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THE LOW-DIMENSIONAL ALGEBRAIC COHOMOLOGY OF INFINITE-DIMENSIONAL LIE ALGEBRAS OF VIRASORO-TYPE

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Abstract

In this doctoral thesis, the low-dimensional algebraic cohomology of infinite-dimensional Lie algebras of Virasoro-type is investigated. The considered Lie algebras include the Witt algebra, the Virasoro algebra and the multipoint Krichever-Novikov vector field algebra. We consider algebraic cohomology, meaning we do not put any constraints of continuity on the cochains. The Lie algebras are considered as abstract Lie algebras in the sense that we do not work with particular realizations of the Lie algebras. The results are thus independent of any underlying choice of topology. The thesis is self-contained, as it starts with a technical chapter introducing the definitions, concepts and methods that are used in the thesis. For motivational purposes, some time is spent on the interpretation of the low-dimensional cohomology. First results include the computation of the first and the third algebraic cohomology of the Witt and the Virasoro algebra with values in the trivial and the adjoint module, the second algebraic cohomology being known already. A canonical link between the low-dimensional cohomology of the Witt and the Virasoro algebra is exhibited by using the Hochschild-Serre spectral sequence. More results are given by the computation of the low-dimensional algebraic cohomology of the Witt and the Virasoro algebra with values in general tensor-densities modules. The study consists of a mix between elementary algebra and algorithmic analysis. Finally, some results concerning the low-dimensional algebraic cohomology of the multipoint Krichever-Novikov vector field algebra are derived. The thesis is concluded with an outlook containing possible short-term goals that could be achieved in the near future as well as some long-term goals.

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Contents

Chapter 1

Introduction

Lie algebras of Virasoro type

The Witt and the Virasoro algebra are the simplest non-trivial examples of infinite-dimensional Lie algebras and as such, they have been the object of extensive studies in mathematics for decades. In fact, the Witt algebra was introduced already in 1909 by Cartan [14]. The central extension of the Witt algebra, given by the Virasoro algebra, was first discovered by Gelfand and Fuks in 1968 [45] for characteristic zero, although it already appeared in positive characteristic in an earlier text by Block in 1966 [12]. The Virasoro algebra is also very prominent in physics. In 1970, during his study of so-called dual resonance models appearing in the sector of strong interaction, Virasoro [120] introduced generating operators of the Virasoro algebra, which were later named as the Virasoro operators. Nowadays, the Virasoro algebra is omnipresent in 2-dimensional conformal field theory and it is also of outermost importance in String Theory, see e.g. the book by Kac, Raina and Rozhkovskaya [67]. The Krichever-Novikov vector field algebra is a generalization of the Witt and the Virasoro algebra and was introduced by Krichever and Novikov in 1987 [72–74]. One concrete realization of the Witt algebra is given by the algebra of meromorphic vector fields on the Riemann sphere $\mathbb{CP}^1$ that are holomorphic outside of 0 and ∞. Krichever and Novikov extended this algebra to compact Riemann surfaces with any genus g and two punctures. In 1990, Schlichenmaier [98–101, 110] further generalized amongst others this vector field algebra to compact Riemann surfaces with any genus g and N punctures, see also the book by Schlichenmaier [108]. This generalization is not only of interest to mathematicians, but also to physicists. In the computation of amplitudes in String Theory, the Riemann surface finds an interpretation as being the worldsheet of closed strings. Higher genus Riemann surfaces correspond to loop diagrams and hence higher order contributions. Moreover, the punctures of the Riemann surface correspond to in-going and out-going strings. Therefore, the generalization to N punctures allows the consideration of several in-going and out-going strings instead of only one in-going and one out-going string. Consequently, creation, annihilation and interaction diagrams can be studied via the N-point Krichever-Novikov algebras. Furthermore, these algebras have also relations to moduli spaces, see e.g. [111].

The cohomology of Lie algebras

The cohomology of Lie algebras goes back to a 1948 paper coined by Chevalley and Eilenberg [20], where they reduce topological questions concerning the de Rahm cohomology of compact Lie groups to algebraic questions concerning Lie algebras. The analogy between the de Rahm cohomology of compact groups and Lie algebra cohomology was pursued further by Koszul in

1950 [71]. Nowadays, the cohomology of finite dimensional Lie algebras is fairly well known. For example, two important results are given by Whitehead's lemmas, stating that for semisimple finite-dimensional Lie algebras $\mathfrak{g}$ over fields of characteristic zero, the first and the second cohomology vanish. These results imply that all derivations from $\mathfrak{g}$ into any representation are inner, and that there are no non-trivial extensions of $\mathfrak{g}$, respectively. These statements go back to papers of Whitehead [134, 135], although they also appear in Hochschild's paper [58]. For more details on these historical aspects and the history of homological algebra in general, see e.g. the text by Weibel [132].

The cohomology of infinite-dimensional Lie algebras is a topic with an entirely different flavor. According to the foreword in Fuks' book [43], the cohomology of infinite-dimensional Lie algebras plays a role à part in mathematics, with it not being covered by traditional branches of mathematics, being characterized by relatively elementary proofs, and finding various applications in many domains. Although the cohomology of infinite-dimensional Lie algebras enjoyed a lot of attention for well over half a century in mathematics, it is still an active and on-going research area. For example, the milestone-breaking theorem of Goncharova from 1972 [49–51], giving the cohomology in characteristic zero of Lie algebras of formal vector fields on the line, received a counterpart in characteristic two only in 2017 by Weinstein [133].

The low-dimensional Lie algebra cohomology is of particular interest both to mathematicians and physicists. In mathematics, the low-dimensional cohomology comes with an easy interpretation in terms of known objects such as invariants, derivations, extensions, deformations, obstructions and crossed modules, see Gerstenhaber [46–48]. The knowledge of these objects leads to a better understanding of the Lie algebra under consideration, while deformations and extensions allow the construction of new Lie algebras. Recently, crossed modules were used to classify Lie-2-algebras, see Baez and Crans [10]. Also in physics, these objects find widely applications. In fact, regularization procedures often lead to the necessity to work with projective representations of Lie algebras. However, the situation can be remedied by using Lie representations of central extensions of the Lie algebra, see e.g. Tuynman and Wiegerinck [118]. Besides, the low-dimensional cohomology appears extensively in the study of anomalies, see e.g. Roger [94].

The Lie algebra cohomology studied in the past half century mainly concerns the so-called *continuous* cohomology of Lie algebras. In fact, the infinite-dimensional Lie algebras studied included mostly the infinite-dimensional Lie algebras of smooth vector fields on a smooth manifold X. The space $\mathrm{Vect}(X)$ corresponds to the space of smooth sections of the tangent bundle on X, and is as such a Lie algebra with respect to the commutator bracket. Elements of $\mathrm{Vect}(X)$ can be represented by first-order differential operators in the ring of smooth functions on X; alternatively, they can be viewed as infinitesimal diffeomorphisms of X. In that case, the group $\mathrm{Diff}(X)$ consisting of all diffeomorphisms of X would constitute the Lie group associated to the Lie algebra $\mathrm{Vect}(X)$, although the relationship between Lie algebras and Lie groups is not as tight in the case of infinite-dimensional Lie algebras as in the case of finite-dimensional Lie algebras. When considering the cohomology of Lie algebras of vector fields over some manifold X, it is natural to consider continuous cohomology, i.e. to consider cochains that are continuous with respect to the topology inherited from the underlying space X. Most results involving the cohomology of infinite-dimensional Lie algebras obtained so far thus concern the continuous cohomology, although the specification "continuous" is often omitted in the literature. In the continuous cohomology, many results about the classical infinite-dimensional Lie algebras such as the Witt algebra are known. In fact, a geometrical realization of the Witt algebra is given by the complexified Lie algebra of polynomial vector fields on the circle, which forms a dense

subalgebra of the complexified Lie algebra of smooth vector fields on the circle, Vect(S^1). The continuous cohomology of vector fields on the circle with values in the trivial module is known, see the results by Gelfand and Fuks [43, 45]. Similarly, based on results of Goncharova [49], Reshetnikov [92] and Tsujishita [117], the continuous cohomology of vector fields on the circle with values in general tensor densities modules was computed by Fialowski and Schlichenmaier in [36]. Since the Witt algebra is a dense subalgebra of Vect(S^1), by density arguments, the results for Vect(S^1) are also valid for the Witt algebra, as long as one uses continuous cochains. For more details, see e.g. Lemma 1 in the article by Wagemann, [127], and also [122]. Related work on the cohomology of conformal algebras was done by Bakalov, Kac and Voronov [11]. Concerning the Krichever-Novikov vector field algebra, the continuous cohomology with values in the trivial module is known. It is given by the so-called Feigin-Novikov conjecture, which was proven by contributions from Wagemann [121, 122, 126, 127] and Kawazumi [69].
Although the Witt algebra has some geometrical interpretations in the form of specific vector fields on the circle S^1 or the Riemann sphere $\mathbb{CP}^1$, it also has prominent algebraic realizations, such as the Lie algebra of derivations of Laurent polynomials. The primary definition of the Witt algebra and its related algebras is given by the Lie bracket and is purely algebraic. In such an algebraic set-up, continuous cohomology has severe limitations, see e.g. Wagemann [125], and is mainly inadequate to correctly describe the Lie algebra cohomology. Instead, the *algebraic* cohomology should be used, which contains the continuous cohomology as a sub-complex. In algebraic cohomology, no continuity constraints are put on the cochains, hence the cochains are arbitrary and purely algebraic. In this thesis, we deal with algebraic cohomology; all proofs involve purely algebraic methods and no results from topology are used. Our results are thus independent from any choice of topology and any concrete realization of the Witt algebra or its related algebras. Moreover, our results are valid for any base field $\mathbb{K}$ with characteristic zero, and not just for $\mathbb{R}$ or $\mathbb{C}$. However, while the computation of continuous cohomology can already be heavily involved, the computation of algebraic cohomology is usually much worse. Consequently, results on the algebraic cohomology of infinite-dimensional Lie algebras are somewhat scarce in the literature so far. Concerning the Witt and the Virasoro algebras, Schlichenmaier showed in [105, 106] the vanishing of the second algebraic cohomology of the Witt and the Virasoro algebra with values in the adjoint module by using elementary algebraic methods; almost at the same time, Fialowski independently showed the result for the Witt algebra, also with elementary algebraic methods, see [34]. Without proof, the result was already announced in [33] by Fialowski . In [119] by Van den Hijligenberg and Kotchetkov, the vanishing of the second algebraic cohomology with values in the adjoint module of the superalgebras $k(1)$, $k^+(1)$ and of their central extensions was proved. In [26–28] by Ecker and Schlichenmaier, the first and the third algebraic cohomology of the Witt and the Virasoro algebra with values in the adjoint module was computed. The computation of the third algebraic cohomology of the Witt and the Virasoro algebra with values in the trivial module was also included in [27]. In [25] by Ecker, results related to the first, second and third algebraic cohomology of the Witt and the Virasoro algebra with values in tensor densities modules were announced, without proof. Concerning the Krichever-Novikov vector field algebra, the Feigin-Novikov conjecture has not yet been proven in the algebraic case. So far, the first algebraic cohomology with values in the trivial module was computed, namely by Schlichenmaier, see [108]. In the two-point case, Millionshchikov proved the finite-dimensionality of the second algebraic cohomology with values in the trivial module, see [83]. In [115], Skryabin obtained a break-through by deriving the second algebraic cohomology with values in the trivial module. Moreover, the second algebraic cohomology with values in the trivial module for the so-called bounded and local cochain classes has been com-

puted by Schlichenmaier [102], see also his book [108]. Local cocycles need to be considered when one wants to obtain meaningful central extensions.

Main results

The main results of this thesis consist of the computation of the first algebraic cohomology of the Witt and the Virasoro algebra over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ with values in the adjoint and general tensor-densities modules,

$$\mathrm{H}^1(\mathscr{W},\mathscr{W}) = \{0\} \qquad \dim \mathrm{H}^1(\mathscr{W},\mathscr{F}^\lambda) = \begin{cases} 2 \text{ if } \lambda = 0 \\ 1 \text{ if } \lambda = 1,2 \\ 0 \text{ else} \end{cases},$$

$$\text{and} \qquad \mathrm{H}^1(\mathscr{V},\mathscr{V}) = \{0\} \qquad \dim \mathrm{H}^1(\mathscr{V},\mathscr{F}^\lambda) = \begin{cases} 2 \text{ if } \lambda = 0 \\ 1 \text{ if } \lambda = 1,2 \\ 0 \text{ else} \end{cases},$$

and the computation of the second algebraic cohomology of the Witt and the Virasoro algebra over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ with values in general tensor-densities modules,

$$\dim \mathrm{H}^2(\mathscr{W},\mathscr{F}^\lambda) = \begin{cases} 2 \text{ if } \lambda = 0,1,2 \\ 1 \text{ if } \lambda = 5,7 \\ 0 \text{ else} \end{cases}, \qquad \dim \mathrm{H}^2(\mathscr{V},\mathscr{F}^\lambda) = \begin{cases} 2 \text{ if } \lambda = 1,2 \\ 1 \text{ if } \lambda = 0,5,7 \\ 0 \text{ else} \end{cases}.$$

In case of the Witt algebra, the adjoint module corresponds to $\mathscr{W} = \mathscr{F}^{-1}$. Hence these results generalize earlier results [106] $\mathrm{H}^2(\mathscr{W},\mathscr{W}) = \{0\}$. By some additional work in [106], it was also shown that $\mathrm{H}^2(\mathscr{V},\mathscr{V}) = \{0\}$.
Moreover, we compute the third algebraic cohomology of the Witt and the Virasoro algebra over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ with values in the trivial, the adjoint and some of the general-tensor densities modules,

$$\dim \mathrm{H}^3(\mathscr{W},\mathbb{K}) = 1, \quad \mathrm{H}^3(\mathscr{W},\mathscr{W}) = \{0\}, \qquad \mathrm{H}^3(\mathscr{W},\mathscr{F}^\lambda) = \{0\} \text{ if } \lambda \in I,$$

$$\dim \mathrm{H}^3(\mathscr{V},\mathbb{K}) = 1, \quad \dim \mathrm{H}^3(\mathscr{V},\mathscr{V}) = 1, \quad \mathrm{H}^3(\mathscr{V},\mathscr{F}^\lambda) = \{0\} \text{ if } \lambda \in I,$$

where $I = \{-100,\dots,-1\} \cup \{6,8,10,14,16,18,20,22,24,26\}$. In the case of the non-vanishing cohomologies, explicit generators will be given. Finally, we also derived an upper bound for the dimension of the third algebraic bounded cohomology of the Krichever-Novikov vector field algebra over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ with values in the trivial module,

$$\dim \mathrm{H}^3_b(\mathscr{K}\mathscr{N},\mathbb{K}) \le K,$$

where K stands for the number of in-points. The various notions appearing here will be introduced in detail later, in Chapter 2.

Techniques used in this thesis

The aim of the present doctoral work is to compute the low-dimensional algebraic cohomology of the Witt, the Virasoro, and as far as possible, the Krichever-Novikov vector field algebra, for various modules. Two simple tools will be mainly used, namely the so-called cocycle condition and the coboundary condition. These will be evaluated on combinations of the basis elements of the Lie algebras. Since we are dealing with infinite-dimensional Lie algebras, the analysis boils down to solving infinite dimensional linear systems, i.e. linear systems with infinitely many variables and equations. To solve these systems, we will aim to create recurrence relations in order to find relations between the variables. Ideally, the infinitely many variables can be expressed in terms of a finite number of variables. This finite number of variables should be as small as possible, as they provide an upper bound for the dimension of the cohomologies. In case the dimension of a cohomological space is different from zero, a lower bound has to be found by constructing explicit non-trivial generators of the cohomology space under consideration.

Most of the proofs in this thesis thus will be elementary algebraic manipulations. In this spirit, we aim to keep the text of the thesis as elementary and self-contained as possible. No preliminary knowledge of the Lie algebra cohomology is needed to understand the majority of the proofs. All the basic notions necessary to understand the background will be introduced in Chapter 2. Concerning known results, we will give a proof only if it is short and simple, or important for motivational purposes, otherwise we provide references. Although the proofs given in this thesis use elementary algebra, the proofs per se are not simple, but rather involved and some are computationally quite heavy. In general though, the proofs will be given with enough details so that they can be understood by simply reading them, without having to do too many additional computations or reasoning.

Although we strive to keep the proofs elementary, we also sometimes use higher tools from homological algebra to derive results for the Virasoro algebra from results for the Witt algebra. In all cases, the results for the Virasoro algebra could have been obtained independently of the results for the Witt algebra in exactly the same way as the results for the Witt algebra were obtained, namely by using elementary algebra in a direct, but lengthy, way. This approach was for example chosen for the corresponding superalgebraic case in [119]. However, the proofs would have been very similar to the proofs for the Witt algebra, and hence writing down elementary proofs à part for the Virasoro algebra would be redundant, cumbersome, and not very interesting for the reader. Instead, a neater and more concise way to obtain results for the Virasoro algebra consists in using the results obtained for the Witt algebra. This can also been done on an elementary level, see for example [106] where it was done for the second algebraic cohomology of the Virasoro algebra with values in the adjoint module. We will also use this method in the present thesis to derive the first algebraic cohomology of the Virasoro algebra.
The proof in [106] can be generalized to the third algebraic cohomology in a straightforward manner, without any unforeseen difficulties showing up. However, we prefer using long exact sequences and spectral sequences, as they allow to shorten the proof and also to obtain a deeper insight and better understanding of the situation than elementary tools.

Structure of the thesis

The thesis is organized as follows.
Chapter 2 is a more technical introductory chapter, which will provide further motivation and known results. We will start by introducing the Witt and the Virasoro algebra as well as the Krichever-Novikov vector field algebra. Aside from giving their algebraic definitions, we will also give some geometrical interpretations of these algebras, for reasons of completeness. Next, we introduce the modules of these algebras that will be studied in this thesis, including the trivial module, the adjoint module and general tensor densities modules. Subsequently, we introduce the Chevalley-Eilenberg cohomology for general Lie algebras. For motivational purposes, we will study the low-dimensional cohomology in more detail, and provide some of its main inter-

pretations. Moreover, the notion of the degree of a cochain is given, and the well-known result given in Theorem [43], important for this thesis, is presented, stating that the cohomology reduces to the degree zero cohomology for internally graded Lie algebras and modules. A brief introduction to spectral sequences and the Hochschild-Serre spectral sequence in particular is given. We will end the chapter by giving a brief summary of the results obtained so far for the low-dimensional algebraic cohomology of the Witt and the Virasoro algebra, as well as the Krichever-Novikov vector field algebra.

In Chapter 3, we will focus on the low-dimensional algebraic cohomology of the Witt and the Virasoro algebra with values in the adjoint and the trivial module. We start by computing the third algebraic cohomology for the trivial module, both for the Witt and the Virasoro algebra. Inspired by the results, we investigate relations between the algebraic cohomology of the Witt and the Virasoro algebra, for arbitrary modules. Next, we focus on the adjoint module. As a warm-up example, we start by computing the first algebraic cohomology of the Witt algebra with values in the adjoint module. The second algebraic cohomology being known already, we continue by computing the third algebraic cohomology with values in the adjoint module for the Witt algebra, with elementary algebra. Using results from the Witt algebra and the trivial module, we can derive the first and the third algebraic cohomology with values in the adjoint module for the Virasoro algebra. Finally, we aim to deepen the investigation of the link between the low-dimensional algebraic cohomology of the Witt and the Virasoro algebra, in the particular cases of the trivial and general tensor densities modules.

Chapter 4 deals with the low-dimensional algebraic cohomology of the Witt and the Virasoro algebra with values in general tensor densities modules. We will first prove the results for the Witt algebra. More precisely, we will compute the first, the second, and partially the third algebraic cohomology. For the third algebraic cohomology, we only have results for some specific tensor densities modules, and not for all of them. The proof of the results for the third algebraic cohomology is very technical, involving an algebraic and a numerical part, with the algebraic part being already 37 pages long. We aim to present it in a way as comprehensive as possible in the main text, although some complementary information can be found in the Appendix A. Concerning the Virasoro algebra, the results can be deduced from the results for the Witt algebra by using the link between their cohomology obtained in Chapter 3.

Chapter 5 deals exclusively with the Krichever-Novikov vector field algebra. In fact, Chapter 5 gives the zeroth algebraic cohomology with values in the general tensor-densities modules, as well as a result for the third algebraic cohomology for bounded cochains with values in the trivial module, of the Krichever-Novikov vector field algebra.

In the conclusion 6, we will provide a summary and an outlook for future work. We will point out some results that possibly could be achieved on the short term, completing some existing results of the thesis. We also offer some other long-term challenges that probably will need more investment in order to be accomplished.

The Appendix A contains some supplementary information concerning the second and the third algebraic cohomology of the Witt algebra with values in general tensor densities modules. More precisely, we provide solutions to recurrence relations showing up in the proofs. The solutions were found by guess.

Finally, we want to point out that parts of the present thesis are already publicly accessible under the form of working papers e-prints [26, 27], proceedings [25, 28] and [29]. The results concerning the Krichever-Novikov vector field algebra in Chapter 5 are not yet publicly available. Also, the results showing up in Chapter 3 related to the differential of the second page of the Hochschild-Serre spectral sequence and the cup product have not been published before. We would also like to point out that the results concerning the first and the third algebraic cohomology of the Witt algebra with values in the adjoint module have already been used by Camacho, Omirov and Kurbanbaev [13] and by Feldvoss and Wagemann [30] to compute the corresponding Leibniz cohomologies.

Chapter 2

The cohomology of infinite-dimensional Lie algebras

2.1 The Witt, the Virasoro and the Krichever-Novikov vector field algebra

In this section, we will introduce the Lie algebras considered in this thesis.

2.1.1 Basic definitions

First of all, we recall some basic definitions concerning Lie algebras, starting with the definition of a Lie algebra. Further definitions will be provided throughout the thesis as needed.

Definition 2.1.1. A *Lie algebra* $\mathcal{L}$ is a vector space over a field $\mathbb{K}$ with $char(\mathbb{K}) \neq 2$ and with a bilinear product $[\cdot,\cdot]$, called *Lie bracket*, satisfying the following properties:

- Skew-symmetry:
$$\forall\ x, y \in \mathcal{L}: \qquad [x, y] = -[y, x]\,, \tag{2.1}$$

- Jacobi identity:
$$\forall\ x, y, z \in \mathcal{L}: \qquad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0\,. \tag{2.2}$$

The *dimension* of a Lie algebra is its dimension as a vector space over $\mathbb{K}$.
A Lie algebra is called *abelian* if its Lie bracket vanishes for all elements in the Lie algebra, i.e. $\forall\ x, y \in \mathcal{L} : [x, y] = 0$.

Next, let us introduce the notion of a Lie algebra homomorphism.

Definition 2.1.2. Let $(\mathcal{L},[\cdot,\cdot])$ and $(\mathcal{L}',[\cdot,\cdot]')$ be two Lie algebras. A linear map $\phi : \mathcal{L} \to \mathcal{L}'$ is called a *Lie homomorphism* if it respects the Lie algebra structure:

$$\forall\ x, y \in \mathcal{L}: \qquad [\phi(x),\phi(y)]' = \phi([x, y])\,.$$

We will also need the notion of a a Lie subalgebra and a Lie ideal.

Definition 2.1.3. Let $\mathcal{L}$ be a Lie algebra.

- A vector subspace $\mathscr{L}'$ of $\mathscr{L}$ is a *Lie subalgebra* of $\mathscr{L}$ if it is closed under the Lie bracket, i.e.

$$[\mathscr{L}', \mathscr{L}'] \subseteq \mathscr{L}' .$$

- A vector subspace $\mathscr{L}'$ of $\mathscr{L}$ is *Lie ideal* of $\mathscr{L}$ if

$$[\mathscr{L}, \mathscr{L}'] \subseteq \mathscr{L}' .$$

A Lie subalgebra is a Lie algebra, and a Lie ideal is a Lie subalgebra. If $\mathscr{I}$ is a Lie ideal of a Lie algebra $\mathscr{L}$, then the quotient $\mathscr{L}/\mathscr{I}$ carries in a natural way a Lie structure and the natural map $\pi : \mathscr{L} \to \mathscr{L}/\mathscr{I}$ is a Lie homomorphism.
Prominent Lie ideals are given by the center of a Lie algebra and the derived subalgebra of an ideal.

Definition 2.1.4.

