PhD-FSTM-2020-36
The Faculty of Sciences, Technology and Medicine

DISSERTATION

Defence held on 23/07/2020 in Esch-sur-Alzette

to obtain the degree of

DOCTEUR DE L'UNIVERSITÉ DU LUXEMBOURG

EN *MATÉMATIQUES*

by

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CONSTANT CURVATURE SURFACES AND VOLUMES OF CONVEX CO-COMPACT HYPERBOLIC MANIFOLDS

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Abstract

We investigate the properties of various notions of volume for convex co-compact hyperbolic 3-manifolds, and their relations with the geometry of the Teichmüller space.

We prove a first order variation formula for the dual volume of the convex core, as a function over the space of quasi-isometric deformations of a convex co-compact hyperbolic 3-manifold.

For quasi-Fuchsian manifolds, we show that the dual volume of the convex core is bounded from above by a linear function of the Weil-Petersson distance between the pair of hyperbolic structures on the boundary of the convex core.

We prove that, as we vary the convex co-compact structure on a fixed hyperbolic 3-manifold with incompressible boundary, the infimum of the dual volume of the convex core coincides with the infimum of the Riemannian volume of the convex core.

We study various properties of the foliation by constant Gaussian curvature surfaces (k-surfaces) of convex co-compact hyperbolic 3-manifolds. We present a description of the renormalized volume of a quasi-Fuchsian manifold in terms of its foliation by k-surfaces. We show the existence of a Hamiltonian flow over the cotangent space of Teichmüller space, whose flow lines corresponds to the immersion data of the k-surfaces sitting inside a fixed hyperbolic end, and we determine a generalization of McMullen's Kleinian reciprocity, again by means of the constant Gaussian curvature surfaces foliation.

Acknowledgements

First I would like to thank my supervisor Jean-Marc Schlenker for his precious help and support during my doctoral studies in Luxembourg. Despite his busy schedule, he patiently guided me in the study of hyperbolic geometry and he constantly encouraged me to participate in many conferences abroad and to meet new people in the academic environment.

I am grateful to Francis Bonahon, for allowing me to visit USC in Los Angeles and to learn much more about the quantum topology world. Similarly, I would like to thank Martin Bridgeman, Jeffrey Brock, Kenneth Bromberg, and Tian Yang, who hosted me in a short but intense second trip around the US this winter, supported by the GEAR Network. A special thanks goes also to my future post-doc mentor Sara Maloni, for the great opportunity she gave me to work together in the next years, and for her help in the last weeks to go through the difficulties of visa applications caused by the COVID pandemic.

I thank Francesco Bonsante, Fanny Kassel, and Frédéric Paulin for accepting to report this thesis, and Thierry Barbot and Hugo Parlier for being part of the defense committee.

This journey would not have been the same without the complaints, the frustrations, the dinners and the laughs I shared with my friends and colleagues during these years. You have all my gratitude, for everything you taught me, and for having made this period so pleasant.

Finally, everything I did in these years would have not been possible without the constant and invaluable support of my family, and in particular of my parents Paolo and Silvia, to whom I dedicate this thesis.

Contents

Introduction

We investigate the notions of dual and renormalized volume for convex co-compact hyperbolic manifolds and their relations with the geometry of the Teichmüller space. The Teichmüller space of a topological surface with negative Euler characteristic can be defined as the space of isotopy classes of either conformal structures, or hyperbolic metrics. The interplay between these two interpretations makes the Teichmüller space an extremely fruitful object of study. The initial approach to the subject via complex analytic methods, developed by Teichmüller, Ahlfors, Bers and Weil, among others, has been tremendously expanded by the work of William Thurston in the 1970's, who further investigated its connections with the study of the topology of 3-manifolds and their geometric structures, leading him to his celebrated Geometrization Conjecture (now Theorem, by Perelman). In this work we will focus our attention on the geometry of convex co-compact hyperbolic 3-manifolds: these are complete Riemannian 3-manifolds of constant sectional curvature equal to -1, which possess a non-empty compact convex subset (here we say that a subset C of a Riemannian manifold M is *convex* if, for every pair of points $p, q \in C$ and *for every* geodesic arc γ joining them, γ in fully contained in C). The smallest compact convex subset of a manifold M is called its *convex core* CM, and it encloses all the geometric information about such M. An example of a rich class of convex co-compact hyperbolic manifolds are *quasi-Fuchsian manifolds*, which are homeomorphic to the product a surface times the real line (here surfaces will always be supposed to be closed and with genus at least 2). Non-closed convex co-compact hyperbolic manifolds can be quasi-isometrically deformed, and their deformation spaces are parametrized by the space of conformal structures on their domain of discontinuity (see e. g. [Ber60], [Sul81a]).

Non-closed convex co-compact hyperbolic manifolds always have infinite Riemannian volume. However, interesting notions of volumes can be introduced also in this context, either by looking at their convex core (which is compact), or by defining suitable renormalization procedures over exhaustions of the manifold, as for the renormalized volume. In Chapter 2 we will study the *dual volume of the convex core* V_C^*, as a function over the deformation space of convex co-compact hyperbolic structures of a given topological type. In particular, we determine a first order variation formula for V_C^*, called the *dual Bonahon-Schläfli formula*. The original *Bonahon-Schläfli formula* expresses the variation of the volume of the convex core of a convex co-compact 3-manifold in terms of the variation of the geometry of its boundary. It takes its name from Schläfli [Sch58], who developed a variation formula for the volume of convex polyhedra inside the elliptic space form $\mathbb{S}^3$, and from Bonahon [Bon98a], who generalized this relation to the context of convex co-compact 3-manifolds. Bonahon's result states that the directional derivative of the volume of the convex core V_C along a smooth family of convex co-compact structures $(M_t)_t$ satisfies

$$\left.\frac{\mathrm{d}V_C(M_t)}{\mathrm{d}t}\right|_{t=0^+} = \frac{1}{2}\ell_m(\dot{\mu}),$$

where $\ell_m(\dot{\mu})$ denotes the length of the *derivative of the bending measure* $\dot{\mu} := \frac{\mathrm{d}\mu_t}{\mathrm{d}t}|_{t=0^+}$ with respect to the hyperbolic metric m of the boundary of the convex core. Here the presence of the directional derivative $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0^+}$ is crucial, because the function V_C is not $\mathscr{C}^1$, but only tangentiable. This statement displays an intrinsic complexity in the study of the function V_C, since it involves the variation of the bending measured lamination $\dot{\mu}$. In Bonahon's work, the understanding of this object passes through the notion of transverse Hölder distributions, and its study inherently requires exceptional care.

Surprisingly, these difficulties can be spared in the study of the *dual volume* of the convex core V_C^*. The notion of the dual volume of a convex set naturally arises from the polarity correspondence between hyperbolic and de Sitter geometries. In the case of the convex core, it coincides with $V_C^* = V_C - \frac{1}{2}\ell_m(\mu)$, where $\ell_m(\mu)$ is the hyperbolic length of the bending measured lamination of the boundary of the convex core. As observed by Krasnov and Schlenker [KS09], a simple application of Bonahon's work proves that the variation of the dual volume satisfies:

Theorem A (Dual Bonahon-Schläfli formula)**.**

$$\mathrm{d}V_C^*(\dot{M}) = -\frac{1}{2}\,\mathrm{d}L_\mu\,(\dot{m}),$$

where L_μ is the analytic function on the Teichmüller space of ∂M that associates, with each hyperbolic structure m, the length of the m-geodesic realization of μ, and $\dot{m}$ is the first order variation of the hyperbolic metric m_t on the boundary of the convex core of M_t.

In contrast to what happens to the standard hyperbolic volume, the derivative of the dual volume involves only the variation of the hyperbolic metric on the convex core, which, at least, does not require exceptional work to be defined. A natural question that arises from this statement is whether it is possible to give a proof of the variation formula of the dual volume without involving the study of the variation of the bending measure.

In Chapter 2 we answer affirmatively to this question. The proof that we provide does not require the application of the Bonahon-Schläfli formula and the study of the transverse Hölder distribution associated with the variation of the bending measure. The tools used are quite elementary and the strategy of the proof leans on purely differential geometric methods. A key ingredient of the analysis is the so-called *differential Schläfli formula*, due to Schlenker and Rivin [RS99], which is an analog of the classical Schläfli formula for compact convex sets with smooth boundary inside Einstein Riemannian (or Lorentzian) manifolds.

In Chapter 3, the local understanding of the dual volume function V_C^* given by the dual Bonahon-Schläfli formula allows us to estimate the growth of V_C^* over the space of quasi-Fuchsian manifolds. The asymptotic behaviour of the "standard" Riemannian volume of the convex core of a quasi-Fuchsian manifold has been described by Brock [Bro03]. In this work, the author proved that the volume of the convex core $V_C(M)$ of a quasi-Fuchsian manifold M is coarsely equivalent to the Weil-Petersson distance between the hyperbolic structures $m^+ = m^+(M)$ and $m^- = m^-(M)$ on the two boundary components of the convex core of M. In particular, for every closed surface Σ of genus larger than 1, we can find two constants $K_\Sigma, N_\Sigma > 0$, depending only on the topology of Σ, such that every quasi-Fuchsian manifold M homeomorphic to $\Sigma \times \mathbb{R}$ satisfies:

$$K_\Sigma^{-1}\ d_{WP}(m^+, m^-) - N_\Sigma \leq V_C(M) \leq K_\Sigma\ d_{WP}(m^+, m^-) + N_\Sigma.$$

The original proof of this result guarantees the existence of the constants K_Σ and M_Σ, but it does not furnish numerical estimates. As anticipated above, the dual Bonahon-Schläfli formula turns out to be very well-suited to find explicit constants satisfying the upper bound of Brock's statement. The first reason is that the dual Bonahon-Schläfli formula involves exactly the variation of the hyperbolic metrics on the boundary of the convex core. In addition, the standard volume and dual volume of the convex core differ by the term $\ell_m(\mu)$, which is known to be bounded by a multiple of the Euler characteristic of Σ, by the work of Bridgeman [Bri98] (and further developments, see e. g. [BBB19]). These properties essentially allow us to reduce Brock's upper bound to the study of the Weil-Petersson norm of $\mathrm{d}L_\mu$, the differential of the length of the bending measured lamination. In particular, in Chapter 3 we prove the following result

Theorem B. *There exists an* explicit *universal constant $C > 0$ such that, for every quasi-Fuchsian manifold M homeomorphic to $\Sigma \times \mathbb{R}$, we have*

$$|V_C^*(M)| \leq C|\chi(\Sigma)|^{1/2}\ d_{WP}(m^+(M), m^-(M)).$$

The approach used here is very different from Brock's original one, which was more combinatorial and based on the study of the complex of pants of the surface Σ. In our analysis, the geometric property that plays the main role is the control of the amount of bending that occurs transversely to the bending lamination, a phenomenon of incompressible hyperbolic ends already observed in the work of Epstein and Marden [EM87].

From its definition $V_C^*(M) := V_C(M) - \frac{1}{2}\ell_m(\mu)$, it is not clear a priori whether the dual volume of a convex co-compact 3-manifold is positive or not. In Chapter 4 we study the infimum of the dual volume function over the space of convex co-compact hyperbolic manifolds with incompressible boundary. In particular, we will see:

Theorem C. *For every convex co-compact hyperbolic 3-manifold M with incompressible boundary we have*

$$\inf_{\mathcal{QD}(M)} V_C^* = \inf_{\mathcal{QD}(M)} V_C,$$

where $\mathcal{QD}(M)$ denotes the space of quasi-isometric deformations of M. Moreover, $V_C^*(M) = V_C(M)$ if and only if the boundary of the convex core of M is totally geodesic.

In particular we deduce that the dual volume of the convex core of a quasi-Fuchsian manifold is always non-negative, and it vanishes only on the Fuchsian locus. The proof that we present follows the same strategy of Bridgeman, Brock, and Bromberg [BBB19], where they observed that the same occurs for the renormalized volume function. It is worth to mention that in the same work, the authors proved that the renormalized volume $V_R(M)$ of a convex co-compact hyperbolic 3-manifold is always larger or equal to the dual volume of the convex core $V_C^*(M)$ (see in particular [BBB19, Theorem 3.7]), therefore Theorem C is actually a strengthening of the analogous result for V_R. The request on M to have incompressible boundary is necessary, indeed it has been shown by Pallete [Pal19] that there exist Schottky groups with negative renormalized volume.

As described by Thurston's work in the study of 3-dimensional hyperbolic geometry, convex co-compact hyperbolic 3-manifolds can be characterized either by the geometric data on the boundary of their convex core, or by the structure of their boundary at infinity. These two descriptions are performed using two different approaches: the first is based on the study of hyperbolic structures and measured laminations over surfaces, while the second has a more "complex-analytical" flavour, involving Riemann surface structures and holomorphic quadratic differentials. In Chapter 5 we investigate the relations between these two descriptions through the notion of constant Gaussian curvature surfaces. By a result of Labourie [Lab91], every hyperbolic end E admits a unique foliation $(\Sigma_k)_k$ by convex k-surfaces, i. e. surfaces of constant Gaussian curvature k, with k that varies in $(-1, 0)$. The leaves Σ_k of the foliation converge to the pleated boundary of E as k goes to -1, and they go towards the conformal boundary at infinity of E when k goes to 0. The k-surface foliations have been used by Labourie [Lab92b] to construct two families of parametrizations of the space of hyperbolic ends $(\Phi_k)_k$, $(\Psi_k)_k$. The map Φ_k associates with E the conformal structure of the second fundamental form of Σ_k, together with a holomorphic quadratic differential q_k naturally associated with Σ_k, while Ψ_k maps E into a pair of hyperbolic metrics, coming from the first and third fundamental forms of Σ_k (the third fundamental form can be interpreted as the first fundamental form of the dual surface of Σ_k in the de Sitter space).

The works of Belraouti [Bel17] and Quinn [Qui20] describe the asymptotic properties of these maps. In particular, we can see that, up to normalization, the maps Φ_k converge to the Schwarzian parametrization as k goes to 0, and the maps Ψ_k converge to the Thurston parametrization as k goes to -1. This phenomenon suggests that the k-surfaces can be the correct notion to interpolate between the structure of the convex core and the one of the conformal boundaries at infinity.

Guided by this interpretation, in Chapter 5 we study the notions of dual volume V_k^* and W-volume W_k of the region of a convex co-compact manifold M enclosed by its k-surfaces. After having developed analogues of the Schläfli formulae for these functions, we give a new description of the notion of renormalized volume of a convex co-compact hyperbolic manifold M in terms of its k-surface foliation. More precisely, if M_k denotes the region of M contained between the k-surfaces of the hyperbolic ends of M, and $W(M_k)$ is its W-volume, then:

Theorem D.

$$V_R(M) = \lim_{k \to 0} \left(W(M_k) - \pi |\chi(\partial M)| \operatorname{arctanh} \sqrt{k+1} \right).$$

This characterization of the renormalized volume has the virtue of being described in terms of a very natural geometric foliation of M. In particular, it does not involve the study of the partial equidistant foliations associated with the metrics in the conformal class at infinity, which greatly simplifies the original approach of Krasnov and Schlenker [KS08].

The Schläfli formulae of V_k^* and W_k turn out to be closely related to the symplectic structure of $T^*\mathcal{T}(\partial M)$, the cotangent space of the Teichmüller space. The maps Φ_k and Ψ_k induce two immersions ϕ_k and ψ_k, respectively, of $\mathcal{QD}(M)$, the space of quasi-isometric deformations of a fixed convex co-compact hyperbolic 3-manifold M, into $T^*\mathcal{T}(\partial M)$. Then, the pullbacks of the Liouville form λ of $T^*\mathcal{T}(\partial M)$ by the maps ϕ_k and ψ_k coincide with the differentials of the functions W_k and V_k^*, respectively. This simple observation immediately implies the following:

Theorem E. *The images of the maps $\phi_k, \psi_k : \mathcal{QD}(M) \to T^*\mathcal{T}(\partial M)$ are Lagrangian submanifolds of $(T^*\mathcal{T}(\partial M), \omega_{cot})$ for every $k \in (-1, 0)$.*

This result extends and generalizes McMullen's Kleinian reciprocity theorem [McM98], replacing the role of the Schwarzian parametrization Sch, appearing in McMullen's original result, with Labourie's parametrizations Φ_k and Ψ_k. The proof of Theorem E is extremely simple and it highlights a series of connections between k-surfaces and the structures of the boundary of the convex core and of the conformal boundary at infinity, which are summarized in Table 1.

Studying Einstein equations of 3-dimensional spacetimes in a constant mean curvature (or, briefly, CMC) gauge, Moncrief [Mon89] proved that CMC-foliations determine a (time-dependent) Hamiltonian flow on $(T^*\mathcal{T}, \omega_{\mathrm{cot}})$, the cotangent space to Teichmüller space with its natural cotangent symplectic structure. This result was later used by Andersson, Moncrief, and Tromba [AMT97] to prove that, if a constant curvature MGHC (maximal globally hyperbolic spatially compact) spacetime in dimension 3 cointains a CMC Cauchy surface, then it admits a CMC-foliation (the general existence of CMC-foliations – without assuming the existence of a CMC Cauchy surface – in any dimension has been extensively studied by Andersson et al. [And+12]).

Using the tools developed for the results described above, we can give an analogous description of the flow determined by Labourie's constant Gaussian curvature foliations of hyperbolic ends. More precisely, if $\dot{\Phi}_k \circ \Phi_k^{-1}$ denotes the vector field of $T^*\mathcal{T}$ given by $\frac{\mathrm{d}}{\mathrm{d}k'} \Phi_{k'} \circ \Phi_k^{-1}|_{k'=k}$, then:

Theorem F. *The k-dependent vector field $\dot{\Phi}_k \circ \Phi_k^{-1}$ is Hamiltonian with respect to the cotangent symplectic structure of $T^*\mathcal{T}$.*

Moreover, the role of the area functional as a Hamiltonian function in Moncrief's work here is replaced by the integral of the mean curvature of Σ_k. A very similar statement holds for the parametrizations Ψ_k. The approach used here is different and more geometric in nature than the one used by Moncrief, which passes through the ADM formalism for the study of the Einstein's equations in 3-dimensional vacuum space-time. In our analysis, the result is a direct consequence of the variation formulae of the W_k volumes mentioned above, and their relations with the symplectic structure of $T^*\mathcal{T}$.

Outline of the thesis

The thesis is organized as follows. In Chapter 1 we give an overview of the preliminary notions that will be used in the rest of the exposition. In Chapter

On ∂CM	On ∂M_k	On $\partial_\infty M$
Intrinsic metric m	Conformal str. $c_k = [I\!I_k]$ & First fund. form I_k	Conformal str. c_∞
Bending measure μ	Foliation $\mathcal{F}_k = \mathrm{Hor}(q_k)$ & Third fund. form $I\!I\!I_k$	Foliation $\mathcal{F}_\infty = \mathrm{Hor}(q_\infty)$
Hyperbolic length $L_\mu(m)$	Extr. length $\mathrm{ext}_{\mathcal{F}_k}(c_k)$ & Mean curvature $\int_{\partial M_k} H_k$	Extr. length $\mathrm{ext}_{\mathcal{F}_\infty}(c_\infty)$
Thurston's param. Th	Parametrization Φ_k & Parametrization Ψ_k	Schwarzian param. Sch
Dual volume V_C^*	W-Volume $W(M_k)$ & Dual volume $V^*(M_k)$	Renormalized volume V_R
Thm A $\dot{V}_C^* = -\frac{1}{2}\,\mathrm{d}L_\mu(\dot{m})$	Thm 5.2.4: $\dot{W}_k = -\frac{1}{2}\,\mathrm{d}\,\mathrm{ext}_{\mathcal{F}_k}(\dot{c}_k)$ & Thm 5.2.8 $\dot{V}_k^* = -\frac{1}{2}\,\mathrm{d}L_{I\!I\!I_k}(\dot{I}_k)$	[Sch17, Thm 1.2] $\dot{V}_R = -\frac{1}{2}\,\mathrm{d}\,\mathrm{ext}_{\mathcal{F}_\infty}(\dot{c}_\infty)$
[KS09] $\mathrm{d}L \circ \mathrm{Th}(\mathcal{QD}(M))$ Lagr.	Thm E $\phi_k(\mathcal{QD}(M))$ Lagrangian & Thm E $\psi_k(\mathcal{QD}(M))$ Lagrangian	McMullen's Kleinian reciprocity [McM98]

Table 1: k-surfaces interpolating between ∂CM and $\partial_\infty M$

2 we develop a proof of the dual Bonahon-Schläfli formula (Theorem A) for convex co-compact hyperbolic 3-manifolds. The material of this chapter can be found in:

Filippo Mazzoli. "The dual Bonahon-Schläfli formula". *arXiv e-prints* (Aug. 2018). To appear, *Algebr. Geom. Topol.*, 2020. arXiv: `1808.08936 [math.DG]`.

Chapter 3 is focused on the derivation of the linear upper bound of the dual volume of the convex core of quasi-Fuchsian manifolds in terms of the Weil-Petersson distance between the hyperbolic structures on the boundary of their convex core (Theorem B). The content of this chapter is in:

Filippo Mazzoli. "The dual volume of quasi-Fuchsian manifolds and the Weil-Petersson distance". *arXiv e-prints* (July 2019). arXiv: `1907.04754 [math.DG]`.

Chapter 4 focuses on the study of the infimum of the dual volume function as we vary the convex co-compact hyperbolic structure on a fixed topological 3-manifold with incompressible boundary (Theorem C), and it is unpublished at the time of writing this thesis.

In Chapter 5 we study the properties of constant Gaussian curvature foliations and how they interpolate the geometry of the boundary of the convex core and of the conformal boundary at infinity. The reader can find in this chapter the proofs of Theorems D, E, and F, which have been described in:

Filippo Mazzoli. “Constant Gaussian curvature foliations and Schläfli formulas of hyperbolic 3-manifolds”. *arXiv e-prints* (Oct. 2019). arXiv: 1910.06203 [math.DG].

Chapter 1

Preliminaries

Outline of the chapter

In this chapter we introduce the objects and the notions that will be used in the rest of the thesis.

In Section 1.1 we present the elementary properties (of certain models) of the hyperbolic and de Sitter spaces $\mathbb{H}^n$ and $\mathrm{d}\mathbb{S}^n$, and we introduce the notion of hyperbolic n-manifolds.

Section 1.2 focuses on hyperbolic surfaces (i. e. $n = 2$), we recall the relations between *conformal* and *hyperbolic* structures on surfaces, and the definition of the *Teichmüller space*. In addition, we describe *holomorphic quadratic differentials*, *measured foliations* and their *extremal length*, on the "Riemann surface" side, and *geodesic laminations* and *measured laminations*, on the "hyperbolic surface" side. We also briefly recall the notions of *harmonic* and *minimal Lagrangian maps* between hyperbolic surfaces, which will be necessary for our exposition in Chapters 3 and 5.

In Section 1.3 we introduce our main object of study, namely *convex co-compact hyperbolic* 3*-manifolds*. We describe the structures of the boundary of their *convex core*, and of their *conformal boundary at infinity*.

Section 1.4 investigates the polarity correspondence between hyperbolic and de Sitter spaces. In particular, we describe the duality between convex sets in these geometries, and the relations between the geometric data of their boundaries. This part of the exposition will be useful for the introduction of the notion of dual volume, done in Section 1.7.

Section 1.5 concerns the notion of constant Gaussian curvature surfaces inside hyperbolic 3-manifolds. We describe the properties of their fundamental forms using the tools of the previous section, and how this class of surfaces relate to the notion of minimal Lagrangian maps.

In Section 1.6 we recall the geometric structure of (geometrically finite) *hyperbolic ends*, and two parametrizations of the deformation of these objects, namely the *Thurston* and *Schwarzian parametrizations*. We also recall Labourie's result concerning foliations by constant Gaussian curvature surfaces of hyperbolic ends.

Finally, Section 1.7 focuses on volumes of convex subsets inside hyperbolic 3-manifolds. We first recall the classical *Schläfli formula* for compact hyperbolic polyhedra, and the *differential Schläfli formula* studied by Rivin and Schlenker [RS99]. The chapter ends with a description of the properties of the *dual volume* for convex bodies in the hyperbolic 3-space. Even if not strictly necessary for the rest of the exposition, this part is indented to give an intrinsic description of the dual volume, and to explain how this notion naturally arises from the duality correspondence between hyperbolic and de Sitter geometries.

1.1 Hyperbolic and de Sitter spaces

Let $\mathbb{R}^{n,1}$ denote the $(n+1)$-dimensional Minkowski space, i. e. the vector space $\mathbb{R}^{n+1}$ endowed with the Lorentzian scalar product $\langle \cdot, \cdot \rangle_{n,1}$, defined as

$$\langle x, x \rangle_{n,1} := x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2,$$

for any $x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n,1}$. We denote by H_1 and H_{-1} the subsets of $\mathbb{R}^{n,1}$ given by the vectors x satisfying $\langle x, x \rangle_{n,1} = 1$ and $\langle x, x \rangle_{n,1} = -1$, respectively.

The subset $H_{-1}^+ := H_{-1} \cap \{x_{n+1} > 0\}$ describes a connected n-manifold embedded in $\mathbb{R}^{n,1}$ and diffeomorphic to $\mathbb{R}^n$. The bilinear form $\langle \cdot, \cdot \rangle_{n,1}$ restricts on each tangent space $T_x H_{-1}^+ = \ker \langle x, \cdot \rangle_{n,1}$ to a positive definite scalar product, which determines a Riemannian metric on H_{-1}^+. We will denote by $\mathbb{H}^n$ the resulting Riemannian manifold. This will be our standard model for the *hyperbolic n-space*.

Similarly, H_1 is diffeomorphic to $S^{n-1} \times \mathbb{R}$ and it admits a structure of Lorentzian n-manifold. Indeed, for every point $x^* \in H_1$, the restriction of the scalar product $\langle \cdot, \cdot \rangle_{n,1}$ to the tangent space $T_{x^*} H_1 = \ker \langle x^*, \cdot \rangle_{n,1}$ defines a bilinear form of signature $(n-1, 1)$. The resulting Lorentzian manifold will be denoted by $\mathrm{d}\mathbb{S}^n$, and it will be called the *de Sitter n-space*.

Definition 1.1.1. A non-trivial tangent vector $v \in T_p N$ to a Lorentzian manifold N (e. g. $\mathbb{R}^{n,1}$ or $\mathrm{d}\mathbb{S}^n$) is called *space-like* if it satisfies $\langle v, v \rangle > 0$, *time-like* if $\langle v, v \rangle < 0$ and *light-like* if $\langle v, v \rangle = 0$. A *time-orientation* of N is the datum of a choice of a connected component of $\{v \in T_x N \mid v \text{ time-like}\}$ at each point x of N, depending continuously on x.

The subgroup $O(n,1)^+$ of isometries of $\mathbb{R}^{n,1}$ that keep H_{-1}^+ invariant obviously preserves the Riemannian structure of $\mathbb{H}^n$. Since this action is transitive and faithful on the bundles of orthonormal frames of $\mathbb{H}^n$, $O(n,1)^+$ identifies with the group of isometries $\mathrm{Iso}(\mathbb{H}^n)$ of $\mathbb{H}^n$. The connected component of the identity $O_o(n,1) \subset O(n,1)^+$ consists of the orientation-preserving isometries of $\mathbb{H}^n$, and it will be also denoted by $\mathrm{Iso}^+(\mathbb{H}^n)$.

For what concerns the de Sitter n-space, the entire group $O(n,1)$ acts on $\mathrm{d}\mathbb{S}^n$ and it identifies with the group of isometries $\mathrm{Iso}(\mathrm{d}\mathbb{S}^n)$. The subgroup $O(n,1)^+$ can be interpreted as the subgroup of $\mathrm{Iso}(\mathrm{d}\mathbb{S}^n)$ that preserves a time-orientation of $\mathrm{d}\mathbb{S}^n$. Similarly, $O_o(n,1)$ consists of the orientation-preserving and time-orientation-preserving isometries of $\mathrm{d}\mathbb{S}^n$.

The hyperbolic space possesses totally geodesic subspaces of any codimension $k \in \{0, \ldots, n\}$, and they are all obtained as the intersection of H_{-1}^+ with codimension k vector subspaces of $\mathbb{R}^{n,1}$ containing a time-like direction. Similarly, the codimension k totally geodesic subspaces of $\mathrm{d}\mathbb{S}^n$ are intersections of H_1 with codimension k vector subspaces containing a space-like direction. In both cases, a codimension 1 subspace is also called *hyperplane* (if $n = 3$ we will simply call it plane). While in the hyperbolic space all totally geodesic subspaces with the same dimension are (ambient) isometric, in the de Sitter space two subspaces are (ambient) isometric if and only if their metrics have the same signature. We say that a subspace of $\mathrm{d}\mathbb{S}^n$ is *space-like* if its induced metric is positive definite. A *half-space* in $\mathrm{d}\mathbb{S}^n$ or in $\mathbb{H}^n$ is the closure of one of the two components of the complementary of a hyperplane.

There are several other ways to describe the hyperbolic n-space, which can be equivalently useful depending on the specific situation. We briefly recall the other models that we will need in the rest of the exposition, and their properties.

The *projective model* of $\mathbb{H}^n$ consists of the open affine subset of the projective n-space $\mathbb{RP}^n$ given by

$$\mathbb{P}\{v \in \mathbb{R}^{n,1} \mid \langle x, x\rangle_{n,1} < 0\}.$$

A peculiarity of this presentation is that, in the affine chart $\{x_{n+1} \neq 0\}$, the totally geodesic subspaces of $\mathbb{H}^n$ are described by euclidean subspaces of $\mathbb{R}^n$ intersected with the open ball $\{v \in \mathbb{R}^n \mid \sum_i v_i^2 < 1\}$. However, the Riemannian structure of $\mathbb{H}^n$ is not conformally equivalent to the flat metric on the affine chart $\{x_{n+1} \neq 0\}$. In particular, in the projective model the "hyperbolic angles" do not coincide with the "euclidean angles".

Conformally flat models for $\mathbb{H}^n$ are the *Poincaré n-disk model*

$$\left(\{v \in \mathbb{R}^n \mid \|v\|_0^2 := \sum_i v_i^2 < 1\}, \left(\frac{2}{1 - \|v\|_0^2}\right)^2 \sum_i \mathrm{d}v_i^2\right),$$

and the *half-space model*

$$\left(\{w = (w_1, \cdots, w_n) \in \mathbb{R}^n \mid w_n > 0\}, \frac{1}{w_n^2} \sum_i \mathrm{d}w_i^2\right).$$

In all these presentations, the action of the isometry group of $\mathbb{H}^n$ extends (uniquely and faithfully) to the *boundary at infinity* $\partial_\infty \mathbb{H}^3$, which can be respectively described as:

- $\mathbb{P}\{x \in \mathbb{R}^{n,1} \mid \langle x, x\rangle_{n,1} = 0\}$ in the projective model, with the natural action of $\mathbb{P}O(n, 1)$;
- $S^{n-1} = \{v \in \mathbb{R}^n \mid \|v\|_0 = 1\}$ in the Poincaré n-disk model, with the natural action of $\mathrm{Conf}(S^{n-1})$, the group of conformal diffeomorphisms of the standard $(n-1)$-sphere;
- $\mathbb{R}^{n-1} \cup \{\infty\} = \{w \in \mathbb{R}^n \mid w_n = 0\} \cup \{\infty\}$ in the half-space model, again with the natural action of $\mathrm{Conf}(S^{n-1}) = \mathrm{Conf}(\mathbb{R}^{n-1} \cup \{\infty\})$, the group of conformal diffeomorphisms of the standard $(n-1)$-sphere.

We will denote by $\overline{\mathbb{H}}^n$ the space $\mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$. Since the hyperbolic n-space is a complete Riemannian manifold, its distance does not extend to the boundary at infinity. However, $\overline{\mathbb{H}}^n$ possesses a natural topology, which coincides with the standard Euclidean topology in the Poincaré n-disk model $\overline{\mathbb{H}}^n \cong \{v \in \mathbb{R}^n \mid \|v\|_0 \leq 1\}$.

We will be particularly interested in the cases of dimension 2 and 3, which have the peculiarity of being intimately related with 1-dimensional complex analysis. We will always consider the Poincaré 2-disk and the half-plane models as sitting inside the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. The first is described by $\Delta := \{z \in \mathbb{C} \mid |z|^2 < 1\}$, endowed with the Riemannian metric $4/(1-|z|^2)^2|\mathrm{d}z|^2$, while the second coincides with $H := \{z = x + iy \in \mathbb{C} \mid y > 0\}$, together with the metric $|\mathrm{d}z|^2/y^2$. In both these presentations, the group $\mathrm{Iso}^+(\mathbb{H}^2)$ identifies with the group of biholomorphisms of Δ and H, which consist of those Möbius transformations of $\mathbb{CP}^1$ that keep them invariant. In particular, in the case of the half-plane model H, this defines a natural isomorphism $\mathrm{Iso}^+(\mathbb{H}^2) \cong \mathbb{P}\,\mathrm{SL}_2(\mathbb{R}) \subset \mathbb{P}\,\mathrm{SL}_2(\mathbb{C})$.

Finally, when $n = 3$, every element of $\mathrm{Iso}^+(\mathbb{H}^3)$ is uniquely determined by its conformal action of the sphere at infinity $\partial_\infty \mathbb{H}^3$, which can be again identified with $\mathbb{CP}^1$. Therefore, $\mathrm{Iso}^+(\mathbb{H}^3)$ coincides with the entire group of Möbius transformations $\mathbb{P}\,\mathrm{SL}_2(\mathbb{C})$, and its subgroup $\mathbb{P}\,\mathrm{SL}_2(\mathbb{R})$ can be interpreted as the set of those orientation-preserving isometries of $\mathbb{H}^3$ that keep invariant some fixed totally geodesic plane of $\mathbb{H}^3$ (and preserve a fixed choice of a normal vector field on it).

Definition 1.1.2. A *complete hyperbolic n-manifold* is a smooth manifold M of dimension n endowed with a complete Riemannian metric of constant sectional curvature -1. If M is connected, then it can be equivalently described as the quotient of the hyperbolic n-space $\mathbb{H}$ by the action of a discrete and torsion-free subgroup Γ of $\mathrm{Iso}(\mathbb{H}^n)$.

The hyperbolic manifold M is orientable if and only if the group Γ is contained in $\mathrm{Iso}^+(\mathbb{H}^n)$, the group of orientation-preserving isometries of $\mathbb{H}^n$. In our exposition we will always consider orientable manifolds.

Definition 1.1.3. A *Fuchsian group* is a discrete and torsion-free subgroup of $\mathrm{Iso}^+(\mathbb{H}^2) \cong \mathbb{P}\,\mathrm{SL}_2(\mathbb{R})$. A *Kleinian group* is a discrete and torsion-free subgroup of $\mathrm{Iso}^+(\mathbb{H}^3) \cong \mathbb{P}\,\mathrm{SL}_2(\mathbb{C})$.

1.2 Hyperbolic surfaces

In our exposition Σ will always be an oriented connected compact smooth surface with empty boundary and with genus $g \geq 2$, unless otherwise stated.

Definition 1.2.1. Let Σ be a surface. Two Riemannian metrics g, g' on Σ are *conformally equivalent* if there exists a smooth function $u \in \mathscr{C}^\infty(\Sigma)$ such that $g' = e^{2u} g$.

A *conformal structure* c on Σ is an equivalence class of Riemannian metrics with respect to the relation above. We will use also the notation $c = [g]$, where g is a representative of c. A surface endowed with a conformal structure is also called a *Riemann surface*.

A *hyperbolic metric* h on Σ is a Riemannian metric with Gaussian curvature constantly equal to -1.

Theorem 1.2.2 (Gauss). *Let (Σ, c) be a Riemann surface. Then for every point $p \in \Sigma$, there exists a local chart $z = x + iy \colon U \to z(U) \subseteq \mathbb{C}$ around p satisfying:*

- *every Riemannian metric g in the conformal class c can be locally expressed as $g = e^{2u}|\mathrm{d}z|^2 = e^{2u}(\mathrm{d}x^2 + \mathrm{d}y^2)$, for some smooth function $u \in \mathscr{C}^\infty(\Sigma)$;*
- *the chart z sends the orientation of $U \subseteq \Sigma$ into the standard orientation of $\mathbb{C}$.*

Local coordinates satisfying the properties above are called conformal coordinates.

Remark 1.2.3. Let (Σ, c) be a Riemann surface. An atlas of conformal coordinates $z_i \colon U_i \to z_i(U_i) \subseteq \mathbb{C}$ satisfies the following property: for every i and j such that $U_i \cap U_j \neq \emptyset$, the change of coordinates

$$z_j \circ z_i^{-1} : z_i(U_i \cap U_j) \longrightarrow z_j(U_i \cap U_j)$$

are biholomorphisms between open sets of $\mathbb{C}$. This explains in particular the equivalence between our definition of conformal structure with the usual one, which endows Σ with a maximal atlas of charts with biholomorphic change of coordinates.

Definition 1.2.4. Let Σ be a surface. The *Teichmüller space* of Σ, denoted by $\mathcal{T}(\Sigma)$, is the space of isotopy classes of conformal structures over Σ. More precisely, we say that two conformal structures c and c' are *Teichmüller-equivalent* if there exists a diffeomorphism f of Σ isotopic to the identity such that $f^* c' = c$. Here $f^* c'$ is the conformal structure of $f^* g'$, where g' is a representative of the conformal class c'.

In light of the Uniformization Theorem, the universal cover of a Riemann surface (Σ, c) with genus $g \geq 2$ is biholomorphic to the unit disk $\Delta := \{z \in \mathbb{C} \mid |z| < 1\}$. As briefly recalled at the beginning of Section 1.1, the group of biholomorphic automorphisms of Δ coincides with the group of orientation-preserving isometries of the metric $g_\Delta = 4/(1 - |z|^2)^2 |\mathrm{d}z|^2$. In particular, every conformal structure on Σ uniquely determines a complete hyperbolic metric $h \in c$. This phenomenon tells us that the Teichmüller space can be considered equivalently as the space of isotopy classes of hyperbolic metrics on Σ. We will write $\mathcal{T}^{\mathfrak{c}}(\Sigma)$ ($\mathfrak{c}$ for *conformal*) when we want to emphasize the first interpretation via conformal structures, and $\mathcal{T}^{\mathfrak{h}}(\Sigma)$ ($\mathfrak{h}$ for *hyperbolic*) in latter case.

1.2.1 Holomorphic quadratic diffentials and measured foliations

Definition 1.2.5. Let $X = (\Sigma, c)$ be a Riemann surface. A *holomorphic quadratic differential* q on X is a holomorphic section of the bundle $T^*X \otimes T^*X$, where T^*X denotes the holomorphic cotangent bundle of X. Equivalently, q can be represented in conformal coordinates (U, z) as $q = f(z)\,\mathrm{d}z^2$, where f is a holomorphic function on U.

Given a conformal structure c on Σ, we denote by $Q(\Sigma, c)$ the space of holomorphic quadratic differentials of (Σ, c). By the Riemann-Roch theorem, $Q(\Sigma, c)$ is a vector space of complex dimension $3g - 3$.

Remark 1.2.6. As shown in [Tro92, p. 45-46], a symmetric bilinear tensor $\sigma(\cdot, \cdot)$ on a Riemannian surface (Σ, g) is equal to the real part of a holomorphic quadratic differential on $(\Sigma, [g])$ if and only if

- it is g-traceless, i. e. $\sigma(e_1, e_1)+\sigma(e_2, e_2) = 0$ for every local g-orthonormal frame e_1, e_2;
- σ is g-divergence-free, i. e. $(\nabla_{e_1}\sigma)(e_1, \cdot)+(\nabla_{e_2}\sigma)(e_2, \cdot) = 0$ for every local g-orthonormal frame e_1, e_2.

Every holomorphic quadratic differential $q = f(z)\,\mathrm{d}z^2$ on a Riemann surface X determines a measured foliation $\mathrm{Hor}(q)$ outside $Z(q)$, the set of zeros of q, which we will call the *horizontal measured foliation* of q. The leaves of the foliation are the (unoriented) maximal curves of Σ tangent to the vectors $v \in T_p\Sigma$ satisfying $q(v, v) \in \mathbb{R}_{>0}$, for every $p \in \Sigma\setminus Z(q)$. Every point $p \notin Z(q)$ admits a local chart (U, w), with $w = u + iv \in w(U) \subseteq \mathbb{C}$, such that the holomorphic quadratic differential q is expressed as $q = \mathrm{d}w^2$ on U. In these coordinates, the foliation is represented by the lines $\{y = \mathrm{const}\} \subset w(U)$. Every arc γ which is properly embedded in U and transverse to the foliation carries a measure $\gamma^*|\mathrm{d}v|$, which records the modulus of the variation of the v-component along γ. Since local charts (U, w) around p satisfying $q = \mathrm{d}w^2$ are essentially unique up to post-composition by translations and π-rotations of $\mathbb{C}$, the procedure just described allow us to define a measure on each arc in $\Sigma \setminus Z(q)$ transverse to the foliation, simply by combining the locally defined measures $\gamma^*|\mathrm{d}y|$. Observe that, if γ_t is a 1-parameter family of curves in $\Sigma \setminus Z(q)$ transverse to $\mathcal{F}$ for every t and such that the endpoints of γ_t lie on the same leaf of $\mathcal{F}$, then the total mass of the measures $\gamma_t^*|\mathrm{d}v|$ is independent of t.

Definition 1.2.7. The *horizontal measured foliation* of q, denoted by $\mathrm{Hor}(q)$, is the datum of the foliation and the transverse measure described above.

A general measured foliation on Σ is a foliation defined outside a finite set of points $\{p_1, \dots, p_k\}$ endowed with a transverse measure $\mathrm{d}m$ that is preserved by deformations of transverse arcs as described above. We also require that, around the singular points p_i, the foliation is topologically equivalent to the horizontal foliation of $z^n\,\mathrm{d}z^2$ around $0 \in \mathbb{C}$, for some $n = n(p_i) \geq 1$. Given γ a closed curve of Σ, and $\mathcal{F}$ a measured foliation with transverse measure $\mathrm{d}m$, we define the geometric intersection $i(\mathcal{F}, \gamma)$ to be $\inf_{\gamma'} \int_{\gamma'} \mathrm{d}m$, as γ' varies among the closed curves homotopic to γ in Σ.

Two measured foliations $\mathcal{F}$ and $\mathcal{F}'$ on Σ, with transverse measures $\mathrm{d}m$ and $\mathrm{d}m'$, respectively, are said to be *equivalent* if for every simple closed curve γ of Σ we have $i(\mathcal{F}, \gamma) = i(\mathcal{F}', \gamma)$. By the work of Thurston, we know that the space of equivalence classes of measured foliations of Σ is homeomorphic to $\mathbb{R}^{6g-6}$, where g is the genus of Σ.

Extremal length of a measured foliation

Let c be a conformal structure on Σ. A *measurable conformal metric* of (Σ, c) is a tensor on Σ of the form $\rho = \rho(z)^2|\mathrm{d}z|^2$, for some locally defined Borel-measurable function $\rho(z) \geq 0$. We define the ρ-area of Σ to be

$$A_\rho := \int_\Sigma \rho(z)^2\,\mathrm{d}x\,\mathrm{d}y\,.$$

Moreover, given γ a simple closed curve on Σ, we set the ρ-length of γ as follows:

$$\ell_\rho(\gamma) := \inf_{\gamma'} \int_{\gamma'} \rho(z)|\mathrm{d}z|,$$

where γ' varies among the closed curves of Σ freely homotopic to γ. Then, the *extremal length* of γ with respect to the conformal structure c is defined as

$$\mathrm{ext}_\gamma(c) := \sup_\rho \frac{\ell_\rho(\gamma)^2}{A_\rho},$$

where the supremum is taken over all measurable conformal metrics ρ of c with non-zero finite ρ-area.

Kerckhoff [Ker80, Proposition 3] extended the notion of extremal length to general measured foliations, using the density of weighted simple closed curves inside the space of measured foliations. As a final remark, we mention the following relation to express the extremal length with respect to c of the horizontal foliation of a holomorphic quadratic differential on (Σ, c):

Theorem 1.2.8 ([Ker80])**.** *Let c be a conformal structure of Σ, and let $q = f(z)\,\mathrm{d}z^2 \in Q(\Sigma, c)$. Then the extremal length of the measured foliation $\mathcal{F} = \mathrm{Hor}(q)$ with respect to the conformal structure c satisfies:*

$$\mathrm{ext}_\mathcal{F}(c) = \int_\Sigma |q| = \int_\Sigma |f|\,\mathrm{d}x\,\mathrm{d}y\,.$$

1.2.2 Geodesic and measured laminations

Definition 1.2.9. Let (M, g) be a Riemannian manifold. A *g-geodesic* is a parametrized curve $\gamma\colon I \to M$, defined on a open interval I of $\mathbb{R}$, satisfying $\nabla_{\dot\gamma}\dot\gamma \equiv 0$, where ∇ denotes the Levi-Civita connection of (M, g). If there is no ambiguity on the Riemannian metric we are considering on M, we will simply call γ a *geodesic* of M.

A geodesic γ of M is *complete* if it is defined on the entire real line. A complete geodesic γ is *simple* if either it is globally injective, or if it is periodic of period $T > 0$ and injective over $[0, T)$.

Definition 1.2.10. Let (Σ, h) be a closed surface endowed with a complete hyperbolic metric. A *geodesic lamination* λ of (Σ, h) is the datum of a closed subset of Σ, together with a foliation by simple geodesics, called the *leaves* of the lamination.

A measured lamination μ of (Σ, h) is the datum of a geodesic lamination λ and of a Borel measure on each arc k transverse to λ, so that every homotopy of arcs $(k_t)_{t\in[0,1]}$, for which k_t is transverse to λ for every t, sends the measure of k_0 to the measure of k_1.

Observe that the invariance of the measures under transverse deformations implies in particular that the support of the measure associated to an arc k is contained in the subset $k \cap \lambda$.

Remark 1.2.11. The definition of geodesic and measured laminations that we gave here have the inconvenience of being dependent on the choice of a hyperbolic structure h on Σ. In fact, it is possible to describe the datum of a geodesic lamination on a closed compact surface in a purely topological way, as briefly summarized in Section 2.3. We will denote by $\mathcal{GL}(\Sigma)$ and $\mathcal{ML}(\Sigma)$ the spaces of geodesic laminations and of measured laminations of Σ, respectively.

1.2.3 Harmonic maps

Definition 1.2.12. Let g, g' be two Riemannian metrics on Σ. The *energy* of a map $f\colon (\Sigma, g) \to (\Sigma, g')$ is defined as

$$E(f) := \int_\Sigma \|\mathrm{d}f_p\|^2_{g,g'} \,\mathrm{d}a_g\,(p),$$

where $\|\mathrm{d}f_p\|_{g,g'}$ is the operator norm of $\mathrm{d}f_p : (T_p\Sigma, g_p) \to (T_p\Sigma, g'_p)$, and $\mathrm{d}a_g\,(p)$ is the area form of g. A simple computation shows that the quantity $E(f)$ is invariant under conformal change of g, in particular it depends only on the conformal class $c = [g]$ and the metric g'. A function f is *harmonic* if it is a critical point of the energy functional, i. e.

$$\frac{\mathrm{d}}{\mathrm{d}t} E(f_t\colon (\Sigma, c) \to (\Sigma, g'))\Big|_{t=0},$$

for every smooth variation f_t of $f_0 = f$.

It turns out that a local diffeomorphism $f\colon (\Sigma, c) \to (\Sigma, g')$ is harmonic if and only if the $(2, 0)$-part of $f^* g'$ with respect to the conformal structure c is a holomorphic quadratic differential (see e. g. [Sam78]). In such case, we call $\mathrm{Hopf}(f) := (f^* g')^{(2,0)}$ the *Hopf differential* of f.

Remark 1.2.13. Given a map $f\colon (\Sigma, c) \to (\Sigma, g')$, the g-traceless part of $f^* g'$ coincides with $2\,\mathrm{Re}(f^* g')^{(2,0)}$, where $(f^* g')^{(2,0)}$ is the $(2, 0)$-part of $f^* g'$ with respect to the conformal class $c = [g]$. Therefore, in light of Remark 1.2.6, a way to verify that a map $f\colon (\Sigma, c) \to (\Sigma, g')$ is harmonic is to show that the g-traceless part of $f^* g'$ has trivial g-divergence, where g is a representative of c. If this is the case, then the g-traceless part of $f^* g'$ is equal to $2\,\mathrm{Re}\,\mathrm{Hopf}(f)$.

Theorem 1.2.14 (See e. g. [Sam78])**.** *Let c be a conformal structure on Σ. Then, for any hyperbolic metric h on Σ, there exists a unique holomorphic quadratic differential $q \in Q(\Sigma, c)$, and a unique diffeomorphism $f\colon (\Sigma, c) \to (\Sigma, h)$ isotopic to the identity, such that f is harmonic with Hopf differential equal to q.*

Theorem 1.2.15 ([Wol89, Theorem 3.1])**.** *Let c be a fixed conformal structure over a surface Σ. For every hyperbolic metric h of Σ, we denote by $q(c, h)$ the Hopf differential of the unique harmonic diffeomorphism from (Σ, c) to (Σ, h) isotopic to the identity. Then the function*

$$\begin{array}{rccc} \varphi_c : & \mathcal{T}^{\mathfrak{h}}(\Sigma) & \longrightarrow & Q(\Sigma, c) \\ & [h] & \longmapsto & q(c, h), \end{array}$$

is well defined and it describes a global diffeomorphism between the Teichmüller space $\mathcal{T}^{\mathfrak{h}}(\Sigma)$ and the space of holomorphic quadratic differentials $Q(\Sigma, c)$ on (Σ, c).

1.2.4 Minimal Lagrangian maps

Definition 1.2.16 (See [BMS13, Proposition 1.3])**.** Let h and h' be two hyperbolic metrics on Σ. A diffeomorphism $f\colon (\Sigma, h) \to (\Sigma, h')$ is *minimal Lagrangian* if it is area-preserving, and its graph is a minimal surface inside $(\Sigma^2, h \oplus h')$.

Equivalently, $f\colon (\Sigma, h) \to (\Sigma, h')$ is minimal Lagrangian if there exists a conformal structure c on Σ such that $f = u \circ v^{-1}$, where u and v are the unique harmonic diffeomorphisms isotopic to the identity from (Σ, c) to (Σ, h'), and from (Σ, c) to (Σ, h), respectively, and if they satisfy $\operatorname{Hopf}(u) = -\operatorname{Hopf}(v) \in Q(\Sigma, c)$ (with the notation introduced in Definition 1.2.12).

Remark 1.2.17. Using the first description of minimal Lagrangian maps, the conformal structure c, appearing in the second definition, can be recovered as the conformal class of the induced metric on the graph of f from the metric $h \oplus h'$ (by identifying the graph of f with Σ using one of the projections onto Σ). Moreover, the projections of the graph of f onto (Σ, h) and (Σ, h') are harmonic maps with respect to c.

As we will explain right after the statement, the following theorem can be interpreted as a result of existence and uniqueness of minimal Lagrangian maps isotopic to the identity between pairs of hyperbolic surfaces:

Theorem 1.2.18 ([Lab92b], [Sch93])**.** *For every hyperbolic metric h and for every isotopy class $m' \in \mathcal{T}^\natural(\Sigma)$, there exists a unique hyperbolic metric $h' \in m'$ and a unique operator $b\colon T\Sigma \to T\Sigma$ such that:*

i) $h' = h(b\cdot, b\cdot)$*;*

ii) b is h-self-adjoint and positive definite,

iii) $\det b = 1$*;*

iv) b is Codazzi with respect to the Levi-Civita connection ∇ of h, i. e. $(\nabla_X b)Y = (\nabla_Y b)X$ for every X and Y.

A pair of hyperbolic metrics h, h' for which we can find such an operator b is called a normalized pair*, and b is called their* Labourie operator*.*

Consider h and h' a normalized pair of hyperbolic metrics with Labourie operator b, and let c be the conformal class of the metric $g := h(b\cdot, \cdot)$. The Levi-Civita connection of g can be expressed as follows:

$$\nabla^g_X Y = \nabla_X Y + \frac{1}{2} b^{-1}(\nabla_X b)Y,$$

where ∇ is the Levi-Civita connection of h. This relation can be proved by checking that the connection of the right-hand side is compatible with g and torsion-free. The first property follows from the fact that b is h-self-adjoint, while the second comes from the fact that b is a Codazzi tensor. Moreover, using the relation $b^2 - \operatorname{tr}(b)\, b + \det b\, \mathbb{1} = 0$ and the fact that b is h-self-adjoint, we can express the g-traceless parts of h and h' as follows:

$$h - \frac{\operatorname{tr}_g h}{2} g = h\left(\left(\mathbb{1} - \frac{\operatorname{tr}(b)}{2} b\right) \cdot, \cdot\right), \quad h' - \frac{\operatorname{tr}_g h'}{2} g = h\left(\left(\frac{\operatorname{tr}(b)}{2} b - \mathbb{1}\right) \cdot, \cdot\right),$$

which are opposite to each other. Finally, using the expression we described above for the Levi-Civita connection of g, we can express the g-divergence of $h - \frac{\operatorname{tr}_g h}{2} g$ as

$$\operatorname{div}_g\left(h - \frac{\operatorname{tr}_g h}{2} g\right) = -\frac{1}{2} \operatorname{d}(\ln \det b)\,,$$

which vanishes, since $\det b = 1$. In other words, in light of Remark 1.2.13, the maps

$$id: (\Sigma, [h(b\cdot, \cdot)]) \longrightarrow (\Sigma, h), \quad id: (\Sigma, [h(b\cdot, \cdot)]) \longrightarrow (\Sigma, h')$$

are harmonic with opposite Hopf differentials, i. e. $id\colon (\Sigma, h) \to (\Sigma, h')$ is minimal Lagrangian. This finally explains the relation between Theorem 1.2.18 and the notion of minimal Lagrangian maps. In light of Theorem 1.2.15, another way to formulate Theorem 1.2.18 that will be useful later on is the following:

Theorem 1.2.19 ([Lab92b], [Sch93]). *The function*

$$\begin{array}{rccc} \mathcal{H}: & T^*\mathcal{T}^{\mathfrak{c}}(\Sigma) & \longrightarrow & \mathcal{T}^{\mathfrak{h}}(\Sigma) \times \mathcal{T}^{\mathfrak{h}}(\Sigma) \\ & (c, q) & \longmapsto & (\varphi_c^{-1}(q), \varphi_c^{-1}(-q)), \end{array}$$

is a diffeomorphism (here φ_c denotes the harmonic parametrization of Theorem 1.2.15).

1.3 Convex co-compact hyperbolic 3-manifolds

Definition 1.3.1. Let M be a complete hyperbolic n-manifold. A non-empty subset C of M is *convex* if, for every pair of points $p, q \in M$ (possibly equal) and *for every* geodesic segment γ of M from p to q, γ is fully contained in C. A hyperbolic n-manifold M is *convex co-compact* if it possesses a non-empty compact convex subset.

Remark 1.3.2. If M is simply connected (i. e. $M \cong \mathbb{H}^n$), then the condition above translates into the usual notion of convexity.

Definition 1.3.3. Given M, M' hyperbolic n-manifolds, a diffeomorphism $M \to M'$ is a *quasi-isometric deformation* of M if it globally bi-Lipschitz. We denote by $\mathcal{QD}(M)$ the space of quasi-isometric deformations of M, where we identify two deformations $M \to M'$ and $M \to M''$ if their pullback metrics are isotopic to each other.

Remark 1.3.4. By a Theorem of Thurston [Thu79, Proposition 8.3.4], two hyperbolic n-manifolds M and M' are quasi-isometric if and only if their fundamental groups Γ, Γ' (as subgroups of $\mathrm{Iso}(\mathbb{H}^n)$) are quasi-conformally conjugated, i. e. there exists a quasi-conformal self-homeomorphism φ of $\partial_\infty \mathbb{H}^n$ such that $\varphi \Gamma \varphi^{-1} = \Gamma'$.

1.3.1 The limit set and the convex core

Let M be a complete hyperbolic n-manifold, and let Γ be a discrete and torsion-free subgroup of $\mathrm{Iso}(\mathbb{H}^n)$ such that M is isometric to $\mathbb{H}^n/\Gamma$. We define the *limit set* of Γ to be

$$\Lambda_\Gamma := \overline{\Gamma \cdot x_0} \cap \partial_\infty \mathbb{H}^n,$$

where $\overline{\Gamma \cdot x_0}$ denotes the closure of the Γ-orbit of x_0 in $\overline{\mathbb{H}}^n := \mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$. It is simple to see that the definition of Λ_Γ does not depend on the choice of the basepoint $x_0 \in \mathbb{H}^n$. If Γ is non-elementary (i. e. it does not have any finite orbit in $\overline{\mathbb{H}}^n$), then Λ_Γ can be characterized as the smallest closed Γ-invariant subset of $\partial_\infty \mathbb{H}^n$ (see e. g. [Rat06, Chapter 12]). The complementary region Ω_Γ of the limit set in $\partial_\infty \mathbb{H}^3$ is called the *domain of discontinuity* of Γ.

If $\pi \colon \mathbb{H}^n \to \mathbb{H}^n/\Gamma \cong M$ stands for the universal cover of M, then a subset C of M is convex if and only if $\pi^{-1}(C)$ is convex in $\mathbb{H}^n$. If Γ is non-elementary, then every non-empty Γ-invariant convex subset of $\mathbb{H}^n$ contains the *convex hull* C_Γ of Γ, which consists of the intersection of all half-spaces H of $\mathbb{H}^n$ satisfying $\overline{H} \supseteq \Lambda_\Gamma$ ($\overline{H}$ stands for the closure of H in $\overline{\mathbb{H}}^n$). The image $CM := \pi(C_\Gamma)$ describes a convex subset of M, called the *convex core* of M, which is minimal among the family of non-empty convex subsets of M.

Let now M be a convex co-compact hyperbolic manifold of dimension 3. The boundary of the convex core ∂CM of M is the union of a finite collection of connected surfaces, each of which is totally geodesic outside a subset of Hausdorff dimension 1. As described in [CEM06], the hyperbolic metrics on the totally geodesic pieces (called *flat pieces*) "merge" together, defining a complete hyperbolic metric m on ∂CM. The locus where ∂CM is not flat is a *geodesic lamination* λ (see Definition 1.2.10), the *bending locus* of ∂CM. The surface ∂CM is bent along λ, and the amount of bending can be described by a measured lamination μ, called the *bending measure* of ∂CM. The μ-measure along an arc k transverse to λ consists of an integral sum of the exterior dihedral angles along the leaves that k meets. A simple example to keep in mind arises when μ is a *rational lamination.* In this case the geodesic lamination λ is the union of a finite number of disjoint simple closed geodesics γ_i, and μ can be considered as a weighted sum $\sum_i \theta_i \, \delta_{\gamma_i}$, where $\theta_i \in (0, \pi]$ is the exterior bending angle along γ_i, and δ_{γ_i} is the transverse measure that counts the geometric intersection with γ_i. From now on, we will denote the transverse measure δ_{γ_i} simply by γ_i, with abuse, so that a rational lamination μ can be represented simply as $\sum_i \theta_i \, \gamma_i$ For a more detailed description we refer to [CEM06, Section II.1.11] (see also Section 2.1 for a description of ∂CM using the notion of *pleated surfaces*).

