $W_{\pm}(\Delta_h, \Delta_g, I)$ implying the corresponding spectra to coincide.

Theorem 4.3. *Let $g, h \in \mathrm{Metr}\, M$, $g \sim h$, and assume that there exists $C < \infty$ such that $\left| \delta_{g,h}^{\nabla} \right| \leqslant C$ and for some $v \in \{g, h\}$, we have $\mathscr{R}_v \in \mathsf{K}(M)$ and*

$$\int \max \left\{ \delta_{g,h}(x)\Xi_g(x,s), \delta_{g,h}^{\nabla}(x) + \Xi_g(x,s), \Xi_v(x,s) \right\} \Phi_v(x,s)\, \mathrm{vol}_v(\mathrm{d}x) < \infty, \qquad s > 0. \tag{4.6}$$

Then the wave operators

$$W_{\pm}(\Delta_h, \Delta_g, I) = \operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{it\Delta_h} I \mathrm{e}^{-it\Delta_g} \mathsf{P}_{\mathrm{ac}}(\Delta_g)$$

exist and are complete. Moreover, $W_{\pm}(\Delta_h, \Delta_g, I)$ are partial isometries with initial space $\mathrm{ran}\, \mathsf{P}_{\mathrm{ac}}(\Delta_g)$ and final space $\mathrm{ran}\, \mathsf{P}_{\mathrm{ac}}(\Delta_h)$, and we have $\sigma_{ac}(\Delta_g) = \sigma_{ac}(\Delta_g)$.

Corollary 4.4. *Let $g, h \in \mathrm{Metr}\, M$, $g \sim h$, and assume that there exists $C < \infty$ such that $\left| \delta_{g,h}^{\nabla} \right| \leqslant C$ and that for some $v \in \{g, h\}$, we have $\mathscr{R}_v \in \mathsf{K}(M)$ and*

$$\int \max \left\{ \delta_{g,h}(x)\Xi_g^{(p)}(x,s), \delta_{g,h}^{\nabla}(x) + \Xi_g^{(p)}(x,s), \Xi_v^{(p)}(x,s) \right\} \Phi_v(x,s)\, \mathrm{vol}_v(\mathrm{d}x) < \infty, \qquad s > 0,$$

where

$$\Xi_v^{(p)}(x,s) := \Xi_v^{(p),+}(x,s) \wedge \Xi_v^{(p),-}(x,s).$$

For all $0 \leqslant p \leqslant m$, let

$$I^{(p)} := I_{g,h}^{(p)} : \Omega_{L^2}^p(M,g) \to \Omega_{L^2}^p(M,h), \quad \alpha \mapsto \bigwedge^p A^{-1/2}(\alpha)$$

be the bounded identification operator acting on p-forms. Then, for all $0 \leqslant p \leqslant m$, the wave operators

$$W_{\pm}(\Delta_h^{(p)}, \Delta_g^{(p)}, I^{(p)}) = \operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{it\Delta_h^{(p)}} I \mathrm{e}^{-it\Delta_g^{(p)}} \mathsf{P}_{\mathrm{ac}}(\Delta_g^{(p)})$$

exist and are complete. Moreover, $W_{\pm}(\Delta_h^{(p)}, \Delta_g^{(p)}, I^{(p)})$ are partial isometries with initial space $\mathrm{ran}\, \mathsf{P}_{\mathrm{ac}}(\Delta_g^{(p)})$ and final space $\mathrm{ran}\, \mathsf{P}_{\mathrm{ac}}(\Delta_h^{(p)})$, and we have $\sigma_{ac}(\Delta_g^{(p)}) = \sigma_{ac}(\Delta_h^{(p)})$.

Proof. We omit the metric in the notation. Set

$$\overline{K}^{(p)}(x) := \max \left\{ \left\langle \mathscr{R}^{(p)}v, v \right\rangle : v \in \bigwedge^p \mathsf{T}_y M, |v| = 1, y \in \mathsf{B}(x,R) \right\},$$

$$\underline{K}^{(p)}(x) := \min \left\{ \left\langle \mathscr{R}^{(p)}v, v \right\rangle : v \in \bigwedge^p \mathsf{T}_y M, |v| = 1, y \in \mathsf{B}(x,R) \right\},$$

for the corresponding constants defined analogously to (3.5) and (3.6), respectively. By similar calculations as in the proofs of Theorem 3.9 and 3.10, we see that for every $\alpha \in \Omega^p_{\mathsf{L}^2}(M)$,

$$\left|(\mathbf{d}^{(p)} P_s \alpha)_x\right|^2 \leqslant \Xi^{(p),+}(x,s)\Phi(x,s)\|\alpha\|^2_{\Omega^p_{\mathsf{L}^2}(M)},$$

$$\left|(\boldsymbol{\delta}^{(p)} P_s \alpha)_x\right|^2 \leqslant \Xi^{(p),-}(x,s)\Phi(x,s)\|\alpha\|^2_{\Omega^p_{\mathsf{L}^2}(M)},$$

$$\left|\langle \nabla P_s \alpha, \xi\rangle\right|^2 \leqslant \max\left\{\Xi^{(p),+}_v(x,s), \Xi^{(p),-}_v(x,s)\right\}\Phi(x,s)\|\alpha\|^2_{\Omega^p_{\mathsf{L}^2}(M)},$$

where

$$\Xi^{(p),\pm}(x,s) := e^{Cs} + (\overline{K}^{(p)}(x)+\underline{K}^{(p\pm1)}(x))^- s/2\left[e^{C(m,R,\underline{K}^{(0),-}(x))s/2}s^{-1} + \max_{y\in\mathsf{B}(x,R)}|\nabla\mathsf{R}(y)|\,s\right].$$

Noticing that

$$\Delta_v = \bigoplus_{p=0}^m \Delta_v^{(p)} \qquad \text{and} \qquad I = \bigoplus_{p=0}^m I^{(p)}$$

this completes the proof. ∎

In the special case of for 0-forms, i.e. functions, and the Hodge-Laplacian acting on 0-forms is the Laplace-Beltrami operator, we recover the main result of [GT20, Theorem A]. Recall that the Weitzenböck curvature endomorphism $\mathscr{R}^{(p)}$ on 1-forms is given by the Ricci curvature, $\mathscr{R}^{(1)} = \mathrm{Ric}$.

Corollary 4.5. *Let $g, h \in \mathrm{Metr}\,M$, $g \sim h$, and assume that for some $s \in (0,\infty)$ and for some $v \in \{g,h\}$ satisfy (4.6). Let $-\Delta_v \geqslant 0$ be the unique self-adjoint extensions of the Laplace-Beltrami operator for $v \in \{g,h\}$. Then the wave operators $W_\pm(-\Delta_h, -\Delta_g, I)$ exist and are complete. Moreover, $W_\pm(\Delta_h, \Delta_g, I)$ are partial isometries with initial space $\mathrm{ran}\,\mathsf{P}_{\mathrm{ac}}(-\Delta_g)$ and final space $\mathrm{ran}\,\mathsf{P}_{\mathrm{ac}}(-\Delta_h)$, and we have $\sigma_{\mathrm{ac}}(-\Delta_h) = \sigma_{\mathrm{ac}}(-\Delta_g)$.*

5 Proof of the main result

Our strategy is to show the assumptions given by a variant of the Belopol'skii-Birman theorem A.1 which is adapted to our special case of two Hilbert space scattering theory originally to be found in [GT20].

The next Lemma shows assumption (2) Belopol'skii-Birman theorem A.1. As we will see in its proof, it therefore necessary for the potentials $\mathscr{R}_v \in \mathsf{K}(M)$ to be in the Kato class, not only $\underline{\mathscr{R}}_v \in \mathsf{K}(M)$, for $v \in \{g,h\}$.