- The *center* of a Lie algebra $\mathscr{L}$ is the Lie ideal defined by:

$$C(\mathscr{L}) := \{x \in \mathscr{L} \mid \forall\ y \in \mathscr{L} : [x, y] = 0\} . \tag{2.3}$$

- The *derived subalgebra* $\tilde{\mathscr{L}}$ of a Lie algebra $\mathscr{L}$ is the Lie ideal defined by:

$$\tilde{\mathscr{L}} := [\mathscr{L}, \mathscr{L}] := \langle [x, y] \mid x, y \in \mathscr{L} \rangle_{\mathbb{K}} . \tag{2.4}$$

It is clear that the center and the derived subalgebras are Lie ideals of the Lie algebra. Another notion we will need is the concept of simple and perfect Lie algebras.

Definition 2.1.5.

- A Lie algebra $\mathscr{L}$ is called *simple* if and only if $\mathscr{L}$ is not abelian and it has no non-trivial ideals, meaning that if $\mathscr{J}$ is an ideal of $\mathscr{L}$, then either $\mathscr{J} = \{0\}$ or $\mathscr{J} = \mathscr{L}$. Since the center and the derived subalgebra are Lie ideals, they must be trivial for a simple Lie algebra, i.e. $C(\mathscr{L}) = \{0\}$ and $[\mathscr{L}, \mathscr{L}] = \mathscr{L}$.

- A *perfect* Lie algebra is a Lie algebra satisfying:

$$[\mathscr{L}, \mathscr{L}] = \mathscr{L} . \tag{2.5}$$

  Clearly, every simple Lie algebra is perfect.

An concept of outermost importance in this thesis is given by the notion of graded Lie algebras.

Definition 2.1.6.

- A Lie algebra $\mathscr{L}$ is called a $\mathbb{Z}$-*graded Lie algebra* if there exist vector subspaces $\mathscr{L}_n \in \mathscr{L}$ $\forall n \in \mathbb{Z}$ such that $\mathscr{L} = \bigoplus_{n\in\mathbb{Z}} \mathscr{L}_n$ as a vector space and

$$[\mathscr{L}_n, \mathscr{L}_m] \subseteq \mathscr{L}_{n+m}, \quad \forall\ n, m \in \mathbb{Z} . \tag{2.6}$$

- The subspaces $\mathscr{L}_n$ are called *homogeneous subspaces*, and the elements of $\mathscr{L}_n$ are called *homogeneous elements of degree n.*

- If there exists $M \in \mathbb{N}$ such that $\dim \mathscr{L}_n < M\ \forall\ n \in \mathbb{Z}$, we say that the Lie algebra $\mathscr{L}$ is *strongly graded.*

An example of a grading is given by the *trivial grading*:

$$\begin{cases} \mathscr{L}_0 := \mathscr{L} \\ \mathscr{L}_n := \{0\} \ \forall \ n \neq 0 \end{cases} . \tag{2.7}$$

Clearly, every Lie algebra is graded with respect to the trivial grading, and every finite dimensional Lie algebra is also strongly graded with respect to the trivial grading. The case of interest is given by infinite-dimensional Lie algebras, since the grading allows to decompose them into a sum of finite-dimensional parts. The Witt and the Virasoro algebra are examples of such Lie algebras, as we will see in the following sections. A weaker concept, which is relevant for the Krichever-Novikov type algebras, is given by the concept of almost grading.

Definition 2.1.7. A Lie algebra $\mathscr{L}$ is called an *almost-graded Lie algebra* if

1. $\mathscr{L}$ can be decomposed as $\mathscr{L} = \bigoplus_{n\in\mathbb{Z}} \mathscr{L}_n$ as a vector space,
2. $\dim \mathscr{L}_n < \infty \ \forall \ n \in \mathbb{Z}$,
3. There exist constants L_1 and L_2 such that:

$$[\mathscr{L}_n, \mathscr{L}_m] \subseteq \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathscr{L}_h, \qquad \forall \ n, m \in \mathbb{Z}. \tag{2.8}$$

The subspaces $\mathscr{L}_n$ are called *homogeneous subspaces*, and the elements of $\mathscr{L}_n$ are called *homogeneous elements of degree n*.
If there exists $M \in \mathbb{N}$ such that $\dim \mathscr{L}_n < M \ \forall \ n \in \mathbb{Z}$, we say that the Lie algebra $\mathscr{L}$ is *strongly almost-graded*.
If we drop condition 2. above, i.e. not all homogeneous subspaces are finite-dimensional, we call the Lie algebra *weakly almost-graded*.

Moreover, we will need the notion of Lie modules.

Definition 2.1.8.

- Let $\mathscr{L}$ be a Lie algebra and M a vector space over $\mathbb{K}$. A *Lie action* of $\mathscr{L}$ on M is a bilinear map $\mathscr{L} \times M \to M$, $(x, m) \mapsto x \cdot m$ fulfilling:

$$\forall \ x, y \in \mathscr{L}, \ \forall \ m \in M: \qquad [x, y] \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m). \tag{2.9}$$

  In that case, the vector space M is called a *$\mathscr{L}$-module*.

- A *trivial* $\mathscr{L}$-module is a module M on which $\mathscr{L}$ acts as zero, i.e. $x \cdot m = 0 \ \forall \ x \in \mathscr{L}, \forall \ m \in M$.

- A Lie module M is *graded* if $M = \bigoplus_{n\in\mathbb{Z}} M_n$ as a vector space and

$$\mathscr{L}_m \cdot M_n \subseteq M_{n+m}, \quad \forall \ n, m \in \mathbb{Z}. \tag{2.10}$$

  The subspaces M_n are called *homogeneous subspaces*, and the elements of M_n are called *homogeneous elements of degree n*.
  If there exists $N \in \mathbb{N}$ such that $\dim M_n < N \ \forall \ n \in \mathbb{Z}$, we say that the Lie module M is *strongly graded*.

- A Lie module M is called an *almost-graded Lie module* if

  1. M can be decomposed as $M = \bigoplus_{n\in\mathbb{Z}} M_n$ as a vector space,

2. $\dim M_n < \infty \ \forall \ n \in \mathbb{Z}$,
3. There exist constants M_1 and M_2 such that:

$$[M_n, M_m] \subseteq \bigoplus_{h=n+m-M_1}^{n+m+M_2} M_h, \qquad \forall \ n, m \in \mathbb{Z}. \tag{2.11}$$

The subspaces M_n are called *homogeneous subspaces*, and the elements of M_n are called *homogeneous elements of degree n*.
If there exists $N \in \mathbb{N}$ such that $\dim M_n < N \ \forall \ n \in \mathbb{Z}$, we say that the Lie module M is *strongly almost-graded*.
If condition 2. is dropped, we call it *weakly almost-graded*.

An example of a grading for a Lie module is again given by the trivial grading (2.7).
Next, we will need the notion of a derivation of a Lie algebra $\mathscr{L}$.

Definition 2.1.9. Let $\mathscr{L}$ be a Lie algebra over $\mathbb{K}$, and M a $\mathscr{L}$-module. A $\mathbb{K}$-linear map $\phi : \mathscr{L} \to M$ is called a *derivation* of $\mathscr{L}$ into a $\mathscr{L}$-module if and only if:

$$\forall \ x, y \in \mathscr{L}: \qquad \phi([x, y]) = x \cdot \phi(y) - y \cdot \phi(x). \tag{2.12}$$

We will denote the space of derivations of $\mathscr{L}$ by:

$$\text{Der}\,(\mathscr{L}, M) := \{\phi : \mathscr{L} \to M \mid \phi \text{ is a derivation of } \mathscr{L}\}.$$

By the very definition of a module (2.9), the module action is always a derivation,

$$\forall a \in M: \qquad \phi_a : \mathscr{L} \to M, \quad x \mapsto x \cdot a. \tag{2.13}$$

Such a derivation is called an *inner* derivation.

In the case $M = \mathscr{L}$ we simply write Der $\mathscr{L}$ for the derivations Der $(\mathscr{L}, \mathscr{L})$. It can be shown that Der $\mathscr{L}$ is a Lie algebra with respect to the commutator bracket of linear maps.
A prominent example of inner derivations is given by the inner derivations of Der $\mathscr{L}$, which correspond to the adjoint action.

Definition 2.1.10. In a Lie algebra $\mathscr{L}$, the *adjoint action* ad_y defined by an element $y \in \mathscr{L}$ is given by,

$$\text{ad}_y : \mathscr{L} \to \mathscr{L}, \qquad x \mapsto [y, x]. \tag{2.14}$$

Finally, let us introduce the notions of a direct sum and semi-direct sum of Lie algebras.

Definition 2.1.11.
- The *direct sum* $\mathscr{L}_1 \oplus \mathscr{L}_2$ of two Lie algebras $(\mathscr{L}_1, [\cdot,\cdot]_1)$ and $(\mathscr{L}_2, [\cdot,\cdot]_2)$ is given by the direct sum $\mathscr{L}_1 \oplus \mathscr{L}_2$ on the level of vector spaces equipped with the following Lie bracket $[\![\cdot,\cdot]\!]$,

$$\text{for } \ x_1, y_1 \in \mathscr{L}_1, \ x_2, y_2 \in \mathscr{L}_2: \quad [\![(x_1, x_2), (y_1, y_2)]\!] := \big([x_1, y_1]_1, [x_2, y_2]_2\big).$$

- Let $(\mathscr{L}_1, [\cdot,\cdot]_1)$ and $(\mathscr{L}_2, [\cdot,\cdot]_2)$ be two Lie algebras, with $\mathscr{L}_2$ acting on $\mathscr{L}_1$ as a derivation $\eta : \mathscr{L}_2 \to \text{Der}\,\mathscr{L}_1, \ x_2 \mapsto \eta(x_2)$ with $\eta(x_2) \in \text{Der}\,\mathscr{L}_1$. The *semi-direct sum* $\mathscr{L}_1 \rtimes \mathscr{L}_2$ is given by the direct sum $\mathscr{L}_1 \oplus \mathscr{L}_2$ on the level of vector spaces equipped with the following Lie bracket $[\![\cdot,\cdot]\!]$,

$$[\![(x_1, x_2), (y_1, y_2)]\!] := \big([x_1, y_1]_1 + \eta(x_2)(y_1) - \eta(y_2)(x_1), [x_2, y_2]_2\big).$$

A straightforward verification shows that $\mathscr{L}_1 \oplus \mathscr{L}_2$ and $\mathscr{L}_1 \rtimes \mathscr{L}_2$ are Lie algebras. In case the action of $\mathscr{L}_2$ on $\mathscr{L}_1$ is trivial, i.e. given by the zero map, the semidirect sum reduces to the direct sum.
We now have the necessary ingredients to introduce the Lie algebras and Lie modules considered in this thesis.

2.1.2 The Witt algebra

For the presentation of the Witt algebra in this section, we mostly follow the lecture notes written by Iena, Leytem and Schlichenmaier [65].
On the level of vector spaces, the Witt algebra $\mathcal{W}$ is generated over a base field $\mathbb{K}$ with characteristic zero by infinitely many basis elements $\{e_n \mid n \in \mathbb{Z}\}$, i.e. $\mathcal{W} = \langle e_n \rangle_{\mathbb{K}}$. On the level of Lie algebras, the Witt algebra is defined by the following Lie algebra structure equation:

$$[e_n, e_m] = (m - n) e_{n+m}, \qquad n, m \in \mathbb{Z}, \tag{2.15}$$

This is the most general and abstract definition of the Witt algebra. In the literature, this version of the Witt algebra is also sometimes called the *two-sided* Witt algebra, or Witt algebra *on the circle*. If one considers only modes bigger or equal to minus one, i.e. $m, n \geq -1$ in Equation (2.15), one refers to the algebra as the *one-sided* Witt algebra, or the Witt algebra *on the line* in the case of continuous cohomology.

The Witt algebra is a strongly $\mathbb{Z}$-graded Lie algebra. In fact, the *degree* of an element e_n can be defined by its index as follows: $\deg e_n := n$. With this definition, the Witt algebra becomes a $\mathbb{Z}$-graded Lie algebra and can be decomposed into infinitely many one-dimensional homogeneous subspaces $\mathcal{W}_n$, i.e. $\mathcal{W} = \bigoplus_{n\in\mathbb{Z}} \mathcal{W}_n$ on the level of vector spaces. Each subspace $\mathcal{W}_n$ is generated as a vector space over a field $\mathbb{K}$ with characteristic zero by one basis element e_n. Obviously, the Lie product between elements of degree n and of degree m produces an element of degree $n+m$, hence (2.6) is fulfilled. Moreover, as each subspace is one-dimensional and thus finite-dimensional, we have that the Witt algebra is indeed a strongly graded Lie algebra.

The grading of the Witt algebra has a peculiarity in that it is given by one of its own elements, namely $e_0 \in \mathcal{W}$. More precisely, from the Witt structure equation (2.15) we see that $[e_0, e_n] = n\, e_n$. This can be given a more concrete meaning. Let us first consider the adjoint action on $\mathcal{W}$ with respect to e_0, which gives according to Definition 2.1.10:

$$\begin{aligned} \mathrm{ad}_{e_0} := [e_0, \cdot] : \ \ & \mathcal{W} \to \mathcal{W} \\ & e_n \to n\, e_n \end{aligned} . \tag{2.16}$$

From $\mathrm{ad}_{e_0}(e_n) = n\, e_n$, we see that e_n is an eigenvector of ad_{e_0} with eigenvalue n. The associated eigenspace corresponds to $\mathcal{W}_n$. In other words, the grading of the Witt algebra corresponds to the eigenspace decomposition of the ad_{e_0}-action on $\mathcal{W}$. The Witt algebra is thus a so-called *internally* graded Lie algebra, as its grading is ensured by one of its own elements.
The Witt algebra is a perfect Lie algebra, i.e. it fulfills (2.5). In fact, all elements e_n with $n \neq 0$ can be easily obtained using e_0 and (2.15), while e_0 can also easily be obtained using for example the pair $e_1,\ e_{-1}$:

$$e_n = \frac{1}{n}[e_0, e_n] \quad n \neq 0, \qquad e_0 = \frac{1}{2}[e_{-1}, e_1]. \tag{2.17}$$

The Witt algebra is also a simple Lie algebra.

In this thesis, we consider the elements e_n as abstract symbolic basis elements satisfying (2.15), meaning we do not work with concrete realizations of the Witt algebra. Nevertheless, we will give a brief description of these in the following, in order to provide some intuitive understanding of the Witt algebra. There are three popular realizations of the Witt algebra, insofar that we can assign concrete objects to the basis elements e_n fulfilling (2.15). One realization is an algebraic one, while the other two realizations are more of an geometrical order.
Let us start with the algebraic realization. Consider the infinite-dimensional $\mathbb{K}$-algebra of Laurent polynomials $\mathcal{A} := \mathbb{K}[Z, Z^{-1}]$. Elements $f \in \mathcal{A}$ of this algebra write as $f = \sum_{n\in\mathbb{Z}} a_n Z^n$, where

only finitely many $a_n \in \mathbb{K}$ are non-zero. This algebra is an associative, commutative algebra with unit. The Witt algebra can then be defined as $\mathscr{W} := \mathrm{Der}\,\mathscr{A}$, where Der is the space of derivations, see Definition 2.1.9. In this realization, the basis elements of the Witt algebra can be written as:

$$e_n = Z^{n+1}\frac{d}{dZ},$$

where $\frac{d}{dZ}$ is the formal derivative with respect to the formal variable Z, defined as $\frac{d}{dZ} : \mathscr{A} \to \mathscr{A} : Z^n \mapsto n\, Z^{n-1}$. This expression for the basis elements satisfies Equation (2.15) when the Lie bracket is taken as the commutator bracket of linear maps. Therefore, the definition $\mathscr{W} := \mathrm{Der}\,\mathscr{A}$ provides indeed a sensible realization of the Witt algebra.

Secondly, a geometrical realization of the Witt algebra is given by the algebra of meromorphic vector fields on the Riemann sphere $\mathbb{CP}^1$ that are holomorphic outside of 0 and ∞. This algebra is a Lie algebra when equipped with the commutator bracket of vector fields. In fact, it can be shown that this Lie algebra is isomorphic to $\mathrm{Der}\,\mathscr{A}$. In this realization, the generators of the Witt algebra are given by:

$$e_n = z^{n+1}\frac{d}{dz},$$

where z is the quasi-global complex coordinate, and $\frac{d}{dz}$ the usual derivative of functions on $\mathbb{CP}^1$. Clearly this realization of the basis elements fulfills Equation (2.15) when the Lie bracket is taken to be the commutator bracket of vector fields.

Finally, another geometrical realization of the Witt algebra can be obtained by complexifying $Vect_{pol}(S^1)$, the Lie algebra of polynomial vector fields on the circle S^1, the Lie bracket being given by the commutator bracket of vector fields. The Lie algebra $Vect_{pol}(S^1) \otimes_{\mathbb{R}} \mathbb{C}$ forms a dense Lie subalgebra of $Vect(S^1) \otimes_{\mathbb{R}} \mathbb{C}$, the Lie algebra of complexified vector fields on the circle S^1. By "polynomial vector fields" we mean the subset of vector fields the decomposition of which into Fourier series is finite, i.e. only finitely many Fourier coefficients will be non-zero. In this realization, the generators of the Witt algebra are given by:

$$e_n = -i e^{in\varphi}\frac{d}{d\varphi},$$

where φ is the angle coordinate along S^1, and $\frac{d}{d\varphi}$ the usual derivative of complexified functions on S^1. Again, this realization of the basis elements fulfills Equation (2.15) when the Lie bracket is taken to be the commutator bracket of vector fields.

2.1.3 The Virasoro algebra

In this section we will introduce the Virasoro algebra. The Virasoro algebra is given by the following sequence of Lie algebra homomorphisms,

$$0 \longrightarrow \mathbb{K} \xrightarrow{\;i\;} \mathscr{V} \xrightarrow{\;\pi\;} \mathscr{W} \longrightarrow 0, \tag{2.18}$$

where $i(\mathbb{K})$ is the center of $\mathscr{V}$. Here, $\mathbb{K}$ is viewed as a trivial Lie algebra with trivial Lie bracket $[\cdot,\cdot]_{\mathbb{K}} = 0$. This sequence is *exact*, meaning that the kernel of each homomorphism is equal to the image of the preceding homomorphism, e.g. $\ker\pi = \mathrm{im}\, i$. We will introduce these notions properly later in Section 2.2.3.

On the level of vector spaces, the Virasoro algebra $\mathscr{V}$ is given as a direct sum $\mathscr{V} = \mathbb{K} \oplus \mathscr{W}$. This results from exactness and a generalization of the Rank Nullity Theorem from linear algebra, the Splitting Lemma, see Remarks 2.2.3 and 2.2.2 given later. Of course, this is only true at the level of vector spaces, it is not true at the level of Lie algebras, as we will see later in Section 2.2.4. However, note that we have $\mathscr{W} \cong \mathscr{V}/_{\mathbb{K}}$ as Lie algebras, which is a consequence of the First Isomorphism Theorem 2.2.5 of Section 2.2.4.

The basis elements of $\mathscr{V}$ are chosen as $\hat{e}_n := (0, e_n)$ and the central element as $t := (1,0)$. Different choices are possible, but they lead to equivalent results, as we will see in Section 2.2.4, in particular in Remark 2.2.2. The defining Lie structure equation is given by:

$$\begin{aligned} [\hat{e}_n, \hat{e}_m] &= (m-n)\hat{e}_{n+m} + \alpha(e_n, e_m)\cdot t \qquad n, m \in \mathbb{Z}, \\ [\hat{e}_n, t] &= [t,t] = 0, \end{aligned} \tag{2.19}$$

where α is an antisymmetric bilinear map $\alpha : \mathscr{W} \wedge \mathscr{W} \to \mathbb{K}$ called Virasoro 2-cocycle, which can be represented by:

$$\alpha(e_n, e_m) = -\frac{1}{12}(n^3 - n)\delta_{n+m,0}. \tag{2.20}$$

The cubic term n^3 is the most important term, while the linear term n is a so-called coboundary term. In the literature, the linear term is thus sometimes omitted. The symbol $\delta_{i,j}$ is the Kronecker Delta, defined as being one if $i = j$ and zero otherwise. The factor $-\frac{1}{12}$ is traditionally chosen in accordance with zeta function regularization methods, see e.g. the book by Green, Schwarz and Witten [52]. We will not derive the Virasoro 2-cocycle explicitly, although it shows up implicitly in some later proofs. For details about its derivation, see e.g. [67].

By defining $deg(\hat{e}_n) := deg(e_n) = n$ and $deg(t) := 0$, the Virasoro algebra becomes also an internally $\mathbb{Z}$-graded Lie algebra.

Moreover, the Virasoro algebra is also a perfect Lie algebra. In fact, the elements $\hat{e}_n$ can be obtained as in Equation (2.17), while the central element t can be obtained for example with the following linear combination:

$$t = 6\,[\hat{e}_{-2}, \hat{e}_2] + 12\,[\hat{e}_1, \hat{e}_{-1}].$$

Contrary to the Witt algebra, the Virasoro algebra is not a simple Lie algebra, as its center is not trivial.

For more details on the Witt and the Virasoro algebras, we refer the reader for example to the book by Guieu and Roger [53].

2.1.4 The Krichever-Novikov vector field algebra

For an introduction to Krichever-Novikov type algebras, we refer the reader to the review by Schlichenmaier [107]. For a complete treatment of the subject, we refer the reader to the book by Schlichenmaier [108]. There are several different Krichever-Novikov type algebras associated to the various classical algebras such as current algebras, affine Lie algebras or superalgebras. Here, we will consider the Krichever-Novikov algebra associated to the Witt algebra, which we will refer to as Krichever-Novikov vector field algebra and denote by $\mathscr{KN}$.

As mentioned already in the introduction, the Krichever-Novikov vector field algebra is a generalization of the Witt algebra to Riemann surfaces of genus g and N punctures. While the Witt algebra can be realized as the Lie algebra of meromorphic vector fields on the Riemann sphere $\mathbb{CP}^1$ that are holomorphic outside of 0 an ∞, the Krichever-Novikov vector field algebra is realized as meromorphic vector fields on a compact Riemann surface of genus g and N punctures, that are holomorphic outside of the N marked points. Let us split the set of the N

punctures into two non-empty disjoint sets of punctures I and O, called the *in-points* and the *out-points*, respectively. We denote by K the number of punctures in the set of in-points, and by M the number of punctures in O. More precisely, we consider disjoint ordered tuples of punctures,

$$I = (P_1, \ldots, P_K) \qquad \text{and} \qquad O = (P_1, \ldots, P_M), \tag{2.21}$$

with $N = K + M$. This splitting is crucial for the introduction of an almost grading. Different splittings will give rise to different non-equivalent almost-grading structures. In the case of the Witt algebra, only two punctures are present given by the points 0 and ∞, hence only one choice for the splitting is possible. Consequently, only one almost-grading is present, which is in fact a grading. For most representation theoretic constructions, the almost-grading of the Krichever-Novikov type algebras is sufficient. Due to the presence of a finite number of non-equivalent almost-gradings, new interesting phenomena show up, compared to the Witt algebra. In string theory, the in-points correspond to in-coming free particle strings, that enter the region of interaction. Similarly, the out-points correspond to particle strings leaving the region of interaction. Hence, this splitting appears quite naturally in physics.

There exist meromorphic vector fields $\{e_{n,p} \mid n \in \mathbb{Z}, p = 1, \ldots, K\}$ such that the zero-order[1] at the point $P_i \in I = \{P_1, \ldots, P_k\}$ of the element $e_{n,p}$ is given by,

$$\mathrm{ord}_{P_i}(e_{n,p}) = n + 2 - \delta_{p,i}, \qquad i = 1, \ldots, K,$$

for the in-points, and some compensating orders at the out-points. Moreover, fixing local coordinates z_i centered at P_i for $i \in \{1, \ldots, K\}$, the basis elements $e_{n,p}$ need to satisfy,

$$e_{n,p} = z_p^{n+1}(1 + \mathcal{O}(z_p))\frac{d}{dz_p} \qquad n \in \mathbb{Z}, \quad p = 1, \ldots, K, \tag{2.22}$$

where $\mathcal{O}(z_p)$ denotes terms of order z_p and higher. The existence of objects $e_{n,p}$ with these properties can be shown by using methods from algebraic geometry, see [108]. The Krichever-Novikov vector field algebra has as basis this set of meromorphic vector fields $\{e_{n,p} \mid n \in \mathbb{Z}, p = 1, \ldots, K\}$.
The basis elements $\{e_{n,p} \mid n \in \mathbb{Z},\ p = 1, \ldots, K\}$ yield the following Lie structure equation:

$$[e_{k,r}, e_{n,s}] = \delta_{r,s}(n-k)e_{k+n,r} + \sum_{h=n+k+1}^{n+k+R} \sum_t c_{(k,r),(n,s)}^{(h,t)} e_{h,t}, \tag{2.23}$$

with $c_{(k,r),(n,s)}^{(h,t)} \in \mathbb{K}$, the second sum going from $t = 1$ to $t = K$, and the constant R depends on the number of punctures and the splitting thereof as well as on the genus g of the Riemann surface. Defining the degree of an element by $deg(e_{n,p}) := n$, we see from Definition 2.1.7 that the Krichever-Novikov vector field algebra is a strongly almost-graded Lie algebra.