1.3.2 The boundary at infinity

Let Γ be a Kleinian group, and let Ω_Γ and C_Γ denote its domain of discontinuity and its convex hull, respectively. The nearest-point retraction r onto C_Γ extends continuously to $\mathbb{H}^3 \cup \Omega_\Gamma$, and it is clearly Γ-invariant. It is not difficult to see that the existence of such map r implies that the action of Γ is free and properly discontinuous on $\mathbb{H}^3 \cup \Omega_\Gamma$ (see e. g. [CEM06]). In addition, since the isometry group $\mathrm{Iso}^+(\mathbb{H}^3)$ acts on the sphere at infinity $\partial_\infty \mathbb{H}^3 \cong \mathbb{CP}^1$ by biholomorphisms, the natural complex structure of the domain of discontinuity $\Omega_\Gamma \subset \mathbb{CP}^1$ is preserved by Γ, and therefore it induces a Riemann surface structure over Ω_Γ/Γ. If M denotes the hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$. then the surface $\partial_\infty M := \Omega_\Gamma/\Gamma$ is called its *conformal boundary at infinity.*

In fact, the structure of the boundary at infinity $\partial_\infty M$ is richer than a standard Riemann surface structure. By construction, $\partial_\infty M$ comes with a $(\mathbb{P}\,\mathrm{SL}_2(\mathbb{C}), \mathbb{CP}^1)$-structure, which is also called a *complex projective structure.*

The space of complex projective structures $\mathcal{CP}(\Sigma)$ over Σ has a natural *forgetful map* π over the Teichmüller space $\mathcal{T}^{\mathfrak{c}}(\Sigma)$, which associates to a projective structure σ its underlying Riemann surface structure. Fuchsian hyperbolic structures are an example of complex projective structures, since they can be described as quotients of the upper half-plane $\mathbb{H}^2 \subset \mathbb{CP}^1$ by a subgroup of $\mathrm{Iso}^+(\mathbb{H}^2) \cong \mathbb{P}\,\mathrm{SL}_2(\mathbb{R})$.

Through of the notion of *Schwarzian derivative*, it is possible to describe the map $\pi\colon \mathcal{CP}(\Sigma) \to \mathcal{T}^{\mathfrak{c}}(\Sigma)$ as an affine bundle over the Teichmüller space. To see this, consider two complex projective structures σ and σ' over Σ that induce the same Riemann surface structure on Σ, with developing maps $D, D'\colon \widetilde{\Sigma} \to \mathbb{CP}^1$, respectively. Even if the functions D, D' may not be globally injective, they still admit local inverses. In particular, we are locally allowed to consider the compositions $D' \circ D^{-1}$, which we can assume to be univalent functions over proper open sets of $\mathbb{C} \subset \mathbb{CP}^1$ (the holomorphicity comes from the fact that $\pi(\sigma) = \pi(\sigma') \in \mathcal{T}^{\mathfrak{c}}(\Sigma)$). Observe that different choices of the local inverses of D make the map $D' \circ D^{-1}$ change by pre-composition of elements in $\mathbb{P}\,\mathrm{SL}_2(\mathbb{C})$. If $f\colon \Omega \subseteq \mathbb{C} \to \mathbb{C}$ is a univalent function, then the *Schwarzian derivative* of f is the holomorphic quadratic differential

$$S(f) := \left(\left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2 \right) \mathrm{d}z^2 .$$

The Schwarzian derivative satisfies:

i) $S(f) \equiv 0$ if and only if f is the restriction of a Möbius transformation of $\mathbb{CP}^1$;

ii) if f and g are two univalent functions for which $f \circ g$ is well-defined, then

$$S(f \circ g) = S(g) + g^* S(f).$$

These two simple properties imply the following fact: the holomorphic quadratic differentials $(D^{-1})^* S(D' \circ D^{-1})$, defined over small open subsets of $\widetilde{\Sigma}$, the universal cover of Σ, *do not depend* on the choices we made of the local inverses of D. In particular, they define a holomorphic quadratic differential on the entire surface $\widetilde{\Sigma}$, which is invariant by the action of the deck transformations of $\widetilde{\Sigma} \to \Sigma$. Moreover, for any complex projective structure σ and for any holomorphic quadratic differential $q \in Q(\Sigma, \pi(\sigma))$, there exists a unique complex projective structure σ' such that $S(\sigma', \sigma) = q$ (we refer to [Dum09] for a more detailed exposition on this topic).

With this procedure, we can associate to each pair of complex projective structures σ, σ' on Σ, belonging to the same fiber of π. a holomorphic quadratic differential $S(\sigma', \sigma) \in Q(\Sigma, \pi(\sigma))$. From the same properties above, we see that

$$S(\sigma'', \sigma) = S(\sigma'', \sigma') + S(\sigma', \sigma) \in Q(\Sigma, c),$$

for every $\sigma, \sigma', \sigma'' \in \pi^{-1}(c)$. In particular, the fibers $\pi^{-1}(c)$ of the forgetful map are naturally endowed with an affine structure over the space of holomorphic quadratic differentials $Q(\Sigma, c)$.

Finally, coming back to the context of Kleinian manifolds:

Definition 1.3.5. The *Schwarzian at infinity* of $\partial_\infty M$ is the holomorphic quadratic differential $q_\infty := S(\sigma_F, \sigma_\infty)$, where σ_∞ stands for the natural complex projective structure of $\partial_\infty M = \Omega_\Gamma/\Gamma$, and σ_F is the Fuchsian uniformization of the *conformal structure at infinity* $c_\infty := \pi(\sigma_\infty)$ of $\partial_\infty M$.

1.3.3 Quasi-Fuchsian manifolds

Definition 1.3.6. A Kleinian group Γ is *quasi-Fuchsian* if its limit set Λ_Γ is a Jordan curve in $\partial_\infty \mathbb{H}^3$ and both components of its domain of discontinuity Ω_Γ are invariant under Γ.

Theorem 1.3.7 ([Mas70]). *Let Γ be a Kleinian group. Then the following are equivalent:*

i) Γ *is quasi-Fuchsian;*

ii) The domain of discontinuity Ω_Γ *has exactly* 2 *connected components, each of which is invariant under* Γ*;*

iii) Γ *is quasi-conformally conjugate to a Fuchsian group, i. e. there exists a quasi-conformal self-homeomorphism* φ *of* $\mathbb{CP}^1$ *and a Fuchsian group* $\Gamma_0 < \mathbb{P}\operatorname{SL}_2(\mathbb{R})$ *such that* $\varphi \Gamma \varphi^{-1} = \Gamma_0$.

Let Σ be closed surface of genus $g \geq 2$. In light of Remark 1.3.4 and Maskit's Theorem, we can define the *space of (marked) quasi-Fuchsian manifolds* homeomorphic to $\Sigma \times \mathbb{R}$, denoted by $\mathcal{QF}(\Sigma)$, to be the quasi-isometric deformation space of $\mathbb{H}^3/\Gamma_0$, for some fixed Fuchsian group Γ_0 isomorphic to $\pi_1(\Sigma)$. Every quasi-Fuchsian manifold $M \in \mathcal{QF}(\Sigma)$ has boundary at infinity homeomorphic to the disjoint union of two copies of Σ, which we will call the upper/lower boundary at infinity $\partial^\pm_\infty M$ of M. Here $\partial^+_\infty M$ will denote the component whose boundary orientation coincides with the one of Σ, while $\partial^-_\infty M$ will be the one coming with the opposite orientation $\overline{\Sigma}$. The boundary components $\partial^\pm_\infty M$ are endowed with two natural complex projective structures $\sigma^\pm_\infty$ (see the previous section for a definition of this notion), and consequently with two induced conformal structures $c^\pm_\infty$.

On the other hand, also the boundary of the convex core has two connected components (unless M is Fuchsian) $\partial^\pm CM$, each of which is endowed with a hyperbolic structure $m^\pm \in \mathcal{T}^{\mathfrak{h}}(\Sigma)$, and a bending measure $\mu^\pm \in \mathcal{ML}(\Sigma)$ (if M is Fuchsian, then we set $m^+ = m^-$ to be the hyperbolic structure of the unique totally geodesic surface lying inside M, and $\mu^\pm = 0$).

A well-known result of Bers [Ber60] states that the map

$$\begin{array}{rccc} B: & \mathcal{QF}(\Sigma) & \longrightarrow & \mathcal{T}^{\mathfrak{c}}(\Sigma) \times \mathcal{T}^{\mathfrak{c}}(\overline{\Sigma}) \\ & M & \longmapsto & (c^+_\infty, c^-_\infty), \end{array}$$

which we will call the *Bers' map*, is a homeomorphism. In fact B is a biholomorphism if we endow $\mathcal{QF}(\Sigma)$ with the complex structure of subset of the character variety $\chi(\pi_1\Sigma, \mathbb{P}\operatorname{SL}_2(\mathbb{C}))$, and the natural complex structure of $\mathcal{T}^{\mathfrak{c}}(\Sigma)$.

Another natural map on $\mathcal{QF}(\Sigma)$ is the following:

$$\begin{array}{rccc} T: & \mathcal{QF}(\Sigma) & \longrightarrow & \mathcal{T}^{\mathfrak{h}}(\Sigma) \times \mathcal{T}^{\mathfrak{h}}(\Sigma) \\ & M & \longmapsto & (m^+, m^-). \end{array}$$

The map T has been conjectured by Thurston to be another parametrization of the space of quasi-Fuchsian manifolds, and this question is still open. Bonahon [Bon98b] proved that the map T is $\mathscr{C}^1$ (and actually not $\mathscr{C}^2$), therefore a first order variation of quasi-Fuchsian structures $\dot{M}$ determines a first order variation of the induced hyperbolic structures $\dot{m}$ on the convex core.

1.4 Convexes and hypersurfaces

In this section we describe a duality between closed convex subsets with non-empty interior and strictly convex hypersurfaces sitting inside the geometric spaces $\mathbb{H}^n$ and $\mathrm{d}\mathbb{S}^n$, which can be interpreted as an "oriented" instance of the polarity of $\mathbb{RP}^n$ with respect to the quadric $\mathbb{P}\{x_1^2 + \cdots + x_n^2 = x_{n+1}^2\}$.

Given a point $x \in \mathbb{H}^n$ and a unitary vector $v \in T^1_x\mathbb{H}^n$, we can associate to (x, v) a corresponding point (x^*, v^*) in the tangent bundle of $\mathrm{d}\mathbb{S}^n$. To see this, we observe that the vector $v \in T^1_x\mathbb{H}^n \subset \mathbb{R}^{n,1}$ verifies $\langle v, v\rangle_{n,1} = 1$ so, as an element of $\mathbb{R}^{n,1}$, it belongs to $\mathrm{d}\mathbb{S}^n$. On the other hand, since v is tangent to $\mathbb{H}^n$ at x, we must have $\langle x, v\rangle_{n,1} = 0$. Therefore x, as element of $\mathbb{R}^{n,1}$, belongs to $\ker\langle v, \cdot\rangle_{n,1}$, which is nothing but $T_v\mathrm{d}\mathbb{S}^n$. In this way, the couple $(x^*, v^*) := (v, x)$ defines a point in $T^{-1}_+\mathrm{d}\mathbb{S}^n$, namely the subset of the unit tangent bundle of $\mathrm{d}\mathbb{S}^n$ given by the pairs (x^*, v^*) of points $x^* \in \mathrm{d}\mathbb{S}^n$ and future-oriented time-like vectors $v^* \in T_{x^*}\mathrm{d}\mathbb{S}^n$ satisfying $\langle v, v\rangle_{n,1} = -1$. In the same way, if (x^*, v^*) is a point in $T^{-1}_+\mathrm{d}\mathbb{S}^n$, the couple $(x, v) := (v^*, x^*)$ defines an element in $T^1\mathbb{H}^n$. This correspondence $T^1\mathbb{H}^n \to T^{-1}_+\mathrm{d}\mathbb{S}^n$ is clearly one-to-one, and it can be interpreted also as a duality between oriented hyperplanes with basepoint of $\mathbb{H}^n$ and future-oriented space-like hyperplanes with basepoint of $\mathrm{d}\mathbb{S}^n$. A couple $(x, v) \in T^1\mathbb{H}^n$ corresponds to a hyperplane (namely $\ker\langle v, \cdot\rangle_{n,1} \cap \mathbb{H}^n$) with basepoint x in $\mathbb{H}^n$, together with the choice of a normal direction v at x. Analogously, a point $(x^*, v^*) \in T^{-1}_+\mathrm{d}\mathbb{S}^n$ is equivalent to the datum of a space-like hyperplane (namely $\ker\langle v^*, \cdot\rangle_{n,1}$) with basepoint $x^* \in \mathrm{d}\mathbb{S}^n$, endowed with its future-directed normal vector v^*.

As first observed by Hodgson and Rivin [HR93], this correspondence induces a duality between closed convex subsets with non-empty interior in the two geometries. We briefly recall now the definitions and the results of [HR93, Section 2] that will be useful in the following.

Definition 1.4.1. A *convex body* in $\mathbb{H}^n$ or $\mathrm{d}\mathbb{S}^n$ is a subset C with non-empty interior, which can be described as the intersection of a family of closed half-spaces.

Observe that any convex body is closed. We define a dual operation between subsets of $\mathbb{H}^n$ and $\mathrm{d}\mathbb{S}^n$ as follows: given $C \subseteq \mathbb{H}^n$, we set

$$C^\wedge := \{v' \in \mathrm{d}\mathbb{S}^n \subset \mathbb{R}^{n,1} \mid \forall w \in C \quad \langle v', w\rangle_{n,1} \geq 0\}.$$

In the same way, if $C' \subseteq \mathrm{d}\mathbb{S}^n$, then

$$(C')^\wedge := \{v \in \mathbb{H}^n \subset \mathbb{R}^{n,1} \mid \forall w' \in C' \quad \langle v, w'\rangle_{n,1} \geq 0\}.$$

Lemma 1.4.2 ([HR93, Section 2])**.** *Let C, D be two subsets of $\mathbb{H}^n$ (of $\mathrm{d}\mathbb{S}^n$). Then*

1. *$C^\wedge$ is a convex body;*
2. *if $C \subseteq D$, then $C^\wedge \supseteq D^\wedge$;*
3. *if C is a convex body, then $C^{\wedge\wedge} = C$.*

Assuming C to be a convex body with regular boundary, we want to investigate in detail the relations between the hypersurfaces ∂C and $\partial(C^\wedge)$. First we need to introduce some notation.

Let $\mathcal{D}$ and $\mathcal{D}^*$ denote the Levi-Civita connections of $\mathbb{H}^n$ and $\mathrm{d}\mathbb{S}^n$, respectively. In what follows, Σ will be an orientable smooth manifold of dimension $n-1$ and $\iota\colon \Sigma \to \mathbb{H}^n$ an immersion of Σ in $\mathbb{H}^n$. The *first fundamental form* I of Σ is given by the restriction of the metric of $\mathbb{H}^n$ to the tangent bundle $T\Sigma$. Fixing a unitary normal vector field $\nu\colon \Sigma \to T^1\mathbb{H}^n$ on Σ, we denote by B its *shape operator*, i. e. the endomorphism of $T\Sigma$ defined by $BU := -\mathcal{D}_U \nu$, for every tangent vector field U of Σ. It is simple to see that the shape operator B is self-adjoint with respect to the first fundamental form I. The trace of the shape operator will be called the *mean curvature* of Σ, and the tensors $I\!I := I(B\cdot,\cdot)$ and $I\!I\!I := I(B\cdot, B\cdot)$ the *second* and *third fundamental forms*, respectively. If ∇ is the Levi-Civita connection of (Σ, I), then we have

$$\nabla_U V = \mathcal{D}_U V - I\!I(U,V)\,\nu,$$

for any tangent vector fields U, V of Σ.

Assume now the second fundamental form $I\!I_p$ to be non-degenerate at every point $p \in \Sigma$. Then we define the dual pair (ι^*, ν^*) of (ι, ν) to be the datum of two maps $\iota^*\colon \Sigma \to \mathrm{d}\mathbb{S}^n$ and $\nu^*\colon \Sigma \to T^{-1}_+\mathrm{d}\mathbb{S}^n$, satisfying $(\iota^*(p), \nu^*(p)) = (\iota(p), \nu(p))^*$ for every $p \in \Sigma$, where $(\iota(p), \nu(p))^*$ denotes the image of the point $(\iota(p), \nu(p))$ of $T^1\mathbb{H}^n$ through the duality with $T^{-1}_+\mathrm{d}\mathbb{S}^n$. In this manner, the Gauss map ν of ι becomes the immersion of a dual hypersurface ι^* in $\mathrm{d}\mathbb{S}^n$, and the immersion ι becomes the Gauss map of the immersed hypersurface ι^* (as we will see in the proof of Proposition 1.4.4, ι^* is an immersion because we assumed $I\!I$ to be non-degenerate).

Instead of referring to the dual hypersurface as the map ι^*, we will often denote it, with abuse, by Σ^*. An immersed hypersurface in $\mathrm{d}\mathbb{S}^n$ is called *space-like* if the restriction I^* of the metric of $\mathrm{d}\mathbb{S}^n$ on its tangent bundle is positive definite. We will see soon that this is always the case when Σ^* arises from the duality procedure we described above. As in the Riemannian case, I^* will be called the *first fundamental form* of Σ^*. We define the *second fundamental form* $I\!I^*$ by requiring:

$$\nabla^*_{U^*} V^* = \mathcal{D}^*_{U^*} V^* - I\!I^*(U^*, V^*)\,\nu^*$$

for any tangent vector fields U^*, V^* of Σ^*, where ∇^* is the Levi-Civita connection of Σ^*. Then the *shape operator* B^* is defined as the I^*-self-adjoint operator associated to $I\!I^*$ ($I\!I^*$ is symmetric because both ∇_* and $\mathcal{D}^*$ are without torsion). Since $\langle \nu^*, \nu^* \rangle_{n,1} = -1$, the shape operator of Σ^* verifies $B^* U^* = +\mathcal{D}^*_{U^*} \nu^*$ (the sign is the opposite of the one in the Riemannian setting). Finally, the *third fundamental form* is defined by setting $I\!I\!I^*(\cdot,\cdot) := I^*(B^*\cdot, B^*\cdot)$.

In the very same way, given a space-like hypersurface $\iota^*\colon \Sigma^* \to \mathrm{d}\mathbb{S}^n$ with future-directed normal vector field ν^* for which $I\!I^*$ is everywhere non-degenerate, we can construct a dual hypersurface Σ^{**} in $\mathbb{H}^3$ with non-degenerate second fundamental form $I\!I$. The process $(\cdot)^*$ is clearly an involution, since it is so at the level of $T^1\mathbb{H}^n$ and $T^{-1}_+\mathrm{d}\mathbb{S}^n$.

When a hypersurface in $\mathbb{H}^n$ with non-degenerate second fundamental form arises as the boundary of a convex domain, we will always choose the normal vector field to point inward (so that the second fundamental is positive definite), while for space-like hypersurfaces in $\mathrm{d}\mathbb{S}^n$ we will always choose the future-oriented normal vector field (the one with positive $(n+1)$-th coordinate in $\mathbb{R}^{n,1}$). In this terms, we can restate the analysis done by Hodgson and Rivin as follows:

Lemma 1.4.3 ([HR93, Section 2]). *Let C be a convex body in $\mathbb{H}^n$ with smooth boundary and with positive definite second fundamental form. Then the boundary of $C^\wedge$ in $\mathrm{d}\mathbb{S}^n$ is parametrized by the dual hypersurface associated to ∂C. In other words, we have*

$$(\partial C)^* = \partial(C^\wedge).$$

In the same way, if C' is a convex body in $\mathrm{d}\mathbb{S}^n$ with smooth space-like boundary and with negative definite second fundamental form, then $(\partial C')^* = \partial(C'^\wedge)$.

The following statement describes explicitly the relations between the fundamental forms of (ι, ν) and (ι^*, ν^*):

Proposition 1.4.4 ([Sch06, Proposition 1.6]). *If Σ is an immersed hypersurface in $\mathbb{H}^n$ with positive definite second fundamental form, then its dual Σ^* is an immersed, space-like hypersurface in $\mathrm{d}\mathbb{S}^n$ with negative definite second fundamental form, and viceversa. Moreover, under the duality correspondence between Σ and Σ^*, we have that:*

- $I = I\!I\!I^*$;
- $I\!I = -I\!I^*$;
- $I\!I\!I = I^*$.

Proof. We denote by $\alpha \colon U \to \Sigma \subset \mathbb{H}^n$ a local parametrization of Σ, where U is an open set in $\mathbb{R}^{n-1}$. Let $(E_k)_k$ be the orthonormal frame on $\mathbb{R}^{n,1}$ corresponding to a fixed orthonormal basis $\mathcal{B}$ of $\mathbb{R}^{n,1}$ (the E_k's are the sections of $T\mathbb{R}^{n,1}$ associated with $\mathcal{B}$ under the identification $T_p\mathbb{R}^{n,1} \cong \mathbb{R}^{n,1}$). For convenience, we introduce the following notation: if f is a map from U to $\mathbb{R}^{n,1}$, we denote by X_f the element of $\Gamma(\alpha^* T\mathbb{R}^{n,1})$ defined as $X_f = \sum_k f^k \, E_k \circ \alpha$, where $\Gamma(\alpha^* T\mathbb{R}^{n,1})$ is the space of sections of the pullback bundle of $T\mathbb{R}^{n,1}$ over α.

Let $\nu \in \Gamma(\alpha^* T\mathbb{R}^{n,1})$ be the unitary normal vector field of Σ. Then, by definition of the duality $T^1\mathbb{H}^n \leftrightarrow T^{-1}_+\mathrm{d}\mathbb{S}^n$, we can construct a parametrization $\alpha^* \colon U \to \mathrm{d}\mathbb{S}^n$ of Σ^* by requiring that $X_{\alpha^*} = \nu$. In other words, the k-th component of α^* with respect to $\mathcal{B}$ coincides with the k-th component of ν with respect to the frame $(E_k)_k$, for all k. Analogously we have $\nu^* = X_\alpha$, where ν^* is the future-directed normal vector field to Σ^*. Since $(E_k)_k$ is a orthonormal frame of parallel vector fields with respect to the Levi-Civita connection D of $\mathbb{R}^{n,1}$, for every coordinate vector field ∂_i of α we have

$$D_{\partial_i} \nu = D_{\partial_i} X_{\alpha^*} = \partial_i^* \tag{1.1}$$

and, in the dual hypersurface

$$D_{\partial_i^*} \nu^* = D_{\partial_i^*} X_\alpha = \partial_i, \tag{1.2}$$

where ∂_i^* is the i-th coordinate vector field of Σ^* associated to the parametrization α^*. Observe also that the normal direction to $\mathbb{H}^n$ at the point $\alpha(p)$ is given by $X_\alpha(p)$. This implies that, if $\mathcal{D}$ is the Levi-Civita connection of $\mathbb{H}^n$, we have

$$\begin{aligned}
\mathcal{D}_{\partial_i}\nu &= D_{\partial_i}\nu - \frac{\langle D_{\partial_i}\nu, X_\alpha\rangle_{n,1}}{\langle X_\alpha, X_\alpha\rangle_{n,1}}\, X_\alpha \\
&= D_{\partial_i}\nu + \langle D_{\partial_i}\nu, X_\alpha\rangle_{n,1}\, X_\alpha && (\langle X_\alpha, X_\alpha\rangle_{n,1} = -1) \\
&= D_{\partial_i}\nu + \langle \partial_i^*, X_\alpha\rangle_{n,1}\, X_\alpha && (\text{relation } (1.1)) \\
&= D_{\partial_i}\nu. && (X_\alpha = \nu^* \perp \Sigma^*)
\end{aligned}$$

This equality, combined with relation (1.1), shows that the shape operator B of Σ verifies

$$B\,\partial_i = -\mathcal{D}_{\partial_i}\nu = -D_{\partial_i}\nu = -\partial_i^*.$$

In the same way we see that $\mathcal{D}^*_{\partial_i^*}\nu^* = D_{\partial_i^*}\nu^*$, with $\mathcal{D}^*$ the connection on $\mathrm{d}\mathbb{S}^n$, so the shape operator B^* of Σ^* verifies

$$B^*\partial_i^* = +\mathcal{D}^*_{\partial_i^*}\nu^* = +D_{\partial_i^*}\nu^* = +\partial_i.$$

The tangent spaces $T_\iota\Sigma$ and $T_{\iota^*}\Sigma^*$, as linear subspaces of $\mathbb{R}^{n,1}$, are both orthogonal to the 2-plane generated by X_α and X_{α^*}, so they must coincide. Therefore, the shape operators B and B^* are both endomorphisms of $T_\iota\Sigma = T_{\iota^*}\Sigma^*$ and, by the relations we just proved, they verify $B^{-1} = -B^*$. All the relations in the statement can be deduced from this equality, in the following we prove $I\!I = -I\!I^*$, the others are analogous:

$$I\!I(\partial_i, \partial_j) = \langle B\,\partial_i, \partial_j\rangle_{n,1} = -\langle \partial_i^*, B^*\partial_i^*\rangle_{n,1} = -I\!I^*(\partial_i^*, \partial_j^*).$$

□

1.5 Constant Gaussian curvature surfaces

Let Σ be a (space-like) surface immersed in a Riemannian (Lorentzian) 3-manifold M of constant sectional curvature $\sec(M)$, with first and second fundamental forms I and $I\!I$, and shape operator B. We denote by K_e its *extrinsic curvature*, i. e. $K_e = \det B$, and by K_i its *intrinsic curvature*, i. e. the Gauss curvature of the Riemannian metric I. For convenience, we define $\mathrm{sgn}(M)$ to be $+1$ if M is a Riemannian manifold, and -1 is M is Lorentzian. Then, the Gauss-Codazzi equations of $(\Sigma, I, I\!I)$ can be expressed as follows:

$$\begin{aligned}
K_i &= \mathrm{sgn}(M)\,K_e + \sec(M), \\
(\nabla_U B)V &= (\nabla_V B)U \quad \forall U, V,
\end{aligned} \tag{1.3}$$

where U and V are tangent vector fields to Σ, and ∇ is the Levi-Civita connection of the metric I. We recall that the *third fundamental form* of Σ is the symmetric 2-tensor $I(B\cdot, B\cdot)$.

If Σ is a surface immersed in a hyperbolic 3-manifold M, then its Gauss equation has the following form:

$$K_i = K_e - 1. \tag{1.4}$$

If the shape operator of Σ is everywhere non-degenerate (equivalently, if K_e never vanishes), then we can give a geometric interpretation of the third fundamental form $I\!I\!I$ applying what observed in Section 1.4. First we lift the immersion of Σ in M to an immersion of $\widetilde{\Sigma}$, the universal cover of Σ, into $\mathbb{H}^3$, which will be equivariant with respect to some representation $\rho\colon \pi_1(\Sigma) \to O_o(3,1)$. Through the polarity correspondence between hyperbolic and de Sitter spaces, we can construct an immersed dual surface $\widetilde{\Sigma}^*$ in $\mathrm{d}\mathbb{S}^3$. Moreover, being $\widetilde{\Sigma}$ equivariant with respect to ρ, the surface $\widetilde{\Sigma}^*$ can be obtained as the lift of a spacelike surface Σ^*, homeomorphic to Σ, sitting inside a 3-dimensional spacetime locally modelled over $\mathrm{d}\mathbb{S}^3$ and with holonomy ρ (see e. g. [Mes07] for details). Then, the first fundamental form I^* of Σ^* coincides with the tensor $I\!I\!I$, and the second fundamental forms $I\!I^*$ and $I\!I$ are the same up to sign (see Proposition 1.4.4). The Gauss equation of the dual surface $(\Sigma^*, I^*, I\!I^*)$ is

$$K_i^* = -K_e^* + 1. \tag{1.5}$$

Since the shape operator B^* of Σ^* coincides with $-B^{-1}$ (again by Proposition 1.4.4), the surface $(\Sigma, I, I\!I)$ has extrinsic curvature K_e if and only if $(\Sigma^*, I^*, I\!I^*)$ has extrinsic curvature $K_e^* = K_e^{-1}$.

Definition 1.5.1. Let Σ be an immersed surface inside a hyperbolic 3-manifold, and let $k \in (-1, 0)$. Σ is a k-surface if the intrinsic (or Gaussian) curvature of its first fundamental form is constantly equal to k.

Assume now Σ to be a k-surface. By equation (1.4), its extrinsic curvature $K_e = k+1$ is strictly positive, so its shape operator is everywhere non-degenerate. Therefore Σ has an immersed dual surface Σ^*, whose extrinsic curvature is equal to $K_e^* = K_e^{-1} = \frac{1}{k+1}$. By the Gauss equation (1.5), the intrinsic curvature of $I^* = I\!I\!I$ is equal to $1 - K_e^* = \frac{k}{k+1}$, which is *constant*. In other words, we have:

Lemma 1.5.2. *If Σ is a k-surface immersed in a hyperbolic 3-manifold M, then its first and third fundamental forms have constant intrinsic curvature equal to k and $\frac{k}{k+1}$, respectively.*

If we define the shape operator of Σ using the normal vector field of Σ that points to the convex side of Σ, then the second fundamental form $I\!I$ of Σ has strictly positive principal curvatures, since $\det B = K_e = k+1 > 0$. Therefore $I\!I$ is a positive definite symmetric bilinear form or, in other words, a Riemannian metric.

Lemma 1.5.3. *Let Σ be a surface immersed in a Riemannian (or Lorentzian) 3-manifold M with constant sectional curvature. Assume that the second fundamental form $I\!I$ of Σ is positive definite. Then, the following are equivalent:*

- *the surface Σ has constant extrinsic curvature;*
- *the identity map $id\colon (\Sigma, I\!I) \to (\Sigma, I)$ is harmonic.*

Proof. The proof of this lemma proceeds similarly to the argument we gave in Section 1.2.4 to describe the relation between the notion of minimal Lagrangian maps and Theorem 1.2.18. The Levi-Civita connection $\nabla^{I\!I}$ of the Riemannian metric $I\!I$ satisfies

$$\nabla^{I\!I}_U V = \nabla_U V + \frac{1}{2} B^{-1}(\nabla_U B)V,$$

where U and V are tangent vector fields to Σ, and ∇ is the Levi-Civita connection of I (this relation is true for any immersed surface Σ, we do not need to require the Gaussian curvature to be constant). As observed in Remark 1.2.13, the map $id\colon (\Sigma, I\!I) \to (\Sigma, I)$ is harmonic if and only if the $I\!I$-traceless part of I, which is equal to $I - \frac{H}{2K_e} I\!I$, is divergence-free with respect to $\nabla^{I\!I}$. Using the expression of $\nabla^{I\!I}$ above, we can prove that

$$\mathrm{div}_{I\!I}\left(I - \frac{H}{2K_e} I\!I\right) = -\frac{1}{2}\,\mathrm{d}(\ln K_e)\,.$$

From this equation the statement is clear. □

As suggested by the proof we gave above, the notion of minimal Lagrangian maps is intimately related with the properties of constant Gaussian curvature surfaces. If $(\Sigma, I, I\!I)$ is a k-surface, we set

$$h := -k\ I, \qquad h' := -\frac{k}{k+1}\ I\!I\!I, \qquad b := \frac{1}{\sqrt{k+1}} B.$$

By construction $I\!I = -\frac{\sqrt{k+1}}{k} h(b\cdot, \cdot)$, $h' = h(b\cdot, b\cdot)$ and $\det b = 1$. By the Codazzi equation, the operator b is Codazzi and, by the choices we made of the multiplicative constants, the metrics h and h' have Gaussian curvature constantly equal to -1. Therefore the pair of metrics h, h' is normalized, and b is their Labourie operator, as in Theorem 1.2.18. This shows, by the same argument of Section 1.2.4, that the identity map

$$id : (\Sigma, h) \longrightarrow (\Sigma, h')$$

is minimal Lagrangian.

We can summarize what we just showed in the following Proposition:

Proposition 1.5.4. *Let $k \in (-1, 0)$. Every k-surface immersed in a hyperbolic 3-manifold satisfies the following properties:*

- *the first and third fundamental forms I and $I\!I\!I$ of Σ are constant Gaussian curvature Riemannian metrics of curvature k and $\frac{k}{k+1}$, respectively;*
- *the second fundamental form is everywhere non-degenerate (without loss of generality, positive definite);*
- *the maps*

$$id : (\Sigma, [I\!I]) \longrightarrow (\Sigma, -k\,I), \qquad id : (\Sigma, [I\!I]) \longrightarrow (\Sigma, -\tfrac{k}{k+1} I\!I\!I)$$

are harmonic, with opposite Hopf differentials. In other words, the map

$$id : (\Sigma, -k\,I) \longrightarrow (\Sigma, -\tfrac{k}{k+1} I\!I\!I)$$

is minimal Lagrangian (see Definition 1.2.16)

1.6 Hyperbolic ends

Definition 1.6.1. Given Σ a closed surface, a *hyperbolic end* E of topological type $\Sigma \times [0, \infty)$ is a hyperbolic 3-manifold with underlying topological space $\Sigma \times (0, \infty)$ and whose metric completion $\overline{E} \cong \Sigma \times [0, \infty)$ is obtained by adding to E a locally concave pleated surface $\Sigma \times \{0\} \subset \Sigma \times [0, \infty)$ (see Section 2.1 for the definition of pleated surface). We will denote by ∂E the locally concave pleated boundary of E.

Two hyperbolic ends $E = (\Sigma \times (0, \infty), g)$ and $E' = (\Sigma \times (0, \infty), g')$ are equivalent if there exists an isometry between them that is isotopic to $id_{\Sigma \times (0,\infty)}$. We set $\mathcal{E}(\Sigma)$ to be the space of equivalence classes of hyperbolic ends of topological type $\Sigma \times [0, \infty)$.

Remark 1.6.2. Typical examples of hyperbolic ends are the connected components of $M \setminus CM$, where M is a convex co-compact hyperbolic 3-manifold and CM is its convex core (see Section 1.3 for the definition of this notion and its properties).

Let E be a hyperbolic end. The manifold $\overline{E} \cong \Sigma \times [0, \infty)$ can be compactified by adding a topological surface "at infinity" $\partial_\infty E := \Sigma \times \{\infty\}$. By the same phenomenon described in Section 1.3.2, the $(\mathrm{Iso}^+(\mathbb{H}^3), \mathbb{H}^3)$-structure on E naturally extends to a *complex projective structure* $\sigma_\infty^E \in \mathcal{CP}(\Sigma)$ on the boundary at infinity $\partial_\infty E$, coming from the action of $\mathrm{Iso}^+(\mathbb{H}^3) \cong \mathbb{P}\mathrm{SL}_2(\mathbb{C})$ by Möbius transformations on $\partial_\infty \mathbb{H}^3 \cong \mathbb{CP}^1$.

By a classical construction due to Thurston, it is possible to invert this process: given a complex projective structure σ on a surface Σ, there exists a hyperbolic end E of topological type $\Sigma \times [0, \infty)$ whose induced complex projective structure on $\partial_\infty E$ coincides with σ. The universal cover $\widetilde{E}$ of E can be locally described as the envelope of those half-spaces H of $\mathbb{H}^3$ satisfying $\overline{H} \cap \partial_\infty \mathbb{H}^3 = D$, where D varies over the developed maximal discs of $(\widetilde{\Sigma}, \tilde{\sigma})$ in $\partial_\infty \mathbb{H}^3 = \mathbb{CP}^1$. This construction establishes a one-to-one correspondence between the space of hyperbolic ends $\mathcal{E}(\Sigma)$ and the deformation space $\mathcal{CP}(\Sigma)$. We refer to [KT92] for a more detailed exposition of Thurston's construction.

1.6.1 The Schwarzian parametrization

Let E be a hyperbolic end. Following the notation introduced above, we denote by $c_\infty = c_\infty^E$ the underlying conformal structure of $\sigma_\infty = \sigma_\infty^E$, and by σ_F the Fuchsian structure of c_∞, i. e. the complex projective structure on $\Sigma = \partial_\infty E$ determined by the uniformization map of $(\widetilde{\Sigma}, \tilde{c}_\infty)$. The space of complex projective structures with underlying conformal structure c_∞ can be interpreted as an affine space over the space of holomorphic quadratic differentials of (Σ, c_∞), and the correspondence sends each element $\sigma_F - \sigma_\infty$ into the *Schwarzian derivative* of σ_F with respect to σ_∞ (see Section 1.3.2 and [Dum09] for details). In particular, the element $\sigma_F - \sigma_\infty$ determines a unique holomorphic quadratic differential $q_\infty = q_\infty^E$ of (Σ, c_∞), called the *Schwarzian at infinity* of E. The resulting map

$$\begin{array}{rccc} \mathrm{Sch}: & \mathcal{E}(\Sigma) & \longrightarrow & T^*\mathcal{T}^{\mathfrak{c}}(\Sigma) \\ & [E] & \longmapsto & (c_\infty^E, q_\infty^E), \end{array}$$

gives a parametrization of the space of hyperbolic ends $\mathcal{E}(\Sigma)$, which we will call the *Schwarzian parametrization*.

1.6.2 The Thurston parametrization

The Schwarzian parametrization of the space of hyperbolic ends $\mathcal{E}(\Sigma)$ uses the geometric structure of the boundary at infinity $\partial_\infty E$ of E. In the following we will describe a analogous construction, due to Thurston (unpublished, described by Kamishima and Tan in [KT92]), involving the shape of the convex pleated boundary ∂E, instead of $\partial_\infty E$.

The surface ∂E is a topologically embedded surface in E, which is almost everywhere totally geodesic. The set of points where ∂E is not locally shaped as an open set of $\mathbb{H}^2$ is a closed subset λ that is disjoint union of simple (not necessarily closed) complete geodesics. The path metric of ∂E is an actual hyperbolic metric $m \in \mathcal{T}^\mathfrak{h}(\Sigma)$, and the structure of the singular locus λ can be described using the notion of measured lamination. In the simple case of λ composed by disjoint simple closed geodesics, each leaf γ_i of λ has an associated exterior dihedral angle $\vartheta_i \in \mathbb{R}_{\geq 0}$, which measures the bending between the totally geodesic portions of ∂E meeting along γ_i. Given any geodesic arc α transverse to λ, we can define the transverse measure $\mu := \sum_i \vartheta_i \ \gamma_i$ along a geodesic segment α to be the sum $\sum_i \vartheta_i \ i(\gamma_i, \alpha)$, where $i(\gamma_i, \alpha)$ is the geometric intersection between α and γ_i. Using an approximation procedure, we can generalize the construction above to a generic support λ, obtaining a measured lamination $\mu \in \mathcal{ML}(\Sigma)$, which measures the amount of bending that occurs transversely to λ. The datum of the hyperbolic metric h and the measured lamination μ is actually sufficient to describe the entire hyperbolic end. In other words, the map

$$\begin{array}{rccc} \mathrm{Th}: & \mathcal{E}(\Sigma) & \longrightarrow & \mathcal{T}^\mathfrak{h}(\Sigma) \times \mathcal{ML}(\Sigma) \\ & [E] & \longmapsto & (m, \mu) \end{array}$$

parametrizes the space of hyperbolic ends (for a detailed proof of this result, see [KT92, Section 2]). We will call Th the *Thurston parametrization* of $\mathcal{E}(\Sigma)$.

Remark 1.6.3. If E is a connected component of $M \setminus CM$, then the data $\mathrm{Th}(E) = (m, \mu)$ are exactly the hyperbolic structure and the bending measure of the component of ∂CM facing E, as described in Section 1.3.2, and similarly $\mathrm{Sch}(E) = (c_\infty, q_\infty)$ determines the data of the component of $\partial_\infty M$ to which E is asymptotic.

However, there are hyperbolic ends in $\mathcal{E}(\Sigma)$ that cannot be realized in such a way, and it can be easily seen from Thurtson's parametrization result. Indeed, the bending measures of those hyperbolic ends that arise as components of $M \setminus CM$, for some convex co-compact hyperbolic manifold M, satisfy certain geometric constraints. For example, a necessary condition that a measured lamination of the form $\theta \cdot \gamma$ (for some simple closed curve γ) must satisfy to be realized as the bending measure of an end of a convex co-compact manifold is $\theta \leq \pi$ (two half-planes meeting along a geodesic in $\mathbb{H}^3$ have exterior angle bounded by π). On the other hand, a general hyperbolic end does not have such constraint (a (not too) heuristic reason to explain this phenomenon is that the developing map D of the hyperbolic structure of E may be not locally injective when extended to the pleated boundary of $\widetilde{E}$, and larger angles can be realized in this way).

Similar constraints arise also in the Schwarzian description. For instance, if E is a hyperbolic end realizable as one of the complementary components of the convex core of a quasi-Fuchsian manifold, then the norm of its Schwarzian at infinity q_∞ (with respect to the hyperbolic metric in c_∞^E) is uniformly bounded by $3/2$. This is a simple consequence of Nehari's bound [Neh49], as described for instance in [GL00, Chapter 6]. This phenomenon arises more generally for incompressible hyperbolic ends of geometrically finite hyperbolic 3-manifolds.

1.6.3 Foliations by k-surfaces

We conclude this description of the properties of hyperbolic ends mentioning a result of Labourie [Lab91], which will play an important role in our study (see Chapters 3 and 5):

Theorem 1.6.4 ([Lab91, Théorème 2])**.** *Every hyperbolic end E is foliated by a family of k-surfaces $(\Sigma_k)_k$, with k that varies in $(-1, 0)$. As k goes to -1, the surface Σ_k converges to the locally concave pleated boundary of E, and as k goes to 0, Σ_k approaches the conformal boundary at infinity $\partial_\infty E$.*

In particular, this fact will be our starting point in the investigation of the properties of k-surfaces that interpolate between the geometries of the local pleated boundary and the conformal boundary at infinity of hyperbolic ends, as mentioned in the last part of the introduction.

1.7 Volumes

1.7.1 Classical and differential Schläfli formulae

The classical Schläfli formula expresses the derivative of the volume along a 1-parameter deformation of polyhedra in terms of the variation of its boundary geometry. It was originally proved by Schläfli [Sch58] in the unit 3-sphere case, and later extended to polyhedra of any dimension sitting inside constant non-zero sectional curvature space forms of any dimension. Here we recall the statement in the 3-dimensional hyperbolic space $\mathbb{H}^3$, which will be our case of interest:

Theorem 1.7.1 ([Sch58],[Mil94],[AVS93])**.** *Let $(P_t)_t$ be a 1-parameter family of convex compact polyhedra in $\mathbb{H}^3$, whose vertices vary smoothly in t, with $P = P_0$. Assume that the boundaries of the polyhedra P_t share the same combinatorial structure for t sufficiently close to 0. Then the function $t \mapsto V_t := \mathrm{Vol}(P_t)$ admits derivative at $t = 0$, and it verifies*

$$\dot{V} = \frac{1}{2} \sum_{\substack{e \text{ edge} \\ \text{of } P}} \ell(e)\, \dot{\theta}(e),$$

where the sum is taken over the set of edges e of P, $\ell(e)$ denotes the length of e in P and $\dot{\theta}(e)$ is the variation of the exterior dihedral angle along e_t in the family $(P_t)_t$ (since the combinatorics of P_t does not change, any edge e of P has a corresponding e_t in P_t).

Rivin and Schlenker [RS99] developed a smooth analogue of Theorem 1.7.1 in the context of open domains with smooth boundary. As in the case of the classical Schläfli, the *differential Schläfli formula* expresses the variation of the volume enclosed by a surface in terms of the variation of the geometry of the surface itself. We present here two similar statements, the first involving a first order variation of the boundary of a region in a *fixed* Riemannian manifold, and in the second case we keep the region fixed and we vary the *ambient metric*.

Theorem 1.7.2 ([RS99, Theorem 1]). *Let M be a Lorentzian or Riemannian Einstein manifold of dimension n with scalar curvature R, and consider Σ a closed embedded $\mathscr{C}^{1,1}$-hypersurface of M which is the boundary of a region $N \subset M$. The choice of a section V of the restriction of TM over Σ determines a first order deformation of Σ inside M. We denote by I, II, H the first and second fundamental forms and the mean curvature of Σ, respectively, defined selecting the inward normal vector field of Σ. If δT denotes the first order variation of the object T under the deformation, then*

$$\frac{R}{n}\,\delta\mathrm{Vol}(N) = -\int_{\Sigma}\left(\delta H + \frac{1}{2}(\delta I, \mathit{II})\right)\mathrm{d}a\,.$$

Remark 1.7.3. The request of $\mathscr{C}^{1,1}$-regularity of the boundary is needed here in order to have a notion of mean curvature. This quantity will be a function in $L^\infty(\Sigma)$, therefore defined almost everywhere. Nevertheless, the relations above still hold and make sense, since the integrals of H and its variation are well defined quantities.

Theorem 1.7.4 ([RS99, Theorem 2]). *Let M be a compact n-manifold with smooth non-empty boundary ∂M, and let g_t be a smooth 1-parameter family of Riemannian Einstein metrics with constant sectional curvature R. Then*

$$\frac{R}{n}\,\frac{\mathrm{dVol}(M, g_t)}{\mathrm{d}t}\bigg|_{t=0} = -\int_{\partial M}\left(\delta H + \frac{1}{2}(\delta g|_{\partial M}, \mathit{II})\right)\mathrm{d}a\,.$$

1.7.2 The dual volume

Thanks to the correspondence between convex bodies in the hyperbolic and de Sitter geometries, it is possible to define a notion of dual volume for convex bodies in $\mathbb{H}^3$. In what follows, we will describe different and complementary ways to introduce this quantity.

Let S be a space-like plane in the de Sitter space $\mathrm{d}\mathbb{S}^3$. We denote by $t_S \colon \mathrm{d}\mathbb{S}^3 \to \mathbb{R}$ the signed future-directed time-like distance from the plane S. Given such a S in $\mathrm{d}\mathbb{S}^3$, we can find global coordinates $(S \times \mathbb{R}, h_S)$ on $\mathrm{d}\mathbb{S}^3$ so that the submanifold $S \times \{0\}$, sitting inside $S \times \mathbb{R}$, corresponds to the space-like plane S, and the $\mathbb{R}$-component of the coordinate system is given by the function t_S defined above. Then the Lorentzian metric of $\mathrm{d}\mathbb{S}^3$ can be written as

$$h_S = -\,\mathrm{d}{t_S}^2 + \cosh^2 t_S\, g_{\mathbb{S}^2},$$

where $g_{\mathbb{S}^2}$ denotes the standard Riemannian metric on the 2-sphere of radius 1. Once we fix an orientation on $\mathrm{d}\mathbb{S}^3$, we can define ω_S to be the 2-form given by

$$\left(\int_0^{t_S}\cosh^2\rho\,\mathrm{d}\rho\right)\mathrm{dvol}_{\mathbb{S}^2},$$

where we are choosing $\mathrm{dvol}_{\mathbb{S}^2}$ so that

$$\mathrm{d}\omega = \cosh^2 t_S\,\mathrm{d}t_S \wedge \mathrm{dvol}_{\mathbb{S}^2} = \mathrm{dvol}_{\mathrm{d}\mathbb{S}^3}\,.$$

Definition 1.7.5. Let C be a compact convex body in $\mathbb{H}^3$ with $\mathscr{C}^{1,1}$-boundary.

- given a fixed point p in the interior of C, we define:
$$V_1^*(C) = \mathrm{Vol}_{\mathrm{d}\mathbb{S}^3}(p^\wedge \setminus C^\wedge),$$
where $p^\wedge$ denotes the convex set of $\mathrm{d}\mathbb{S}^3$ dual of $\{p\} \subset \mathbb{H}^3$;
- given a fixed space-like plane S in $\mathrm{d}\mathbb{S}^3$, we define:
$$V_2^*(C) = -\int_{\partial(C^\wedge)} \omega_S,$$
where $(\partial C)^* = \partial(C^\wedge) \subset \mathrm{d}\mathbb{S}^3$ is future-oriented;
- choosing as normal vector field to ∂C the one pointing inward, we define:
$$V_3^*(C) = -\mathrm{Vol}_{\mathbb{H}^3}(C) + \frac{1}{2}\int_{\partial C} H \,\mathrm{d}a\,.$$

Remark 1.7.6. Given a point p in $\mathbb{H}^3$, the set $p^\wedge$ coincides with the lower (i. e. past-directed) half-space bounded by the polar space-like plane of p. If C is a compact convex body and p lies in the interior of C, then there exists a radius $r > 0$ such that the ball B_r of radius r centered at p is contained in C. By Lemma 1.4.2 we deduce that $p^\wedge \supset B_r^\wedge \supseteq C^\wedge$. This implies in particular that $C^\wedge$ lies in the interior of $p^\wedge$. The subset $p^\wedge \setminus C^\wedge$ is the region of $\mathrm{d}\mathbb{S}^3$ bounded from below by $\partial(C^\wedge)$ and from above by the polar plane to p. Since C is compact, we can find a R-ball B_R at p containing C. Again by Lemma 1.4.2 we have
$$p^\wedge \setminus B_r^\wedge \subseteq p^\wedge \setminus C^\wedge \subseteq p^\wedge \setminus B_R^\wedge.$$
It is immediate to check that $p^\wedge \setminus B_R^\wedge$ is compact, therefore the same holds for $p^\wedge \setminus C^\wedge$. This proves that $0 < V_1^*(C) < \infty$. In fact, the same kind of argument shows that V_1^* is *monotonic increasing* with respect to the inclusion. Contrary to the standard hyperbolic volume, V_1^* is not additive, as one can easily see by considering, for instance, two simplices glued along a face to build a convex polytope (see relation (1.8) below).

We will see in Remark 1.7.10 a proof of the independence of V_1^* and V_2^* on the chosen point p and plane S, respectively. The request of $\mathscr{C}^{1,1}$-regularity of the boundary is technical and it will appear later when we will consider variation formulae. Observe that all the results in the previous subsection hold also in the $\mathscr{C}^{1,1}$-case, up to replacing any equality with an equality almost everywhere whenever order 2 derivatives are involved (e. g. H, B, $I\!I$ and $I\!I\!I$).

The remainder of this subsection will be dedicated to the proof of the equivalence of these quantities. More precisely, we will see in Proposition 1.7.13 that, for every compact convex body in $\mathbb{H}^3$ with $\mathscr{C}^{1,1}$-boundary and with positive definite second fundamental form, we have
$$V_1^*(C) = V_2^*(C) = V_3^*(C).$$
Therefore, combining this with Proposition 1.7.14, we will be allowed to give the following definition:

Definition 1.7.7. Let C be a compact convex body in $\mathbb{H}^3$. We define the *dual volume* of C to be $\mathrm{Vol}^*(C) := V_1^*(C)$. If C has $\mathscr{C}^{1,1}$-boundary, we can equivalently set $\mathrm{Vol}^*(C) := V_i^*(C)$, $i = 1, 2, 3$.

Before going through the details, we want to make some remarks about the convenience of these different descriptions. The definition V_1^* is useful because it does not require the convex body to have $\mathscr{C}^{1,1}$-boundary. The expression V_2^* will be convenient to show the independence of V_1^* on the chosen point p. Lastly, the third definition gives an explicit link between the notions of dual and standard volumes in terms of the geometry of the boundary of the domain. In addition, V_3^* can be trivially extended to the case of convex subsets with regular boundary sitting inside a general 3-dimensional hyperbolic manifold, as we will do in Definition 2.2.1.

Lemma 1.7.8. *For any choice of space-like planes S, S' we have:*

$$\int_S \omega_{S'} = 0.$$

Proof. Let $F \colon \mathrm{d}\mathbb{S}^3 \to \mathrm{d}\mathbb{S}^3$ be the antipodal map, i. e. $F(v) = -v$ for all $v \in \mathrm{d}\mathbb{S}^3$. Since the subspaces of $\mathrm{d}\mathbb{S}^3$ are intersections of vector subspaces of $\mathbb{R}^{3,1}$ with $\mathrm{d}\mathbb{S}^3$, every subspace of $\mathrm{d}\mathbb{S}^3$ is invariant under F. The degree of F as a diffeomorphism of $\mathrm{d}\mathbb{S}^3$ is equal to $(-1)^4 = 1$, while the degree of the restriction of F on a plane in $\mathrm{d}\mathbb{S}^3$ is equal to $(-1)^3 = -1$. Moreover, we observe that, if t_S is the signed distance from S', then we have $t_{S'} \circ F = -t_{S'}$. Then

$$\begin{aligned} F^* \omega_{S'} &= \left(\int_0^{t_S \circ F} \cosh^2 \rho \, \mathrm{d}\rho \right) F^* \, \mathrm{dvol}_{\mathbb{S}^2} \\ &= \left(\int_0^{-t_S} \cosh^2 \rho \, \mathrm{d}\rho \right) (-1) \, \mathrm{dvol}_{\mathbb{S}^2} \\ &= \omega_{S'}. \end{aligned}$$

Now, using this relation and the fact that F has degree -1 on S, we get

$$\int_S \omega_{S'} = - \int_S F^* \omega_{S'} = - \int_S \omega_{S'},$$

and so $\int_S \omega_{S'} = 0$, as desired. ☐

Corollary 1.7.9. *For every compact convex body C in $\mathbb{H}^3$ with $\mathscr{C}^{1,1}$-boundary we have*

$$V_1^*(C) = V_2^*(C).$$

Proof. The proof goes as follows:

$$\begin{aligned} V_1^*(C) &:= \int_{p^\wedge \setminus C^\wedge} \mathrm{dvol}_{\mathrm{d}\mathbb{S}^3} = \int_{p^\wedge \setminus C^\wedge} \mathrm{d}\omega_S \\ &= \int_{\partial_+ (p^\wedge) \sqcup \partial_- (C^\wedge)} \omega_S = - \int_{\partial_+ (C^\wedge)} \omega_S =: V_2^*(C). \end{aligned}$$

The first equality holds by definition of the 2-form ω_S; the second one is simply an application of the Stokes' Theorem, where the signs $+$ and $-$ stand for future and past-oriented, respectively; in the third one we are using the fact that $\partial(p^\wedge)$ is a plane, therefore $\int_{\partial_+(p^\wedge)} \omega_S$ vanishes by Lemma 1.7.8. ☐

Remark 1.7.10. The chain of equalities in the previous proof shows at the same time that $V_1^*(C)$ does not depend on the choice of p (since it is equal to $-\int_{\partial(C^\wedge)} \omega_S$), and $V_2^*(C)$ does not depend on the choice of S (since it is equal to $\int_{p^\wedge \setminus C^\wedge} \mathrm{dvol}_{\mathrm{d}\mathbb{S}^3}$).

In fact, the proof of Corollary 1.7.9 generalizes to $\mathbb{H}^n$, $\mathrm{d}\mathbb{S}^n$ for any $n \geq 3$. On the contrary, the equality between V_3^* and the other two definitions is specific of the 3-dimensional case (see [SS03] for higher dimensional analogues).

In order to prove that V_3^* coincides with $V_1^* = V_2^*$, we will use an analytic approach based on the differential Schläfli formula (see Theorem 1.7.2). In particular, we will need the following:

Lemma 1.7.11. *Let $(\Sigma_t)_t$ be a smooth 1-parameter family of embedded surfaces in $\mathbb{H}^3$, with positive definite second fundamental forms, and let Σ_t^* be the dual surface of Σ_t^*, obtained following the construction described in Section 1.4. Then the variation of the volume in $\mathrm{d}\mathbb{S}^3$ bounded by the surfaces Σ_t^* can be expressed as*

$$\delta \mathrm{Vol}_{\mathrm{d}\mathbb{S}^3} = \frac{1}{4} \int_{\Sigma} (\delta I, HI - I\!I) \, \mathrm{d}a \, ,$$

where $\Sigma = \Sigma_0$.

Proof. By Theorem 1.7.2 we have:

$$\delta \mathrm{Vol}_{\mathrm{d}\mathbb{S}^3} = -\frac{1}{2} \int_{\Sigma^*} \left(\delta H^* + \frac{1}{2} (\delta I^*, I\!I^*)^* \right) \mathrm{d}a^* \, .$$

Here we are using the fact that the de Sitter space has constant sectional curvature equal to $+1$. To prove the statement, we will apply Proposition 1.4.4 and we will translate this expression on Σ^* in terms of one on Σ. By Proposition 1.4.4 we have that $H^* = -\operatorname{tr}(B^{-1}) = -\frac{H}{\det B}$ and $\mathrm{d}a^* = \det B \, \mathrm{d}a$ ($\det B$ is everywhere different from 0 because the second fundamental form is non-degenerate). Therefore, we can compute the variation of the mean curvature as follows:

$$\begin{aligned} \delta H^* &= -\delta \left(\frac{H}{\det B} \right) = -\frac{\operatorname{tr}(\delta B)}{\det B} + \frac{\operatorname{tr}(B) \operatorname{tr}(B^{-1} \delta B)}{\det B} \\ &= \det B^{-1} \left(\operatorname{tr}(B^{-1} \delta B) \operatorname{tr}(B) - \operatorname{tr}(\delta B) \right) \\ &= \operatorname{tr}(B^{-1} \delta B \, B^{-1}), \end{aligned}$$

where in the last step we used the identity

$$\operatorname{tr}(M^{-1} N) = \det M^{-1} \left(\operatorname{tr}(M) \operatorname{tr}(N) - \operatorname{tr}(MN) \right) \qquad \forall M, N \in \mathrm{GL}(2, \mathbb{R}), \quad (1.6)$$

for $M = B$ and $N = B^{-1} \delta B$. Using the relation $\operatorname{tr}(MN) = \operatorname{tr}(NM)$ and the fact that B is I-selfadjoint, we see that

$$(\delta I^*, I\!I^*)^* = -2 \operatorname{tr}(B^{-1} \delta B \, B^{-1}) - \operatorname{tr}(B^{-1} I^{-1} \delta I).$$

On the other hand, we have:

$$\begin{aligned} (\delta I, HI - I\!I) &= \operatorname{tr}(I^{-1} \delta I \, I^{-1} (HI - I\!I)) \\ &= \operatorname{tr}(B) \operatorname{tr}(I^{-1} \delta I) - \operatorname{tr}(I^{-1} \delta I \, B) \\ &= \det B \operatorname{tr}(B^{-1} I^{-1} \delta I), \end{aligned}$$

where in the last step we used again the relation (1.6) with $M = B$ and $N = I^{-1}\delta I$. Now, putting these equalities together, we see that

$$\begin{aligned} -\frac{1}{2}\left(\delta H^* + \frac{1}{2}(\delta I^*, I\!I^*)^*\right) \mathrm{d}a^* &= -\frac{1}{2}\left(-\frac{1}{2}\operatorname{tr}(B^{-1}I^{-1}\delta I)\right) \mathrm{d}a^* \\ &= \frac{1}{4}\operatorname{tr}(B^{-1}I^{-1}\delta I)\det B \,\mathrm{d}a \\ &= \frac{1}{4}(\delta I, HI - I\!I)\,\mathrm{d}a\,, \end{aligned}$$

as desired. □

Remark 1.7.12. We observe that the variation formula from Lemma 1.7.11, as the differential Schläfli formulae in Section 1.7.1, actually make sense for infinite volume domains, as long as their boundary is compact. In such case, the relations express the variation of the volume, which is still finite.

