ARITHMETIC BILLIARDS

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ABSTRACT. Arithmetic billiards show a nice interplay between arithmetics and geometry. The billiard table is a rectangle with integer side lengths. A pointwise ball moves with constant speed along segments making a $45°$ angle with the sides and bounces on these. In the classical setting, the ball is shooted from a corner and lands in a corner. We allow the ball to start at any point with integer distances from the sides: either the ball lands in a corner or the trajectory is periodic. The length of the path and of certain segments in the path are precisely (up to the factor $\sqrt{2}$ or $2\sqrt{2}$) the least common multiple and the greatest common divisor of the side lengths.

1. INTRODUCTION

Arithmetic billiards, also known under the name *Paper Pool*, show a nice interplay between arithmetics and geometry. They are a mathematical model for a billiard with which one can visualize the greatest common divisor and the least common multiple of two natural numbers (more general models for billiards exist in the branch of mathematics called dynamical systems).

The billiard table is a rectangle with integer side lengths. The ball is a point that bounces on the billiard sides and moves with constant speed on segments that make a $45°$ angle with the sides. Only the path matters.

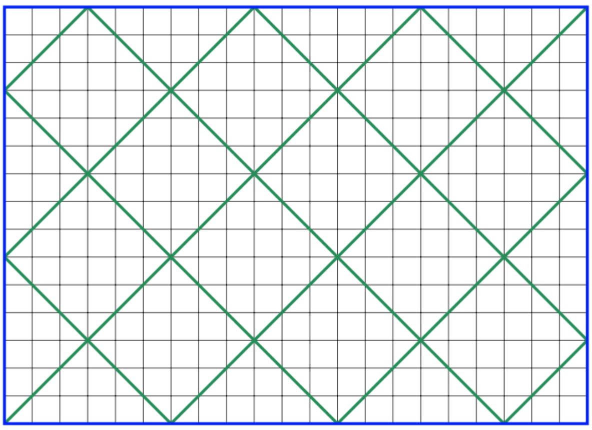

FIGURE 1. Example of corner path for the 21×15 billiard table.

In the classical setting, the ball is shooted from a corner of the billiard table, and the ball necessarily lands in a different corner. We call the resulting path *corner path*: these have been studied by various authors including Martin Gardner [1, 6, 7, 5]. If a and b are the side lengths of the billiard table, then a corner path is the intersection of the billiard table with a grid of squares of side length $\sqrt{2}\,\mathrm{gcd}(a,b)$ (the grid is diagonally oriented, and the starting corner is a vertex for one of the squares), and the length of a corner path is $\sqrt{2}\,\mathrm{lcm}(a,b)$.

We investigate the analogous paths that start at any point of the billiard table with integer distances from the sides. If the starting point does not belong to a corner path, then the ball does not land in a corner but it periodically bounces on the billiard sides: we call such paths *closed paths*.

With a closed path, we can again visualize the greatest common divisor and the least common multiple of the side lengths of the billiard table. Indeed, closed paths have the following properties:

- The length of the path is $2\sqrt{2}\,\mathrm{lcm}(a,b)$.
- The path is the intersection of the billiard table with two grids of squares of side length $\sqrt{2}\,\mathrm{gcd}(a,b)$.
- The two grids are diagonally oriented, and they only differ by a translation parallel to the rectangle sides; vertices for the squares in the grids may be found on the billiard sides.
- The path is symmetric (point-symmetric w.r.t. the center of the billiard table or symmetric with respect to the perpendicular bisector of two billiard sides).

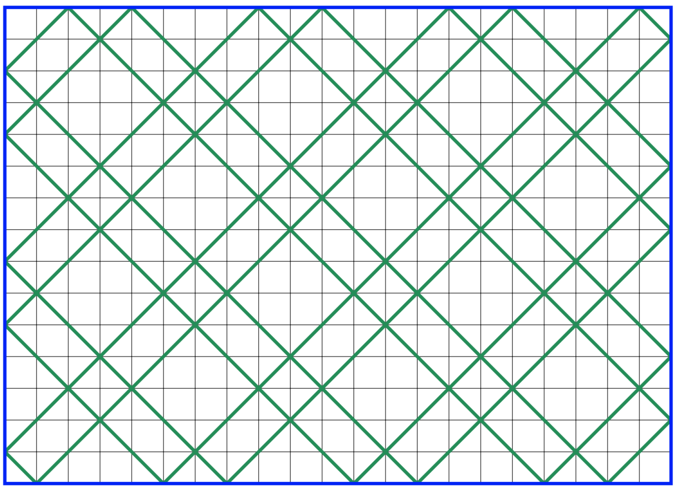

FIGURE 2. Example of closed path for the 21×15 billiard table.