Let us point out that, just like the Witt algebra, the Krichever-Novikov vector field algebra is a simple and hence a perfect Lie algebra. The proof involves methods from algebraic geometry, hence we refrain from presenting it here. A proof can be found in [108], Proposition 6.99. on page 153.

Finally, note that it is not a trivial matter to recover the Lie structure equation (2.23) from (2.22). To compute (2.23) via residues, the pole order of basis elements around out-points is needed, as well as the so-called Krichever-Novikov pairing, and more sophisticated methods. For the details, we refer the reader to [108].

[1]Recall that the order of a meromorphic vector field is defined to be the order of a local representing function for it with respect to a local coordinate. It is independent of the chosen local coordinate. Note that a negative zero-order of $-n$ corresponds to a pole of order n.

2.1.5 The trivial, the adjoint and the tensor-densities modules

In this section, we will introduce the modules M considered in this thesis for the Witt, the Virasoro and the Krichever-Novikov vector field algebra.
We will start with the *trivial* module, given by $M = \mathbb{K}$, the base field. The trivial module $M = \mathbb{K}$ is a trivial $\mathscr{L}$-module in the sense given in the second bullet point of Definition 2.1.8,

$$\forall\, x \in \mathscr{L},\ \forall\, m \in \mathbb{K}: \qquad x \cdot m := 0\,, \tag{2.24}$$

where $\mathscr{L}$ stands for the Witt, the Virasoro or the Krichever-Novikov vector field algebra. The module $\mathbb{K}$ comes with a trivial grading as in (2.7), i.e. we have $\mathbb{K} = \bigoplus_{n\in\mathbb{Z}} \mathbb{K}_n$ with $\mathbb{K}_0 = \mathbb{K}$ and $\mathbb{K}_n = \{0\}$ for $n \neq 0$.

The next important module is given by the *adjoint* module, which corresponds to taking the Lie algebra itself as module, i.e. $M = \mathscr{L}$, where $\mathscr{L}$ in our case corresponds to either the Witt, the Virasoro, or the Krichever-Novikov vector field algebra. The module structure is simply given by the Lie bracket,

$$\forall\, x \in \mathscr{L},\ \forall\, m \in \mathscr{L}: \qquad x \cdot m := [x, m]\,. \tag{2.25}$$

This is clearly an $\mathscr{L}$-module, as the action satisfies (2.9) because $\mathscr{L}$ fulfills the Jacobi identity (2.2). Obviously, it is also a(n almost) graded module, the (almost) grading being the same as for the Lie algebra itself.

Finally, we will introduce the general *tensor densities* modules $\mathscr{F}^\lambda$, with $\lambda \in \mathbb{C}$. The vector space $\mathscr{F}^\lambda = \langle f_m^\lambda \mid m \in \mathbb{Z}\rangle_{\mathbb{K}}$ is generated by the symbolic basis elements f_m^λ over $\mathbb{K}$. We will first consider the action of the Witt algebra on these modules, which is given by the action of its basis elements e_n on $\mathscr{F}^\lambda$:

$$\forall\, n, m \in \mathbb{Z},\ \forall\, \lambda \in \mathbb{C}: \qquad e_n \cdot f_m^\lambda := (m + \lambda n)\, f_{n+m}^\lambda\,. \tag{2.26}$$

A direct computation (see e.g. [65]) shows that this action fulfills (2.9), hence $\mathscr{F}^\lambda$ is indeed a $\mathscr{W}$-module. Moreover, by defining $deg(f_m^\lambda) := m$, it becomes an internally graded $\mathscr{W}$-module, and the grading is ensured by the same element that gives the grading of the Lie algebra itself, namely, $e_0 \cdot f_m^\lambda = m\, f_m^\lambda$, as can be seen from (2.26).
In this thesis, we do not work with specific realizations of the elements f_m^λ. Instead, these elements are just considered as abstract symbolic basis elements satisfying 2.26. As we remain on the abstract algebraic level and do not work with any geometrical interpretation, we will mostly consider $\lambda \in \mathbb{C}$, to get results as general as possible. Nevertheless, for motivational purposes, we provide a geometrical interpretation of the elements f_m^λ in the following.
When working with the realization of the Witt algebra as vector fields on $\mathbb{CP}^1$, one can give also to the symbols f_m^λ a geometrical meaning. In fact, the symbols f_m^λ can be interpreted as meromorphic differential forms of weight λ that are holomorphic outside of 0 and ∞, a basis of which is given by:

$$f_m^\lambda = z^{m-\lambda}(dz)^\lambda \qquad \lambda \in \mathbb{Z}\,, \tag{2.27}$$

where z is the quasi-global coordinate on $\mathbb{CP}^1$. The quantity $(dz)^\lambda$ denotes the frame $dz\otimes\cdots\otimes dz$ on the tensor product of λ dual tangent spaces $TM^* \otimes\cdots\otimes TM^*$ for $\lambda > 0$, the frame $\frac{d}{dz}\otimes\cdots\otimes\frac{d}{dz}$ on the tensor product of λ tangent spaces $TM\otimes\cdots\otimes TM$ for $\lambda < 0$, and for $\lambda = 0$, we simply obtain functions on $\mathbb{CP}^1$. For $\lambda = 1$ we thus obtain differential 1-forms, for $\lambda = 2$ we obtain quadratic differential forms, and for $\lambda = -1$ we obtain vector fields. In fact, taking $\lambda = -1$ in

(2.26), we see that the action corresponds simply to the Lie bracket. Therefore, $\lambda = -1$ corresponds to the adjoint module and we have $\mathscr{F}^{-1} = \mathscr{W}$. For general $\lambda \in \mathbb{Z}$, we obtain general tensor densities, thence the name of the module. It can be shown that the realization (2.27) satisfies (2.26), and that the action corresponds to the Lie derivative of differential forms of weight λ with respect to meromorphic vector fields, see e.g. [65]. Note that in (2.27), λ can also taken to be an half-integer by fixing a square root of the canonical line bundle, see [108], which is relevant in the case of superalgebras.

The tensor densities $\mathscr{W}$-modules $\mathscr{F}^\lambda$ extend to $\mathscr{V}$-modules in a canonical way,

$$\hat{e}_n \cdot f_m^\lambda := \pi(\hat{e}_n) \cdot f_m^\lambda = e_n \cdot f_m^\lambda \qquad \text{and} \qquad t \cdot f_m^\lambda := \pi(t) \cdot f_m^\lambda = 0\,, \tag{2.28}$$

where π is the projection appearing in the central extension (2.18). The fact that this is a $\mathscr{V}$-module is ensured by π being a Lie algebra homomorphism.

Remark 2.1.1. Any $\mathscr{W}$-module M is a $\mathscr{V}$-module via the quotient map $\mathscr{V} \to \mathscr{W}$. Also, considering $\mathbb{K}$ as a Lie algebra with trivial Lie bracket $[\cdot,\cdot]_{\mathbb{K}} = 0$, we have that every $\mathscr{W}$-module M is a trivial $\mathbb{K}$-module.

The modules $\mathscr{F}^\lambda$ are also $\mathscr{KN}$-modules. In this thesis, we will work mostly with the trivial module in the case of the Krichever-Novikov vector field algebra, hence we will not provide all the details here. To give some intuition, let us point out that the basis elements of $\mathscr{F}^\lambda$ as $\mathscr{KN}$-modules are given by the symbols $\{f_{m,p}^\lambda \mid m \in \mathbb{Z},\ p = 1, \dots K\}$, which can locally be represented as:

$$f_{m,p}^\lambda = z_p^{m-\lambda}(1 + \mathscr{O}(z_p))(dz_p)^\lambda \qquad \lambda \in \mathbb{Z}\,, \tag{2.29}$$

where z_p is the local coordinate around the puncture P_p. The basis elements $f_{m,p}^\lambda$ can be interpreted as meromorphic differential forms of weight λ on a Riemann surface that are holomorphic outside of N marked points. As before, the superscript λ can also be taken to be half-integer. For more details, we refer the reader to [108].

2.2 The Chevalley-Eilenberg cohomology

In this section, we will introduce the cohomology of Lie algebras, and derive interpretations for the low-dimensional cohomology. Also, we will give an overview of the results known in the case of the low-dimensional algebraic cohomology of the Witt, the Virasoro and the Krichever-Novikov vector field algebra.

2.2.1 The Chevalley-Eilenberg complex

The cohomology of Lie algebras, called the Chevalley-Eilenberg cohomology, is the counterpart to the Hochschild cohomology [59] of associative algebras. In general, Lie algebra cohomology can be approached via functorial methods, such as right derived functors and the Ext functor. Alternatively, it can be described in a less categorical way by basic homological algebra. In this thesis, we will choose the second approach, in order to respect the spirit of simplicity of the thesis. We recall the basic definitions needed from homological algebra below. The functorial approach can be found e.g. in [131].

Definition 2.2.1.
- A *cochain complex* is a sequence of vector spaces V^k, $k \in \mathbb{Z}$,
$$\cdots \xrightarrow{\delta^{k-2}} V^{k-1} \xrightarrow{\delta^{k-1}} V^k \xrightarrow{\delta^k} V^{k+1} \xrightarrow{\delta^{k+1}} \dots,$$
together with a linear map δ such that $\delta^2 = \delta^k \circ \delta^{k-1} = 0$.
- The operator δ is called *coboundary operator*, *cohomology operator* or *differential*.
- The elements of V are called *cochains*, the elements of $\ker\delta$ are called *cocycles*, and the elements of $\mathrm{im}\,\delta$ are referred to as *coboundaries*.
Since $\delta^k \circ \delta^{k-1} = 0$, we have $\mathrm{im}\,\delta^{k-1} \subseteq \ker\delta^k$, hence it is sensible to consider the quotient $\frac{\ker\delta^k}{\mathrm{im}\,\delta^{k-1}}$.
- The *k-th cohomology* of (V,δ) is given by the quotient vector space,
$$\mathrm{H}^k(V,\delta) := {}^{\ker\delta^k}\!\big/_{\mathrm{im}\,\delta^{k-1}},$$
and the *cohomology* of the cochain complex (V,δ) is given by the graded vector space
$$\mathrm{H}(V,\delta) := \bigoplus_k \mathrm{H}^k(V,\delta)\,. \tag{2.30}$$

Let $\mathscr{L}$ be a Lie algebra and M a $\mathscr{L}$-module. The vector spaces V^q of the Definition 2.2.1 will in our case be given by $C^q(\mathscr{L},M)$, the vector space of q-multilinear alternating maps with values in M,
$$C^q(\mathscr{L},M) := \mathrm{Hom}_{\mathbb{K}}(\wedge^q\mathscr{L},M)\,.$$
The elements of the space $C^q(\mathscr{L},M)$ are called *q-cochains*, in accordance with Definition 2.2.1. By convention, we set $C^0(\mathscr{L},M) := M$. Next, let us introduce the family of coboundary operators δ_q defined by:
$$\forall q \in \mathbb{N}, \quad \delta_q : C^q(\mathscr{L},M) \to C^{q+1}(\mathscr{L},M) : \psi \mapsto \delta_q\psi\,,$$
$$(\delta_q\psi)(x_1,\dots x_{q+1}) : \ = \textstyle\sum_{1\le i<j\le q+1}(-1)^{i+j+1}\, \psi([x_i,x_j],x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{q+1}) \tag{2.31}$$
$$+\textstyle\sum_{i=1}^{q+1}(-1)^i\, x_i\cdot\psi(x_1,\dots,\hat{x}_i,\dots,x_{q+1})\,.$$
with $x_1,\dots,x_{q+1} \in \mathscr{L}$, $\hat{x}_i$ meaning that the entry x_i is omitted and the dot $\cdot$ stands for the module structure. In our case, we consider the trivial, the adjoint and the tensor densities modules, the module structures of which have been defined in (2.24), (2.25) and (2.26), respectively. It can be shown that the coboundary operators applied twice give zero, $\delta_{q+1}\circ\delta_q = 0\ \forall\ q \in \mathbb{N}$, see e.g. Proposition 4.1 in the book by Knapp [70], or Lemma 3.1. in the lecture notes by Wagemann [128], hence we obtain the following cochain complex of vector spaces:
$$\{0\} \xrightarrow{\delta_{-1}} M \xrightarrow{\delta_0} C^1(\mathscr{L},M) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{q-2}} C^{q-1}(\mathscr{L},M)$$
$$\xrightarrow{\delta_{q-1}} C^q(\mathscr{L},M) \xrightarrow{\delta_{q+1}} C^{q+1}(\mathscr{L},M) \xrightarrow{\delta_{q+1}} \dots \longrightarrow \dots$$
where $\delta_{-1} := 0$. This cochain complex is called the *Chevalley-Eilenberg complex*. The elements of the vector space $Z^q(\mathscr{L},M) :=\ \ker\delta_q$ are called *q-cocycles*, while the elements of the vector space $B^q(\mathscr{L},M) :=\ \mathrm{im}\,\delta_{q-1}$ are called *q-coboundaries*, as in Definition 2.2.1. The q^{th} Lie algebra

cohomology of $\mathscr{L}$ with values in M is obtained by taking the quotient of the q-cocycles by the q-coboundaries:

$$\mathrm{H}^q(\mathscr{L},M) := {Z^q(\mathscr{L},M)}\big/{B^q(\mathscr{L},M)}.$$

The total cohomology space is called the *Chevalley-Eilenberg cohomology* [20], associated to the Lie algebra $\mathscr{L}$ with values in M:

$$\mathrm{H}(\mathscr{L},M) := \bigoplus_{q=0}^{\infty} \mathrm{H}^q(\mathscr{L},M).$$

We will refer to the spaces $\mathrm{H}^q(\mathscr{L},M)$ with cohomological degree $q = 0,1,2,3$ as the *low-dimensional* cohomology of $\mathscr{L}$ with values in M.
In the case of graded Lie algebras such as the Witt or the Virasoro algebra, it is sensible to introduce the notion of a degree for cochains as a helpful tool. This is presented in the next section.

2.2.2 The degree of a homogeneous cochain

Let $\mathscr{L} = \bigoplus_{n\in\mathbb{Z}} \mathscr{L}_n$ be a $\mathbb{Z}$-graded Lie algebra, and $M = \bigoplus_{n\in\mathbb{Z}} M_n$ a $\mathbb{Z}$-graded $\mathscr{L}$-module. It is quite natural to translate this graded structure to the cochain complex and also the cohomology complex. We say that a q-cochain is homogeneous of degree d if it sends homogeneous elements of degree p to homogeneous elements of degree $p+d$.

Definition 2.2.2. A q-cochain $\psi : \wedge^q\mathscr{L} \to M$ is *homogeneous of degree d*, if there exists a number $d \in \mathbb{Z}$ such that for all q-tuple $x_1,\ldots,x_q$ of homogeneous elements $x_i \in \mathscr{L}_{deg(x_i)}$, we have:

$$\psi(x_1,\ldots,x_q) \in M_n \text{ with } n = \sum_{i=1}^{q} deg(x_i) + d.$$

We denote by $C^q_{(d)}(\mathscr{L},M)$ the subspace of homogeneous q-cochains of degree d. The subspaces $C^q_{(d)}(\mathscr{L},M)$ assemble to a subcomplex of the complex formed by the spaces $C^q(\mathscr{L},M)$. Every q-cochain can be written as a formal infinite sum,

$$\psi = \sum_{d\in\mathbb{Z}} \psi_{(d)}, \qquad \psi_{(d)} \in C^q_{(d)}(\mathscr{L},M),$$

where only a finite number of terms are non-zero when applied to a fixed q-tuple of elements. The coboundary operators are of degree zero. Consequently, if $\psi = \sum_d \psi_{(d)}$ is a q-cocycle, then $\delta_q\psi = \sum_d \delta_q\psi_{(d)} = 0$ implies that all the individual terms of the sum must be zero, i.e. every $\psi_{(d)}$ must be a $q-$cocycle. Actually, the terms in the sum cannot cancel each other, as the sum goes over cochains $\psi_{(d)}$ of different degree and the coboundary operator δ_q does not change the degrees of the $\psi_{(d)}$. Besides, suppose ψ is a q-coboundary of degree d, i.e. $\psi = \delta_{q-1}\phi$ for ϕ some $(q-1)$-cochain ϕ. A priori, ϕ need not be of degree d, as it could be a $(q-1)$-cocycle. However, it is always possible to perform a cohomological change such that $\psi = \delta_{q-1}\phi'$ where ϕ' is a $(q-1)$-cochain of degree d. Therefore, the decomposition can be performed on the level of equivalence classes $[\psi] \in \mathrm{H}^q(\mathscr{L},M)$,

$$[\psi] = \sum_{d\in\mathbb{Z}} [\psi_{(d)}], \qquad [\psi_{(d)}] \in \mathrm{H}^q_{(d)}(\mathscr{L},M),$$

where $\mathrm{H}^q_{(d)}(\mathscr{L},M)$ is the subspace consisting of classes of q-cocycles of degree d divided by q-coboundaries of degree d. Thus, we obtain the decomposition on the level of cohomology:

$$\mathrm{H}^q(\mathscr{L},M) = \bigoplus_{d\in\mathbb{Z}} \mathrm{H}^q_{(d)}(\mathscr{L},M)\,,$$

so that the study of $\mathrm{H}^q(\mathscr{L},M)$ can be reduced to the analysis of each of its components $\mathrm{H}^q_{(d)}(\mathscr{L},M)$, which simplifies the analysis.
An important well-known result says that for internally $\mathbb{Z}$-graded Lie algebras and modules, the cohomology reduces to the degree zero cohomology, see Theorem 1.5.2a. in [43], page 46. A proof of this result can be found on page 45 of [43], Theorem 1.5.2, where it was derived in the case of the trivial module. The generalization to general $\mathbb{Z}$-graded modules, given by Theorem 1.5.2a. of [43], is immediate. Since the result is important and used throughout this thesis, and since the proof is rather short, and because we want to keep this thesis as self-contained as possible, we will give a proof of this result with general $\mathbb{Z}$-graded modules below.

Theorem 2.2.1. *Let $\mathscr{L}$ be an internally graded Lie algebra with respect to a grading element e_0, and M a graded $\mathscr{L}$- module with respect to the same grading element e_0. Then the cohomology $\mathrm{H}(\mathscr{L},M)$ is zero except for the degree zero cohomology $\mathrm{H}_{(0)}(\mathscr{L},M)$:*

$$\forall\, q\in\mathbb{N}: \qquad \mathrm{H}^q_{(d)}(\mathscr{L},M) = \{0\} \;\; \text{for} \;\; d\neq 0\,, \tag{2.32a}$$

$$\forall\, q\in\mathbb{N}: \qquad \mathrm{H}^q(\mathscr{L},M) = \mathrm{H}^q_{(0)}(\mathscr{L},M)\,. \tag{2.32b}$$

Proof. Let $\mathscr{L}$ be an internally graded Lie algebra with respect to a grading element e_0. Let M be a graded $\mathscr{L}$- module with respect to the same grading element e_0, i.e.

$$M = \bigoplus_{n\in\mathbb{Z}} M_n\,, \qquad \text{with} \qquad M_n = \{x\in M \mid e_0\cdot x = n\,x\}\,.$$

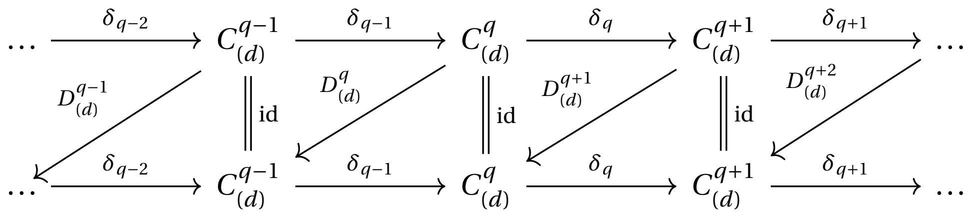
For $d\neq 0$, let us define the following map:

$$D^q_{(d)} : C^q_{(d)}(\mathscr{L},M) \to C^{q-1}_{(d)}(\mathscr{L},M)\,,$$

as follows:

$$(D^q_{(d)}\psi)(x_1,\ldots,x_{q-1}) \mapsto \psi(e_0,x_1,\ldots,x_{q-1})\,.$$

The situation is illustrated in the diagram below,



Let ψ be a q-cochain of degree $d\neq 0$, i.e. $\psi\in C^q_{(d)}(\mathscr{L},M)$. Let $\psi(x_1,\ldots,x_q)\in M_n$ with $n = \sum_{i=1}^q d_i + d$, with $d_i := deg(x_i)$. Let $\tilde{d} := \sum_{i=1}^q d_i$. Let us apply D on a coboundary:

$$\begin{aligned}
&\left(D^{q+1}_{(d)}(\delta_q\psi)\right)(x_1,\ldots,x_q) = (\delta_q\psi)(e_0,x_1,\ldots,x_q)\\
&= \sum_{i=1}^{q}(-1)^{i-1}\psi([e_0,x_i],x_1,\ldots,\hat{x}_i,\ldots,x_q)\\
&- \sum_{1\le i<j\le q}(-1)^{i+j-1}\psi(e_0,[x_i,x_j],x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_q)\\
&+ \sum_{i=0}^{q}(-1)^{i+1}x_i\cdot\psi(e_0,x_1,\ldots,\hat{x}_i,\ldots,x_q)
\end{aligned} \tag{2.33}$$

The minus sign in front of the second sum appears because the element e_0 was switched with the bracket. Using $[e_0, x_i] = d_i x_i$, the first sum can be rewritten as follows:

$$\sum_{i=1}^{q}(-1)^{i-1}\psi([e_0,x_i],x_1,\ldots,\hat{x}_i,\ldots,x_q) = \tilde{d}\,\psi(x_1,\ldots,x_q)\,.$$

The second sum can be rewritten by using the definition of $D^q_{(d)}$:

$$\begin{aligned}&\sum_{1\le i<j\le q}(-1)^{i+j-1}\psi(e_0,[x_i,x_j],x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_q)\\ &= \sum_{1\le i<j\le q}(-1)^{i+j-1}\left(D^q_{(d)}\psi\right)([x_i,x_j],x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_q)\,.\end{aligned}$$

The last sum is due to the fact that we are dealing with an arbitrary module and not a trivial module. The last sum can be rewritten as follows:

$$\begin{aligned}&\sum_{i=0}^{q}(-1)^{i+1}x_i\cdot\psi(e_0,x_1,\ldots,\hat{x}_i,\ldots,x_q)\\ &= -\sum_{i=1}^{q}(-1)^{i}x_i\cdot\psi(e_0,x_1,\ldots,\hat{x}_i,\ldots,x_q) - e_0\cdot\psi(x_1,\ldots,x_q)\\ &= -\sum_{i=1}^{q}(-1)^{i}x_i\cdot\left(D^q_{(d)}\psi\right)(x_1,\ldots,\hat{x}_i,\ldots,x_q) - (\tilde{d}+d)\psi(x_1,\ldots,x_q)\,.\end{aligned}$$

Inserting everything back into (2.33), we obtain:

$$\begin{aligned}\left(D^{q+1}_{(d)}(\delta_q\psi)\right)(x_1,\ldots,x_q) &= \tilde{d}\,\psi(x_1,\ldots,x_q) - (\tilde{d}+d)\psi(x_1,\ldots,x_q)\\ &\quad - \sum_{1\le i<j\le q}(-1)^{i+j-1}\left(D^q_{(d)}\psi\right)([x_i,x_j],x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_q)\\ &\quad - \sum_{i=1}^{q}(-1)^{i}x_i\cdot\left(D^q_{(d)}\psi\right)(x_1,\ldots,\hat{x}_i,\ldots,x_q)\\ \Leftrightarrow \left(D^{q+1}_{(d)}(\delta_q\psi)\right)(x_1,\ldots,x_q) &= -d\psi(x_1,\ldots,x_q) - \delta_{q-1}\left(\left(D^q_{(d)}\psi\right)(x_1,\ldots,x_q)\right)\,.\end{aligned}$$

Next, let us consider cohomology, i.e. suppose ψ is a degree-zero cocycle, $\psi\in Z^q_{(d)}(\mathscr{L},M)$. Then $(\delta_q\psi)=0$, and we obtain,

$$\psi(x_1,\ldots,x_q) = -\frac{\delta_{q-1}\left(\left(D^q_{(d)}\psi\right)(x_1,\ldots,x_q)\right)}{d}\,.$$

We obtain that ψ is necessary a coboundary. This means that all cocycles of degree different from zero are coboundaries, which proves Theorem 2.2.1. □

2.2.3 Exact sequences

In this section, we recall some definitions and results from homological algebra about exact sequences, for the convenience of the reader. The notions introduced here will be used in the next Section 2.2.4.