Proposition 1.7.13. *The three definitions of the dual volume given above coincide on all compact convex bodies in $\mathbb{H}^3$ with $\mathscr{C}^{1,1}$-boundary and positive definite second fundamental form.*

Proof. In order to prove the remaining equality, we first show that V_1^* and V_3^* have the same variation formula. Let C_t be a differentiable family of compact convex bodies in $\mathbb{H}^3$ with $\mathscr{C}^{1,1}$-boundary and positive definite second fundamental forms. If p lies in the interior of C_0, then it is an internal point of C_t for small values of t. In particular p can be used to define $V_1^*(C_t) = \mathrm{Vol}_{\mathrm{d}\mathbb{S}^3}(p^\wedge \setminus (C_t)^\wedge)$ whenever t is sufficiently close to 0. Since p is fixed, the only component of the boundary that is varying is $\partial(C_t^\wedge)$. Applying Lemma 1.7.11, we get

$$\left.\frac{\mathrm{d}V_1^*(C_t)}{\mathrm{d}t}\right|_{t=0} = \frac{1}{4}\int_{\partial C_0} (\delta I, HI - I\!I)\,\mathrm{d}a\,.$$

On the other side, by Theorem 1.7.2, the variation of V_3^* is

$$\left.\frac{\mathrm{d}V_3^*(C_t)}{\mathrm{d}t}\right|_{t=0} = -\frac{1}{2}\int_{\partial C_0}\left(\delta H + \frac{1}{2}(\delta I, I\!I)\right)\mathrm{d}a + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial C_t} H_t\,\mathrm{d}a_t\bigg|_{t=0}.$$

In local coordinates (x^1, x^2) the volume form can be written as $\sqrt{\det((g_t)_{ij})}\,\mathrm{d}x^1\wedge \mathrm{d}x^2$, where $\det((g_t)_{ij})$ denotes the determinant of the matrix representing g_t with respect to the basis ∂_1, ∂_2. The differential of the function det at a point $A \in \mathrm{GL}(2,\mathbb{R})$ verifies $\mathrm{d}(\det)_A(H) = \det A \operatorname{tr}(A^{-1}H)$. Using this expression combined with the relation $(\delta I, I) = \operatorname{tr}(I^{-1}\delta I)$, we see that the variation of $H_t\,\mathrm{d}a_t$ is given by $(\delta H + \frac{H}{2}(\delta I, I))\,\mathrm{d}a$. Therefore we obtain

$$\left.\frac{\mathrm{d}V_3^*(C_t)}{\mathrm{d}t}\right|_{t=0} = \frac{1}{4}\int_{\partial C_0} (\delta I, HI - I\!I)\,\mathrm{d}a\,,$$

which proves the equality between the derivatives in t of $V_1^*(C_t)$ and $V_3^*(C_t)$.

Assuming that any convex body with $\mathscr{C}^{1,1}$-boundary and $I\!I > 0$ can be differentiably deformed, through convex bodies with $\mathscr{C}^{1,1}$-boundary and $I\!I > 0$, into a small geodesic ball, it would be enough to show that V_1^* and V_3^* coincide on any geodesic ball of $\mathbb{H}^3$.

A way to prove the first claim is to reduce the problem to the Euclidean setting, and then perform the deformation using convex combinations. To do so, we work in the projective model of the hyperbolic space, instead of the hyperboloid model described at the beginning of Section 1.4. We recall that the projective model can be described, in a suitable affine chart of $\mathbb{RP}^n$, as the interior of a Euclidean open ball B centered at the origin. In this description, the half-spaces are nothing but intersections of B with Euclidean half-spaces. It follows that any compact convex body C of $\mathbb{H}^3$ corresponds to an Euclidean compact convex body lying inside B. Up to acting by isometries of $\mathbb{H}^3$, we can always assume the origin $0 \in B$ to be contained the interior of C. It is enough to show that there exists a differentiable deformation of convex $\mathscr{C}^{1,1}$-surfaces $(\Sigma_t)_t$ such that $I\!I_t > 0$, $\Sigma_0 = \partial C$ and $\Sigma_1 = \partial D_r$, where D_r is a small closed disk of radius r centered at 0 and contained in the interior of C. To do so, we consider $t \mapsto N^E_{tr}((1-t) \cdot C)$ (where $s \cdot C := \{sx \mid x \in C\} \subset \mathbb{R}^3$), as t varies in $I = [0,1]$. Here $N^E_\varepsilon(X)$ stands for the ε-Euclidean neighborhood of X in $\mathbb{R}^3$. Since the boundary of $(1-t) \cdot C$ is $\mathscr{C}^{1,1}$ and it has positive definite second fundamental form for all $t \neq 1$, the same properties hold for boundary of $N^E_{tr}((1-t) \cdot C)$. At time $t = 0$ we have $N_0 C = C$, and at $t = 1$ $N^E_r(0) = D_r$. It is not difficult to see that the boundaries $\partial N^E_{tr}((1-t) \cdot C)$ are varying differentiably in t, and therefore that this deformation satisfies the required conditions.

It remains to show that for any geodesic ball $B_\varepsilon = B_\varepsilon(p)$ of radius ε in $\mathbb{H}^3$ we have $V_1^*(B_\varepsilon) = V_3^*(B_\varepsilon)$. Working in the hyperboloid model of $\mathbb{H}^3$ introduced at the beginning of Section 1.4, we can assume p to be equal to $e_4 \in \mathbb{H}^3 \subset \mathbb{R}^{3,1}$. In what follows, we work in the coordinate system $(x,t) \in S^2 \times \mathbb{R}$ of $\mathrm{d}\mathbb{S}^3$ introduced in Section 1.7.2, with $S = S^2 \times \{0\} \subset \mathrm{d}\mathbb{S}^3$ and $t = t_S$. A simple computation shows that the dual convexes $p^\wedge$ and $B_\varepsilon^\wedge$ satisfy the following:

$$\begin{aligned} p^\wedge &= \{x \in \mathrm{d}\mathbb{S}^3 \mid t(x) < 0\}, \\ B_\varepsilon^\wedge &= \{x \in \mathrm{d}\mathbb{S}^3 \mid t(x) < -\varepsilon\}. \end{aligned}$$

Using the equality $\mathrm{dvol}_{\mathrm{d}\mathbb{S}^3} = \cosh^2 t \, \mathrm{dvol}_{\mathbb{S}^2}$, we obtain:

$$\begin{aligned} V_1^*(B_\varepsilon) &= \mathrm{Vol}_{\mathrm{d}\mathbb{S}^3}(p^\wedge \setminus B_\varepsilon^\wedge) \\ &= \int_{-\varepsilon}^0 \cosh^2 t \, \mathrm{d}t \ \ \mathrm{Vol}(\mathbb{S}^2) \\ &= \frac{1}{2}\left(\frac{\sinh 2\varepsilon}{2} + \varepsilon\right) \mathrm{Vol}(\mathbb{S}^2). \end{aligned}$$

On the other side, choosing the normal vector field on ∂B_ε to point inward, we see that the following relations hold:

$$\begin{aligned} I_\varepsilon &= \sinh^2 \varepsilon \, g_{\mathbb{S}^2}, \\ I\!I_\varepsilon &= \coth \varepsilon \, I_\varepsilon, \\ \mathrm{Vol}(B_\varepsilon) &= \int_0^\varepsilon \sinh^2 t \, \mathrm{d}t \ \ \mathrm{Vol}(\mathbb{S}^2). \end{aligned}$$

Therefore we have:

$$\begin{aligned} V_3^*(B_\varepsilon) &= -\int_0^\varepsilon \sinh^2 t \, \mathrm{d}t \ \ \mathrm{Vol}(\mathbb{S}^2) + \frac{1}{2}\,(2\coth\varepsilon)\sinh^2\varepsilon \ \mathrm{Vol}(\mathbb{S}^2) \\ &= \frac{1}{2}\left(-\frac{\sinh 2\varepsilon}{2} + \varepsilon + \sinh 2\varepsilon\right)\mathrm{Vol}(\mathbb{S}^2), \end{aligned}$$

and therefore $V_1^*(B_\varepsilon) = V_3^*(B_\varepsilon)$ for every $\varepsilon > 0$. □

Let $\mathcal{CB}_c$ denote the family of compact convex bodies of $\mathbb{H}^3$ endowed with the Hausdorff distance $d_\mathcal{H}$, defined as:

$$d_\mathcal{H}(C, D) := \inf\{\varepsilon > 0 \mid N_\varepsilon C \supseteq D \text{ and } C \subseteq N_\varepsilon D\},$$

where $N_\varepsilon X$ stands for the ε-neighborhood of X.

Proposition 1.7.14. *The function $V_1^* \colon \mathcal{CB}_c \to \mathbb{R}_{\geq 0}$ is continuous.*

Proof. By definition of the Hausdorff distance, it is enough to prove that, for any compact convex body C we have

$$\lim_{\varepsilon \to 0} V_1^*(N_\varepsilon C) = V_1^*(C).$$

Since $(N_\varepsilon C)^\wedge$ is the ε-neighborhood of $C^\wedge$ with respect to the time-like distance from $C^\wedge$, the fact follows from the continuity of $\mathrm{Vol}_{\mathrm{d}\mathbb{S}^3}$ with respect to the Hausdorff distance in $\mathrm{d}\mathbb{S}^3$. Alternatively, the same argument of [BBB19, Proposition 3.4] applies, where now the corresponding metric at infinity is defined on the full Riemann sphere $\mathbb{CP}^1$. □

Proposition 1.7.14 implies that the dual volume of a compact convex body can be approximated by the dual volume of strictly convex bodies with $\mathscr{C}^{1,1}$-boundary which converge to C with respect to the Hausdorff distance. For the existence of such a sequence, we can consider $C_n := N_{1/n} C$, for $n \in \mathbb{N} \setminus \{0\}$, as observed in Remark 2.1.2. This shows the consistence of the different definitions V_i^* we gave initially.

1.7.3 The dual Schläfli formula

Applying Proposition 1.7.13, we can easily deduce a *dual Schläfli formula* (see [San04] and [Suá00]) for the dual volume of a polyhedron in $\mathbb{H}^3$. Let $(P_t)_{t \in (-\varepsilon, \varepsilon)}$ be a 1-parameter family of convex compact polyhedra, whose vertices are varying smoothly in t. Consider the convex body $N_\varepsilon P_t$ given by the set of points at distance $\leq \varepsilon$ from P_t (which has $\mathscr{C}^{1,1}$ boundary). In [RS00] it is proved that:

$$\lim_{\varepsilon \to 0} \int_{(P_t)_\varepsilon} H_{t,\varepsilon} \, \mathrm{d}a_{t,\varepsilon} = \sum_{\substack{e_t \text{ edge} \\ \text{of } P_t}} \ell(e_t)\, \theta(e_t). \tag{1.7}$$

Therefore the integral of the mean curvature can be considered as the analogous, in the $\mathscr{C}^{1,1}$-case, of the weighted length of the codimension 1 bending locus of ∂P, where the weights are given by the exterior dihedral angles along the edges.

Using the relation above and the description of the dual volume given by V_3^*, we deduce that

$$V_t^* := \mathrm{Vol}^*(P_t) = -\mathrm{Vol}(P_t) + \frac{1}{2} \sum_{\substack{e \text{ edge} \\ \text{of } P_t}} \ell(e_t)\, \theta(e_t). \tag{1.8}$$

Now, differentiating this relation in t and applying the classical Schläfli formula from Theorem 1.7.1, we obtain

$$\begin{aligned} \dot{V}^* &= -\frac{1}{2} \sum_{\substack{e \text{ edge} \\ \text{of } P}} \ell(e)\, \dot{\theta}(e) + \frac{1}{2} \sum_{\substack{e \text{ edge} \\ \text{of } P}} \left(\dot{\ell}(e)\, \theta(e) + \ell(e)\, \dot{\theta}(e) \right) \\ &= \frac{1}{2} \sum_{\substack{e \text{ edge} \\ \text{of } P}} \dot{\ell}(e)\, \theta(e). \end{aligned}$$

Therefore, the variation of the dual volume of $(P_t)_t$ is in fact the "dual" of the variation of the hyperbolic volume of $(P_t)_t$, in the sense that, instead of involving the variation of the angles along the edge $e = e_0$ and the length of e, we have the variation of the length of e and the angle along e.

Remark 1.7.15. We highlight that the expression found above for $\dot{V}^*$ holds true also for variations of polyhedra $(P_t)_t$ along which the combinatorial structure is not preserved. Indeed, if an edge e_t of P_t collapses into a face at $t = 0$, its dihedral angle $\theta(e)$ in $P = P_0$ is 0, and therefore the variation $\dot{\theta}(e)$ does not contribute to $\dot{V}^*$.

We summarize the observations above in the following statement:

Theorem 1.7.16 (Dual Schläfli formula)**.** *Let $(P_t)_t$ be a 1-parameter family of convex compact polyhedra in $\mathbb{H}^3$, whose vertices vary smoothly in t, with $P = P_0$. Then the function $t \mapsto V_t^* := \mathrm{Vol}^*(P_t)$ admits derivative at $t = 0$, and it verifies*

$$\dot{V}^* = \frac{1}{2} \sum_{\substack{e \text{ edge} \\ \text{of } P}} \dot{\ell}(e)\, \theta(e),$$

where the sum is taken over the set of edges e of P, $\theta(e)$ denotes the exterior dihedral angle along e in P and $\dot{\ell}(e)$ is the variation of the length of e_t P_t at $t = 0$.

Chapter 2

The dual Bonahon-Schläfli formula

Outline of the chapter

This chapter is dedicated to the proof of the *dual Bonahon-Schläfli formula*:

Theorem A. *Let $(M_t)_t$ be a smooth 1-parameter family of quasi-isometric convex co-compact hyperbolic structures on a fixed underlying topological 3-manifold. Then there exists the derivative of the dual volume of the convex core along the path $(M_t)_t$, and it satisfies:*

$$\mathrm{d}V_C^*(\dot{M}) = -\frac{1}{2}\,\mathrm{d}L_\mu(\dot{m}),$$

where μ is the bending measure of the boundary of the convex core of $M = M_0$, and $\dot{m}$ denotes the variation of the hyperbolic structures on the boundary of the convex cores of M_t at $t = 0$.

In the following we describe the strategy of the proof and we outline the structure of the chapter. The general idea will be to deduce the statement from the combination of the so-called *differential Schläfli formula*, proved by Rivin and Schlenker [RS99], with a careful approximation argument of the boundary of the convex core by smoother surfaces. As already mentioned in Section 1.3.1, the boundary of convex core of a convex co-compact hyperbolic manifold M is far from being smooth, and the understanding of the variation of its geometry is a subtle problem, which intrinsically involves technical difficulties. In order to do not go through the same sophisticated (but necessary) analysis of Bonahon in the study of the standard volume function (see [Bon98a]), it will be essential to make use of the peculiarities of the dual volume. In particular, we highlight two phenomena which will play an important role in our argument:

- the dual volume of the ε-neighborhood of the convex core $\mathrm{Vol}^*(N_\varepsilon CM)$ satisfies
$$\mathrm{Vol}^*(N_\varepsilon CM) - \mathrm{Vol}^*(CM) = O(\varepsilon^2),$$
rather than $O(\varepsilon)$, as happens for the standard hyperbolic volume;

- the variation of the dual volume enclosed by a surface S depends on the variation of the first fundamental form δI of S, but *not* on the variation of its higher order geometric quantities, as the mean curvature, the extrinsic curvature or the second fundamental form (compare Theorem 1.7.2 or 1.7.4 with Proposition 2.2.5).

The first property is specific of the convex core, and it is due to the fact that the boundary of the convex core is almost everywhere totally geodesic, and it is bend along complete geodesics (in particular, there are no "vertices"). The second property is reminiscent of a feature of the dual Schläfli formula, already highlighted in Remark 1.7.15: if we want to express the variation of the dual volume of a family of polyhedra $(P_t)_t$, we do not need to require the combinatorial structure of P_t to be preserved along the deformation. This suggests a higher flexibility of the dual volume function, which will be useful in our approximation procedure (see in particular Proposition 3.3.1).

The chapter is organized as follows. In Section 2.1 we will briefly describe the geometry of equidistant surfaces from a totally geodesic plane and from a line, which are the basic ingredients to understand the shape of equidistant surfaces from the boundary of the convex core CM. We will also give a more detailed description of the geometry of the boundary of the convex core ∂CM, through the notion of *pleated surfaces*. This will allow us to give a fairly technical but useful procedure to locally approximate ∂CM by finitely bent surfaces. Section 2.2 is dedicated to the notion of dual volume of convex domains of M, and the description of its properties. In Section 2.3 we develop a formula that expresses $\frac{\mathrm{d}}{\mathrm{d}t}\ell_{M_t}(\mu)$, the derivative of the length of the realization of a fixed measured lamination μ inside a 1-parameter family $(M_t)_t$ of convex co-compact manifolds (see Proposition 2.3.3).

Section 2.4 is the central part of our proof. Firstly we will approximate the convex cores CM_t by their ε-neighborhoods $N_\varepsilon CM_t$. Fixing the underlying topological space and varying the hyperbolic structures M_t regularly enough, we will study for which values of ε and t the surfaces $N_\varepsilon CM_0$ remain convex with respect to the structure of M_t. This will allow us to estimate the dual volumes of the convex cores CM_t with the dual volumes of the regions $N_\varepsilon CM_0$ (see Lemma 2.4.1). Here the key properties that will play a role are the minimality of the convex core among all convex subsets, and the monotonicity of the dual volume with respect to the inclusion. In this way we will be able to deduce the variation of the dual volume of the convex core from the one of a more regular family of convex regions, on which in particular we will be allowed to apply the differential Schläfli formula of [RS99]. At this point we will see how the two properties that we mentioned above will play a role. The final outcome of this argument will be Proposition 3.3.1, which states that there exists the derivative of the dual volume function $V_C^*(M_t)$, and it satisfies

$$\mathrm{d}V_C^*(\dot{M}) = -\frac{1}{2}\left.\frac{\mathrm{d}\ell_{M_t}(\mu)}{\mathrm{d}t}\right|_{t=0},$$

where $\mu = \mu_0$ is the bending measure of the boundary of the convex core of $M = M_0$, and $\ell_{M_t}(\mu)$ represents the length of the realization of μ inside the 3-manifold M_t. This result will be achieved without making *any* use of Bonahon's Hölder cocycles machinery.

At the very end, in order to relate the term $\frac{\mathrm{d}}{\mathrm{d}t}\ell_{M_t}(\mu)|_{t=0}$ with $\mathrm{d}L_\mu(\dot{m})$, i. e. the differential of the μ-length function over the Teichmüller space $\mathcal{T}^{\mathfrak{h}}(\partial CM)$ applied to the variation of the hyperbolic structures on ∂CM, we will need Bonahon's results about the $\mathscr{C}^1$-dependence of the hyperbolic metric on the boundary of the convex core with respect to the convex co-compact structure of M (see [Bon98b, Theorem 1]).

2.1 Convex co-compact manifolds

In this section, we will state some geometric properties of equidistant surfaces from planes and lines in $\mathbb{H}^3$, we will introduce the notion of pleated surfaces and we will describe a procedure to locally approximate the the boundary of the (lift to $\mathbb{H}^3$ of the) convex core of a convex co-compact hyperbolic manifold by finitely bent surfaces. These will be useful technical ingredients for the rest of our exposition. We refer to Section 1.3 for an introduction to the notion of convex co-compact hyperbolic manifolds and the properties of their convex cores.

Definition 2.1.1. If A is a subset of a metric space (X, d), the *ε-neighborhood of A in X*, which will be denoted by $N_\varepsilon A$, is the set of points of X at distance $\leq \varepsilon$ from A. The *ε-surface of A in X*, which will be denoted by $S_\varepsilon A$, is the set of points of X at distance ε from A.

Remark 2.1.2. If C is a closed convex subset in $\mathbb{H}^3$, then the surfaces $S_\varepsilon C$ are strictly convex $\mathscr{C}^{1,1}$-surfaces. Indeed, the distance function $d(C, \cdot)\colon \mathbb{H}^3 \to \mathbb{R}_{\geq 0}$ is continuously differentiable on $\mathbb{H}^3 \setminus C$ (see [CEM06, Lemma II.1.3.6]) and its gradient is uniformly Lipschitz on

$$\overline{N_\varepsilon C \setminus N_{\varepsilon'} C}$$

for all $\varepsilon > \varepsilon' > 0$ (see [CEM06, Section II.2.11]). In particular, the equidistant surfaces from the convex core of a convex co-compact hyperbolic manifold M are $\mathscr{C}^{1,1}$-surfaces.

Given Σ an immersed surface inside a hyperbolic 3-manifold M, we denote by I and $I\!I$ its first and second fundamental forms, respectively, as introduced in Section 1.4. Wherever we have to deal with surfaces that are boundaries of domains or with portions of ε-surfaces, we will always endow them with the interior normal vector field pointing towards the domain or the ε-neighborhood, respectively.

Lines and *planes* in $\mathbb{H}^3$ are 1 and 2-dimensional totally geodesic subspaces of $\mathbb{H}^3$, respectively. A *half-space* is the closure on one of the complementary regions of a plane inside $\mathbb{H}^3$. In the following we recall the geometric data of the equidistant surfaces from a plane and a line, respectively. For a proof of them, we refer for instance to [CEM06, Chapter II.2].

Lemma 2.1.3. *Let P be a plane in $\mathbb{H}^3$, and fix ν a unit normal vector field on P. Then the map $\eta_\varepsilon\colon P \to \mathbb{H}^3$, defined by*

$$\eta_\varepsilon(p) := \exp_p(\varepsilon \nu(p)),$$

parametrizes a connected component of the ε-surface from the hyperbolic plane P in $\mathbb{H}^3$, and in these coordinates we have

$$I_\varepsilon = \cosh^2 \varepsilon \, g_P,$$
$$I\!I_\varepsilon = \frac{\sinh 2\varepsilon}{2} g_P = \tanh \varepsilon \, I_\varepsilon,$$

where we are choosing as unit normal vector field the one pointing towards P.

Lemma 2.1.4. *Let $\tilde{\gamma} \colon \mathbb{R} \to \mathbb{H}^3$ be a unit speed complete geodesic, and denote by $e_1(s)$, $e_2(s) \in T_{\tilde{\gamma}(s)}\mathbb{H}^3$ the tangent vectors at $\tilde{\gamma}(s)$ obtained by parallel transport of a fixed orthonormal basis e_1, e_2 of $\tilde{\gamma}'(0)^\perp \subset T_{\tilde{\gamma}(0)}\mathbb{H}^3$. Then the map $\psi_\varepsilon \colon \mathbb{R} \times S^1 \to \mathbb{H}^3$, defined by*

$$\psi_\varepsilon(s, e^{i\theta}) := \exp_{\tilde{\gamma}(s)}(\varepsilon(\cos\theta \, e_1(s) + \sin\theta \, e_2(s))),$$

parametrizes the ε-surface from the line $\tilde{\gamma}$ and in these coordinates we have

$$I_\varepsilon = \cosh^2 \varepsilon \, \mathrm{d}s^2 + \sinh^2 \varepsilon \, \mathrm{d}\theta^2 \, ,$$
$$I\!I_\varepsilon = \cosh \varepsilon \sinh \varepsilon \, (\mathrm{d}s^2 + \mathrm{d}\theta^2),$$

where we are choosing as unit normal vector field the one pointing outwards the ε-neighborhood of $\tilde{\gamma}$.

Let now M denote a convex co-compact hyperbolic 3-manifold with convex core CM (see Section 1.3). We want to give a more precise description of the structure of the boundary of the convex core and, to do so, we need to recall the following notion:

Definition 2.1.5 ([Bon96])**.** Let S be a topological surface. A *(abstract) pleated surface* with topological type S is a pair $(\tilde{f}, \rho)$, where $\tilde{f} \colon \widetilde{S} \to \mathbb{H}^3$ is a continuous map from the universal cover $\widetilde{S}$ of S to $\mathbb{H}^3$, and $\rho \colon \pi_1(S) \to \mathrm{Iso}^+(\mathbb{H}^3)$ is a homomorphism, verifying the following properties:

1. $\tilde{f}$ is ρ-equivariant;
2. the path metric on $\widetilde{S}$, obtained by pullback of the metric on $\mathbb{H}^3$ under $\tilde{f}$, induces a hyperbolic metric m on S;
3. there exists a m-geodesic lamination on S such that $\tilde{f}$ sends every leaf of its preimage $\tilde{\lambda} \subset \widetilde{S}$ in a geodesic of $\mathbb{H}^3$, and $\tilde{f}$ is totally geodesic embedding on each complementary region of $\tilde{\lambda}$ in $\widetilde{S}$.

Let $\widetilde{C}$ be the preimage of the convex core CM inside $\mathbb{H}^3 \cong \widetilde{M}$. Its boundary $\partial\widetilde{C}$ is parametrized by a pleated surface $\tilde{f} \colon \widetilde{S} \to \mathbb{H}^3$ with bending locus $\tilde{\lambda}$, where $\widetilde{S}$ is the universal cover of ∂CM, and with holonomy ρ given by the composition of the homomorphism induced by the inclusion $\partial CM \to M$ and the holonomy representation of M. In this situation, the pleated surface $\tilde{f}$ is *locally convex*, in the sense that the bending occurs always in the same direction, making $\tilde{f}$ locally bound a convex region (see also [CEM06, Section II.1.11]). In general $\tilde{f}$ is a covering of $\partial\widetilde{C}$, which is non-trivial whenever CM has compressible boundary.

It will be useful in our analysis to have a way to locally approximate ∂CM by finitely bent surfaces. We briefly recall a procedure described in [Bon96, Section 7] which is well suited for our purpose. We start by considering an arc k in $\widetilde{S}$ transverse to the bending lamination $\tilde{\lambda}$, having endpoints in two different flat pieces P and Q of $\widetilde{S} \setminus \tilde{\lambda}$. We will assume k to be short enough, so that we can find an open neighborhood U of k on which $\tilde{f}$ is a topological embedding, and all the leaves of $\tilde{\lambda}$ meeting U intersect k. When this happens, we say that $\tilde{f}$ is a *nice embedding near* k. Let $\mathcal{P}_{PQ}$ be the set of those flat pieces in $\widetilde{S} \setminus \tilde{\lambda}$ that separate P from Q. For every finite subset $\mathcal{P}$ of $\mathcal{P}_{PQ}$, we label its elements by $P_0, \dots, P_{n+1}$ following the order from $P = P_0$ to $Q = P_{n+1}$. Let Σ_i be the closure of the region in $\widetilde{S}$ which lies between P_i and P_{i+1}, for $i = 0, \dots, n$. If we orient the two leaves γ_i γ_i' lying in $\partial \Sigma_i$ accordingly, so that they can be deformed continuously from one to the other though oriented geodesics in Σ_i, then we call *diagonals* of Σ_i the two unoriented lines in Σ_i that connect two opposite endpoints of γ_i and γ_i'.

We denote by $\tilde{\lambda}_{\mathcal{P}}$ the geodesic lamination of $\widetilde{S}$ obtained from $\tilde{\lambda}$ as follows: we maintain the geodesic lamination as it is outside $\bigcup_i \Sigma_i$ and, for every $i = 0, \dots, n$, we erase all the leaves lying in the interior of the strip Σ_i and we replace them by one of the two diagonals of Σ_i, say d_i. Now we define a pleated surface $\tilde{f}_{\mathcal{P}} \colon \widetilde{S} \to \mathbb{H}^3$, with bending locus $\tilde{\lambda}_{\mathcal{P}}$, so that it coincides with $\tilde{f}$ outside the strips, and inside any Σ_i it sends the chosen d_i in the geodesic of $\mathbb{H}^3$ joining the endpoints of $\tilde{f}(\partial \Sigma_i)$ corresponding to the endpoints of d_i. Once we make a choice of a diagonal d_i for any i, there is a unique way to extend $\tilde{f}_{\mathcal{P}}$ on $\widetilde{S}$ so that is becomes a pleated surface bent along $\tilde{\lambda}_{\mathcal{P}}$. Moreover, if the strips Σ_i are thin enough and if the starting $\tilde{f}$ is locally convex, then we can make a choice of the diagonals $d_0, \dots, d_n$ so that the resulting $\tilde{f}_{\mathcal{P}}$ is still locally convex. Such $\tilde{f}_{\mathcal{P}}$ will not be equivariant anymore under the action of the holonomy of $\tilde{f}$, but it will approximate the restriction of $\tilde{f}$ on U.

Now, choose a sequence of increasing subsets $\mathcal{P}_n$ exhausting $\mathcal{P}_{PQ}$ and construct a corresponding sequence of convex pleated surfaces $\tilde{f}_n := \tilde{f}_{\mathcal{P}_n}$ as above. Every such $\tilde{f}_n$ is finitely bent on the neighborhood U. Following the construction, we see that, given any P' flat piece of $\widetilde{S}$ intersecting k, there exists a large $N \in \mathbb{N}$ so that $\tilde{f}_n(P') = \tilde{f}(P') \subset \partial \widetilde{C}$ for every $n \geq N$. In particular, the functions $\tilde{f}_n$ are approximating $\tilde{f}$ over the open set U. Moreover, following the proof of [Bon96, Lemma 22], we see that the bending measures $\mu_n(k)$ of $\tilde{f}_n$ on the arc k are converging to $\mu(k)$, the bending measure of k in $\partial \widetilde{C}$.

Let now $r \colon \mathbb{H}^3 \to \widetilde{C}$ denote the metric retraction of $\mathbb{H}^3$ over the convex set $\widetilde{C}$ and let $d \colon \mathbb{H}^3 \to \mathbb{R}_{\geq 0}$ be the distance from $\widetilde{C}$. We select an open neighborhood V of k so that $\overline{V} \subset U$ and, fixed $\rho > 0$, we define $W = W(V, \rho) := r^{-1}(V) \cap N_\rho \widetilde{C}$. The surfaces $\tilde{f}_n(U)$ lie behind $\tilde{f}(U) \subset \partial \widetilde{C}$ if seen from W. Denote by $d_n \colon W \to \mathbb{R}_{\geq 0}$ the distance function from $\tilde{f}_n(U)$ on W. Since the surfaces $\tilde{f}_n(U)$ are convex, for every point $p \in W$ there exists a unique $q_n \in \tilde{f}_n(U)$ realizing $d_n(p) = d(p, q_n)$. Therefore, it makes sense to consider the metric retractions $r_n \colon W \to \tilde{f}_n(U)$, which will converge to r over the compact sets of W thanks to the convergence properties previously observed of the $\tilde{f}_n$'s. By the same argument as [CEM06, Lemma II.2.11.1], the distance functions d_n are converging $\mathscr{C}^{1,1}$-uniformly to d on any compact set of W (i. e. the gradients $\operatorname{grad} d_n$ are uniformly Lipschitz and they converge to $\operatorname{grad} d$). This shows that for every $\varepsilon < \rho$, the surface $d^{-1}(\varepsilon) \cap W = S_\varepsilon \widetilde{C} \cap W$ is $\mathscr{C}^{1,1}$-approximated by the sequence of surfaces $(d_n^{-1}(\varepsilon))_n \subset W$. Moreover, such surfaces $d_n^{-1}(\varepsilon) \subset W$

are the ε-equidistant surfaces from finitely bent convex pleated surfaces having bending measures on k converging to $\mu(k)$.

Definition 2.1.6. Given k an arc on which $\tilde{f}$ is a nice embedding, we say that the sequence $\tilde{f}_n$ defined above is a *standard approximation of* $\partial \widetilde{C}$ *near* k and that the the sequence of surfaces $S_{\varepsilon,n}$ is a *standard approximation of* $S_\varepsilon \widetilde{C}$ *over* k.

2.2 The dual volume of the convex core

This section is devoted to the definition of dual volume on convex sets sitting inside a convex co-compact 3-manifold, and its main properties.

Definition 2.2.1. Let M be a convex co-compact hyperbolic manifold. If N is a compact convex subset of M with $\mathscr{C}^{1,1}$-boundary, we define the *dual volume* of N to be

$$\mathrm{Vol}^*(N) := \mathrm{Vol}(N) - \frac{1}{2} \int_{\partial N} H \, \mathrm{d}a \, .$$

If $N = CM$, then we set $\mathrm{Vol}^*(CM) := \mathrm{Vol}(CM) - \frac{1}{2}\ell_m(\mu)$, where m and μ are the hyperbolic metric and the bending measure of ∂CM, respectively.

Remark 2.2.2. When ∂N is only $\mathscr{C}^{1,1}$, the mean curvature function is defined almost everywhere and it belongs to $L^\infty(\partial N)$ (here ∂N is endowed with the measure induced by the Riemannian volume form of its induced metric), in particular the integral $\int_{\partial N} H \, \mathrm{d}a$ is a well-defined quantity.

Observe that the definition we are using here has opposite sign with respect to the one in Chapter 1. This choice is intentional, and it is justified by the following observation. If $M \cong \Sigma \times \mathbb{R}$ is quasi-Fuchsian, the length of the bending measure $\ell_m(\mu)$ is bounded from above by a constant depending only on the genus of Σ (see Theorem 3.3.5). Consequently, we choose the sign convention in the definition of the dual volume so that $\mathrm{Vol}^*(CM)$, as a function of the space of quasi-Fuchsian manifolds, is bounded from below (instead than from above).

There is a relation between the notions of dual volume and of *W-volume*, defined in [KS08] and used to introduce the *renormalized volume* of a convex co-compact hyperbolic manifold. If N is a compact convex subset with $\mathscr{C}^{1,1}$-boundary in a convex co-compact manifold M, the W-volume of N is defined as

$$W(N) := \mathrm{Vol}(N) - \frac{1}{4} \int_{\partial N} H \, \mathrm{d}a = \frac{1}{2} \left(\mathrm{Vol}(N) + \mathrm{Vol}^*(N) \right) .$$

In addition, we mention that in [BBB19, Lemma 3.3] the authors described a way to characterize the quantity $\int_{\partial N} H \, \mathrm{d}a$ in terms of the *metric at infinity* ρ_N associated to the equidistant foliation $(S_\varepsilon N)_\varepsilon$. In this way the definition of dual volume (and of W-volume) can be given without any regularity assumption on ∂N. More precisely, they showed that

$$\int_{\partial N} H \, \mathrm{d}a = \mathrm{Area}(\rho_N) - 2 \, \mathrm{Area}(\partial N) - 2\pi \chi(\partial M).$$

We recall that the mean curvature here is the trace of the shape operator B, which is defined using the interior normal vector field to ∂N; this explains why the relation above differ by a factor 2 from the one in [BBB19]. In particular, the proof of [BBB19, Proposition 3.4] shows also:

Proposition 2.2.3. *The dual volume is continuous on the space of compact convex subsets of M with the Hausdorff topology.*

In light of this fact, the following Proposition, besides its future usefulness, justifies the definition we gave of $\mathrm{Vol}^*(CM)$.

Proposition 2.2.4. *Let M be a convex co-compact hyperbolic manifold, with convex core CM, bending lamination $\mu \in \mathcal{ML}(\partial CM)$ and hyperbolic metric m on the boundary of CM. Then, for every $\varepsilon > 0$ we have*

$$\mathrm{Vol}^*(N_\varepsilon CM) = \mathrm{Vol}^*(CM) - \frac{\ell_m(\mu)}{4}(\cosh 2\varepsilon - 1) - \frac{\pi}{2}|\chi(\partial CM)|(\sinh 2\varepsilon - 2\varepsilon).$$

As a consequence, we have

$$\mathrm{Vol}^*(N_\varepsilon CM) = \mathrm{Vol}^*(CM) + O(|\chi(\partial CM)|, \ell_m(\mu); \varepsilon^2).$$

Proof. First we study $\mathrm{Vol}(N_\varepsilon CM) - \mathrm{Vol}(CM)$. Let λ be the support of μ and let $r' \colon N_\varepsilon CM \to CM$ be the restriction of the metric retraction. We divide $N_\varepsilon CM \setminus CM$ in two regions, $(r')^{-1}(\partial CM \setminus \lambda)$ and $(r')^{-1}(\lambda)$.

If F is the interior of a flat piece in ∂CM, then the portion of $N_\varepsilon CM$ which retracts onto F through r' has volume equal to

$$\int_0^\varepsilon \int_F \cosh^2 t \, \mathrm{dvol}_{\mathbb{H}^2} \, \mathrm{d}t = \frac{\mathrm{Area}(F)}{2} \left(\frac{\sinh 2\varepsilon}{2} + \varepsilon \right),$$

where we are making use of the coordinates described in Lemma 2.1.3. Since the lamination λ has Lebesgue measure 0 inside ∂CM, the sum of the areas of the flat pieces is $\mathrm{Area}(\partial CM) = 2\pi|\chi(\partial CM)|$. Therefore the region in $N_\varepsilon CM \setminus CM$ which retracts over $\partial CM \setminus \lambda$ has volume $\pi|\chi(\partial CM)| \left(\frac{\sinh 2\varepsilon}{2} + \varepsilon \right)$.

Let D be the closed convex subset in $\mathbb{H}^3$ obtained as the intersection of two half-spaces whose boundary planes meet with an exterior dihedral angle equal to θ_0 and select γ a geodesic arc lying inside the line along which ∂D is bent. Then, the region in $N_\varepsilon D$ which retracts over γ has volume equal to

$$\int_0^\varepsilon \int_0^{\theta_0} \int_\gamma \cosh t \sinh t \, \mathrm{d}\ell \, \mathrm{d}\theta \, \mathrm{d}t = \frac{\theta_0 \, \ell(\gamma)}{4}(\cosh \varepsilon - 1). \tag{2.1}$$

An immediate consequence of this relation is that whenever ∂CM is finitely bent, the volume of $(r')^{-1}(\lambda)$ coincides with $\frac{\ell_m(\mu)}{4}(\cosh \varepsilon - 1)$, where m is the hyperbolic metric of ∂CM. In the general case, we can select a suitable covering of ∂CM by open sets on which we can apply the standard approximation argument of Definition 2.1.6. With this procedure, it is straightforward to see that the relation $\mathrm{Vol}((r')^{-1}(\lambda)) = \frac{\ell_m(\mu)}{4}(\cosh \varepsilon - 1)$ extends to the general case. Combining the relations we found, we obtain

$$\mathrm{Vol}(N_\varepsilon CM \setminus CM) = \pi|\chi(\partial CM)| \left(\frac{\sinh 2\varepsilon}{2} + \varepsilon \right) + \frac{\ell_m(\mu)}{4}(\cosh 2\varepsilon - 1).$$

Now we want to compute $\int_{S_\varepsilon CM} H_\varepsilon \, \mathrm{d}a_\varepsilon$. Using Lemmas 2.1.4 and 2.1.3 we immediately see that, in the finitely bent case the following holds:

$$\int_{S_\varepsilon CM} H_\varepsilon \, \mathrm{d}a_\varepsilon = 2\pi|\chi(\partial CM)| \sinh 2\varepsilon + \ell_m(\mu) \cosh 2\varepsilon.$$

The standard approximation procedure (see Definition 2.1.6) allows us again to prove this relation in the general case, with the only difference that the $\mathscr{C}^{1,1}$-convergence is now crucial, because the expression of the mean curvature in chart involves the second derivatives in the coordinates system. Combining the relations we proved with the equality $\mathrm{Vol}^*(CM) = \mathrm{Vol}(CM) - \ell_m(\mu)/2$, we deduce the relation in the statement. □

As we will see in a moment, it will be convenient for us to differentiate the dual volume enclosed in a differentiable 1-parameter family of $\mathscr{C}^{1,1}$-surfaces. In particular, we will make use of the following result, which is a corollary of the differential Schläfli formulae of Theorems 1.7.2 and 1.7.4:

Proposition 2.2.5. *Let $M_t = (N, g_t)$ be a smooth 1-parameter family of complete convex co-compact hyperbolic structures on N. Consider C a compact convex subset of $N \setminus \partial N$ with $\mathscr{C}^{1,1}$-boundary. Then the variation of the dual volume of C in M_t exists and can be expressed as:*

$$\left.\frac{\mathrm{dVol}^*_{M_t}(C)}{\mathrm{d}t}\right|_{t=0} = \frac{1}{4}\int_{\partial C} (\delta g|_{\partial C}, I\!I - HI)\,\mathrm{d}a\,,$$

where I, $I\!I$, H are the first and second fundamental forms and the mean curvature of the surface ∂C, and $(\cdot,\cdot)$ is the scalar product induced by I on the space of 2-tensors on ∂C.

Similarly, if we fix the hyperbolic 3-manifold $M = M_0$ and we consider a 1-parameter family of convex subsets C_t with $\mathscr{C}^{1,1}$-boundaries ∂C_t varying smoothly in t, then:

$$\left.\frac{\mathrm{dVol}^*_{M}(C_t)}{\mathrm{d}t}\right|_{t=0} = \frac{1}{4}\int_{\partial C} (\delta I, I\!I - HI)\,\mathrm{d}a\,.$$

Proof. The strategy used in Proposition 1.7.13 to compute the derivative of V_3^* applies verbatim to both cases, using Theorems 1.7.4 and 1.7.2, respectively. The difference of sign between these relations and the one for δV_3^* is due to the different convention in the definition of dual volume, as observed in Remark 2.2.2. □

Contrary to the case of the hyperbolic volume, it is not clear whether the dual volume of a convex set is positive or not. However, Vol^* shares with the usual notion of volume the property of being monotonic (in fact *decreasing*) with respect to the inclusion, as we see in the following:

Proposition 2.2.6. *Let C, C' be two compact convex subsets inside a convex co-compact manifold M. If $C \subseteq C'$, then $\mathrm{Vol}^*(C) \geq \mathrm{Vol}^*(C')$.*

Proof. Thanks to Proposition 2.2.3, up to considering ε-neighborhoods and passing to the limit as ε goes to 0, we can assume that C and C' are compact convex subsets with $\mathscr{C}^{1,1}$-boundary. We will make use of the variation formula of Proposition 2.2.5. Assume that $\Sigma\colon I \times S \to M$ is a differentiable 1-parameter family of convex $\mathscr{C}^{1,1}$-surfaces $\Sigma_t := \Sigma(t,\cdot)$, which parametrize the boundaries of an increasing family of compact convex subsets $(C_t)_{t\in I}$ inside M. Let V_t be the infinitesimal generator of the deformation at time t, i. e. V_t is the vector field over S defined by $V_t := \frac{\mathrm{d}\Sigma_t}{\mathrm{d}t}$. The tangential component of V_t

does not contribute to the variation of the dual volume (compare with [RS99, Theorem 1]). Consequently, in order to compute the derivative of $\mathrm{Vol}^*(C_t)$, we can assume V_t to be along the interior normal vector field ν_t of ∂C_t. Moreover, since the deformation $(N_t)_t$ is increasing with respect to the inclusion, V_t is of the form $f_t \nu_t$, for some $f_t \colon S \to \mathbb{R}$, $f_t \leq 0$. Under this condition, the variation of the first fundamental form of ∂C_t is $\delta I_t = -2 f_t I\!I_t$ (again, compare with [RS99, Theorem 1]). If $k_1(t)$, $k_2(t)$ denote the principal curvatures of ∂C_t, we obtain that

$$\begin{aligned}(\delta I_t, I\!I_t - H_t I_t) &= -2 f_t (I\!I_t, I\!I_t - H_t I_t) \\ &= -2 f_t (k_1(t)^2 + k_2(t)^2 - (k_1(t) + k_2(t))^2) \\ &= +4 f_t k_1(t) k_2(t) \leq 0,\end{aligned}$$

where, in the last step, we used the fact that the extrinsic curvature $K_e(t) = k_1(t) k_2(t)$ is non-negative since ∂C_t is convex. By Proposition 2.2.5, we deduce that Vol^* is non-increasing along the deformation $(C_t)_t$.

It remains to show that, if C, C' are two convex subsets of M with $\mathscr{C}^{1,1}$-boundary and such that $C \subseteq C'$, we can find a differentiable 1-parameter family, indexed by $t \in [0,1]$, of increasing convex subsets C_t with $\mathscr{C}^{1,1}$-boundary so that $C_0 = C$ and $C_1 = C'$. A way to produce such a path is described in the proof of [Sch13, Lemma 3.14], we briefly recall the ideas involved in the construction. Given any convex set C with $\mathscr{C}^{1,1}$-boundary in M, the asymptotic expansion of the first fundamental forms of the equidistant surfaces from C determines a unique Riemannian metric h_C belonging to the conformal class at infinity of $\partial_\infty M$. Moreover, the surface ∂C can be recovered from h_C as the envelope of a family of horoballs determined by h_C, thanks to a construction due to Epstein (∂C is the so-called *Epstein surface* associated to the metric h_C, see [Eps84]). This correspondence behaves well with respect to the inclusion, in the sense that if C and C' are convex sets as above and $C \subseteq C'$, then $h_C \leq h_{C'}$. Being h_C and $h_{C'}$ elements of the same conformal class, there exists a non-negative function u on $\partial_\infty M$ such that $h_{C'} = e^{2u} h_C$. If we set now $h_t := e^{2tu} h_C$, then the Epstein surfaces associated to h_t turn out to be the boundaries of an increasing family of convex subsets C_t satisfying the desired requirements (see [Sch13, Lemma 3.14] for a more detailed exposition). □

2.3 The derivative of the length

From now on, S will be a fixed closed surface of genus $g \geq 2$. We briefly recall the notions of [Bon88] that we will need. Given m a hyperbolic metric on S, the universal cover $\widetilde{S}$, endowed with the lifted metric $\tilde{m}$, is isometric to $\mathbb{H}^2$. As the topological boundary of the Poincaré disk sits at infinity of $\mathbb{H}^2$, also $\widetilde{S}$ can be compactified by adding a topological circle $\partial_\infty \widetilde{S}$ at infinity, and the resulting space does not depend on the chosen identification between them. The fundamental group naturally acts by isometries on $\widetilde{S} \cong \mathbb{H}^2$, and since the isometries of $\mathbb{H}^2$ extend to $\partial_\infty \mathbb{H}^2$, the action extends to $\partial_\infty \widetilde{S}$. It turns out that the topological space $\partial_\infty \widetilde{S}$, together with its action of $\pi_1(S)$, is independent of the hyperbolic metric m we chose. In particular, all the spaces we are going to describe are intrinsically associated to the topological surface S, without prescribing any additional structure. Since a geodesic in $\widetilde{S}$ is determined by its (distinct) endpoints in $\partial_\infty \widetilde{S}$, the space $\mathcal{G}(\widetilde{S})$ of unoriented geodesics of $\widetilde{S}$ can be naturally identified with

$$\left(\partial_\infty \widetilde{S} \times \partial_\infty \widetilde{S} \setminus \Delta\right) / \mathbb{Z}_2,$$

where Δ denotes the diagonal subspace of $(\partial_\infty \widetilde{S})^2$, and the action of $\mathbb{Z}_2$ exchanges the two coordinates in $(\partial_\infty \widetilde{S})^2$. Therefore, a *geodesic lamination* λ of S is identified with a closed, $\pi_1(S)$-invariant subset $\widetilde{\lambda}$ of disjoint geodesics in $\mathcal{G}(\widetilde{S})$. In the same spirit, a *measured lamination* of S corresponds to a $\pi_1(S)$-invariant, locally finite Borel measure on $\mathcal{G}(\widetilde{S})$ with support contained in a geodesic lamination λ of S. We denote by $\mathcal{GL}(S)$ and $\mathcal{ML}(S)$ the spaces of geodesic laminations and measured laminations on S, respectively (see also Section 1.2.2 for an alternative description of these objects).

In the following, we recall the notion of length of measured laminations realized inside a fixed hyperbolic 3-manifold M from [Bon97, Section 7]. As in the case of S, we can define the space of unoriented geodesics of M, making use of the natural compactification of $\mathbb{H}^3$. The substantial difference is that the dynamical properties of the action of $\pi_1(M)$ do depend in general on the hyperbolic metric we are considering on M. However, our interest will be to apply these notions to quasi-isometric deformations of hyperbolic manifolds. In this case, the holonomy representations turn out to be quasi-conformally conjugated in $\partial_\infty \mathbb{H}^3$, therefore the qualitative properties of the action of $\pi_1(M)$ on $\mathcal{G}(\widetilde{M})$ are preserved. Fix now a homotopy class of maps $[f_0 \colon S \to M]$.

Definition 2.3.1. A geodesic lamination λ on S is *realizable* inside M in the homotopy class $[f_0]$ if there exists a representative $f \colon S \to M$ of $[f_0]$ which sends each geodesic of λ homeomorphically in a geodesic of M. In such case, we say that λ is *realized* by f.

In order to talk about the realization of a *measured* lamination μ, we need to find a way to push-forward the measure μ to a measure on $\mathcal{G}(\widetilde{M})$. Let λ be a geodesic lamination on S realized by a map f, and let $\rho \colon \pi_1(S) \to \pi_1(M)$ be the homomorphism induced by $[f_0]$ on the fundamental groups. Fixed a lift $\tilde{f}$ of f to the universal covers, we can construct a function $r \colon \tilde{\lambda} \to \mathcal{G}(\widetilde{M})$, associating to each leaf g of $\tilde{\lambda}$ the geodesic $\tilde{f}(g)$ sitting inside $\widetilde{M}$. The map r is ρ-equivariant and continuous with respect to the topologies of $\tilde{\lambda}$ as subset of $\mathcal{G}(\widetilde{S})$ and of $\mathcal{G}(\widetilde{M})$ (compare with [Bon97, Section 7]). It is easy to prove that r depends only on the homotopy class $[f]$ and on the choice of a lift of *any* representative of $[f]$ realizing λ. To see this, let $F_0 = f$ and $f_1 = f'$ be two such maps in $[f]$ homotopic through $(F_t)_{t \in I}$ (here I denotes the interval $[0,1]$). Once we choose a lift $\tilde{f}$ of f, there exists a unique lift $\widetilde{F}_t$ of the homotopy so that $\widetilde{F}_0 = \tilde{f}$. This gives a preferred lift of f', namely $\tilde{f}' := \widetilde{F}_1$. Because of the compactness of S and the existence of a homotopy $\widetilde{F}_t$ between them, the lifts $\tilde{f}$ and $\tilde{f}'$ must agree (up to reparametrization) on any leaf g of $\tilde{\lambda}$, since the geodesics $\tilde{f}(g)$ and $\tilde{f}'(g)$ are necessarily at bounded distance in $\mathbb{H}^3$ (see [Thu79, Proposition 8.10.2]). This implies that the definitions of r obtained using $\tilde{f}$ and $\tilde{f}'$ coincide. Moreover, different choices of lifts $\tilde{f}$ produce maps r, r' which differ by post-composition by an element in $\pi_1(M)$. The same argument as above shows that, if λ_1, λ_2 are two geodesic laminations realized by the maps f_1, f_2 respectively, which both contain the lamination λ, then the two realizations f_1 and f_2 coincide on λ.

We are finally ready to describe the definition of the length of the realization of a measured lamination inside M. Let α be a measured lamination on S with support contained in λ. We denote by $\bar{\alpha} := r_* \alpha$ the push-forward of α under the map r. $\bar{\alpha}$ is a measure on $\mathcal{G}(\widetilde{M})$ with support $r(\operatorname{supp} \alpha)$, depending only on $\alpha \in \mathcal{ML}(S)$, on the homotopy class $[f]$ and on the choice of a lift of f. Assume that $f(\lambda)$ lies inside some compact set K of M and let $\mathcal{F}$, $\widetilde{\mathcal{F}}$ denote the geodesic foliations of the projective tangent bundles $\mathbb{P}TM$, $\mathbb{P}T\widetilde{M}$, respectively. We can cover the preimage of K in $\mathbb{P}TM$ by finitely many $\mathcal{F}$-flow boxes $\sigma_j \colon D_j \times I \to B_j$. Here D_j is some topological space and σ_j is a homeomorphism sending each subset $\{p\} \times I \subset D_j \times I$ in a subarc of a leaf in $\mathcal{F}$, for any $p \in D_j$. In addition, we fix a collection $\{\xi_j\}_j$ of smooth functions with supports $\operatorname{supp} \xi_j$ contained in the interior of B_j for every j, and such that $\sum_j \xi_j = 1$ over the preimage of K in $\mathbb{P}TM$. If σ_j is a $\mathcal{F}$-flow box that meets $f(\operatorname{supp} \alpha)$, we can lift it to a $\widetilde{\mathcal{F}}$-flow box $\tilde{\sigma}_j \colon D_j \times I \to \mathbb{P}T\widetilde{M}$ accordingly with the choice of the lift $\tilde{f}$. The lift $\tilde{\sigma}_j$ induces an identification between the space D_j with a subset in $\mathcal{G}(\widetilde{M})$. Namely, a point $p \in D_j$ corresponds to the complete leaf in $\widetilde{\mathcal{F}}$ extending the arc $\tilde{\sigma}_j(\{p\} \times I)$. Through this identification, it makes sense to integrate the D_j-component of $\tilde{\sigma}_j$ with respect to the measure $\bar{\alpha}$ previously defined on $\mathcal{G}(\widetilde{M})$. If σ_j does not meet $f(\operatorname{supp} \alpha)$, then we choose an arbitrary lift $\tilde{\sigma}_j$. Finally, we select lifts $\tilde{\xi}_j$'s of the ξ_j's according with the choices of the lifts $\tilde{\sigma}_j$. The *length of the realization of* α *in* M (in the homotopy class $[f]$) is

$$\ell_M(\alpha) = \iint_\lambda \mathrm{d}\ell \, \mathrm{d}\alpha := \sum_j \int_{D_j} \int_0^1 \tilde{\xi}_j(\tilde{\sigma}_j(p,s)) \, \mathrm{d}\ell \, (s) \, \mathrm{d}\bar{\alpha} \, (p), \tag{2.2}$$

where $d\ell$ denotes the length-measure along the leaves of $\widetilde{\mathcal{F}}$.

Remark 2.3.2. By invariance of the length under reparametrization and by linearity of the integral, the choices of the functions $\{\xi_j\}_j$ and the chosen $\mathcal{F}$-flow boxes $\{\sigma_j\}_j$ are irrelevant; moreover, different lifts of f produce maps r which are conjugated by isometries in $\pi_1(M)$. Therefore, the quantity $\ell_M(\alpha)$ only depends on the measured lamination α, the hyperbolic metric on M and the homotopy class $[f \colon S \to M]$. The notion makes sense as long as there exists a realizable geodesic lamination λ in the homotopy class $[f]$ which contains $\operatorname{supp} \alpha$. Moreover, by what we observed before, this quantity does not depend on the specific representable lamination λ we chose, but it is determined only by $\operatorname{supp} \alpha$.

We are now ready to produce a variation formula for the length of the realization of a measured lamination inside a 1-parameter family of quasi-isometric convex co-compact hyperbolic manifolds $(M_t)_t$. For convenience, we think of $(M_t)_t$ as a differentiable 1-parameter family of complete hyperbolic metrics g_t on a fixed 3-manifold X, so that the identity map, from $M_0 = (X, g_0)$ to $M_t = (X, g_t)$, is a quasi-isometric diffeomorphism for any t. Let $\alpha \in \mathcal{ML}(S)$ be a measured lamination and $[f_0 \colon S \to X]$ a homotopy class of maps. In the convex co-compact case, all finite laminations are realizable and their realizations are necessarily contained in the convex core CM_t. Therefore, by [CEM06, Corollary I.5.2.13] and [CEM06, Theorem I.5.3.6], any geodesic lamination on

S is realizable in the homotopy class $[f_0]$, and their realizations lie inside a fixed compact subset K of X (where K contains CM_t for every small t). Let now λ be *any* geodesic lamination containing $\operatorname{supp}\alpha$ and assume that it is realized inside M_t by a certain map $f_t \colon S \to M_t$, for any t. By the above, we are allowed to consider the length of the realization of α inside M_t for every t. Let $\{\sigma_j\}_j$, $\{\xi_j\}_j$, $\{\tilde{\sigma}_j\}_j$, $\{\tilde{\xi}_j\}_j$ be a collection of functions as in the definition of $\ell_M(\alpha)$. Then, in the same notations as above, we set

$$\iint_\lambda \mathrm{d}\dot{\ell}\,\mathrm{d}\alpha := \sum_j \int_{D_j} \int_0^1 \tilde{\xi}_j(\tilde{\sigma}_j(p,s))\, \frac{\dot{g}\left(\partial_s\tilde{\sigma}_j(p,s), \partial_s\tilde{\sigma}_j(p,s)\right)}{2g\left(\partial_s\tilde{\sigma}_j(p,s), \partial_s\tilde{\sigma}_j(p,s)\right)}\, \mathrm{d}\ell\,\mathrm{d}\bar{\alpha}\,(p),$$

where $\partial_s\tilde{\sigma}_j = \frac{\partial\tilde{\sigma}_j}{\partial s}$, $g = g_0$ and $\dot{g} = \frac{\mathrm{d}g}{\mathrm{d}t}|_{t=0}$. The result we want to prove is the following:

Proposition 2.3.3. *Let $(g_t)_t$ be a 1-parameter family of convex co-compact hyperbolic metrics on a 3-manifold X, which are quasi-isometric to each other via the identity map of X. Let α be a measured lamination on a surface S and let $[f \colon S \to X]$ be a fixed homotopy class. Then α is realizable in M_t for all values of t, and the variation of its length verifies*

$$\left.\frac{\mathrm{d}\ell_{M_t}(\alpha)}{\mathrm{d}t}\right|_{t=0} = \iint_\lambda \mathrm{d}\dot{\ell}\,\mathrm{d}\alpha\,, \tag{2.3}$$

where λ is a geodesic lamination of S containing $\operatorname{supp}\alpha$.