We denote by $\mathbf{q}_v$ the nonnegative closed sesquilinear form corresponding to Δ_v, i.e. $\mathbf{q}_v(\alpha) = \langle \Delta_v \alpha, \alpha\rangle = \|\mathbf{D}_v\alpha\|^2_v$ with $\mathrm{dom}\,\mathbf{q}_v = \mathrm{dom}\,\sqrt{\Delta_v}$ for any $v \in \{g,h\}$.

Lemma 5.1. *Let $g, h \in \mathrm{Metr}\,M$, $g \sim h$, and assume that there exists $C < \infty$ such that $\left|\delta^\nabla_{g,h}\right| \leqslant C$ and for some $v \in \{g,h\}$, we have $\mathscr{R}_v \in \mathsf{K}(M)$. Then*

$$I_{g,h}\,\mathrm{dom}\,\mathbf{q}_g = \mathrm{dom}\,\mathbf{q}_h.$$

Proof. Note that, for any $v \in \{g, h\}$, dom $\mathbf{q}_v$ is the closure of compactly supported forms $\Omega_{C_c^\infty}(M, v)$ with respect to the Dirac graph norm

$$\alpha \mapsto \left(\|\alpha\|_v^2 + \|\mathbf{D}_v \alpha\|_v^2 \right)^{1/2}.$$

Moreover let $\mathbf{q}^\nabla$ be the nonnegative closed sesquilinear form corresponding to the horizontal Laplacian $\square_v = (\nabla^v)^* \nabla^v$, i.e. $\mathbf{q}_v^\nabla(\alpha) = \langle \square_v \alpha, \alpha \rangle_v = \|\nabla^v \alpha\|_v^2$ with $\mathrm{dom}(\mathbf{q}_v^\nabla)$ its natural domain of definition. Recall that by the Weitzenböck formula, we thus have the relation

$$\Delta_v = \square_v - \mathscr{R}_v.$$

As the Weitzenböck curvature term $\mathscr{R}_v$ is in the Kato class, it is well-known [Gün17, Lemma VII.4.] that the corresponding form domains

$$\mathrm{dom}\, \mathbf{q}_v = \mathrm{dom}\, \mathbf{q}_v^\nabla$$

coincide. To this end, it suffices to show that

$$I \,\mathrm{dom}\, \mathbf{q}_g^\nabla = \mathrm{dom}\, \mathbf{q}_h^\nabla.$$

For all compactly supported $\alpha \in \Omega_{C_c^\infty}(M, g)$, we write

$$\nabla^h(I\alpha) = \nabla^h(\mathscr{A}^{-1/2}\alpha) = \nabla^h \mathscr{A}^{-1/2}(\alpha) + \mathscr{A}^{-1/2} \nabla^h \alpha$$
$$= \left(\nabla^h - \nabla^g \right) \mathscr{A}^{-1/2}(\alpha) + \nabla^g \mathscr{A}^{-1/2}(\alpha) + \mathscr{A}^{-1/2} \left(\nabla^h - \nabla^g \right) \alpha + \mathscr{A}^{-1/2} \nabla^g \alpha.$$

Moreover,

$$\left| \left(\nabla_X^h - \nabla_X^g \right) \mathscr{A}^{-1/2} \right|_h = \left| \mathscr{A}^{1/2} \left(\nabla_X^h - \nabla_X^g \right) \mathscr{A}^{-1/2} \right|_g$$
$$\leqslant \left| \mathscr{A}^{1/2} \right| \left| \nabla_X^h - \nabla_X^g \right|_g \left| \mathscr{A}^{-1/2} \right|$$

and

$$\left| \mathscr{A}^{-1/2} \left(\nabla_X^h - \nabla_X^g \right) \right|_h = \left| \nabla_X^h - \nabla_X^g \right|_g.$$

Thus we can estimate as follows

$$\left\| \nabla^h(I\alpha) \right\|_h^2 = \int_M \left| \nabla^h(\mathscr{A}^{-1/2}\alpha) \right|_h^2 \mathrm{d}\,\mathrm{vol}_h$$
$$= \int_M \left| \left(\nabla^h - \nabla^g \right) \mathscr{A}^{-1/2}(\alpha) + \nabla^g \mathscr{A}^{-1/2}(\alpha) + \mathscr{A}^{-1/2} \left(\nabla^h - \nabla^g \right)(\alpha) + \mathscr{A}^{-1/2} \nabla^g \alpha \right|_h^2 \mathrm{d}\,\mathrm{vol}_h$$
$$\leqslant C \int_M \left(\left| \left(\nabla^h - \nabla^g \right) \mathscr{A}^{-1/2}(\alpha) \right|_h^2 + \left| \nabla^g \mathscr{A}^{-1/2}(\alpha) \right|_h^2 + \right.$$
$$\left. \left| \mathscr{A}^{-1/2} \left(\nabla^h - \nabla^g \right)(\alpha) \right|_h^2 + \left| \mathscr{A}^{-1/2} \nabla^g \alpha \right|_h^2 \right) \rho_{g,h} \,\mathrm{d}\,\mathrm{vol}_g$$
$$\leqslant C \int_M \left(\left| \mathscr{A}^{1/2} \left(\nabla^h - \nabla^g \right) \mathscr{A}^{-1/2}(\alpha) \right|_g^2 + \left| \mathscr{A}^{1/2} \nabla^g \mathscr{A}^{-1/2}(\alpha) \right|_g^2 + \right.$$
$$\left. \left| \left(\nabla^h - \nabla^g \right)(\alpha) \right|_g^2 + \left| \nabla^g \alpha \right|_g^2 \right) \mathrm{d}\,\mathrm{vol}_g$$

$$\leqslant C \int_M \left(\left| \mathscr{A}^{1/2} \right|^2 \left| \nabla^h - \nabla^g \right|_g^2 \left| \mathscr{A}^{-1/2} \right|^2 |\alpha|_g^2 + \left| \mathscr{A}^{1/2} \right|^2 |\mathscr{A}|^2 \left| \mathscr{A}^{-1/2} \right|^6 \left| \nabla_X^h - \nabla_X^g \right|_g^2 |\alpha|_g^2 + \right.$$

$$\left. \left| \nabla^h - \nabla^g \right|_g^2 |\alpha|_g^2 + |\nabla^g \alpha|_g^2 \right) \mathrm{d\,vol}_g$$

$$\leqslant C \int_M \left(\left| \nabla^h - \nabla^g \right|_g^2 |\alpha|_g^2 + |\nabla^g \alpha|_g^2 \right) \mathrm{d\,vol}_g$$

$$\leqslant C \int_M \left(\left\| \delta_{g,h}^\nabla \right\|_\infty |\alpha|_g^2 + |\nabla^g \alpha|_g^2 \right) \mathrm{d\,vol}_g$$

$$\leqslant C \left(\|\alpha\|_g^2 + \|\nabla^g \alpha\|_g^2 \right),$$

using the elementary inequality $(a+b)^c \leqslant 2^{c-1}(a^c + b^c)$ multiple times and that δ^∇ is bounded by assumption.