Definition 2.2.3.
- An *exact* sequence of Lie algebras is a sequence of Lie algebras $\mathscr{L}_i$ and Lie algebra homomorphisms f_i,

$$\cdots \xrightarrow{f_{i-2}} \mathscr{L}_{i-1} \xrightarrow{f_{i-1}} \mathscr{L}_{l_i} \xrightarrow{f_i} \mathscr{L}_{i+1} \xrightarrow{f_{i+1}} \dots ,$$

that satisfy $\ker f_i = \operatorname{im} f_{i-1}$ for all i.

- A *short* exact sequence is an exact sequence consisting of three terms,

$$0 \hookrightarrow \mathscr{L}_1 \overset{i}{\hookrightarrow} \mathscr{L}_2 \overset{\pi}{\twoheadrightarrow} \mathscr{L}_3 \twoheadrightarrow 0, \tag{2.34}$$

where 0 is the zero Lie algebra. We say that $\mathscr{L}_2$ is an *extension* of $\mathscr{L}_3$ by $\mathscr{L}_1$.

- A *long* exact sequence is an exact sequence containing more than three terms.

- Two extensions $\mathscr{L}_2$ and $\mathscr{L}_2'$ of a Lie algebra $\mathscr{L}_3$ by a Lie algebra $\mathscr{L}_1$ are *equivalent* if there is an Lie algebra isomorphism $\varphi : \mathscr{L}_2 \to \mathscr{L}_2'$ such that the following diagram commutes,

$$\begin{array}{ccccccccc} 0 & \hookrightarrow & \mathscr{L}_1 & \overset{i}{\hookrightarrow} & \mathscr{L}_2 & \overset{\pi}{\twoheadrightarrow} & \mathscr{L}_3 & \twoheadrightarrow & 0 \\ & & \Big\| \text{id} & & \downarrow \varphi & & \Big\| \text{id} & & \\ 0 & \hookrightarrow & \mathscr{L}_1 & \overset{i'}{\hookrightarrow} & \mathscr{L}_2' & \overset{\pi'}{\twoheadrightarrow} & \mathscr{L}_3 & \twoheadrightarrow & 0 \end{array} \tag{2.35}$$

i.e. $\varphi \circ i = i'$ and $\pi' \circ \varphi = \pi$.

Remark [Origin of short exact sequences] 2.2.1. There are two sources for short exact sequences. Let $\phi : \mathscr{L} \to \mathscr{L}'$ be a Lie algebra homomorphism. Due to the first isomorphism theorem, which will be recalled later in the Remark 2.2.5, we have that $\ker\phi$ is a Lie ideal of $\mathscr{L}$ and $\operatorname{im}\phi$ a Lie subalgebra of $\mathscr{L}'$. We thus obtain the following canonical short exact sequence of Lie algebras,

$$0 \longrightarrow \ker\phi \overset{i}{\longrightarrow} \mathscr{L} \overset{\phi}{\longrightarrow} \operatorname{im}\phi \longrightarrow 0. \tag{2.36}$$

Another origin of short exact sequences is given by Lie ideals of $\mathscr{L}$. Let $\mathscr{J}$ be a Lie ideal of $\mathscr{L}$, then we obtain the following natural short exact sequence of Lie algebras,

$$0 \longrightarrow \mathscr{J} \overset{i}{\longrightarrow} \mathscr{L} \overset{\pi}{\longrightarrow} \mathscr{L}/\mathscr{J} \longrightarrow 0. \tag{2.37}$$

Every short exact sequence will be of these types, up to isomorphism. In fact, the map π in (2.34) is surjective, due to exactness at $\mathscr{L}_3$, hence $\operatorname{im}\pi = \mathscr{L}_3$. Moreover, $\ker\pi = \operatorname{im} i = i(\mathscr{L}_1)$ due to exactness at $\mathscr{L}_2$, and $i(\mathscr{L}_1) \cong \mathscr{L}_1$ since i is injective due to exactness at $\mathscr{L}_1$. Hence, every given short exact sequence of Lie algebras is of type (2.36). Besides, if we set $\ker\phi =: \mathscr{J}$ in (2.36), then we have $\operatorname{im}\phi = \dfrac{\mathscr{L}}{\ker\phi}$ due to the first isomorphism Theorem 2.2.5, and we see that the short exact sequence (2.36) can be regarded as being a short exact sequence of type (2.37).

Remark [Splitting and gluing of exact sequences] 2.2.1. Short exact sequences are useful because any long exact sequence can be decomposed into short exact sequences and conversely, short exact sequences can be glued into long exact sequences.

- Let $0 \longrightarrow \mathscr{L}_1 \xrightarrow{f_1} \mathscr{L}_2 \xrightarrow{f_2} \mathscr{L}_3 \xrightarrow{f_3} \mathscr{L}_4 \longrightarrow 0$ be a 4-term exact sequence of Lie algebras. Then the two short sequences

$$0 \longrightarrow \mathscr{L}_1 \xrightarrow{f_1} \mathscr{L}_2 \xrightarrow{f_2} \mathscr{N} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathscr{N} \longrightarrow \mathscr{L}_3 \xrightarrow{f_3} \mathscr{L}_4 \longrightarrow 0,$$

  are also exact, with $\mathscr{N} = \operatorname{im} f_2 = \ker f_3$

- Let $0 \longrightarrow \mathscr{L}_1 \xrightarrow{f_1} \mathscr{L}_2 \xrightarrow{f_2} \mathscr{N} \longrightarrow 0$ and $0 \longrightarrow \mathscr{N} \longrightarrow \mathscr{L}_3 \xrightarrow{f_3} \mathscr{L}_4 \longrightarrow 0$ be two exact sequences, with $\mathscr{N}$ a Lie subalgebra of $\mathscr{L}_3$. The 4-term sequence

$$0 \longrightarrow \mathscr{L}_1 \xrightarrow{f_1} \mathscr{L}_2 \xrightarrow{f_2} \mathscr{L}_3 \xrightarrow{f_3} \mathscr{L}_4 \longrightarrow 0,$$

  is also exact.

This result allows to split and glue exact sequences of arbitrary length. See the lecture notes by Gathmann [44] for the proofs and more details.

Next, we introduce the concept of a split short exact sequence or split extension.

Definition 2.2.4. Consider a short exact sequence of objects A, B, C and morphisms f, g,

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

A short exact sequence is

(1) a *split exact sequence* if there exist isomorphisms ϕ_1, ϕ_2 and ϕ_3 such that the following diagram commutes,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & A \oplus C & \xrightarrow{\pi} & C & \longrightarrow & 0 \end{array}.$$

(2) *left split* if there exists a morphism $r : B \to A$, called *retraction map*, such that $r \circ f = \mathrm{id}_A$,

$$0 \longrightarrow A \underset{r}{\overset{f}{\rightleftarrows}} B \xrightarrow{g} C \longrightarrow 0.$$

(3) *right split* if there exists a morphism $s : C \to B$, called *splitting map*, such that $g \circ s = \mathrm{id}_C$,

$$0 \longrightarrow A \xrightarrow{f} B \underset{s}{\overset{g}{\rightleftarrows}} C \longrightarrow 0.$$

Remark 2.2.2. A well-known result states that every short exact sequence of vector spaces over a field $\mathbb{K}$ is a split exact sequence. The Theorem holds both in the finite-dimensional and the infinite-dimensional case.

Remark [Splitting Lemma] 2.2.3. Let us briefly introduce the Splitting Lemma for the objects considered in this thesis. The Splitting Lemma is a generalization of the Rank Nullity Theorem to categorical aspects.
For short exact sequences of vector spaces, we have in Definition 2.2.4 (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Hence, due to Remark 2.2.2, every short exact sequence of vector spaces is also left split and right split.
In the case of short exact sequences of Lie algebras, we get (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). Hence, a short exact sequence is not necessarily split if there is a splitting map. In general, B will only be the semi-direct sum of A and C.
In the case of central extensions of Lie algebras though, we get (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

Extensions can be classified into central extensions, abelian and non-abelian extensions. In this thesis, we focus only on abelian and central extensions. Therefore, we provide definitions only for abelian and central extensions below in 2.2.5. For details on non-abelian extensions, see e.g. the articles by Frégier [42] or by Alekseevsky, Michor and Ruppert [3].

Definition 2.2.5.
- A Lie algebra extension $0 \hookrightarrow \mathscr{L}_1 \overset{i}{\hookrightarrow} \mathscr{L}_2 \overset{\pi}{\twoheadrightarrow} \mathscr{L}_3 \twoheadrightarrow 0$ is *abelian* if $\mathscr{L}_1$ is an abelian Lie algebra.
- A Lie algebra extension $0 \hookrightarrow \mathscr{L}_1 \overset{i}{\hookrightarrow} \mathscr{L}_2 \overset{\pi}{\twoheadrightarrow} \mathscr{L}_3 \twoheadrightarrow 0$ is *central* if $\operatorname{im} i = \ker \pi$ is contained in the center of $\mathscr{L}_2$, i.e. $\operatorname{im} i \subseteq C(\mathscr{L}_2)$.
- The *trivial* central extension is defined as being the extension $0 \hookrightarrow \mathscr{L}_1 \overset{i}{\hookrightarrow} \mathscr{L}_1 \oplus \mathscr{L}_3 \overset{\pi}{\twoheadrightarrow} \mathscr{L}_3 \twoheadrightarrow 0$, where $\mathscr{L}_1 \oplus \mathscr{L}_3$ corresponds to the direct sum on the level of Lie algebras, given in Definition 2.1.11. Similarly, we obtain the trivial abelian extension, by replacing the direct sum $\mathscr{L}_1 \oplus \mathscr{L}_3$ by the semi-direct sum $\mathscr{L}_1 \rtimes \mathscr{L}_3$, for a fixed action of $\mathscr{L}_3$ on $\mathscr{L}_1$.
- A central Lie extension $0 \hookrightarrow \mathscr{L}_1 \overset{i}{\hookrightarrow} \mathscr{L}_2 \overset{\pi}{\twoheadrightarrow} \mathscr{L}_3 \twoheadrightarrow 0$ is *universal* if for every other central extension $0 \hookrightarrow \mathscr{L}_1' \overset{i}{\hookrightarrow} \mathscr{L}_2' \overset{\pi}{\twoheadrightarrow} \mathscr{L}_3' \twoheadrightarrow 0$ there exist unique homomorphisms $f_1 : \mathscr{L}_1 \to \mathscr{L}_1'$ and $f_2 : \mathscr{L}_2 \to \mathscr{L}_2'$ such that the following diagram commutes,

$$\begin{array}{ccccccccc} 0 & \hookrightarrow & \mathscr{L}_1 & \overset{i}{\hookrightarrow} & \mathscr{L}_2 & \overset{\pi}{\twoheadrightarrow} & \mathscr{L}_3 & \twoheadrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_2 & & \| \,\mathrm{id} & & \\ 0 & \hookrightarrow & \mathscr{L}_1' & \overset{i'}{\hookrightarrow} & \mathscr{L}_2' & \overset{\pi'}{\twoheadrightarrow} & \mathscr{L}_3' & \twoheadrightarrow & 0 \end{array}$$

  i.e. $i' \circ f_1 = f_2 \circ i$ and $\pi' \circ f_2 = \pi$.

On the level of vector spaces, all the extensions can be given by a direct sum, see the Remarks 2.2.3 and 2.2.2 above. On the level of Lie algebras, the extensions we will encounter in this thesis are based on direct sums and semi-direct sums twisted by a 2-cocycle $\alpha : \mathscr{L}_3 \wedge \mathscr{L}_3 \to \mathscr{L}_1$, with an action of $\mathscr{L}_3$ on $\mathscr{L}_1$, i.e. $\mathscr{L}_1$ is an $\mathscr{L}_3$-module. We will see later in Section 2.2.1 as to why α has to be a 2-cocycle.
An abelian extension $\mathscr{L}_2$ is given as a semi-direct sum of $\mathscr{L}_1$ and $\mathscr{L}_3$ twisted by a 2-cocycle $\alpha : \mathscr{L}_3 \wedge \mathscr{L}_3 \to \mathscr{L}_1$, with bracket given by, $\forall x_1, y_1 \in \mathscr{L}_1$, $x_3, y_3 \in \mathscr{L}_3$:

$$\big[(x_1, x_3), (y_1, y_3)\big]_{\mathscr{L}_2} = \big(\eta(x_3)y_1 - \eta(y_3)x_1 + \alpha(x_3, y_3), [x_3, y_3]_{\mathscr{L}_3}\big) \,, \tag{2.38}$$

and the action of $\mathscr{L}_3$ on $\mathscr{L}_1$ is given by a Lie homomorphism $\eta : \mathscr{L}_3 \to \operatorname{Der} \mathscr{L}_1$. In the abelian extensions considered in this thesis, the derivation η appearing in (2.38) will correspond to the

inner derivation given by the module structure.
In the case of central extensions, we have that $\mathscr{L}_1$ is abelian and the action of $\mathscr{L}_3$ on $\mathscr{L}_1$ is trivial. The central extension $\mathscr{L}_2$ is then given as a direct sum of $\mathscr{L}_1$ and $\mathscr{L}_3$ twisted by a 2-cocycle $\alpha : \mathscr{L}_3 \wedge \mathscr{L}_3 \longrightarrow \mathscr{L}_1$, with bracket given by, $\forall x_1, y_1 \in \mathscr{L}_1$, $x_3, y_3 \in \mathscr{L}_3$:

$$\big[(x_1, x_3), (y_1, y_3)\big]_{\mathscr{L}_2} = \big(\alpha(x_3, y_3), [x_3, y_3]_{\mathscr{L}_3}\big) . \tag{2.39}$$

We will derive the explicit expression for the Lie bracket given above in (2.39) later in Section 2.2.4.
We see that the definition of a central extension (2.39) coincides with the definition of the Virasoro algebra introduced in Section 2.1.3. In fact, the Virasoro algebra corresponds, up to equivalence and rescaling, to the unique non-trivial one-dimensional central extension of the Witt algebra by the base field $\mathbb{K}$. Actually, it is a universal central extension. Traditionally, the Virasoro bracket is written as a sum using the central element t like in 2.19, but clearly it can also be written as a couple as in (2.39).
In the case of a trivial extension, the 2-cocycle $\alpha : \mathscr{L}_3 \wedge \mathscr{L}_3 \longrightarrow \mathscr{L}_1$ in (2.38) and (2.39) is the zero 2-cocycle.

Finally, another important result of homological algebra states that every short exact sequence of Lie algebra modules gives rise to a long exact sequence in cohomology. Before we state the result, we briefly need to introduce the concept of a cochain map.

Definition 2.2.6. Let (V, δ_V) and (W, δ_W) be two cochain complexes. A *morphism of cochain complexes* or *cochain map* $\phi : (V, \delta_V) \longrightarrow (W, \delta_W)$ is a family of linear maps respecting the cohomological grading $\phi^k : V^k \longrightarrow W^k$ such that:

$$\delta_W^k \phi^k = \phi^{k+1} \delta_V^k ,$$

which is often abbreviated[2] as $\delta_W \phi = \phi \delta_V$. Alternatively, the definition can be given by the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\delta_V^{k-2}} & V^{k-1} & \xrightarrow{\delta_V^{k-1}} & V^k & \xrightarrow{\delta_V^{k}} & V^{k+1} & \xrightarrow{\delta_V^{k+1}} & \cdots \\
 & \circlearrowright & \big\downarrow \phi k-1 & \circlearrowleft & \big\downarrow \phi k & \circlearrowright & \big\downarrow \phi k+1 & \circlearrowleft & \\
\cdots & \xrightarrow{\delta_W^{k-2}} & W^{k-1} & \xrightarrow{\delta_W^{k-1}} & W^k & \xrightarrow{\delta_W^{k}} & W^{k+1} & \xrightarrow{\delta_W^{k+1}} & \cdots
\end{array}$$

The symbol H in (2.30) acts not only on cochain complexes (V, δ_V) yielding graded vector spaces $\mathrm{H}(V, \delta_V)$, but it also acts on cochain maps $\phi : (V, \delta_V) \longrightarrow (W, \delta_W)$, inducing graded vector space maps $\mathrm{H}(\phi) : \mathrm{H}(V, \delta_V) \longrightarrow \mathrm{H}(W, \delta_W)$ defined by:

$$\mathrm{H}(\phi)[v^k] := [\phi^k v^k], \text{ for } v^k \in V^k . \tag{2.40}$$

The well-known result from homological algebra mentioned above is stated in Theorem 2.2.2 below.

Theorem 2.2.2. *Let $\mathscr{L}$ be a Lie algebra and A, B, C be $\mathscr{L}$-modules. Let be given the short exact sequence of $\mathscr{L}$-modules,*

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ \pi\ } C \longrightarrow 0 ,$$

[2] Normally, we denote the k-coboundary operator with a lower index δ_k. In the following though, we denote it with an upper index δ^k to increase readability.

and consider the induced cochain complexes with cochain maps ϕ^k and ψ^k,

$$0 \longrightarrow C^k(\mathscr{L},A) \xrightarrow{\phi^k} C^k(\mathscr{L},B) \xrightarrow{\psi^k} C^k(\mathscr{L},C) \longrightarrow 0\,, \qquad k \in \mathbb{N}\,.$$

Then there is a long exact sequence in cohomology given by,

$$\ldots \longrightarrow \mathrm{H}^k(\mathscr{L},A) \xrightarrow{\mathrm{H}(\phi^k)} \mathrm{H}^k(\mathscr{L},B) \xrightarrow{\mathrm{H}(\psi^k)} \mathrm{H}^k(\mathscr{L},C) \xrightarrow{\Delta^k} \mathrm{H}^{k+1}(\mathscr{L},A) \xrightarrow{\mathrm{H}(\phi^{k+1})} \mathrm{H}^{k+1}(\mathscr{L},B) \xrightarrow{\mathrm{H}(\psi^{k+1})} \mathrm{H}^{k+1}(\mathscr{L},C) \longrightarrow \ldots$$

where the Δ^k is called the connecting homomorphism.

Proof. This is a standard result in homological algebra. A proof can be found for example in the book by M. Scott Osborne [89], page 48, Theorem 3.3. □

Remark 2.2.4. Sometimes it is useful to know an explicit expression for the connecting homomorphism Δ. Actually, it is defined in a most natural way. Consider $[\gamma] \in \mathrm{H}^k(\mathscr{L},C)$ with representing k-cocycle $\gamma : \wedge^k\mathscr{L} \to C$, and a lift of γ to $\tilde{\gamma} : \wedge^k\mathscr{L} \to B$,

$$\begin{array}{ccc} & & B \\ & \overset{\tilde{\gamma}}{\nearrow} & \downarrow \pi \\ \wedge^k\mathscr{L} & \xrightarrow{\gamma} & C \end{array}$$

The lift $\tilde{\gamma}$ exists because there always exists a section of π on the level of vector spaces. As γ is a k-cocycle, we have $\delta^k_C\gamma = 0$, and hence $\delta^k_B\tilde{\gamma}$ takes values in $A \subseteq B$, since $\pi(\delta^k_B\tilde{\gamma}) = \delta^k_C\pi(\tilde{\gamma}) = \delta^k_C\gamma = 0$. One therefore defines:

$$\Delta^k : \mathrm{H}^k(\mathscr{L},C) \to \mathrm{H}^{k+1}(\mathscr{L},A)\,, \qquad [\gamma] \mapsto [\delta^k_B\tilde{\gamma}]\,.$$

Note that indeed we have $\delta^k_B\tilde{\gamma} \in Z^{k+1}(\mathscr{L},A)$. In fact, as $\delta^k_B\tilde{\gamma}$ takes values in $A \subseteq B$, it has a preimage in A under i, denoted by $i^{-1}\delta^k_B\tilde{\gamma}$. Thus, $i\delta^{k+1}_A i^{-1}\delta^k_B\tilde{\gamma} = \delta^{k+1}_B i i^{-1}\delta^k_B\tilde{\gamma} = \delta^{k+1}_B\delta^k_B\tilde{\gamma} = 0$, which is enough to conclude due to the injectivity of i. Moreover, clearly Δ^k is independent of the choice of the lift $\tilde{\gamma}$. For more details on the construction, see e.g. Addendum 1.3.3 of [131].

2.2.4 Interpretation of the low-dimensional cohomology

In this section, we derive some interpretations of the low-dimensional cohomology for various modules, for motivational purposes.

The zeroth cohomology

We will start with the zeroth cohomology of a Lie algebra $\mathscr{L}$ with values in a general $\mathscr{L}$-module M. Recall that 0-cochains are elements of the module, i.e. $\psi \in C^0(\mathscr{L},M) = M$. The 0-coboundaries and the 0-cocycles are given by, see (2.31):

$$\begin{aligned} B^0(\mathscr{L},M) &= \operatorname{im}\delta_{-1} = \{0\} \qquad \text{and} \\ Z^0(\mathscr{L},M) &= \ker\delta_0 = \{\psi \in M \mid (\delta_0\psi)(x) = 0\ \forall\ x \in \mathscr{L}\} \\ &= \{\psi \in M \mid -x\cdot\psi = 0\ \forall\ x \in \mathscr{L}\}\,. \end{aligned}$$

The zeroth cohomology is given by the quotient Z^0/B^0,

$$\mathrm{H}^0(\mathscr{L}, M) = {}^{\ker\delta_0}\!\big/_{\mathrm{im}\,\delta_{-1}} = \{m \in M \mid x\cdot m = 0 \;\forall\; x \in \mathscr{L}\} =: {}^{\mathscr{L}}M. \tag{2.41}$$

The space ${}^{\mathscr{L}}M$ is called the space of $\mathscr{L}$*-invariants*[3] of the module M.
We can make this more explicit if we consider the trivial $M = \mathbb{K}$ and the adjoint module $M = \mathscr{L}$. For the former, every element $m \in \mathbb{K}$ is $\mathscr{L}$-invariant, as the action is trivial, i.e. we have $x\cdot m = 0$ for all $x \in \mathscr{L}$ and $m \in \mathbb{K}$. For the latter, the elements ${}^{\mathscr{L}}\mathscr{L}$ correspond to the space $\{y \in \mathscr{L} \mid [x, y] = 0 \;\forall\; x \in \mathscr{L}\}$, which is exactly the center of $\mathscr{L}$, see the Definition 2.1.4. We thus obtain:

$$\mathrm{H}^0(\mathscr{L},\mathbb{K}) = \mathbb{K} \qquad \text{and} \qquad \mathrm{H}^0(\mathscr{L},\mathscr{L}) = C(\mathscr{L}). \tag{2.42}$$

The first cohomology

Let us start by computing the 1-coboundaries and the 1-cocycles for a general module M. From (2.31), we see that the coboundaries and cocycles are given by:

$$B^1(\mathscr{L}, M) = \mathrm{im}\,\delta_0 = \{(\delta_0\phi)(x) = -x\cdot\phi,\; \phi \in C^0(\mathscr{L}, M) = M,\; x \in \mathscr{L}\}, \tag{2.43}$$

$$\begin{aligned} Z^1(\mathscr{L}, M) &= \ker\delta_1 = \{\psi \in C^1(\mathscr{L}, M) \mid (\delta_1\psi)(x, y) = 0 \;\forall\; x, y \in \mathscr{L}\} \\ &= \{\psi \in C^1(\mathscr{L}, M) \mid \psi([x, y]) - x\cdot\psi(y) + y\cdot\psi(x) = 0 \;\forall\; x, y \in \mathscr{L}\}. \end{aligned} \tag{2.44}$$

We can render these expressions more concrete in the case of the trivial and the adjoint module.