Up to a symmetry of the billiard table, there is exactly one corner path in a given billiard table. However there can be many non symmetric closed paths (and there are no closed paths if a and b are coprime).

- The number of closed paths up to symmetry is the integer part of $\gcd(a, b)/2$.
- Let x be an integer in the range from 1 to $\gcd(a, b) - 1$, and consider the point P_x on a fixed billiard side at distance x from one fixed corner at that side. Then any closed path contains precisely one of the points P_x.

There are several quantities that are the same for all closed paths inside a given billiard table. We have already mentioned the length of the path and the length of the squares in the grid. But there is also — as we will see — the number of *boundary points*, i.e. the points of the path which are on the billiard sides. And the number of *self-intersection points*, i.e. the points where the path crosses itself. Moreover, the path partitions the billiard table into rectangles and triangles: also the number of rectangles and the number of triangles do not depend on the closed path.

Key to our investigation is the following: we can write a formula for the coordinates of the boundary points.

The exploration of arithmetic billiards is a source of activities for pupils [2, 8, 9, 3, 4]. Indeed, the pupils are asked to find out by themselves some of the known results. It is also possible to go one step further and investigate open questions: as a research direction we suggest to consider other shapes for the billiard table (for example, an L-shaped figure with an axial symmetry, or a square with a square hole in the middle).

2. Preliminary remarks, Corner paths

Setting. We fix two positive integers a and b and call g their greatest common divisor. We take as a *billiard table* a rectangle with side lengths a and b, and we choose coordinates by placing the origin in a billiard corner and letting the opposite corner be the point (a, b).

We consider the trajectory of one point (the *ball*) inside the billiard table such that the path consists of segments that make a $45°$ angle with the sides. The speed does not matter, so we may suppose that it is constant. The ball bounces on the billiard sides (making either a left or a right $90°$ turn) and stops only if it lands in a corner. A step in the path results when the ball moves from a point with integer coordinates to the next one (each coordinate changes by 1).

There are *corner paths*, where the ball is shooted from a billiard corner and necessarily lands in a different corner. If we start at a point with integer coordinates that is not on a corner path, then we get a *closed path*, which corresponds to a periodic trajectory.

We call *boundary points* the points of the path which are on the rectangle sides. Most paths have *self-intersection points*, where two path segments cross perpendicularly.

Corner paths. A corner path starts in any billiard corner, and we can predict what the ending corner will be: if a/g and b/g are odd, then the starting corner and the ending corner are opposite; if a/g is even and b/g is odd, then the starting and the ending corner are adjacent to one a-side; if a/g is odd and b/g is even, then the starting and the ending corner are adjacent to one b-side.

Neglecting their orientation, there are two corner paths. Moreover, there is a symmetry of the billiard table mapping one path to the other, namely the symmetry mapping starting and ending corner of one path to those of the other. The length of the path is $\sqrt{2}\,\mathrm{lcm}(a,b)$ (because there are $\mathrm{lcm}(a,b)$ steps) and the path crosses $\mathrm{lcm}(a,b)$ unit squares.

The path is symmetric: if the starting and the ending corner are opposite, then the path is point symmetric w.r.t. the center of the rectangle, else it is symmetric with respect to the perpendicular bisector of the side connecting the starting and the ending corner.

There are a/g boundary points (including the corners) on the two a-sides, and b/g on the two b-sides. Moreover, the boundary points are evenly distributed along the rectangle perimeter: the distance along the perimeter (i.e. possibly going around the corner) between two such neighbouring points equals $2g$.

The corner path starting at the origin is the intersection of the billiard table with the grid of squares whose corners are the points (xg, yg), where x, y are integers with the same parity (the squares are oriented at 45° w.r.t. the billiard sides).

Unless a is a multiple or a divisor of b there are self-intersection points, and more precisely there are $(a-g)(b-g)/2g^2$ of them (to derive this formula consider that every g steps there is a boundary point or a self-intersection point, and we find each self-intersection point twice). Moreover, there are self-intersection points on the first segment of the path, and the ball arrives at such a point after g steps: the least distance between a corner and a self-intersection point is $\sqrt{2}\,\mathrm{gcd}(a,b)$. Unless $a = b$ the integer g is the least distance between a corner and a boundary point which is not a corner (if $a = b$, then the path is just a diagonal of the billiard table).

If we would let the ball bounce at the corners, then a corner path would correspond to a periodic trajectory: the ball would go twice through the path (forwards and backwards) in every period.

3. Boundary points for closed paths

We now turn our attention to closed paths. These do not contain corners, and the ball never stops. The trajectory is periodic because the path consists of finitely many segments, and we concentrate on one period.

Notice that if $g = 1$, then all points in the billiard table with integer coordinates lie on the corner paths and there is no closed path, so *we may suppose that* $g > 1$.