We will prove the Proposition using an approximation argument. Firstly we deal with the rational case:

Lemma 2.3.4. *When $\alpha \in \mathcal{ML}(S)$ is a rational lamination, Proposition 2.3.3 holds.*

Proof. Let c be a free homotopy class of simple closed curves in X and assume that c admits a geodesic representative in M_0. Since we are considering a quasi-isometric deformation of convex co-compact manifolds, the homotopy class c will admit a geodesic representative for all values of t. Moreover, we can find parametrizations γ^t of the geodesic of c in M_t depending smoothly on t, because of the smooth dependence of the holonomy representation $\mathrm{hol}_t(c)$. In other words, we can find a smooth map $\Sigma \colon (-\varepsilon, \varepsilon) \times I \to X$ such that $\Sigma(t,s) = \gamma^t(s)$ for every t and $s \in I$. Let $\|\cdot\|_t$ denote the norm with respect to the metric g_t, and let $\gamma = \gamma^0$. We have

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_s\gamma^t\|_t\right|_{t=0} &= \frac{\dot{g}(\partial_s\gamma, \partial_s\gamma) + 2g(\mathcal{D}_{\partial_t}\partial_s\Sigma|_{t=0}\,, \partial_s\gamma)}{2\|\partial_s\gamma\|_0} \\
&= \frac{\dot{g}(\partial_s\gamma, \partial_s\gamma)}{2\|\partial_s\gamma\|_0} + g\left(\mathcal{D}_{\partial_s}\ \partial_t\Sigma|_{t=0}\,, \frac{\partial_s\gamma}{\|\partial_s\gamma\|_0}\right) \\
&= \frac{\dot{g}(\partial_s\gamma, \partial_s\gamma)}{2\|\partial_s\gamma\|_0} + \frac{\mathrm{d}}{\mathrm{d}s}\left[g\left(\partial_t\Sigma|_{t=0}\,, \frac{\partial_s\gamma}{\|\partial_s\gamma\|_0}\right)\right],
\end{aligned}$$

where in the last step we used the fact that γ parametrizes a geodesic in $M = M_0$, and consequently the covariant derivative of $\frac{\partial_s\gamma}{\|\partial_s\gamma\|_0}$ vanishes. Once we integrate the last term in $t \in [0,1]$ we get 0, because the function of which we are taking the derivative coincides at the extremes (since the geodesics γ^t are closed). Hence we obtain

$$\frac{\mathrm{d}\ell_{M_t}(c)}{\mathrm{d}t}\bigg|_{t=0} = \int_0^1 \frac{\dot g(\partial_s\gamma,\partial_s\gamma)}{2\|\partial_s\gamma\|_0}\,\mathrm{d}s = \int_0^1 \frac{\dot g(\partial_s\gamma,\partial_s\gamma)}{2g(\partial_s\gamma,\partial_s\gamma)}\,\mathrm{d}\ell\,.$$

Take now a rational lamination $\alpha \in \mathcal{ML}(S)$, i. e. the measure α is the weighted sum $\sum_i u_i\,\delta_{d_i}$, where the d_i are homotopy classes of simple closed curves, the u_i are positive weights, and δ_{d_i} is the transverse measure which counts the geometric intersection of an arc transverse to d_i with d_i. Assume that α is realizable in M or, equivalently, that the curves $c_i = f_0(d_i)$ admit a geodesic representative γ_i in M. The same argument given above shows that the lamination α is realizable in M_t for all t. Applying the definition of $\ell_{M_t}(\alpha)$, and denoting by $\gamma_i^t \colon I \to M_t$ the geodesic representative of c_i, we see that

$$\ell_{M_t}(\alpha) := \sum_i u_i \left(\int_0^1 \left\|\partial_s\gamma^t(s)\right\|_t \mathrm{d}s\right).$$

Hence, taking the derivative in t and using what observed above, we get

$$\frac{\mathrm{d}\ell_{M_t}(\alpha)}{\mathrm{d}t}\bigg|_{t=0} = \sum_i u_i\left(\int_0^1 \frac{\dot g(\partial_s\gamma,\partial_s\gamma)}{2\|\partial_s\gamma\|_0}\,\mathrm{d}s\right) = \iint_\lambda \mathrm{d}\dot\ell\,\mathrm{d}\alpha\,,$$

where $\lambda = \operatorname{supp}\alpha = \bigcup_i d_i$. □

We are now ready to deal with the proof of Proposition 2.3.3.

Proof of Proposition 2.3.3. Let T be a train track in S carrying α and consider a sequence of rational laminations α_n carried by T and converging to α as measured laminations (see [Thu79, Proposition 8.10.7]). Up to passing to a subsequence, we can assume that the laminations $\operatorname{supp}\alpha_n$ converge in the Hausdorff topology to a lamination λ carried by T. Since α_n is converging to α, we must have $\lambda \supseteq \operatorname{supp}\alpha$. We denote by $f_t \colon S \to X$ a realization of λ in the homotopy class $[f]$ with respect to the metric g_t, and by $\tilde f_t \colon \widetilde S \to \widetilde M$ lifts of the f_t's so that $t \mapsto \tilde f_t$ is continuous with respect to the compact-open topology of $\mathscr{C}^0(\widetilde S, \widetilde X)$.

Let now K be a large compact set of X containing all the convex cores CM_t for small values of t. Then, if $\mathcal{F}_t$ is the geodesic foliation of $\mathbb{P}M_t$, we can choose $\mathcal{F}_t$-flow boxes $\{\sigma_j^t\}_j$ whose union of images contain the preimage of K in $\mathbb{P}TM_t$, and hence the realizations $f_t(\lambda)$. We consequently construct maps $\{\tilde\sigma_j^t\}_j$, $\{\xi_j^t\}_j$, $\{\tilde\xi_j^t\}_j$ as in the definition of $\ell_{M_t}(\cdot)$. We can ask these functions to vary smoothly in the parameter t, since the hyperbolic metrics depends smoothly in t. Now, we define

$$\varphi_j^t(\cdot) := \int_0^1 \tilde\xi_j^t(\tilde\sigma_j^t(\cdot,s))\,\mathrm{d}\ell_t\,(s).$$

In this notation, the length of the realization of α_n in M_t can be expressed as

$$\ell_{M_t}(\alpha_n) = \sum_j \int_{D_j} \varphi_j^t\,\mathrm{d}\bar\alpha_n\,.$$

From this relation is clear that, as n goes to ∞, $\ell_{M_t}(\alpha_n)$ converges uniformly to $\ell_{M_t}(\alpha)$ on a small interval $(-\varepsilon, \varepsilon)$ of the parameter t. In the same way we see that $\iint \mathrm{d}\dot{\ell}_0 \, \mathrm{d}\alpha_n$ converges to $\iint \mathrm{d}\dot{\ell}_0 \, \mathrm{d}\alpha$ (here is even easier, because there is no dependence on t). Thanks to Lemma 2.3.4, the only thing left to conclude the proof is to show that

$$\lim_{n \to \infty} \frac{\mathrm{d}}{\mathrm{d}t} \ell_{M_t}(\alpha_n) \bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \ell_{M_t}(\alpha) \bigg|_{t=0} .$$

Here we can argue as follows: the length of a homotopy class c of non-parabolic type can be expressed as the real part of its *complex length* $\ell^{\mathbb{C}}_{\bullet}(c) \in \mathbb{C}/2\pi i \mathbb{Z}$, which is holomorphic in the holonomy representation. The argument described above shows that the real lengths $\ell_{\bullet}(\alpha_n)$ are converging uniformly in a small neighborhood of hol_0 (see also [Sul81b, Theorem 2]). Since the real part of a holomorphic function determines (up to imaginary constant) the holomorphic function itself, we deduce that also the complex lengths $\ell^{\mathbb{C}}_{\bullet}(\alpha_n)$ are converging uniformly, and hence $\mathscr{C}^\infty$-uniformly. In particular this proves the convergence of the derivatives in t. □

2.4 The dual Bonahon-Schläfli formula

In this section we will describe the proof of Theorem A. The first subsection will be dedicated to the study of the convexity of the equidistant surfaces from the convex core while we vary the hyperbolic structure. Afterwards we will introduce an auxiliary function on which we can apply the differential Schläfli formula (Proposition 2.2.5). This is the step in which the variation of the length of the bending measure arises (see Proposition 2.4.5). In Proposition 2.4.4 we will relate this with the actual variation of the dual volume of the convex core. In the end of the section we will use Bonahon's results about the dependence of the metric of the convex core in terms of the convex co-compact hyperbolic structure to finally prove Theorem A.

Let $(M_t)_t$ be a smooth family of quasi-isometric convex co-compact manifolds, parametrized by $t \in (-t_0, t_0)$. We can choose diffeomorphisms $\varphi_t \colon M_0 \to M_t$ so that the following properties hold:

1. φ_t is a quasi-isometric diffeomorphism for any t, and $\varphi_0 = id$;

2. fixed identifications of the universal covers of M_t with $\mathbb{H}^3$ for every t, we can find lifts $\tilde{\varphi}_t \colon \mathbb{H}^3 \to \mathbb{H}^3$ of φ_t so that $\tilde{\varphi}_0 = id_{\mathbb{H}^3}$ and so that the map $\tilde{\varphi}$, defined by $\tilde{\varphi}(t, \cdot) := \tilde{\varphi}_t(\cdot)$, is smooth as a map from $(-t_0, t_0) \times \mathbb{H}^3$ to $\mathbb{H}^3$.

2.4.1 Convexity of equidistant surfaces

In order to prove Theorem A, it will be important for us to understand for which values of t and $\varepsilon \leq \varepsilon_0$ the surfaces $\varphi_t(S_\varepsilon CM_0)$ and $\varphi_t^{-1}(S_\varepsilon CM_t)$ remain convex. This is the most technical part of our argument and it will require special care. We want to prove the following fact:

Lemma 2.4.1. *There exist constants K, $\tau > 0$, with $0 < \tau \leq t_0$, which depend only on the quasi-isometric deformation $(M_t)_t$ and on the fixed family of diffeomorphims $(\varphi_t)_t$, such that, for every $t \in (-\tau, \tau)$ the regions $\varphi_t(N_{K|t|} CM_0)$ and $\varphi_t^{-1}(N_{K|t|} CM_t)$ are convex in M_t and M_0, respectively. As a consequence, we have*

$$\varphi_t(N_{K|t|} CM_0) \supset CM_t \qquad \text{and} \qquad N_{K|t|} CM_t \supset \varphi_t(CM_0).$$

We denote by $\pi_t \colon \mathbb{H}^3 \to M_t$ the universal cover of M_t, and by $\widetilde{C}_t \subset \mathbb{H}^3$ the preimage of the convex core CM_t under π_t. Fixed q_0 a basepoint in $\mathbb{H}^3$, we can find a large $R > 0$ so that the metric ball $B_R = B(q_0, R)$ in $\mathbb{H}^3$ verifies

$$\pi_t \tilde{\varphi}_t(B_R) = \varphi_t \pi_0(B_R) \supseteq N_{\varepsilon_0} CM_t$$

and $\varphi_t(\overline{B}_R) \subseteq B_{R+1}$, whenever t is small enough. This follows from the fact that the convex cores CM_t are compact and they vary continuously in the parameter t. Clearly Lemma 2.4.1 reduces to the study of the surfaces $\tilde{\varphi}_t(S_\varepsilon \widetilde{C}_0 \cap B_R)$ and $\tilde{\varphi}_t^{-1}(S_\varepsilon \widetilde{C}_t \cap B_R)$ in $\mathbb{H}^3$. However, instead of dealing directly with equidistant surfaces from $\widetilde{C}_0$, which are only $\mathscr{C}^{1,1}$, we will rather focus our study on the family of ε-surfaces from half-spaces of $\mathbb{H}^3$, which are more regular and can be used as "support surfaces" for $S_\varepsilon \widetilde{C}_0$. The strategy will be to understand how the convexity of their image under $\tilde{\varphi}_t$ behave, and from this to deduce the convexity of the surfaces $\tilde{\varphi}_t(S_\varepsilon \widetilde{C}_0 \cap B_R)$ (and similarly for $\tilde{\varphi}_t^{-1}(S_\varepsilon \widetilde{C}_t \cap B_R)$).

In order to clarify this idea, we need to introduce some notation. Let r_t be the nearest point retraction of $\mathbb{H}^3$ onto the convex subset $\widetilde{C}_t$. Given a point q of $S_\varepsilon \widetilde{C}_t$, we denote by $\mathcal{H}_{t,q}$ the unique support half-space of $\widetilde{C}_t$ at $r_t(q)$ whose boundary $\partial \mathcal{H}_{t,q} = H_{t,q}$ is orthogonal to the geodesic segment connecting $r_t(q)$ to q (see Figure 2.1). By construction, we have the inclusion $N_\varepsilon \mathcal{H}_{t,q} \supseteq N_\varepsilon \widetilde{C}_t$, and the surfaces $S_\varepsilon \mathcal{H}_{t,q}$, $S_\varepsilon \widetilde{C}_t$ are tangent to each other at the point q. In other words, given $q \in S_\varepsilon \widetilde{C}_t$, the surface $S_\varepsilon \mathcal{H}_{t,q}$ lies outside $\operatorname{int}(N_\varepsilon \widetilde{C}_t)$, it approximates $S_\varepsilon \widetilde{C}_t$ at first order at q and it is strictly convex, with second fundamental form described in Lemma 2.1.3. Therefore, if for every $q \in S_\varepsilon \widetilde{C}_0 \cap B_R$ and $t \in (-t_0, t_0)$ the surface $\tilde{\varphi}_t(S_\varepsilon \mathcal{H}_{0,q})$ remains convex at $\tilde{\varphi}_t(q)$, then $\tilde{\varphi}_t(S_\varepsilon \widetilde{C}_0 \cap B_R)$ has to be convex too. Analogously, the convexity of the surfaces $\tilde{\varphi}_t^{-1}(S_\varepsilon \mathcal{H}_{t,q})$ at $\tilde{\varphi}^{-1}(q)$, as q varies in $S_\varepsilon \widetilde{C}_t \cap B_R$, implies the convexity of $\varphi_t^{-1}(S_\varepsilon CM_t)$.

In what follows, we state the technical result about equidistant surfaces from which Lemma 2.4.1 will follow. Given U an open set of $\mathbb{H}^3$, we denote by $\mathcal{S}(U, \varepsilon_0)$ the collection of those surfaces embedded in U that are obtained by intersecting U with an equidistant surface $S_\varepsilon \mathcal{H}$, for some $\mathcal{H}$ half-space of $\mathbb{H}^3$ meeting U and for some $0 < \varepsilon \leq \varepsilon_0$. We remark that, using the notation introduced above, for every $\varepsilon \leq \varepsilon_0$ and for every $q \in S_\varepsilon \widetilde{C}_t$, the surface $S_\varepsilon \mathcal{H}_{t,q} \cap B_R$ belongs to the family $\mathcal{S}(B_R, \varepsilon_0)$.

By considering the Poincaré disk model, we can identify $\mathbb{H}^3$ with the open unit ball Δ of $\mathbb{R}^3$, and functions $f \colon \mathbb{H}^3 \to \mathbb{H}^3$ as maps from $\Delta \subset \mathbb{R}^3$ to itself. If U is an open set of $\mathbb{R}^n$, $K \subset U$ is compact and $f \colon U \to \mathbb{R}^m$ is a smooth map,

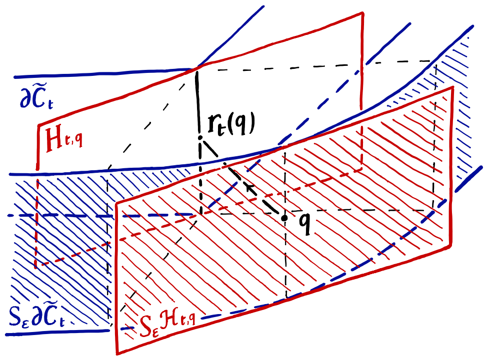

Figure 2.1: A schematic picture of the surface $S_\varepsilon \mathcal{H}_{\varepsilon,q}$

we define

$$\|f\|_{\mathscr{C}^0(K)} := \max_{p\in K} \|f(p)\|_0,$$

$$\|f\|_{\mathscr{C}^k(K)} := \|f\|_{\mathscr{C}^0(K)} + \sum_{h=1}^{k} \|D^h f\|_{\mathscr{C}^0(K)}$$

for $k \geq 1$, where $\|\cdot\|_0$ is the Euclidean (operator) norm and D is the Levi-Civita connection of the Euclidean metric of $\mathbb{R}^n$ (if $X = \sum_i X^i e_i$ and $Y = \sum_j Y^j e_j$ are two vector fields, then $D_X Y = \sum_{i,j} X^i \partial_i Y^j e_j$). Then we have:

Lemma 2.4.2. *Let B be an open ball in $\mathbb{H}^3$, let $F\colon (-t_0, t_0) \times \mathbb{H}^3 \to \mathbb{H}^3$ be a smooth family of diffeomorphisms $F_t = F(t,\cdot)$, satisfying $F_0 = id_{\mathbb{H}^3}$ and $\|F\|_{\mathscr{C}^4((-t_0,t_0)\times\overline{B})} < \infty$, and let ε_0 be a positive number. Given $\Sigma \in \mathcal{S}(B, \varepsilon_0)$, we denote by I_t^Σ and $I\!I_t^\Sigma$ the first and second fundamental forms of $F_t(\Sigma)$, respectively, as t varies in $(-t_0, t_0)$. Then we can find $t_0' \in (0, t_0]$ and $D > 0$, depending only on the ball $\overline{B}$ and on $\|F\|_{\mathscr{C}^4((-t_0,t_0)\times\overline{B})}$, such that, for every surface $\Sigma = S_\varepsilon \mathcal{H} \cap B$ in $\mathcal{S}(B, \varepsilon_0)$, we have*

$$I\!I_t^\Sigma - \tanh\varepsilon\, I_t^\Sigma \geq -D|t|\, I_t^\Sigma, \tag{2.4}$$

where we are considering the unit normal vector field on $F_t(\Sigma)$ pointing toward $F_t(N_\varepsilon \mathcal{H} \cap B)$.

Assuming momentarily this fact, we can prove Lemma 2.4.1:

Proof of Lemma 2.4.1. First we study the surfaces $\varphi_t(S_\varepsilon C M_0)$. Following the argument described above, we need to measure the convexity of the surfaces $\tilde{\varphi}_t(S_\varepsilon \mathcal{H}_{0,q} \cap B_R)$. We apply Lemma 2.4.2 to $F_t := \tilde{\varphi}_t$ and $B := B_R$, obtaining two positive constants $t_0' \leq t_0$ and D, which depend only on $\|\tilde{\varphi}\|_{\mathscr{C}^4((-t_0,t_0)\times\overline{B}_R)}$, so that the relation (2.4) holds for every $\Sigma \in \mathcal{S}(B_R, \varepsilon_0)$. Now we choose K_1, $\tau_1 > 0$, which will depend only on D and t_0', so that $\tau_1 < t_0'$, $K_1\tau_1 \leq \varepsilon_0$ and

$$\frac{\tanh K_1|t|}{2} - D|t| \geq 0 \qquad \text{for every } t \in (-\tau_1, \tau_1).$$

We want to show that $\varphi_t(S_{K_1|t|}CM_0)$ is convex for every $t \in (-\tau_1, \tau_1)$. Let t be in $(-\tau_1, \tau_1)$ and consider $\varepsilon = K_1|t|$. By the choices we made, if q is a point in $S_{K_1|t|}\widetilde{C}_0 \cap B_R$, then the surface $S_{K_1|t|}\mathcal{H}_{0,q} \cap B_R$ belongs to $\mathcal{S}(B_R, \varepsilon_0)$. In particular, the first and second fundamental forms I_t, $I\!I_t$ of $\tilde{\varphi}_t(S_\varepsilon \mathcal{H}_{0,q} \cap B_R)$ verify the relation (2.4) with $\varepsilon = K_1|t|$, which can be rewritten as

$$I\!I_t - \frac{\tanh K_1|t|}{2} I_t \geq \left(\frac{\tanh K_1|t|}{2} - D|t| \right) I_t.$$

Because of the choices we made, the right hand side is positive semi-definite. Therefore we have

$$I\!I_t \geq \frac{\tanh K_1|t|}{2} I_t.$$

In particular, the surface $\tilde{\varphi}_t(S_{K_1|t|}\mathcal{H}_{0,q} \cap B_R)$ is strictly convex at the point $\tilde{\varphi}_t(q)$. Since the choice of $q \in S_{K_1|t|}\widetilde{C}_0 \cap B_R$ was arbitrary and the surface $\tilde{\varphi}_t(S_{K_1|t|}\mathcal{H}_{0,q} \cap B_R)$ locally contains $\tilde{\varphi}_t(S_{K_1|t|}\widetilde{C}_0)$, the argument previously mentioned proves the convexity of $\varphi_t(S_{K_1|t|}CM_0)$ for every $t \in (-\tau_1, \tau_1)$.

Now we have to deal with the case of $\varphi_t^{-1}(S_\varepsilon CM_t)$. Fixed $t \in (-t_0, t_0)$, we define

$$M_s^{(t)} := M_{t+s},$$
$$\psi_s^{(t)} := \varphi_{t+s} \circ \varphi_t^{-1} : M_0' = M_t \longrightarrow M_s' = M_{t+s}$$

for every $s \in (-s_0, s_0)$, with $s_0 = s_0(t) = t_0 - |t|$. Then we apply Lemma 2.4.2 to the 1-parameter family of diffeomorphisms $(\tilde{\psi}_s^{(t)})_s$, where $\tilde{\psi}_s^{(t)} := \tilde{\varphi}_{t+s} \circ \tilde{\varphi}_t^{-1}$. By construction, the constants s_0' and D' only depend on $\overline{B}_{R+1}$ and $\|\tilde{\psi}^{(t)}\|_{\mathscr{C}^4((-s_0,s_0)\times \overline{B}_{R+1})}$. Since we can find a uniform upper bound for $\|\tilde{\psi}^{(t)}\|_{\mathscr{C}^4((-s_0,s_0)\times \overline{B}_{R+1})}$, we can assume that s_0' and D' are independent of $t \in (-\tau_1, \tau_1)$. Therefore, applying the argument of the previous case to the 1-parameter deformation $(M_s^{(t)})_s$ and the diffeomorphisms $(\psi_s^{(t)})_s$, we can select $\tau \leq s_0'$ and K, both independent of t, so that the surfaces $\psi_s^{(t)}(S_{K|s|}CM_0^{(t)})$ are convex for every $s \in (-\tau, \tau)$. Moreover, it is not restrictive to ask that $\tau \leq \tau_1$ and $K \geq K_1$ (this ensures that K and τ work also for $\varphi_t(S_{K|t|}CM_0)$). Therefore, if $t \in (-\tau, \tau)$, then $s = -t \in (-\tau, \tau)$ and the surface

$$\psi_s^{(t)}(S_{K|s|}CM_0^{(t)})\Big|_{s=-t} = \varphi_t^{-1}(S_{K|t|}CM_t)$$

is convex, as desired. The second part of the statement follows because of the minimality of the convex core in the family of convex subsets. □

It remains to prove Lemma 2.4.2:

Proof of Lemma 2.4.2. Let α be a curve lying on some surface $\Sigma = S_\varepsilon \mathcal{H} \cap B \in \mathcal{S}(B, \varepsilon_0)$. We denote by α_t the curve $F_t \circ \alpha$, by ν_t the unit normal vector field of $F_t(\Sigma)$ pointing toward $F_t(N_\varepsilon \mathcal{H} \cap B)$, and by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ the norm and the scalar product in the hyperbolic metric of $\mathbb{H}^3$.

Assume momentarily that we could find two universal constants C_1, $C_2 > 0$ (depending only on the ball $\overline{B} \subset \mathbb{H}^3$) and a $\bar{t}_0 > 0$ (depending only on $\overline{B}$ and on the family $(F_t)_t)$, such that

$$\left| \|\alpha'_t\|^2 - \|\alpha'\|^2 \right| \leq C_1 \|\alpha'_t\|^2 \|F_t - id\|_{\mathscr{C}^1(\overline{B})},$$

$$\left| \langle \mathcal{D}_{\alpha'_t} \nu_t, \alpha'_t \rangle - \langle \mathcal{D}_{\alpha'} \nu_0, \alpha' \rangle \right| = \left| \langle \mathcal{D}_{\alpha'_t} \nu_t, \alpha'_t \rangle + \tanh \varepsilon \, \|\alpha'\|^2 \right| \leq C_2 \|\alpha'_t\|^2 \|F_t - id\|_{\mathscr{C}^2(\overline{B})}$$

for all $t \in (-\bar{t}_0, \bar{t}_0)$ (in the last line we used the fact that $S_\varepsilon \mathcal{H}$ has second fundamental form as in Lemma 2.1.3). With such estimates, we deduce that

$$\begin{aligned}
(I\!I_t^\Sigma - \tanh \varepsilon \, I_t^\Sigma)(\alpha'_t, \alpha'_t) &= -\langle \mathcal{D}_{\alpha'_t} \nu_t, \alpha'_t \rangle - \tanh \varepsilon \, \|\alpha'_t\|^2 \\
&\geq \tanh \varepsilon \, \|\alpha'\|^2 - C_2 \|\alpha'_t\|^2 \|F_t - id\|_{\mathscr{C}^2(\overline{B})} - \tanh \varepsilon \, \|\alpha'\|^2 + \\
&\qquad - C_1 \tanh \varepsilon \, \|\alpha'_t\|^2 \|F_t - id\|_{\mathscr{C}^1(\overline{B})} \\
&\geq -(C_1 + C_2) \, \|F_t - id\|_{\mathscr{C}^2(\overline{B})} \, I_t^\Sigma(\alpha'_t, \alpha'_t)
\end{aligned}$$

and therefore that $I\!I_t^\Sigma - \tanh \varepsilon \, I_t^\Sigma \geq -(C_1 + C_2) \, \|F_t - id\|_{\mathscr{C}^2(\overline{B})} \, I_t^\Sigma$ for every $t \in (-\bar{t}_0, \bar{t}_0)$. Since the map F is regular in t, where $F_t = F(t, \cdot)$, we can find two constants t'_0 and D, depending only on $\|F\|_{\mathscr{C}^4((-t_0,t_0) \times \overline{B})}$ and $\overline{B}$, for which the final statement holds (for this it is definitively enough to control the derivatives of order ≤ 2 in t and of order ≤ 2 in $p \in \overline{B}$).

The only thing left is to prove the two relations above. Let g_0 denote the Euclidean metric of $\mathbb{R}^3$ and g the hyperbolic metric on $\Delta \cong \mathbb{H}^3$. Identifying $\mathbb{H}^3$ with an open set of $\mathbb{R}^3$, it make sense to compute a tensor T_p at p on vectors (or forms) lying in the tangent (or cotangent) space at a different point q, via the identifications $T_p\mathbb{H}^3 \cong T_p\mathbb{R}^3 \cong T_q\mathbb{R}^3 \cong T_q\mathbb{H}^3$. Therefore we can write:

$$\begin{aligned}
\left| \|\alpha'_t\|^2 - \|\alpha'\|^2 \right| &\leq |(g \circ F_t)(D_{\alpha'} F_t, D_{\alpha'} F_t) - g(\alpha', \alpha')| \\
&\leq |(g \circ F_t)(D_{\alpha'} F_t, D_{\alpha'} F_t - \alpha')| + |(g \circ F_t)(D_{\alpha'} F_t - \alpha', \alpha')| + \\
&\qquad + |(g \circ F_t)(\alpha', \alpha') - g(\alpha', \alpha')| \\
&\leq (\|g \circ F_t\|_0 \|D.F_t\|_0 \|D.F_t - D.id\|_0 + \|g \circ F_t\|_0 \|D.F_t - D.id\|_0 + \\
&\qquad + \|g \circ F_t - g\|_0) \, \|\alpha'\|_0^2,
\end{aligned}$$

where $\|\cdot\|_0$ is the operator norm with respect to the Euclidean metric in $\mathbb{R}^3$. The terms $\|D.F_t - D.id\|_0$ and $\|g \circ F_t - g\|_0$ can be bounded by some universal constant multiplied by $\|F_t - id\|_{\mathscr{C}^1(\overline{B})}$. The terms $\|g \circ F_t\|_0$, $\|D.F_t\|_0$ are controlled, since F_t is $\mathscr{C}^1$-close to id. Since $\overline{B}$ is compact and the F_t's are diffeomorphisms $\mathscr{C}^1$-close to id, the norms $\|\cdot\|_0$, $\|D.F_t\|$ and $\|\cdot\|$ are uniformly equivalent between each other on $\overline{B}$. Combining these facts together we obtain the first inequality.

For the second relation, we can proceed similarly decomposing the expression in the following way:

$$\begin{aligned}
\left| \langle \mathcal{D}_{\alpha'_t} \nu_t, \alpha'_t \rangle - \langle \mathcal{D}_{\alpha'} \nu_0, \alpha' \rangle \right| &\leq \left| (g \circ F_t)(\mathcal{D}_{\alpha'_t} \nu_t, \alpha'_t - \alpha') \right| + \left| (g \circ F_t)(\mathcal{D}_{\alpha'_t - \alpha'} \nu_t, \alpha') \right| + \\
&\qquad + |(g \circ F_t)(\mathcal{D}_{\alpha'} \nu_t - \mathcal{D}_{\alpha'} \nu_0, \alpha')| + \\
&\qquad + |(g \circ F_t)(\mathcal{D}_{\alpha'} \nu_0, \alpha') - g(\mathcal{D}_{\alpha'} \nu_0, \alpha')| \\
&\leq 2 \|g \circ F_t\|_0 \|\mathcal{D}.\nu_t\|_0 \|D.F_t - D.id\|_0 \|\alpha'\|_0^2 + \\
&\qquad + \|g \circ F_t\|_0 \|\mathcal{D}.\nu_t - \mathcal{D}.\nu_0\|_0 \|\alpha'\|_0^2 + \\
&\qquad + \|g \circ F_t - g\|_0 \|\mathcal{D}.\nu_0\|_0 \|\alpha'\|_0^2.
\end{aligned}$$

The vector field ν_0 is the restriction to Σ of the gradient $\operatorname{grad} d$ of the signed distance from the plane $\partial\mathcal{H}$ (oriented in the suitable way), *independently on* ε. We can find two vector fields V_1, V_2 on a neighborhood of $\partial\mathcal{H}$ so that V_1, V_2 span the tangent space of the surface $S_\varepsilon\mathcal{H}$ for every $\varepsilon \leq \varepsilon_0$. The vector fields V_1, V_2 and $\operatorname{grad} d$ have covariant derivatives which are uniformly bounded, as we vary $\mathcal{H}$, since the half-spaces $\mathcal{H}$ must meet $\overline{B}$. The vector field ν_t can be obtained as

$$\frac{(F_t)_*(V_1) \times (F_t)_*(V_2)}{\|(F_t)_*(V_1) \times (F_t)_*(V_2)\|},$$

where $\times$ denotes the vector product. Therefore the first derivatives of ν_t are close to the ones of $\nu_0 = V_1 \times V_2 / \|V_1 \times V_2\|$, again uniformly in the half-space $\mathcal{H}$ meeting $\overline{B}$. This implies that the terms $\|\mathcal{D}.\nu_0\|_0$, $\|\mathcal{D}.\nu_t\|_0$ are uniformly controlled, and that $\|\mathcal{D}.\nu_t - \mathcal{D}.\nu_0\|_0$ can be bounded by some universal constant multiplied by $\|F_t - id\|_{\mathscr{C}^2(\overline{B})}$. Combining these observations with what previously done for the first inequality, we deduce the second claimed inequality. □

2.4.2 The variation of the dual volume

Given $\varepsilon \in [0, \varepsilon_0]$ and $t \in (-t_0, t_0)$, we define

$$v_\varepsilon^*(t) := \operatorname{Vol}^*_{M_t}(N_\varepsilon C M_t), \qquad u_\varepsilon^*(t) := \operatorname{Vol}^*_{M_t}(\varphi_t(N_\varepsilon C M_0)).$$

Our proof of Theorem A will be divided in some steps. The function that needs to be differentiated at $t = 0$ is $V_C^*(M_t) = v_0^*(t)$, in the notation above. However, this quantity is not easy to handle directly, because the variation of the geometric structure of CM_t is complicated. To overcome this problem, we will first study the family of functions u_ε^* in Lemma 2.4.3, and the limit $\lim_\varepsilon (u_\varepsilon^*)'(0)$ in Proposition 2.4.4. Here we will see how the differential of the length of the bending measure comes into play. Afterwards we will use the properties of the dual volume to relate $\lim_\varepsilon (u_\varepsilon^*)'(0)$ to the actual derivative $(v_0^*)'(0)$ in Proposition 2.4.5. In this manner we will conclude that the variation of the dual volume coincides, up to multiplicative constant, with the variation of the length of the realization of the bending measure of the convex core $\mu = \mu_0$. The last part of this subsection will be dedicated to relating this result with the differential of the length function of μ over the Teichmüller space.

Lemma 2.4.3. *The functions $u_\varepsilon^* \colon (-t_0, t_0) \to \mathbb{R}$ are smooth in t, and they converge $\mathscr{C}^\infty$-uniformly to u_0^* as ε goes to 0. Moreover, they satisfy*

$$(u_\varepsilon^*)'(0) = \frac{1}{4} \int_{S_\varepsilon C M_0} (\delta I_\varepsilon, I\!I_\varepsilon - H_\varepsilon I_\varepsilon)_\varepsilon \, \mathrm{d}a_\varepsilon \,,$$

where $(\cdot,\cdot)_\varepsilon$ denotes the scalar product on the space of 2-tensors induced by I_ε.

Proof. Let $u_\varepsilon(t)$ be $\operatorname{Vol}_{M_t}(\varphi_t(N_\varepsilon C M_0))$. Then the functions u_ε^* can be expressed as

$$u_\varepsilon^*(t) = u_\varepsilon(t) - \frac{1}{2} \int_{\varphi_t(S_\varepsilon C M_0)} H \, \mathrm{d}a \,.$$

We prove the regularity of u_ε^* in t by focusing on the two terms separately. By the choice we made of the family of diffeomorphisms $(\varphi_t)_t$ at the beginning of Section 2.4, the pullback $\varphi_t^* \, \mathrm{dvol}_{M_t}$ of the volume forms of M_t vary smoothly in t, and they can be expressed in the form $\varphi_t^* \operatorname{dvol}_{M_t} = f(t,\cdot)\operatorname{dvol}_{M_0}$, for some smooth function $f\colon (-t_0,t_0)\times M_0 \to \mathbb{R}$. If we denote by A_ε the subset $N_\varepsilon C M_0$ of M_0 for every $\varepsilon \in]0,\varepsilon_0]$, then the functions u_ε satisfy:

$$u_\varepsilon(t) = \int_{M_0} \mathbb{1}_{A_\varepsilon} f(t,\cdot)\operatorname{dvol}_{M_0},$$

where $\mathbb{1}_{A_\varepsilon}$ stands for the characteristic function of the set A_ε (i. e. $\mathbb{1}_{A_\varepsilon}(p) = 1$ if $p \in A_\varepsilon$, and $\mathbb{1}_{A_\varepsilon}(p) = 0$ otherwise). Observe that the sets A_ε are compact and they decrease, as ε goes to 0, to CM_0. As a consequence of the regularity of f in t, a simple application of the Lebesgue's dominated convergence theorem (see e. g. [Roy88]) proves the smoothness of the functions u_ε in t and their $\mathscr{C}^\infty$-uniform convergence to u_0.

To show the regularity of the second term of u_ε^*, we will describe a way to express the integral of the mean curvature as the integral of a suitable 2-form, from which the dependence in t and ε will be clearer.

Consider (M,g) an oriented Riemannian 3-manifold with volume form dvol_M. Given any point (p,v) of the tangent bundle TM, the Levi-Civita connection ∇ of M determines a natural splitting of the tangent space $T_{(p,v)}TM$ of the form $T_{(p,v)}TM = U_{(p,v)} \oplus W_{(p,v)}$, where $U_{(p,v)}$ is the vector subspace of $T_{(p,v)}TM$ tangent to the space of ∇-parallel vector fields at p, and $W_{(p,v)}$ is the tangent space at (p,v) to the fiber $T_pM \subset TM$, which can be naturally identified with T_pM. The differential of the bundle map $TM \to M$ at (p,v) has kernel equal to $W_{(p,v)}$, and it restricts to an isomorphism from $U_{(p,v)}$ to T_pM. This procedure determines a natural identification between $T_{(p,v)}TM$ and $(T_pM)^2$, which we will implicitly use in what follows. We define a 2-form ω_M over T^1M, the unit tangent bundle of M, as follows:

$$(\omega_M)_{(p,v)}((\dot p,\dot v),(\dot p',\dot v')) := \langle v, \dot p' \times \dot v - \dot p \times \dot v'\rangle$$

where $(p,v) \in T^1M$, $(\dot p,\dot v),(\dot p',\dot v') \in T_{(p,v)}T^1M \subset T_{(p,v)}M$, and $\langle\cdot,\cdot\rangle$ denotes the scalar product over T_pM. If S is an embedded surface in M, then the choice of a normal vector field on S determines a lift $\iota\colon S \to T^1M$, given by $\iota(p) = (p,n_p)$. Consider now e_1,e_2 a local orthonormal frame of S diagonalizing the shape operator B of S, i. e. $Be_i = -D_{e_i}n = \lambda_i e_i$ for $i = 1,2$, and locally satisfying $e_1 \times e_2 = n$. Then we have:

$$\begin{aligned}(\iota^*\omega_M)(e_1,e_2) &= \omega_M((e_1,D_{e_1}n),(e_2,D_{e_2}n))\\ &\langle n, e_2 \times (-Be_1) - e_1 \times (-Be_2)\rangle\\ &= \langle n, -\lambda_1\, e_2 \times e_1 + \lambda_2\, e_1 \times e_2\rangle\\ &= \lambda_1 + \lambda_2 = H.\end{aligned}$$

This shows in particular that, given any surface $S \subset M$, the integral of its mean curvature can be expressed as the integral over S of the 2-form $\iota^*\omega_M$, where ι is the lift of S to T^1M determined by its normal vector field. Consider now $\psi\colon M \to N$ a diffeomorphism between two Riemannian manifolds M and N, and define an induced map on the unit tangent bundles $\hat\psi\colon T^1M \to T^1N$ as follows:

$$\hat\psi(p,v) := \left(\psi(p), \frac{(\mathrm{d}\psi^{-1})^{\mathrm{ad}}_{\psi(p)}(v)}{\left\|(\mathrm{d}(\psi^{-1})^{\mathrm{ad}}_{\psi(p)}\,(v)\right\|}\right),$$

where ad stands for the adjoint map with respect to the scalar products on $T_p M$ and $T_{\psi(p)} N$. Given $v \in T^1_p M$, the vector $(\mathrm{d}\psi^{-1})^{\mathrm{ad}}_{\psi(p)}(v)$ is orthogonal (with respect to the metric of N) to the image under $\mathrm{d}\psi_p$ of the subspace $\langle v \rangle^\perp \subset T_p M$. This property implies that, if $\iota \colon S \to M$ is the lift of S to $T^1 M$, then $\hat{\psi} \circ \iota$ parametrizes the lift of $\psi(S)$ in $T^1 N$. In particular, combining this remark with what previously observed, we see that

$$\int_{\psi(S)} H \,\mathrm{d}a = \int_S (\hat{\psi} \circ \iota)^* \omega_N,$$

for every embedded surface $S \subset M$ and for every diffeomorphism $\psi \colon M \to N$ (up to sign for the choice of the normal direction).

The claimed regularity of the term in the mean curvature will now follows from this simple relation. To see this, let E be the subset of $T^1 M_0$ given by the pairs (p, ν) where $p \in \partial C M_0$ and ν is the exterior normal direction to a support half-space of $C M_0$ at p. Observe that, if p lies on an atomic leaf of the bending measured lamination with weight α, then there is a 1-parameter family of unit tangent vectors $(\nu_\vartheta)_{\vartheta \in [0,\alpha]}$ in $T^1_p M_0$ satisfying $(p, \nu_\vartheta) \in E$. The subset E describes a surface in the unit tangent bundle of M_0, which in a sense generalizes the notion of normal bundle to the singular surface $\partial C M_0$. If $\exp_t$ denotes the geodesic flow at time t on the unit tangent bundle of M_0, then the lifts of the surfaces $S_\varepsilon C M_0$ in $T^1 M_0$ are parametrized by the maps $\iota_\varepsilon \colon E \to T^1 M_0$, with $\iota_\varepsilon(p, \nu) = \exp_\varepsilon(p, \nu)$ (here the resulting normal vector field is the exterior one). The lift of a fixed surface $S_{\varepsilon_0} C M_0$ is $\mathscr{C}^{0,1}$, with Lipschitz constant of the first derivatives that a priori depends on ε_0. However, since the geodesic flow $(\exp_{-\varepsilon})_{\varepsilon \leq \varepsilon_0}$ is uniformly $\mathscr{C}^2$ over the compact set $T^1 M_0|_{N_{2\varepsilon_0} C M_0}$, and since $\exp_{-\varepsilon'} \circ \iota_\varepsilon = \iota_{\varepsilon - \varepsilon'}$ for all $\varepsilon' < \varepsilon$, the Lipschitz constants of the first-order derivatives of the *lifts* of surfaces $S_\varepsilon C M_0$ are uniformly bounded in $\varepsilon \in [0, \varepsilon_0]$ (observe that this is not the case if we look at the second-order derivatives of $S_\varepsilon C M_0$ before lifting them to the unitary tangent bundle). This remark shows in particular that the surface E is $\mathscr{C}^{0,1}$, and that the functions ι_ε converge $\mathscr{C}^{0,1}$-uniformly to id_E as ε goes to 0. Let now $\omega_t = \omega_{M_t}$ denote the natural 2-form over the manifold $T^1 M_t$ described as above. Then, by the formula we showed, we have:

$$-\int_{\varphi_t(S_\varepsilon C M_0)} H \,\mathrm{d}a = \int_E (\hat{\varphi}_t \circ \iota_\varepsilon)^* \omega_t = \int_E \iota_\varepsilon^* (\hat{\varphi}_t^* \omega_t).$$

Since the maps ι_ε are uniformly $\mathscr{C}^{0,1}$, the forms $(\hat{\varphi}_t \circ \iota_\varepsilon)^* \omega_t$ are $L^\infty(\Sigma, \mathrm{d}a_E)$ uniformly in ε and smooth in t, for fixed area form $\mathrm{d}a_E$ on E (area forms on a $\mathscr{C}^{0,1}$-surface are defined almost everywhere). In particular, by applying again the Lebesgue's dominated convergence theorem we see that the quantity $\int_{\varphi_t(S_\varepsilon C M_0)} H \,\mathrm{d}a$ is smooth in t and it converges $\mathscr{C}^\infty$-uniformly as ε goes to 0.

Finally, the first-order variation at $t = 0$ in the statement is an immediate consequence of the differential Schläfli formula in Proposition 2.2.5, and the fact that $\varphi_0 = id$. □

Proposition 2.4.4. *Assume that $(M_t)_t$ is a 1-parameter family of convex co-compact manifolds as above. Then we have:*

$$(u_0^*)'(0) = \lim_{\varepsilon \to 0} (u_\varepsilon^*)'(0) = -\frac{1}{2} \iint_\lambda \mathrm{d}\dot{\ell} \,\mathrm{d}\mu \,,$$

where μ is the bending measure of $\partial CM = \partial CM_0$ and λ is a geodesic lamination containing $\operatorname{supp}\mu$.

Proof. As already observed, we can divide the surface $S_\varepsilon CM = S_\varepsilon CM_0$ in two regions:

- the open set $S_\varepsilon^f := r^{-1}(\partial CM \setminus \lambda) \cap S_\varepsilon CM$ (f stands for flat), namely the portion of $S_\varepsilon CM$ that projects onto the union of the interior of the flat pieces of ∂CM;
- the closed set $S_\varepsilon^b := r^{-1}(\lambda)$ (b stands for bent), namely the portion of $S_\varepsilon CM$ that projects onto the bending lamination.

On the portion S_ε^f we have an explicit description of all the geometric quantities, by Lemma 2.1.3. In particular, we can write the integral in terms of the hyperbolic metric on the flat parts, obtaining

$$\begin{aligned}\int_{S_\varepsilon^f} (\delta I_\varepsilon, I\!I_\varepsilon - H_\varepsilon I_\varepsilon)_\varepsilon \, \mathrm{d}a_\varepsilon &= \sum_{F \subset \partial CM_0 \setminus \lambda} \int_F ((\delta I_\varepsilon, -\tanh\varepsilon \, I_\varepsilon)_\varepsilon \circ r) \cosh^2\varepsilon \, \mathrm{d}a_F \\ &= -\sinh\varepsilon \cosh\varepsilon \int_{\partial CM_0 \setminus \lambda} (\delta I_\varepsilon, I_\varepsilon)_\varepsilon \circ r \, \mathrm{d}a \,,\end{aligned}$$

where the sum is taken over all the flat pieces F in $\partial CM \setminus \lambda$. The variation of the first fundamental form δI_ε is the restriction of $\dot g = \frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* g_{M_t}\big|_{t=0}$ to the tangent space of $S_\varepsilon CM$. In particular, since $S_\varepsilon CM$ lies in a compact set K of $M = M_0$, the function $(\delta I_\varepsilon, I_\varepsilon)_\varepsilon$ is uniformly bounded. In conclusion, we obtain

$$\lim_{\varepsilon \to 0} \int_{S_\varepsilon^f} (\delta I_\varepsilon, I\!I_\varepsilon - H_\varepsilon I_\varepsilon)_\varepsilon \, \mathrm{d}a_\varepsilon = -\lim_{\varepsilon \to 0} \sinh\varepsilon \cosh\varepsilon \int_{\partial CM \setminus \mu} (\delta I_\varepsilon, I_\varepsilon)_\varepsilon \circ r \, \mathrm{d}a = 0.$$

Therefore, the only contribution to $\lim (u_\varepsilon^*)'(0)$ is given by S_ε^b.

For convenience, we lift our study to the universal cover $\pi \colon \widetilde{M} \cong \mathbb{H}^3 \to M$. We will first set our notation. The convex subset $\widetilde{C} := \pi^{-1}(CM)$ has a metric projection $\tilde r \colon \mathbb{H}^3 \to \widetilde{C}$. Its boundary $\partial \widetilde{C}$ is bent along the lamination $\tilde\lambda := \pi^{-1}(\lambda)$, and it is parametrized by a locally convex pleated surface $\tilde f \colon \widetilde{S} \to \mathbb{H}^3$, having bending locus $\tilde f^{-1}(\tilde\lambda)$. The preimage $\pi^{-1}(S_\varepsilon^b)$, which coincides with $S_\varepsilon \widetilde{C} \cap \tilde r^{-1}(\tilde\lambda)$, will be denoted by $\widetilde{S}_\varepsilon^b$. Consider a short arc k in $\widetilde{S}$ with a neighborhood U on which $\tilde f$ is a nice embedding and set $W := \operatorname{int}(\tilde r^{-1} \tilde f(U)) \subseteq \mathbb{H}^3 \setminus \widetilde{C}$. Our actual goal is to compute

$$\lim_{\varepsilon \to 0} \int_{W \cap \widetilde{S}_\varepsilon^b} (\delta I_\varepsilon, I\!I_\varepsilon - H_\varepsilon I_\varepsilon)_\varepsilon \, \mathrm{d}a_\varepsilon \,. \tag{2.5}$$

We will make use of a construction described in [CEM06, Section II.2.4]: there the authors illustrate an explicit way to extend the lamination $\tilde\lambda$ to a partial foliation $\mathcal{L} = \mathcal{L}_\eta$ of $\partial\widetilde{C}$, defined in the η-neighborhood (with respect its hyperbolic path metric) of $\tilde\lambda$, for any fixed $\eta < \log 3/2$. We briefly recall here the idea of the construction. Let T be an ideal triangle in $\mathbb{H}^2$, and denote by U_η the η-neighborhood of ∂T in T, with η small. Then the region of those points in U_η that are very close to exactly two edges of T, sharing an ideal vertex v, can be foliated using geodesic arcs asymptotic to v, while the region of those points that are very close to exactly one edge e of T can be foliated by equidistant curves from e. Defining a proper extension of this foliation in the regions of transition between these two behaviors in U_η, we can build a foliation on U_η that extends the geodesic lamination of ∂T. Applying this construction to each ideal triangle in the pleated boundary of $\widetilde{C}$, we can construct the desired extension $\mathcal{L}$ (see [CEM06, Section II.2.4] for a more precise description).

Up to taking a smaller neighborhood U of k, we can assume that $\tilde{f}(U) \subset \bigcup \mathcal{L}$ and we can choose a continuous orientation of the foliation $\mathcal{L} \cap \tilde{f}(U)$. Analogously to what is done in [CEM06, Section II.2.11], we define three orthonormal vector fields on W as follows:

1. the first vector field ν is given by the opposite of the gradient of the distance from $\widetilde{C}$;

2. the second vector field E_1 is defined in terms of the oriented foliation $\mathcal{L} \cap \tilde{f}(U)$. If p lies in W, its projection $r(p)$ belongs to an oriented leaf $\tilde{f}(\gamma)$ of $\mathcal{L} \cap \tilde{f}(U)$. We denote by w the unitary vector of $T_{r(p)}\mathbb{H}^3$ tangent to $\tilde{f}(\gamma)$, and we define $E_1(p)$ to be the parallel translation of w along the geodesic arc in $\mathbb{H}^3$ connecting $r(p)$ to p.

3. the last vector field E_2 is defined requiring that (E_1, E_2, ν) is a positively oriented orthonormal frame of $T\mathbb{H}^3$ in W (assume we have fixed an orientation of $\mathbb{H}^3$ since the beginning).

Observe that the E_i's are tangent to the surfaces $S_\varepsilon\widetilde{C} \cap W$, since they are orthogonal to the gradient of the distance. Therefore, they define two orthogonal oriented foliations on $S_\varepsilon\widetilde{C} \cap W$ for every ε. Moreover, if $r(p) \in \tilde{\lambda}$, then $E_1(p)$ is a principal direction for the equidistant surface $S_\varepsilon\widetilde{C}$ passing through p. In particular, we have that $I\!I_\varepsilon(E_1, E_1) \equiv \tanh\varepsilon$ (it is a direct consequence of the relations in Lemma 2.1.4). Expanding the expression $(\delta I_\varepsilon, I\!I_\varepsilon - H_\varepsilon I_\varepsilon)_\varepsilon$ in terms of this orthonormal frame over $W \cap \widetilde{S}^b_\varepsilon$ we have

$$\begin{aligned}(\delta I_\varepsilon, I\!I_\varepsilon - H_\varepsilon I_\varepsilon)_\varepsilon &= -(\delta I_\varepsilon)(E_1, E_1) I\!I_\varepsilon(E_2, E_2) - (\delta I_\varepsilon)(E_2, E_2) I\!I_\varepsilon(E_1, E_1) \\ &= -(\delta I_\varepsilon)(E_1, E_1) I\!I_\varepsilon(E_2, E_2) + O(\dot{g}|_K; \varepsilon).\end{aligned}$$

Since the area of $W \cap \widetilde{S}^b_\varepsilon$ goes to 0 as ε goes to 0, the integral of the term $O(\dot{g}|_K; \varepsilon)$ in the expression (2.5) has limit 0. In the end, it remains to study

$$\lim_{\varepsilon \to 0} \int_{W \cap \widetilde{S}^b_\varepsilon} (\delta I_\varepsilon)(E_1, E_1) I\!I_\varepsilon(E_2, E_2)\, \mathrm{d}a_\varepsilon = \lim_{\varepsilon \to 0} \int_{W \cap \widetilde{S}^b_\varepsilon} (\delta I_\varepsilon)_{11} (I\!I_\varepsilon)_{22}\, \mathrm{d}a_\varepsilon\, .$$

We denote by $\mathcal{L}^1_\varepsilon$, $\mathcal{L}^2_\varepsilon$ the foliations on $\widetilde{S}^b_\varepsilon \cap W$ tangent to E_1, E_2, and by $\mathrm{d}\ell^1_\varepsilon$, $\mathrm{d}\ell^2_\varepsilon$ their length elements, respectively. Then we can write

$$\int_{W \cap \widetilde{S}^b_\varepsilon} (\delta I_\varepsilon)_{11} (I\!I_\varepsilon)_{22}\, \mathrm{d}a_\varepsilon = \int_{\mathcal{L}^2_\varepsilon} \left(\int_{\mathcal{L}^1_\varepsilon} (\delta I_\varepsilon)_{11}\, \mathrm{d}\ell^1_\varepsilon \right) (I\!I_\varepsilon)_{22}\, \mathrm{d}\ell^2_\varepsilon\, . \tag{2.6}$$

Now it is time to see how this expression behaves in the finitely bent case. Assume that $\tilde{f}(U)$ meets a unique geodesic arc γ in $\tilde{\lambda}$ with bending angle θ_0. Then, in the coordinates described in Lemma 2.1.4, the vector fields E_1 and

E_2 can be written as $E_1 = (\cosh\varepsilon)^{-1}\partial_s^\varepsilon$, $E_2 = (\sinh\varepsilon)^{-1}\partial_\theta^\varepsilon$. Therefore the following relations hold

$$(\delta I_\varepsilon)_{11}\,\mathrm{d}\ell_\varepsilon^1 = \frac{\dot g(\partial_s^\varepsilon,\partial_s^\varepsilon)}{\cosh^2\varepsilon}\,\mathrm{d}(\cosh\varepsilon\, s) \qquad (I\!I_\varepsilon)_{22}\,\mathrm{d}\ell_\varepsilon^2 = \frac{\cosh\varepsilon}{\sinh\varepsilon}\,\mathrm{d}(\sinh\varepsilon\,\theta)\,.$$

where $\dot g = \dot g_0$. In particular, the limit as $\varepsilon \to 0$ of the expression (2.6) becomes

$$\lim_{\varepsilon\to 0}\int_{\mathcal{L}_\varepsilon^2}\left(\int_{\mathcal{L}_\varepsilon^1}(\delta I_\varepsilon)_{11}\,\mathrm{d}\ell_\varepsilon^1\right)(I\!I_\varepsilon)_{22}\,\mathrm{d}\ell_\varepsilon^2 = \theta_0\int_\gamma \dot g(\gamma',\gamma')\,\mathrm{d}\ell = 2\iint_{\tilde\lambda\cap W}\mathrm{d}\dot\ell\,\mathrm{d}\mu\,.$$

To prove this relation in the general case, we make use of the standard approximations of Definition 2.1.6. The bending measures along the arc k of the finitely bent approximations $\tilde f_n$ weak*-converge to μ along k; the ε-surfaces from the $\tilde f_n$'s converge $\mathscr{C}^{1,1}$-uniformly to $W \cap S_\varepsilon\widetilde{C}$; the vector fields $E_{1,n}$, $E_{2,n}$ and ν_n, defined from the surface $\tilde f_n(U)$, converge uniformly to E_1, E_2 and ν over all the compact subsets of W. From these properties, the relation we proved in the finitely bent case extends to the general one.

Finally, a suitable choice of a partition of unity on a neighborhood of the bending lamination μ, combined with Lemma 2.4.3, proves the statement. □

Proposition 2.4.5. *Assume $(M_t)_t$ is a 1-parameter family of convex co-compact manifolds as above. Then there exists the derivative of $V_C^*(M_t)$ at $t = 0$ and it verifies*

$$\mathrm{d}V_C^*(\dot M) = -\frac{1}{2}\iint_\lambda \mathrm{d}\dot\ell\,\mathrm{d}\mu\,.$$

Proof. The left-hand side is nothing but the limit of the incremental ratio of the function v_0^* at $t = 0$. Let K, τ be the constants furnished by Lemma 2.4.1. We split our incremental ratio as follows:

$$\frac{v_0^*(t)-v_0^*(0)}{t} = \underbrace{\frac{u_{K|t|}^*(t)-u_{K|t|}^*(0)}{t}}_{\text{term 1}} + \underbrace{\frac{v_{K|t|}^*(0)-v_0^*(0)}{t}}_{\text{term 2}} - \underbrace{\frac{u_{K|t|}^*(t)-v_0^*(t)}{t}}_{\text{term 3}},$$

where we used the fact that $u_\varepsilon^*(0) = v_\varepsilon^*(0)$ for all $\varepsilon > 0$. In Lemma 2.4.3, we showed that the functions u_ε^* are smooth in t and that they converge $\mathscr{C}^\infty$-uniformly to u_0^* as ε goes to 0. Using the first-order expansion of u_ε^* at $t = 0$ and evaluating for $\varepsilon = K|t|$, we have:

$$\frac{u_{K|t|}^*(t)-u_{K|t|}^*(0)}{t} = (u_{K|t|}^*)'(0) + O((u_{K|t|})''(\xi_t);t),$$

where the constant involved in the $O(t)$ depends a priori on the value of $(u_{K|t|}^*)''$ in a point ξ_t close to 0. However, thanks to the $\mathscr{C}^\infty$-uniform convergence of the functions $(u_\varepsilon^*)_\varepsilon$, the second derivatives $(u_\varepsilon^*)''$ can be bounded uniformly in ε over a small neighborhood of 0, so the term $O((u_{K|t|})''(\xi_t);t)$ is an actual $O(t)$. By Proposition 2.4.4 we conclude that the limit of the first term in the decomposition above is equal to $-\frac{1}{2}\iint_\lambda \mathrm{d}\dot\ell\,\mathrm{d}\mu$. In what follows we will show that the second and third terms of the splitting of the incremental ratio are converging to 0 as t goes to 0.

By Proposition 2.2.4 applied to the 3-manifold M_s, for every $\varepsilon > 0$ we have

$$v^*_\varepsilon(s) - v^*_0(s) = \mathrm{Vol}^*_{M_s}(N_\varepsilon CM_s) - \mathrm{Vol}^*_{M_s}(CM_s) = O(\ell_{m_s}(\mu_s), \chi(\partial CM_s); \varepsilon^2). \tag{2.7}$$

In particular, for $s = 0$ and $\varepsilon = K|t|$, this relation proves that the second term goes to 0.