The trivial module Let us start with the trivial module $M = \mathbb{K}$. In the case of the trivial module, we see from (2.43) that $B^1(\mathscr{L},\mathbb{K}) = \{0\}$, hence $\mathrm{H}^1(\mathscr{L},\mathbb{K}) = Z^1(\mathscr{L},\mathbb{K})$. From (2.44), we see that we are looking at 1-cochains $\psi : \mathscr{L} \to \mathbb{K}$ with the property that $\psi|_{[\mathscr{L},\mathscr{L}]} = 0$. In the present case, this boils down to looking at cochains $\overline{\psi} : \dfrac{\mathscr{L}}{[\mathscr{L},\mathscr{L}]} \to \mathbb{K}$. Actually, the maps ψ and $\overline{\psi}$ are in one-to-one correspondence. This is based on the First Isomorphism Theorem, sometimes also called the Factorization Theorem, which we recall in the remark below.

Remark [First Isomorphism Theorem] 2.2.5. Let $\mathscr{L}$ and $\mathscr{L}'$ be Lie algebras, and $\varphi : \mathscr{L} \to \mathscr{L}'$ be a Lie algebra homomorphism. Then $\ker\varphi$ is a Lie ideal in $\mathscr{L}$ and $\mathrm{im}\,\varphi$ is a Lie subalgebra in $\mathscr{L}'$. Moreover, $\mathscr{L}/\ker\varphi \cong \mathrm{im}\,\varphi$. Let $\mathscr{I}$ be any Lie ideal of $\mathscr{L}$ such that $\mathscr{I} \subseteq \ker\varphi$. Then φ factors uniquely through the canonical projection $\pi : \mathscr{L} \to \mathscr{L}/\mathscr{I}$, i.e. there exists a unique homomorphism $\overline{\varphi} : \mathscr{L}/\mathscr{I} \to \mathscr{L}'$ such that $\varphi = \overline{\varphi}\circ\pi$.

In our case, we have $\phi = \psi$ and $\mathscr{I} = [\mathscr{L},\mathscr{L}]$. Recall from Definition 2.1.4 that the derived subalgebra $\mathscr{I} = [\mathscr{L},\mathscr{L}]$ is indeed a Lie ideal of $\mathscr{L}$. Moreover, we consider cochains $\psi : \mathscr{L} \to \mathbb{K}$ with $\psi|_{\mathscr{I}} = 0$, hence $\mathscr{I} \subseteq \ker\psi$. Using the Theorem 2.2.5 above, we can instead consider cochains $\overline{\psi} : \dfrac{\mathscr{L}}{\mathscr{I}} \to \mathbb{K}$ defined by $(x \bmod j) \mapsto \overline{\psi}(x \bmod j) := \psi(x)$, with $x \in \mathscr{L}$ and $j \in \mathscr{I}$.
Hence, $\mathrm{H}^1(\mathscr{L},\mathbb{K})$ is given by the space $\mathrm{Hom}_{\mathbb{K}}\left(\dfrac{\mathscr{L}}{[\mathscr{L},\mathscr{L}]},\mathbb{K}\right) =: \left(\dfrac{\mathscr{L}}{[\mathscr{L},\mathscr{L}]}\right)^*$,

$$\mathrm{H}^1(\mathscr{L},\mathbb{K}) = \left(\frac{\mathscr{L}}{[\mathscr{L},\mathscr{L}]}\right)^*, \tag{2.45}$$

where $*$ stands for the dual space.

[3]We write ${}^{\mathscr{L}}M$ instead of $M^{\mathscr{L}}$ because we consider left modules, i.e. $\mathscr{L}$ acts from the left on M. In case of Lie algebras though, all left modules are also right modules, thus the distinction between the notations is not relevant in our case.

The adjoint module Let us now consider the first cohomology with values in the adjoint module, $M = \mathscr{L}$. In order to find the 1-coboundaries and 1-cocycles, the image of δ_0 and the kernel of δ_1 have to be computed. From the Equations (2.43) and (2.44), we derive the 1-coboundaries and 1-cocycles for the adjoint module,

$$B^1(\mathscr{L},\mathscr{L}) = \operatorname{im}\delta_0 = \{(\delta_0\phi)(x) = [\phi, x]\,, \phi, x \in \mathscr{L}\} = \{\mathrm{ad}_\phi(x)\,, \phi, x \in \mathscr{L}\}\,, \tag{2.46}$$
$$Z^1(\mathscr{L},\mathscr{L}) = \ker\delta_1 = \{\psi \in C^1(\mathscr{L},\mathscr{L}) \mid \psi([x,y]) = [x,\psi(y)] + [\psi(x),y]\,, \forall\; x,y \in \mathscr{L}\}\,. \tag{2.47}$$

We see that the 1-coboundaries correspond exactly to the inner derivations of $\mathscr{L}$ into $\mathscr{L}$, see (2.14), while the 1-cocycles satisfy the Leibniz rule and are thus derivations, see the Definition 2.1.9. The first cohomology with values in the adjoint module is thus given by the quotient of derivations by inner derivations of $\mathscr{L}$ into $\mathscr{L}$, yielding the so-called *outer derivations* $\mathrm{Out}(\mathscr{L})$ of $\mathscr{L}$ into $\mathscr{L}$,

$$\mathrm{H}^1(\mathscr{L},\mathscr{L}) = {}^{\mathrm{Der}\,\mathscr{L}}\!\big/_{\mathrm{ad}_{\mathscr{L}}} = \mathrm{Out}(\mathscr{L})\,. \tag{2.48}$$

Remark 2.2.6. The first cohomology with values in the adjoint module $\mathrm{H}^1(\mathscr{L},\mathscr{L})$ also classifies, up to equivalence, *right* extensions $(\hat{\mathscr{L}}, [\![\cdot,\cdot]\!])$ of a Lie algebra $(\mathscr{L}, [\cdot,\cdot])$, which are given by the short exact sequence of Lie algebras $0 \longrightarrow \mathscr{L} \longrightarrow \hat{\mathscr{L}} \longrightarrow \mathbb{K} \longrightarrow 0$. The Lie algebra structure $[\![\cdot,\cdot]\!]$ in $\hat{\mathscr{L}}$, with $\hat{\mathscr{L}}$ isomorphic to $\mathscr{L} \oplus \mathbb{K}$ as vector space, is given by $[\![(x,\lambda),(y,\mu)]\!] = [x,y] + \mu\psi(x) - \lambda\psi(y)$, where $x, y \in \mathscr{L}$, $\lambda, \mu \in \mathbb{K}$ and $\psi \in \mathrm{H}^1(\mathscr{L},\mathscr{L})$. We will not give details for this correspondence, since the considerations are very similar to the correspondence between $\mathrm{H}^2(\mathscr{L},\mathbb{K})$ and central extensions, which we will describe below in detail. For more details on right extensions, see e.g. [43], pages 31-32.

The general tensor-densities modules The reasoning we did for $\mathrm{H}^1(\mathscr{L},\mathscr{L})$ is also valid for $\mathrm{H}^1(\mathscr{L},\mathscr{F}^\lambda)$, or any module M, $\mathrm{H}^1(\mathscr{L},M)$, see the definition of a general derivation, 2.1.9. Thus, for the general tensor-densities modules or any $\mathscr{L}$-module M, we also obtain that the first cohomology corresponds to outer derivations of $\mathscr{L}$ into $\mathscr{F}^\lambda$ or M,

$$\mathrm{H}^1(\mathscr{L},\mathscr{F}^\lambda) = \frac{\mathrm{Der}(\mathscr{L},\mathscr{F}^\lambda)}{\mathrm{IDer}(\mathscr{L},\mathscr{F}^\lambda)} = \mathrm{Out}(\mathscr{L},\mathscr{F}^\lambda)\,,$$

where $\mathrm{IDer}(\mathscr{L},\mathscr{F}^\lambda)$ stands for the inner derivations of $\mathscr{L}$ into $\mathscr{F}^\lambda$, given by the module action of $\mathscr{L}$ on $\mathscr{F}^\lambda$. See for example Hilton and Stammbach, p.234 [57].
In the same spirit of Remark 2.2.6, the first cohomology with values in $\mathscr{F}^\lambda$ or any $\mathscr{L}$-module M also has an interpretation in terms of extensions. In fact, $\mathrm{H}^1(\mathscr{L},\mathscr{F}^\lambda)$ (or $\mathrm{H}^1(\mathscr{L},M)$) classifies extensions of $\mathscr{L}$-modules of the form,

$$0 \longrightarrow \mathscr{F}^\lambda \longrightarrow N \longrightarrow \mathbb{K} \longrightarrow 0\,. \tag{2.49}$$

We will not provide details here, since we are going to discuss central extensions in detail below. For more details, see e.g. Weibel, Exercise 7.4.5 p. 232 [131].

The second cohomology

Let us start by writing down the 2-coboundaries and the 2-cocycles for a module M. Equation (2.31) yields for $q = 1$ and $q = 2$ respectively,

$$B^2(\mathcal{L}, M) = \operatorname{im}\delta_1 = \{(\delta_1\phi)(x,y) = \phi([x,y]) - x\cdot\phi(y) + y\cdot\phi(x)\,, \quad x,y\in\mathcal{L}\,,\phi\in C^1(\mathcal{L},M)\}\,, \tag{2.50}$$

$$Z^2(\mathcal{L}, M) = \ker\delta_2 = \{\psi\in C^2(\mathcal{L},M)\mid \psi([x,y],z)-\psi([x,z],y)+\psi([y,z],x) -x\cdot\psi(y,z)+y\cdot\psi(x,z)-z\cdot\psi(x,y)=0\,,\forall\ x,y,z\in\mathcal{L}\}\,. \tag{2.51}$$

We will give concrete interpretations of the second cohomology for the trivial and the adjoint module, since these are the most popular. Also, because these interpretations are very important interpretations, we provide some details about the proofs and computations. The analysis for the general tensor densities modules is very similar to the one for the trivial module, hence we will not give details for these.

The trivial module Let us start with the trivial module. For the trivial module, the 2-coboundaries (2.50) and 2-cocycles (2.51) become:

$$B^2(\mathcal{L},\mathbb{K}) = \{(\delta_1\phi)(x,y) = \phi([x,y])\,, x,y\in\mathcal{L}\,,\phi\in C^1(\mathcal{L},M)\}\,, \tag{2.52}$$

$$Z^2(\mathcal{L},\mathbb{K}) = \{\psi\in C^2(\mathcal{L},\mathbb{K})\mid \psi([x,y],z)-\psi([x,z],y)+\psi([y,z],x)=0\,,\forall\ x,y,z\in\mathcal{L}\}\,. \tag{2.53}$$

The second cohomology with values in the trivial module describes central extensions of $\mathcal{L}$ by the base field $\mathbb{K}$. The Virasoro algebra is such an extension. We will present this cohomology in detail, as it allows for a better understanding of the Virasoro algebra, which is in the focus of this thesis. We follow here mostly the presentation given in the lecture notes by Iena, Leytem and Schlichenmaier [65].
Consider the following central extension of Lie algebras,

$$0\longrightarrow\mathbb{K}\overset{i}{\longrightarrow}\hat{\mathcal{L}}\overset{\pi}{\longrightarrow}\mathcal{L}\longrightarrow 0\,, \tag{2.54}$$

with $i(\mathbb{K})\subseteq C(\hat{\mathcal{L}})$, $i:\mathbb{K}\to\hat{\mathcal{L}}\,, m\mapsto(m,0)$ and $\pi:\hat{\mathcal{L}}\to\mathcal{L}\,,(\ell,\ell')\mapsto(0,\ell')$. Since this is in particular also an exact sequence of vector spaces, due to the theorem mentioned in the Remark 2.2.2, we already know that $\hat{\mathcal{L}}=\mathbb{K}\oplus\mathcal{L}$ as vector spaces. This corresponds to the following splitting map s_0,

$$s_0:\mathcal{L}\to\hat{\mathcal{L}},\qquad x\mapsto(0,x)\,. \tag{2.55}$$

According to the Remark 2.2.2 and the Remark 2.2.3, we have splitting maps s on the level of vector spaces. We do not necessary have them on the level of Lie algebras, i.e. they are in general not Lie algebra homomorphisms. The general form of the splitting maps $s:\mathcal{L}\to\hat{\mathcal{L}}$ is given by,

$$s:\mathcal{L}\to\hat{\mathcal{L}},\qquad x\mapsto(f(x),x)\,,$$

for some linear form $f:\mathcal{L}\to\mathbb{K}$. Clearly s is linear and satisfies $\pi\circ s=\mathrm{id}_{\mathcal{L}}$. The splitting maps could not be of the form $s(x)=(f(x),g(x))$ for some linear $f:\mathcal{L}\to\mathbb{K}$ and $g:\mathcal{L}\to\mathcal{L}$, since this does not satisfy $\pi\circ s=\mathrm{id}_{\mathcal{L}}$. A priori, there is no canonical choice for the splitting map. However, we will see later that the non-canonical choice of s disappears at the cohomological level. We can already show now that the choice disappears at the level of the Lie bracket. In fact, we have for all $\ell,\ell'\in\hat{\mathcal{L}}$ that $[\![\ell,\ell']\!]=[\![s_0(x),s_0(y)]\!]$ for some $x,y\in\mathcal{L}$, where $[\![\cdot,\cdot]\!]$ stands for

the Lie bracket on $\hat{\mathscr{L}}$. To see this, let $t := i(1) = (1,0)$ and let $\ell, \ell' \in \hat{\mathscr{L}}$. Then there exist $m, n \in \mathbb{K}$ and $x, y \in \mathscr{L}$ such that $\ell = (m, x) = s_0(x) + m\ t$ and $\ell' = (n, y) = s_0(y) + n\ t$. The bracket becomes $[\![\ell, \ell']\!] = [\![s_0(x), s_0(y)]\!] + n[\![s_0(x), t]\!] + m[\![s_0(y), t]\!] +\ m\ n[\![t, t]\!] = [\![s_0(x), s_0(y)]\!]$, since $t \in C(\hat{\mathscr{L}})$.
The goal is now to discover what $\hat{\mathscr{L}}$ looks like at the level of Lie algebras, i.e. we need to know what the Lie bracket on $\hat{\mathscr{L}}$ looks like. We will denote the Lie bracket on $\hat{\mathscr{L}}$ by $[\![\cdot,\cdot]\!]$ and the Lie bracket on $\mathscr{L}$ by $[\cdot,\cdot]$ for the time being.

Observation 2.2.1. In a first step, we will show that every central extension $(\hat{\mathscr{L}}, [\![\cdot,\cdot]\!])$ of a Lie algebra $(\mathscr{L}, [\cdot,\cdot])$ by $\mathbb{K}$ as in (2.54), together with a choice of a linear splitting map s, gives rise to a 2-cocycle of $Z^2(\mathscr{L},\mathbb{K})$. In a second step, we prove that every 2-cocycle of $Z^2(\mathscr{L},\mathbb{K})$ yields a central extension of $\mathscr{L}$. The correspondence on the level of extensions alone is not well-defined due to the non-canonical choice of the section s, as we pointed out before. We will come back to this later in Observations 2.2.2 and 2.2.3.
Let $t = i(1)$. We start by showing that there exists a 2-cocycle $\alpha_s \in Z^2(\mathscr{L},\mathbb{K})$ such that

$$x, y \in \mathscr{L}: \qquad [\![s(x), s(y)]\!] = s([x, y]) + \alpha_s(x, y)\ t\,, \tag{2.56}$$

where $s : \mathscr{L} \to \hat{\mathscr{L}}$ is a splitting map on the level of vector spaces. We see that $\alpha_s(x, y)$ characterizes the failure of s to be a Lie algebra homomorphism. We consider the quantity $[\![s(x), s(y)]\!] - s([x, y])$ and apply π to it. Since (2.54) is a central extension of Lie algebras, π must be a Lie algebra homomorphism. Moreover, we have $\pi \circ s = \mathrm{id}_{\mathscr{L}}$. We obtain:

$$\pi([\![s(x), s(y)]\!]) - \pi(s([x, y])) = [\pi \circ s(x), \pi \circ s(y)] - \pi \circ s([x, y]) = 0\,,$$

hence $\big([\![s(x), s(y)]\!] - s([x, y])\big) \in \ker\pi$ and due to exactness also $\big([\![s(x), s(y)]\!] - s([x, y])\big) \in \mathrm{im}\,\pi$. Therefore, we must have some constant $\alpha_s(x, y) \in \mathbb{K}$ such that $([\![s(x), s(y)]\!] - s([x, y])) = \alpha_s(x, y)\ t$. Besides, $\hat{\mathscr{L}}$ must be a Lie algebra, meaning that $[\![\cdot,\cdot]\!]$ must be bilinear, skew-symmetric and satisfy the Jacobi identity (2.2). From $([\![s(x), s(y)]\!] - s([x, y])) = \alpha_s(x, y)\ t$, we see that the linearity of s as well as the bilinearity and skew-symmetry of the Lie brackets implies that α_s must be bilinear and skew-symmetric, i.e. $\alpha_s \in C^2(\mathscr{L},\mathbb{K})$. There remains the Jacobi identity. The first term of the Jacobi identity on $\hat{\mathscr{L}}$ yields:

$$[\![[\![s(x), s(y)]\!], s(z)]\!] = [\![s([x, y]) + \alpha_s(x, y)t, s(z)]\!] = [\![s([x, y]), s(z)]\!] = s\big([[x, y], z]\big) + \alpha_s\big([x, y], z\big)\, t\,.$$

Adding the cyclic permutations, we obtain that the condition of $[\![\cdot,\cdot]\!]$ satisfying the Jacobi identity leads to:

$$s\Big(\underline{[[x, y], z] + [[z, x], y] + [[y, z], x]}\Big) + \Big(\alpha_s\big([x, y], z\big) + \alpha_s\big([y, z], x\big) + \alpha_s\big([z, x], y\big)\Big)\, t = 0\,.$$

The underlined terms are zero due to the Jacobi identity on $\mathscr{L}$. Hence, the remaining terms involving α_s need to cancel each other. This corresponds exactly to the 2-cocycle condition with values in the trivial module $\mathbb{K}$, see (2.51). This shows that every central extension gives rise to a 2-cocycle with values in $\mathbb{K}$.
Vice-versa, every 2-cocycle in $Z^2(\mathscr{L},\mathbb{K})$ also gives rise to a central extension. Let $\alpha \in Z^2(\mathscr{L},\mathbb{K})$ be given. Then we can simply define a Lie algebra structure on $\hat{\mathscr{L}} = \mathbb{K} \oplus \mathscr{L}$ using s_0 from (2.55) as follows,

$$\begin{aligned} [\![s_0(x), s_0(y)]\!] &:= s_0\big([x, y]\big) + \alpha(x, y)\ t\,, \quad \forall\ x, y \in \mathscr{L}\,, \\ [\![t, \hat{\mathscr{L}}]\!] &:= 0\,. \end{aligned} \tag{2.57}$$

A direct computation shows that this is indeed a Lie bracket on $\hat{\mathscr{L}}$. This concludes the proof.

Remark 2.2.7. Note that if we define $\alpha := 0$ in (2.57), we obtain the central extension $\hat{\mathscr{L}}$ given by the Lie direct sum of the trivial Lie algebra $\mathbb{K}$ and the Lie algebra $\mathscr{L}$, s_0 will be a Lie algebra homomorphism for this extension and the short exact sequence associated to $\hat{\mathscr{L}}$ splits. This central extension is called the *trivial central extension*, see the third bullet point in Definition 2.2.5.

Observation 2.2.2. We already mentioned that the choice of a splitting map is non-canonical, but that the choice is not visible at the cohomological level. Actually, two different splitting maps give rise to cohomologous 2-cocycles in (2.56), i.e. if s and s' are two different splitting maps in (2.54), then in (2.56) the difference $\alpha_s - \alpha_{s'}$ is a 2-coboundary. This can be seen as follows. Let $t = i(1)$. The maps s and s' satisfy $\pi \circ s = \mathrm{id}_{\mathscr{L}} = \pi \circ s'$, hence $\pi \circ (s - s') = 0$, meaning that $s - s'$ is in the kernel of π, and by exactness, also in the image of i, $\forall\ x \in \mathscr{L}, \quad s(x) - s'(x) \in \ker \pi = \mathrm{im}\, i$. Therefore, there exists a linear form $f : \mathscr{L} \to \mathbb{K}$ such that, $\forall\ x \in \mathscr{L}, \quad s(x) - s'(x) = f(x)\ t$. Next, we insert this relation into the expression (2.56) of the corresponding 2-cocycles, and we obtain for $x, y \in \mathscr{L}$, $\alpha_{s'}(x,y)\ t = [\![s'(x), s'(y)]\!] - s'([x,y]) = [\![s(x), s(y)]\!] - s([x,y]) + f([x,y])\ t = \big(\alpha_s(x,y) + f([x,y])\big)\ t = \big(\alpha_s(x,y) + (\delta_1 f)(x,y)\big)\ t$. The last equality was obtained by observing that the 2-coboundaries are exactly maps of the form $f([x,y])$, see (2.52). Thus, α_s and $\alpha_{s'}$ are cohomologous.

In the last Observation 2.2.3, we will show that two cohomologous 2-cocycles in (2.56) give rise to equivalent central extensions in the sense of the definition in (2.35). Therefore, we will find that equivalence classes of central extensions are in one-to-one correspondence with the second cohomology $\mathrm{H}^2(\mathscr{L}, \mathbb{K})$.

Observation 2.2.3. Two central extensions $\hat{\mathscr{L}}_1$ and $\hat{\mathscr{L}}_2$ of a Lie algebra $\mathscr{L}$ are equivalent if and only if their defining 2-cocycles α_1 and α_2 are cohomologous.
$\boxed{\Leftarrow}$ Suppose we have two 2-cocycles $\alpha_1, \alpha_2 \in \mathrm{H}^2(\mathscr{L}, \mathbb{K})$ that are cohomologous, i.e. they differ by a coboundary,

$$(\alpha_1 - \alpha_2)(x,y) = (\delta_1 c)(x,y) \overset{(2.52)}{=} c\big([x,y]\big) \quad \forall\ x, y \in \mathscr{L},$$

and let them be defining cocycles of two central extensions $(\hat{\mathscr{L}}_1, [\![\cdot,\cdot]\!]_1)$ and $(\hat{\mathscr{L}}_2, [\![\cdot,\cdot]\!]_2)$, respectively. In order to show that they are equivalent, we need to construct a Lie algebra isomorphism φ satisfying the definition in (2.35), i.e.

$$\begin{array}{ccccccccc} 0 & \hookrightarrow & \mathbb{K} & \overset{i_1}{\hookrightarrow} & \hat{\mathscr{L}}_1 & \overset{\pi_1}{\twoheadrightarrow} & \mathscr{L} & \twoheadrightarrow & 0 \\ & & \Big\| \mathrm{id} & & \downarrow \varphi & & \Big\| \mathrm{id} & & \\ 0 & \hookrightarrow & \mathbb{K} & \overset{i_2}{\hookrightarrow} & \hat{\mathscr{L}}_2 & \overset{\pi_2}{\twoheadrightarrow} & \mathscr{L} & \twoheadrightarrow & 0 \end{array} \tag{2.58}$$

with $\varphi \circ i_1 = i_2$ and $\pi_2 \circ \varphi = \pi_1$. Let the splitting maps be $s_1(x) = (f(x), x)$ for some linear form $f : \mathscr{L} \to \mathbb{K}$ and $s_2(x) = (g(x), x)$ with $g : \mathscr{L} \to \mathbb{K}$. Then the following map φ,

$$\varphi : \hat{\mathscr{L}}_1 \to \hat{\mathscr{L}}_2, \quad (\lambda, x) \mapsto (\lambda + (g - c)(x), x),$$

fulfills (2.58). In fact, it is bijective and direct computation shows $\varphi \circ i_1 = i_2$ and $\pi_2 \circ \varphi = \pi_1$, and also that $\varphi \circ s_1$ is a splitting map for $\hat{\mathscr{L}}_2$. It remains to show that φ is a Lie algebra homomorphism, i.e. that it satisfies $\varphi\big([\![s_1(x), s_1(y)]\!]_1\big) = [\![\varphi(s_1(x)), \varphi(s_1(y))]\!]_2$ for $x, y \in \mathscr{L}$. To do this, one can use $s_2(x) - \varphi(s_1(x)) \in \ker \pi_2 = \mathrm{im}\, i_2$ in order to write $\varphi(s_1(x)) = s_2(x) + \beta(x)\ t$ for $\beta(x) \in \mathbb{K}$ and $t = i_2(1)$, and then use the fact that t is central in $\hat{\mathscr{L}}_2$.