Length of the path, number of boundary points. There are boundary points on each billiard side so we may suppose that the starting point is on the bottom a-side and the starting direction is rightwards, i.e. both coordinates are increasing.

• *The length of a closed path is* $2\sqrt{2}\,\mathrm{lcm}(a,b)$. Indeed, we are back to the bottom a-side after any number of steps which is a multiple of $2b$. Moreover, since we start and end the path by going rightwards, then we can be back to the starting point only after a number of steps which is a multiple of $2a$. So the total number of steps is $2\,\mathrm{lcm}(a,b)$.

• *There are a/g boundary points on each a-side and b/g boundary points on each b-side.* Indeed, we touch one same a-side every $2b$ steps, and $2\,\mathrm{lcm}(a,b)/2b = a/g$. For the b-sides, we reason analogously.

The boundary points. In what follows we determine the set of boundary points. The formulas for the coordinates of these points depend on a, b, and the smallest positive integer r such that the point $(r,0)$ belongs to the path.

Let r be an integer in the range from 1 to $g-1$. The boundary points are as in the following tables, where we specify the x-coordinate for the a-sides and the y-coordinate for the b-sides. Keep in mind that a/g and b/g cannot be both even and that, up to exchanging the role of a and b, we may suppose that a/g is odd.

bottom a-side	$r, 2g-r, \ldots, n2g+r, (n+1)2g-r, \ldots, \frac{a-g}{2g}2g+r$
right b-side	$g-r, g+r, \ldots, n2g+g-r, n2g+g+r, \ldots, \frac{b-g}{2g}2g+g-r$
upper a-side	$g-r, g+r, \ldots, n2g+g-r, n2g+g+r, \ldots, \frac{a-g}{2g}2g+g-r$
left b-side	$r, 2g-r, \ldots, n2g+r, (n+1)2g-r, \ldots, \frac{b-g}{2g}2g+r$

FIGURE 3. Boundary points if a/g and b/g are odd.

bottom a-side	$r, 2g-r, \ldots, n2g+r, (n+1)2g-r, \ldots, \frac{a-g}{2g}2g+r$
right b-side	$g-r, g+r, \ldots, (\frac{b}{2g}-1)2g+g-r, (\frac{b}{2g}-1)2g+g+r$
upper a-side	$r, 2g-r, \ldots, n2g+r, (n+1)2g-r, \ldots, \frac{a-g}{2g}2g+r$
left b-side	$r, 2g-r, \ldots, (\frac{b}{2g}-1)2g+r, (\frac{b}{2g}-1)2g+2g-r$

FIGURE 4. Boundary points if a/g is odd and b/g is even.

The boundary points are thus the points on the sides whose coordinate leaves remainder r or $2g-r$ (respectively, $g-r$ or $g+r$) after division by $2g$. By varying r we obtain all points of the sides whose coordinate is not a multiple of g (those other points lie on the corner paths). Also notice that the boundary points are evenly distributed along the rectangle perimeter (i.e. possibly going around the corner) because the distance between any two of them is alternatively $2r$ and $2g-2r$.

How can we prove that we have written down the correct set of boundary points? Since in the tables we have the correct amount of boundary points, it suffices to show that the next boundary point is again in the set. For example, suppose that a/g and b/g are odd and consider the boundary point $(p,0)$, where p is any integer from 1 to $a-1$ whose remainder after division by $2g$ is r or $2g-r$: the next boundary point can only be one of $(a, a-p)$, $(p \pm b, b)$, $(0,p)$ so it belongs to the given set.

From the distribution of the boundary points, we may deduce that a closed path is symmetric. Indeed, if a/g and b/g are odd, then the path is point-symmetric w.r.t. the center of the billiard table while if w.l.o.g. a/g is odd and b/g is even, then the path is symmetric with respect to the perpendicular bisector of the b-sides.

4. Shape of a closed path

Consider the closed path containing the point $(r, 0)$, where r is an integer from 1 to $g-1$, and recall that we know the boundary points.

Notice that, in the very special case $a = b$, then the path is the rectangle with corners $(r, 0)$, $(a, a-r)$, $(a-r, a)$, $(0, r)$.

The grid structure. The segments which form the closed path are the segments with slope 1 or -1 connecting two boundary points. The distances between parallel path segments are alternatively $\sqrt{2}r$ and $\sqrt{2}(g-r)$. Then the path segments form a grid which partitions the billiard table into squares having side lengths $\sqrt{2}r$ and $\sqrt{2}(g-r)$, rectangles with side lengths $\sqrt{2}r$ and $\sqrt{2}(g-r)$, triangles around the border which are half of one of the squares, and triangles at the corners which are a quarter of one of the squares. We call *corner triangles* the triangles containing the corners and the further triangles along the boundary *side triangles*.