Let $L > 1$ be a constant so that all the diffeomorphisms φ_t are L-Lipschitz on a large compact set in M_0 containing the convex core CM_0. It is immediate to see that the following properties hold:

$$\varphi_t(N_\varepsilon CM_0) \subseteq N_{L\varepsilon}\varphi_t(CM_0) \qquad \text{for every } \varepsilon > 0,$$
$$N_{\varepsilon'} N_\varepsilon CM_t \subseteq N_{\varepsilon'+\varepsilon} CM_t \qquad \text{for every } \varepsilon',\, \varepsilon > 0.$$

Applying Lemma 2.4.1 to the 3-manifold M_t and using the inclusion relations above, we obtain the following chain:

$$CM_t \subseteq \varphi_t(N_{K|t|}CM_0) \subseteq N_{LK|t|}\varphi_t(CM_0) \subseteq N_{LK|t|}N_{Kt}CM_t \subseteq N_{(L+1)K|t|}CM_t.$$

for all $t \in (-\tau, \tau)$. All the submanifolds involved are compact convex subsets of M_t, hence we are allowed to consider their dual volumes. Using the monotonicity of $\mathrm{Vol}^*_{M_t}$, proved in Proposition 2.2.6, we get

$$v^*_0(t) \geq u^*_{K|t|}(t) \geq v^*_{(L+1)K|t|}(t) \qquad \text{for all } t \in (-\tau, \tau).$$

Applying this to estimate the third term, we obtain

$$0 \geq \frac{u^*_{K|t|}(t) - v^*_0(t)}{t} \geq \frac{v^*_{(L+1)K|t|}(t) - v^*_0(t)}{t}. \tag{2.8}$$

Since the constants K and L only depend on the family $(\varphi_t)_t$, if we apply the equation (2.7) with $s = t$ and $\varepsilon = (L+1)K|t|$, we get

$$v^*_{(L+1)K|t|}(t) - v^*_0(t) = O((\varphi_t)_t, \ell_{m_t}(\mu_t), \chi(\partial CM_t); t^2).$$

Consequently, the right side in the inequality (2.8) goes to 0 as t goes to 0, and so does the third term, which concludes the proof. □

Given $\mu \in \mathcal{ML}(S)$, we define the *length function of* μ as the map $L_\mu \colon \mathcal{T}(S) \to \mathbb{R}_{\geq 0}$ from the Teichmüller space of S to $\mathbb{R}_{\geq 0}$ which associates to the hyperbolic metric $m \in \mathcal{T}(S)$ the length of μ with respect to the metric m. The functions L_μ are real-analytic, since they are restrictions of holomorphic functions over the set of quasi-Fuchsian groups (see [Ker85, Corollary 2.2]).

The dependence of the geometry of the convex core CM on the hyperbolic structure of M is a subtle problem. In [KS95] the authors established the continuity of the hyperbolic metric and the bending measure of ∂CM with respect to the structure of M. A much more sophisticated analysis, involving the notion of *Hölder cocycles*, allowed Bonahon to describe more precisely the regularity of these maps, as done in [Bon98b]. In the following, we recall a parametrization result from [Bon96], which was an essential tool in the study of [Bon98b].

Fixed a maximal lamination λ on a surface S, we say that a representation ρ of $\pi_1(S)$ in $\mathrm{Iso}^+(\mathbb{H}^3)$ realizes λ if there exists a pleated surface $\tilde{f}$ with holonomy

ρ and pleating locus contained in λ. Let $\mathcal{R}(\lambda)$ be the set of conjugacy classes of homomorphisms realizing λ, which is open in the character variety of $\pi_1(S)$ and in bijection with the space of pleated surfaces with bending locus λ, up to a natural equivalence relation. [Bon96, Theorem 31] describes a biholomorphic parametrization of $\mathcal{R}(\lambda)$ in terms of the hyperbolic metric and the bending cocycle of the pleated surface realizing $\rho \in \mathcal{R}(\lambda)$. In particular, we denote by $\psi_\lambda \colon \mathcal{R}(\lambda) \to \mathcal{T}(S)$ the map associating to $[\rho]$ the hyperbolic metric of the pleated surface with holonomy ρ.

Now, let M be a hyperbolic convex co-compact manifold. Denote by $\mathcal{QD}(M)$ the space of quasi-isometric deformations of M, and by $\mathcal{R}(\partial CM)$ the representation variety of $\pi_1(\partial CM)$ in $\mathrm{Iso}^+(\mathbb{H}^3)$. We have a natural map $R \colon \mathcal{QD}(M) \to \mathcal{R}(\partial CM)$ which associates to a convex co-compact hyperbolic structure M' on M the conjugacy class of the holonomy $[\rho']$ of $\partial CM'$. If λ is a maximal lamination of $\partial CM'$ extending the support of the bending measure of $\partial CM'$, then ψ_λ is defined on a open neighborhood of $[\rho']$, therefore we are allowed to consider the map $\psi_\lambda \circ R$. The result of [Bon98b] we need is the following:

Theorem 2.4.6 ([Bon98b, Theorem 1])**.** *Let M be a hyperbolic convex co-compact manifold and denote by $\mathcal{QD}(M)$ the space of quasi-isometric deformations of M. Then the map $Q \colon \mathcal{QD}(M) \to \mathcal{T}(\partial CM)$ associating to the structure M' the hyperbolic metric on $\partial CM'$, is continuously differentiable. Moreover, given any maximal lamination extending the support of the bending measure of CM', the differential of Q at M' coincides with the differential of the map $\psi_\lambda \circ R$ at M'.*

We are finally ready to prove the variation formula for the dual volume of the convex core of a convex co-compact hyperbolic manifold:

Theorem A. *Let $(M_t)_t$ be a smooth 1-parameter family of quasi-isometric hyperbolic convex co-compact manifolds, with $M_0 = M$. Denote by $\mu \in \mathcal{ML}(\partial CM)$ the bending measure of the convex core of M and let $t \mapsto m_t \in \mathcal{T}(\partial CM)$ be the family of hyperbolic metrics m_t associated to the boundary of the convex core CM_t at the time t. Then the dual volume of the convex core $V_C^*(M_t)$ admits derivative at $t = 0$, and it verifies*

$$\mathrm{d}V_C^* (\dot{M}) = -\frac{1}{2} \, \mathrm{d}L_\mu (\dot{m}).$$

Proof. By Proposition 2.4.5, the derivative of $V_C^*(M_t)$ at $t = 0$ exists and it coincides with $\lim_\varepsilon (u_\varepsilon^*)'(0)$. By Proposition 2.4.4, we have the equality

$$\mathrm{d}V_C^* (\dot{M}) = -\frac{1}{2} \iint_\lambda \mathrm{d}\dot{\ell} \, \mathrm{d}\mu \, ,$$

where $\lambda = \operatorname{supp} \mu$. By Theorem 2.4.6, given a maximal lamination λ containing $\lambda = \operatorname{supp} \mu$, the variation of the hyperbolic metric $\tilde{m}_t$ of the pleated surface in M_t realizing λ coincides with the variation of the hyperbolic metric m_t on the boundary of the convex core CM_t. By definition, the quantity $\iint \mathrm{d}\dot{\ell} \, \mathrm{d}\mu$ coincides with $\frac{\mathrm{d}}{\mathrm{d}t} L_\mu(\tilde{m}_t)|_{t=0}$. Therefore, we obtain that

$$\mathrm{d}L_\mu (\dot{m}) = \iint_\lambda \mathrm{d}\dot{\ell} \, \mathrm{d}\mu \, ,$$

which proves the statement. □

Chapter 3

The dual volume of quasi-Fuchsian manifolds and the Weil-Petersson distance

Outline of the chapter

The aim of this Chapter is to prove Theorem B, which we recall here for convenience:

Theorem B. *There exists an explicit positive constant $C \approx 7.3459$ such that, for every quasi-Fuchsian manifold M homeomorphic to $\Sigma \times \mathbb{R}$, we have*

$$|V_C^*(M)| \leq C\ (g-1)^{1/2}\ d_{WP}(m^-(M), m^+(M)),$$

where $V_C^*(M)$ denotes the dual volume of the convex core CM of M, and $m^\pm(M)$ are the hyperbolic structures on the upper/lower components of CM.

Before describing the structure of the chapter, we remark some consequences and observations concerning this statement. The dual volume and the hyperbolic volume of the convex core differ by the term $\frac{1}{2}L_\mu(m)$, which is bounded by $6\pi|\chi(\Sigma)|$, as shown in [BBB19]. Moreover, the structures $m^\pm(M)$ and the conformal structures at infinity $c^\pm(M)$ of M are at bounded Weil-Petersson distance from each other, by the works of Linch [Lin74] and Sullivan [Sul81b] (see also Epstein and Marden [CEM06, Part II]). Therefore, Theorem B can be used to give an alternative proof of Brock's upper bound in [Bro03] and to exhibit *explicit constants* satisfying the inequality, with a fairly simple argument.

Our way to proceed is analogous to the one used by Schlenker [Sch13] to obtain a bound of the renormalized volume $V_R(M)$ in terms of the Weil-Petersson distance between the conformal structures at infinity of M. The key ingredients in the work [Sch13] are the variation formula of the renormalized volume $V_R(M)$ and the Nehari's bound of the norm of the Schwarzian derivative of the complex projective structures at infinity of $\partial_\infty M$. In particular, the author showed that, for every quasi-Fuchsian manifold M, we have:

$$V_R(M) \leq 3\sqrt{\pi}(g-1)^{1/2}\ d_{WP}(c^+(M), c^-(M)). \tag{3.1}$$

We remark that the multiplicative constant C appearing in our statement is larger than the one obtained using the renormalized volume, $3\sqrt{\pi} \approx 5.3174 < 7.3459 \approx C$. Therefore, the inequality (3.1) is more efficient in terms of coarse estimates.

Nevertheless, Theorem B carries more information than its implications concerning the coarse Weil-Petersson geometry, in particular when we consider quasi-Fuchsian structures that are close to the Fuchsian locus. In this case, Theorem B and the inequality (3.1) furnish complementary insights, since they involve the Weil-Petersson distance between the hyperbolic structures, on one side, and the conformal structures at infinity on the other. Moreover, Proposition 3.2.4 and its application for the bound of the dual volume show that the multiplicative constant in Theorem B can be improved if we have a better control of $L_\mu(m)$ than the one from [BBB19], exactly as the inequality (3.1) can be improved if we have a better control of the L^∞-norm of the Schwarzian at infinity than the Nehari's bound.

The chapter is organized as follows. In Section 3.1 we recall the description of the tangent and cotangent bundles of the Teichmüller space $\mathcal{T}(\Sigma)$ of a surface Σ, first using Beltrami differentials and holomorphic quadratic differentials, and afterwards, following [Tro92], using traceless and divergence free (also called transverse traceless) symmetric tensors. The Section ends with a simple Lemma describing the relation between the two equivalent interpretations and between their natural norms.

Section 3.2 is dedicated to the proof of Proposition 3.2.4, in which we produce a uniform bound of the differential of $L_\mu \colon \mathcal{T}^{\mathfrak{h}}(\Sigma) \to \mathbb{R}$, the hyperbolic length function of a measured lamination μ over the Teichmüller space. This is the main "quantitative" ingredient for the proof of Theorem B. The proof proceeds as follows. We represent a variation of hyperbolic metrics $\dot{m}$ as the real part of a holomorphic quadratic differential q. Using standard properties of holomorphic functions, the pointwise norm of q at x can be bounded by the L^p-norm of q over some embedded geodesic ball in Σ centered at x. The variation of L_μ can be expressed as an integral over the support of μ of the product of the variation of the length measure of $\dot{m}$ times the transverse measure of μ. Then the result will follow using the pointwise estimation and a Fubini's exchange of integration over a suitable finite cover of Σ.

In Section 3.3 we obtain a uniform control of the differential of V_C^*, the dual volume of the convex core function over the space of quasi-Fuchsian manifolds, in terms of the norm of the variation of the hyperbolic metrics on ∂CM. To do so, we will apply the works of Bridgeman, Canary, and Yarmola [BCY16] and Bridgeman, Brock, and Bromberg [BBB19], which give universal controls of the bending measure of the convex core. These results are to the dual volume as the Nehari's bound of the norm of the Schwarzian derivative is to the renormalized volume (the bounds obtained in [BBB19] are actually proved *using* Nehari's bound). The dual Bonahon-Schläfli formula (Theorem A) relates the variation of V_C^* with the differential of the length of the bending measured lamination, and the mentioned universal bounds combined with Proposition 3.2.4 will produce the desired control of $\mathrm{d}V_C^*$ (see Corollary 3.3.6).

In Section 3.4 we will finally give a proof of Theorem B. Contrary to what happens for the conformal structures at infinity, the hyperbolic structures on ∂CM are only conjecturally thought to give a parametrization of the space of quasi-Fuchsian manifolds. Because of this, proving Theorem B from Corollary 3.3.6 is not as immediate as it is for the renormalized volume using its variation formula. Our procedure to overcome to this difficulty passes through the foliation of hyperbolic ends by constant Gaussian curvature surfaces Σ_k, with $k \in (-1, 0)$, and the notion of landslide, which is a "smoother" analogue of earthquakes between hyperbolic metrics on Σ introduced by Bonsante, Mondello, and Schlenker [BMS13] (see also [BMS15]). By the work of Schlenker [Sch06] and Labourie [Lab92a], the data of the metrics on the surfaces Σ_k parametrize the space of quasi-Fuchsian manifolds (see Theorem 3.4.1). Therefore, the strategy will roughly be to:

i) approximate the dual volume of the convex core $V_C^*(M)$ by the dual volume of the region enclosed by the k-surfaces of M, which we denote by $V_k^*(M)$;

ii) prove that the differentials of the functions V_k^* converge to the differential of V_C^* as k goes to -1, i. e. as the surfaces Σ_k get closer to the convex core CM;

iii) use the parametrization result for the metrics of Σ_k to deduce the statement of Theorem B via an approximation argument.

For point (ii), which is the most delicate part of our argument, we will highlight a connection between the differential of the functions V_k^* and the infinitesimal smooth grafting, introduced in [BMS13]. As described by McMullen [McM98], the earthquake map can be complexified using the notion of grafting along a measured lamination. In the same way the landslide admits a complex extension via the *smooth grafting map*. Moreover, the complex earthquake can be actually recovered by a suitable limit of complex landslides. Using this convergence procedure, we will be able to show that $\mathrm{d}V_C^*$ is the limit of the differentials $\mathrm{d}V_k^*$, in the sense described by Proposition 3.4.2. The rest of the proof of Theorem B will be an elementary application of the results from the previous section, similarly to what done in [Sch13] for the renormalized volume.

3.1 The Weil-Petersson metric

In the following, we will recall the definition of the Weil-Petersson Riemannian metric on the Teichmüller space (see Section 1.2 for the definition of the Teichmüller space).

Let X be a Riemann structure on Σ. A *Beltrami differential* on X is a $(1,1)$-tensor ν that can be expressed in local coordinates as $\nu = n\ \partial_z \otimes \mathrm{d}\bar{z}$, where n is a measurable complex-valued function. If $h = \rho|\mathrm{d}z|^2$ is the unique hyperbolic metric in the conformal class c, then for any $p \in [1, \infty)$ we define the L^p-norm of the Beltrami differential $\nu = n\ \partial_z \otimes \mathrm{d}\bar{z}$ to be

$$\|\nu\|_{B,p} := \left(\int_\Sigma |n|^p \rho \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p} .$$

When $p = \infty$, we set $\|\nu\|_{B,\infty} := \operatorname{ess\,sup}_\Sigma |n|$. We will denote by $B(X)$ the space of Beltrami differentials of X with finite L^∞-norm. Observe that the norm $\|\cdot\|_{B,2}$ on $B(X)$ is induced by the hermitian scalar product

$$\langle \nu, \mu \rangle_{B,2} = \int_\Sigma \bar{n} m\ \rho \,\mathrm{d}x\,\mathrm{d}y \,,$$

where $\nu = n\, \partial_z \otimes \mathrm{d}\bar{z}$ and $\mu = m\, \partial_z \otimes \mathrm{d}\bar{z}$.

A holomorphic quadratic differential on X is a symmetric 2-covariant tensor that can be locally written as $q = f\,\mathrm{d}z^2$, where f is holomorphic. In analogy to what was done above, for every $p \in [1, \infty)$ we define the L^p-norm of q to be

$$\|q\|_{Q,p} := \left(\int_\Sigma \frac{|f|^p}{\rho^{p-1}} \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p} .$$

When $p = \infty$, we set $\|q\|_{Q,\infty} := \operatorname{ess\,sup}_\Sigma |f|/\rho$. When $p = 2$, the norm $\|\cdot\|_{Q,2}$ is induced by a scalar product, defined as follows:

$$\langle q, q' \rangle_{Q,2} := \int_\Sigma \frac{f \bar{f}'}{\rho} \,\mathrm{d}x\,\mathrm{d}y \,.$$

There is a natural pairing between the space of bounded Beltrami differentials $B(X)$ and the space of holomorphic quadratic differentials $Q(X)$: given a Beltrami differential $\nu = n\, \partial_z \otimes \mathrm{d}\bar{z}$ and a holomorphic quadratic differential $q = f\,\mathrm{d}z^2$, we define

$$(q, \nu) := \int_\Sigma f n \,\mathrm{d}x\,\mathrm{d}y \,.$$

A Beltrami differential $\nu \in B(X)$ is *harmonic* if there exists a holomorphic quadratic differential $q = f\,\mathrm{d}z^2$ such that $\nu = \bar{f}/\rho\, \partial_z \otimes \mathrm{d}\bar{z}$. We denote by $B_h(X)$ the space of harmonic Beltrami differentials on X.

Let $N(X)$ be the subspace of $B(X)$ of those Beltrami differentials ν verifying $(q, \nu) = 0$ for every $q \in Q(X)$. As described in [GL00], the space $B_h(X)$ and $N(X)$ are in direct sum, and the quotient of $B(X)$ by the subspace $N(X)$ identifies with the tangent space to the Teichmüller space $T_X \mathcal{T}^{\mathfrak{c}}(\Sigma)$ (here we denote by X the isotopy class of the conformal structure, with abuse). Moreover, the pairing $(\cdot, \cdot)$ determines a natural isomorphism between the dual of $T_X \mathcal{T}^{\mathfrak{c}}(\Sigma)$ and the space of holomorphic quadratic differentials $Q(X)$, which is consequently identified with the cotangent space $T_X^* \mathcal{T}^{\mathfrak{c}}(\Sigma)$. The scalar product g_{WP} on $T_X \mathcal{T}^{\mathfrak{c}}(\Sigma)$ induced by $\mathrm{Re}\langle \cdot, \cdot \rangle_{B,2}$ defines the *Weil-Petersson metric* of the Teichmüller space $\mathcal{T}^{\mathfrak{c}}(\Sigma)$, and $\mathrm{Re}\langle \cdot, \cdot \rangle_{Q,2}$ determines the corresponding metric on the cotangent bundle to Teichmüller space. The skew-symmetric bilinear form $\omega_{WP} := \mathrm{Re}\langle \cdot, i \cdot \rangle_{B,2}$ is actually a symplectic structure, i. e. $\mathrm{d}\omega_{WP} = 0$ (see e. g. [Ahl61]) or, in other words, the complex manifold $(\mathcal{T}^{\mathfrak{c}}(\Sigma), g_{WP}, \omega_{WP})$ is Kähler.

Lemma 3.1.1. *For every $q \in Q(X)$ we have:*

$$\|q\|_{Q,2} = \sup_{\nu \in B(X) \setminus \{0\}} \frac{|(q, \nu)|}{\|\nu\|_{B,2}} = \sup_{\nu \in B_h(X) \setminus \{0\}} \frac{|(q, \nu)|}{\|\nu\|_{B,2}} .$$

Proof. By the Cauchy-Schwarz inequality we have $|(q, \nu)| \leq \|q\|_{Q,2} \|\nu\|_{B,2}$, with equality realized by the harmonic Beltrami differential ν_q, which satisfies $(q, \nu_q) = \|q\|_{Q,2}^2 = \|\nu_q\|_{B,2}^2$. Therefore we get:

$$\|q\|_{Q,2} \geq \sup_{\nu \in B(X) \setminus \{0\}} \frac{|(q, \nu)|}{\|\nu\|_{B,2}} \geq \sup_{\nu \in B_h(X) \setminus \{0\}} \frac{|(q, \nu)|}{\|\nu\|_{B,2}} \geq \|q\|_{Q,2} .$$

The first inequality holds because of Cauchy-Schwarz, the second one because $B_h(X) \subset B(X)$, and the last one by taking $\nu = \nu_q$. □

We recall now the Riemannian description of the Teichmüller space as developed in [Tro92]. Let $S^2(\Sigma)$ be the bundle of symmetric 2-tensors on Σ, and let $\Gamma(S^2(\Sigma))$ denote the space of its smooth sections, which is an infinite dimensional vector space. The space $\mathcal{M}$ of smooth Riemannian metrics on Σ identifies with an open convex subset of $\Gamma(S^2(\Sigma))$. Therefore, given any Riemannian metric g on Σ, the tangent space $T_g\mathcal{M}$ is canonically isomorphic to $\Gamma(S^2(\Sigma))$. The metric g determines a scalar product on $T_g\mathcal{M}$, which can be expressed as $(\sigma,\tau)_g := g^{ik}g^{jh}\sigma_{ij}\tau_{kh}$, for σ, τ in $\Gamma(S^2(\Sigma))$. The norm induced by this scalar product will be denoted by $\|\sigma\|_g^2 := \langle\sigma,\sigma\rangle_g$. Given $\sigma \in \Gamma(S^2(\Sigma))$, we define the g-divergence of σ to be the 1-form $(\operatorname{div}_g \sigma)(V) := \operatorname{tr}_g(\nabla_*\sigma)(*,V)$, for any V tangent vector field to Σ. Now we set

$$S^2_{tt}(\Sigma,g) := \{h \in \Gamma(S^2(\Sigma)) \mid \sigma \text{ is } g\text{-traceless and } \operatorname{div}_g \sigma = 0\}.$$

An element of $S^2_{tt}(\Sigma,g)$ is usually called a *tranverse traceless* tensor (with respect to the metric g). As shown in [Tro92], every element of $S^2_{tt}(\Sigma,g)$ can be written (uniquely) as the real part of a holomorphic quadratic differential $q \in Q(\Sigma,[g])$, and vice versa for every q, the tensor $\operatorname{Re} q$ belongs to $S^2_{tt}(\Sigma,g)$. In particular, the space $S^2_{tt}(\Sigma,g)$ depends only on the conformal class of the metric g. If g is a hyperbolic metric, then $S^2_{tt}(\Sigma,g)$ is tangent to the space $\mathcal{M}_{-1}$ of hyperbolic metrics on Σ, and it is transverse to the orbit of g by the action of the group of diffeomorphisms isotopic to the identity. Therefore, the tangent space of the Teichmüller space at the isotopy class of g can be identified with $S^2_{tt}(\Sigma,g)$.

For any open set $\Omega \subseteq \Sigma$ and any $p \in [1,\infty)$, the Fischer-Tromba p-norm of $\sigma \in S^2_{tt}(\Sigma,g)$ is defined as

$$\|\sigma\|_{FT,L^p(\Omega)} := \left(\int_\Omega \|\sigma\|_g^p \,\mathrm{dvol}_g\right)^{1/p},$$

where dvol_g is the area form induced by g. When $p=\infty$, we set $\|\sigma\|_{FT,L^\infty(\Omega)} := \sup_\Omega \|h\|_g$. If $\Omega = \Sigma$, we simply write $\|\cdot\|_{FT,p}$.

Let now m be a point of the Teichmüller space, and let g be a hyperbolic metric in the equivalence class m, with associated Riemann surface structure X.

Lemma 3.1.2. *The vector spaces $B_h(X)$ and $S^2_{tt}(\Sigma,g)$ are identified to $T_X\mathcal{T}^{\mathfrak{c}}(\Sigma) \cong T_m\mathcal{T}^{\mathfrak{h}}(\Sigma)$ through the linear isomorphism*

$$\begin{array}{ccc} B_h(X) & \longrightarrow & S^2_{tt}(\Sigma,g) \\ \nu_q & \longmapsto & 2\operatorname{Re} q. \end{array}$$

Moreover, for every $q \in Q(X)$ we have

$$\|\nu_q\|_{B,p} = \frac{1}{2\sqrt{2}}\|2\operatorname{Re} q\|_{FT,p}.$$

Proof. Let $g_t = \rho_t|\mathrm{d}z_t|^2$ be a smooth 1-parameter family of Riemannian metrics on Σ, with $g_0 = g$, and let $q = f\,\mathrm{d}z_0{}^2$ be a holomorphic quadratic differential on the Riemann surface $X = (\Sigma,[g])$. If we require the identity map $(\Sigma,g) \to (\Sigma,g_t)$ to be quasi-conformal with harmonic Beltrami differential

$$\nu^0_{tq} := \frac{t\bar{f}}{\rho_0}\,\partial_{z_0} \otimes \mathrm{d}\bar{z}_0\,,$$

then the Riemannian metric g_t can be expressed as

$$g_t = \rho_t \left| \frac{\partial z_t}{\partial z_0} \right|^2 |\mathrm{d}z_0|^2 + 2t\rho_t \left| \frac{\partial z_t}{\partial z_0} \right|^2 \operatorname{Re}\left(\frac{f}{\rho_0} \, \mathrm{d}z_0{}^2 \right) + O(t^2).$$

Therefore the first order variation of g_t at $t = 0$ coincides with

$$\dot{g} = \left(\frac{\mathrm{d}}{\mathrm{d}t} \, \rho_t \left| \frac{\partial z_t}{\partial z_0} \right|^2 \bigg|_{t=0} |\mathrm{d}z_0|^2 \right) + 2 \operatorname{Re} q.$$

The quantity $\dot{g}$ identifies with a tangent vector to the space $\mathcal{M}$ of Riemannian metrics over Σ at the point g. The first term in the expression above is conformal to the Riemannian metric g, hence it is tangent to the conformal class $[g] \subset \mathcal{M}$. The remaining term $2 \operatorname{Re} q$ is a symmetric, g-traceless and divergence-free tensor, so it lies in the subspace $S^2_{tt}(\Sigma, g)$ of $T_g \mathcal{M}$.

The computation above proves that the harmonic Beltrami differential ν_q, seen as an element of $T_X \mathcal{T}^{\mathfrak{c}}(\Sigma)$, corresponds to $2 \operatorname{Re} q \in S^2_{tt}(\Sigma, g) \cong T_m \mathcal{T}^{\mathfrak{h}}(\Sigma)$. Finally, an explicit computation shows the relation between the norms $\|\cdot\|_{B,p}$ and $\|\cdot\|_{FT,p}$. □

3.2 A bound of the differential of the length

Let $\mathcal{ML}(\Sigma)$ denote the space of measured laminations of Σ (see Section 1.2.2 and Section 2.3 for the definition of this notion). The aim of this section is to produce, given $\mu \in \mathcal{ML}(\Sigma)$, a quantitative upper bound of the differential of the length function $L_\mu \colon \mathcal{T}^{\mathfrak{h}}(\Sigma) \to \mathbb{R}$, which associates to every class of hyperbolic metrics $m \in \mathcal{T}^{\mathfrak{h}}(\Sigma)$ the length of the m-geodesic realization of μ. This estimate is the content of Proposition 3.2.4, which will be our main technical ingredient to produce the upper bound of the dual volume in terms of the Weil-Petersson distance between the hyperbolic metrics on the convex core of a quasi-Fuchsian manifold.

We briefly sketch the structure of this section: Lemma 3.2.1 describes a natural way to express the differential of L_μ applied to a first order variation of hyperbolic metrics $\dot{g}$. Lemma 3.2.2 uses the properties of holomorphic functions to bound the pointwise value of a holomorphic quadratic differential at $x \in \Sigma$ with its L^p-norm on the ball centered at x. Then Proposition 3.2.4 will follow by selecting a first order variation $\dot{g}$ in $S^2_{tt}(\Sigma, g)$ and then carefully applying the bound of Lemma 3.2.2 in the expression found in Lemma 3.2.1.

Let $m \in \mathcal{T}^{\mathfrak{h}}(\Sigma)$ and $\mu \in \mathcal{ML}(\Sigma)$. Given a hyperbolic metric g in the equivalence class m, we identify the measured lamination μ with its g-geodesic realization inside (Σ, g). If λ is a g-geodesic lamination of Σ containing the support of μ, we can cover λ by finitely many flow boxes $\sigma_j \colon I \times I \to B_j$, where $I = [0, 1]$ and σ_j is a homeomorphism verifying $\sigma_j^{-1}(\lambda) = D_j \times I$, for some closed subset D_j of I. We select also a collection $\{\xi_j\}_j$ of smooth functions with supports contained in the interior of B_j for every j, and such that $\sum_j \xi_j = 1$ over λ. Since the arcs $\sigma_j(I \times \{s\})$ are transverse to λ, it makes sense to integrate the first component of σ_j with respect to the measure μ. We set the *length of*

μ with respect to m to be the quantity

$$L_\mu(m) := \sum_j \int_{D_j} \int_0^1 \xi_j(\sigma_j(p,\cdot))\, \mathrm{d}\ell(\cdot)\, \mathrm{d}\mu\,(p),$$

where $\mathrm{d}\ell(s) = \|\partial_s \sigma_j(p,s)\|_g \, \mathrm{d}s$. More generally, given a measurable function f defined on a neighborhood of λ, we define

$$\iint_\lambda f \,\mathrm{d}\ell\, \mathrm{d}\mu := \sum_j \int_{D_j} \int_0^1 \xi_j(\sigma_j(p,\cdot)) f(\sigma_j(p,\cdot))\, \mathrm{d}\ell(\cdot)\, \mathrm{d}\mu\,(p).$$

The quantity $L_\mu(m)$ does not depend on the choices we made of σ_j, ξ_j and the hyperbolic metric g in the equivalence class $m \in \mathcal{T}^\mathfrak{h}(\Sigma)$ (see e. g. [Bon96]). Therefore, any measured lamination μ of Σ determines a positive function L_μ on the Teichmüller space $\mathcal{T}^\mathfrak{h}(\Sigma)$, which associates to any $m \in \mathcal{T}^\mathfrak{h}(\Sigma)$ the length of the geodesic realization of μ in m.

Similarly, if $(g_t)_t$ is a smooth 1-parameter family of hyperbolic metrics on Σ, with $g_0 = g$ and $\dot{g}_0 = \dot{g}$, we set

$$\iint_\lambda \mathrm{d}\dot{\ell}\, \mathrm{d}\mu := \frac{1}{2} \sum_j \int_{D_j} \int_0^1 \xi_j(\sigma_j(p,\cdot)) \frac{\dot{g}\,(\partial_s \sigma_j(p,\cdot), \partial_s \sigma_j(p,\cdot))}{g\,(\partial_s \sigma_j(p,\cdot), \partial_s \sigma_j(p,\cdot))}\, \mathrm{d}\ell(\cdot)\, \mathrm{d}\mu\,(p).$$

Lemma 3.2.1. *Let μ be a measured lamination of Σ, and let $(m_t)_t$ be a smooth path in $\mathcal{T}^\mathfrak{h}(\Sigma)$ verifying $m_0 = m$ and $\dot{m}_0 = \dot{m} \in T_m \mathcal{T}^\mathfrak{h}(\Sigma)$. Then we have*

$$\mathrm{d}(L_\mu)_m\,(\dot{m}) = \iint_\lambda \mathrm{d}\dot{\ell}\, \mathrm{d}\mu\,,$$

where $\iint_\lambda \mathrm{d}\dot{\ell}\, \mathrm{d}\mu$ is defined as above by selecting a smooth path $t \mapsto g_t$ of hyperbolic metrics representing $t \mapsto m_t$.

Proof. First we prove the statement when μ is a weight 1 simple closed curve γ in Σ. Let $\gamma_t \colon [0,1] \to \Sigma$ denote a parametrization of the geodesic representative of γ with respect to the hyperbolic metric g_t, which can be chosen to depend differentiably in t. Then the length of γ_t with respect to the metric g_t can be expressed as

$$L_\gamma(m_t) = \int_0^1 \sqrt{g_t(\gamma_t'(s), \gamma_t'(s))}\, \mathrm{d}s\,.$$

Now, by taking the derivative of this expression in t and using the fact that $\nabla \dot{\gamma}_0 \equiv 0$ (with ∇ being the Levi-Civita connection of g_0), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} L_\gamma(m_t)\Big|_{t=0} = \frac{1}{2} \int_0^1 \frac{\dot{g}_0(\gamma_0'(s), \gamma_0'(s))}{\sqrt{g_0(\gamma_0'(s), \gamma_0'(s))}}\, \mathrm{d}s\,,$$

which coincides with the quantity $\iint_\gamma \mathrm{d}\dot{\ell}\, \mathrm{d}\mu$. By linearity we deduce the statement for any rational lamination $\mu = \sum_i a_i \gamma_i$.

Now, if μ is a general measured lamination, we select a sequence of rational laminations $(\mu_n)_n$ converging to μ. As shown in [Ker85], the functions L_{μ_n} are real analytic over $\mathcal{T}^\mathfrak{h}(\Sigma)$ and they converge in the $\mathscr{C}^\infty$-topology on compact sets to L_μ. In particular the terms $\mathrm{d}(L_{\mu_n})_m\,(\dot{m})$ converge to $\mathrm{d}(L_\mu)_m\,(\dot{m})$. Since the expression $\iint_\lambda \mathrm{d}\dot{\ell}\, \mathrm{d}\mu$ can be proved to be continuous in the measured lamination $\mu \in \mathcal{ML}(\Sigma)$, the statement follows by an approximation argument. □

Before stating Lemma 3.2.2, we define for convenience the following quantities: for every $p \in [1, \infty)$ and $r > 0$, we set

$$C(r,p) := \left(\frac{2p-1}{4\pi} \frac{(\cosh(r/2))^{4p-2}}{(\cosh(r/2))^{4p-2}-1} \right)^{1/p} . \tag{3.2}$$

When $p = \infty$, we define $C(r, \infty) := 1$ for every $r > 0$.

Lemma 3.2.2. *Let (Σ, g) be a hyperbolic surface. Given $x \in \Sigma$ and $r < \operatorname{injrad}_g(x)$, we denote by $B_r(x)$ the metric ball of radius r centered at $x \in \Sigma$. Then, for every $p \in [1, \infty]$ and for every holomorphic quadratic differential on $(\Sigma, [g])$, we have*

$$\|\operatorname{Re} q(x)\| \leq C(r,p) \, \|\operatorname{Re} q\|_{FT, L^p(B_r(x))}.$$

where $\|\operatorname{Re} q(x)\|$ is the pointwise norm of the tensor $\operatorname{Re} q$ at x.

Proof. If $p = \infty$, the statement is clear. Consider $p < \infty$. By passing to the universal cover, we can assume the surface to be $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ and x to be $0 \in \Delta$. The hyperbolic metric of Δ is of the form

$$g_\Delta = \left(\frac{2}{1 - |z|^2} \right)^2 |\mathrm{d}z|^2,$$

where $z \in \Delta$ is the natural coordinate of $\Delta \subset \mathbb{C}$. In what follows, we will denote by $\|\cdot\|$ the norm induced by the hyperbolic metric, and by $\|\cdot\|_0$ the one induced by the standard Euclidean metric $|\mathrm{d}z|^2$.

If $q = f \, \mathrm{d}z^2$ is a holomorphic quadratic differential, then for any $\rho \in (0,1)$ the residue theorem tells us that

$$f(0) = \frac{1}{2\pi i} \int_{\partial B^E_\rho} \frac{f(z)}{z} \, \mathrm{d}z \, ,$$

where $B^E_\rho = B^E_\rho(0) = \{z \in \Delta \mid |z| < \rho\}$ (here E stands for "Euclidean"). In particular we have

$$|f(0)|^p \leq \left(\frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})| \, \mathrm{d}\theta \right)^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^p \, \mathrm{d}\theta \, , \tag{3.3}$$

where in the last step we used the Hölder inequality. At $z = \rho e^{i\theta}$, the hyperbolic norm of $\operatorname{Re} q(z)$ can be expressed as follows:

$$\|\operatorname{Re} q(z)\| = \frac{1}{\sqrt{2}} |f(\rho e^{i\theta})| \left(\frac{2}{1-\rho^2} \right)^{-2} \|\mathrm{d}z^2\|_0.$$

It is easy to check that the metric ball B_r centered at 0 with respect to the hyperbolic distance coincides with $B^E_{\tanh(r/2)}$, and that the hyperbolic volume form dvol is given by $\rho(2/(1-\rho^2))^2 \, \mathrm{d}\rho \, \mathrm{d}\theta$. Combining all these facts, if we multiply the inequality (3.3) by $\rho(2/(1-\rho^2))^{2-2p}$ and we integrate in $\int_0^{\tanh r/2} \mathrm{d}\rho$, we deduce that

$$\begin{aligned}
\int_{B_r} \|\mathrm{Re}\, q\|^p \,\mathrm{dvol} &= 2^{-p/2} \left\| \mathrm{d}z^2 \right\|_0^p \int_0^{\tanh r/2} \rho \left(\frac{2}{1-\rho^2} \right)^{2-2p} \int_0^{2\pi} \left| f(\rho e^{i\theta}) \right|^p \mathrm{d}\theta \,\mathrm{d}\rho \\
&\geq 2\pi |f(0)|^p \, 2^{-p/2-2(p-1)} \left\| \mathrm{d}z^2 \right\|_0^p \int_0^{\tanh r/2} \rho (1-\rho^2)^{2(p-1)} \,\mathrm{d}\rho \\
&= 4\pi \|\mathrm{Re}\, q(0)\|^p \, \frac{1}{2p-1} \left(1 - \frac{1}{(\cosh(r/2))^{4p-2}} \right) \\
&= C(r,p)^{-p} \|\mathrm{Re}\, q(0)\|^p,
\end{aligned}$$

which proves the assertion. □

We state here another useful fact that we will use in the proof of Proposition 3.2.4:

Lemma 3.2.3. *Let (Σ, g) be a hyperbolic surface and let μ be a measured lamination on Σ. Then, for every L^1-function $f\colon N_r(\mu) \to \mathbb{R}$ defined on the r-neighborhood of μ in Σ, we have*

$$\iint_\lambda \left(\int_{B_r(\cdot)} f \,\mathrm{dvol}_g \right) \mathrm{d}\ell \,\mathrm{d}\mu = \int_\Sigma \left(\iint_{\lambda \cap B_r(\cdot)} \mathrm{d}\ell \,\mathrm{d}\mu \right) f \,\mathrm{dvol}_g \,.$$

Proof. Assume that μ is a 1-weighted simple closed curve $\gamma\colon [0,1] \to \Sigma$, and let $\tilde{f}$ denote the extension of the function f to Σ verifying $\tilde{f}(x) = 0$ for all $x \in \Sigma \setminus N_r(\gamma)$. We set $\xi\colon \Sigma^2 \to \mathbb{R}$ to be the function taking value $\xi(x,y) = 1$ if the distance between x and y is less than r, and $\xi(x,y) = 0$ otherwise. Then the integral on the left can be expressed as

$$\int_0^1 \int_\Sigma \tilde{f}(x) \; \xi(x, \gamma(t)) \,\mathrm{dvol}_g(x) \,\mathrm{d}\ell(t) \,.$$

Applying Fubini's theorem we obtain

$$\begin{aligned}
\int_0^1 \int_\Sigma \tilde{f}(x) \xi(x, \gamma(t)) \,\mathrm{dvol}_g(x) \,\mathrm{d}\ell(t) &= \int_\Sigma \int_0^1 \xi(x, \gamma(t)) \,\mathrm{d}\ell(t) \, \tilde{f}(x) \,\mathrm{dvol}_g(x) \\
&= \int_\Sigma \left(\int_{\gamma^{-1}(B_r(x))} \mathrm{d}\ell(t) \right) \tilde{f}(x) \,\mathrm{dvol}_g(x).
\end{aligned}$$

The last term coincides with the right term of the equality in the statement in the case $\mu = \gamma$. By linearity we deduce the statement when μ a rational lamination, and by continuity of the two integrals in the statement with respect to μ we obtain the result for any general measured lamination. □

Let $m \in \mathcal{T}^{\mathfrak{h}}(\Sigma)$ and $\mu \in \mathcal{ML}(\Sigma)$, and select a hyperbolic metric g in the equivalence class m. If $(\widetilde{\Sigma}, \tilde{g})$ denotes the universal cover of (Σ, g), we define

$$D(m, \mu, r) := \sup_{\tilde{x} \in \widetilde{\Sigma}} \iint_{\tilde{\lambda} \cap B_r(\tilde{x})} \mathrm{d}\tilde{\ell} \,\mathrm{d}\tilde{\mu} < \infty.$$

where $\tilde{\lambda}$ denotes the support of the measured lamination $\tilde{\mu}$. In other words, $D(m, \mu, r)$ is the supremum, over the points $\tilde{x}$ in the universal cover $\widetilde{\Sigma}$, of the length of the portion of $\tilde{\mu}$ contained in the ball centered at $\tilde{x}$ of radius r.

Proposition 3.2.4. *For any $r > 0$ and for any $p \in [1, \infty]$ we have*

$$\left|\mathrm{d}(L_\mu)_m(\dot{m})\right| \leq L_\mu(m)^{1/p}\ C(r,p')\ D(m,\mu,r)^{1/p'} \|\nu\|_{B,p'},$$

where p and p' are conjugate exponents, i. e. $\frac{1}{p} + \frac{1}{p'} = 1$, and ν denotes the harmonic Beltrami differential representing the tangent direction $\dot{m} \in T_m \mathcal{T}^{\mathfrak{h}}(\Sigma)$. In particular, for $p = 2$, we have

$$\left\|\mathrm{d}(L_\mu)_m\right\|_{Q,2} \leq C(r,2)\sqrt{L_\mu(m)\ D(m,\mu,r)}.$$

Proof. As described in [Tro92], there exists a unique symmetric transverse-traceless tensor $\sigma \in S^2_{tt}(\Sigma, g)$ representing the tangent vector $\dot{m} \in T_m \mathcal{T}^{\mathfrak{h}}(\Sigma)$, which is of the form $\operatorname{Re} q = \sigma$ for some holomorphic quadratic differential q on $(\Sigma, [g])$. We start by making use of Lemma 3.2.1. From the definition of $\iint_\lambda \mathrm{d}\dot{\ell}\,\mathrm{d}\mu$ and the inequality $|\sigma(v,v)| \leq \frac{1}{\sqrt{2}}\|\sigma\|_g \|v\|_g^2$, we see that

$$\left|\mathrm{d}(L_\mu)_m(\dot{m})\right| = \left|\iint_\lambda \mathrm{d}\dot{\ell}\,\mathrm{d}\mu\right| \leq \frac{1}{2\sqrt{2}}\iint_\lambda \|\sigma\|_g\,\mathrm{d}\ell\,\mathrm{d}\mu\,.$$

By applying the Hölder inequality on the right-side integral, we get

$$\left|\mathrm{d}(L_\mu)_m(\dot{m})\right| \leq \frac{1}{2\sqrt{2}}\iint_\lambda \|\sigma\|_g\,\mathrm{d}\ell\,\mathrm{d}\mu \leq \frac{L_\mu(m)^{1/p}}{2\sqrt{2}}\left(\iint_\lambda \|\sigma\|_g^{p'}\,\mathrm{d}\ell\,\mathrm{d}\mu\right)^{1/p'}. \tag{3.4}$$

Now we estimate the integral $\iint_\lambda \|\sigma\|_g^{p'}\,\mathrm{d}\ell\,\mathrm{d}\mu$ by lifting it to a suitable covering of Σ, and then applying Lemma 3.2.2. More precisely, let $(\widehat{\Sigma}, \hat{g}) \to (\Sigma, g)$ be a N-index covering so that $\operatorname{injrad}(\widehat{\Sigma}, \hat{g}) > r$, for some $N \in \mathbb{N}$. We denote by $\hat{\bullet}$ the lift of the object $\bullet$ on $\widehat{\Sigma}$. It is immediate to check that the following relation holds

$$\iint_\lambda \|\sigma\|_g^{p'}\,\mathrm{d}\ell\,\mathrm{d}\mu = \frac{1}{N}\iint_{\hat{\lambda}} \|\hat{\sigma}\|_{\hat{g}}^{p'}\,\mathrm{d}\hat{\ell}\,\mathrm{d}\hat{\mu}\,.$$

Then, by applying Lemma 3.2.2 on the surface $(\widehat{\Sigma}, \hat{g})$ and at each point $\hat{x} \in \hat{\lambda}$, we get

$$\begin{aligned}
\iint_\lambda \|\sigma\|_g^{p'}\,\mathrm{d}\ell\,\mathrm{d}\mu &= \frac{1}{N}\iint_{\hat{\lambda}} \|\hat{\sigma}\|_{\hat{g}}^{p'}\,\mathrm{d}\hat{\ell}\,\mathrm{d}\hat{\mu} \\
&\leq \frac{C(r,p')^{p'}}{N}\iint_{\hat{\lambda}} \|\hat{\sigma}\|_{FT,L^{p'}(B_r(\cdot))}^{p'}\,\mathrm{d}\hat{\ell}\,\mathrm{d}\hat{\mu} \\
&= \frac{C(r,p')^{p'}}{N}\iint_{\hat{\lambda}} \left(\int_{B_r(\cdot)} \|\hat{\sigma}\|_{\hat{g}}^{p'}\,\mathrm{dvol}_{\hat{g}}\right)\mathrm{d}\hat{\ell}\,\mathrm{d}\hat{\mu}\,.
\end{aligned}$$

Using Lemma 3.2.3 and the definition of $D(m,\mu,r)$, we obtain

$$\begin{aligned}
\iint_{\hat{\lambda}} \left(\int_{B_r(\cdot)} \|\hat{\sigma}\|_{\hat{g}}^{p'}\,\mathrm{dvol}_{\hat{g}}\right)\mathrm{d}\hat{\ell}\,\mathrm{d}\hat{\mu} &= \int_{\widehat{\Sigma}} \left(\iint_{\hat{\lambda}\cap B_r(\cdot)} \mathrm{d}\hat{\ell}\,\mathrm{d}\hat{\mu}\right) \|\hat{\sigma}\|_{\hat{g}}^{p'}\,\mathrm{dvol}_{\hat{g}} \\
&\leq D(m,\mu,r)\int_{\widehat{\Sigma}} \|\hat{\sigma}\|_{\hat{g}}^{p'}\,\mathrm{dvol}_{\hat{g}} \\
&= N\ D(m,\mu,r)\ \|\sigma\|_{FT,p'}^{p'},
\end{aligned}$$

where, in the last step, we are using again the fact that $(\Sigma, g) \to (\widehat{\Sigma}, \hat{g})$ is a N-index covering. Combining the last two estimates, we obtain

$$\iint_{\lambda} \|\sigma\|_g^{p'} \, \mathrm{d}\ell \, \mathrm{d}\mu \leq C(r,p')^{p'} \; D(m,\mu,r) \; \|\sigma\|_{FT,p'}^{p'}. \tag{3.5}$$

Using the inequalities (3.4) and (3.5), we have shown that

$$\left|\mathrm{d}(L_\mu)_m \, (\dot{m})\right| \leq \frac{L_\mu(m)^{1/p} \; C(r,p') \; D(m,\mu,r)^{1/p'}}{2\sqrt{2}} \; \|\sigma\|_{FT,p'}.$$

Finally, by applying Lemma 3.1.2, we obtain

$$\left|\mathrm{d}(L_\mu)_m \, (\dot{m})\right| \leq L_\mu(m)^{1/p} \; C(r,p') \; D(m,\mu,r)^{1/p'} \, \|\nu\|_{B,p'}.$$

The last assertion follows from the estimate we just proved for $p = 2$ and from Lemma 3.1.1. □

3.3 The differential of the dual volume

Let $V_C^* \colon \mathcal{QF}(\Sigma) \to \mathbb{R}$ denote the function that associates, to each quasi-Fuchsian manifold M homeomorphic to $\Sigma \times \mathbb{R}$, the dual volume of its convex core (see Section 2.2 for the definition of dual volume). The aim of this section is to produce a uniform bound of the differential of V_C^* in terms of the Weil-Petersson norm of the variation of the hyperbolic metric on the boundary of the convex core.

An immediate consequence of Proposition 3.2.4 and Theorem A is the following:

Proposition 3.3.1. *Let $(M_t)_t$ be a smooth 1-parameter family of quasi-Fuchsian manifolds, with $M = M_0$. Then for every $r > 0$ and for every $p \in [1, \infty]$ we have*

$$\left|\mathrm{d}V_C^* \, (\dot{M})\right| \leq \frac{1}{2} \; L_\mu(m)^{1/p} \; C(r,p') \; D(m,\mu,r)^{1/p'} \; \|\nu\|_{B,p'},$$

where $C(r,p')$ and $D(m,\mu,r)$ are the constants defined in the previous section, p and p' are conjugated exponents, and ν denotes the harmonic Beltrami differential representing the variation of the hyperbolic metric of the boundary of the convex core of M.

Let M be a quasi-Fuchsian manifold, obtained as the quotient of the hyperbolic space $\mathbb{H}^3$ by the action of a discrete and torsion-free subgroup of isometries. As described in Section 1.3, the lift of the boundary of the convex core of M to $\mathbb{H}^3$ is the union of two *embedded* locally bent pleated planes $H^\pm$. This property turns out to determine uniform upper bounds of the quantities $L_\mu(m)$ and $D(m,\mu,r)$ appearing in the statement of Proposition 3.2.4. The first results in this direction have been developed by Epstein and Marden in [CEM06, Part II]. In our exposition, we will recall and make use of the works of Bridgeman, Brock, and Bromberg [BBB19] and Bridgeman, Canary, and Yarmola [BCY16], which will give us separate bounds for $L_\mu(m)$ and $D(m,\mu,r)$, respectively. We will also require r to be less than $\ln(3)/2$. This restriction simplifies our argument in the proof of Corollary 3.3.4. However, we do not exclude the possibility that a joint study of the quantity $L_\mu(m)^{1/p} D(m,\mu,r)^{1/p'}$ and a careful choice of r might improve the multiplicative constants obtained here.

First we focus on the term $D(m,\mu,r)$, which we defined before the statement of Proposition 3.2.4. Let $\tilde{\lambda}$ denote the geodesic lamination in $\widetilde{\Sigma}$ given by the lift of the support of the measured lamination μ. Let Q be a component of $\widetilde{\Sigma} \setminus \tilde{\lambda}$ and let l_1, l_2, l_3 be three boundary components of Q. We will use the following fact:

Lemma 3.3.2 ([CEM06, Corollary II.2.4.3])**.** *Let $r < \ln(3)/2 = \operatorname{arcsinh}(1/\sqrt{3})$, and suppose we have a point $x \in Q$ which is at distance $\leq \operatorname{arcsinh}(e^{-r})$ from both l_2 and l_3. Then its distance from l_1 is $> r$.*

Following [BCY16], given $\tilde{\mu}$ a measured lamination on $\mathbb{H}^2$, we denote by $\|\tilde{\mu}\|_s$ the supremum over α of the transverse measure of $\tilde{\mu}$ along α, where α varies among the geodesic arcs in $\mathbb{H}^3$ of length $s > 0$ which are transverse to the support of $\tilde{\mu}$.

Theorem 3.3.3 ([BCY16])**.** *Let $s \in (0, 2\operatorname{arcsinh} 1)$ and let $\tilde{\mu}$ be a measured lamination of $\mathbb{H}^2$ so that the pleated plane with bending measure $\tilde{\mu}$ is embedded inside $\mathbb{H}^3$. Then*

$$\|\tilde{\mu}\|_s \leq 2\arccos\left(-\sinh(s/2)\right).$$

Corollary 3.3.4. *Let $\mu \in \mathcal{ML}(\Sigma)$ and $m \in \mathcal{T}^{\mathfrak{h}}(\Sigma)$ be the bending measure and the hyperbolic metric, respectively, of the boundary of an incompressible hyperbolic end inside a hyperbolic convex co-compact 3-manifold. Then for every $r < \ln(3)/2$ we have*

$$D(m,\mu,r) \leq 4r \arccos\left(-\sinh r\right).$$

Moreover, for every $\varepsilon > 0$ there exists $m_\varepsilon \in \mathcal{T}^{\mathfrak{h}}(\Sigma)$ and $\mu_\varepsilon \in \mathcal{ML}(\Sigma)$ as above verifying

$$D(m_\varepsilon, \mu_\varepsilon, r) \geq 2(\pi - \varepsilon) r \qquad \forall r > 0.$$

Proof. Let g be a hyperbolic metric in the equivalence class $m \in \mathcal{T}^{\mathfrak{h}}(\Sigma)$. We denote by $(\widetilde{\Sigma}, \tilde{g}) \to (\Sigma, g)$ the Riemannian universal cover of (Σ, g) and by $\tilde{\lambda}$ the support of the lift $\tilde{\mu}$ of the measured lamination μ to $\widetilde{\Sigma}$. Given a point $\tilde{x}$ in $\widetilde{\Sigma}$ and a positive $r < \ln(3)/2$, we are looking for an upper bound of the length of $\tilde{\mu} \cap B_r(\tilde{x})$, where $B_r(\tilde{x})$ denotes the metric ball of radius r at $\tilde{x}$.

The convenience of considering $r < \ln(3)/2$ comes from Lemma 3.3.2: under this hypothesis, any plaque Q of $\tilde{\lambda}$ at distance less than r from x has at most two components of its boundary intersecting $B_r(x)$. A simple argument proves that, if this happens, we can find a geodesic path α of length $< 2r$ that intersects all the leaves of $\tilde{\lambda} \cap B_r(\tilde{x})$. Each leaf of $\tilde{\lambda} \cap B_r(\tilde{x})$ has length $< 2r$, therefore the length of $\tilde{\mu} \cap B_r(\tilde{x})$ is bounded by $2r$ (the length of each leaf) times the total mass $\tilde{\mu}(\alpha)$, which can be estimated applying Theorem 3.3.3 with $s = 2r < \ln 3 < 2\operatorname{arcsinh} 1$. This proves the first part of the statement[1].

For what concerns the last part of the assertion, we fix a simple closed curve γ and we assign it the weight $\pi - \varepsilon$. By the work of Bonahon and Otal [BO04], we can find a quasi-Fuchsian manifold M_ε realizing $(\pi - \varepsilon)\gamma$ as the bending

[1] See Remark 3.3.8.

lamination of the upper component of the boundary of the convex core $\partial^+ CM_\varepsilon$. It is immediate to check that, if m_ε is the hyperbolic metric of $\partial^+ CM_\varepsilon$, then $D(m_\varepsilon, \mu_\varepsilon, r) \geq 2(\pi - \varepsilon) r$ for all $r > 0$. □

For the bound of the term $L_\mu(m)$, we will apply the following result:

Theorem 3.3.5 ([BBB19, Theorem 2.16]). *Let $\mu \in \mathcal{ML}(\Sigma)$ and $m \in \mathcal{T}^\natural(\Sigma)$ be the bending measure and the hyperbolic metric, respectively, of the boundary of an incompressible hyperbolic end inside a hyperbolic convex co-compact 3-manifold. Then*

$$L_\mu(m) \leq 6\pi |\chi(\Sigma)|.$$

Finally, given $p \in (1, \infty)$ and $r < \ln(3)/2$, we set

$$\begin{aligned} K(r,p) &:= \frac{1}{2}(24\pi)^{1/p}\ C(r,p')\ (4r \arccos(-\sinh r))^{1/p'} \\ &= \frac{1}{2}(24\pi)^{1/p} \left(\frac{2p'-1}{\pi} \frac{(\cosh(r/2))^{4p'-2}}{(\cosh(r/2))^{4p'-2} - 1}\, r \arccos(-\sinh r) \right)^{1/p'}, \end{aligned}$$

where $C(r,p')$ was defined in equation (3.2), and p' is the conjugate exponent of p. We define also

$$K(r,1) = 12\pi, \qquad K(r,\infty) = \frac{r \arccos(-\sinh r)}{2\pi \tanh^2(r/2)}.$$

Corollary 3.3.6. *In the same notations of Proposition 3.3.1, for every $p \in [1,\infty]$ we have*

$$\left| \mathrm{d}V_C^*(\dot{M}) \right| \leq K(p)(g-1)^{1/p} \|\nu\|_{B,p'},$$

where $K(p) := K(\ln(3)/2, p)$ and ν denotes the harmonic Beltrami differential representing the variation of the hyperbolic metrics on the boundary of the convex core ∂CM of M. We have in particular that $K(2) \approx 10.3887$.

Proof. We combine Proposition 3.3.1, Corollary 3.3.4 and Theorem 3.3.5 on the upper and lower components of $\partial CM = \partial CM_0$, and then we take the limit as r goes to $\ln(3)/2$. □

We can compare this statement with the analogous bound for the differential of the renormalized volume:

Theorem 3.3.7 ([Sch13]). *Let $V_R \colon \mathcal{QF}(\Sigma) \to \mathbb{R}$ denote the function associating to each quasi-Fuchsian manifold M its renormalized volume $V_R(M)$. Then for every $p \in [1,\infty]$ we have*

$$\mathrm{d}V_R\,(\dot{M}) \leq H(p)(g-1)^{1/p} \|\dot{c}\|_{B,p'},$$

where $\dot{c}$ denotes the variation of the conformal structures at infinity of M, and $H(p) := \frac{3}{2}(8\pi)^{1/p}$.

Remark 3.3.8. From the first part of the proof of Corollary 3.3.4 it is clear that our estimate of the constant $D(m,\mu,r)$ is far from being optimal. However, using the second part of the assertion, it is easy to see that the possible improvement of the constant $K(2)$ is not enough to make the multiplicative constant in Theorem B to be less than $3\sqrt{\pi}$, which is the one appearing in the analogous statement for the renormalized volume. Because of this, we preferred to present a simpler but rougher argument.

3.4 Dual volume and Weil-Petersson distance

This section is dedicated to the proof of the linear upper bound of the dual volume of a quasi-Fuchsian manifold M in terms of the Weil-Petersson distance between the hyperbolic structures on the boundary of its convex core CM. As we mentioned in Section 1.3, the data of the hyperbolic metrics of ∂CM is only conjectured to give a parametrization of the space of quasi-Fuchsian manifolds, contrary to what happens with the conformal structures at infinity. In particular, the same strategy used in [Sch13] to bound the renormalized volume cannot be immediately applied.

In order to overcome this problem, we will take advantage of the foliation by k-surfaces of $M \setminus CM$, described in Section 1.6.3 (see also Remark 3.4.9). The space of hyperbolic structures with strictly convex boundary on $\Sigma \times [0, 1]$ is parametrized by the data of the metrics on its boundary, as proved in [Sch06]. In particular, the Teichmüller classes of the metrics of the upper and lower k-surfaces parametrize the space of quasi-Fuchsian structures of topological type $\Sigma \times \mathbb{R}$ (see Theorem 3.4.1). Moreover, the first order variation of the dual volume of the region M_k encosed between the two k-surfaces is intimately related to the notion of landslide, which was first introduced and studied in [BMS13], [BMS15]. This connection will be very useful to relate the first order variation of $V_C^*(M)$ and of $V_k^*(M) := \mathrm{Vol}^*(M_k)$, as k goes to -1, allowing us to prove Theorem B using an approximation argument, together with the bounds obtained in the previous Section.

3.4.1 Constant Gaussian curvature surfaces

In every quasi-Fuchsian manifold M, the subset $M \setminus CM$ has exactly two connected components E^+ and E^-, each of which is homeomorphic to $\Sigma \times (0, \infty)$ (these are the *hyperbolic ends* of M, as in Definition 1.6.1). By Theorem 1.6.4, the sets $E^\pm$ are foliated by k-surfaces $(\Sigma_k^\pm)_k$, with k that varies in $(-1, 0)$. The surfaces $\Sigma_k^\pm$ approach the pleated boundaries $\partial^\pm CM$ of the convex core of M as k goes to -1, and the conformal boundaries at infinity $\partial_\infty^\pm M$ as k goes to 0.