Hence, we arrive at the estimate

$$\|I\alpha\|_h^2 + \left\| \nabla^h I\alpha \right\|_h^2 \leqslant C \left(\|\alpha\|_g^2 + \|\nabla^g \alpha\|_g^2 \right),$$

proving

$$I \operatorname{dom} \mathbf{q}_g \subset \operatorname{dom} \mathbf{q}_h.$$

Since $I^{-1} = I_{g,h}^{-1} = I_{h,g}$ and the arguments above are symmetric in g and h, this shows the claim. ■

Next, we denote by $|\cdot| : \mathbb{C} \to \mathbb{R}$ the absolute value function and by $\operatorname{sgn} : \mathbb{C} \to \mathbb{C}$ the sign-function with $\operatorname{sgn}(0) = 1$. We note that [Wei80] if P is normal operator (e.g. positive or diagonalisable), we get the (pointwise) polar decomposition $P = |P| (\operatorname{sgn} P)$, where $|P|(x) = |P(x)| \geqslant 0$ and $|\operatorname{sgn} P(x)| = 1$, and where $|P|(x)$ is a non-negative endomorphism and $\operatorname{sgn} P(x)$ is unitary. More precisely, by the spectral theorem choosing $f(\lambda) := |\lambda|$, we have an endomophism

$$f\left(\hat{\mathbf{S}}_{g,h}(x) \right) : \bigwedge \mathsf{T}^* M \to \bigwedge \mathsf{T}^* M$$

giving rise to a decomposition

$$\hat{\mathbf{S}}_{g,h}(x) = \left| \hat{\mathbf{S}}_{g,h}(x) \right| \operatorname{sgn} \hat{\mathbf{S}}_{g,h}(x) : \bigwedge \mathsf{T}^* M \to \bigwedge \mathsf{T}^* M.$$

For the proof of Theorem 4.3, we now introduce sections

$$\mathbf{S}_{g,h} : M \to \mathbb{R}$$

$$\mathbf{S}_{g,h}(x) := \rho_{g,h}(x)^{1/2} - \rho_{g,h}(x)^{-1/2} = 2 \sinh \frac{1}{2} \log(\rho_{g,h}(x)),$$

$$\hat{\mathbf{S}}_{g,h} : M \to \operatorname{End}\left(\bigwedge \mathsf{T}^* M \right)$$

$$\hat{\mathbf{S}}_{g,h}(x) := \left(\rho_{g,h}(x) \mathscr{A}_{g,h}(x) \right)^{1/2} - \left(\rho_{g,h}(x) \mathscr{A}_{g,h}(x) \right)^{-1/2} = 2 \sinh \frac{1}{2} \log(\rho_{g,h}(x) \mathscr{A}_{g,h}(x)),$$

$$\hat{\mathbf{S}}_{g,h;\nu} : \Omega_{\mathsf{L}^2}(M, \nu) \to \Omega_{\mathsf{L}^2}(M, \nu)$$

$$\hat{\mathbf{S}}_{g,h;\nu}(x) := \left| \hat{\mathbf{S}}_{g,h}(x) \right|^{1/2} \alpha(x),$$

$$\mathbf{U}_{g,h} : \Omega_{\mathsf{L}^2}(M, g) \to \Omega_{\mathsf{L}^2}(M, h)$$

$$\mathbf{U}_{g,h}(x) := \mathscr{A}_{g,h}(x)^{-1/2}\alpha(x),$$

$$\hat{\mathbf{U}}_{g,h} : \Omega_{\mathsf{L}^2}(M,g) \to \Omega_{\mathsf{L}^2}(M,h)$$
$$\hat{\mathbf{U}}_{g,h}(x) := (\rho_{g,h}(x)\mathscr{A}_{g,h}(x))^{-1/2}\alpha(x),$$

$$\widetilde{\mathbf{U}}_{g,h} : \Omega_{\mathsf{L}^2}(M,g) \to \Omega_{\mathsf{L}^2}(M,h)$$
$$\widetilde{\mathbf{U}}_{g,h}(x) := (\mathrm{sgn}\,\hat{\mathbf{S}}_{g,h}(x))(\rho_{g,h}(x)\mathscr{A}_{g,h}(x))^{-1/2}\alpha(x).$$

By quasi-isometry, $g \sim h$, the operators $\hat{\mathbf{S}}_{g,h;v}$, $\mathbf{U}_{g,h}$, $\hat{\mathbf{U}}_{g,h}$ and $\widetilde{\mathbf{U}}_{g,h}$ are bounded. Moreover, we get the pointwise estimate [HPW14, Lemma 3.3], [GT20, (4.1)].

Lemma 5.2. *For $v \in \{g, h\}$, we have the pointwise estimate*

$$\max\left\{\left|\mathbf{S}_{g,h}(x)\right|, \sigma_{\max}\left(\left|\hat{\mathbf{S}}_{g,h}(x)\right|\right)\right\} \leqslant \delta_{g,h}(x) \qquad \text{for all } x \in M.$$

Proof. We write $\rho = \rho_{g,h}$ and $\mathscr{A} = \mathscr{A}_{g,h}$ for short. By definition, we have

$$\left|\hat{\mathbf{S}}_{g,h}\right| = \left|(\rho\mathscr{A})^{1/2} - (\rho\mathscr{A})^{-1/2}\right| = 2\sinh\left|\frac{1}{2}\log(\rho\mathscr{A})\right|,$$

and the ith eigenvalue of $\log(\rho\mathscr{A})$ is given by

$$-\sum_{k=1}^{m}\frac{\log\lambda_k}{2} + \log\lambda_i.$$

If we choose k_0 such that $|\log\lambda_{k_0}| = \max_k|\log\lambda_k|$, then

$$\left|-\sum_{k=1}^{m}\frac{\log\lambda_k}{2} + \log\lambda_i\right| \leqslant \frac{m}{2}\left|\log\lambda_{k_0}\right|.$$

Hence,

$$\sigma_{\max}\left(\left|\hat{\mathbf{S}}_{g,h}\right|\right) \leqslant 2\sinh\frac{n}{4}\left|\log\lambda_{k_0}\right| = \delta_{g,h}(x),$$

justifying the definition of $\delta_{g,h}$. A similar calculation shows the assertion for $\mathbf{S}_{g,h}$. ∎

The following lemma provides the trace class operator required in the decomposition formula in assumption (4) of the Belopol'skii-Birman Theorem A.

Lemma 5.3. *Let $g, h \in \mathrm{Metr}\,M$, $g \sim h$. We define the unbounded operator*

$$T_s^{g,h} : \Omega_{\mathsf{L}^2}(M,g) \to \Omega_{\mathsf{L}^2}(M,h)$$
$$T_s^{g,h} := (\hat{P}_s^h)^*\mathbf{U}_{g,h}\hat{P}_s^{g,h} - (\hat{P}_s^{h,g})^*\hat{\mathbf{U}}_{g,h}\hat{P}_s^g + (\check{P}_s^h)^*\mathbf{U}_{g,h}\check{P}_s^{h,g} - (\check{P}_s^{g,h})^*\hat{\mathbf{U}}_{g,h}\check{P}_s^g$$
$$- P_s^h\hat{\mathbf{S}}_{g,h;h}\widetilde{\mathbf{U}}_{g,h}\hat{\mathbf{S}}_{g,h;g}P_{s/2}^g\Delta_g P_{s/2}^g.$$

Then we get, for $\alpha_1 \in \mathrm{dom}\,\Delta_g$, $\alpha_2 \in \mathrm{dom}\,\Delta_h$ and $s > 0$,

$$\left\langle\alpha_2, T_s^{g,h}\alpha_1\right\rangle_h = \left\langle\Delta_h\alpha_2, P_s^h I P_s^g \alpha_1\right\rangle_h - \left\langle\alpha_2, P_s^h I P_s^g \Delta_g\alpha_1\right\rangle_h.$$

Proof. First note that

$$\mathbf{d}^2 = 0 \qquad \text{and} \qquad \boldsymbol{\delta}^2 = 0. \tag{5.1}$$