$\boxed{\Rightarrow}$ Let $(\hat{\mathscr{L}}_1, [\![\cdot,\cdot]\!]_1)$ and $(\hat{\mathscr{L}}_2, [\![\cdot,\cdot]\!]_2)$ be two equivalent central extensions of $\mathscr{L}$, and let φ be a Lie algebra isomorphism such that we have (2.58). Using the facts that $\varphi(s_1(x)) = s_2(x) + \beta(x)\ t$ for $\beta(x) \in \mathbb{K}$, $\varphi\left([\![s_1(x), s_1(y)]\!]_1\right) = [\![\varphi(s_1(x)), \varphi(s_1(y))]\!]_2$ for $x, y \in \mathscr{L}$ and the definition in (2.56), we obtain

$$-\beta([x,y])\ t + \varphi \circ \alpha_1\ t = \alpha_2\ t, \tag{2.59}$$

where $\alpha_1, \alpha_2 \in \mathrm{H}^2(\mathscr{L}, \mathbb{K})$ are the defining 2-cocycles of $\hat{\mathscr{L}}_1$ and $\hat{\mathscr{L}}_2$, respectively. As the Diagram (2.58) commutes, we have $\varphi \circ i_1(\alpha_1) = i_2(\alpha_1)$. In our case, both inclusions i_1 and i_2 are the canonical inclusion given by multiplication by $t = i_1(1) = i_2(1) = (1,0)$, see (2.54), hence we have $\varphi \circ \alpha_1\ t = \alpha_1\ t$. From (2.59), we thus immediately obtain that α_1 and α_2 must be cohomologous.

Theorem 2.2.3. *The set of equivalence classes of central extensions of a Lie algebra $\mathscr{L}$ by the base field $\mathbb{K}$ is in one-to-one correspondence with the second cohomology $\mathrm{H}^2(\mathscr{L}, \mathbb{K})$.*

Proof. This results immediately from the Observations 2.2.1, 2.2.2 and 2.2.3. □

Remark 2.2.8. We already mentioned in Remark 2.2.7 that the cocycle $\alpha = 0$ corresponds to the trivial central extension. The cocycle $\alpha = 0$ up to coboundaries, given by the cohomological equivalence class $[0]$ in $\mathrm{H}^2(\mathscr{L}, \mathbb{K})$, thus gives rise to central extensions equivalent to the trivial central extension. Moreover, a central extension $\hat{\mathscr{L}}$ is equivalent to the trivial central extension if and only if $\hat{\mathscr{L}}$ is a split exact sequence of Lie algebras as given in Definition 2.2.4.

Remark 2.2.9. Rescaling the defining cocycles in (2.56) by a factor $\lambda \in \mathbb{K} \setminus \{0\}$ does not lead to equivalent central extensions, i.e. the cocycles α and $\lambda\alpha$ do not yield equivalent extensions. However, they give rise to isomorphic central extensions. We say that two central extensions $\hat{\mathscr{L}}_1$ and $\hat{\mathscr{L}}_2$ are *isomorphic* if they are isomorphic as short exact sequences of Lie algebras. In the particular case of rescaling, the classification is given by the projective space $\mathbb{P}\mathrm{H}^2(\mathscr{L}, \mathbb{K})$, i.e. non-trivial central extensions are, up to equivalence and rescaling, in one-to-one correspondence with $\mathbb{P}\mathrm{H}^2(\mathscr{L}, \mathbb{K})$. In practice, one is often interested in central extensions up to equivalence and rescaling only.

The general tensor-densities modules We saw above how central extensions of Lie algebras are classified by the second cohomology. The same holds true for abelian extensions, meaning that abelian extensions of a Lie algebra $\mathscr{L}$ by an abelian Lie algebra $\mathscr{H}$ are classified by $\mathrm{H}^2(\mathscr{L}, \mathscr{H})$. The proof is the same as for the central extensions, see also e.g. the lecture notes by Wagemann [128] for more details. We will therefore not present it here. In general, $\mathscr{F}^\lambda$ for fixed λ is not a Lie algebra. However, we can always equip the modules $\mathscr{F}^\lambda$ with the trivial Lie bracket, turning them into abelian Lie algebras. Therefore, $\mathrm{H}^2(\mathscr{L}, \mathscr{F}^\lambda)$ corresponds to abelian Lie algebra extensions $0 \to \mathscr{F}^\lambda \to \mathscr{E} \to \mathscr{L} \to 0$. In that case, we consider $\eta : \mathscr{L} \to \mathrm{Der}\, \mathscr{F}^\lambda$ in 2.38 to yield the inner derivation given by the module action of $\mathscr{L}$ on $\mathscr{F}^\lambda$ as defined in Section 2.1.5. More precisely, we have $\forall x \in \mathscr{L}$, $\forall f^\lambda \in \mathscr{F}^\lambda$, $\eta(x)(f^\lambda) := \phi_{f^\lambda}(x) = x \cdot f^\lambda$, where ϕ_{f^λ} is the inner derivation as defined in (2.13).

The adjoint module Next, we focus on the second cohomology with values in the adjoint module, $\mathrm{H}^2(\mathscr{L}, \mathscr{L})$. Let us start by writing down the expression for the 2-coboundaries (2.50) and the 2-cocycles (2.51) in the case of the adjoint module,

$$B^2(\mathscr{L},M) = \operatorname{im}\delta_1 = \big\{(\delta_1\phi)(x,y) = \phi([x,y]) - [x,\phi(y)] + [y,\phi(x)]\,, \quad x,y \in \mathscr{L}\,, \phi \in C^1(\mathscr{L},M)\big\}\,, \tag{2.60}$$

$$Z^2(\mathscr{L},M) = \ker\delta_2 = \big\{\psi \in C^2(\mathscr{L},M) \mid \psi([x,y],z) - \psi([x,z],y) + \psi([y,z],x) - [x,\psi(y,z)] + [y,\psi(x,z)] - [z,\psi(x,y)] = 0\,, \forall\ x,y,z \in \mathscr{L}\big\}\,. \tag{2.61}$$

In the following, we will show that $\mathrm{H}^2(\mathscr{L},\mathscr{L})$ characterizes infinitesimal deformations of the Lie algebra $\mathscr{L}$. We follow here mostly the presentation of [26, 106, 108].
Let $(\mathscr{L},[\cdot,\cdot])$ be a Lie algebra. In general, the Lie bracket $[\cdot,\cdot]$ of a Lie algebra $\mathscr{L}$ can be written as an antisymmetric bilinear map, i.e. a 2-cochain ψ_0,

$$\psi_0 : \mathscr{L}\times\mathscr{L} \to \mathscr{L}\,, \qquad (x_1,x_2) \mapsto \psi_0(x_1,x_2) = [x_1,x_2]\,.$$

Besides antisymmetry and bilinearity, the map ψ_0 also has to satisfy the Jacobi identity (2.2), in order for $\mathscr{L}$ to be a Lie algebra. Next, we consider on the same vector space $\mathscr{L}$ is defined on, the following family of Lie algebra structures:

$$\mu_t = \psi_0 + \psi_1\, t + \psi_2\, t^2 + \dots\,. \tag{2.62}$$

The product μ_t should be a Lie bracket, i.e. the maps $\psi_i : \mathscr{L}\times\mathscr{L} \to \mathscr{L}$ must be antisymmetric, bilinear and they must be such that the Lie algebra structure μ_t fulfills the Jacobi identity. In that case, we obtain a family of Lie algebras $\mathscr{L}_t := (\mathscr{L},\mu_t)$, with the original Lie algebra $\mathscr{L}_0 = (\mathscr{L},\psi_0)$ being given by $t = 0$. We say that the family $\{\mathscr{L}_t\}$ is a *deformation* of $\mathscr{L}_0$. Regarding the deformation parameter t, we need to distinguish different cases:

1. The parameter t can be taken as a variable over $\mathbb{K}$. In this case, $\mathscr{L}_\alpha$ with $\alpha \in \mathbb{K}$ is a Lie algebra for every α for which the expression (2.62) exists with respect to issues of convergence. In general though, this set-up is very hard to analyze.

2. The parameter t can be viewed as a formal variable, in which case we consider the family $\mathscr{L}_t$ not over $\mathbb{K}$, but over the ring of formal power series $\mathbb{K}[\![t]\!]$. Moreover, the underlying vector space of $\mathscr{L}_t$ will no longer be the vector space V of $\mathscr{L}$, but rather the formal power series $V[\![t]\!]$. If we take the formal variable t over $\mathbb{K}$, we obtain a deformation in the previous sense only if the series (2.62) converges. However, it is possible that the series converges only when plugging in zero for t. We call a deformation with the formal power series as parameter space a *formal deformation*.

3. A deformation $\mathscr{L}_t$ can also be taken over the quotient $\mathbb{K}[\![X]\!]/(X^{n+1})$. In that case, the deformation is called a *n-deformation*, and the sum in (2.62) contains maximally $n+1$ terms. The particular case given by the parameter t being considered as an infinitesimal variable with $t^2 = 0$ is called *infinitesimal deformation*, and corresponds to taking $n = 1$ in $\mathbb{K}[\![X]\!]/(X^{n+1})$.

More complicated parameter spaces can be envisaged, some of them yielding unexpected properties in the case of infinite-dimensional Lie algebras, see the work by Fialowski and Schlichenmaier [36–38, 103, 104].

We will now show how infinitesimal deformations relate to $\mathrm{H}^2(\mathscr{L},\mathscr{L})$.

Proposition 2.2.1. *The set of infinitesimal deformations of a Lie algebra $\mathscr{L}$ is in one-to-one correspondence with $Z^2(\mathscr{L},\mathscr{L})$.*

Proof. In order for $\mathscr{L}_t$ to be a Lie algebra, the family μ_t in (2.62) must fulfill the Jacobi identity up to all orders in t, i.e.

$$\begin{aligned} &\mu_t(\mu_t(x_1,x_2),x_3) + \text{ cyclic permutations of } (x_1,x_2,x_3) = 0 \\ \Leftrightarrow &\sum_{i,j\geq 0} \psi_i(\psi_j(x_1,x_2),x_3)\ t^{i+j} + \text{ cyclic permutations of } (x_1,x_2,x_3) = 0 \end{aligned} \tag{2.63}$$

has to be fulfilled for all t. For infinitesimal deformations, we have $t^2 = 0$ and hence we only obtain two conditions, namely the condition for order zero t^0 and the condition for order one t^1. The former condition simply corresponds to the original Jacobi identity of ψ_0 on $\mathscr{L}$. The condition for order one t^1 is:

$$\psi_1([x_1,x_2],x_3) + \text{ cycl. perm. } + \left[\psi_1(x_1,x_2),x_3\right] + \text{ cycl. perm. } = 0\,. \tag{2.64}$$

No terms of higher order have to be verified since $t^2 = 0$. Comparing the equation (2.64) above to the 2-cocycle condition with values in the adjoint module given in (2.61), we see that they coincide exactly. We therefore obtain that the family $\mu_t = \psi_0 + \psi_1\ t$ is an infinitesimal deformation if and only if ψ_1 is a Lie algebra 2-cocycle with values in the adjoint module, i.e. $\psi_1 \in Z^2(\mathscr{L},\mathscr{L})$. This concludes the proof. □

Remark 2.2.10. In general, a necessary condition for a general family μ_t to be a deformation is given by the first non-vanishing coefficient ψ_i having to be a 2-cocycle.

Next, we consider aspects of equivalence. Generally, two deformations μ_t and μ'_t of the Lie bracket ψ_0 are called *equivalent* if there exists a linear automorphism ψ_t such that:

$$\psi_t = \mathrm{id} + \alpha_1\ t + \alpha_2\ t^2 + \dots, \tag{2.65}$$

with linear maps $\alpha_i : \mathscr{L} \to \mathscr{L}$ fulfilling:

$$\mu'_t(x_1,x_2) = \psi_t^{-1}(\mu_t(\psi_t(x_1),\psi_t(x_2)))\,. \tag{2.66}$$

Proposition 2.2.2. *Two infinitesimal deformations $\mu_t = \psi_0 + \psi_1\ t$ and $\mu'_t = \psi_0 + \psi'_1\ t$ are equivalent if and only if ψ_1 and ψ'_1 are cohomologous in* $\mathrm{H}^2(\mathscr{L},\mathscr{L})$.

Proof. $\boxed{\Rightarrow}$
Let $\mu_t = \psi_0 + \psi_1\ t$ and $\mu'_t = \psi_0 + \psi'_1\ t$ be two equivalent infinitesimal deformations. Using $\psi_t^{-1} = \mathrm{id} - \alpha_1\ t + \mathscr{O}(t^2)$, taking (2.62) and (2.65) and inserting them into (2.66), we obtain after a power expansion in t for the linear term in t,

$$\psi'_1(x,y) = \psi_1(x,y) + [\alpha_1(x),y] + [x,\alpha_1(y)] - \alpha_1[x,y]\,.$$

Comparing this expression to the form of a coboundary given in (2.60), we see that $\psi'_1 - \psi_1$ is a coboundary $\delta_1\alpha_1$.
$\boxed{\Leftarrow}$
Let $\mu_t = \psi_0 + \psi_1\ t$ and $\mu'_t = \psi_0 + \psi'_1\ t$ be two infinitesimal deformations such that $\psi'_1 - \psi_1 = \delta\phi$ for $\phi \in C^1(\mathscr{L},\mathscr{L})$. We define: $\psi_t = \mathrm{id} + \phi\ t$. Clearly this is a linear automorphism. The same computation as above shows that it also satisfies (2.66) for order zero and order one. □

Remark 2.2.11. In the Remark 2.2.10, we mentioned that a necessary condition for a general family to be a deformation is that the first non-vanishing coefficient ψ_i has to be a 2-cocycle. Similarly, if two general deformations μ_t and μ'_t are equivalent, then the corresponding ψ_i and ψ'_i are cohomologous in $\mathrm{H}^2(\mathscr{L},\mathscr{L})$.

Theorem 2.2.4. *The set of equivalence classes of infinitesimal deformations is in one-to-one correspondence with* $\mathrm{H}^2(\mathscr{L},\mathscr{L})$.

Proof. This results directly from the Propositions 2.2.1 and 2.2.2. □

Remark 2.2.12. We say that a Lie algebra $(\mathscr{L},\psi_0)$ is *rigid* if every deformation μ_t of ψ_0 is locally equivalent to the *trivial deformation*, given by $\mu_t = \psi_0$ for all values of t. Locally means that t has to be "close to zero". This, of course, depends of the type of deformation under consideration. Intuitively, rigidity means that a Lie algebra cannot be deformed. Depending on the type of the deformation and the Lie algebra under consideration, the vanishing of the second cohomology implies different properties,

- if $\dim \mathscr{L} < \infty$, $\mathrm{H}^2(\mathscr{L},\mathscr{L}) = 0$ implies that $\mathscr{L}$ is rigid with respect to any deformation [46–48, 86].
- if $\mathrm{H}^2(\mathscr{L},\mathscr{L}) = 0$, then the Lie algebra $\mathscr{L}$ is rigid with respect to infinitesimal and formal deformations, see Fialowski and Fuchs [35], Fialowski [32, 33], Gerstenhaber [46–48], and Nijenhuis and Richardson [87, 88]. However, contrary to finite-dimensional Lie algebras, the vanishing of $\mathrm{H}^2(\mathscr{L},\mathscr{L})$ does not imply rigidity with respect to other parameter spaces in the case of infinite-dimensional Lie algebras, see [36–38, 103, 104].

For the sake of completeness, let us point out that if $\mathrm{H}^2(\mathscr{L},\mathscr{L}) < \infty$, then there is a family of infinitesimal deformations that is universal [46–48]. In addition, if $\mathrm{H}^2(\mathscr{L},\mathscr{L}) < \infty$, then there is a family of formal deformations that is versal, i.e. it induces all other non-equivalent formal deformations [35].

In general, it is possible to obtain an infinitesimal deformation from a general deformation. A straightforward way to do this is to simply truncate the general deformation by putting $t^2 = 0$ in (2.62). The deformation thus obtained is called *differential of the deformation*. Hence, every deformation yields an infinitesimal deformation. The opposite is not always true, as obstructions will arise given in terms of elements of the third cohomology, see [31, 46–48]. We will see this in more detail when discussing the third cohomology with values in the adjoint module below.

Third Cohomology

We start by writing down the expressions for the 3-coboundaries and the 3-cocycles for a general module M. Taking $q = 2$ and $q = 3$ in Equation (2.31) gives respectively,

$$\begin{aligned} B^3(\mathscr{L},M) = \operatorname{im}\delta_2 = \big\{&(\delta_2\phi)(x,y,z) = \phi([x,y],z) - \phi([x,z],y) + \phi([y,z],x) \\ &-x\cdot\phi(y,z) + y\cdot\phi(x,z) - z\cdot\phi(x,y)\,, x,y,z\in\mathscr{L}\,, \phi\in C^2(\mathscr{L},M)\big\}\,, \end{aligned} \tag{2.67}$$

$$\begin{aligned} Z^3(\mathscr{L},M) = \ker\delta_3 = \big\{&\psi\in C^3(\mathscr{L},M) \mid \psi([x,y],z,w) - \psi([x,z],y,w) + \psi([x,w],y,z) \\ &+\psi([y,z],x,w) - \psi([y,w],x,z) + \psi([z,w],x,y) - x\cdot\psi(y,z,w) \\ &+y\cdot\psi(x,z,w) - z\cdot\psi(x,y,w) + w\cdot\psi(x,y,z) = 0\,, \forall\ x,y,z\in\mathscr{L}\big\}\,. \end{aligned} \tag{2.68}$$

The adjoint module We will start by analyzing the particular case of $M = \mathscr{L}$ given by the ajoint module.

For the adjoint module, the Equations (2.67) and (2.68) become,

$$B^3(\mathscr{L},\mathscr{L}) = \operatorname{im}\delta_2 = \{(\delta_2\phi)(x,y,z) = \phi([x,y],z) - \phi([x,z],y) + \phi([y,z],x) \\ -[x,\phi(y,z)] + [y,\phi(x,z)] - [z,\phi(x,y)]\,, x,y,z \in \mathscr{L}, \phi \in C^2(\mathscr{L},\mathscr{L})\} \tag{2.69}$$

$$Z^3(\mathscr{L},\mathscr{L}) = \ker\delta_3 = \{\psi \in C^3(\mathscr{L},\mathscr{L}) \mid \psi([x,y],z,w) - \psi([x,z],y,w) + \psi([x,w],y,z) \\ + \psi([y,z],x,w) - \psi([y,w],x,z) + \psi([z,w],x,y) - [x,\psi(y,z,w)] \\ + [y,\psi(x,z,w)] - [z,\psi(x,y,w)] + [w,\psi(x,y,z)] = 0\,, \forall\; x,y,z \in \mathscr{L}\}\,. \tag{2.70}$$

A sensible question to consider is, whether a given infinitesimal deformation can be extended to a general deformation. This will not be possible in general, since the Jacobi identity has to be satisfied at all orders of t, and also aspects of convergence in (2.62) need to be considered. Usually, it is not an easy exercise to determine which infinitesimal deformations can be lifted to genuine deformations. Slightly more accessible is the task of determining which infinitesimal deformations allow an extension at least on the formal level, i.e. which ones can be given by a formal power series to all orders of t. This problem boils down to a step-by-step approach given by a step n to a step $n+1$ lifting property. Generally, obstructions to this lifting will appear, which are given by elements of $\mathrm{H}^3(\mathscr{L},\mathscr{L})$.

Let us consider a n-deformation given by $\mu_t = \sum_{i=0}^{n} \psi_i\; t^i$, hence the sum in (2.62) contains maximally $n+1$ terms. The question now is whether it can be lifted to a $n+1$-deformation $\mu'_t = \sum_{i=0}^{n+1} \psi_i\; t^i$. The Jacobi identity in (2.63) is fulfilled to all orders if the coefficients of t^k are zero for all k, i.e. the following conditions hold:

$$\sum_{i+j=k,i,j\geq 0} \psi_i(\psi_j(x_1,x_2),x_3) + \text{ cycl. perm.} = 0 \qquad 0 \leq k \leq n+1\,,$$

where "cycl. perm." stands for cyclic permutations of x_1,x_2,x_3. The n-deformation is given, hence the equations above are satisfied for $0 \leq k \leq n$. Therefore, the n-deformation can be extended to a $n+1$-deformation if the remaining equation of order $k = n+1$ is fulfilled:

$$\begin{aligned} & \sum_{i+j=n+1,i,j\geq 0} \psi_i(\psi_j(x_1,x_2),x_3) + \text{ cycl. perm.} = 0 \\ \Leftrightarrow\; & \left[\big(\psi_0(\psi_{n+1}(x_1,x_2),x_3) + \psi_{n+1}(\psi_0(x_1,x_2),x_3)\big) + \text{ cycl. perm.}\right] \\ & + \left[\sum_{i+j=n+1,i,j>0} \big(\psi_i(\psi_j(x_1,x_2),x_3)\big) + \text{ cycl. perm.}\right] = 0 \\ \Leftrightarrow\; & (\delta_2\psi_{n+1})(x_1,x_2,x_3) + \left[\sum_{i+j=n+1,i,j>0} \psi_i(\psi_j(x_1,x_2),x_3) + \text{ cycl. perm.}\right] = 0\,. \end{aligned}$$

The second line was obtained by separating the sum into terms with $i = 0, j = 0$ and $i,j > 0$. To obtain the last line, recall that ψ_0 is the original bracket $[\cdot,\cdot]$. The terms with ψ_0 then give rise to the 3-coboundary term $\delta_2\psi_{n+1}$, see the expression (2.69). Thus, the lifting problem of the n-deformation μ_t to the $n+1$-deformation μ'_t boils down to a condition on a 3-coboundary term for ψ_{n+1} plus an extra quantity which is called *obstruction*, given by:

$$\Psi_{n+1} := \sum_{i+j=n+1,i,j>0} \psi_i(\psi_j(x_1,x_2),x_3) + \text{ cycl. perm.}.$$

A straightforward computation yields $\delta_3\Psi_{n+1} = 0$, i.e. $\Psi_{n+1} \in Z^3(\mathscr{L},\mathscr{L})$ is a 3-cocycle [46–48]. An obstruction to a deformation can be interpreted in terms of the equivalence class $[\Psi_{n+1}]$

in $\mathrm{H}^3(\mathscr{L},\mathscr{L})$ as follows: A n-deformation can be lifted to an $n+1$-deformation if and only if $[\Psi_{n+1}] = 0$ in $\mathrm{H}^3(\mathscr{L},\mathscr{L})$, as in that case there exists a ψ_{n+1} yielding a coboundary term that cancels the obstruction, i.e. $\delta_2\psi_{n+1} = -\Psi_{n+1}$. In particular, the vanishing of all obstructions at all levels, i.e. $\mathrm{H}^3(\mathscr{L},\mathscr{L}) = \{0\}$, is a sufficient condition for the extension of each infinitesimal deformation to a formal deformation. For more details, see e.g. [24].

The trivial module and the general tensor-densities modules Next, we focus on modules $M = V$, where V is an abelian Lie algebra. This is for example the case for $V = \mathbb{K}$ or $V = \mathscr{F}^\lambda$ equipped with the trivial Lie bracket. We will show that $\mathrm{H}^3(\mathscr{L}, V)$ is in one-to-one correspondence with crossed modules, which we define below. This correspondence appeared first in the articles by Gerstenhaber [46–48], and was analyzed in more detail by Wagemann [123, 124].
A major part of the present thesis concentrates on the third cohomology with values in the trivial and general tensor densities modules. Hence, it is paramount to motivate this analysis, so we will introduce crossed modules and the link to the third cohomology in some detail. Thus, we will reproduce the relevant proofs here instead of just pointing to references. We will follow here the lecture notes by Wagemann [128], in which the interested reader can find more details.

Definition 2.2.7. A *crossed module* of Lie algebras is a homomorphism of Lie algebras $\mu : \mathscr{M} \to \mathscr{N}$ together with a Lie action of $\mathscr{N}$ on $\mathscr{M}$ via derivations, $\eta : \mathscr{N} \to \mathrm{Der}\,\mathscr{M}$, $n \mapsto \eta(n)$ with $\eta(n) \in \mathrm{Der}\,\mathscr{M}$, such that $\forall\ m, m' \in \mathscr{M}$ and $\forall\ n \in \mathscr{N}$:

1. $\mu(\eta(n)(m)) = [n, \mu(m)]_{\mathscr{N}}$,
2. $\eta(\mu(m))(m') = [m, m']_{\mathscr{M}}$,

where $[\cdot,\cdot]_{\mathscr{N}}$ and $[\cdot,\cdot]_{\mathscr{M}}$ are the Lie brackets in $\mathscr{N}$ and $\mathscr{M}$, respectively.