We denote by $m_k^\pm(M) \in \mathcal{T}^\mathfrak{h}(\Sigma)$ the isotopy classes of the hyperbolic metrics $(-k)\, I_k^\pm$, where $I_k^\pm$ is the first fundamental form of the upper/lower k-surface $\Sigma_k^\pm$ of M. Then for every $k \in (-1, 0)$ we have maps

$$\begin{array}{rccc} T_k : & \mathcal{QF}(\Sigma) & \longrightarrow & \mathcal{T}^\mathfrak{h}(\Sigma) \times \mathcal{T}^\mathfrak{h}(\Sigma) \\ & M & \longmapsto & (m_k^+(M), m_k^-(M)). \end{array}$$

The family of functions $(T_k)_k$ is clearly related to the maps T and B that we introduced in Section 1.3. As k goes to -1, $T_k(M)$ converges to $T(M)$, and as k goes to 0, $T_k(M)$ converges to $B(M)$. The convenience in considering the foliation by k-surfaces relies in the following result, based on the works of Labourie [Lab92a] and Schlenker [Sch06]:

Theorem 3.4.1. *The map T_k is a $\mathscr{C}^1$-diffeomorphism for every $k \in (-1, 0)$.*

Proof. Let $(N, \partial N)$ be a compact connected 3-manifold admitting a hyperbolic structure with convex boundary. Schlenker [Sch06] proved that any Riemannian metric with Gaussian curvature > -1 on ∂N is uniquely realized as the restriction to the boundary of a hyperbolic metric on N with smooth strictly convex boundary. In other words, if $\mathscr{G}$ and $\mathscr{H}$ denote the spaces of isotopy classes of metrics on N with strictly convex boundary and of metrics on ∂N with Gaussian curvature > -1, respectively, then the restriction map

$$\begin{array}{rccc} r: & \mathscr{G} & \longrightarrow & \mathscr{H} \\ & [g] & \longmapsto & [g|_{\partial N}] \end{array}$$

is a homeomorphism. The surjectivity had already been showed by Labourie in [Lab92a], therefore the proof proceeds by showing the local injectivity of r. To do so, the strategy in [Sch06] is to apply the Nash-Moser implicit function theorem.

Let us fix now a $k \in (-1, 0)$, and consider $N = \Sigma \times I$. If $\mathscr{G}_k$ is the space of hyperbolic structures on N with boundary having constant Gaussian curvature equal to k, then $\mathscr{G}_k$ identifies with the space of quasi-Fuchsian manifolds $\mathcal{QF}(\Sigma)$, thanks to Theorem 1.6.4 and the fact that any hyperbolic structure with convex boundary on N uniquely extends to a quasi-Fuchsian structure (see e. g. [CEM06, Theorem I.2.4.1]). In addition, the space $\mathscr{H}_k$ of constant k Gaussian curvature structures on ∂N clearly identifies with the product of two copies of the Teichmüller space $\mathcal{T}_h(\Sigma)$, one for each component of ∂N. Therefore the function r restricts to $r_k \colon \mathscr{G}_k \to \mathscr{H}_k$, which can be identified with T_k thanks to what we just observed. The map r_k is now a function between finite dimensional differential manifolds. The fact that r verifies the hypotheses to apply the Nash-Moser inverse function theorem implies in particular that r_k verifies the hypotheses to apply the ordinary inverse function theorem between finite dimensional manifolds. In particular, this shows that r_k is a $\mathscr{C}^1$-diffeomorphism, for any $k \in (-1, 0)$, as desired. □

3.4.2 The proof of Theorem B

In the following we outline the proof of Theorem B. Let $V_k^*(M)$ denote the dual volume of the convex subset enclosed by the two k-surfaces in the quasi-Fuchsian manifold M. With abuse, we will continue to denote by V_k^* the composition $V_k^* \circ T_k^{-1} : \mathcal{T}^{\mathfrak{h}}(\Sigma)^2 \to \mathbb{R}$. An immediate corollary of Theorem 3.4.1 is that the function V_k^* is $\mathscr{C}^1$ for every $k \in (-1, 0)$.

Fix now a quasi-Fuchsian manifold M, with hyperbolic structures $m_k^\pm = m_k^\pm(M)$ on its k-surfaces. Since the Teichmuller space endowed with the Weil-Petersson metric is a unique geodesic space [Wol87], there exists a unique Weil-Petersson geodesic $\beta_k \colon [0, 1] \to \mathcal{T}^{\mathfrak{h}}(\Sigma)$ verifying $\beta_k(0) = m_k^-$ and $\beta_k(1) = m_k^+$. We set γ_k to be the path in $\mathcal{T}^{\mathfrak{h}}(\Sigma)^2$ given by $\gamma_k(t) = (\beta_k(t), m_k^-)$. By construction $T_k^{-1}(\gamma_k(0))$ is a Fuchsian manifold for every $k \in (-1, 0)$ and $T_k^{-1}(\gamma_k(1)) = M$. We decompose the differential of the function V_k^* as follows

$$\mathrm{d}V_k^* = \mathrm{d}V_k^{*,+} + \mathrm{d}V_k^{*,-} \in T^*\mathcal{T}^{\mathfrak{h}}(\Sigma) \oplus T^*\mathcal{T}^{\mathfrak{h}}(\Sigma).$$

Now we observe that

$$\begin{aligned}|V_k^*(\gamma_k(1)) - V_k^*(\gamma_k(0))| &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} V_k^*(\gamma_k(t))\,\mathrm{d}t \right| \\ &\leq \int_0^1 \|\mathrm{d}V_k^{*,+}\|_{\gamma_k(t)} \, \|\beta_k'(t)\|\,\mathrm{d}t \\ &\leq \max_{t\in[0,1]} \|\mathrm{d}V_k^{*,+}\|_{\gamma_k(t)} \, \ell_{WP}(\beta_k) \\ &= \max_{t\in[0,1]} \|\mathrm{d}V_k^{*,+}\|_{\gamma_k(t)} \, d_{WP}(m_k^+, m_k^-),\end{aligned}$$

where $\|\cdot\|_x$ denotes the Weil-Petersson norm on $T_x^*\mathcal{T}^{\mathfrak{h}}(\Sigma)$. The step from the first to the second line follows from the fact that the second component of the curve γ_k does not depend on t, and in the last step we used that β_{WP} is a Weil-Petersson geodesic. Since the dual volume of the convex core of a Fuchsian manifold vanishes, we have that

$$\lim_{k\to -1} V_k^*(\gamma_k(1)) - V_k^*(\gamma_k(0)) = V_C^*(M).$$

By Theorem 1.6.4 we have

$$\lim_{k\to -1} d_{WP}(m_k^+, m_k^-) = d_{WP}(m^+, m^-)$$

where m^+, m^- are the hyperbolic metrics of the upper and lower components of ∂CM, respectively. Therefore, taking the limit as k goes to -1 of the inequality above we obtain

$$|V_C^*(M)| \leq \liminf_{k\to -1} \max_{t\in[0,1]} \|\mathrm{d}V_k^{*,+}\|_{\gamma_k(t)} \, d_{WP}(m^+, m^-). \tag{3.6}$$

If $\pi^+ \colon \mathcal{T}^{\mathfrak{h}}(\Sigma)^2 \to \mathcal{T}^{\mathfrak{h}}(\Sigma)$ denotes the projection onto the first component (the one concerning the upper k-surface Σ_k^+), then the function $\mathrm{d}V_k^{*,+} \circ T_k$ is a section of the bundle $(\pi^+ \circ T_k)^*(T^*\mathcal{T}^{\mathfrak{h}}(\Sigma))$. In order to simplify the notation, we will set $\mathrm{d}L_{\mu^+}$ to be the map

$$\mathcal{QF}(\Sigma) \ni M \longmapsto \mathrm{d}(L_{\mu^+(M)})_{\pi^+\circ T(M)} \in T^*\mathcal{T}^{\mathfrak{h}}(\Sigma).$$

Assuming that the sections $(\mathrm{d}V_k^{*,+} \circ T_k)_k$ converge to $\mathrm{d}L_{\mu^+}$ uniformly over compact sets of $\mathcal{QF}(\Sigma)$ as k goes to -1, then Theorem B easily follows:

Proof of Theorem B. The paths $T_k^{-1}(\gamma_k)$ considered above lie inside a common compact subset of $\mathcal{QF}(\Sigma)$. Following the proof of Corollary 3.3.6, we observe that $\|\mathrm{d}L_{\mu^+}\|$ is bounded by $K(2)/\sqrt{2}$ (the factor $1/\sqrt{2}$ appears because we consider only the upper component of the bending measure). Therefore, by uniform convergence we have

$$\liminf_{k\to -1} \max_{t\in[0,1]} \|\mathrm{d}V_k^{*,+}\|_{\gamma_k(t)} \leq K(2)/\sqrt{2} \approx 7.3459,$$

which, combined with the inequality (3.6), implies the statement. □

Therefore, the last ingredient left to prove is the following:

Proposition 3.4.2. *The sections* $(\mathrm{d}V_k^{*,+} \circ T_k)_k$ *converge uniformly to* $\mathrm{d}L_{\mu^+}$ *over compact sets of* $\mathcal{QF}(\Sigma)$ *as* k *goes to* -1

We will deduce this fact from the *dual differential Schläfli formula*, stated in Proposition 2.2.5, and from the connection between the first order variation of the volumes V_k^* and the notion of landslides introduced in [BMS13], [BMS15].

3.4.3 Earthquakes and landslides

We briefly recall the definition of landslide flow, introduced in Bonsante, Mondello, and Schlenker [BMS13], and the properties that we will need for the proof of Proposition 3.4.2. Landslides are described by a map

$$\begin{array}{rccc} \mathscr{L}: & S^1 \times \mathcal{T}^{\mathfrak{h}}(\Sigma)^2 & \longrightarrow & \mathcal{T}^{\mathfrak{h}}(\Sigma)^2 \\ & (e^{i\theta}, m, m') & \longmapsto & \mathscr{L}_{e^{i\theta}}(m, m'). \end{array}$$

The first component of $\mathscr{L}_{e^{i\theta}}(m, m')$, which we will denote by $\mathscr{L}^1_{e^{i\theta}}(m, m')$, is called the *landslide of* m *with respect to* m' *with parameter* $e^{i\theta}$. The map $\mathscr{L}$ is defined through the existence and uniqueness of minimal Lagrangian maps, as described in Theorems 1.2.18 and 1.2.19. We refer to Section 1.2.4 for the relative terminology. In the following, we will identify, with abuse, a pair of isotopy classes m, $m' \in \mathcal{T}^{\mathfrak{h}}(\Sigma)$ with a pair of hyperbolic metrics h, h' satisfying the conclusions of Theorem 1.2.18. Given $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and two metrics h, h' with Labourie operator b, we denote by b^θ the endomorphism $\cos(\theta/2)\mathbb{1} + \sin(\theta/2) J b$, where J is the almost complex structure of h, and we set $h^\theta := h(b^\theta \cdot, b^\theta \cdot)$. Then the function $\mathscr{L}$ is defined as follows:

$$\mathscr{L}_{e^{i\theta}}(h, h') := (h^\theta, h^{\pi+\theta}).$$

It turns out that, for any θ, the metric h^θ is hyperbolic, and $\mathscr{L}$ actually defines a flow, in the sense that it satisfies $\mathscr{L}_{e^{i\theta}} \circ \mathscr{L}_{e^{i\theta'}} = \mathscr{L}_{e^{i(\theta+\theta')}}$ for all θ, θ'.

Bonsante, Mondello, and Schlenker [BMS13] proved that, as earthquakes extend to *complex earthquakes* (see [McM98]), a similar phenomenon happens for landslides. More precisely, fixed $h, h' \in \mathcal{T}^{\mathfrak{h}}(\Sigma)$, the map $\mathscr{L}^1_\bullet(h, h')$ extends to a holomorphic function $C_\bullet(h, h')$ defined on a open neighborhood of the closure of the unit disc Δ in $\mathbb{C}$. If $\zeta = \exp(s + i\theta) \in \overline{\Delta}$, then C_ζ can be written as

$$C_\zeta(h, h') = \mathrm{sgr}_s \circ \mathscr{L}_{e^{i\theta}}(h, h'),$$

where $\mathrm{sgr}_s \colon \mathcal{T}^{\mathfrak{h}}(\Sigma)^2 \to \mathcal{T}^{\mathfrak{c}}(\Sigma)$ is called the *smooth grafting map*, first introduced and described in [BMS13]. If $s = 0$, then $\mathrm{sgr}_0 \circ \mathscr{L}_{e^{i\theta}} = \mathscr{L}^1_{e^{i\theta}}$. We mentioned the existence of this complex extension for completeness, but we will not need to describe the smooth grafting map for the rest of our exposition, the interested reader can find its definition and properties in [BMS13, Section 5].

Fixed h', we set $l^1(h, h')$ to be the infinitesimal generator of the landslide flow with respect to the hyperbolic metric h' at the point $h \in \mathcal{T}^{\mathfrak{h}}(\Sigma)$. In other words,

$$l^1(h, h') := \frac{\mathrm{d}}{\mathrm{d}\theta} \mathscr{L}^1_{e^{i\theta}}(h, h')\Big|_{\theta=0} \in T_h \mathcal{T}^{\mathfrak{h}}(\Sigma).$$

Landslides extend the notion of earthquake in the sense explained by the following Theorem:

Theorem 3.4.3 ([BMS13, Proposition 6.8]). *Let $(h_n)_n$ and $(h'_n)_n$ be two sequences of hyperbolic metrics on Σ such that $(h_n)_n$ converges to $h \in \mathcal{T}^{\mathfrak{h}}(\Sigma)$, and $(h'_n)_n$ converges to a projective class of measured lamination $[\mu]$ in the Thurston boundary of Teichmüller space. If $(\theta_n)_n$ is a sequence of positive numbers such that $\theta_n \ell_{h'_n}$ converges to $\iota(\mu, \cdot)$, then $\mathscr{L}^1_{e^{i\theta_n}}(h_n, h'_n)$ converges to the left earthquake $\mathscr{E}_{\mu/2}(h)$, and $\theta_n \cdot l^1(h_n, h'_n)|_{h_n}$ converges to $\frac{1}{2} e_\mu|_h = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_{t\mu/2}(h)$.*

Remark 3.4.4. The last part of the assertion follows from the fact that the functions $e^{i\theta} \mapsto \mathscr{L}^1_{e^{i\theta}}(h, h')$ extend to holomorphic functions $\zeta \mapsto C_\zeta(h, h')$, where ζ varies in a neighborhood of $\overline{\Delta}$. In particular, the uniform convergence of the complex landslides $C_\bullet(h_n, h'_n)$ to the complex earthquake map implies uniform convergence in the $\mathscr{C}^\infty$-topology with respect to the complex parameter ζ.

In order to prove the relation between the differential of V^*_k and the landslide flow, it will be useful to have an explicit expression to compute the variation of the hyperbolic length of a simple closed curve α of Σ along the infinitesimal landslide $l^1(h, h')$.

Lemma 3.4.5. *Let α be a simple closed curve in Σ. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}\theta} L_\alpha(\mathscr{L}^1_{e^{i\theta}}(h, h'))\big|_{\theta=0} = -\int_\alpha \frac{h(b\alpha', J\alpha')}{2\|\alpha'\|^2_h} \,\mathrm{d}\ell_h \,,$$

where J is the complex structure of h and b is the Labourie operator of the pair h, h'.

Proof. With abuse, we denote the h-geodesic realization of α by α itself. By definition of landslide we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \mathscr{L}^1_{e^{i\theta}}(h, h')(\alpha', \alpha')\big|_{t=0} = \dot{h}(\alpha', \alpha') = h(\alpha', Jb\alpha').$$

Since J is h-skew-symmetric, we deduce that $\dot{h}(\alpha', \alpha') = -h(b\alpha', J\alpha')$. Combining this relation with Proposition 3.2.1 we obtain the statement. ☐

We recall, from Section 1.5, that both the *first* and *third* fundamental forms of a k-surface immersed in a hyperbolic 3-manifold are Riemannian metrics with constant Gaussian curvature (the curvature of the first fundamental form is k, while the curvature of the third is $\frac{k}{k+1}$). In what follows, we will denote by

$$h^\pm_k := -k \; I^\pm_k, \qquad {h'}^\pm_k := -\frac{k}{k+1} \; I\!I\!I^\pm_k$$

the hyperbolic metrics associated to first and third fundamental forms of the k-surfaces $\Sigma^\pm_k$ sitting inside a quasi-Fuchsian manifold M.

The relation between landslides and the dual volume of the region enclosed by the two k-surfaces is described by the following statement:

Proposition 3.4.6. *For every $k \in (-1, 0)$ and for every quasi-Fuchsian manifold M we have*

$$\mathrm{d}V^*_k \circ T_k(M) = \sqrt{-\frac{k+1}{k}} \; \hat{\omega}_{WP}(l^1(h^+_k, {h'}^+_k) \oplus l^1(h^-_k, {h'}^-_k), \cdot) \in T^*_{T_k(M)} \mathcal{T}^{\mathfrak{h}}(\Sigma)^2,$$

where $\hat{\omega}_{WP} = \omega_{WP} \oplus \omega_{WP}$ is the direct sum of the Weil-Petersson symplectic structures on $\mathcal{T}^{\mathfrak{h}}(\Sigma)^2$.

Proof. In order to simplify the notation, we will denote by h_k the hyperbolic metric $h_k^+ \sqcup h_k^-$ on $\Sigma_k := \Sigma_k^+ \sqcup \Sigma_k^-$, and similarly for h_k'.

Given a simple closed curve α in Σ_k, let e_α be the infinitesimal generator of the left earthquake flow along α on $\mathcal{T}^{\mathfrak{h}}(\Sigma_k) = \mathcal{T}^{\mathfrak{h}}(\Sigma)^2$. We will prove the statement by showing that, for every simple closed curve α, we have:

$$\mathrm{d}(V_k^*)_{T_k(M)}(e_\alpha) = \sqrt{-\frac{k+1}{k}}\; \hat{\omega}_{WP}(l^1(h_k, h_k'), e_\alpha). \tag{3.7}$$

Since the constant k will be fixed, from now on we will not write the dependence on k in the objects involved in the argument. By Theorem 3.4.1, for every first order variation of metrics δI on $\Sigma^+ \sqcup \Sigma^-$, we can find a variation δg of hyperbolic metrics on M satisfying $\delta g|_\Sigma = \delta I$. Our first step will be to construct an explicit variation δI corresponding to the vector field e_α, and then to apply Proposition 3.4.2 to compute $\mathrm{d}V_k^*(e_\alpha)$.

We will identify the curve α with its I-geodesic parametrization of length L_α and at speed 1. Let J denote the almost complex structure of I, and set V to be the vector field along α given by $-J\alpha'$. We can find a $\varepsilon > 0$ so that the map

$$\begin{array}{rccc} \xi : & \mathbb{R}/L_\alpha\mathbb{Z} \times [0,\varepsilon] & \longrightarrow & \Sigma \\ & (s,r) & \longmapsto & \exp_{\alpha(s)}(rV(s)) \end{array}$$

is a diffeomorphism onto its image (here exp is the exponential map with respect to I). The image of ξ is a closed cylinder in Σ having α as left boundary component. Observe that the metric I equals $\mathrm{d}r^2 + \cosh^2 r\, \mathrm{d}s^2$ in the coordinates defined by ξ^{-1}. We also choose a smooth function $\eta\colon [0,\varepsilon] \to [0,1]$ that coincides with 1 in a neighborhood of 0, and with 0 in a neighborhood of ε. Now define

$$\begin{array}{rccc} f_t : & \mathbb{R}/L_\alpha\mathbb{Z} \times [0,\varepsilon] & \longrightarrow & \mathbb{R}/L_\alpha\mathbb{Z} \times [0,\varepsilon] \\ & (s,r) & \longmapsto & (s + t\eta(r), r). \end{array}$$

The maps $u_t := \xi \circ f_t \circ \xi^{-1}$ give a smooth isotopy of the strip $\operatorname{Im}\xi$ adjacent to α, with $u_0 = id$. Finally we set

$$\delta I := \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u_t^* I\big|_{t=0} = 2\eta'(r)\cosh^2 r \;\; \mathrm{d}r\,\mathrm{d}s & \text{inside } \operatorname{Im}\xi, \\ 0 & \text{elsewhere,} \end{cases}$$

where here $2\,\mathrm{d}s\,\mathrm{d}r = \mathrm{d}s \otimes \mathrm{d}r + \mathrm{d}r \otimes \mathrm{d}s$. Thanks to our choice of the function η, δI is a smooth symmetric tensor of Σ_k that represents the first order variation of I along the infinitesimal left earthquake e_α. By Proposition 3.4.2, we have that

$$\mathrm{d}V_k^*(\delta g) = \frac{1}{4}\int_{\Sigma_k} (\delta g|_{\Sigma_k}, HI - I\!I)\,\mathrm{da} = -\frac{1}{4}\int_0^{L_\alpha}\int_0^{\varepsilon} (\delta I, I\!I)\cosh r\,\mathrm{d}r\,\mathrm{d}s\,,$$

where the last step follows from the fact that δI is I-traceless. Let ∇ denote the Levi-Civita connection of I. Then the coordinate vector fields of ξ^{-1} satisfy:

$$\nabla_{\partial_r}\partial_r = 0, \qquad \nabla_{\partial_s}\partial_r = \nabla_{\partial_r}\partial_s = \tanh r\;\partial_s, \qquad \nabla_{\partial_s}\partial_s = -\sinh r\cosh r\;\partial_r.$$

By definition, $(\delta I, I\!I) = 2\, I^{rr}\, I^{ss}\, \delta I_{rs}\, I\!I_{rs} = 2\eta'\, I\!I_{rs}$. If we set $f(r) := \int_0^{L_\alpha} I\!I_{rs}\, \mathrm{d}s$, then, integrating by parts and recalling that $\eta(\varepsilon) = 0$, we get

$$\begin{aligned}\mathrm{d}V_k^*\,(\delta g) &= -\frac{1}{2}\int_0^\varepsilon \eta'(r) f(r)\cosh r\,\mathrm{d}r \\ &= \frac{1}{2} f(0) + \frac{1}{2}\int_0^\varepsilon \eta(r)(f'(r)\cosh r + f(r)\sinh r)\,\mathrm{d}r \qquad (\star)\end{aligned}$$

Being the shape operator a Codazzi tensor, we have $(\nabla_{\partial_r} I\!I)_{rs} = (\nabla_{\partial_s} I\!I)_{rr}$. Using the expressions of the connection given above, this relation can be rephrased as $\partial_r I\!I_{rs} = \partial_s I\!I_{rr} - \tanh r\ I\!I_{sr}$. Hence we deduce

$$f'(r) = \int_0^{L_\alpha} (\partial_s I\!I_{rr} - \tanh r\ I\!I_{sr})\,\mathrm{d}s = -\tanh r\ f(r),$$

where the first summand vanishes because α is a closed curve. Therefore the integral in the relation $(\star)$ equals 0, and we end up with the equation

$$\mathrm{d}V_k^*\,(\delta g) = \frac{1}{2}\int_0^{L_\alpha} I\!I_{rs}\,\mathrm{d}s = -\frac{1}{2}\int_0^{L_\alpha} I(B\alpha', J\alpha')\,\mathrm{d}s \qquad (3.8)$$

since $\partial_r|_{r=0} = V = -J\alpha'$ and $\partial_s|_{r=0} = \alpha'$.

Now we apply Lemma 3.4.5 to α, the hyperbolic metrics $h = -k\ I$, $h' = -\frac{k}{k+1}\ I\!I\!I$ and the operator $b = \frac{1}{\sqrt{k+1}}B$ (here B is the shape operator of Σ_k), obtaining

$$\mathrm{d}(L_\alpha)_h\,(l^1(h,h')) = -\frac{1}{2}\sqrt{-\frac{k}{k+1}}\int_0^{L_\alpha} I(B\alpha', J\alpha')\,\mathrm{d}s\,.$$

This relation, combined with (3.8), proves that

$$\mathrm{d}V_k^*\,(\delta g) = \sqrt{-\frac{k+1}{k}}\ \mathrm{d}(L_\alpha)_h\,(l^1(h,h'))$$

By the work of Wolpert [Wol83], we have $\mathrm{d}L_\alpha = \hat{\omega}_{WP}(\cdot, e_\alpha)$, which proves relation (3.7), and therefore the statement. □

Since the complex landislide is holomorphic with respect to the complex structure of $\mathcal{T}^{\mathfrak{h}}(\Sigma)^2$, an equivalent way to state Proposition 3.4.6 is the following:

Proposition 3.4.7. *Let M be a quasi-Fuchsian manifold and let h_k, h'_k denote the hyperbolic metrics $-k\ I_k$ and $-k(k+1)^{-1}\ I\!I\!I_k$ on $\Sigma_k^+ \sqcup \Sigma_k^-$. Then the Weil-Petersson gradient of V_k^* coincides, up to a multiplicative factor, with the infinitesimal grafting with respect to the couple (h_k, h'_k). In other words,*

$$\operatorname{grad}_{WP} V_k^* = \sqrt{-\frac{k+1}{k}}\ \frac{\mathrm{d}}{\mathrm{d}s}\ \mathrm{sgr}_s(h_k, h'_k)|_{s=0}\,.$$

The behavior of the third fundamental forms $I\!I\!I_k$ of the k-surfaces, as k approaches -1, is well understood and described by the following Theorem:

Theorem 3.4.8. *Let $(E_n)_n$ be a sequence of hyperbolic ends converging to a hyperbolic end E homeomorphic to $\Sigma \times \mathbb{R}_{\geq 0}$, and let $(k_n)_n$ be any decreasing sequence of numbers converging to -1. Then $\ell_{I\!I\!I_n}$ converges to $\iota(\mu, \cdot)$, where $I\!I\!I_n$ denotes the third fundamental form of the k_n-surface of E_n, and μ is the bending measured lamination of the concave boundary of E.*

Remark 3.4.9. Theorem 3.4.8 is in fact a restatement of [Bel17, Theorem 2.10]. In [Bel17] the author works with *maximal global hyperbolic spatially compact* (MGHC) *de Sitter spacetimes*, which connect to the world of hyperbolic ends through the duality between the de Sitter and the hyperbolic space-forms described in Section 1.4 (see also Theorem 5.1.4 and its proof). In particular, this phenomenon allowed Barbot, Béguin, and Zeghib [BBZ11] to give an alternative proof of the existence of the foliation by k-surfaces.

Finally, we have all the elements to give a proof of Proposition 3.4.2:

Proof of Proposition 3.4.2. Let $(M_n)_n$ be a sequence of quasi-Fuchsian manifolds converging to M, and let $(k_n)_n$ be a decreasing sequence converging to -1. We denote my m_n and m_n' the isotopy classes of the hyperbolic metrics

$$h_n := -k_n I_{k_n}, \qquad h_n' := -\frac{k_n}{1+k_n} I\!I\!I_{k_n},$$

where I_{k_n} and $I\!I\!I_{k_n}$ are the first and second fundamental forms of the k_n-surface $\Sigma_{k_n}^+ \sqcup \Sigma_{k_n}^-$ sitting inside M_n. The k_n-surface is at distance $< \operatorname{arctanh}(\sqrt{k_n+1})$ from the convex core of M_n (apply the same argument of [BMS13, Lemma 6.14] in the hyperbolic setting), therefore the metrics m_n converge to the metric m on the boundary of the convex core of M. If we take

$$\theta_n := \sqrt{-\frac{1+k_n}{k_n}},$$

then, by Theorem 3.4.8, the length spectrum of $\theta_n \ell_{m_n'}$ converges to the bending measure μ of the boundary of the convex core of M. Therefore, applying Theorem 3.4.3 we obtain that $l_1(m_n, m_n')|_{T_{k_n}(M_n)}$ converges to $1/2\ e_\mu|_m$. Combining this with Proposition 3.4.6, we prove that

$$\lim_{n\to\infty} \mathrm{d}V_{k_n}^* \circ T_{k_n}(M_n) = \frac{1}{2}\hat{\omega}_{WP}(e_\mu, \cdot) = -\frac{1}{2}\,\mathrm{d}(L_\mu)_m(\cdot),$$

where the last step follows from [Wol83]. This concludes the proof. □

Chapter 4

The infimum of the dual volume

Outline of the chapter

The aim of this chapter is to study the infimum of the dual volume of the convex core as we vary the convex co-compact hyperbolic structure on a fixed underlying topological 3-manifold with incompressible boundary. In particular, we will see that:

Theorem C. *For every convex co-compact hyperbolic 3-manifold M with incompressible boundary we have*

$$\inf_{\mathcal{QD}(M)} V_C^* = \inf_{\mathcal{QD}(M)} V_C .$$

Moreover, $V_C^*(M) = V_C(M)$ if and only if the boundary of the convex core of M is totally geodesic.

This statement is the analogue for the dual volume of a result due to Bridgeman, Brock, and Bromberg [BBB19], where the authors studied the infimum of the *renormalized volume* function. More precisely, they showed:

Theorem ([BBB19, Theorem 3.10])**.** *For every convex co-compact hyperbolic 3-manifold M with incompressible boundary we have*

$$\inf_{\mathcal{QD}(M)} V_R = \inf_{\mathcal{QD}(M)} V_C .$$

Moreover, $V_R(M) = V_C(M)$ if and only if the boundary of the convex core of M is totally geodesic.

Dual volume, renormalized volume and Riemannian volume of the convex core are related by the following chain of inequality:

$$V_C^*(M) := V_C(M) - \frac{1}{2}\ell_m(\mu) \leq V_R(M) \leq V_C(M) - \frac{1}{4}\ell_m(\mu).$$

Here the upper bound is originally due to Schlenker [Sch02], and the lower bound is proved again in [BBB19, Theorem 3.7]. In particular, Theorem C can be considered as a strengthening of [BBB19, Theorem 3.10].

The proof we will present is analogous to the one developed in [BBB19] but, as happened already in the previous Chapter, we will need to pass through an approximation procedure, using the properties of *constant Gaussian curvature surfaces*. The necessity of this process is due to the fact that, in contrast to the conformal structures at infinity, the hyperbolic metrics on the boundary of the convex core are not known to parametrize the space of quasi-isometric convex co-compact structures of M.

To be more precise, we need to introduce some notation. Let M_k denote the compact region of M enclosed by the k-surface on M, which has one connected component in each end on M. We define the map

$$\begin{array}{rccc} T_k : & \mathcal{QD}(M) & \longrightarrow & \mathcal{T}^{\mathfrak{h}}(\partial M) \\ & M' & \longmapsto & m_k(M'), \end{array}$$

which associates, to each convex co-compact structure M', the isotopy class $m_k(M')$ of the hyperbolic metric $h_k = (-k)I_k$ on the k-surface ∂M_k of M. By the works of Labourie [Lab92a] and Schlenker [Sch06], if M has incompressible boundary the function T_k is a diffeomorphism for every $k \in (-1, 0)$ (the same argument presented in Theorem 3.4.1 applies to this more general setting). For every hyperbolic structure $m \in \mathcal{T}^{\mathfrak{h}}(\partial M)$, we define $V_k^*(m)$ to be the dual volume of the region M'_k of $M' = T_k^{-1}(m)$ enclosed by its k-surface.

We briefly summarize the strategy of the proof. Given any $k \in (-1, 0)$, we will estimate the infimum of V_k^* by moving along the flow of its Weil-Petersson gradient $\operatorname{grad}_{WP} V_k^*$. In order to prove the existence of such flow for every time, we will show that the L^∞-norm of $\operatorname{grad}_{WP} V_k^*$ is uniformly bounded over $\mathcal{T}^{\mathfrak{h}}(\partial M)$. The technical estimates for this purpose will be developed in Section 4.1. From the uniform control of $\|\operatorname{grad}_{WP} V_k^*\|_\infty$, the existence of the flow will easily follow (see Corollary 4.2.6).

The second key ingredient will be a bound from below of the Weil-Petersson norm of $\operatorname{grad}_{WP} V_k^*$ in terms of the integral of the mean curvature of ∂M_k. This will be achieved in Section 4.2, and in particular in Lemma 4.2.4. Through these observations, in the last section we will follow the same formal procedure of [BBB19, Theorem 3.10] to determine a bound from below of the functions V_k^*. Then the final statement of Theorem C will be achieved by taking a limit for k that goes to -1, concluding the approximation procedure of the dual volume of the convex core V_C^* through the functions V_k^*.

4.1 Some useful estimates

In this section we develop estimates for the solution $u_k : \partial M_k \to \mathbb{R}$ of a certain elliptic PDE (see relation (4.1)) over the k-surface ∂M_k of a convex co-compact hyperbolic 3-manifold M with incompressible boundary. The function u_k will be involved in the description of the Weil-Petersson gradient $\operatorname{grad}_{WP} V_k^*$ of Proposition 4.2.2.

As already observed by Bonsante et al. [Bon+19], the incompressibility of the boundary ∂M_k determines (non-explicit) bounds on the mean curvature function H_k depending only on the curvature $k \in (-1, 0)$ and, in particular, not on the geometry of M. Being the function u_k determined by a simple equation involving H_k, these controls will imply uniform bounds on the $\mathscr{C}^2$-norms of u_k.

First we introduce some notation. If (N, g) is a Riemannian manifold, we denote by $H^n(N, \mathrm{d}a_g)$ the Sobolev space of real-valued functions f on M with $L^2(N, \mathrm{d}a_g)$-integrable weak derivatives $({}^g\nabla)^i f$ for all $i \leq n$, where ${}^g\nabla$ and $\mathrm{d}a_g$ are the Levi-Civita connection and the volume form of (N, g), respectively. The space $H^n(N, \mathrm{d}a_g)$ is a Hilbert space if endowed with the scalar product

$$(f, f') := \sum_{i=0}^{n} \int_N (({}^g\nabla)^i f, ({}^g\nabla)^i f')_g \, \mathrm{d}a_g \,, \qquad f, f' \in H^n(N, \mathrm{d}a_g).$$

Finally, given $f \colon N \to \mathbb{R}$ a $\mathscr{C}^n$-function, we define its $\mathscr{C}^n(N, g)$-norm as

$$\|f\|_{\mathscr{C}^n(N,g)} := \sum_{i=0}^{n} \sup_{p \in N} \left\| ({}^g\nabla)^i f\big|_p \right\|_g .$$

In the following, we will denote by ${}^k\nabla$ and Δ_k the Levi-Civita connection and the Laplace-Beltrami operator $\Delta_k u = \mathrm{tr}_{h_k}({}^k\nabla^2 u)$ with respect to the hyperbolic metric $h_k := (-k) I_k$ on the k-surface ∂M_k. We define the following linear differential operator:

$$L_k u := (\Delta_k - 2\,\mathbb{1})u = \Delta_k u - 2u.$$

Let A be the symmetric bilinear form over the Hilbert space $H^1(\partial M_k, \mathrm{d}a_k)$ given by

$$A(u, u) := \int_\Sigma (\|\mathrm{d}u\|_k^2 + 2u^2) \, \mathrm{d}a_k \,,$$

where $\|\cdot\|_k$ and $\mathrm{d}a_k$ denote the norm and the area form of h_k, respectively. A simple application of the Lax-Milgram's theorem (see e. g. [Bre11, Corollary 5.8]) applied to the Sobolev space $H^1(\partial M_k, \mathrm{d}a_k)$ and to the coercive symmetric bilinear form A shows that, for every $f \in L^2(\partial M_k, \mathrm{d}a_k)$, there exists a unique weak solution $u \in H^1(\partial M_k, \mathrm{d}a_k)$ of the equation $L_k u = f$. We will denote by u_k the solution of the equation

$$L_k u_k = -k^{-1} H_k \Leftrightarrow \Delta_{I_k} u_k + 2k u_k = H_k, \tag{4.1}$$

where H_k denotes the mean curvature function $\mathrm{tr}\big(I_k^{-1} I\!I_k\big)$ of the k-surface ∂M_k. We will always consider the second fundamental form defined by the normal vector field on ∂M_k pointing towards M_k, so that $I\!I_k$ is positive definite, and H_k is a positive function.

By the classical regularity theory for linear elliptic PDE's (see e. g. [Eva98, Section 6.3]), the smoothness of the mean curvature H_k and the compactness of ∂M_k imply that the function u_k is smooth and it is a strong solution of equation (4.1).

It has been shown by Rosenberg and Spruck [RS94, Theorem 4] that, for every Jordan curve c in $\partial_\infty \mathbb{H}^3$, there exist exactly two k-surfaces $\widetilde{\Sigma}_k^\pm(c)$ asymptotic to c. A fundamental property of k-surfaces, which will crucial in Lemma 4.1.3, is the following:

Proposition 4.1.1 ([Bon+19, Proposition 3.8])**.** *Let $k \in (-1, 0)$ and $n \in \mathbb{N}$. Then there exists a constant $N_{k,n} > 0$ such that, for every Jordan curve c in $\partial_\infty \mathbb{H}^3$, the mean curvature $H_{c,k}$ of the k-surface $\widetilde{\Sigma}_k(c) = \widetilde{\Sigma}_k^+(c) \sqcup \widetilde{\Sigma}_k^-(c)$ asymptotic to c satisfies*

$$\|H_{c,k}\|_{\mathscr{C}^n(\widetilde{\Sigma}_k(c))} \leq N_{n,k}.$$

Proof. For completeness, we briefly recall here the proof of this statement. k-surfaces satisfy the following compactness criterion:

Proposition 4.1.2 ([Bon+19, Proposition 3.6])**.** *Let $k \in (-1, 0)$, and consider $f_n \colon \mathbb{H}^2_k \to \mathbb{H}^3$ a sequence of proper isometric embeddings of the hyperbolic plane $\mathbb{H}^2_k$ with constant Gaussian curvature k. If there exists a point $p \in \mathbb{H}^2$ such that $(f_n(p))_n$ is precompact, then there exists a subsequence of $(f_n)_n$ that converges $\mathscr{C}^\infty$-uniformly on compact sets to an isometric immersion $f \colon \mathbb{H}^2_k \to \mathbb{H}^3$.*

Fixed $k \in (-1, 0)$ and $n \in \mathbb{N}$, assume by contradiction that there exists a sequence of Jordan curves $(c_m)_m$ such that the mean curvatures $H_m = H_{c_m,k}$ of the k-surface $\widetilde{\Sigma}_k(c_m)$ satisfy $\|H_m\|_{\mathscr{C}^n(\widetilde{\Sigma}_k(c_m))} > m$. Up to extract a subsequence, there exists an $i \leq n$ such that for every $m \in \mathbb{N}$

$$\sup_{\widetilde{\Sigma}_k(c_m)} \left\| ({}^k\nabla)^i H_m \right\| > \frac{m}{n+1} = C_n \, m.$$

Now choose $q_m \in \widetilde{\Sigma}_k(c_m)$ for which the norm of $({}^k\nabla)^i H_m$ at q_m is $\geq C_n \, m$. Since each component of $\widetilde{\Sigma}_k(c_m)$ is embedded and isometric to the hyperbolic plane $\mathbb{H}^2_k$ (which is homogeneous), we can find a sequence of proper isometric embeddings $f_m \colon \mathbb{H}^2_k \to \mathbb{H}^3$, parametrizing a component of $\widetilde{\Sigma}_k(c_m)$, such that $f_m(\bar{p}) = q_m$ for some fixed basepoint $\bar{p} \in \mathbb{H}^2_k$. Up to post-composing f_m by an isometry of $\mathbb{H}^3$, we can assume that $f_m(\bar{p}) = \bar{q}$ is fixed. In this way, we have found a sequence of proper isometric embeddings $f_m \colon \mathbb{H}^2_k \to \mathbb{H}^3$ satisfying

- $f_m(\bar{p}) = \bar{q} \in \mathbb{H}^3$ is independent of $m \in \mathbb{N}$;
- the mean curvature of the surfaces $f_m(\mathbb{H}^2_k)$ at $\bar{q}$ has some i-th order derivative that is unbounded as m goes to ∞.

This clearly contradicts the compactness criterion mentioned above. □

From this result we can now obtain a uniform control on u_k:

Lemma 4.1.3. *Let M be a convex co-compact hyperbolic 3-manifold with incompressible boundary. Then, the function $u_k \colon \partial M_k \to \mathbb{R}$, solution of (4.1), satisfies*

$$\frac{\max_{\partial M_k} H_k}{2k} \leq u_k \leq \frac{\min_{\partial M_k} H_k}{2k} = \frac{\sqrt{k+1}}{k} < 0.$$

Moreover, if M has incompressible boundary, then there exists a constant $C_k > 0$ depending only on the intrinsic curvature $k \in (-1, 0)$, and in particular not on the hyperbolic structure of M, such that

$$\max_{\partial M_k} \left\| {}^k\nabla^2 u_k \right\|_k \leq C_k.$$

Proof. The first assertion is an immediate consequence of the maximum principle applied to u_k as a solution of the PDE (4.1). Moreover, since the product of the principal curvatures (i. e. the eigenvalues of the shape operator) of a k-surface is everywhere equal to $k+1$, the trace of the shape operator is bounded from below by $2\sqrt{k+1}$ (see also Remark 4.1.5 below).

The proof of the second part of the assertion requires more care. Let Σ_k be a connected component of the k-surface ∂M_k, and let $\widetilde{M} \cong \mathbb{H}^3$ denote the universal cover of M. Since M is a convex co-compact hyperbolic 3-manifold with incompressible boundary, every component $\widetilde{\Sigma}_k$ of the preimage of Σ_k in $\widetilde{M}$ is stabilized by a subgroup $\Gamma \cong \pi_1(\Sigma_k)$ of the fundamental group of M, acting by isometries on $\widetilde{M}$. Each of these subgroups Γ is quasi-Fuchsian (see e. g. [Kap09, Corollary 4.112 and Theorem 8.17] for a proof of this assertion), and the surface $\widetilde{\Sigma}_k$ is a k-surface asymptotic to some Jordan curve in $\partial_\infty \widetilde{M} \cong \partial_\infty \mathbb{H}^3$. In particular, by Proposition 4.1.1, we can find a universal constant $N_k = N_{2,k} > 0$ that satisfies

$$\|\tilde{H}_k\|_{\mathscr{C}^2(\widetilde{\Sigma}_k)} \leq N_k. \tag{4.2}$$

Here we stress that the constant N_k does not depend on the hyperbolic structure of M, or Σ_k, but only on the value of $k \in (-1, 0)$.

Our goal is now to make use of this control to obtain a uniform bound of the norm of the Hessian of u_k. For this purpose, we will need the following classical result of regularity for linear elliptic differential equations:

Theorem 4.1.4 ([Eva98, Theorem 2, page 314]). *Let $m, n \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ a bounded open set. We consider a differential operator L of the form:*

$$Lf = -\sum_{i,j=1}^{n} a^{ij}(x)\, \partial^2_{x_i,x_j} f + \sum_{i=0}^{n} b^i(x)\, \partial_{x_i} f + c(x) f,$$

where $a^{ij} = a^{ji}, b^i, c \in \mathscr{C}^{m+1}(U, \mathbb{R})$. Assume that L is uniformly elliptic, i. e. there exists a constant $\varepsilon > 0$ such that $\sum_{i,j} a^{ij}(x) v_i v_j \geq \varepsilon \|v\|^2$ for all $v \in \mathbb{R}^n$. If $f \in H^1(U)$ is a weak solution of the equation $Lf = \lambda$, for some $\lambda \in H^m(U)$, then for every bounded open set V with closure contained in U, there exists a constant C, depending only on m, U, V and the functions a^{ij}, b^i, c, such that

$$\|f\|_{H^{m+2}(V)} \leq C(\|\lambda\|_{H^m(U)} + \|f\|_{L^2(U)}).$$

The surface $\widetilde{\Sigma}_k$ endowed with the lift of the hyperbolic metric h_k of Σ_k is isometric to the hyperbolic plane $\mathbb{H}^2$. In the following, we will identify $\widetilde{\Sigma}_k$ with the Poincaré disk model $\mathbb{H}^2 := (B_1, g_P)$, where B_1 is the Euclidean ball of radius 1 and center 0 in $\mathbb{C}$, and

$$g_P = \left(\frac{2}{1 - |z|^2} \right)^2 |\mathrm{d}z|^2.$$

Now we choose U and V to be the g_P-geodesic balls of center $0 \in B_1$ and hyperbolic radius equal to 2 and 1, respectively. The lift of the operator $-L_k$ over U is clearly uniformly elliptic, because of the compactness of $\overline{U}$ and because of the following expression in coordinates:

$$-L_k f = -g_P^{ij}(\partial^2_{ij} f - \Gamma^h_{ij}(g_P)\, \partial_h f) + 2f,$$

where $\Gamma^h_{ij}(g_P)$ denote the Christoffel symbols of g_P. Again by the compactness of $\overline{U}$ and $\overline{V}$, the norms of the Sobolev spaces $\|\cdot\|_{H^j(U)}$ and $\|\cdot\|_{H^j(V)}$, computed with respect to the flat connection of $B_1 \subset \mathbb{R}^2$ and the Euclidean volume form, are equivalent to the norms of the corresponding Sobolev spaces defined using the Levi-Civita connection of g_P and the g_P-volume form. Moreover, the bi-

Lipschitz constants involved in the equivalence only depend on a bound of the $\mathscr{C}^{j+1}$-norm of g_P over U, therefore they can be chosen to depend only on $j \in \mathbb{N}$. From now on, we will always consider the norms on the spaces $H^j(U)$ and $H^j(V)$ to be defined using the metric g_P and its connection.

Now we apply Theorem 4.1.4 to $m = n = 2$, the operator $-L_k$ and the functions $f = \tilde{u}_k$, $\lambda = -k^{-1}\tilde{H}_k$, where $\tilde{F}$ denotes the lift of the function F over $\widetilde{\Sigma}_k$. Therefore we can find a universal constant $C > 0$ (depending only on the open sets U, V, that we chose once for all, and on the metric $g_P|_U$) such that:

$$\|\tilde{u}_k\|_{H^4(V)} \leq C(-k^{-1}\|\tilde{H}_k\|_{H^2(U)} + \|\tilde{u}_k\|_{L^2(U)}).$$

By the first part of the Lemma 4.1.3, $\|\tilde{u}_k\|_{\mathscr{C}^0(U)} \leq -(2k)^{-1}\|\tilde{H}_k\|_{\mathscr{C}^0(\mathbb{H}^2)}$. In addition, we have:

$$\|\tilde{u}_k\|_{L^2(U)} \leq \operatorname{Area}(U, g_P)^{1/2}\|\tilde{u}_k\|_{\mathscr{C}^0(U)} \leq -(2k)^{-1}\operatorname{Area}(U, g_P)^{1/2}\|\tilde{H}_k\|_{\mathscr{C}^0(\mathbb{H}^2)},$$

and

$$\|\tilde{H}_k\|_{H^2(U)} \leq \operatorname{Area}(U, g_P)^{1/2}\|\tilde{H}_k\|_{\mathscr{C}^2(\mathbb{H}^2)}.$$

In conclusion, we deduce that

$$\|\tilde{u}_k\|_{H^4(V)} \leq -2k^{-1}C\operatorname{Area}(U, g_P)^{1/2}\|\tilde{H}_k\|_{\mathscr{C}^2(\mathbb{H}^2)}.$$

By the Sobolev embedding theorem (see e. g. [Bre11, Corollary 9.13, page 283]), given W an open set satisfying $0 \in W \subset \overline{W} \subset V$, the $\mathscr{C}^2(W)$-norm of $\tilde{u}_k$ (again, computed with respect to the Levi-Civita connection of g_P) is controlled by a multiple of its H^4-norm over V, and the multiplicative factor depends only on W and V. Therefore, if we choose for instance $W = B_{\mathbb{H}^2}(0, 1/2)$ we get:

$$\|{}^k\nabla\tilde{u}_k\|_{\mathscr{C}^0(W)} \leq C'(k)\,\|\tilde{H}_k\|_{\mathscr{C}^2(\mathbb{H}^2)}.$$

Now the desired statement easily follows. From relation (4.2) and the last inequality, we obtain a uniform bound of the Hessian of $\tilde{u}_k$ over $W \ni 0$. Let now q be any other point of $\mathbb{H}^2$, and choose a g_P-isometry $\varphi_q \colon B_1 \to B_1$ such that $\varphi_q(0) = q$. If we replace $\tilde{u}_k$ and $\tilde{H}_k$ with $\tilde{u}_k \circ \varphi_q$ and $\tilde{H}_k \circ \varphi_q$, respectively, the exact same argument above applies, since the operator L_k and the norms $\|\cdot\|_{H^j}$, $\|\cdot\|_{\mathscr{C}^l}$ are invariant under the action of the isometry group of $\mathbb{H}^2$ (and since $\|\tilde{H}_k\|_{\mathscr{C}^2(\mathbb{H}^2)} = \|\tilde{H}_k \circ \varphi_q\|_{\mathscr{C}^2(\mathbb{H}^2)}$). In particular, this gives us a control of the norm of ${}^k\nabla\tilde{u}_k$ over $\varphi_q(W)$ for any point $q \in \mathbb{H}^2$, and the last part of our assertion follows. □

Remark 4.1.5. The minimum of the mean curvature $2\sqrt{k+1}$ is always realized. As described in Section 1.5, whenever we have a k-surface Σ_k with first and second fundamental forms I_k and $I\!I_k$, respectively, the identity map *id*: $(\Sigma_k, I\!I_k) \to (\Sigma_k, I_k)$ is harmonic, with Hopf differential ψ_k satisfying

$$2\operatorname{Re}\psi_k = I_k - \frac{H_k}{2(k+1)} I\!I_k.$$

Its squared norm with respect to $I\!I_k$ can be expressed as follows

$$\|2\operatorname{Re}\psi_k\|^2_{I\!I_k} = \frac{H_k^2 - 4(k+1)}{(k+1)^2}.$$

In particular, at each zero of ψ_k (which necessarily exist because $\chi(\Sigma_k) < 0$) we have $H_k = 2\sqrt{k+1}$.

We stress that, even if the maximum of the mean curvature H_k will clearly depend on the hyperbolic structure of M, Proposition 4.1.1 guarantees that $\max H_k$ is controlled by a function of k independent on the geometry of M, as long as ∂M is incompressible.

We will make use of the upper bound $u_k \leq \frac{\sqrt{k+1}}{k}$ in Lemma 4.2.4, where we will determine a lower bound of the Weil-Petersson norm of the differential of V_k^* in terms of the integral of the mean curvature.

4.2 The gradient of the dual volume

The aim of this section is to describe the gradient of the dual volume function V_k^* with respect to the Weil-Petersson metric on the Teichmüller space of ∂M in terms of the function u_k studied in the previous section.

First we introduce the necessary notation for the "Riemannian geometric tools" that will be used. Let (N, g) be a Riemannian manifold, and consider $(e_i)_i$ a local g-orthonormal frame. Given $S \in \Gamma(S^2(N))$ a symmetric 2-tensor, we define the g-divergence of S as the 1-form $\operatorname{div}_g S$ defined by:

$$(\operatorname{div}_g S)(X) := \sum_i ({}^g\nabla_{e_i} S)(e_i, X),$$

for every tangent vector field X. Similarly, the g-divergence of a vector field X is the function

$$\operatorname{div}_g X = \sum_i g({}^g\nabla_{e_i} X, e_i).$$

The Laplace-Beltrami operator can be expressed as $\Delta_g f = \operatorname{div}_g \operatorname{grad}_g f$. Given two symmetric tensors S, T, their scalar product is defined as

$$(S, T)_g := g^{ij}\, g^{hk}\, S_{ih}\, T_{jk} = \operatorname{tr}\big(g^{-1} S g^{-1} T\big).$$

In particular, we set $\operatorname{tr}_g S := (g, S)_g = \operatorname{tr}\big(g^{-1} S\big)$. It will also be useful to keep in mind the way that these operators change if with replace g with λg, for some positive constant λ:

$$\operatorname{div}_{\lambda g} S = \lambda^{-1} \operatorname{div}_g S, \quad \Delta_{\lambda g} f = \lambda^{-1} \Delta_g f, \quad \mathrm{d}a_{\lambda g} = \lambda^{n/2}\, \mathrm{d}a_g\,,$$
$$(S, T)_{\lambda g} = \lambda^{-2} (S, T)_g, \qquad \operatorname{tr}_{\lambda g} S = \lambda^{-1} \operatorname{tr}_g S,$$

if $\dim N = n$.

We recall that, by the dual differential Schläfli formula (see Proposition 2.2.5), we have:

Proposition 4.2.1.

$$\begin{aligned}\mathrm{d}(V_k^* \circ T_k)\,(\dot{M}) &= \frac{1}{4} \int_{\partial M_k} (\dot{I}_k, I\!I_k - H_k I_k)_{I_k}\, \mathrm{d}a_{I_k} \\ &= \frac{1}{4} \int_{\partial M_k} (\dot{h}_k, I\!I_k + k^{-1} H_k h_k)_{h_k}\, \mathrm{d}a_{h_k}\,,\end{aligned}$$

where $\dot{I}_k = -k^{-1}\dot{h}_k$ *is the first order variation of the first fundamental form on* ∂M_k *along the variation* $\dot{M}$*, and* $T_k \colon \mathcal{QD}(M) \to \mathcal{T}^\mathfrak{h}(\partial M)$ *is the diffeomorphism introduced at the beginning of the chapter.*

From this variation formula, we can give an explicit description of the Weil-Petersson gradient of the dual volume functions V_k^*, which will turn out to be useful for the study of its flow.

Proposition 4.2.2. *The vector field* $\operatorname{grad}_{WP} V_k^*$ *is represented by the harmonic Beltrami differential associated to* ϕ_k*, where* ϕ_k *is the (unique) holomorphic quadratic differential satisfying*

$$\operatorname{Re}\phi_k = I\!I_k - {}^k\nabla^2 u_k + u_k h_k,$$

where u_k *denotes the solution of equation* (4.1)*.*

Proof. Let $\dot{m}_k$ denote a tangent vector to the Teichmüller space of ∂M at m_k. As described by Tromba [Tro92], given any hyperbolic metric h_k representing the isotopy class $m_k \in \mathcal{T}^\mathfrak{h}(\partial M)$, we can find a unique symmetric tensor $\dot{h}_k$ representing $\dot{m}_k$ that is h_k-traceless and h_k-divergence-free (also called *transverse traceless*). This analytic condition turns out to be equivalent to ask that the tensor $\dot{h}_k$ coincides with the real part of a holomorphic quadratic differential (see Remark 1.2.6).

Assume for a moment that we can find a decomposition of the symmetric tensor $I\!I_k + k^{-1}H_k h_k$ of the following form:

$$I\!I_k + k^{-1}H_k h_k = S_{tt} + \mathcal{L}_X h_k + \lambda\, h_k,$$

where S_{tt} is a transverse traceless tensor with respect to h_k, X is a vector field and λ is a smooth function on ∂M. Then, by Proposition 4.2.1, we could express the variation of the dual volume V_k^* along a transverse traceless variation $\dot{h}_k$ as follows:

$$\mathrm{d}V_k^*(\dot{h}_k) = \frac{1}{4}\int_{\partial M_k} (\dot{h}_k, S_{tt} + \mathcal{L}_X h_k + \lambda\, h_k)_{h_k}\, \mathrm{d}a_{h_k}.$$

Since $\dot{h}_k$ is traceless, the scalar product $(\dot{h}_k, h_k) = \operatorname{tr}_{h_k}(\dot{h}_k)$ vanishes identically. Moreover, the L^2-scalar product between $\dot{h}_k$ and $\mathcal{L}_X h_k$ vanishes too, because the condition of being h_k-divergence-free is equivalent to be L^2-orthogonal to the vector space $\{\mathcal{L}_X h_k \mid X \in \Gamma(T\partial M)\}$ (see [Tro92, Theorem 1.4.2] or relation (4.6) for a proof of this last assertion). In particular, we must have

$$\mathrm{d}V_k^*(\dot{h}_k) = \frac{1}{4}\int_{\partial M_k} (\dot{h}_k, S_{tt})_{h_k}\, \mathrm{d}a_{h_k}.$$

On the other hand, if ϕ_k is the holomorphic quadratic differential associated to the harmonic Beltrami differential representing $\operatorname{grad}_{WP} V_k^*$, then by Lemma 3.1.2 we have

$$\mathrm{d}V_k^*(\dot{h}_k) = \frac{1}{8}\int_{\partial M_k} (\dot{h}_k, 2\operatorname{Re}\phi_k)_{h_k}\, \mathrm{d}a_{h_k}.$$

Therefore, by varying the tangent vector $\dot{m}_k \in T_{m_k}\mathcal{T}(\partial M)$, we deduce that the tensor S_{tt} and the holomorphic quadratic differential ϕ_k must satisfy $\operatorname{Re}\phi_k = S_{tt}$.