Since Δ_ν is essentially self-adjoint, for $\nu \in \{g, h\}$, we can assume $\alpha_1 \in \Omega_{C_c^\infty}(M, g)$ and $\alpha_2 \in \Omega_{C_c^\infty}(M, h)$ to be compactly supported. Then

$$\left\langle \Delta_h \alpha_2, P_s^h I P_s^g \alpha_1 \right\rangle_h - \left\langle \alpha_2, P_s^h I P_s^g \Delta_g \alpha_1 \right\rangle_h$$

$$= \left\langle \Delta_h P_s^h \alpha_2, I P_s^g \alpha_1 \right\rangle_h - \left\langle \Delta_g I^{-1} P_s^h \alpha_2, P_s^g \alpha_1 \right\rangle_h - \left\langle P_s^h \alpha_2, \left(I - (I^{-1})^*\right) P_s^g \Delta_g \alpha_1 \right\rangle_h$$

$$= \left\langle (\mathbf{d} + \boldsymbol{\delta}_h) P_s^h \alpha_2, (\mathbf{d} + \boldsymbol{\delta}_h) I P_s^g \alpha_1 \right\rangle_h - \left\langle (\mathbf{d} + \boldsymbol{\delta}_g) I^{-1} P_s^h \alpha_2, (\mathbf{d} + \boldsymbol{\delta}_g) P_s^g \alpha_1 \right\rangle_h$$

$$\qquad - \left\langle P_s^h \alpha_2, \left(I - (I^{-1})^*\right) P_s^g \Delta_g \alpha_1 \right\rangle_h$$

$$\stackrel{(5.1)}{=} \left\langle \mathbf{d} P_s^h \alpha_2, \mathbf{d} I P_s^g \alpha_1 \right\rangle_h + \left\langle \boldsymbol{\delta}_h P_s^h \alpha_2, \boldsymbol{\delta}_h I P_s^g \alpha_1 \right\rangle_h$$

$$\qquad - \left\langle \mathbf{d} I^{-1} P_s^h \alpha_2, \mathbf{d} P_s^g \alpha_1 \right\rangle_h - \left\langle \boldsymbol{\delta}_g I^{-1} P_s^h \alpha_2, \boldsymbol{\delta}_g P_s^g \alpha_1 \right\rangle_h \tag{5.2}$$

$$\qquad - \left\langle P_s^h \alpha_2, \left(I - (I^{-1})^*\right) P_s^g \Delta_g \alpha_1 \right\rangle_h.$$

Let us treat the terms separately. For the last term in (5.2),

$$\left\langle P_s^h \alpha_2, \left(I - (I^{-1})^*\right) P_s^g \Delta_g \alpha_1 \right\rangle_h$$

$$= \int_M \left\langle P_s^h \alpha_2, \left(\mathscr{A}^{-1/2} - (\rho^{-1} \mathscr{A}^{-1/2}\right) P_s^g \Delta_g \alpha_1 \right\rangle \mathrm{d} \operatorname{vol}_h$$

$$= \int_M \left\langle P_s^h \alpha_2, \left(1 - \rho^{-1}\right) \mathscr{A}^{-1/2} P_s^g \Delta_g \alpha_1 \right\rangle \mathrm{d} \operatorname{vol}_h$$

$$= \int_M \left\langle P_s^h \alpha_2, \hat{\mathbf{S}}_{g,h} \rho^{-1/2} \mathscr{A}^{-1/2} P_s^g \Delta_g \alpha_1 \right\rangle \mathrm{d} \operatorname{vol}_h$$

$$= \int_M \left\langle P_s^h \alpha_2, \left|\hat{\mathbf{S}}_{g,h}\right|^{1/2} (\operatorname{sgn} \hat{\mathbf{S}}_{g,h})(\rho \mathscr{A})^{-1/2} \left|\hat{\mathbf{S}}_{g,h}\right|^{1/2} P_s^g \Delta_g \alpha_1 \right\rangle \mathrm{d} \operatorname{vol}_h$$

$$= \left\langle \alpha_2, P_s^h \hat{\mathbf{S}}_{g,h;h} \widetilde{\mathbf{U}}_{g,h} \hat{\mathbf{S}}_{g,h;g} P_{s/2}^g \Delta_g P_{s/2}^g \alpha_1 \right\rangle_h.$$

For the first and third term involving the exterior derivative $\mathbf{d}$, we get

$$\left\langle \mathbf{d} P_s^h \alpha_2, \mathbf{d} I P_s^g \alpha_1 \right\rangle_h - \left\langle \mathbf{d} I^{-1} P_s^h \alpha_2, \mathbf{d} P_s^g \alpha_1 \right\rangle_h$$

$$= \left\langle \mathbf{d} P_s^h \alpha_2, I(I^{-1} \mathbf{d} I) P_s^g \alpha_1 \right\rangle_h - \left\langle (I \mathbf{d} I^{-1}) P_s^h \alpha_2, (I^{-1})^* \mathbf{d} P_s^g \alpha_1 \right\rangle_h$$

$$= \left\langle \mathbf{d} P_s^h \alpha_2, I \hat{P}_s^{g,h} \alpha_1 \right\rangle_h - \left\langle \hat{P}_s^{h,g} \alpha_2, (I^{-1})^* \mathbf{d} P_s^g \alpha_1 \right\rangle_h$$

$$= \left\langle \alpha_2, (\mathbf{d} P_s^h)^* I \hat{P}_s^{g,h} - (\hat{P}_s^{h,g})^* (I^{-1})^* \mathbf{d} P_s^g \alpha_1 \right\rangle_h$$

$$= \int \left(\alpha_2, (\mathbf{d} P_s^h)^* \mathscr{A}^{-1/2} \hat{P}_s^{g,h} - (\hat{P}_s^{h,g})^* \rho^{-1} \mathscr{A}^{-1/2} \mathbf{d} P_s^g \alpha_1 \right) \mathrm{d} \operatorname{vol}_h$$

$$= \left\langle \alpha_2, \left((\mathbf{d} P_s^h)^* \mathbf{U}_{g,h} \hat{P}_s^{g,h} - (\hat{P}_s^{h,g})^* \hat{\mathbf{U}}_{g,h} \mathbf{d} P_s^g \right) \alpha_1 \right\rangle_h$$

$$= \left\langle \alpha_2, \left((\hat{P}_s^h)^* \mathbf{U}_{g,h} \hat{P}_s^{g,h} - (\hat{P}_s^{h,g})^* \hat{\mathbf{U}}_{g,h} \hat{P}_s^g \right) \alpha_1 \right\rangle_h.$$

Similarly, for the codifferential $\boldsymbol{\delta}$,

$$\left\langle \boldsymbol{\delta}_h P_s^h \alpha_2, \boldsymbol{\delta}_h I P_s^g \alpha_1 \right\rangle_h - \left\langle \boldsymbol{\delta}_g I^{-1} P_s^h \alpha_2, \boldsymbol{\delta}_g P_s^g \alpha_1 \right\rangle_h$$

$$= \left\langle \delta_h P_s^h \alpha_2, I(I^{-1}\delta_h I)P_s^g \alpha_1 \right\rangle_h - \left\langle (I\delta_g I^{-1})P_s^h \alpha_2, (I^{-1})^* \delta_g P_s^g \alpha_1 \right\rangle_h$$

$$= \left\langle \delta_h P_s^h \alpha_2, I \check{P}_s^{h,g} \alpha_1 \right\rangle_h - \left\langle \check{P}_s^{g,h} \alpha_2, (I^{-1})^* \delta_h P_s^g \alpha_1 \right\rangle_h$$

$$= \left\langle \alpha_2, \left((\delta_h P_s^h)^* \mathbf{U}_{g,h} \check{P}_s^{h,g} - (\check{P}_s^{g,h})^* \hat{\mathbf{U}}_{g,h} \delta_h P_s^g \right) \alpha_1 \right\rangle_h$$

$$= \left\langle \alpha_2, \left((\check{P}_s^h)^* \mathbf{U}_{g,h} \check{P}_s^{h,g} - (\check{P}_s^{g,h})^* \hat{\mathbf{U}}_{g,h} \check{P}_s^g \right) \alpha_1 \right\rangle_h. \qquad \blacksquare$$

We are finally in the position to proof our main result.