To each crossed module is associated a four-term exact sequence,

$$0 \to V \xrightarrow{i} \mathscr{M} \xrightarrow{\mu} \mathscr{N} \xrightarrow{\pi} \mathscr{L} \to 0, \tag{2.71}$$

where $\ker\mu = i(V)$ and $\mathscr{L} = \operatorname{coker}\mu = \dfrac{\mathscr{N}}{\operatorname{im}\mu}$. By the second property in Definition 2.2.7, we see that V must be central in $\mathscr{M}$, and also abelian.
We have that V is a $\mathscr{L}$-module, but $\mathscr{M}$ and $\mathscr{N}$ in general are not well-defined $\mathscr{L}$-modules. In fact, taking different sections ρ and ρ' of π gives different elements of $\mathscr{L}$ on $\mathscr{M}$, their difference being an inner derivation, $\eta((\rho-\rho')(x))(m) = [m', m]$ with $x \in \mathscr{L}$, $m \in \mathscr{M}$ and some $m' \in \mathscr{M}$. Actually, $\eta\circ\rho$ satisfies the conditions of being an action (2.9) up to inner derivations, $\eta\big([\rho(x),\rho(y)]_{\mathscr{N}} - \rho([x,y]_{\mathscr{L}})\big)(m) = [m', m]_{\mathscr{M}}$, hence it can be seen as an outer action. The action of $\mathscr{L}$ on V induced by the outer action though is well-defined, as we have that $\eta\big([\rho(x),\rho(y)]_{\mathscr{N}} - \rho([x,y]_{\mathscr{L}})\big)(m)|_{i(V)} = 0$ because V is central in $\mathscr{M}$.
If V is a trivial $\mathscr{L}$-module, the crossed module is called *central* crossed module. The *trivial* crossed module for kernel V and cokernel $\mathscr{L}$ is given by,

$$0 \longrightarrow V \xrightarrow{\mathrm{id}_V} V \xrightarrow{0} \mathscr{L} \xrightarrow{\mathrm{id}_{\mathscr{L}}} \mathscr{L} \longrightarrow 0. \tag{2.72}$$

Next, let us introduce the notion of equivalent crossed modules.

Definition 2.2.8. Two crossed modules (μ,η) and (μ',η') are *elementary equivalent* if there are Lie algebra morphisms $\varphi : \mathscr{M} \to \mathscr{M}'$ and $\psi : \mathscr{N} \to \mathscr{N}'$ compatible with the actions

$$\varphi\big(\eta(n)(m)\big) = \eta'\big(\psi(n)\big)\big(\varphi(m)\big), \tag{2.73}$$

such that the following diagram commutes:

$$\begin{array}{ccccccccccc} 0 & \hookrightarrow & V & \xhookrightarrow{i} & \mathscr{M} & \xrightarrow{\mu} & \mathscr{N} & \xrightarrow{\pi} & \mathscr{L} & \twoheadrightarrow & 0 \\ & & \| \text{id} & & \downarrow \varphi & & \downarrow \psi & & \| \text{id} & & \\ 0 & \hookrightarrow & V & \xhookrightarrow{i'} & \mathscr{M}' & \xrightarrow{\mu'} & \mathscr{N}' & \xrightarrow{\pi'} & \mathscr{L} & \twoheadrightarrow & 0 \end{array} \quad . \tag{2.74}$$

The *equivalence* of crossed modules is generated by elementary equivalences.

Let us denote by crmod $(\mathscr{L}, V)$ the space of equivalence classes of crossed modules associated to $\mathscr{L}$ and V. The Theorem to show is given below.

Theorem [Gerstenhaber] 2.2.5. *The space crmod* $(\mathscr{L}, V)$ *is in one-to-one correspondence with the third cohomology* $\mathrm{H}^3(\mathscr{L}, V)$*:*

$$\Phi: \qquad \mathit{crmod}\,(\mathscr{L}, V) \cong \mathrm{H}^3(\mathscr{L}, V)\,.$$

Remark 2.2.13. The isomorphism Φ also holds on the level of abelian groups.

In order to show the one-to-one correspondence between equivalence classes of $\mathrm{H}^3(\mathscr{L}, V)$ and the equivalence classes of crossed modules associated to $\mathscr{L}$ and V, we start by presenting how to a given crossed module (2.71) can be associated a 3-cocycle of $\mathrm{H}^3(\mathscr{L}, V)$, and prove that Φ is injective. The proof is taken from [128].

Proposition 2.2.3. *There exists an injective map* $\Phi : \mathit{crmod}\,(\mathscr{L}, V) \to \mathrm{H}^3(\mathscr{L}, V)$.

Proof. Three steps need to be shown. First, we prove the existence of the map by showing that every crossed module yields a 3-cocycle. Secondly, we show that equivalent crossed modules give rise to cohomologously equivalent 3-cocycles. Finally, we show that Φ is injective, i.e. $\ker \Phi = 0$.

Let us consider a crossed module as in (2.71). Taking ρ to be a section of π, we consider again the failure of ρ to be a Lie algebra homomorphism, just as in the case of central extensions, $\alpha(x, y) := \left[\rho(x), \rho(y)\right]_{\mathscr{N}} - \rho\left([x, y]_{\mathscr{L}}\right)$ with $x, y \in \mathscr{L}$. Clearly, $\pi \circ \alpha(x, y) = 0$ as π is a Lie algebra homomorphism, hence $\alpha(x, y) \in \ker \pi = \operatorname{im} \mu$ and there exists $\beta(x, y) \in \mathscr{M}$ such that $\mu(\beta(x, y)) = \alpha(x, y)$. We can choose a section σ on $\operatorname{im} \mu$ such that $\beta(x, y) = \sigma \circ \alpha(x, y)$, showing that β is bilinear and antisymmetric, $\beta \in C^2(\mathscr{L}, \mathscr{M})$. Using the first property of a crossed module in the Definition 2.2.7 as well as the Jacobi identities in $\mathscr{L}$ and $\mathscr{N}$, one obtains $\mu\big((\delta_{\mathscr{M}}\beta)(x, y, z)\big) = 0\ \forall\ x, y, z \in \mathscr{L}$, where $\delta_{\mathscr{M}}$ is the formal coboundary operator (2.31) with values in $\mathscr{M}$, the action of $\mathscr{L}$ on $\mathscr{M}$ being given by $\eta \circ \rho$, which is an outer action. Therefore, $(\delta_{\mathscr{M}}\beta)(x, y, z) \in \ker \mu = \operatorname{im} i$ and there exists a $\gamma(x, y, z) \in V$ such that $(\delta_{\mathscr{M}}\beta)(x, y, z) = i\big(\gamma(x, y, z)\big)$. We can choose a section τ on $\operatorname{im} i$ to obtain $\gamma(x, y, z) = \tau\left((\delta_{\mathscr{M}}\beta)\right)(x, y, z)$, showing that γ is trilinear and skew-symmetric, $\gamma \in C^3(\mathscr{L}, V)$. By writing down the expressions explicitly, and by recalling that the action of $\mathscr{L}$ on V is induced by the outer action of $\mathscr{N}$ on $\mathscr{M}$, one finds that $i((\delta_V\gamma)(x, y, z, w)) = (\delta_{\mathscr{M}} i\gamma)(x, y, z, w)$. Writing $i\gamma = \delta_{\mathscr{M}}\beta$, one finds after some computational manipulations that $(\delta_{\mathscr{M}} i\gamma)(x, y, z, w) = 0\ \forall\ x, y, z, w \in \mathscr{L}$, which is enough to conclude that $(\delta_V\gamma)(x, y, z, w) = 0\ \forall\ x, y, z, w \in \mathscr{L}$ due to the injectivity of i. Note that $\delta_{\mathscr{M}} \circ \delta_{\mathscr{M}}$ is not automatically zero, as there is only an outer action. We obtain that every crossed module gives rise to a 3-cocycle.

Next, let us show that equivalent crossed modules give rise to equivalent 3-cocycles. First, we need to show the analogue of Remark 2.2.2, i.e. different choices of sections ρ, σ, τ give rise

to 3-cocycles differing by a coboundary. Let ρ and ρ' be different sections of π. As $(\rho'-\rho)(x) \in \ker\pi$, we can write $\rho'(x) = \rho(x) + c(x)$ for $c : \mathscr{L} \to \ker\pi$. The default α' of ρ' being a Lie algebra homomorphism can then be written as $\alpha'(x,y) = \alpha(x,y) + (\delta_{\mathscr{N}} c)(x,y) + [c(x),c(y)]_{\mathscr{N}}$, where $\delta_{\mathscr{N}}$ is the formal coboundary (2.31) with values in $\mathscr{N}$ and $c : \mathscr{L} \to \mathscr{N}$ is considered as 1-cochain. Since $(\delta_{\mathscr{N}} c)(x,y)$ and $[c(x),c(y)]_{\mathscr{N}}$ lie in $\ker\pi = \operatorname{im}\mu$, there exists $\epsilon(x,y) \in \mathscr{M}$ and $\theta(x,y) \in \mathscr{M}$ such that $(\delta_{\mathscr{N}} c)(x,y) = (\mu\circ\epsilon)(x,y)$ and $[c(x),c(y)]_{\mathscr{N}} = (\mu\circ\theta)(x,y)$, respectively. We therefore obtain $\alpha'(x,y) = (\mu\circ\beta')(x,y) = (\mu\circ\beta)(x,y) + (\mu\circ\epsilon)(x,y) + (\mu\circ\theta)(x,y)$. Lifting this expression from $\mathscr{M}$ to $\mathscr{N}$, one obtains $\beta'(x,y) = \beta(x,y) + \epsilon(x,y) + \theta(x,y) + \zeta(x,y)$, for some $\zeta(x,y) \in \ker\mu = \operatorname{im} i$. To obtain $i(\gamma'-\gamma) = \delta_{\mathscr{M}}(\beta'-\beta)$, we need to apply $\delta_{\mathscr{M}}$ to the obtained expression. The aim is then to show that the obtained terms will be coboundary terms. The term $(\delta_{\mathscr{M}}\zeta)(x,y)$ yields a coboundary term with values in V, since $\zeta(x,y) \in \operatorname{im} i$ hence $(\delta_{\mathscr{M}}\zeta)(x,y) = (\delta_{\mathscr{M}} i(\lambda))(x,y) = i(\delta_V\lambda)(x,y)$ for some $\lambda(x,y) \in V$. Concerning the term with ϵ, we have $\epsilon(x,y) = (\sigma\delta_{\mathscr{N}} c)(x,y)$ and a direct computation using the first property in Definition 2.2.7 yields $\mu\big((\sigma\delta_{\mathscr{N}} c)(x,y) - (\delta_{\mathscr{M}}\sigma c)(x,y)\big) = 0$, hence $(\sigma\delta_{\mathscr{N}} c)(x,y) - (\delta_{\mathscr{M}}\sigma c)(x,y) \in \ker\mu = \operatorname{im} i$ and there exists some $\lambda(x,y) \in V$ such that $\epsilon(x,y) - (\delta_{\mathscr{M}}\sigma c)(x,y) = i(\lambda(x,y))$. Applying $\delta_{\mathscr{M}}$ to this last expression yields after some algebraic manipulations that $\delta_{\mathscr{M}}\epsilon$ must be a coboundary. The same reasoning holds true for the term with θ. Hence $i(\gamma'-\gamma)$ is a coboundary and changing the section ρ does not affect the cohomology class.
Next, let us consider the sections σ, σ'. These will give two different $\beta = \sigma\alpha$ and $\beta' = \sigma'\alpha$. We have $\beta - \beta' \in \ker\mu = \operatorname{im} i$, hence there exists $\lambda \in V$ such that $i(\lambda) = \beta - \beta'$. Applying the coboundary operator $\delta_{\mathscr{M}}$, we obtain that $i(\gamma'-\gamma)$ is a coboundary by the same arguments as before. Concerning two sections τ and τ' of i, they are needed only on $\operatorname{im} i$, where they must be the same because i is an isomorphism from V to $\operatorname{im} i$. Therefore, the choice of different sections does not change the cohomology class.
Now, let (μ,η) and (μ',η') be two equivalent crossed modules as in Definition 2.2.8. We need to show that the associated 3-cocycles γ and γ' differ by a coboundary. Let ρ,σ,τ and ρ',σ',τ' be the sections associated to the crossed modules (μ,η) and (μ',η') respectively. As the diagram in (2.74) commutes, we have $\pi'(\psi\circ\rho) = \pi\circ\rho = \mathrm{id}_{\mathscr{L}}$, hence $\tilde{\rho}' := \psi\circ\rho$ is a section of π'. Thus, ρ' and $\tilde{\rho}'$ are two different sections of π'. As we saw above, they will yield cohomologous 3-cocycles, and we can work with $\tilde{\rho}'$ instead of ρ' since we are interested in cohomology classes, not cocycles. Let $\tilde{\alpha}'$ be the default of $\tilde{\rho}'$ to be a Lie algebra homomorphism. Let $\tilde{\beta}' = \sigma'\tilde{\alpha}'$ and $\beta = \sigma\alpha$ as before. Using the fact that the commutativity of the diagram (2.74) yields $\mu'\varphi = \psi\mu$ and that σ' and σ are sections of μ' and μ respectively, we have that $(\tilde{\beta}' - \varphi\circ\beta)(x,y) \in \ker\mu' = \operatorname{im} i'$, hence there exists $\lambda'(x,y) \in V$ such that $(\tilde{\beta}' - \varphi\circ\beta)(x,y) = i'\lambda'(x,y)$. Next, we consider $(\delta_{\mathscr{M}'}(\tilde{\beta}' - \varphi\circ\beta))(x,y,z) = (\delta_{\mathscr{M}'} i'\lambda')(x,y,z)$, where $\delta_{\mathscr{M}'}$ is the coboundary (2.31) with values in $\mathscr{M}'$ and the formal action $\eta'\circ\psi\circ\rho$. By computation and using the equivalence relation (2.73), one finds that $\delta_{\mathscr{M}'}\varphi\circ\beta = \varphi\delta_{\mathscr{M}}\beta$. Moreover, direct verification shows that we can exchange $\delta_{\mathscr{M}'} i'\lambda' = i'\delta_V\lambda'$, yielding at most a coboundary. With $i'\gamma' = \delta_{\mathscr{M}'}\tilde{\beta}'$ and $i\gamma = \delta_{\mathscr{M}}\beta$, we then obtain $i'\gamma' - \varphi i\gamma = i'\delta_V\lambda'$. As the diagram in (2.74) commutes, we furthermore obtain $i'\gamma' - i'\gamma = i'\delta_V\lambda'$. Taking the section τ', we obtain that γ' and γ are cohomologous.

It remains to show the injectivity of Φ. We want to show $\ker\Phi = 0$, hence whenever $[\gamma] = 0$ then the associated crossed module corresponds to the trivial crossed module of Definition 2.72. We proceed in three steps. First, we show that a crossed module associated to the zero cohomology class $[\gamma] = 0$ gives rise to an extension. This extension needs not be central nor abelian. In a second step, one shows that the map between the extension and the crossed module associated to $[\gamma] = 0$ is a morphism of crossed modules . In a third step, one shows that if a

crossed module admits such a morphism, then it is equivalent to the trivial crossed module.
Let (μ,η) be the crossed module associated to the zero cohomology class $[\gamma]=0$. If $[\gamma]=0$, then γ is a coboundary, i.e. there exists $\omega\in C^2(\mathscr{L},V)$ such that $\gamma=\delta_V\omega$. We define $\zeta(x,y):=\beta(x,y)-i(\omega(x,y))$. Using $i\gamma=\delta_{\mathscr{M}}\beta$, we obtain $\delta_{\mathscr{M}}\zeta=\delta_{\mathscr{M}}\beta-(\delta_{\mathscr{M}}i\circ\omega)=\delta_{\mathscr{M}}\beta-i(\delta_V\omega)=\delta_{\mathscr{M}}\beta-i(i^{-1}(\delta_{\mathscr{M}}\beta))=0$. Hence, $\zeta\in Z^2(\mathscr{L},\mathscr{M})$ is a 2-cocycle with values in $\mathscr{M}$ and thus defines an extension of $\mathscr{L}$ by $\mathscr{M}$,

$$0\hookrightarrow\mathscr{M}\hookrightarrow\mathscr{E}\twoheadrightarrow\mathscr{L}\twoheadrightarrow 0\,. \tag{2.75}$$

Hence, every crossed module associated to the zero cohomology class $[\gamma]=0$ gives rise to an extension.
Next, we construct the morphism. As a vector space, we have $\mathscr{E}=\mathscr{M}\oplus\mathscr{L}$, see Remark 2.2.2. Moreover, we have $\mathscr{N}=\operatorname{im}\mu\oplus\mathscr{L}$ on the level of vector spaces. In fact, the exact sequence associated to the crossed module (2.71) can be decomposed into two short exact sequences as in Remark 2.2.1, yielding:

$$0\to V\xrightarrow{i}\mathscr{M}\xrightarrow{\mu}\operatorname{im}\mu\to 0\qquad\text{and}\qquad 0\to\operatorname{im}\mu\xrightarrow{i}\mathscr{N}\xrightarrow{\pi}\mathscr{L}\to 0\,.$$

By the Remark 2.2.2, we then obtain from the second extension, $\mathscr{N}=\operatorname{im}\mu\oplus\mathscr{L}$. The following map m,

$$m:\quad\mathscr{M}\oplus\mathscr{L}\to\operatorname{im}\mu\oplus\mathscr{L},\qquad (a,x)\mapsto(\mu(a),x)\,, \tag{2.76}$$

induces a map of crossed modules,

$$\begin{array}{ccccccccc} & & 0 & \hookrightarrow & \mathscr{M} & \longrightarrow & \mathscr{E} & \twoheadrightarrow & \mathscr{L} & \twoheadrightarrow & 0 \\ & & & & \|\,\text{id} & & \downarrow m & & \|\,\text{id} & & \\ 0 & \hookrightarrow & V & \xhookrightarrow{i} & \mathscr{M} & \xrightarrow{\mu} & \mathscr{N} & \xrightarrow{\pi} & \mathscr{L} & \twoheadrightarrow & 0 \end{array}\,. \tag{2.77}$$

Let us check whether m is a morphism of crossed modules. The Lie bracket in the extension $\mathscr{N}$ is given by, for $a,b\in\operatorname{im}\mu\subset\mathscr{N}$ and $x,y\in\mathscr{L}$,

$$[(a,x),(b,y)]_{\mathscr{N}}=([a,b]_{\mathscr{N}}|_{\operatorname{im}\mu}+x\cdot b-y\cdot a+\alpha(x,y),[x,y]_{\mathscr{L}})\,, \tag{2.78}$$

where the dot in e.g. $x\cdot b$ denotes the action of $\mathscr{L}$ on $\mathscr{N}$, given by $[\rho(x),b]_{\mathscr{N}}$. Similarly, the Lie bracket in the extension $\mathscr{E}$ is given by, for $a,b\in\mathscr{M}$ and $x,y\in\mathscr{L}$:

$$[(a,x),(b,y)]_{\mathscr{E}}=([a,b]_{\mathscr{M}}+x\cdot b-y\cdot a+\zeta(x,y),[x,y]_{\mathscr{L}})\,, \tag{2.79}$$

where the dot in e.g. $x\cdot b$ denotes the action of $\mathscr{L}$ on $\mathscr{M}$, given by $\eta(\rho(x))(b)$. Using the properties of a crossed module 2.2.7, and $\mu\zeta=\mu(\beta-i\omega)=\alpha$ as $\mu\beta=\mu\sigma\alpha=\alpha$ and $\ker\mu=\operatorname{im}i$, a direct computation shows that $m\big([(a,x),(b,y)]_{\mathscr{E}}\big)=\big[m((a,x)),m\big((b,y)\big)\big]_{\mathscr{N}}$. In conclusion, we have that m is a morphism of crossed modules.
Finally, one can show that if a crossed module admits a morphism m,

$$\begin{array}{ccccccccc} & & 0 & \hookrightarrow & \mathscr{M} & \xrightarrow{\varphi} & \mathscr{E} & \xrightarrow{\psi} & \mathscr{L} & \twoheadrightarrow & 0 \\ & & & & \|\,\text{id} & & \downarrow m & & \|\,\text{id} & & \\ 0 & \hookrightarrow & V & \xhookrightarrow{i} & \mathscr{M} & \xrightarrow{\mu} & \mathscr{N} & \xrightarrow{\pi} & \mathscr{L} & \twoheadrightarrow & 0 \end{array}\,, \tag{2.80}$$

then it represents the trivial equivalence class. In fact, if m exists, then one can construct the following commutative diagram,

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & V & = & V & \xrightarrow{0} & \mathscr{L} & = & \mathscr{L} & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle \mathrm{proj}_1} & & \uparrow{\scriptstyle \psi} & & \| & & \\
0 & \longrightarrow & V & \xrightarrow{\mathrm{incl}_1} & V \oplus \mathscr{M} & \xrightarrow{(0,\varphi)} & \mathscr{E} & \xrightarrow{\psi} & \mathscr{L} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle (i,\mathrm{proj}_2)} & & \downarrow{\scriptstyle m} & & \| & & \\
0 & \longrightarrow & V & \xrightarrow{i} & \mathscr{M} & \xrightarrow{\mu} & \mathscr{N} & \xrightarrow{\pi} & \mathscr{L} & \longrightarrow & 0
\end{array} \quad . \tag{2.81}$$

This is an equivalence of crossed modules. Therefore, μ represents the zero map, meaning that the crossed module is equivalent to the trivial crossed module as defined in (2.72). □

The proof of the surjectivity of Φ is taken from [124, 128].

Proposition 2.2.4. *The map* $\Phi : \mathit{crmod}\,(\mathscr{L}, V) \to \mathrm{H}^3(\mathscr{L}, V)$ *is surjective.*

Proof. We want to show the surjectivity of Φ, i.e. we want to show that for each 3-cocycle class $[\gamma] \in \mathrm{H}^3(\mathscr{L}, V)$, there is an associated equivalence class of crossed modules given by the pre-image of Φ. One way to do this is to fix $\pi : \mathscr{N} \to \mathscr{L}$ in (2.71) and hence also $\mathscr{N}$. With three terms fixed in the four-term exact sequence (2.71), this boils down to constructing $\mathscr{M}$ as a semidirect sum with a twist, and finding a suitable action η. Basically, this boils down to an extension problem similar to the ones we saw previously in this section. This procedure can be found in the text by Kassel and Loday [68]. However, in the present text we prefer to present the more general setting and not fix $\mathscr{N}$, hence we will follow the construction by Wagemann given in [124, 128], which he called the *Principal Construction.*
The plan of the proof is as follows:

1. The 3-cocycle class $[\gamma] \in \mathrm{H}^3(\mathscr{L}, V)$ is given, and with $\mathscr{L}$ and V given, we construct a short exact sequence of $\mathscr{L}$-modules.

2. The short exact sequence of point 1. gives rise to a long exact sequence in cohomology in accordance with Theorem 2.2.2.

3. Using the long exact sequence, we construct a 2-cocycle $[\alpha]$ starting from $[\gamma]$.

4. Using Theorem 2.2.3, we can construct an abelian extension with $[\alpha]$.

5. Gluing the short exact sequences of points 1. and 4. together, we obtain a four-term exact sequence.

6. We construct an action η and a Lie homomorphism μ and verify that together with the four-term exact sequence of point 5., they verify the properties of a crossed module as given in Definition 2.2.7.

7. Proceeding as in the proof of Proposition 2.2.3, we construct the 3-cocycle class $[\gamma']$ associated to the crossed module we constructed in point 6.

8. We have a closer look at the constructions used in the proof and see that $[\gamma] = [\gamma']$. Therefore, starting with $[\gamma]$, we constructed a crossed module whose associated 3-cocycle is $[\gamma]$, i.e. the crossed module is the pre-image of $[\gamma]$ under Φ, which allows to conclude.