Notice that the path is the intersection with the billiard table of two parallel grids of squares of side length $\sqrt{2}g$, one grid being the vertical shift of the other by $2r$.

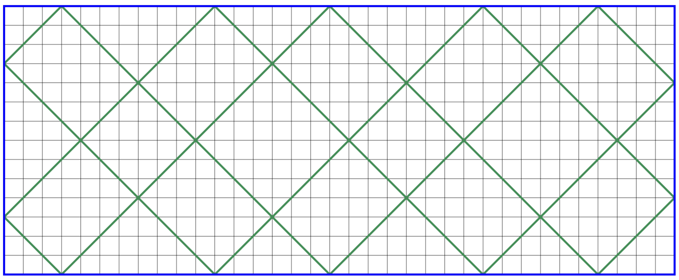

FIGURE 5. Example of closed path for the 35×14 billiard table.

The triangles. In the special case where $r = g/2$ all four corner triangles have legs r. Moreover, all side triangles have hypothenuse g: there are $a/g - 1$ side triangles along each a-side and $b/g - 1$ side triangles along each b-side.

Now suppose that $r \neq g/2$. In this case, two corner triangles have legs r, the other two have legs $g - r$, and we have: if a/g and b/g are odd, then the corner triangles at two opposite corners are congruent; if w.l.o.g. a/g is odd and b/g is even, then the corner triangles adjacent to one same b-side are congruent.

The side triangles have hypothenuse $2r$ and $2g-2r$. There are $a/g-1$ side triangles along each a-side, and their size alternates. If a/g is odd, then there are $(a-g)/2g$ side triangles of each type on each a-side; if a/g is even, then there are $a/g-1$ side triangles of each type on the two a-sides (more precisely there are $a/2g-1$ large side triangles on the side whose triangle corners are small, and $a/2g$ on the other). The analogous formulas hold for the b-sides.

The rectangles. Suppose that $r \neq g/2$. Then, in the partition of the billiard table given by the closed path, there are rectangles that are not squares and there are squares of two sizes. Up to exchanging the role of a and b we suppose that a/g is odd. Consider a stripe of squares whose centers have the same b-coordinate: smaller and larger squares alternate, and there are $(a-g)/2g$ squares of each type. Since there are $(b-g)/g$ stripes, the total number of squares of each type is $(a-g)(b-g)/2g^2$. Moreover, there are ab/g^2 non-square rectangles (consider their centers: there are b/g possibilities for the b-coordinate, and a/g possibilities for the a-coordinate). In total there are

$$\frac{2ab}{g^2}-\frac{a+b}{g}+1$$

rectangles in the partition, and a similar calculation shows that this is also the case if $r=g/2$.

The self-intersection points. If $a=b$, then there are no self-intersection points because the path is a rectangle. On the other hand if $a\neq b$, then there are self-intersection points. Indeed, supposing w.l.o.g. that $a>b$, the path contains the segment from $(r,0)$ to $(r+b,b)$: this segment cuts the billiard table into two parts and there are self-intersection points on it.

We count the self-intersection points together with the boundary points as the vertices of the non-square rectangles in the partition (if $r=g/2$, then we have no such rectangles but we may easily adapt the reasoning). Consider stripes of non-square rectangles whose centers have the same b-coordinate: each stripe contains a/g rectangles and there are b/g such stripes. The rectangles in each stripe have $3a/g+1$ distinct vertices, and any two rectangle stripes have a/g common vertices. We deduce that the total number of vertices is $2ab/g^2+(a+b)/g$ and hence the number of self-intersection points is

$$\frac{2ab}{g^2}-\frac{a+b}{g}\,.$$

We partition the self-intersection points into two easier sets: for one set the x-coordinate (for the other set, the y-coordinate) equals ng, where n ranges from 1 to $a/g-1$ (for the other set, $b/g-1$). The other coordinate of the self-intersection points is

$$g-r, g+r, 3g-r, 3g+r, 5g-r, 5g+r, \ldots \text{(if } n \text{ is odd)}$$

$$r, 2g-r, 2g+r, 4g-r, 4g+r, 6g-r, \ldots \text{(if } n \text{ is even)}\,.$$

The number of closed paths. We have given parametric formulas for the boundary points of a closed path, where the parameter r is in the range from 1 to $g-1$. In this way we can parametrize the closed paths inside a given billiard table. In particular, we can easily deduce that there are $g-1$ closed paths.

For example, if $g=2$, then there is only one closed path, which consists of the grid of squares whose corners are the points (x,y) in the billiard table such that $x+y$ is odd.

However, if we count the closed paths up to a symmetry of the billiard table, then their amount is the integer part of $g/2$. To see this, notice that with a symmetry we may replace r by $g-r$.

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