Proof of Theorem 4.3. We check the assumptions of the Belopol'skii-Birman theorem A.1.

Since $g \sim h$, the operator $I \equiv I_{g,h}$ is well-defined and bounded and has a bounded inverse $I^{-1} \equiv I_{h,g}$, so (1) follows. By Lemma 5.1, also assumption (2) is satisfied.

Recalling that by (2.10), $I_{g,h}^* = \rho_{g,h} I_{g,h}^{-1}$, we see that the operator $(I^* I - 1)e^{-s\Delta_g}$ has the integral kernel

$$\left[(I^* I - 1)e^{-s\Delta_g} \right](x,y) = \left(\rho_{g,h} - 1 \right) p_s^g(x,y)$$
$$= \rho_{g,h}^{1/2}(\operatorname{sgn} \mathbf{S}_{g,h}) \left| \mathbf{S}_{g,h} \right|^{1/2} \left| \mathbf{S}_{g,h} \right|^{1/2} p_s^g(x,y).$$

Thus by Lemma 3.5 again, for some $s > 0$,

$$\int \left| \left[(I^* I - 1)e^{-s\Delta_g} \right](x,y) \right|^2 \operatorname{vol}_g(\mathrm{d}y) \leqslant \left\| \rho_{g,h}^{1/2} \mathbf{S}_{g,h} \right\|_\infty \left| \mathbf{S}_{g,h} \right| \int p_s^g(x,y)^2 \operatorname{vol}_g(\mathrm{d}y)$$
$$\leqslant C(\gamma, c_\gamma, s) \left\| \rho_{g,h}^{1/2} \mathbf{S}_{g,h} \right\|_\infty \left| \mathbf{S}_{g,h} \right| \Phi_g(x,s) \int p_s^{g,(0)}(x,y) \operatorname{vol}_g(\mathrm{d}y),$$

and we arrive at the Hilbert-Schmidt estimate

$$\iint \left| \left[(I^* I - 1)e^{-s\Delta_g} \right](x,y) \right|^2 \operatorname{vol}_g(\mathrm{d}y) \operatorname{vol}_g(\mathrm{d}x) \lesssim \int \delta_{g,h}(x) \Xi_g(x,s) \Phi_g(x,s) \operatorname{vol}_g(\mathrm{d}x) < \infty.$$

So, the operator $(I^* I - 1)e^{-s\Delta_g}$ is Hilbert-Schmidt, hence compact, proving assumption (3).

Finally, we prove (4). Using Lemma 5.3 it remains to show that $T_s^{g,h}$ is trace class. Since the product of Hilbert-Schmidt operators is trace class, we prove that the operators $\hat{P}_s^g$, $\check{P}_s^g$, $\hat{P}_s^{g,h}$ and $\check{P}_s^{g,h}$ are Hilbert-Schmidt, where $v \in \{g, h\}$. Recall that $\hat{p}_s^g(x,y)$, $\check{p}_s^g(x,y)$, $\hat{p}_s^{g,h}(x,y)$ and $\check{p}_s^{g,h}(x,y)$ are the corresponding jointly smooth integral kernel of $\hat{P}_s^g$, $\check{P}_s^g$, $\hat{P}_s^{g,h}$ and $\check{P}_s^{g,h}$, respectively.

Then, by (4.1),

$$\iint \left| \hat{p}_s^v(x,y) \right|_v^2 \operatorname{vol}_v(\mathrm{d}y) \operatorname{vol}_v(\mathrm{d}x) \leqslant \int \Psi_v(x,s) \Phi_v(x,s) \operatorname{vol}_v(\mathrm{d}x)$$
$$\lesssim \int \Xi_v(x,s) \Phi_v(x,s) \operatorname{vol}_v(\mathrm{d}x)$$

and, by (4.2),

$$\iint \left| \check{p}_s^v(x,y) \right|_v^2 \operatorname{vol}_v(\mathrm{d}y) \operatorname{vol}_v(\mathrm{d}x) \leqslant \int \Psi_v(x,s) \Phi_v(x,s) \operatorname{vol}_v(\mathrm{d}x)$$
$$\lesssim \int \Xi_v(x,s) \Phi_v(x,s) \operatorname{vol}_v(\mathrm{d}x).$$

We recall that $\delta_{g,h}^{\nabla}$ is bounded by assumption, i.e. $\left|\delta_{g,h}^{\nabla}\right| \leqslant C$ for some $C < \infty$. So similarly, we have by (4.3)

$$\iint \left|\hat{p}_s^{g,h}(x,y)\right|_g^2 \mathrm{vol}_g(\mathrm{d}y)\,\mathrm{vol}_g(\mathrm{d}x) \lesssim \int \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s)\right) \Phi_g(x,s)\,\mathrm{vol}_g(\mathrm{d}x)$$

and by (4.4)

$$\iint \left|\breve{p}_s^{g,h}(x,y)\right|_g^2 \mathrm{vol}_g(\mathrm{d}y)\,\mathrm{vol}_g(\mathrm{d}x) \lesssim \int \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s)\right) \Phi_g(x,s)\,\mathrm{vol}_g(\mathrm{d}x)$$

Finally, by (4.5),

$$\iint \left|\check{p}_s^{h,g}(x,y)\right|_g^2 \mathrm{vol}_g(\mathrm{d}y)\,\mathrm{vol}_g(\mathrm{d}x) \lesssim \int \left(\delta_{g,h}^{\nabla}(x) + \Xi_g(x,s)\right) \Phi_g(x,s)\,\mathrm{vol}_g(\mathrm{d}x)$$

This completes the proof. ∎

6 Applications and examples

6.1 Ricci flow

We first generalise a result, given in [GT20], concerning the stability of the absolutely continuous spectrum of a family of metrics evolving under a Ricci flow. Let therefore R_g be the Riemannian curvature tensor with respect to the metric g.

Corollary 6.1. *Let $S > 0$, $\lambda \in \mathbb{R}$ and assume that*

(a) the family $(g_s)_{0 \leqslant s \leqslant S} \subset \mathrm{Metr}\,M$ evolves under a Ricci-type flow

$$\partial_s g_s = \lambda \,\mathrm{Ric}_{g_s}, \qquad \text{for all } 0 \leqslant s \leqslant S$$

(b) the initial metric g_0 is geodesically complete

(c) there is some $C > 0$ such that $\left|\mathrm{R}_{g_s}\right|_{g_s}, \left|\nabla^{g_s}\mathrm{R}_{g_s}\right|_{g_s} \leqslant C \qquad$ for all $0 \leqslant s \leqslant S$.