In conclusion, this argument shows us that, in order to prove our assertion, we need to determine a decomposition of the tensor $I\!I_k + k^{-1} H_k h_k$ of the form we described above, with $S_{tt} = I\!I_k - {}^k\nabla^2 u_k + u_k h_k$. For this purpose, we consider the following expression:

$$\begin{aligned} I\!I_k + k^{-1} H_k h_k &= (I\!I_k - {}^k\nabla^2 u_k + u_k h_k) + {}^k\nabla^2 u_k + (k^{-1} H_k - u_k) h_k \\ &= (I\!I_k - {}^k\nabla^2 u_k + u_k h_k) + \frac{1}{2} \mathcal{L}_{\mathrm{grad}_{h_k} u_k} h_k + (k^{-1} H_k - u_k) h_k, \end{aligned}$$

where we used the relation $\mathcal{L}_{\mathrm{grad}_{h_k} u_k} h_k = 2\, {}^k\nabla^2 u_k$. In this expression, the second term of the sum is of the type $\mathcal{L}_X h_k$, while the third term has the form $\lambda\, h_k$. Then, it is enough to show that the first term is h_k-traceless and h_k-divergence-free. The trace of $I\!I_k - {}^k\nabla^2 u_k + u_k h_k$ satisfies

$$\mathrm{tr}_{h_k}(I\!I_k - {}^k\nabla^2 u_k + u_k h_k) = -k^{-1} H_k - \Delta_k u_k + 2 u_k .$$

This expression vanishes because u_k is a solution of equation (4.1). In order to compute the divergence of our tensor, we will need the following relations:

$$\mathrm{div}_{h_k} I\!I_k = -k^{-1}\, \mathrm{d} H_k \,, \qquad \mathrm{div}_g({}^g\nabla^2 f) = \mathrm{d}(\Delta_g f) + \mathrm{Ric}_g(\mathrm{grad}_g f, \cdot).$$

The first equality follows from the Codazzi equation $({}^k\nabla_X B_k) Y = ({}^k\nabla_Y B_k) X$ satisfied by the shape operator B_k of ∂M_k (the Levi-Civita connections of h_k and the first fundamental form I_k are the same, since they differ by a multiplicative constant). The second relation is true for any Riemannian metric g, and we will apply it in the case $g = h_k$ and $f = u_k$. Since h_k is a hyperbolic metric on a 2-manifold, we have $\mathrm{Ric}_{h_k} = -h_k$. Therefore

$$\begin{aligned} \mathrm{div}_{h_k}(I\!I_k - \nabla_k^2 u_k + u_k h_k) &= -k^{-1}\, \mathrm{d} H_k - \mathrm{d}(\Delta_k u_k) + \mathrm{d} u_k + \mathrm{d} u_k \\ &= \mathrm{d}\big(-k^{-1} H_k - \Delta_k u_k + 2 u_k\big) \,, \end{aligned}$$

where we used the relation $\mathrm{div}_g(f\, g) = \mathrm{d} f$. Again, the expression above vanishes because u_k solves equation (4.1). Then we have shown that $I\!I_k - {}^k\nabla^2 u_k + u_k h_k$ is a transverse traceless tensor, as desired. □

Remark 4.2.3. In fact, the decomposition we presented for the tensor $I\!I_k + k^{-1} H_k h_k$ is related to the orthogonal decomposition of the space of symmetric tensors due to Fischer and Marsden [FM75]. Given g a hyperbolic metric, every symmetric tensor S admits an orthogonal decomposition of the following form:

$$S = S_{tt} + \mathcal{L}_X g + ((-\Delta_g f + f) g + {}^g\nabla^2 f),$$

where

- S_{tt} is transverse traceless with respect to g;
- $S_{tt} + \mathcal{L}_X g$ is tangent to the space of Riemannian metrics with constant Gaussian curvature equal to -1. In other words, if $g' \mapsto K(g')$ denotes the operator that associates to the Riemannian metric g' its Gaussian curvature, then $S_{tt} + \mathcal{L}_X g \in \ker \mathrm{d} K_g$;
- $(-\Delta_g f + f) g + {}^g\nabla^2 f$ lies in the L^2-orthogonal of $\ker \mathrm{d} K_g$.

Then, the expression

$$\begin{aligned} I\!I_k + k^{-1} H_k h_k &= (I\!I_k - {}^k\nabla^2 u_k + u_k h_k) + 0 + ((k^{-1} H_k - u_k) h_k + {}^k\nabla^2 u_k) \\ &= (I\!I_k - {}^k\nabla^2 u_k + u_k h_k) + 0 + ((-\Delta_k u_k + u_k) h_k + {}^k\nabla^2 u_k) \end{aligned}$$

is the Fischer-Marsden decomposition of $I\!I_k + k^{-1} H_k h_k$, where $f = u_k$, $X = 0$ and $S_{tt} = (I\!I_k - {}^k\nabla^2 u_k + u_k h_k)$.

Using this explicit description of the Weil-Petersson gradient of the dual volume function V_k^*, we can determine a lower bound of its norm in terms of the integral of the mean curvature:

Lemma 4.2.4. *For every $k \in (-1, 0)$ we have*

$$\|\mathrm{d}V_k^*\|_{WP}^2 \geq -\frac{\sqrt{k+1}}{2k} \int_{\partial M_k} H_k \,\mathrm{d}a_{I_k} - \frac{2\pi(k+1)}{k^2} |\chi(\partial M)|.$$

Proof. In what follows, we will prove the following expression:

$$\left\| I\!I_k - \nabla_k^2 u_k + u_k h_k \right\|_{I_k}^2 = k u_k H_k - 2(k+1) + \mathrm{div}_{I_k} W, \tag{4.3}$$

for some tangent vector field W on ∂M_k. Assuming for the moment this relation, we deduce that

$$\begin{aligned} \|\mathrm{d}V_k^*\|_{WP}^2 &= \frac{1}{2} \int_{\partial M_k} \|\mathrm{Re}\,\phi_k\|_{h_k}^2 \,\mathrm{d}a_{h_k} && \text{(Prop. 4.2.2 and Lemma 3.1.2)} \\ &= \frac{1}{2} \int_{\partial M_k} (-k)^{-2} \|\mathrm{Re}\,\phi_k\|_{I_k}^2 (-k) \,\mathrm{d}a_{I_k} \\ &= -\frac{1}{2k} \int_{\partial M_k} (k u_k H_k - 2(k+1)) \,\mathrm{d}a_{I_k}, && \text{(relation (4.3))} \end{aligned}$$

where we used the relation $h_k = (-k) I_k$, and that the integral of the term $\mathrm{div}_{I_k} W$ vanishes by the divergence theorem. By Lemma 4.1.3, we have $u_k \leq \frac{\sqrt{k+1}}{k}$, therefore we obtain

$$\|\mathrm{d}V_k^*\|_{WP}^2 \geq -\frac{\sqrt{k+1}}{2k} \int_{\partial M_k} H_k \,\mathrm{d}a_{I_k} - \frac{2\pi(k+1)}{k^2} |\chi(\partial M)|,$$

where we applied the Gauss-Bonnet theorem to say that the area of ∂M_k with respect to I_k is equal to $-2\pi k^{-1} |\chi(\partial M)|$.

The only ingredient left to prove is relation (4.3). For this computation, we will use the *Bochner's formula* (see e. g. [Lee18, page 223]):

$$\frac{1}{2} \Delta_g \|\mathrm{d}f\|_g^2 = \left\| {}^g\nabla^2 f \right\|_g^2 + g(\mathrm{grad}_g f, \mathrm{grad}_g \Delta_g f) + \mathrm{Ric}_g(\mathrm{grad}_g f, \mathrm{grad}_g f), \tag{4.4}$$

and the following expressions:

$$\mathrm{div}_g(fX) = g(\mathrm{grad}_g f, X) + f \,\mathrm{div}_g X, \tag{4.5}$$

$$\frac{1}{2} (\mathcal{L}_X g, T)_g = -(\mathrm{div}_g T)(X) + \mathrm{div}_g Y, \tag{4.6}$$

where X is a tangent vector field, f is a smooth function, T is a symmetric 2-tensor, and $Y = T(X, \cdot)^\sharp$ is the vector field defined by requiring that $g(Y, Z) =$

$T(X, Z)$ for all vector fields Z. From now on, we will omit everywhere the dependence of the connections, norms, and the Laplace-Beltrami operator on the Riemannian metric g, and everything has to be interpreted as associated to $g = I_k$. Observe also that the Levi-Civita connection of I_k and h_k are equal, since these metrics differ by the multiplication by a constant and, in particular, the h_k- and I_k-Hessians coincide. Then we have:

$$\begin{aligned}\left\|I\!I_k - \nabla^2 u_k + u_k h_k\right\|^2 &= \left\|I\!I_k - \nabla^2 u_k - k\, u_k I_k\right\|^2 \\ &= \left\|I\!I_k\right\|^2 + \left\|\nabla^2 u_k\right\|^2 + k^2\, u_k^2 \left\|I_k\right\|^2 - 2(I\!I_k, \nabla^2 u_k) + \\ &\qquad - 2k\, u_k (I\!I_k, I_k) + 2k\, u_k (\nabla^2 u_k, I_k).\end{aligned} \tag{4.7}$$

First we focus our attention on the terms $\left\|\nabla^2 u_k\right\|^2$ and $(I\!I_k, \nabla^2 u_k)$. In order to simplify the notation, we say that two functions a and b on ∂M_k are equal "modulo divergence", and we write $a \equiv_{\text{div}} b$, if their difference coincides with the divergence of some smooth vector field. Then, we have:

$$\begin{aligned}\left\|\nabla^2 u_k\right\|^2 &= \frac{1}{2}\Delta\|\mathrm{d}u_k\|^2 - \langle \operatorname{grad} u_k, \operatorname{grad} \Delta u_k\rangle - k\|\mathrm{d}u_k\|^2 && \text{(relation (4.4))} \\ &\equiv_{\text{div}} -\langle \operatorname{grad} u_k, \operatorname{grad} \Delta u_k\rangle - k\|\mathrm{d}u_k\|^2 && (\Delta_g f = \operatorname{div}_g \operatorname{grad}_g f) \\ &= -\operatorname{div}(\Delta u_k \operatorname{grad} u_k) + (\Delta u_k)^2 - k\|\mathrm{d}u_k\|^2 && \text{(relation (4.5))} \\ &\equiv_{\text{div}} (\Delta u_k)^2 - k \operatorname{div}(u_k \operatorname{grad} u_k) + k u_k \Delta u_k && \text{(relation (4.5))} \\ &\equiv_{\text{div}} \Delta u_k(\Delta u_k + k u_k),\end{aligned}$$

$$\begin{aligned}(I\!I_k, \nabla^2 u_k) &= \frac{1}{2}(I\!I_k, \mathcal{L}_{\operatorname{grad} u_k} I_k) && (\mathcal{L}_{\operatorname{grad}_g f}\, g = 2\, {}^g\nabla^2 f) \\ &\equiv_{\text{div}} -(\operatorname{div} I\!I_k)(\operatorname{grad} u_k) && \text{(relation (4.6))} \\ &= -\langle \operatorname{grad} H_k, \operatorname{grad} u_k\rangle && (\operatorname{div} I\!I_k = \mathrm{d}H_k) \\ &= -\operatorname{div}(H_k \operatorname{grad} u_k) + H_k \Delta u_k && \text{(relation (4.5))} \\ &\equiv_{\text{div}} H_k \Delta u_k.\end{aligned}$$

The other terms in equation (4.7) are simpler to handle. In particular we have:

$$\begin{aligned}\left\|I\!I_k\right\|^2 &= H_k^2 - 2(k+1), \\ \left\|I_k\right\|^2 &= 2, \\ (I\!I_k, I_k) &= H_k, \\ (\nabla^2 u_k, I_k) &= \Delta u_k.\end{aligned}$$

Replacing all the relations we found in equation (4.7), we obtain:

$$\begin{aligned}\left\|I\!I_k - \nabla^2 u_k + u_k h_k\right\|^2 &\equiv_{\text{div}} H_k^2 - 2(k+1) + \Delta u_k(\Delta u_k + k u_k) + 2k^2 u_k^2 + \\ &\qquad - 2H_k \Delta u_k - 2k u_k H_k + 2k u_k \Delta u_k \\ &= H_k^2 - 2(k+1) + 2k^2 u_k^2 - 2k u_k H_k + \\ &\qquad + \Delta u_k(\Delta u_k + 3k u_k - 2H_k)\end{aligned}$$

Finally, by replacing the expression of $\Delta u_k = \Delta_{I_k} u_k$ from equation (4.1) in the equality above, we find that:

$$\left\| I\!I_k - \nabla^2 u_k + u_k h_k \right\|^2 \equiv_{\mathrm{div}} k u_k H_k - 2(k+1),$$

which is equivalent to relation (4.3). □

Since the Weil-Petersson metric of the Teichmüller space is non-complete, a control from above of the quantity $\|\mathrm{d}V_k^*\|_{WP}$ would not suffice to guarantee the existence of the flow for every time. For this purpose, we rather study the L^∞-norm of the Beltrami differentials equivalent to $\mathrm{grad}_{WP} V_k^*$, which gives a control with respect to the Teichmüller metric (that is complete). At this point, the estimates determined in Lemma 4.1.3 will play an essential role.

Proposition 4.2.5. *There exists a constant $D_k > 0$ depending only on the intrinsic curvature $k \in (-1, 0)$ such that*

$$\|\mathrm{grad}_{WP} V_k^*\|_{\mathcal{T}} \leq D_k,$$

where $\|\cdot\|_{\mathcal{T}}$ denotes the Teichmüller norm on $T\mathcal{T}(\partial M)$.

Proof. Let m_k be a point of the Teichmüller space $\mathcal{T}^{\mathfrak{h}}(\partial M)$, interpreted as an isotopy class of hyperbolic metrics on ∂M_k. The Teichmüller norm of a tangent vector $\dot{m}_k \in T_{m_k}\mathcal{T}(\partial M)$ is the infimum of the L^∞-norms of the Beltrami differentials representing $\dot{m}_k$. In Proposition 4.2.2, we showed that the vector field $\mathrm{grad}_{WP} V_k^*$ at a point $m_k \in \mathcal{T}^{\mathfrak{h}}(\partial M)$ is represented by the harmonic Beltrami differential associated to ϕ_k. Let now h_k denote a representative of the isotopy class of hyperbolic metrics m_k, and let ν_{ϕ_k} be the harmonic Beltrami differential on $(\partial M_k, h_k)$ associated to the holomorphic quadratic differential ϕ_k from Proposition 4.2.2. Therefore, by Lemma 3.1.2, we have that

$$\|\mathrm{grad}_{WP} V_k^*\|_{\mathcal{T}} \leq \|\nu_{\phi_k}\|_{B,\infty} = \frac{1}{\sqrt{2}} \|\mathrm{Re}\,\phi_k\|_{FT,\infty} = \frac{1}{\sqrt{2}} \sup_{\partial M_k} \|\mathrm{Re}\,\phi_k\|_{h_k}.$$

Therefore it is enough to show that the norm $\left\| I\!I_k - {}^k\nabla^2 u_k + u_k h_k \right\|_{h_k}$ is uniformly bounded by a constant depending only on k. The norm of $I\!I_k$ is equal to $-k^{-1}\sqrt{H_k^2 - 2(k+1)}$, and $\|u_k h_k\|_{h_k} = \sqrt{2}\,|u_k|$. Therefore we have

$$\left\| I\!I_k - {}^k\nabla^2 u_k + u_k h_k \right\|_{h_k} \leq -k^{-1}\sqrt{\|H_k\|_{\mathscr{C}^0}^2 - 2(k+1)} + \left\| {}^k\nabla^2 u_k \right\|_{h_k} + \sqrt{2}\,\|u_k\|_{\mathscr{C}^0}.$$

Our assertion is now an immediate consequence of Proposition 4.1.1 and of Lemma 4.1.3. □

Corollary 4.2.6. *The flow Θ_t of the vector field $-\mathrm{grad}_{WP} V_k^*$ over $\mathcal{T}(\partial M)$ is defined for all times $t \in \mathbb{R}$.*

Proof. The assertion follows from the fact that the Teichmüller distance is complete, and on the bound shown in Proposition 4.2.5. □

4.3 The proof of Theorem C

The last ingredient that we will need for the study of the infimum of the dual volume is the existence of some lower bound for the dual volume on the space of quasi-isometric deformations $\mathcal{QD}(M)$ of M. To do so, we will make use of the properties of the dual volume proved in Chapter 2, and of the upper bound described by Bridgeman, Brock, and Bromberg [BBB19] for the length of the bending measure of the boundary of the convex core of a convex co-compact manifold with incompressible boundary, which we recalled in Theorem 3.3.5.

Lemma 4.3.1. *For every $k \in (-1, 0)$ and for every convex co-compact hyperbolic 3-manifold M with incompressible boundary we have:*

$$V_k^*(M) \geq F(k, \chi(\partial M)),$$

where F is an explicit function of the curvature $k \in (-1, 0)$ and the Euler characteristic of ∂M.

Proof. Since the k-surfaces foliate the complementary of the convex core CM, a simple application of the geometric maximum principle (see for instance [Lab00, Lemme 2.5.1]) shows that the k-surface ∂M_k is contained in $N_{\varepsilon_k} CM$, the ε_k-neighborhood of the convex core CM, for $\varepsilon_k = \operatorname{arctanh} \sqrt{k+1}$. Moreover, by Proposition 2.2.6, the dual volume is a decreasing function with respect to the inclusion, therefore the quantity $V_k^*(M)$ is bounded from below by the dual volume of the ε_k-neighborhood of the convex core. We showed in Proposition 2.2.4 that, for every $\varepsilon > 0$, we have

$$\operatorname{Vol}^*(N_\varepsilon CM) = \operatorname{Vol}(CM) - \frac{\ell_m(\mu)}{4}(\cosh 2\varepsilon + 1) - \frac{\pi}{2}|\chi(\partial CM)|(\sinh 2\varepsilon - 2\varepsilon),$$

where $\ell_m(\mu)$ denotes the length of the bending measured lamination on the boundary of the convex core of M. By Theorem 3.3.5, the term $\ell_m(\mu)$ is less or equal to $6\pi|\chi(\partial M)|$. Combining these observations, we deduce that

$$\begin{aligned} V_k^*(M) &\geq \operatorname{Vol}^*(N_{\varepsilon_k} CM) \\ &= \operatorname{Vol}(CM) - \frac{\ell_m(\mu)}{4}(\cosh 2\varepsilon_k + 1) - \frac{\pi}{2}|\chi(\partial CM)|(\sinh 2\varepsilon_k - 2\varepsilon_k) \\ &\geq -\frac{\ell_m(\mu)}{4}(\cosh 2\varepsilon_k + 1) - \frac{\pi}{2}|\chi(\partial CM)|(\sinh 2\varepsilon_k - 2\varepsilon_k) \\ &\geq -\frac{\pi}{2}|\chi(\partial M)|(3\cosh \varepsilon_k + 3 + \sinh 2\varepsilon_k - 2\varepsilon_k), \end{aligned}$$

which proves the desired inequality. □

We are finally ready to present the proof of Theorem C:

Proof of Theorem C. Let M be a convex co-compact hyperbolic 3-manifold with incompressible boundary. We denote by $M_t := \Theta_t(M)$ the hyperbolic 3-manifold obtained by following the flow of the vector field $-\operatorname{grad}_{WP} V_k^*$, which is defined for every $t \in \mathbb{R}$ in light of Corollary 4.2.6. In order to simplify the notation, we will continue to denote by V_k^* the k-dual volume as a function over the space of quasi-isometric deformations of M. This abuse is justified by the fact that, for every $k \in (-1, 0)$, a convex co-compact manifold is uniquely determined by the hyperbolic structures on its k-surfaces, as mentioned at the beginning of the chapter. We have

$$V_k^*(M) - V_k^*(M_t) = \int_0^t \|\mathrm{d}V_k^*\|_{M_s}^2 \, \mathrm{d}s \, .$$

By Lemma 4.3.1, the left hand side of the relation is bounded from above with respect to t. In particular, the integral on the right side has to converge as t goes to $+\infty$. Therefore we can find an unbounded increasing sequence $(t_n)_n$ for which the Weil-Petersson norm $\|\mathrm{d}V_k^*\|^2$ evaluated at M_{t_n} goes to 0 as n goes to ∞. Then, by Lemma 4.2.4, we have

$$\limsup_{n\to\infty} \int_{\partial M_{t_n,k}} H_k \, \mathrm{d}a_{I_k} \leq -4\pi k^{-1}\sqrt{k+1}|\chi(\partial M)|,$$

where $M_{t_n,k}$ stands for $(M_{t_n})_k$, the region of M_{t_n} enclosed by its k-surfaces. Therefore we deduce:

$$\begin{aligned} V_k^*(M) &\geq \lim_{n\to\infty} V_k^*(M_{t_n}) = \lim_{n\to\infty} \left(V_k(M_{t_n}) - \frac{1}{2}\int_{\partial M_{t_n.k}} H_k \, \mathrm{d}a_{I_k} \right) \\ &\geq \inf_{M'\in\mathcal{QD}(M)} V_k(M') - \frac{1}{2}\liminf_{n\to\infty}\int_{\partial M_{t_n,k}} H_k \, \mathrm{d}a_{I_k} \\ &\geq \inf_{M'\in\mathcal{QD}(M)} V_k(M') + 2\pi k^{-1}\sqrt{k+1}|\chi(\partial M)|, \end{aligned}$$

where $V_k(M')$ denotes the Riemannian volume of the region M'_k of M' enclosed by its k-surface. Observe that the term $2\pi k^{-1}\sqrt{k+1}|\chi\partial M|$ is equal to $-\frac{1}{2}\int_{\partial M'_k} H_k \, \mathrm{d}a_{I_k}$ when the boundary of the convex core of M' is totally geodesic.

Finally, by taking the limit as k goes to $(-1)^+$, we obtain that $V_C^*(M) \geq \inf_{M'} V_C(M')$ for every convex co-compact structure M. This proves that

$$\inf_{\mathcal{QD}(M)} V_C^* \geq \inf_{\mathcal{QD}(M)} V_C.$$

On the other hand, the dual volume $V_C^*(M) := V_C(M) - \frac{1}{2}\ell_m(\mu)$ is always smaller or equal to $V_C(M)$, so the other inequality between the infima is clearly satisfied.

If $V_C^*(M) = V_C(M)$, then the length of the bending measured lamination μ of the convex core of M has to vanish, therefore $\mu = 0$ or, in other words, ∂CM is totally geodesic. □

Chapter 5

Constant Gaussian curvature surfaces in hyperbolic 3-manifolds

Outline of the chapter

The aim of this chapter is to investigate the properties of constant Gaussian curvature surfaces inside hyperbolic ends, and to show how their geometry interpolates the structure of the locally pleated boundary on one side, and of the conformal boundary at infinity on the other. We refer to Sections 1.5 and 1.6 for the necessary background concerning these notions.

In Section 5.1 we recall the definitions of two families of parametrizations of the space of hyperbolic ends $(\Phi_k)_k$, $(\Psi_k)_k$, taking values into the cotangent space to Teichmüller space $\mathcal{T}(\Sigma)$, and constructed in terms of the geometric data of the k-surface foliations (see Section 1.6.3). These maps have been firstly introduced by Labourie [Lab92b], and further investigated by Bonsante, Mondello, and Schlenker [BMS13], [BMS15]. Our first goal will be to determine the relations between the asymptotic of the families $(\Phi_k)_k$, $(\Psi_k)_k$ and the classical *Schwarzian* and *Thurston parametrizations* Sch and Th, respectively (see Sections 1.6.1 and 1.6.2 for their definitions), and it will be achieved in Corollary 5.1.3 and Proposition 5.1.6. The proof of these facts will be based on the works of Quinn [Qui20] and Belraouti [Bel17], which describe the limits of the geometric quantities associated to the k-surfaces as they approach the conformal boundary at infinity, and the locally concave pleated boundary, respectively.

Section 5.2 focuses on the variation formulae of two notions of volumes for quasi-Fuchsian manifolds M. The first, that will be denoted by $V_k^*(M)$, coincides with the *dual volume* of the region M_k of M bounded by its k-surfaces, and similarly $W_k(M)$ is equal to the W*-volume* of M_k. As shown in Lemma 5.2.2 and Theorem 5.2.8, these *Schläfli formulae* are closely related to the variation formulae of the renormalized volume V_R and of the dual volume of the convex core V_C^*, respectively. As a corollary of these relations and the connection between the parametrizations Φ_k and Sch, we will obtain a new and simple description of the renormalized volume function of M in terms of the foliation by k-surfaces of its hyperbolic ends:

Theorem D. *For every quasi-Fuchsian manifold M, the renormalized volume $V_R(M)$ can be expressed as follows:*

$$V_R(M) = \lim_{k \to 0} \left(\mathrm{Vol}(M_k) - \frac{1}{4} \int_{\partial M_k} H_k \, \mathrm{d}a_k - \pi |\chi(\partial M)| \operatorname{arctanh} \sqrt{k+1} \right).$$

As highlighted by the work of Krasnov and Schlenker [KS09], the Schläfli-type variation formulae of the dual volume V_C^* and the renormalized volume V_R have strong implications with respect to the symplectic geometry of the spaces $T^*\mathcal{T}^{\mathfrak{c}}(\Sigma)$ and $T^*\mathcal{T}^{\mathfrak{h}}(\Sigma)$, endowed with the symplectic structures $\omega^{\mathfrak{c}}$ and $\omega^{\mathfrak{h}}$ of cotangent manifolds, respectively. Here we develop the same ideas applied to the volumes V_k^* and W_k^*, and the Labourie's parametrizations Φ_k and Ψ_k through their Schläfli formulae. In particular, in Section 5.4 we will prove:

Theorem G. *For every $k, k' \in (-1, 0)$, the function*

$$\Phi_k \circ \Psi_{k'}^{-1} \colon (T^*\mathcal{T}^{\mathfrak{h}}(\Sigma), \omega^{\mathfrak{h}}) \to (T^*\mathcal{T}^{\mathfrak{c}}(\Sigma), 2\omega^{\mathfrak{c}})$$

is a symplectomorphism.

Another surprisingly simple consequence of the variation formulae of the volumes W_k and V_k^* is the following generalization of (Krasnov and Schlenker's reformulation from [KS09] of) McMullen's Kleinian reciprocity Theorem:

Theorem E. *Let M be a convex co-compact hyperbolic 3-manifold, and denote by $\mathcal{QD}(M)$ the space of quasi-isometric deformations of M. We set*

$$\phi_k : \mathcal{QD}(M) \longrightarrow T^*\mathcal{T}^{\mathfrak{c}}(\partial M), \qquad \psi_k : \mathcal{QD}(M) \longrightarrow T^*\mathcal{T}^{\mathfrak{h}}(\partial M)$$

to be the maps that associate, to a convex co-compact hyperbolic structure M', the points of $T^*\mathcal{T}(\partial M)$ given by the vectors $(\Phi_k(E_i))_i$ and $(\Psi_k(E_i))_i$, respectively, where E_i varies among the set of hyperbolic ends of M'. Then, for every $k \in (-1, 0)$, the images $\phi_k(\mathcal{QD}(M))$ and $\psi_k(\mathcal{QD}(M))$ are Lagrangian submanifolds of $(T^*\mathcal{T}^{\mathfrak{c}}(\partial M), \omega^{\mathfrak{c}})$ and $(T^*\mathcal{T}^{\mathfrak{h}}(\partial M), \omega^{\mathfrak{h}})$, respectively.

In Section 5.4.1 we will discuss the relations between the original McMullen's formulation of the quasi-Fuchsian reciprocity (in terms of adjoint maps) and the statement we have presented here.

In Section 5.5, as last application of the tools developed here, we prove that the k-surface foliations of hyperbolic ends correspond to integral curves of k-dependent Hamiltonian vector fields on $T^*\mathcal{T}(\Sigma)$. This phenomenon can be interpreted as the analogous of what observed by Moncrief [Mon89] for constant mean curvature foliations in 3-dimensional Lorenzian space-times. If $\dot{\Phi}_k$ and $\dot{\Psi}_k$ denote the vector fields $\frac{\mathrm{d}}{\mathrm{d}k} \Phi_k$ and $\frac{\mathrm{d}}{\mathrm{d}k} \Psi_k$, respectively, then we will prove:

Theorem F. *The k-dependent vector fields $\dot{\Phi}_k \circ \Phi_k^{-1}$ and $\dot{\Psi}_k \circ \Psi_k^{-1}$ are Hamiltonian with respect to the cotangent symplectic structure of $T^*\mathcal{T}(\Sigma)$.*

We will observe that the role of the area functional in [Mon89] as Hamiltonian function here is replaced by the integral of the mean curvature, up to a multiplicative constant depending only of the curvature k.

5.1 Foliations by k-surfaces

This Section is mainly devoted to the description of two families of parametrizations of the space of hyperbolic ends $\mathcal{E}(\Sigma)$, denoted by $(\Phi_k)_k$ and $(\Psi_k)_k$, firstly introduced by Labourie [Lab92b], and further investigated by Bonsante, Mondello and Schlenker in [BMS13] and [BMS15]. Our main goal will be to establish a connection between the asymptotic of these maps and the classical Schwarzian and Thurston parametrizations (described in Section 1.6), applying the recent works of Quinn [Qui20] and Belraouti [Bel17], respectively.

In order to define the maps $(\Phi_k)_k$ and $(\Psi_k)_k$, we need to introduce some notation. By Theorem 1.6.4, every hyperbolic end E (see Definition 1.6.1) admits a foliation by k-surfaces $(\Sigma_k)_k$, with intrinsic curvature $k \in (-1, 0)$. Let I_k, $I\!I_k$ and $I\!I\!I_k$ denote the first, second and third fundamental forms of the k-surface Σ_k of E. We set h_k and h'_k to be the hyperbolic metrics $-k\ I_k$ and $-\frac{k}{k+1}\ I\!I\!I_k$, respectively, and c_k to be the conformal class of $I\!I_k$. As observed in Section 1.5, the identity maps

$$id : (\Sigma_k, c_k) \to (\Sigma_k, h_k), \qquad id : (\Sigma_k, c_k) \to (\Sigma_k, h'_k)$$

are harmonic with opposite Hopf differentials (see Definition 1.2.12). We will denote by q_k the holomorphic quadratic differential

$$-\frac{2\sqrt{k+1}}{k} \operatorname{Hopf}((\Sigma_k, c_k) \to (\Sigma_k, h_k)) = \frac{2\sqrt{k+1}}{k} \operatorname{Hopf}((\Sigma_k, c_k) \to (\Sigma_k, h'_k)).$$

The choice of the multiplicative constant in the definition of q_k may look arbitrary at this point of the exposition, but it will be crucial in the following (see for instance Corollary 5.1.3 and Remark 5.5.3). The holomorphic quadratic differential q_k satisfies

$$2 \operatorname{Re} q_k = 2\sqrt{k+1} \left(I_k - \frac{H_k}{2(k+1)} I\!I_k \right) = -\frac{2}{\sqrt{k+1}} \left(I\!I\!I_k - \frac{H_k}{2} I\!I_k \right). \tag{5.1}$$

For future references, we also observe that the area forms with respect to I_k and $I\!I_k$ differ by a multiplicative constant, as follows:

$$\mathrm{d}a_{I_k} = \frac{1}{\sqrt{\det B_k}} \mathrm{d}a_{I\!I_k} = \frac{1}{\sqrt{k+1}} \mathrm{d}a_{I\!I_k}\,. \tag{5.2}$$

5.1.1 The parametrizations Φ_k

The first class of parametrizations described by Labourie [Lab92b] is given by the following maps: for every $k \in (-1, 0)$ we define the function

$$\begin{array}{rccc} \Phi_k : & \mathcal{E}(\Sigma) & \longrightarrow & T^*\mathcal{T}(\Sigma)^{\mathfrak{c}} \\ & [E] & \longmapsto & (c_k, q_k), \end{array}$$

which associates, to every hyperbolic end E, the point of the cotangent space to Teichmüller space (c_k, q_k) determined by the unique k-surface Σ_k contained in E, as above. We have:

Theorem 5.1.1 ([Lab92b, Théorème 3.1]). *The function Φ_k is a diffeomorphism for every $k \in (-1, 0)$.*

In the following we will see how the maps Φ_k relate to the Schwarzian parametrization Sch, defined in Section 1.6.1. Using the hyperbolic Gauss map (see e. g. [Lab91]), we can think about the families $(I_k)_k$, $(I\!I_k)_k$ and $(I\!I\!I_k)_k$ as paths in the space of symmetric 2-tensors over the surface $\partial_\infty E$, which does not depend on k. In this way we can study the asymptotic of these geometric quantities as k goes to 0.

In a recent work [Qui20], Quinn introduced the notion of *asymptotically Poincaré families of surfaces* inside a hyperbolic end E, and he determined a connection between their geometric properties and the complex projective structure at infinity of E. The foliation by k-surfaces is an example of such families and the asymptotic of their fundamental forms is understood. In order to do not introduce more notions, we specialize the results of [Qui20] in the form that we will need:

Theorem 5.1.2 ([Qui20])**.** *For every hyperbolic end $E \in \mathcal{E}(\Sigma)$ we have*

$$\lim_{k \to 0^-} h_k = \lim_{k \to 0^-} (-k) I\!I_k = h_\infty,$$

where h_∞ is the hyperbolic metric in the conformal class at infinity c_∞ of E. Moreover

$$\left.\frac{\mathrm{d}h_k}{\mathrm{d}k}\right|_{k=0^-} = -\operatorname{Re} q_\infty, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}k}(-k) I\!I_k\right|_{k=0^-} = \frac{1}{2} h_\infty,$$

where q_∞ is the Schwarzian at infinity of E (see Section 1.6.1).

Corollary 5.1.3. *The maps $(\Phi_k)_k$ converge to Sch $\mathscr{C}^1$-uniformly over compact subsets, as k goes to 0.*

Proof. First we prove the pointwise convergence. Let E be a hyperbolic end, and consider the path $(\Phi_k(E))_k$ in $T^* \mathcal{T}^{\mathfrak{c}}(\Sigma)$. We define $g_k := (-k) I\!I_k$. Then, the relations of Theorem 5.1.2 can be rewritten as follows:

$$\lim_{k \to 0^-} g_k = h_\infty, \qquad \left.\frac{\mathrm{d}h_k}{\mathrm{d}k}\right|_{k=0^-} = -\operatorname{Re} q_\infty, \qquad \left.\frac{\mathrm{d}g_k}{\mathrm{d}k}\right|_{k=0^-} = \frac{1}{2} h_\infty.$$

The first relation proves that the conformal classes c_k converge to the conformal structure of $\partial_\infty E$. We need to show that the holomorphic quadratic differentials q_k converge to the Schwarzian differential q_∞. This is a simple application of the relations above, we briefly summarize the steps in the following. First we observe that

$$\begin{aligned}
\lim_{k \to 0^-} 2 \operatorname{Re} q_k &= \lim_{k \to 0^-} -\frac{2\sqrt{k+1}}{k} \left(h_k - \frac{\operatorname{tr}_{g_k}(h_k)}{2} g_k \right) \\
&= \lim_{k \to 0^-} -2\sqrt{k+1}\, \frac{h_k - h_\infty}{k} + \sqrt{k+1}\, \frac{\operatorname{tr}_{g_k}(h_k)\, g_k - 2h_\infty}{k} \\
&= -2 \left.\frac{\mathrm{d}h_k}{\mathrm{d}k}\right|_{k=0^-} + \left.\frac{\mathrm{d}}{\mathrm{d}k} \operatorname{tr}_{g_k}(h_k)\, g_k\right|_{k=0^-},
\end{aligned}$$

where, in the last step, we are using that $\lim_{k \to 0^-} \operatorname{tr}_{g_k}(h_k)\, g_k = 2h_\infty$. Applying the relations in Theorem 5.1.2, we see that $\frac{\mathrm{d}}{\mathrm{d}k} \operatorname{tr}_{g_k}(h_k)|_{k=0} = -1$. Combining this with the relation above we obtain

$$\lim_{k \to 0^-} 2 \operatorname{Re} q_k = -2(-\operatorname{Re} q_\infty) - h_\infty + 2 \left.\frac{\mathrm{d}g_k}{\mathrm{d}k}\right|_{k=0^-} = 2 \operatorname{Re} q_\infty,$$

which was our claim.

In [Qui20], the author gave an alternative proof of the existence of the k-surface foliation, for k close to 0. The strategy of his proof is to apply the Banach implicit function theorem to a function

$$F : (-1, 0] \times \mathrm{Conf}^s(\Sigma, c) \longrightarrow \mathrm{Conf}^s(\Sigma, c),$$

which satisfies $F(k, \tau) = 0$ if and only if τ is (a proper multiple of) the metric at infinity associated to the k-surface. Here $\mathrm{Conf}^s(\Sigma, c)$ denotes the space of Sobolev metrics in the conformal class c (see [Qui20, Theorem 5.1] for details). The map F depends smoothly on k and also on the complex projective structure at infinity (c, q). In particular, the implicit function theorem guarantees the smooth regularity of the metric at infinity τ_k, associated to the k-surface Σ_k, with respect to $k \in (-1, 0]$ and $(c, q) \in T^*\mathcal{T}^{\mathfrak{c}}(\Sigma)$. Since the tensors I_k and $I\!I_k$ are smooth functions of τ_k and (c, q), the function $\Phi(k; c, q) := \Phi_k \circ \mathrm{Sch}^{-1}(c, q)$ is smooth in all its arguments. This properties imply the higher order convergence. □

5.1.2 The parametrizations Ψ_k

The diffeomorphism $\mathcal{H}$ from Theorem 1.2.19 allows us to convert the family of parametrizations $(\Phi_k)_k$, which take values in $T^*\mathcal{T}^{\mathfrak{c}}(\Sigma)$, into a family of parametrizations $(\hat{\Psi}_k)_k$ with values in $\mathcal{T}^{\mathfrak{h}}(\Sigma)^2$. Indeed, the functions

$$\begin{array}{rccc} \hat{\Psi}_k := \mathcal{H} \circ \Phi_k : & \mathcal{E}(\Sigma) & \longrightarrow & \mathcal{T}^{\mathfrak{h}}(\Sigma)^2 \\ & [E] & \longmapsto & (h_k, h'_k), \end{array}$$

associate to each hyperbolic end E, the pair of hyperbolic metrics $h_k = (-k) I_k$ and $h'_k = -\frac{k}{k+1} I\!I\!I_k$ coming from the first and third fundamental forms of the k-surface Σ_k of E. This expression for the map $\hat{\Psi}_k$ follows from the definition of the map $\mathcal{H}$ (see Theorem 1.2.19) and the link between k-surfaces and minimal Lagrangian maps, observed at the end of Section 1.5.

The maps $\hat{\Psi}_k$ have been the main object of study of Bonsante, Mondello, and Schlenker in [BMS13], [BMS15]. In these works, the authors introduced the notions of *landslide flow* and of *smooth grafting* SGr'_s, and studied their convergence to the classical earthquake flow and grafting map Gr. Our functions $\hat{\Psi}_k$ are actually the inverses of the maps SGr'_s, in the notation of [BMS13] (the relation between our parameter k and the s used in [BMS13] is $k = -\frac{1}{\cosh^2(s/2)}$).

As the Schwarzian parametrization can be recovered from the limit of the maps Φ_k when $k \to 0$, the Thurston parametrization, defined in Section 1.6.2, can be recovered from the limit of the maps $\hat{\Psi}_k$ when $k \to -1$. Indeed, we have:

Theorem 5.1.4. *The maps $\hat{\Psi}_k$ converge to* Th, *as k goes to -1, in the following sense: if E is a hyperbolic end, then the length spectrum of $I\!I\!I_k$ converges to $\iota(\cdot, \mu)$, where $\iota(\cdot, \cdot)$ denotes the geometric intersection of currents. Moreover, the first fundamental forms I_k converge to the hyperbolic metric of the locally concave pleated boundary ∂E.*

Proof. Let E be a fixed hyperbolic end. The convergence of the first fundamental forms I_k is a direct consequence of Theorem 1.6.4.

By the correspondence between hyperbolic ends and maximal global hyperbolic spatially compact (MGHC) de Sitter spacetimes (see e. g. [Mes07] and the duality described in Section 1.4), the foliation by k-surfaces of E determines a constant curvature surfaces foliation of the MGHC de Sitter spacetime E^* dual of E. Through this correspondence, the third fundamental form $I\!I\!I_k$ of the leaf Σ_k in E can be interpreted as the first fundamental form of its dual surface Σ_k^* in E^*, which has constant intrinsic curvature equal to $\frac{k}{k+1}$. Moreover, the initial singularity of E^* is dual of the bending measured lamination μ of the pleated boundary ∂E, as shown by Benedetti and Bonsante [BB09, Chapter 3].

In [Bel17], the author studied the intrinsic metrics of families of surfaces which foliate a neighborhood of the initial singularity in E^*. In particular, Belraouti [Bel17, Theorem 2.10] proved that, for a wide class of such foliations, the intrinsic metrics of the surfaces converge, with respect to the Gromov equivariant topology, to the real tree dual of the measured lamination μ, as the surfaces approach the initial singularity of E^*. By applying this result to the constant curvature foliation of E^*, and interpreting $I\!I\!I_k$ as the first fundamental forms of its leaves, we deduce the convergence of the length spectrum of $I\!I\!I_k$ to $\iota(\cdot, \mu)$. □

5.1.3 Hyperbolic length functions

Following [BMS15], we define

$$\begin{array}{rccc} j: & \mathcal{T}^{\mathfrak{h}}(\Sigma)^2 & \longrightarrow & \mathbb{R} \\ & (h, h') & \longmapsto & \int_\Sigma \operatorname{tr} b \, \mathrm{d}a_h \, , \end{array}$$

which associates, to a normalized pair of hyperbolic metrics h, h' with Labourie operator $b \colon T\Sigma \to T\Sigma$ (see Theorem 1.2.18), the integral of the trace of b with respect to the area measure of h (here we are identifying, with abuse, the hyperbolic metrics h and h' with their isotopy classes). The quantity $j(h, h')$ satisfies

$$j(h, h') = 2\ E(id \colon (\Sigma, c) \to (\Sigma, h)) = 2\ E(id \colon (\Sigma, c) \to (\Sigma, h')),$$

where c is the conformal class of $h(b\cdot, \cdot)$, and $E(\cdot)$ denotes the energy functional (see [BMS15, Section 1.2]). This shows in particular that j is symmetric, i. e. $j(h, h') = j(h', h)$.

For any hyperbolic metric h', we define $L_{h'} \colon \mathcal{T}^{\mathfrak{h}}(\Sigma) \to \mathbb{R}$ to be $L_{h'}(h) := j(h, h')$. The functions $L_{h'}$, which are real analytic by [BMS15, Proposition 1.2], can be interpreted as generalizations of length functions, in light of the following fact:

Proposition 5.1.5. *Let $(h_n)_n$, $(h'_n)_n$ be two sequences of hyperbolic metrics. Suppose that $(h_n)_n$ converges to $h \in \mathcal{T}^{\mathfrak{h}}(\Sigma)$, and that there exists a sequence of positive numbers $(\vartheta_n)_n$ such that the length spectrum of $\varepsilon_n^2\ h'_n$ converges to $\iota(\cdot, \mu)$, for some measured lamination $\mu \in \mathcal{ML}(\Sigma)$. Then*

$$\lim_{n \to \infty} \varepsilon_n\ L_{h'_n}(h_n) = L_\mu(h).$$

Proof. Using the interpretation via k-surfaces, we can easily prove this statement, which is purely 2-dimensional, using 3-dimensional hyperbolic geometry.

First we observe that, since the injectivity radius of h'_n is going to 0, the sequence ε_n must converge to 0. In particular, the limit of $k_n := -(\cosh^2 \varepsilon_n)^{-1}$

is equal to -1, as n goes to infinity. In [BMS13, Proposition 6.2], the authors proved that, under our hypotheses, the sequence of hyperbolic ends $(E_n)_n$ given by $E_n := \hat{\Psi}_{k_n}^{-1}(h_n, h'_n)$ (which, in the notation of [BMS13], coincides with $\mathrm{SGr}'_{2\varepsilon_n}(h_n, h'_n)$), converges to $E := \mathrm{Gr}_\mu(h)$. Recalling the definitions of h_n, h'_n, we see that

$$L_{h'_n}(h_n) = -\frac{k_n}{\sqrt{k_n+1}} \int_{\Sigma_{k_n}} H_{k_n} \, \mathrm{d}a_{I_{k_n}} ,$$

where Σ_{k_n} is the k_n-surface inside E_n, and I_{k_n} and H_{k_n} are its first fundamental form and mean curvature, respectively.

Since E_n goes to $E = \mathrm{Gr}_\mu(h)$, and k_n goes to -1, the intrinsic metrics of the surfaces Σ_{k_n} converge to the hyperbolic metric h of the pleated boundary ∂E, and the bending measures of ∂E_n converge to μ. In particular, the integral of the mean curvature of Σ_{k_n} converges to $L_\mu(h)$, the length of the bending measure of ∂E (see for instance Section 2.2). From the relation between k_n and ε_n, we see that

$$\lim_{n\to\infty} \varepsilon_n \left(-\frac{k_n}{\sqrt{k_n+1}} \right) = 1.$$

The combination of these two facts implies the statement. □

As done in [BMS15], instead of working directly with $\hat{\Psi}_k$, we we will introduce a family of maps $(\Psi_k)_k$ that have the advantage of taking values in the cotangent space $T^*\mathcal{T}^{\mathfrak{h}}(\Sigma)$. This will be more convenient for the rest of our paper, since we investigate the properties of these parametrizations with respect to the cotangent symplectic structure of $T^*\mathcal{T}^{\mathfrak{h}}(\Sigma)$ and $T^*\mathcal{T}^{\mathfrak{c}}(\Sigma)$. The functions Ψ_k are defined as follows:

$$\begin{array}{rccc} \Psi_k : & \mathcal{E}(\Sigma) & \longrightarrow & T^*\mathcal{T}^{\mathfrak{h}}(\Sigma) \\ & [E] & \longmapsto & (h_k, -\frac{\sqrt{k+1}}{k} \, \mathrm{d}(L_{h'_k})_{h_k}), \end{array}$$

where $\mathrm{d}(L_{h'_k})_{h_k}$ denotes the differential of the function $L_{h'_k}$, defined as before, at the point h_k. We also consider the function

$$\begin{array}{rccc} \mathrm{d}L : & \mathcal{T}^{\mathfrak{h}}(\Sigma) \times \mathcal{ML}(\Sigma) & \longrightarrow & T^*\mathcal{T}^{\mathfrak{h}}(\Sigma) \\ & (h, \mu) & \longmapsto & (h, \mathrm{d}(L_\mu)_h). \end{array}$$

Proposition 5.1.6. *The functions*

$$\mathrm{d}L \circ \mathrm{Th} : \mathcal{E}(\Sigma) \longrightarrow T^*\mathcal{T}^{\mathfrak{h}}(\Sigma) \quad \textit{and} \quad \Psi_k : \mathcal{E}(\Sigma) \to T^*\mathcal{T}^{\mathfrak{h}}(\Sigma)$$

are $\mathscr{C}^1$ diffeomorphisms, for every $k \in (-1, 0)$. Moreover, the functions Ψ_k converge pointwisely to $\mathrm{d}L \circ \mathrm{Th}$ as k goes to -1.

Proof. A proof of the $\mathscr{C}^1$-regularity of $\mathrm{d}L\circ\mathrm{Th}$ can be found in [KS09, Lemma 1.1]. The smoothness of the maps $\hat{\Psi}_k$ follows from the original work of Labourie [Lab92b]. Up to scalar multiplication in the fiber, the functions Ψ_k are equal to the composition of the $\hat{\Psi}_k$'s with the map

$$\begin{array}{ccc} \mathcal{T}^{\mathfrak{h}}(\Sigma) \times \mathcal{T}^{\mathfrak{h}}(\Sigma) & \longrightarrow & T^*\mathcal{T}^{\mathfrak{h}}(\Sigma) \\ (h, h') & \longrightarrow & (h, \mathrm{d}(L_{h'})_h). \end{array}$$

This function has been proved to be a diffeomorphism in [BMS15, Proposition 1.10]. This shows that Ψ_k is a diffeomorphism for every $k \in (-1, 0)$. The pointwise convergence of the functions Ψ_k follows from Theorem 5.1.4, Proposition 5.1.5 and the analyticity of the functions $L_{h'}$, established in [BMS15, Proposition 1.2]. □

5.2 Volumes and Schläfli formulae

In this section we define two families of volume functions for convex co-compact hyperbolic 3-manifolds: the W_k-volumes, related to the notion of *W-volume* introduced in [KS08], and the V_k^*-volumes, related the notion of *dual volume* introduced in [KS09]. For both these families we will prove a Schläfli-type variation formula, involving the extremal length, in the case of W_k, and the hyperbolic length functions $L_{h'}$ introduced in the previous section, in the case of V_k^* (see also Section 3.4). We also describe a simple way to compute the *renormalized volume* V_R of a convex co-compact hyperbolic manifold using the volumes W_k.

5.2.1 W_k-volumes

Let M be a convex co-compact hyperbolic 3-manifold and let $\mathcal{QD}(M)$ denote the space of quasi-isometric deformations of M (introduced in Section 2.1). We define

$$W_k(M') := W(M'_k) = \mathrm{Vol}(M'_k) - \frac{1}{4}\int_{\partial M'_k} H'_k \, \mathrm{d}a_{I'_k} ,$$

where M'_k denotes the compact region of $M' \in \mathcal{QD}(M)$ bounded by the union of the k-surfaces sitting inside the ends of M'. The quantities I_k, $I\!I_k$, $I\!I\!I_k$, c_k and q_k of ∂M_k are defined using the conventions of the previous section.

First we need a way to express the variation of the W-volume:

Proposition 5.2.1. *Let (N, g) be a compact hyperbolic 3-manifold with smooth boundary ∂N having positive definite second fundamental form, and let $(g_t)_t$ be a smooth 1-parameter family of hyperbolic metrics on N, with $g_0 = g$. Then we have:*

$$\left.\frac{\mathrm{d}W(N, g_t)}{\mathrm{d}t}\right|_{t=0} = \frac{1}{4}\int_{\partial N}\left(\left(\delta I\!I, I\!I\!I - \frac{H}{2} I\!I\right)_{I\!I} + \frac{\delta K_e}{2K_e} H\right) \mathrm{d}a_I ,$$

where $\delta I\!I$, δK_e denote the variations of the second fundamental form and of the extrinsic curvature of ∂N, respectively, and $(\cdot,\cdot)_{I\!I}$ denotes the scalar product induced by $I\!I$ on the space of $(2,0)$-tensors on ∂N.

Proof. We will apply the same strategy used in Proposition 1.7.13 to compute the variation formula of V_3^*. From the definition of W-volume and Theorem 1.7.4 we have:

$$\begin{aligned}\left.\frac{\mathrm{d}W(N, g_t)}{\mathrm{d}t}\right|_{t=0} &= \left.\frac{\mathrm{dVol}(N, g_t)}{\mathrm{d}t}\right|_{t=0} - \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial N} H \, \mathrm{d}a_I\bigg|_{t=0} \\ &= \frac{1}{2}\int_{\partial N}\left(\delta H + \frac{1}{2}(\delta I, I\!I)_I\right)\mathrm{d}a_I - \frac{1}{4}\int_{\partial N}(\delta H \, \mathrm{d}a_I + H\delta(\mathrm{d}a_I)) .\end{aligned}$$

Since $I = I\!I\, B^{-1}$, we have $\delta I = \delta I\!I\, B^{-1} - I\!I\, B^{-1}\, \delta B\, B^{-1}$. Therefore

$$\begin{aligned}(\delta I, I\!I)_I &= \operatorname{tr}\big(I^{-1}\, \delta I\, I^{-1}\, I\!I\big)\\ &= \operatorname{tr}\big(B\, I\!I^{-1}(\delta I\!I\, B^{-1} - I\!I\, B^{-1}\, \delta B\, B^{-1})B\, I\!I^{-1}\, I\!I\big)\\ &= \operatorname{tr}\big(B\, I\!I^{-1}\, \delta I\!I\big) - \operatorname{tr}(\delta B)\\ &= \operatorname{tr}\big(I\!I^{-1}\, I\!I\!I\, I\!I^{-1}\, \delta I\!I\big) - \delta H\\ &= (I\!I\!I, \delta I\!I)_{I\!I} - \delta H.\end{aligned}$$

Using the expression in local coordinates $\mathrm{d}a_g = \sqrt{\det(g_{ij})}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2$, we find $\delta(\mathrm{d}a_g) = \frac{1}{2}(\delta g, g)_g\, \mathrm{d}a_g$. Hence we have:

$$\begin{aligned}\delta\left(\mathrm{d}a_I\right) &= \delta\left(\frac{\mathrm{d}a_{I\!I}}{\sqrt{K_e}}\right)\\ &= -\frac{\delta K_e}{2(K_e)^{3/2}}\, \mathrm{d}a_{I\!I} + \frac{1}{2\sqrt{K_e}}(\delta I\!I, I\!I)_{I\!I}\, \mathrm{d}a_{I\!I}\\ &= \left(-\frac{\delta K_e}{2K_e} + \frac{1}{2}(\delta I\!I, I\!I)_{I\!I}\right) \mathrm{d}a_I\, .\end{aligned}$$

Combining the relations above, we obtain:

$$\begin{aligned}\left.\frac{\mathrm{d}W(N, g_t)}{\mathrm{d}t}\right|_{t=0} &= \frac{1}{2}\int_{\partial N}\left(\delta H + \frac{1}{2}(\delta I, I\!I)_I\right)\mathrm{d}a_I - \frac{1}{4}\int_{\partial N}\left(\delta H\, \mathrm{d}a_I + H\delta(\mathrm{d}a_I)\right)\\ &= \frac{1}{4}\int_{\partial N}\left(2\delta H + (I\!I\!I, \delta I\!I)_{I\!I} - \delta H - \delta H - H\left(-\frac{\delta K_e}{2K_e} + \frac{1}{2}(\delta I\!I, I\!I)_{I\!I}\right)\right)\mathrm{d}a_I\\ &= \frac{1}{4}\int_{\partial N}\left(\left(\delta I\!I, I\!I\!I - \frac{H}{2} I\!I\right)_{I\!I} + \frac{\delta K_e}{2K_e} H\right)\mathrm{d}a_I\, ,\end{aligned}$$

which proves the statement. □

An alternative way to express the variation of the W-volume can be found in [KS08, Relation (7)]. The expression we found in Proposition 5.2.1 is very convenient when applied to variations of metrics $(g_t)_t$ for which the boundary ∂N has constant extrinsic curvature K_e independent of t, as displayed by the following Lemma:

Lemma 5.2.2. *The function $W_k \colon \mathcal{QD}(M) \to \mathbb{R}$ satisfies*

$$\mathrm{d}W_k(\delta M) = -\operatorname{Re}\langle q_k, \delta c_k\rangle,$$

where δc_k denotes the variation of the conformal class c_k along the variation δM.

Proof. We apply the variation formula of the W-volume, proved in Proposition 5.2.1. Since the boundary of M_k is a k-surface for every convex co-compact structure M, the term involving δK_e vanishes. Therefore we have:

$$\begin{aligned} \mathrm{d}W_k(\delta M) &= \frac{1}{4}\int_{\partial M_k}\left(\delta I\!I_k, I\!I\!I_k - \frac{H_k}{2} I\!I_k\right)_{I\!I_k} \mathrm{d}a_{I_k} \\ &= \frac{1}{4\sqrt{k+1}}\int_{\partial M_k}\left(\delta I\!I_k, I\!I\!I_k - \frac{H_k}{2} I\!I_k\right)_{I\!I_k} \mathrm{d}a_{I\!I_k} && \text{(eq. (5.2))} \\ &= -\frac{1}{8}\int_{\partial M_k}(\delta I\!I_k, 2\operatorname{Re} q_k)_{I\!I_k} \mathrm{d}a_{I\!I_k} && \text{(eq. (5.1))} \\ &= -\operatorname{Re}\langle q_k, \delta c_k\rangle. && \text{(Lemma 5.3.1)} \end{aligned}$$

□

Starting from Lemma 5.2.2, the proof of the Schläfli formula for the volumes W_k proceeds in analogy to what done by Schlenker [Sch17] for the Schläfli formula for the renormalized volume, thanks to the following result:

Theorem 5.2.3 (Gardiner's formula, [Gar84, Theorem 8])**.** *Let (Σ, c) be a Riemann surface, and let $\mathcal{F}$ denote the horizontal foliation of a homorphic quadratic differential q of (Σ, c). Then the extremal length function $\operatorname{ext}_{\mathcal{F}} \colon \mathcal{T}^{\mathfrak{c}}(\Sigma) \to \mathbb{R}$ satisfies*

$$\mathrm{d}\operatorname{ext}_{\mathcal{F}}(\delta c) = 2\operatorname{Re}\langle q, \delta c\rangle .$$

The combination of Lemma 5.2.2 and the Gardiner's formula immediately implies:

Theorem 5.2.4 (Schläfli formula for W_k)**.** *The differential of the function W_k can be expressed as follows:*

$$\mathrm{d}W_k(\delta M) = -\frac{1}{2}\mathrm{d}\operatorname{ext}_{\mathcal{F}_k}(\delta c_k),$$

where $\mathcal{F}_k$ denotes the horizontal foliation of the holomorphic quadratic differential q_k.

5.2.2 The renormalized volume

The definition of renormalized volume $V_R(M)$ of a conformally compact Einstein manifold M is motivated by the $\mathbb{A}\mathrm{d}\mathbb{S}$/CFT correspondence of string theory [Wit98], [Gra00]. Krasnov and Schlenker [KS08] enlightened its geometrical meaning in the context of convex co-compact hyperbolic 3-manifolds, describing a regularization procedure based on equidistant foliations from convex subsets of M. In relation with the study of the geometry of the Teichmüller space, the renormalized volume furnishes a Kähler potential for the Weil-Petersson metric of the Teichmüller space, and it allows to give a remarkably simple proof of McMullen's Kleinian reciprocity (see [KS08] and Section 5.4). Moreover, its variation formula has been used by Schlenker [Sch13] to give a quantitative version of Brock's upper bound of the volume of the convex core of a quasi-Fuchsian manifold in terms of the Weil-Petersson distance between the hyperbolic metrics on the boundary of the convex core.

The aim of this Section is to describe a new and simpler way to define the renormalized volume of a quasi-Fuchsian manifold in terms of the asymptotic of its foliation by k-surfaces.

First we recall the Schläfli-type formula of the renormalized volume:

Theorem 5.2.5 ([KS08, Lemmas 8.3, 8.5], [Sch17, Theorem 1.2])**.** *The differential of the renormalized volume $V_R\colon \mathcal{QD}(M) \to \mathbb{R}$ can be expressed as follows:*

$$\mathrm{d}V_R(\delta M) = -\operatorname{Re}\langle q_\infty, \delta c_\infty \rangle = -\frac{1}{2} \mathrm{d}\,\mathrm{ext}_{\mathcal{F}_\infty}(\delta c_\infty).$$

The combination of Corollary 5.1.3 and Theorem 5.2.4 allows us to give the following description of the renormalized volume $V_R(M)$:

Theorem D. *The renormalized volume of a quasi-Fuchsian manifold M satisfies*

$$V_R(M) = \lim_{k \to 0^-} \left(W_k(M) - \pi |\chi(\partial M)| \operatorname{arctanh} \sqrt{k+1} \right).$$

Proof. Let $\widetilde{W}_k(M) := W_k(M) - \pi|\chi(\partial M)| \operatorname{arctanh} \sqrt{k+1}$. We will prove the assertion by showing the following facts:

i) the differentials of the functions $\widetilde{W}_k$ converge, uniformly over compact subsets of $\mathcal{QF}(\Sigma)$, to the differential of the renormalized volume V_R;

ii) the limit, as k goes to 0, of $\widetilde{W}_k(M)$ coincides with $V_R(M)$ whenever M is Fuchsian.

Then the assertion will follow from the connectedness of the space $\mathcal{QF}(\Sigma)$.

The first step easily follows from our previous observations. By Corollary 5.1.3 and Theorem 5.2.4, $\mathrm{d}\widetilde{W}_k = \mathrm{d}W_k$ converges, uniformly over compact subsets of $\mathcal{QF}(\Sigma)$, to $-\frac{1}{2}\mathrm{d}\,\mathrm{ext}_{\mathcal{F}_\infty}(\delta c_\infty)$, where $\mathcal{F}_\infty$ is the horizontal foliation of the Schwarzian differential at infinity q_∞, and δc_∞ is the variation of the conformal structure of $\partial_\infty M$. By Theorem 5.2.5, this coincides with $\mathrm{d}V_R$.