We set, for all $x \in M$,

$$M_1(x) := \sup\left\{ \left|\mathrm{Ric}_{g_s}(v,v)\right| : 0 \leqslant s \leqslant S,\ v \in \mathsf{T}_x M,\ |v|_{g_s} \leqslant 1 \right\},$$

$$M_2(x) := \sup\left\{ \left|\nabla_v^{g_s}\mathrm{Ric}_{g_s}(u,w) + \nabla_u^{g_s}\mathrm{Ric}_{g_s}(v,w) + \nabla_w^{g_s}\mathrm{Ric}_{g_s}(u,v)\right| : 0 \leqslant s \leqslant S, \right.$$

$$\left. u,v,w \in \mathsf{T}_x M,\ |u|_{g_s},|v|_{g_s},|w|_{g_s} \leqslant 1 \right\}.$$

Let $\mathsf{B}_g(x,R)$ denote the open geodesic ball. If

$$\int \mathrm{vol}_{g_0}(\mathsf{B}_g(x,1))^{-1} \max\left\{ \sinh\left(\frac{m}{4}S\,|\lambda|\,M_1(x)\right), M_2(x) \right\} \mathrm{vol}_{g_0}(\mathrm{d}x) < \infty,$$

then $\sigma_{\mathrm{ac}}(\Delta_{g_s}) = \sigma_{\mathrm{ac}}(\Delta_{g_0})$ for all $0 \leqslant s \leqslant S$.

Proof. The Ricci flow equation together with (i) implies that $g_s \sim g_0$ for all $0 \leqslant s \leqslant S$ and all g_s are complete. Assumption (c) assures that $\Xi(x,s)$ is bounded. By the same arguments as in [GT20, Corollary B],

$$\delta_{g_s,g_0} \leqslant \sinh\left(\frac{m}{4}S\,|\lambda|\,M_1(x)\right)$$

and as in the proof of [BG20, Theorem 6.1]

$$\delta^{\nabla}_{g_s,g_0}(x) \leqslant C M_2(x)$$

and so the claim follows. ∎

6.2 Conformal perturbations

We study the important case of conformally equivalent metrics: Given a smooth function $\psi : M \to \mathbb{R}$, we define another metric by $g_{\psi} := e^{2\psi} g$. Note that g and g_{ψ} are quasi-isometric, if and only if ψ is bounded (cf. Example 2.7 above).

The bounded identification operator is now given by

$$I := I_{g,h} : \Omega_{L^2}(M, g) \to \Omega_{L^2}(M, g_{\psi})$$
$$I_{g,h}\eta(x) \mapsto \mathscr{A}^{-1/2}(x)\eta(x).$$

Given a smooth function ψ on M, we define

$$\tau := \bigoplus_{p=0}^{m} (m - 2p)\mathbf{1}_{\bigwedge^p T^*M} \in \mathscr{D}^{(0)}(M; \textstyle\bigwedge T^*M),$$

$$e^{\psi\tau} := \bigoplus_{p=0}^{m} e^{(m-2p)\psi}\mathbf{1}_{\bigwedge^p T^*M} \in \mathscr{D}^{(0)}(M; \textstyle\bigwedge T^*M).$$

Next, we collect some useful transformation rules for the conformal metric g_{ψ} in terms of g. A standard reference for various invariants of conformal metric change in part (a) is [Bes87, 1.159 Theorem].

Proposition 6.2. *Let $\psi : M \to \mathbb{R}$ be smooth.*

(a) We have

$$(\cdot,\cdot)^{(p)}_{g_{\psi}} = e^{-2p\psi} (\cdot,\cdot)^{(p)}_g \qquad \text{for all } p \in \{0, \dots, m\} \tag{6.1a}$$

$$d\,\mathrm{vol}_{g_{\psi}} = e^{m\psi} d\,\mathrm{vol}_g \tag{6.1b}$$

$$\bullet \lrcorner_{g_{\psi}} \alpha = e^{-2\psi} \left(\bullet \lrcorner_g \alpha \right) \qquad \text{for all } \alpha \in \Omega^1(M)$$

$$\nabla^{g_{\psi}}_X Y = \nabla^g_X Y + \mathbf{d}\psi(X)Y + \mathbf{d}\psi(Y)X - (X,Y)_g \,\mathrm{grad}_g\,\psi \quad \text{for all } X,Y \in \Gamma_{C^\infty}(TM) \tag{6.1c}$$

$$\boldsymbol{\delta}_{g_{\psi}} \alpha = e^{-2\psi}(\boldsymbol{\delta}_g \alpha - \tau\,\mathbf{d}\psi \lrcorner_g \alpha) \qquad \text{for all } \alpha \in \Omega^p(M) \tag{6.1d}$$

(b) If ψ is bounded, then

$$I^* = e^{m\psi} I^{-1}.$$

Remark 6.3. We note that the canonical musical isomorphisms $\sharp$ and $\flat$ between TM and T^*M do not agree for g and g_{ψ}.

Proof. (a) We show (6.1d). Using (6.1a) and (6.1b), for any $\eta_1 \in \Omega^{p-1}(M)$, $\eta_2 \in \Omega^p(M)$,

$$
\begin{aligned}
\left\langle \eta_1, \boldsymbol{\delta}_{g_\psi} \eta_2 \right\rangle_{g_\psi} &= \left\langle \eta_1, \mathrm{e}^{-2\psi} \left(\boldsymbol{\delta}_g \eta_2 + (m-2p)\mathbf{d}\psi \lrcorner \eta_2 \right) \right\rangle_{g_\psi} \\
&= \left\langle \mathrm{e}^{(m-2(p-1))\psi} \eta_1, \mathrm{e}^{-2\psi} \boldsymbol{\delta}_g \eta_2 \right\rangle_g + \left\langle \mathrm{e}^{(m-2(p-1))\psi} \eta_1, \mathrm{e}^{-2\psi}(m-2p)\mathbf{d}\psi \lrcorner \eta_2 \right\rangle_g \\
&= \left\langle \mathbf{d}\left(\mathrm{e}^{(m-2p)\psi} \eta_1 \right), \eta_2 \right\rangle_g + \left\langle (m-2p)\mathrm{e}^{(m-2p)\psi}\eta_1, \mathbf{d}\psi \lrcorner \eta_2 \right\rangle_g \\
&= \left\langle \mathrm{e}^{(m-2p)\psi}\mathbf{d}\eta_1, \eta_2 \right\rangle_g + \left\langle \mathbf{d}\left(\mathrm{e}^{(m-2p)\psi} \right) \wedge \eta_1, \eta_2 \right\rangle_g - \left\langle (m-2p)\mathrm{e}^{(m-2p)\psi}\mathbf{d}\psi \wedge \eta_1, \eta_2 \right\rangle_g \\
&= \left\langle \mathbf{d}\eta_1, \eta_2 \right\rangle_{g_\psi}.
\end{aligned}
$$

(b) Follows from (2.10). ∎

Theorem 6.4. *Let $\psi : M \to \mathbb{R}$ be smooth with ψ bounded, and assume that $g, g_\psi \in \mathrm{Metr}\, M$ with $g_\psi = \mathrm{e}^{2\psi} g$ such that $\left| \delta^\nabla_{g,g_\psi} \right| \leqslant C$ for some $C < \infty$ and that for some $v \in \{g, h\}$, we have $\mathcal{R}_v \in \mathsf{K}(M)$ and*

$$
\int \max \left\{ \sinh \left| \frac{m}{4} \psi(x) \right| \Xi_g(x,s), \delta^\nabla_{g,h}(x) + \Xi_g(x,s), \Xi_v(x,s) \right\} \Phi_v(x,s) \, \mathrm{vol}_v(\mathrm{d}x) < \infty, \qquad s > 0.
$$

Then the wave operators

$$
W_\pm(\Delta_{g_\psi}, \Delta_g, I) = \operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{it\Delta_{g_\psi}} I \mathrm{e}^{-it\Delta_g} \mathsf{P}_{\mathrm{ac}}(\Delta_g)
$$

exist and are complete. Moreover, $W_\pm(\Delta_{g_\psi}, \Delta_g, I)$ are partial isometries with initial space $\mathrm{ran}\, \mathsf{P}_{\mathrm{ac}}(\Delta_g)$ and final space $\mathrm{ran}\, \mathsf{P}_{\mathrm{ac}}(\Delta_{g_\psi})$, and we have $\sigma_{\mathrm{ac}}(\Delta_g) = \sigma_{\mathrm{ac}}(\Delta_g)$.

Proof. Using Example 2.7, we have $\delta_{g,g_\psi} = 2 \sinh \frac{m}{4} |\psi|$ and

$$
g \sim g_\psi \qquad \Longleftrightarrow \qquad \psi \text{ bounded.}
$$

Hence the claim follows from our main result, Theorem 4.3. ∎

By the same argument as in the poof of Theorem 4.3, we get the following consequence for the wave operators acting on p-forms but with appropriate localised constants respecting the degree of the differential form (cf. Proof of Corollary 4.4 above).

Corollary 6.5. *Let $\psi : M \to \mathbb{R}$ be smooth with ψ bounded, and assume that $g, g_\psi \in \mathrm{Metr}\, M$ with $g_\psi = \mathrm{e}^{2\psi} g$ such that $\left| \delta^\nabla_{g,g_\psi} \right| \leqslant C$ for some $C < \infty$ and that for some $v \in \{g, h\}$, we have $\mathcal{R}_v \in \mathsf{K}(M)$ and*

$$
\int \max \left\{ \sinh \left| \frac{m}{4} \psi(x) \right| \Xi_g(x,s), \delta^\nabla_{g,h}(x) + \Xi_g^{(p)}(x,s), \Xi_v^{(p)}(x,s) \right\} \Phi_v(x,s) \, \mathrm{vol}_v(\mathrm{d}x) < \infty, \qquad s > 0,
$$

where

$$
\Xi_v^{(p)}(x,s) := \Xi_v^{(p),+}(x,s) \wedge \Xi_v^{(p),-}(x,s).
$$

For all $0 \leqslant p \leqslant m$, let

$$
I^{(p)} := I^{(p)}_{g,g_\psi} : \Omega^p_{\mathsf{L}^2}(M,g) \to \Omega^p_{\mathsf{L}^2}(M,g_\psi), \quad \alpha \mapsto \bigwedge^p A^{-1/2}(\alpha)
$$

be the bounded identification operator acting on p-forms. Then, for all $0 \leqslant p \leqslant m$, the wave operators

$$W_{\pm}(\Delta_{g_\psi}^{(p)}, \Delta_g^{(p)}, I^{(p)}) = \underset{t \to \pm\infty}{\text{s-lim}}\, e^{it\Delta_{g_\psi}^{(p)}} I e^{-it\Delta_g^{(p)}} P_{ac}(\Delta_g^{(p)})$$

exist and are complete. Moreover, $W_{\pm}(\Delta_{g_\psi}^{(p)}, \Delta_g^{(p)}, I^{(p)})$ are partial isometries with initial space $\operatorname{ran} P_{ac}(\Delta_g^{(p)})$ *and final space* $\operatorname{ran} P_{ac}(\Delta_{g_\psi}^{(p)})$, *and we have* $\sigma_{ac}(\Delta_g^{(p)}) = \sigma_{ac}(\Delta_{g_\psi}^{(p)})$.

6.3 Global curvature bounds

Let R_ν be the Riemannian curvature tensor with respect to the metric $\nu \in \{g, h\}$. Then the curvature operator

$$Q_\nu \in \mathscr{D}^{(0)}\left(M; \textstyle\bigwedge^2 T^*M\right)$$

is self-adjoint and uniquely determined by the equation

$$\left(Q_\nu(X \wedge Y), U \wedge V\right)_\nu = \left(R_\nu(X, Y)U, V\right)_\nu$$

for all smooth vector fields $X, Y, U, V \in \Gamma_{C^\infty}(TM)$.

By the Gallot–Meyer estimate [GM75], a global bound $Q_\nu \geqslant -K$, for some constant $K > 0$, already implies that curvature endomorphism in the Weitzenböck formula (3.1) is globally bounded by

$$\mathscr{R}_\nu^{(p)} \geqslant -Kp(m - p).$$

Remark 6.6. In particular, if $Q_\nu \geqslant -K$, for some constant $K > 0$, then on 0-forms, i.e. functions,

$$\operatorname{Ric}_\nu \geqslant -K(m - 1).$$

Hence, the following two corollaries include the special case of 0-forms, i.e. functions.

Theorem 6.7. *Let $Q_\nu \geqslant -K$, for some constant $K > 0$ ($\nu \in \{g, h\}$). Let $g, h \in \operatorname{Metr} M$ such that $g \sim h$ and assume that there exists $C < \infty$ such that $\left|\delta_{g,h}^\nabla\right| \leqslant C$ and that for some $\nu \in \{g, h\}$*

$$\int \max\left\{\delta_{g,h}(x)\Xi_g(x, s), \delta_{g,h}^\nabla(x) + \Xi_g(x, s), \Xi_\nu(x, s)\right\} \Phi_\nu(x, s)\, \operatorname{vol}_\nu(dx) < \infty, \qquad s > 0.$$

Then, the wave operators $W_{\pm}(\Delta_h, \Delta_g, I)$ exist and are complete. Moreover, $W_{\pm}(\Delta_h, \Delta_g, I)$ are partial isometries with initial space $\operatorname{ran} P_{ac}(\Delta_g)$ *and final space* $\operatorname{ran} P_{ac}(\Delta_h)$, *and we have* $\sigma_{ac}(\Delta_g) = \sigma_{ac}(\Delta_h)$.

Corollary 6.8. *Let $Q_\nu \geqslant -K$, for some constant $K > 0$ ($\nu \in \{g, h\}$). Let $g, h \in \operatorname{Metr} M$ such that $g \sim h$ and assume that there exists $C < \infty$ such that $\left|\delta_{g,h}^\nabla\right| \leqslant C$ and that for some $\nu \in \{g, h\}$*

$$\int \max\left\{\delta_{g,h}(x)\Xi_g(x, s), \delta_{g,h}^\nabla(x) + \Xi_g^{(p)}(x, s), \Xi_\nu^{(p)}(x, s)\right\} \Phi_\nu(x, s)\, \operatorname{vol}_\nu(dx) < \infty, \qquad s > 0,$$

where

$$\Xi_\nu^{(p)}(x, s) := \Xi_\nu^{(p),+}(x, s) \wedge \Xi_\nu^{(p),-}(x, s).$$

Then, for all $0 \leqslant p \leqslant m$, the wave operators $W_\pm(\Delta_h^{(p)}, \Delta_g^{(p)}, I^{(p)})$ exist and are complete. Moreover, $W_\pm(\Delta_h^{(p)}, \Delta_g^{(p)}, I^{(p)})$ are partial isometries with initial space $\operatorname{ran} \mathsf{P}_{ac}(\Delta_g^{(p)})$ and final space $\operatorname{ran} \mathsf{P}_{ac}(\Delta_h^{(p)})$, and we have $\sigma_{ac}(\Delta_g^{(p)}) = \sigma_{ac}(\Delta_h^{(p)})$.

Remark 6.9. Following the arguments in the previous Subsection 6.2, we may easily elaborate Theorem 6.4 and Corollary 6.5 for the special case of conformal metric change in the same manner.

6.4 ε-close Riemannian metrics

In this section, we denote by κ_g the sectional curvature with respect to a smooth, complete Riemannian metric g.