It remains to prove the second part of the statement. Let M be a Fuchsian manifold. The equidistant surfaces from the convex core of M at distance $\varepsilon(k) := \operatorname{arctanh} \sqrt{k+1}$ are the two k-surfaces of M. Their fundamental forms can be expressed as follows:

$$I_k = -\frac{1}{k} h, \qquad I\!I_k = -\frac{k}{\sqrt{k+1}} h, \qquad I\!I\!I_k = -\frac{k}{k+1} h,$$

where h is the hyperbolic metric on the totally geodesic surface sitting inside M. From here, we easily see that

$$\int_{\Sigma_k} H_k \, \mathrm{d}a_{I_k} = 2\pi |\chi(\partial M)| \sinh 2\varepsilon(k), \qquad V(M_k) = \pi|\chi(\partial M)| \left(\frac{\sinh 2\varepsilon(k)}{2} + \varepsilon(k) \right).$$

In particular, for every Fuchsian manifold M, we have

$$W_k(M) = V(M_k) - \frac{1}{4} \int_{\Sigma_k} H_k \, \mathrm{d}a_{I_k} = \pi|\chi(\partial M)| \operatorname{arctanh} \sqrt{k+1}.$$

Therefore the functions $\widetilde{W}_k$ vanish identically over the Fuchsian locus, and the same happens for $V_R(M)$. This concludes the proof of the second step, and therefore of the statement. □

Remark 5.2.6. The quantity $\operatorname{arctanh} \sqrt{k+1}$ is equal to the distance of the k-surface from the convex core in the Fuchsian case. For a generic quasi-Fuchsian manifold M, the geometric maximum principle [Lab00, Lemme 2.5.1] shows that the k-surface is at distance less or equal than $\operatorname{arctanh} \sqrt{k+1}$ from the convex core CM.

In the proof that we gave above, we *assumed* the existence of the renormalized volume function V_R and we proved the convergence of the functions W_k to V_R. In fact, with some additional work, it is possible to show that the sequence of functions $(\widetilde{W}_k)_k$ is convergent *without* assuming the existence of the function V_R. In other words, we can *define* the renormalized volume $V_R(M)$ of a quasi-Fuchsian manifold M as the limit of the sequence $(\widetilde{W}_k(M))_k$.

5.2.3 V_k^*-volumes

In analogy to what done for the W_k-volumes, we define

$$V_k^*(M') := \mathrm{Vol}^*(M'_k) = \mathrm{Vol}(M'_k) - \frac{1}{2}\int_{\partial M'_k} H'_k \, \mathrm{d}a_{I'_k} \, ,$$

for every $M' \in \mathcal{QD}(M)$. The Schläfli formula for V_k^* is a direct consequence of the variation formula for the dual volume (Proposition 2.2.5) and the following expression for the variation of the length function $L_{h'}$:

Lemma 5.2.7 ([BMS15, Lemma 7.9])**.**

$$\mathrm{d}(L_{h'})\,(\delta h) = -\frac{1}{2}\int_{\Sigma} (\delta h, h(b\cdot,\cdot) - \mathrm{tr}(b)h)_h \, \mathrm{d}a_h \, .$$

In order to simplify the next statement, we extend the definition of the function j to constant curvature metrics, not necessarily hyperbolic. In particular, if g and g' are Riemannian metrics of with constant Gaussian curvatures K and K', then we set $j(g, g')$ to be $(KK')^{-1/2} j((-K)g, (-K')g')$ (observe that $(-K)g$ and $(-K')g'$ are hyperbolic). In this way, the function j is 1/2-homogeneous in both its arguments, As before, L_g will denote the function $j(g, \cdot)$.

Theorem 5.2.8 (Schläfli formula for V_k^*)**.** *The differential of the function V_k^* can be expressed as follows:*

$$\mathrm{d}V_k^*(\delta M) = -\frac{1}{2}\,\mathrm{d}L_{I\!I\!I_k}\,(\delta I_k).$$

Proof. By Proposition 2.2.5, the variation of V_k^* verifies

$$\mathrm{d}V_k^*\,(\delta M) = \frac{1}{4}\int_{\Sigma_k} (\delta I_k, I\!I_k - H_k I_k)_{I_k} \, \mathrm{d}a_{I_k} \, .$$

Using the definitions of h_k, h'_k, we can rephrase the expression above as follows:

$$\mathrm{d}V_k^*\,(\delta M) = -\frac{\sqrt{k+1}}{4k}\int_{\Sigma_k} (\delta h_k, h_k(b_k\cdot,\cdot) - \mathrm{tr}(b_k)h_k)_{h_k} \, \mathrm{d}a_{h_k} \, ,$$

where $b_k = \frac{1}{\sqrt{k+1}} B_k$ is the Labourie operator between h_k and h'_k (see Theorem 1.2.18 and Section 1.5). By Lemma 5.2.7, the expression above is equal to

$$\frac{\sqrt{k+1}}{2k}\,\mathrm{d}L_{h'_k}\,(\delta h_k) = -\frac{1}{2}\,\mathrm{d}L_{I\!I\!I_k}\,(\delta I_k),$$

which proves the statement. □

We introduced the notation in order to emphasize the similarities between the Schläfli formula of V_k^* and the dual Bonahon-Schläfli formula from Theorem A. Observe in particular that formally Theorem A can be obtained as the limit of the relation of Theorem 5.2.8, in light of Proposition 5.1.5.

5.3 Volumes and symplectomorphisms

The aim of this section is to study the properties of the maps Φ_k and $\Psi_{k'}$. In particular, we will prove that the diffeomorphisms $\Phi_k \circ \Psi_{k'}^{-1} \colon T^*\mathcal{T}^{\mathfrak{h}}(\Sigma) \to T^*\mathcal{T}^{\mathfrak{c}}(\Sigma)$ are symplectic with respect to the cotangent symplectic structures of $T^*\mathcal{T}^{\mathfrak{h}}(\Sigma)$ and $T^*\mathcal{T}^{\mathfrak{c}}(\Sigma)$, up to a multiplicative factor. This fact extends the results of Krasnov and Schlenker [KS09] and Bonsante, Mondello, and Schlenker [BMS15] concerning the grafting map Gr and the smooth grafting map SGr, respectively.

5.3.1 Relative volumes

Let E be a hyperbolic end (see Definition 1.6.1). In light of Theorem 1.6.4, we denote by E_k the portion of E that is in between the concave pleated boundary ∂E and the k-surface Σ_k of E. Now we define

$$w_k(E) := \mathrm{Vol}(E_k) - \frac{1}{4}\int_{\Sigma_k} H_k \, \mathrm{d}a_k + \frac{1}{2} L_\mu(m),$$

where H_k and $\mathrm{d}a_k$ are the mean curvature and the I_k-area form of Σ_k, and $L_\mu(m)$ is the length of the bending measure μ with respect to the hyperbolic metric m of ∂E. Similarly, we define

$$v_k^*(E) := \mathrm{Vol}(E_k) - \frac{1}{2}\int_{\Sigma_k} H_k \, \mathrm{d}a_k + \frac{1}{2} L_\mu(m).$$

The functions w_k and v_k^* can be considered as the relative versions of the W_k-volume and V_k^*-volume, respectively.

5.3.2 Cotangent symplectic structures

Let M be a smooth n-manifold, with cotangent bundle $\pi \colon T^*M \to M$. The *Liouville form* λ of T^*M is the 1-form defined by:

$$\lambda_{(p,\alpha)}(v) := \alpha(\mathrm{d}\pi_{(p,\alpha)}(v))$$

for every $(p, \alpha) \in T^*M$ and $v \in T_{(p,\alpha)}T^*M$. The 2-form $\omega := \mathrm{d}\lambda$ is non-degenerate and it defines a natural symplectic structure on the total space T^*M.

In the following, $\lambda^{\mathfrak{h}}$, $\lambda^{\mathfrak{c}}$ will denote the Liouville forms of $T^*\mathcal{T}^{\mathfrak{h}}(\Sigma)$, $T^*\mathcal{T}^{\mathfrak{c}}(\Sigma)$, respectively, and $\omega^{\mathfrak{h}}$, $\omega^{\mathfrak{c}}$ their associated symplectic forms. As before, Th stands for the Thurston parametrization, defined in Section 1.6.2. The reader can find the necessary notation concerning the geometry of k-surfaces in Section 2.1, and the definitions of the parametrizations Φ_k and Ψ_k in Sections 5.1.1 and 5.1.2, respectively.

The first step of our analysis will be to describe the pullback of the Liouville forms $\lambda^{\mathfrak{c}}$ and $\lambda^{\mathfrak{h}}$ by the maps Φ_k and $\mathrm{d}L \circ \mathrm{Th}$, Ψ_k, respectively. First we will need the following technical lemma:

Lemma 5.3.1. *Let $(g_t)_t$ be a 1-parameter family of Riemannian metrics on Σ, with conformal classes $c_t = [g_t]$. If δc denotes the Beltrami differential* *representing the variation of the conformal classes* $(c_t)_t$*, and* δg *the variation of the Riemannian metrics* $(g_t)_t$*, then we have*

$$\operatorname{Re}\langle q, \delta c\rangle = \frac{1}{4}\int_\Sigma (\delta g, \operatorname{Re} q)_g \, \mathrm{d}a_g \, .$$

Proof. Let $(g_t)_t$ be a smooth 1-parameter family of metrics so that the conformal class of $g = g_0$ is equal to $c = c_0$, and the derivative at $t = 0$ of the conformal class c_t of g_t coincides with δc. If X_t is an object that depends on t, then δX will denote its derivative with respect to t at $t = 0$. Let J_t be the almost complex structure of g_t for every t. As shown in [BMS15, Section 2.1], the Beltrami differential ν_t of the map $id \colon (\Sigma, c) \to (\Sigma, c_t)$ satisfies

$$\nu_t = (\mathbb{1} - J_t J)^{-1}(1 + J_t J),$$

In particular its derivative $\delta\nu$ can be expressed as $\frac{1}{2}\delta J \, J$. The almost complex structure J_t of g_t is characterized by the relation $\mathrm{d}a_t\,(\cdot, \cdot) = g_t(J_t \cdot, \cdot)$, where $\mathrm{d}a_t$ is the area form of the metric g_t. Taking the derivative of this identity, and using the fact that $\mathrm{d}a_g = \sqrt{\det(g_{ij})}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$ in local coordinates, we obtain

$$\frac{1}{2}(\delta g, g)_g \, \mathrm{d}a = \frac{1}{2}\operatorname{tr}\big(g^{-1}\delta g\big)\,\mathrm{d}a = \delta(\mathrm{d}a_t) = \delta(g_t(J_t \cdot, \cdot)) = \delta g(J\cdot, \cdot) + g(\delta J \cdot, \cdot).$$

If $\delta g = g(A\cdot, \cdot)$, with A g-self-adjoint, then from the relation above we see that

$$\delta J \; J = A - \frac{1}{2}\operatorname{tr}(A)\mathbb{1} = A_0,$$

where A_0 stands for the traceless part of A. In particular, this proves that $\delta\nu = \frac{1}{2}A_0$. The pairing between Beltrami differentials and holomorphic quadratic differentials can be described as follows:

$$\langle q, \mu\rangle := \int_\Sigma q \bullet \mu,$$

where $q \bullet \mu$ is the $\mathbb{C}$-valued 2-form given by

$$(q \bullet \mu)(u, w) := \frac{1}{2i}(q(\mu(u), w) - q(u, \mu(w))).$$

Again, we refer to [BMS15, Section 2.1] for a more detailed description. Let now B be the traceless and g-self-adjoint operator satisfying $\operatorname{Re} q(\cdot, \cdot) = g(B\cdot, \cdot)$. Given any unit vector u, the basis u, Ju is orthonormal and positive oriented. In particular, since $q \bullet \delta\nu$ is a multiple of the volume form $\mathrm{d}a_g$ (Σ is a 2-manifold), we must have $q \bullet \delta\nu = (q \bullet \delta\nu)(u, Ju)\, \mathrm{d}a_g$. Now we observe:

$$\begin{aligned}
\operatorname{Re}(q \bullet \delta\nu)(u, Ju) &= \operatorname{Re} \frac{1}{2i}(q(\delta\nu(u), Ju) - q(u, \delta\nu(Ju))) \\
&= \operatorname{Re} \frac{1}{2i}(iq(\delta\nu(u), u) + q(J^2 u, \delta\nu(Ju))) \\
&\qquad\qquad (q\ \mathbb{C}\text{-linear and } J^2 = -\mathbb{1}) \\
&= \operatorname{Re} \frac{1}{2}(q(\delta\nu(u), u) + q(\delta\nu(Ju), Ju)) \qquad (q\ \mathbb{C}\text{-linear}) \\
&= \frac{1}{4}(g(BA_0 u, u) + g(BA_0 Ju, Ju)) \\
&\qquad\qquad (\text{def. of } B \text{ and } \delta\nu = \tfrac{1}{2}A_0) \\
&= \frac{1}{4}\operatorname{tr}(BA_0) \qquad (u,\ Ju \text{ orthon. basis}) \\
&= \frac{1}{4}\operatorname{tr}(BA) \qquad (B \text{ traceless}) \\
&= \frac{1}{4}(\operatorname{Re} q, \delta g) \qquad (\text{def's of } A \text{ and } B)
\end{aligned}$$

Combining what we have proved so far, we obtain that

$$\operatorname{Re}\langle q, \delta c\rangle = \int_\Sigma \operatorname{Re} q \bullet \delta\nu = \frac{1}{4}\int_\Sigma (\operatorname{Re} q, \delta g)\, \mathrm{d}a_g\,,$$

which is our desired relation. □

We are now ready to study the pullback of the Liouville forms under the maps Φ_k, $\mathrm{d}L \circ \mathrm{Th}$ and Ψ_k:

Lemma 5.3.2. *The following relations hold:*

$$\Phi_k^* \lambda^{\mathfrak{c}}(\delta E) = \frac{1}{4}\int_{\Sigma_k} (\delta I\!I_k, \operatorname{Re} q_k)_{I\!I_k}\, \mathrm{d}a_{I\!I_k}\,, \tag{5.3}$$

$$(\mathrm{d}L \circ \mathrm{Th})^* \lambda^{\mathfrak{h}}(\delta E) = \mathrm{d}L_\mu\,(\delta m), \tag{5.4}$$

$$\Psi_k^* \lambda^{\mathfrak{h}}(\delta E) = -\frac{1}{2}\int_{\Sigma_k} (\delta I_k, I\!I_k - H_k I_k)_{I_k}\, \mathrm{d}a_{I_k}\,, \tag{5.5}$$

where δI_k and $\delta I\!I_k$ represent the variations of the first and second fundamental forms of the k-surface, respectively.

Proof. The Liouville form $\lambda^{\mathfrak{c}}$ of $T^*\mathcal{T}^{\mathfrak{c}}(\Sigma)$ satisfies

$$(\Phi_k^* \lambda^{\mathfrak{c}})_E(\delta E) = \lambda^{\mathfrak{c}}_{(c_k, q_k)}(\mathrm{d}(\Phi_k)_E\,(\delta E)) = \operatorname{Re}\langle q_k, \delta c_k\rangle,$$

where δc_k is the Beltrami differential representing the variation of c_k as we deform the hyperbolic end along the direction δE. Then relation (5.3) follows from Lemma 5.3.1.

Relation (5.4) has been originally shown by Krasnov and Schlenker in the proof of [KS09, Theorem 1.2]. First observe that the 1-form $(\mathrm{d}L \circ \mathrm{Th})^* \lambda^{\mathfrak{h}}$ is well defined since the function $\mathrm{d}L \circ \mathrm{Th}$ is $\mathscr{C}^1$ (see Proposition 5.1.6). Similarly to what done above, we see that

$$((\mathrm{d}L \circ \mathrm{Th})^* \lambda^{\mathfrak{h}})_E(\delta E) = \lambda^{\mathfrak{h}}_{(m, \mathrm{d}(L_\mu)_m)}(\mathrm{d}(\mathrm{d}L \circ \mathrm{Th})_E\,(\delta E)) = \mathrm{d}(L_\mu)_m\,(\delta m),$$

where δm denotes the first order variation of the hyperbolic metric of the concave pleated surface ∂E along the direction δE.

Finally, the Liouville form $\lambda^{\mathfrak{h}}$ satisfies

$$(\Psi_k^* \lambda^{\mathfrak{h}})_E(\delta E) = -\frac{\sqrt{k+1}}{k} \, \mathrm{d}(L_{h_k'})_{h_k}(\delta h_k).$$

Therefore, relation (5.5) follows from Lemma 5.2.7 by backtracking the multiplicative factors involved in the definitions of all the quantities. □

Similarly to what done in Section 5.2, we can describe the first order variation of the relative volume functions w_k and v_k^* as follows:

Lemma 5.3.3. *The relative volumes w_k and v_k^* satisfy:*

$$\mathrm{d}w_k(\delta E) = \frac{1}{4}\int_{\Sigma_k} \left(\delta I\!I_k, I\!I\!I_k - \frac{H_k}{2} I\!I_k\right)_{I\!I_k} \mathrm{d}a_{I_k} + \frac{1}{2}\mathrm{d}L_\mu(\delta m),$$

$$\mathrm{d}v_k^*(\delta E) = \frac{1}{4}\int_{\Sigma_k} (\delta I_k, I\!I_k - H_k I_k)_{I_k} \, \mathrm{d}a_{I_k} + \frac{1}{2}\mathrm{d}L_\mu(\delta m).$$

Proof. Both the relations can be proved by applying the same strategy of [KS09, Proposition 4.3]. Let $(g_t)_t$ be a differentiable 1-parameter family of hyperbolic metrics on $\Sigma \times (0,\infty)$ so that the first order variation of $E_t = (\Sigma \times (0,\infty), g_t)$ coincides with δE. For any t, we choose an embedded surface S in $\Sigma \times (0,\infty)$ that lies below the k-surface of E_t (i. e. it is contained in the interior of the region $(E_t)_k$) for all small values of t. Now we decompose the quantity $w_k(E)$ in two terms:

$$w_k(E) = \left(\mathrm{Vol}(N(S, \Sigma_{t,k})) - \frac{1}{4}\int_{\Sigma_{t,k}} H_{t,k}\, \mathrm{d}a_{k,t}\right) + \left(\mathrm{Vol}(N(\partial E_t, S)) + \frac{1}{2} L_{\mu_t}(m_t)\right)$$

where $\Sigma_{t,k}$ is the k-surface of E_t, and $N(S', S'')$ denotes the region of E bounded by S' from below and S'' from above.

Following step by step the proof of Proposition 5.2.1, we see that the variation of the first term equals

$$\frac{1}{4}\int_{\Sigma_k} \left(\delta I\!I_k, I\!I\!I_k - \frac{H_k}{2} I\!I_k\right)_{I\!I_k} \mathrm{d}a_{I_k} + \frac{1}{2}\int_S \left(\delta H + \frac{1}{2}(\delta I, I\!I)\right) \mathrm{d}a\,,$$

where the mean curvature H and the second fundamental form $I\!I$ of S are defined with respect to the normal vector field of S pointing *towards* the k-surface Σ_k.

The variation formula of the right term can be computed with the exact same argument of Chapter 2, the only difference is that we are looking at a region bounded by a smooth surface and a locally *concave* pleated surface, while in Chapter 2 we were considering the convex core, which is a region bounded by convex pleated surfaces. This leads to the following variation:

$$\frac{1}{2}\mathrm{d}(L_\mu)_m(\delta m) + \frac{1}{2}\int_S \left(\delta(-H) - \frac{1}{2}(\delta I, -I\!I)\right) \mathrm{d}a\,.$$

The signs multiplying H and $I\!I$ are due to the fact that we need to consider the mean curvature and the second fundamental form defined with the normal vector field pointing *inside* $N(\partial E, S)$, which is the opposite of the one considered above. In particular, when we look at the sum of the two terms, the integrals over S simplify, and we are left with the first relation of our statement.

The second relation follows by an analogous argument, replacing the use of Proposition 5.2.1 with Proposition 2.2.5. □

Lemma 5.3.4. *For every $k \in (-1.0)$, we have*

$$\begin{aligned} \mathrm{d}w_k &= -\Phi_k^* \lambda^{\mathfrak{c}} + \frac{1}{2}(\mathrm{d}L \circ \mathrm{Th})^* \lambda^{\mathfrak{h}}, \\ \mathrm{d}v_k^* &= -\frac{1}{2}\Psi_k^* \lambda^{\mathfrak{h}} + \frac{1}{2}(\mathrm{d}L \circ \mathrm{Th})^* \lambda^{\mathfrak{h}}. \end{aligned}$$

Proof. The statement is a direct consequence of Lemma 5.3.2 and Lemma 5.3.3. □

Taking the differential of the identities in Lemma 5.3.4, and remembering that $\mathrm{d}^2 = 0$, we immediately conclude the following:

Theorem 5.3.5. *For every $k \in (-1.0)$, the maps*

$$\begin{aligned} \Phi_k \circ (\mathrm{d}L \circ \mathrm{Th})^{-1} &: (T^*\mathcal{T}^{\mathfrak{h}}(\Sigma), \omega^{\mathfrak{h}}) \longrightarrow (T^*\mathcal{T}^{\mathfrak{c}}(\Sigma), 2\omega^{\mathfrak{c}}), \\ \Psi_k \circ (\mathrm{d}L \circ \mathrm{Th})^{-1} &: (T^*\mathcal{T}^{\mathfrak{h}}(\Sigma), \omega^{\mathfrak{h}}) \longrightarrow (T^*\mathcal{T}^{\mathfrak{h}}(\Sigma), \omega^{\mathfrak{h}}) \end{aligned}$$

are symplectomorphisms.

Observe that Theorem G is a direct consequence of what we just observed.

Remark 5.3.6. Theorem 5.3.5, combined with Corollary 5.1.3, implies that the map

$$\mathrm{Sch} \circ (\mathrm{d}L \circ \mathrm{Th})^{-1} : (T^*\mathcal{T}^{\mathfrak{h}}(\Sigma), \omega^{\mathfrak{h}}) \longrightarrow (T^*\mathcal{T}^{\mathfrak{c}}(\Sigma), 2\omega^{\mathfrak{c}})$$

is a symplectomorphism, which has been originally shown in [KS09, Theorem 1.2]. In addition, [KS09, Theorem 1.2] and Theorem G imply also [BMS15, Theorem 1.11], which states that the function

$$\mathrm{Sch} \circ \Psi_k^{-1} : (T^*\mathcal{T}^{\mathfrak{h}}(\Sigma), \omega^{\mathfrak{h}}) \longrightarrow (T^*\mathcal{T}^{\mathfrak{c}}(\Sigma), 2\omega^{\mathfrak{c}})$$

is a symplectomorphism for every $k \in (-1, 0)$. Finally, by applying Theorem G to the case $k = k'$, and taking care of the multiplicative factors involved in the definitions of Φ_k and Ψ_k, we deduce that the function

$$\begin{array}{cccc} \hat{\mathcal{H}}: & (T^*\mathcal{T}^{\mathfrak{c}}(\Sigma), \omega^{\mathfrak{c}}) & \longrightarrow & (T^*\mathcal{T}^{\mathfrak{h}}(\Sigma), \omega^{\mathfrak{h}}) \\ & (c, q) & \longrightarrow & (h(c,q), \mathrm{d}(L_{h(c,-q)})) \end{array}$$

is a symplectomorphism, where $h(c, \pm q) = \varphi_c^{-1}(\pm q)$ is the hyperbolic metric of Σ for which the identity map $(\Sigma, c) \to (\Sigma, h(c, \pm q))$ is harmonic with Hopf differential equal to $\pm q$ (see Theorem 1.2.15).

5.4 Kleinian reciprocities

Let M be a convex co-compact hyperbolic 3-manifold and let $\mathcal{QD}(M)$ denote the space of quasi-isometric deformations of M. Any isotopy class of hyperbolic metrics $[g] \in \mathcal{QD}(M)$ has a collection of k-surfaces, each one sitting inside a hyperbolic end E_i of (M, g). In this way, we can define a function

$$\phi_k : \mathcal{QD}(M) \longrightarrow T^*\mathcal{T}^{\mathfrak{c}}(\partial M),$$

which associates to any class $[g]$ the data $(\Phi_k(E_i))_i$ of its k-surfaces. Similarly, we define the function $\psi_k \colon \mathcal{QD}(M) \to T^*\mathcal{T}^{\mathfrak{h}}(\partial M)$, sending $[g]$ into the data $(\Psi_k(E_i))_i$.

Theorem E. *For every $k \in (-1, 0)$, the image $\phi_k(\mathcal{QD}(M))$ (resp. $\psi_k(\mathcal{QD}(M))$) is a Lagrangian submanifold of $T^*\mathcal{T}^{\mathfrak{c}}(\partial M)$ (resp. $T^*\mathcal{T}^{\mathfrak{h}}(\partial M)$).*

Proof. The statement is a consequence of Lemma 5.3.4 and of the variation formula of the dual volume of a convex co-compact hyperbolic manifold. To see this, first we apply Lemma 5.3.4 to each end of M:

$$\mathrm{d}w_{k,i}\,(\delta E_i) = -\Phi_{k,i}^* \lambda_i^{\mathfrak{c}}(\delta E_i) + \frac{1}{2}(\mathrm{d}L_i \circ \mathrm{Th}_i)^* \lambda_i^{\mathfrak{h}}(\delta E_i).$$

By the dual Bonahon-Schläfli formula (Theorem A), we have that

$$\mathrm{d}V_C^*\,(\delta M) = -\frac{1}{2}\sum_i \mathrm{d}L_{\mu_i}\,(\delta m_i) = -\frac{1}{2}\sum_i (\mathrm{d}L_i \circ \mathrm{Th}_i)^* \lambda_i^{\mathfrak{h}}(\delta E_i).$$

Therefore we deduce that

$$\mathrm{d}\left(\sum_i w_{k,i} + V_C^*\right) = \sum_i \mathrm{d}w_{k,i} + \mathrm{d}V_C^* = -\sum_i \Phi_{k,i}^* \lambda_i^{\mathfrak{c}} = -\phi_k^* \lambda^{\mathfrak{c}}.$$

The function $\sum_i w_{k,i} + V_C^*$ is in fact equal to the W-volume of M_k, the portion of M contained in the union of the k-surfaces of the ends $(E_i)_i$. Indeed:

$$\begin{aligned}
\sum_i w_{k,i}(E_i) + V_C^*(M) &= \sum_i \left(\mathrm{Vol}(E_{k,i}) - \frac{1}{4}\int_{\Sigma_{k,i}} H_{k,i}\,\mathrm{d}a_{k,i} + \frac{1}{2}L_{\mu_i}(m_i)\right) + \\
&\qquad + \mathrm{Vol}(CM) - \frac{1}{2}L_\mu(m) \\
&= \mathrm{Vol}(CM) + \sum_i \mathrm{Vol}(E_{k,i}) - \frac{1}{4}\sum_i \int_{\Sigma_{k,i}} H_{k,i}\,\mathrm{d}a_{k,i} + \\
&\qquad + \frac{1}{2}L_\mu(m) - \frac{1}{2}L_\mu(m) \\
&= \mathrm{Vol}(M_k) - \frac{1}{4}\int_{\partial M_k} H_k\,\mathrm{d}a_k \\
&= W_k(M).
\end{aligned}$$

Therefore we have proved that $\mathrm{d}W_k = -\phi_k^* \lambda^{\mathfrak{c}}$. Taking the differential of this identity we obtain that $\phi_k^* \omega^{\mathfrak{c}} = 0$. This implies the statement, since ϕ_k is an embedding and $2\dim \mathcal{QD}(M) = \dim T^*\mathcal{T}^{\mathfrak{c}}(\partial M)$.

In an analogous manner we can prove that $\psi_k^* \lambda^{\mathfrak{h}} = -2\,\mathrm{d}V_k^*$. To see this, it is enough to replace the role of the relative W-volumes $w_{k,i}$ with the dual volumes $v_{k,i}^*$ and then proceed in the exact same way. Again, by taking the differential of the identity $\psi_k^* \lambda^{\mathfrak{h}} = -2\,\mathrm{d}V_k^*$, we obtain the second part of the statement. □

Theorem E is a generalization of Krasnov and Schlenker's reformulation of McMullen's Kleinian reciprocity Theorem [KS09, Theorem 1.5], and their result can be recovered by taking the limit of the identity $\phi_k^* \omega^{\mathfrak{c}} = 0$ and applying Corollary 5.1.3 to each hyperbolic end of M. Moreover, Krasnov and Schlenker [KS09, Theorem 1.4] proved that the image of the function $\mathrm{d}L \circ \mathrm{Th}$ is Lagrangian inside $(T^* \mathcal{T}^{\mathfrak{h}}(\Sigma), \omega^{\mathfrak{h}})$. Since the map $\mathrm{d}L \circ \mathrm{Th}$ is the limit of the ψ_k's, the part of the statement concerning the maps ψ_k can be similarly seen as an extension of Krasnov and Schlenker's original result.

5.4.1 Quasi-Fuchsian reciprocities

In this section we present a generalization of McMullen's quasi-Fuchsian reciprocity Theorem in its original formulation from [McM98]. First we will recall McMullen's original statement, and then we will see how to formulate Theorem E is a similar manner. We define the Bers' embeddings to be the maps:

$$\begin{array}{rccc} \beta_X : & \mathcal{T}^{\mathfrak{c}}(\overline{\Sigma}) & \longrightarrow & T_X^* \mathcal{T}^{\mathfrak{c}}(\Sigma) \\ & Y & \longmapsto & \mathrm{Sch}(Q(X,Y))^+ \end{array} \qquad \begin{array}{rccc} \beta_Y : & \mathcal{T}^{\mathfrak{c}}(\Sigma) & \longrightarrow & T_Y^* \mathcal{T}^{\mathfrak{c}}(\overline{\Sigma}) \\ & Y & \longmapsto & \mathrm{Sch}(Q(X,Y))^- \end{array}$$

where $Q(X,Y)$ denotes the unique quasi-Fuchsian manifold with conformal classes at infinity (X,Y), and $\mathrm{Sch}(Q(X,Y))^{\pm}$ are the Schwarzian differentials at infinity on the upper and lower boundaries at infinity. McMullen's original formulation of the quasi-Fuchsian reciprocity Theorem is the following:

Theorem 5.4.1 ([McM98, Theorem 1.6])**.** *Given $(X,Y) \in \mathcal{T}^{\mathfrak{c}}(\Sigma) \times \mathcal{T}^{\mathfrak{c}}(\overline{\Sigma})$, the differentials of the Bers' embeddings*

$$\mathrm{d}(\beta_X)_Y : T_Y \mathcal{T}^{\mathfrak{c}}(\overline{\Sigma}) \longrightarrow T_X^* \mathcal{T}^{\mathfrak{c}}(\Sigma), \qquad \mathrm{d}(\beta_Y)_X : T_X \mathcal{T}^{\mathfrak{c}}(\Sigma) \longrightarrow T_Y^* \mathcal{T}^{\mathfrak{c}}(\overline{\Sigma})$$

are adjoint linear operators. In other words, $\mathrm{d}(\beta_X)_Y = \mathrm{d}(\beta_Y)_X^*$.

We want to describe analogous statements in the case in which Sch is replaced by ϕ_k or ψ_k. For every $k \in (-1,0)$, let B_k and T_k be the maps

$$\begin{array}{rccc} B_k : & \mathcal{QF}(\Sigma) & \longrightarrow & \mathcal{T}^{\mathfrak{c}}(\Sigma) \times \mathcal{T}^{\mathfrak{c}}(\overline{\Sigma}) \\ & M & \longmapsto & (c_k^+, c_k^-) \end{array} \qquad \begin{array}{rccc} T_k : & \mathcal{QF}(\Sigma) & \longrightarrow & \mathcal{T}^{\mathfrak{h}}(\Sigma) \times \mathcal{T}^{\mathfrak{h}}(\overline{\Sigma}) \\ & M & \longmapsto & (h_k^+, h_k^-) \end{array}$$

where $c_k^{\pm}$ are the conformal classes of the second fundamental forms of the upper and lower k-surface of M, respectively, and $h_k^{\pm}$ are the hyperbolic metrics $(-k) I_k^{\pm}$ of the upper and lower k-surface of M, respectively.

A consequence of Labourie and Schlenker's works [Lab92a], [Sch06] (see Theorem 3.4.1) is that the function T_k is a diffeomorphism for every $k \in (-1,0)$. We do not know if the same is true for B_k, we will assume this to be true for the rest of this section. In analogy to Bers' embeddings, we define the following maps:

$$\begin{array}{rcccl} \beta_{k,X}: & \mathcal{T}^{\mathfrak{c}}(\overline{\Sigma}) & \longrightarrow & T^*_X\mathcal{T}^{\mathfrak{c}}(\Sigma) \\ & Y & \longmapsto & \phi_k^+ \circ B_k^{-1}(X,Y) \end{array} \qquad \begin{array}{rcccl} \beta_{k,Y}: & \mathcal{T}^{\mathfrak{c}}(\Sigma) & \longrightarrow & T^*_Y\mathcal{T}^{\mathfrak{c}}(\overline{\Sigma}) \\ & Y & \longmapsto & \phi_k^- \circ B_k^{-1}(X,Y) \end{array}$$

$$\begin{array}{rcccl} \tau_{k,X}: & \mathcal{T}^{\mathfrak{h}}(\overline{\Sigma}) & \longrightarrow & T^*_X\mathcal{T}^{\mathfrak{h}}(\Sigma) \\ & Y & \longmapsto & \psi_k^+ \circ T_k^{-1}(X,Y) \end{array} \qquad \begin{array}{rcccl} \tau_{k,Y}: & \mathcal{T}^{\mathfrak{h}}(\Sigma) & \longrightarrow & T^*_Y\mathcal{T}^{\mathfrak{h}}(\overline{\Sigma}) \\ & Y & \longmapsto & \psi_k^- \circ T_k^{-1}(X,Y) \end{array}$$

where:

a) $B_k^{-1}(X,Y)$ is the conjecturally unique quasi-Fuchsian manifold whose upper and lower k-surfaces have X and Y as conformal classes of their second fundamental forms, respectively;

b) $T_k^{-1}(X,Y)$ is the unique quasi-Fuchsian manifold whose upper and lower k-surfaces have X and Y as hyperbolic structures induced by their first fundamental forms, respectively;

c) $\phi_k^{\pm} \circ B_k^{-1}(X,Y)$ are the holomorphic quadratic differentials $q_k^{\pm}$ on the upper and lower k-surfaces (as defined in Section 5.1.1);

d) $\psi_k^{\pm} \circ T_k^{-1}(X,Y)$ are the 1-forms $\mathrm{d}(L_{I\!I\!I_k^{\pm}})_{I_k^{\pm}}$ on the upper and lower k-surfaces (as defined in Section 5.1.2).

Now that we have introduced all the notation, we are ready to state the formulations of the quasi-Fuchsian reciprocity Theorems that follow from Theorem E:

Theorem 5.4.2. *For every $(X,Y) \in \mathcal{T}^{\mathfrak{h}}(\Sigma) \times \mathcal{T}^{\mathfrak{h}}(\overline{\Sigma})$, the differentials of the maps*

$$\mathrm{d}(\tau_{k,X})_Y : T_Y\mathcal{T}^{\mathfrak{h}}(\overline{\Sigma}) \longrightarrow T^*_X\mathcal{T}^{\mathfrak{h}}(\Sigma), \qquad \mathrm{d}(\tau_{k,Y})_X : T_X\mathcal{T}^{\mathfrak{h}}(\Sigma) \longrightarrow T^*_Y\mathcal{T}^{\mathfrak{h}}(\overline{\Sigma})$$

are adjoint linear operators.

Theorem 5.4.3. *If the map B_k is a diffeomorphism, then for every $(X,Y) \in \mathcal{T}^{\mathfrak{c}}(\Sigma) \times \mathcal{T}^{\mathfrak{c}}(\overline{\Sigma})$, the differentials of the maps*

$$\mathrm{d}(\beta_{k,X})_Y : T_Y\mathcal{T}^{\mathfrak{c}}(\overline{\Sigma}) \longrightarrow T^*_X\mathcal{T}^{\mathfrak{c}}(\Sigma), \qquad \mathrm{d}(\beta_{k,Y})_X : T_X\mathcal{T}^{\mathfrak{c}}(\Sigma) \longrightarrow T^*_Y\mathcal{T}^{\mathfrak{c}}(\overline{\Sigma})$$

are adjoint linear operators.

Proof of Theorems 5.4.2 and 5.4.3. Let $F\colon N^+ \times N^- \to T^*(N^+ \times N^-)$ be a smooth function satisfying $\pi \circ F = id$, where $\pi\colon T^*(N^+ \times N^-) \to N^+ \times N^-$ is the cotangent bundle projection. For every X in N^+, we set $F_X^+\colon N^- \to T^*_X N^+$ to be $F_X^+(Y) := F(X,Y)^+$, where $F(X,Y)^+$ is the component of $F(X,Y)$ in the fiber $T^*_X N^+$, and, for every $Y \in N_-$, we set $F_Y^-\colon N^+ \to T^*_Y N^-$ to be $F_Y^-(X) := F(X,Y)^-$, where $F(X,Y)^-$ is the component of $F(X,Y)$ in the fiber $T^*_Y N^-$. Then, the following relation holds:

$$\langle \mathrm{d}(F_Y^-)_X(u), v\rangle - \langle \mathrm{d}(F_X^+)_Y(v), u\rangle = (F^*\omega)_{(X,Y)}((u,0),(0,v)),$$

for all $(X,Y) \in N^+ \times N^-$, $u \in T_X N^+$, $v \in T_Y N^-$. A proof of this relation can be found in [KS12, Section 5.2.1] for the function $F = \mathrm{Sch}$, the proof of the general case is formally identical. Now, using this relation for the maps $F = \phi_k \circ B_k^{-1}$ and $F = \psi_k \circ T_k^{-1}$, and applying Theorem E, we obtained the desired statement. □

5.5 The k-flows are Hamiltonian

In this section we show that the k-surface foliation of a hyperbolic end can be described as the integral curve of a time-dependent Hamiltonian vector field with respect to the symplectic structure $2\Phi_k^*\omega^{\mathfrak{c}} = \Psi_k^*\omega^{\mathfrak{h}}$ on $\mathcal{E}(\Sigma)$, which does not depend on k in light of Theorem G. The vector fields we will look at are defined in terms of the diffeomorphisms $(\Phi_k)_k$ and $(\Psi_k)_k$ as follows:

$$X_k := \frac{\mathrm{d}}{\mathrm{d}h}\, \Phi_{k+h} \circ \Phi_k^{-1}\big|_{h=0} \in \Gamma(TT^*\mathcal{T}^{\mathfrak{c}}(\Sigma)),$$
$$Y_k := \frac{\mathrm{d}}{\mathrm{d}h}\, \Psi_{k+h} \circ \Psi_k^{-1}\big|_{h=0} \in \Gamma(TT^*\mathcal{T}^{\mathfrak{h}}(\Sigma)).$$

In order to simplify the notation, whenever we have an object X that depends on the curvature k, we will denote by $\dot{X}$ its derivative with respect to k. We denote by $m_k \colon \mathcal{E}(\Sigma) \to \mathbb{R}$ the function

$$m_k(E) := \int_{\Sigma_k} H_k \, \mathrm{d}a_{I_k} \, .$$

Lemma 5.5.1. *For every $k \in (-1, 0)$, we have*

$$\lambda^{\mathfrak{c}}(X_k) \circ \Phi_k = -\dot{w}_k + \frac{1}{8(k+1)}\, m_k, \tag{5.6}$$
$$\lambda^{\mathfrak{h}}(Y_k) \circ \Psi_k = -2\dot{v}_k^* + \frac{1}{k}\, m_k. \tag{5.7}$$

Proof. Let E be a fixed hyperbolic end. If (c_k, q_k) denotes the point $\Phi_k(E) \in T^*\mathcal{T}^{\mathfrak{c}}(\Sigma)$, then the Liouville form $\lambda^{\mathfrak{c}}$ satisfies

$$\lambda^{\mathfrak{c}}(X_k) \circ \Phi_k(E) = (\lambda^{\mathfrak{c}})_{\Phi_k(E)} \left(\frac{\mathrm{d}}{\mathrm{d}h}\, \Phi_{k+h}(E)|_{h=0} \right) = \operatorname{Re}\langle q_k, \dot{c}_k \rangle.$$

By Proposition 5.2.1, we have

$$\begin{aligned}
\dot{w}_k(E) &= \frac{1}{4} \int_{\Sigma_k} \left(\left(\dot{I\!I}_k, I\!I\!I_k - \frac{H_k}{2} I\!I_k \right)_{I\!I_k} + \frac{H_k}{2(k+1)} \right) \mathrm{d}a_{I_k} \\
&= -\operatorname{Re}\langle q_k, \dot{c}_k \rangle + \frac{1}{8(k+1)} \int_{\Sigma_k} H_k \, \mathrm{d}a_{I_k} \\
&= -\operatorname{Re}\langle q_k, \dot{c}_k \rangle + \frac{1}{8(k+1)}\, m_k(E).
\end{aligned}$$

Combining these two relations we obtain the first part of the statement. Similarly, we see that

$$\lambda^{\mathfrak{h}}(Y_k) \circ \Psi_k(E) = -\frac{\sqrt{k+1}}{k}\, \mathrm{d}(L_{h_k'})_{h_k}(\dot{h}_k).$$

By definition of h_k, we have $\dot{h}_k = -I_k - k\, \dot{I}_k$. Using Lemma 5.2.7, we obtain

$$\begin{aligned}
\lambda^{\mathfrak{h}}(Y_k) \circ \Psi_k(E) &= \frac{1}{2k} \int_{\Sigma_k} (-I_k - k\, \dot{I}_k, I\!I_k - H_k I_k)_{I_k} \, \mathrm{d}a_{I_k} \\
&= -\frac{1}{2} \int_{\Sigma_k} (\dot{I}_k, I\!I_k - H_k I_k)_{I_k} \, \mathrm{d}a_{I_k} + \frac{1}{k} \int_{\Sigma_k} H_k \, \mathrm{d}a_{I_k} \\
&= -2\dot{v}_k^*(E) + \frac{1}{k}\, m_k(E),
\end{aligned}$$

where, in the last step, we used Proposition 2.2.5. □

Lemma 5.5.2. *Let M and N be a n- and a $2n$-manifold, respectively, and let $\varphi_t \colon N \to T^*M$ be a 1-parameter family of diffeomorphisms, indexed by a variable t varying in an open interval J of $\mathbb{R}$. Denote by λ the Liouville form of T^*M, and set V_t to be the vector field of T^*M given by*

$$V_t := \frac{\mathrm{d}}{\mathrm{d}h} \varphi_{t+h} \circ \varphi_t^{-1}\big|_{h=0} ,$$

for any $t \in J$. Then we have

$$(\varphi_t^{-1})^* \left(\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* \lambda \right) = \iota_{V_t} \omega + \mathrm{d}(\iota_{V_t} \lambda) ,$$

for every $t \in J$.

Proof. The statement is a consequence of Cartan formula. The time-dependent family of vector fields $(V_t)_t$ corresponds to a ordinary vector field $\tilde{V}$ on the manifold $J \times T^*M$, by setting

$$\tilde{V}(t, \cdot) := \partial_t + V_t(\cdot) \in T_t J \times T_\cdot(T^*M) \cong T_{(t,\cdot)}(J \times T^*M).$$

An intergral curve $\gamma = \gamma(t)$ of $(V_t)_t$ in T^*M corresponds to the integral curve $t \mapsto (t, \gamma(t))$ of $\tilde{V}$ in $J \times T^*M$. Let π denote the projection of $J \times T^*M$ onto its second component. We apply Cartan formula to the 1-form $\pi^*\lambda$ and the vector field $\tilde{V}$, obtaining

$$\mathcal{L}_{\tilde{V}} \pi^* \lambda = \iota_{\tilde{V}} \, \mathrm{d}(\pi^* \lambda) + \mathrm{d}(\iota_{\tilde{V}} \pi^* \lambda) , \tag{5.8}$$

where $\mathcal{L}_{\tilde{V}} \pi^* \lambda$ denotes the Lie derivative of the 1-form $\pi^*\lambda$ along the vector field $\tilde{V}$. A straightforward computation proves the following relations:

$$\begin{aligned} \iota_{\tilde{V}} \, \mathrm{d}(\pi^*\lambda)|_{(t,\cdot)} &= \pi^*(\iota_{V_t} \mathrm{d}\lambda)|_{(t,\cdot)} , \\ (\iota_{\tilde{V}} \pi^* \lambda)(t, \cdot) &= (\iota_{V_t} \lambda \circ \pi)(t, \cdot), \\ (\varphi_t^{-1} \circ \pi)^* \left(\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* \lambda \right) \bigg|_{(t,\cdot)} &= \mathcal{L}_{\tilde{V}} \pi^* \lambda|_{(t,\cdot)} . \end{aligned}$$

Replacing these expressions in the equation (5.8), we obtain that, for every $t \in J$

$$\pi^* \left((\varphi_t^{-1})^* \left(\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* \lambda \right) - \iota_{V_t} \omega - \mathrm{d}(\iota_{V_t} \lambda) \right) \bigg|_{(t,\cdot)} = 0.$$

Since $\mathrm{d}\pi_{(t,\cdot)}$ is surjective, the pullback by π at $(t, \cdot)$ is injective on k-forms. In particular, for every $t \in J$ we must have

$$(\varphi_t^{-1})^* \left(\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* \lambda \right) - \iota_{V_t} \omega - \mathrm{d}(\iota_{V_t} \lambda) = 0,$$

which proves the statement. □

Theorem F. *For every $k \in (-1, 0)$, the vector field X_k of $T^*\mathcal{T}^{\mathfrak{c}}(\Sigma)$ is Hamiltonian with respect to the symplectic structure $\omega^{\mathfrak{c}}$, with Hamiltonian function $-\frac{1}{8(k+1)} \, m_k \circ \Phi_k^{-1}$. Similarly, the vector field Y_k of $T^*\mathcal{T}^{\mathfrak{h}}(\Sigma)$ is Hamiltonian with respect to the symplectic structure $\omega^{\mathfrak{h}}$, with Hamiltonian function $-\frac{1}{k} \, m_k \circ \Psi_k^{-1}$.*

Proof. From Lemma 5.3.4 we see that

$$\frac{\mathrm{d}}{\mathrm{d}k}\Phi_k^*\lambda^{\mathfrak{c}} = \frac{\mathrm{d}}{\mathrm{d}k}\left[-\,\mathrm{d}w_k + \frac{1}{2}(\mathrm{d}L\circ \mathrm{Th})^*\lambda^{\mathfrak{h}}\right] = -\,\mathrm{d}\dot{w}_k\,. \tag{5.9}$$

Applying Lemma 5.5.2 to $N = \mathcal{E}(\Sigma)$, $M = \mathcal{T}^{\mathfrak{c}}(\Sigma)$ and $\varphi_t = \Phi_k$, we get

$$(\Phi_k^{-1})^*\left(\frac{\mathrm{d}}{\mathrm{d}k}\Phi_k^*\lambda^{\mathfrak{c}}\right) = \iota_{X_k}\omega^{\mathfrak{c}} + \mathrm{d}(\iota_{X_k}\lambda^{\mathfrak{c}})\,. \tag{5.10}$$

Now, putting everything together, we obtain

$$\begin{aligned}
\iota_{X_k}\omega^{\mathfrak{c}} &= (\Phi_k^{-1})^*\left(\frac{\mathrm{d}}{\mathrm{d}k}\Phi_k^*\lambda^{\mathfrak{c}}\right) - \mathrm{d}(\iota_{X_k}\lambda^{\mathfrak{c}}) && \text{(eq. (5.10))}\\
&= -(\Phi_k^{-1})^*\,\mathrm{d}\dot{w}_k - \mathrm{d}\left(-\dot{w}_k\circ\Phi_k^{-1} + \frac{1}{8(k+1)}\,m_k\circ\Phi_k^{-1}\right) && \text{(eq (5.6) and (5.9))}\\
&= -\,\mathrm{d}\big(\dot{w}_k\circ\Phi_k^{-1}\big) + \mathrm{d}\big(\dot{w}_k\circ\Phi_k^{-1}\big) - \frac{1}{8(k+1)}\,\mathrm{d}\big(m_k\circ\Phi_k^{-1}\big)\\
&= -\frac{1}{8(k+1)}\,\mathrm{d}\big(m_k\circ\Phi_k^{-1}\big)\,,
\end{aligned}$$

which proves the first part of the statement. With the exact same strategy we can prove the assertion concerning the vector fields $(Y_k)_k$. □

Remark 5.5.3. It can be easily checked that the choice of the multiplicative constant in the definition of q_k, and consequently of Φ_k, becomes relevant for Theorem F to hold. The same holds for the multiplicative constant in the definition of Ψ_k.

Possible developments

In the following, we describe a series of questions that are related to (or that arise from) our results, on which we hope to work on in the near future.

Prescriptions on k-surfaces

In Chapter 3, and more specifically in the proof of Theorem 3.4.1, we made use of a deep result, due to Labourie and Schlenker, which we recall here:

Theorem 5.5.4 ([Lab92a], [Sch06]). *Let g be a hyperbolic metric on a compact 3-manifold M with smooth and strictly convex boundary. Then the induced metric I on ∂M has Gaussian curvature > -1. Every Riemannian metric on ∂M with Gaussian curvature > -1 is realized as the induced metric on ∂M by a unique hyperbolic metric on M with strictly convex boundary.*

In [Sch06], it is also presented a similar result concerning the third fundamental forms on ∂M:

Theorem 5.5.5 ([Sch06]). *Let g be a hyperbolic metric on a compact 3-manifold M with smooth and strictly convex boundary. Then the third fundamental form $I\!I\!I$ on ∂M has curvature < -1, and its closed geodesics which are contractible in M have length $> 2\pi$. Moreover, every such metric is realized by a unique g.*

In particular, these results imply that, for every $k \in (-1, 0)$, we have:

- for every smooth metric on ∂M with constant Gaussian curvature k, there exists a unique hyperbolic metric on M with such first fundamental form on its boundary (see Theorem 3.4.1);
- for every smooth metric on ∂M with constant Gaussian curvature $\frac{k}{k+1}$, and such that its closed geodesics which are contractible in M have length $> 2\pi$, there exists a unique hyperbolic metric on M with such third fundamental form on its boundary.

Theorems 5.5.4 and 5.5.5 are related to two questions, asked by William Thurston:

Conjecture 1 (Thurston). *Is the space of quasi-Fuchsian manifolds $\mathcal{QF}(\Sigma)$ parametrized by the hyperbolic metrics $(m^+, m^-) \in \mathcal{T}^\mathfrak{h}(\Sigma)^2$ on the boundary of the convex core?*

It is known that every pair of hyperbolic metrics can be realized as the hyperbolic structures on the boundary of the convex core of a quasi-Fuchsian manifold, but the uniqueness has not been proved (or disproved) yet.

Let $\mathcal{FML}_{<\pi}(\Sigma)$ denote the subset of $\mathcal{ML}(\Sigma)^2$ given by the pairs of measured laminations that are filling Σ and without simple closed curves with weight $\geq \pi$.

Conjecture 2 (Thurston)**.** *Is the space of quasi-Fuchsian manifolds $\mathcal{QF}(\Sigma)$ parametrized by the bending measured laminations $(\mu^+, \mu^-) \in \mathcal{FML}_{<\pi}(\Sigma)$ on the boundary of the convex core?*

It has been proved by Bonahon and Otal [BO04] that every pair of filling measured laminations in $\mathcal{FML}_{<\pi}(\Sigma)$ is realizable, and that the uniqueness holds for pairs of rational laminations lying in $\mathcal{FML}_{<\pi}(\Sigma)$.

In light of the analogies between k-surfaces, the boundary at infinity and the boundary of the convex core that we observed in the introduction, a natural question that arises is the following:

Question 1. *Let $k \in (-1, 0)$. Is the space of quasi-Fuchsian manifolds $\mathcal{QF}(\Sigma)$ parametrized by the conformal structures of the second fundamental forms of its k-surfaces?*

An affirmative answer to this question would extend the classical work of Bers [Ber60], which states that the space of quasi-Fuchsian manifolds is parametrized by the pair of conformal structures at infinity. Similarly we can ask:

Question 2. *Let $k \in (-1, 0)$. Is it possible to prescribe the pair of measured foliations of the Hopf differentials q_k associated to the k-surfaces of a quasi-Fuchsian manifold?*

Even in the case of the boundary at infinity, it is not known whether the pair of measured foliations of the Schwarzian at infinity are filling, and which are the candidate necessary and sufficient conditions of the foliations to be realized.

Constant curvature foliations

In [Mon89], Moncrief proved the existence of a Hamiltonian flow corresponding to constant mean curvature (briefly CMC) foliations in constant sectional curvature Lorentzian spacetimes. His method is based on the *ADM formalism* (named after its authors Arnowitt, Deser and Misner), which describes a Hamiltonian formulation of the general theory of relativity [ADM59].

Question 3. *Is it possible to show that the CMC-flow in de-Sitter, Minkowski, and anti-de Sitter spacetimes is Hamiltonian through the study of suitable notions of volumes?*

Moncrief's results were later used by Andersson, Moncrief, and Tromba [AMT97] to develop a proof of the existence of CMC-foliations for those MGHC spacetimes that contain at least one CMC Cauchy surface.

Question 4. *Is it possible to prove the existence of constant Gaussian curvature foliations for de-Sitter, Minkowski, and anti-de Sitter spacetimes, originally due to Barbot, Béguin and Zeghib [BBZ11], using the Hamiltonian k-surface flow described in Section 5.5?*

Such results would clarify the similarities between CMC and constant Gaussian curvature foliations, and they would furnish a simple and unified strategy to approach these problems.

A modification of the argument used in Theorem F proves the existence of a Hamiltonian flow over the space of germs of CMC-surfaces satisfying a suitable natural bound of their principal curvatures. Since a generic quasi-Fuchsian manifold M may possess several minimal surfaces (see [HW13, Section 4]), there is no hope to have a natural and unique foliation by constant mean curvature for any quasi-Fuchsian structure. However, this question, first asked by Thurston, still makes sense when we restrict ourselves to the space of almost Fuchsian manifolds, which possess a unique minimal surface with principal curvatures contained in $(-1, 1)$:

Conjecture 3 (Thurston)**.** *Is an almost-Fuchsian manifold foliated by CMC-surfaces?*

A natural question that arises from this picture is whether an approach similar to the one of Andersson, Moncrief, and Tromba [AMT97] in the Lorentzian context could be developed in the hyperbolic setting, at least to determine sufficient conditions for the existence of CMC-foliations, based on bounds on the geometry of the minimal surfaces.

Extensions to higher Teichmüller theories

A general powerful tool in the study of representations of surface groups is to determine equivariant immersions of surfaces that are natural, in some sense, with respect to the geometry to the target space. A typical "natural condition" to require for these maps is to be minimal. However, there are cases in which the study through minimal surfaces displays complications, as it happens for quasi-Fuchsian manifolds, which many contain several minimal surfaces.

The same phenomenon does not occur for constant Gaussian curvature surfaces: every hyperbolic end possesses exactly one k-surface for every $k \in (-1, 0)$. Equivalently, for every complex projective structure σ on Σ there exists a unique k-surface that is equivariant by the action of the holonomy of σ, which takes value into $\mathbb{P}\,\mathrm{SL}_2(\mathbb{C})$. A possible analogous of k-surfaces for convex projective structures (so for the Hitchin component of $\mathrm{SL}(3, \mathbb{R})$) that seem to be promising are *affine spheres* and *constant affine curvature surfaces*, as described by the works of Labourie [Lab07], and independently of Loftin [Lof01]. A general interesting question is to investigate the connections between these notions, and possibly to determine a proper general framework for such classes of surfaces in higher rank Lie groups.

Para-quaternionic structures on the space of AdS manifolds

A celebrated result by Donaldson [Don03] asserts that the space of almost-Fuchsian manifolds admits a natural hyper-Kähler structure, invariant under the action of the mapping class group. Donaldson's construction proceeds by applying an infinite-dimensional version of the symplectic reduction on the space of sections of a certain bundle over Σ, which possess a formal natural hyper-Kähler structure.

An interesting problem, on which I am working together with Andrea Seppi and Andrea Tamburelli, is to understand whether a similar phenomenon occurs for the space of globally hyperbolic maximal compact Anti-de-Sitter spacetimes. In this setting, the promising notion to look at is the one of *para-hyperKähler structure*, in which a complex structure ($J_1^2 = -id$) coexists with a pair of *para-complex structures* ($J_2^2 = J_3^2 = id$) and a pseudo-Riemannian metric, verifying certain compatibility relations.

Notation

Σ : oriented connected compact smooth surface with empty boundary and of genus $g \geq 2$ ($\overline{\Sigma}$ if endowed with the opposite orientation);

$\mathbb{H}^n$: the hyperbolic space of dimension n;

$\mathrm{d}\mathbb{S}^n$: the de Sitter space of dimension n;

$\mathbb{A}\mathrm{d}\mathbb{S}^n$: the anti de Sitter space of dimension n;

$\mathcal{T}(\Sigma)$: the Teichmüller space of Σ ($\mathcal{T}^{\mathfrak{h}}$ for hyperbolic structures, $\mathcal{T}^{\mathfrak{c}}$ for conformal structures);

I, II, III : the first, second and third fundamental forms;

ℓ_m : the length function over $\mathcal{ML}(\Sigma)$ with respect to the hyperbolic metric m;

L_μ : the length function over $\mathcal{T}^{\mathfrak{h}}(\Sigma)$ of the measured lamination μ;

CM : the convex convex core of M;

$N_\varepsilon X$: the ε-neighbourhood of X;

$S_\varepsilon X$: the ε-equidistant surface from X;

$\mathbb{1}$: the identity endomorphism;

$\mathcal{MCG}(\Sigma)$: the mapping class group of Σ;

$\mathcal{GL}(\Sigma)$: the space of geodesic laminations of Σ;

$\mathcal{ML}(\Sigma)$: the space of measured geodesic laminations of Σ;

$\mathcal{QD}(M)$: the space of quasi-isometric deformations of a complete hyperbolic 3-manifold M;

$\mathcal{QF}(\Sigma)$: the space of quasi-Fuchsian manifolds homeomorphic to $\Sigma \times \mathbb{R}$;

$\mathcal{E}(\Sigma)$: the space of hyperbolic ends homeomorphic to $\Sigma \times (0, \infty)$;

V_R : the renormalized volume function;

V_C : the volume of the convex core function;

V_C^* : the dual volume of the convex core function;

Sch : the Schwarzian parametrization of $\mathcal{E}(\Sigma)$;

Th : the Thurston parametrization of $\mathcal{E}(\Sigma)$;

$\mathrm{ext}_{\mathcal{F}}$: the extremal length function of the measured foliation $\mathcal{F}$.

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