In [CFG92, Theorem 1.3 & 1.7], Cheeger, Fukaya and Gromov show what is also known as Cheeger-Gromov's thick/thin decomposition:

Theorem 6.10. *For each $m\ (= \dim M)$ there is a constant $C(m)$, such that for any $0 < \varepsilon \leqslant C(m)$ and any complete Riemannian m-manifold (M, g) with $|\kappa_g| \leqslant 1$, there exists a Riemannian metric g_ε on M such that*

(i) the Riemannian metric g_ε is ε-quasi-isometric to g, i.e. $(1/C^\varepsilon)g_\varepsilon \leqslant g \leqslant C^\varepsilon g_\varepsilon$

(ii) it has bounded covariant derivatives $|\nabla^{g_\varepsilon} - \nabla^g| < \varepsilon$

(iii) $|(\nabla^{g_\varepsilon})^k \mathsf{R}_{g_\varepsilon}| < C(m, k, \varepsilon)$, where the constant C depends in addition on the order of derivative k and ε.

Assuming that the sectional curvature κ_g is bounded by 1, implies that the Riemannian curvature tensor R_g is bounded, and hence, the curvature operator Q_g. Following our results in section 6.3, we may get

Theorem 6.11. *Let $|\kappa_g| \leqslant 1$. For each $m\ (= \dim M)$ there is a constant $C(m)$, such that for any $0 < \varepsilon \leqslant C(m)$ and there exists a Riemannian metric g_ε that is ε-quasi-isometric metric to g. If for some $v \in \{g, g_\varepsilon\}$*

$$\int \delta_{g,g_\varepsilon}(x)\Phi_v(x, s)\,\mathrm{vol}_v(\mathrm{d}x) < \infty, \qquad s > 0,$$

then the wave operators $W_\pm(\Delta_{g_\varepsilon}, \Delta_g, I)$ exist and are complete. Moreover, $W_\pm(\Delta_{g_\varepsilon}, \Delta_g, I)$ are partial isometries with initial space $\operatorname{ran} \mathsf{P}_{ac}(\Delta_g)$ and final space $\operatorname{ran} \mathsf{P}_{ac}(\Delta_{g_\varepsilon})$, and we have $\sigma_{ac}(\Delta_g) = \sigma_{ac}(\Delta_{g_\varepsilon})$.

Proof. By the previous Theorem 6.10 (i), the assumption $|\kappa_g| \leqslant 1$ assures that for any $\varepsilon > 0$ there exists a Riemannian metric g_ε that is ε-quasi-isometric metric to g. Hence,

$$\sup \delta_{g,g_\varepsilon}(x) < \infty \quad \Longleftrightarrow \quad g \sim g_\varepsilon.$$

By Theorem 6.10 (ii), the covariant derivatives are bounded so that

$$\delta_{g,g_\varepsilon}^\nabla = |\nabla^{g_\varepsilon} - \nabla^g|_g < \varepsilon. \qquad \blacksquare$$

A Belopol'skii-Birman theorem

We will use a variant of the Belopol'skii-Birman theorem [RS79; Wei80] which is adapted to our special case of two Hilbert space scattering theory originally to be found in [GT20].

Given a self-adjoint operator H in a Hilbert space $\mathscr{H}$ with its operator valued spectral measure E_H, one defines the H-**absolutely continuous subspace** $\mathscr{H}_{ac}(H)$ **of** $\mathscr{H}$ to be the space of all $f \in \mathscr{H}$ such that the Borel measure $\left\| E_H(\cdot)f \right\|^2$ on $\mathbb{R}$ is absolutely continuous with respect to the Lebesgue measure. Then $\mathscr{H}_{ac}(H)$ becomes a closed subspace of $\mathscr{H}$ and the restriction H_{ac} of H to $\mathscr{H}_{ac}(H)$ is a well-defined self-adjoint operator. The **absolutely continuous spectrum** $\sigma_{ac}(H)$ **of** H is defined to be the spectrum of H_{ac}.

Theorem A.1 (Belopol'skii-Birman). *For $k = 1, 2$, let $H_k \geqslant 0$ be self-adjoint operators in a Hilbert space $\mathscr{H}_k$ and $\mathsf{P}_{ac}(H_k)$ the projection onto the absolutely continuous subspace of $\mathscr{H}_k$ corresponding to H_k. Assume that $I \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ is a bounded operator such that the following assumptions hold:*

(1) I has a two-sided bounded inverse

(2) We have either $I \operatorname{dom} \sqrt{H_1} = \operatorname{dom} \sqrt{H_2}$ or $I \operatorname{dom} H_1 = \operatorname{dom} H_2$

(3) The operator $(I^*I - 1)\mathrm{e}^{-sH_1} : \mathscr{H}_1 \to \mathscr{H}_1$ is compact for some $s > 0$

(4) There is a trace class operator $T : \mathscr{H}_1 \to \mathscr{H}_2$ and a number $s > 0$ such that for all $\alpha_1 \in \operatorname{dom} H_1$, $\alpha_2 \in \operatorname{dom} H_2$ we have

$$\langle \alpha_2, T\alpha_1 \rangle_{\mathscr{H}_2} = \left\langle H_2\alpha_2, \mathrm{e}^{-sH_2} I \mathrm{e}^{-sH_1} \alpha_1 \right\rangle_{\mathscr{H}_2} - \left\langle \alpha_2, \mathrm{e}^{-sH_2} I \mathrm{e}^{-sH_1} H_1 \alpha_1 \right\rangle_{\mathscr{H}_2}.$$

Then the wave operators

$$W_{\pm}(H_2, H_1, I) = \operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{itH_2} I \mathrm{e}^{-itH_1} \mathsf{P}_{ac}(H_1)$$

exist and are complete, where completeness means that

$$\left(\ker W_{\pm}(H_2, H_1, I) \right)^{\perp} = \operatorname{ran} \mathsf{P}_{ac}(H_1), \qquad \overline{\operatorname{ran} W_{\pm}(H_2, H_1, I)} = \operatorname{ran} \mathsf{P}_{ac}(H_2).$$

Moreover, $W_{\pm}(H_2, H_1, I)$ are partial isometries with initial space $\operatorname{ran} \mathsf{P}_{ac}(H_1)$ and final space $\operatorname{ran} \mathsf{P}_{ac}(H_2)$, and we have $\sigma_{ac}(H_1) = \sigma_{ac}(H_2)$.

Proof. In view of Theorem XI.13 from [RS79] and its proof, it remains to show that for every bounded interval $\mathbb{I}$ the operator $(I^*I - 1)E_1(\mathbb{I})$ is compact, and that there exists a trace class operator $D \in \mathscr{J}^1(\mathscr{H}_1, \mathscr{H}_2)$ such that for every bounded interval $\mathbb{I}$ and all α_1, α_2 as above we have

$$\langle \varphi, D\alpha_1 \rangle_{\mathscr{H}_2} = \langle H_2\alpha_2, E_2(\mathbb{I}) I E_1(\mathbb{I})\alpha_1 \rangle_{\mathscr{H}_2} - \langle \alpha_2, E_2(\mathbb{I}) I E_1(\mathbb{I}) H_1\alpha_1 \rangle_{\mathscr{H}_2}.$$

However, using that for all self-adjoint operators A and all Borel functions $\varphi, \varphi' : \mathbb{R} \to \mathbb{C}$ we have

$$\varphi(A)\varphi'(A) \subset (\varphi \cdot \varphi')(A), \quad \operatorname{dom}(\varphi(A)\varphi'(A)) = \operatorname{dom}(\varphi(A)\varphi'(A)) \cap \operatorname{dom} \varphi'(A),$$

the required compactness becomes obvious, and furthermore it is easily justified that

$$D := \mathrm{e}^{sH_2} E_2(\mathbb{I}) I \mathrm{e}^{sH_1} E_1(\mathbb{I})$$

has the required trace class property. ∎

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