Let $[\gamma] \in \mathrm{H}^3(\mathscr{L},V)$ be a given 3-cocycle class. Thus, we also have a Lie algebra $\mathscr{L}$ and a $\mathscr{L}$-module V given. In the following, we will use some notions and results from more advanced algebraic cohomology. We will not introduce these in detail here, because they will not be used in the rest of the thesis. The important result is that the category of $\mathscr{L}$-modules possesses enough so-called injectives, which guarantees the existence of a so-called injective $\mathscr{L}$-module I and an injective homomorphism $i: V \hookrightarrow I$. For more details on this well-known result, see e.g. Corollary 7.3.4 of [131]. With this data, we can construct a short exact sequence of the following form,

$$0 \longrightarrow V \overset{i}{\hookrightarrow} I \overset{\pi}{\longrightarrow} Q \longrightarrow 0, \tag{2.82}$$

where $Q = \dfrac{I}{\operatorname{im} i}$ is the cokernel of i. This is a short exact sequence of $\mathscr{L}$-modules and thus induces a short exact sequence of cochain complexes:

$$0 \longrightarrow C^*(\mathscr{L},V) \overset{i}{\hookrightarrow} C^*(\mathscr{L},I) \overset{\pi}{\longrightarrow} C^*(\mathscr{L},Q) \longrightarrow 0.$$

According to Theorem 2.2.2, this gives rise to a long exact sequence in cohomology,

$$\ldots \longrightarrow \mathrm{H}^2(\mathscr{L},V) \xrightarrow{\mathrm{H}(i)} \mathrm{H}^2(\mathscr{L},I) \xrightarrow{\mathrm{H}(\pi)} \mathrm{H}^2(\mathscr{L},Q) \xrightarrow{\Delta} \mathrm{H}^3(\mathscr{L},V) \xrightarrow{\mathrm{H}(i)} \mathrm{H}^3(\mathscr{L},I) \xrightarrow{\mathrm{H}(\pi)} \mathrm{H}^3(\mathscr{L},Q) \longrightarrow \ldots$$

A standard result of homological algebra states that $\mathrm{H}^q(\mathscr{L},I) = 0$ for $q > 0$ if I is injective, see e.g. chapters 2 and 3 of [131], in particular exercise 2.5.1. Therefore, due to exactness, we have that the connecting homomorphism Δ is an isomorphism. From there, we can obtain a 2-cocycle class $[\alpha] \in \mathrm{H}^2(\mathscr{L},Q)$ by taking the pre-image of $[\gamma]$ under Δ, $[\alpha] = \Delta^{-1}[\gamma]$. Moreover, the $\mathscr{L}$-module Q can be considered as a Lie algebra equipped with the trivial Lie bracket, i.e. it will be an abelian Lie algebra. Therefore, with $[\alpha]$, we can construct an abelian extension $\mathscr{E}$ of $\mathscr{L}$ by Q due to Theorem 2.2.3,

$$0 \longrightarrow Q \overset{i'}{\longrightarrow} \mathscr{E} \overset{\pi'}{\longrightarrow} \mathscr{L} \longrightarrow 0. \tag{2.83}$$

Next, we glue the short exact sequences (2.82) and (2.83) together as in Remark 2.2.1 to obtain a four-term exact sequence,

$$0 \longrightarrow V \longrightarrow I \overset{\mu}{\longrightarrow} \mathscr{E} \longrightarrow \mathscr{L} \longrightarrow 0. \tag{2.84}$$

The map μ is induced by π and i' and given by $\mu: I \to \mathscr{E}$, $(q,v) \mapsto (q,0)$, where we have $I = Q \oplus V$ and $\mathscr{E} = Q \oplus \mathscr{L}$ on the level of vector spaces. The η-action of $\mathscr{E} = Q \oplus \mathscr{L}$ on I is induced by the action of $\mathscr{L}$ on I, $\eta(e)(i) = \eta(q,x)(i) := x \cdot i$, where $e \in \mathscr{E}$, $i \in I$, $q \in Q$ and $x \in \mathscr{L}$. Considering the $\mathscr{L}$-module I as a Lie algebra with trivial bracket, we obtain that the second condition in Definition 2.2.7 is trivially satisfied. Using the definition of the Lie bracket (2.38) of an abelian extension, a direct verification shows that the first condition in Definition 2.2.7 is also satisfied. Therefore, the sequence (2.84) corresponds to a crossed module.
Finally, we need to check whether the associated 3-cocycle γ' is in the cohomology class of the 3-cocycle γ we started with. We already saw how to obtain the 3-cocycle associated to a given crossed module in the proof of Proposition 2.2.3. In fact, the 3-cocycle γ' is given by: $\gamma' = i^{-1}\delta_I\beta'$ with $\beta' = \sigma\alpha'$, where σ is a section of μ and $\beta' \in \mathrm{H}^2(\mathscr{L},I)$, $\alpha' \in \mathrm{H}^2(\mathscr{L},\mathscr{E})$. To see whether $[\gamma'] = [\gamma]$, we need to have a closer look at the definition of Δ in order to obtain an

explicit expression for $[\gamma]$. In fact, the connecting homomorphism Δ is defined as follows, see the Remark 2.2.4:

$$
\begin{array}{ccccccc}
 & & & & [\beta] & \xleftarrow{\;\mathrm{H}(\pi^{-1})\;} & [\alpha] \\
 & & & & \Big\uparrow & & \rotatebox{90}{$\in$} \\
\cdots \longrightarrow & \mathrm{H}^2(\mathscr{L},V) & \xhookrightarrow{\mathrm{H}(i)} & & \mathrm{H}^2(\mathscr{L},I) & \xrightarrow{\mathrm{H}(\pi)} & \mathrm{H}^2(\mathscr{L},Q) \\
 & & & & \Big\downarrow \delta^2_I & & \\
 & \mathrm{H}^3(\mathscr{L},V) & \xhookrightarrow{\mathrm{H}(i)} & & \mathrm{H}^3(\mathscr{L},I) & \xrightarrow{\mathrm{H}(\pi)} & \mathrm{H}^3(\mathscr{L},Q) \longrightarrow \cdots \\
 & \rotatebox{90}{$\ni$} & & & \Big\downarrow & & \\
 & [\gamma] = \Delta[\alpha] & \xleftarrow[\mathrm{H}(i^{-1})]{} & & [\delta^2_I \beta] & &
\end{array}
\tag{2.85}
$$

In the last step, the pre-image of $\delta^2_I\beta$ under i exists since $\delta^2_Q\alpha = 0$ and hence $\delta^2_I\beta$ takes values in $V \subseteq I = V \oplus Q$. The map π in the diagram (2.85) above, extended by the zero map on the second factor, actually corresponds to the map μ, i.e. $(\pi,0) = \mu$. We see that this construction, starting from $[\alpha] \in \mathrm{H}^2(\mathscr{L},Q)$, corresponds exactly to the construction in the proof of Proposition 2.2.3, recalled above the diagram (2.85), starting with $[\alpha'] \in \mathrm{H}^2(\mathscr{L},\mathscr{E})$, where $\mathscr{E} = Q \oplus \mathscr{L}$. Hence, we can choose as starting cocycle class $[\alpha'] = [(\alpha,0)]$, where $(\alpha,0)$ is the 3-cocycle α corresponding to the abelian extension (2.83) extended by the zero map on the second factor. We then obtain $[\gamma'] = [\gamma]$. Note that we are working directly with cohomology classes and not cocycles, hence the addition of coboundaries leads to at most equivalent crossed modules.

□

Remark 2.2.14. The motivation to use two extensions to construct the crossed module in the proof of Proposition 2.2.4 stems from the fact that a crossed module can be split into an abelian and a general, i.e. not necessarily abelian, extension. Actually, in accordance with the Remark 2.2.1, a crossed module as in (2.71) can be split into

$$0 \longrightarrow V \xrightarrow{\;i\;} \mathscr{M} \xrightarrow{\;\mu\;} \operatorname{im}\mu \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \operatorname{im}\mu \xrightarrow{\;\mu\;} \mathscr{N} \xrightarrow{\;\pi\;} \mathscr{L} \longrightarrow 0\,, \tag{2.86}$$

where the first extension is an abelian extension and the second one is a general extension. In fact, in the proof of Proposition 2.2.4, a central extension (2.82) and an abelian extension (2.83) were used. The $\mathscr{L}$-modules V, I and Q in (2.82) are indeed seen as Lie algebras with trivial Lie bracket.

Corollary 2.2.1. *Every crossed module is equivalent to a crossed module stemming from the Principal Construction.*

Crossed modules are useful to classify the so-called Lie-2-algebras, which is a categorified version of Lie algebras, see the original article by Baez and Crans [10], and also e.g. [128]. Moreover, crossed modules and $\mathrm{H}^3(\mathscr{L},V)$ with V abelian also classify so-called double central extensions, which are a kind of two-dimensional central extensions, see the article by Rodelo and Van der Linden [93].

2.2.5 Spectral sequences

One of the aims of spectral sequences is to avoid having to compute the cohomology of a cochain complex in one go. Instead, the computation is broken down into small steps, which is more manageable. Spectral sequences were first introduced by Leray [77]; their significance in algebraic topology was discovered by Serre [112]. Nowadays, spectral sequences are the main computational tool in homological algebra.

Spectral sequences based on filtrations

The main source of spectral sequences are spectral sequences arising from filtrations. An example of a spectral sequence based on filtration is the Hochschild-Serre spectral sequence, which we will introduce later since we will use it. We will follow here mostly the presentation of spectral sequences as given in [39].

Definition 2.2.9.
- Let C be an abelian group. A *filtration* of C is a family of subgroups $F^pC \subseteq C$, $p \in \mathbb{Z}$, such that $F^p \subseteq F^q$ if $q < p$.
- A filtration $\{F^pC\}$ is called *positive* if $F^pC = C$ for $p \leq 0$.
- A filtration $\{F^pC\}$ is called *finite* if only finitely many terms F^pC of the filtration are different from 0 and C.
- The *adjoint graded* group **Gr** C of C is defined as **Gr** $C = \bigoplus_p \dfrac{F^pC}{F^{p+1}C}$.

With these definitions, a finite positive filtration is thus of the form,

$$C = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^n \supseteq F^{n+1} = 0\,, \tag{2.87}$$

with $F^pC = C$ for $p < 0$ and $F^pC = 0$ for $p > n+1$. In our case, we are interested in C being a cochain complex of Lie algebras $C^q(\mathscr{L}, M)$. Thus, let C have a grading, $C = \bigoplus_q C^q$, and a differential $\delta : C^q \to C^{q+1}$ such that $\delta(F^pC) \subseteq F^pC$. The grading and the filtration must be compatible, i.e. one must have $F^pC = \bigoplus_q F^pC^q$ with $F^pC^q = F^pC \cap C^q$.
The filtration $(F^pC, \delta|_{F^pC})$ of the complex (C, δ) gives rise to a filtration of the corresponding cohomology, $\mathrm{H}(F^pC, \delta|_{F^pC})$, which is denoted by $F^p\mathrm{H}$,

$$F^p\mathrm{H} := \mathrm{H}(F^pC, \delta|_{F^pC}) = \frac{F^pC \cap \ker\delta}{F^pC \cap \delta(C)}\,. \tag{2.88}$$

Concerning the adjoint graded complex **Gr** C appearing in the Definition 2.2.9, it is interesting because it has many properties in common with the original complex C. For example, if C is a finite-dimensional vector space, then **Gr** C has the same dimension as C.
Again, the filtration in the adjoint graded complex gives rise to a adjoint graded complex in cohomology,

$$\begin{aligned}\mathbf{Gr}\,\mathrm{H}(C) = \bigoplus_p \frac{F^p\mathrm{H}}{F^{p+1}\mathrm{H}} &= \bigoplus_p \frac{F^pC \cap \ker\delta}{F^pC \cap \delta(C)}\,\frac{F^{p+1}C \cap \delta(C)}{F^{p+1}C \cap \ker\delta} \\ &= \bigoplus_p \frac{F^pC \cap \ker\delta}{(F^pC \cap \delta(C)) + (F^{p+1}C \cap \ker\delta)}\,.\end{aligned} \tag{2.89}$$

Again, **Gr** $\mathrm{H}(C)$ and $\mathrm{H}(C)$ are closely related, hence knowing **Gr** $\mathrm{H}(C)$ will provide substantial information on $\mathrm{H}(C)$.
Next, we can introduce the notion of a spectral sequence, which is given by so-called pages, defined below.

Definition 2.2.10. Let $\{F^p C\}$ be a filtration of a graded abelian group $C = \bigoplus_q C^q$. Then the *rth page* of a spectral sequence is given by the following space,

$$E_r^{pq} = \frac{F^p C^{p+q} \cap \delta^{-1}(F^{p+r} C^{p+q+1})}{\left[F^{p+1} C^{p+q} \cap \delta^{-1}(F^{p+r} C^{p+q+1})\right] + \left[F^p C^{p+q} \cap \delta(F^{p-r+1} C^{p+q-1})\right]} \qquad r, p, q \in \mathbb{N}, \tag{2.90}$$

equipped with a differential defined as follows,

$$d_r^{pq} : E_r^{pq} \to E_r^{p+r, q-r+1}. \tag{2.91}$$

The index p is called the *filtration degree*, the index q is called the *complementary degree*, and $p+q$ is called the *full* degree.

It can be shown, see Leray [77], that d_r^{pq} is well defined, i.e. the differential does not depend of the choice of the representative in $F^p C^{p+q} \cap \delta^{-1}(F^{p+r} C^{p+q+1})$. Moreover, the differentials satisfy $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ [77].
Let us have a closer look at the first two pages of a spectral sequence. The zeroth page is given by taking $r = 0$ in (2.90), yielding,

$$\begin{aligned} E_0^{pq} &= \frac{F^p C^{p+q} \cap \delta^{-1}(F^p C^{p+q+1})}{\left[F^{p+1} C^{p+q} \cap \delta^{-1}(F^p C^{p+q+1})\right] + \left[F^p C^{p+q} \cap \delta(F^{p+1} C^{p+q-1})\right]} \\ &= \frac{F^p C^{p+q}}{F^{p+1} C^{p+q} + \delta(F^{p+1} C^{p+q-1})} = \frac{F^p C^{p+q}}{F^{p+1} C^{p+q}}. \end{aligned}$$

We see that the zeroth page is given by the adjoint graded complex. Since the adjoint graded complex is closely related to the original complex, we see that the starting point of the spectral sequence is almost the original complex. The first page is obtained by taking $r = 1$ in (2.90):

$$\begin{aligned} E_1^{pq} &= \frac{F^p C^{p+q} \cap \delta^{-1}(F^{p+1} C^{p+q+1})}{\left[F^{p+1} C^{p+q} \cap \delta^{-1}(F^{p+1} C^{p+q+1})\right] + \left[F^p C^{p+q} \cap \delta(F^p C^{p+q-1})\right]} \\ &= \frac{F^p C^{p+q} \cap \delta^{-1}(F^{p+1} C^{p+q+1})}{F^{p+1} C^{p+q} + \delta(F^p C^{p+q-1})} = \frac{\ker d_0^{p,q}}{\operatorname{im} d_0^{p,q-1}}, \end{aligned} \tag{2.92}$$

where $d_0^{p,q}$ is the differential of the zeroth page, induced by the differential of the complex,

$$d_0^{p,q} = \delta^{p+q}\big|_{\frac{F^p C}{F^{p+1} C}} : \quad \frac{F^p C^{p+q}}{F^{p+1} C^{p+q}} \to \frac{F^p C^{p+q+1}}{F^{p+1} C^{p+q+1}}.$$

We see that the first page gives us the cohomology of the adjoint graded complex with respect to the differential d_0^{pq}, i.e. $\mathrm{H}(\mathbf{Gr}\, C, d^0)$. One could hope that this would be the same as $\mathbf{Gr}\, \mathrm{H}(C, \delta)$, so that one was able to compute the cohomology of the original complex in two steps. However, in general $\mathrm{H}(\mathbf{Gr}\, C)$ is bigger than $\mathbf{Gr}\, \mathrm{H}(C)$, thus these two spaces are not equal in general. Actually, the original differential δ takes values in different filtration levels, whereas the differential d_0 takes values only in the filtration level of its argument. Nevertheless, spectral sequences allow for a transition from $\mathrm{H}(\mathbf{Gr}\, C)$ to $\mathbf{Gr}\, \mathrm{H}(C)$, but more steps are needed. In fact, the differential δ of the original complex also induces a differential d_1 on the first page,

$$d_1^{pq} : \quad \frac{F^p C^{p+q} \cap \delta^{-1}(F^{p+1} C^{p+q+1})}{F^{p+1} C^{p+q} + \delta(F^p C^{p+q-1})} \to \frac{F^{p+1} C^{p+q+1} \cap \delta^{-1}(F^{p+2} C^{p+q+2})}{F^{p+2} C^{p+q+1} + \delta(F^{p+1} C^{p+q})}. \tag{2.93}$$

We see that d_1 goes down one level in the filtration (2.87). Together with d_0, this provides already a better approximation of δ, which can go down to any filtration level. In (2.92) we see that the first page E_1 is obtained by taking the cohomology of the zeroth page E_0 with respect to d_0. Continuing the reasoning, it is natural to take the cohomology of the first page with respect to d_1, in order to get a second page E_2. This will provide an even better approximation of **Gr** H(C). On the second page, we have a differential d_2 induced by δ, which will go down two levels in the filtration. Taking again the cohomology, one can obtain a third page, and so on. Actually, one can prove that the $(r+1)th$ page as defined in (2.90) can be obtained by taking the cohomology of the rth page with respect to the induced differential d_r as defined in (2.91), i.e. $E^{pq}_{r+1} \cong \ker d^{p,q}_r / \operatorname{im} d^{p-r,q+r-1}_r$, see [77].
The next aspect to consider is the end of the spectral sequence. If we take r in (2.90) big enough, we will have $F^{p+r}C = 0$ and $F^{p-r+1}C = C$, hence E_r will become independent of r. The page where the independence of r occurs is called the *infinity* page and denoted by E_∞. One says that the spectral sequence *converges* to E_∞. Taking r big in (2.90) , the infinity page reduces to:

$$E^{pq}_\infty = \frac{F^pC^{p+q} \cap \ker\delta}{\left[F^{p+1}C^{p+q} \cap \ker\delta\right] + \left[F^pC^{p+q} \cap \delta(C^{p+q-1})\right]} = \frac{F^p\mathrm{H}^{p+q}(C)}{F^{p+1}\mathrm{H}^{p+q}(C)}, \tag{2.94}$$

see (2.89) to obtain the last equality. We see that the infinity page gives the desired result, namely the adjoint graded complex in cohomology **Gr** H(C). To summarize, starting with the adjoint graded complex **Gr** C, one takes successively the cohomology to finally obtain the adjoint graded complex in cohomology **Gr** H(C). Thus, instead of trying to compute **Gr** H(C) in one go, one divides the computation of the cohomology into smaller, simpler steps. In practice though, the expressions for the pages and the differentials become complicated so fast, that, normally, one does not have explicit expressions for the differentials. Spectral sequences that converge rapidly though, are a very powerful tool for computing cohomology. One spectral sequence we will introduce in some more detail below will be the Hochschild-Serre spectral sequence.

Before continuing though, it is important to point out that each page of a spectral sequence has a standard two-dimensional graphical representation. The filtration degree is given on the horizontal axis, while the complementary degree is indicated on the vertical axis. The entries E^{pq}_r are placed into the cells (p,q), and the differentials are represented by arrows. For example, the zeroth and the first page come with vertical and horizontal arrows, respectively, see Figure 2.1, while the second page comes with differentials going two entries to the right and one entry down, see Figure 2.2.

The relation between E_∞ and **Gr** H is given by Equation (2.94), and it can also be represented graphically. For example, elements of full degree $n \in \mathbb{N}$, i.e. elements of **Gr** H^n, are given by the direct sum of the elements on the nth anti-diagonal of the graph of E_∞, see Figure 2.3.

This concludes our introduction to general spectral sequences, as we introduced all the notions we will need in this thesis.
For a more profound understanding of the basic ideas and functioning of spectral sequences, we refer the reader to the texts by Chow[21] and Mitchell [84]. For some examples of computations using spectral sequences, we refer the reader to the lecture notes by Diaz Ramos [23]. A historical overview of spectral sequences is provided by McCleary [81]. Finally, for a more thorough introduction on spectral sequences, see the textbook by McCleary [82].

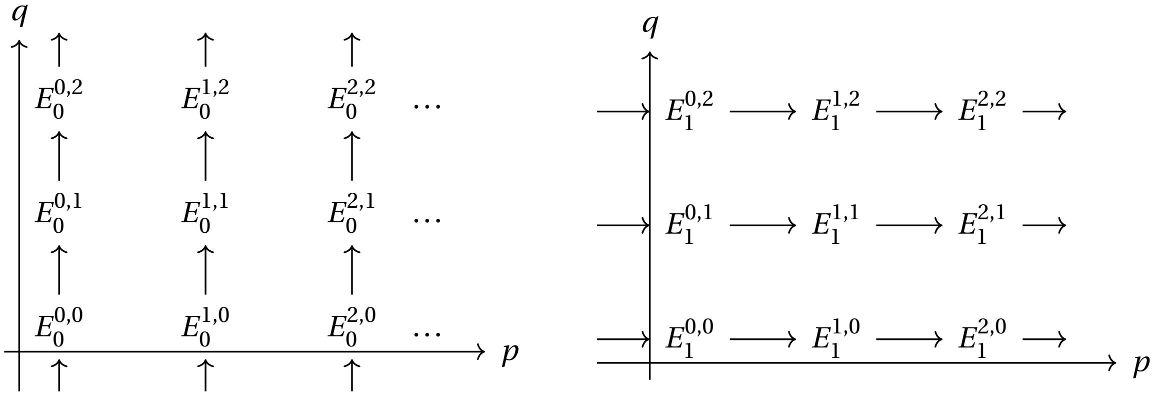


Figure 2.1: The zeroth page on the left has entries $E_0^{pq} = \dfrac{F^p C^{p+q}}{F^{p+1} C^{p+q}}$ and arrows $d_0^{pq} : E_0^{pq} \to E_0^{p,q+1}$, while the first page on the right has entries $E_1^{pq} = \ker d_0^{pq} / \operatorname{im} d_0^{p,q-1}$ and arrows $d_1^{pq} : E_1^{pq} \to E_1^{p+1,q}$.

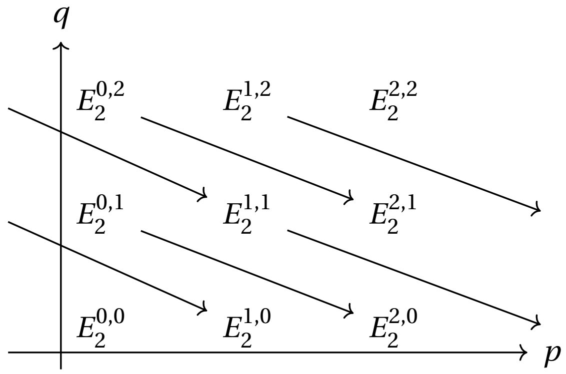


Figure 2.2: The second page has entries given by $E_2^{pq} = \ker d_1^{pq} / \operatorname{im} d_1^{p-1,q}$ and differentials $d_2^{pq} : E_2^{pq} \to E_2^{p+2,q-1}$.

The Hochschild-Serre spectral sequence

We will finish this section by briefly presenting the Hochschild-Serre spectral sequence, which will be the spectral sequence used in this thesis. The spectral sequence used in this thesis is a spectral sequence in the theory of Lie algebra cohomology. Therefore, the starting point will be the Chevalley-Eilenberg complex $C(\mathscr{L}, M)$, where $\mathscr{L}$ is a Lie algebra and M a $\mathscr{L}$-module. We denote by ψ^q the q-cochains of $C^q(\mathscr{L}, M)$. The zeroth page will be given by the adjoint graded complex associated to a certain filtration of the complex. By taking successively the cohomology, a spectral sequence is constructed that will converge to $\mathbf{Gr}\,\mathrm{H}(\mathscr{L}, M)$. Let $\mathscr{H} \subset \mathscr{L}$ be an ideal of $\mathscr{L}$, and let us consider an extension,

$$0 \longrightarrow \mathscr{H} \longrightarrow \mathscr{L} \longrightarrow \mathscr{E} \longrightarrow 0 .$$

The filtration used in the Hochschild-Serre spectral sequence is given by,

$$F^p C^{p+q}(\mathscr{L}, M) = \left\{ \psi^{p+q} \in C^{p+q}(\mathscr{L}, M) \mid \psi^{p+q}(x_1, \dots, x_{p+q}) = 0, \text{ if } x_1, \dots, x_{q+1} \in \mathscr{H} \right\} .$$

By definition,

$$C^n(\mathscr{L}, M) = F^0 C^n(\mathscr{L}, M) \supseteq \cdots \supseteq F^n C^n(\mathscr{L}, M) \supseteq F^{n+1} C^n(\mathscr{L}, M) = 0 ,$$

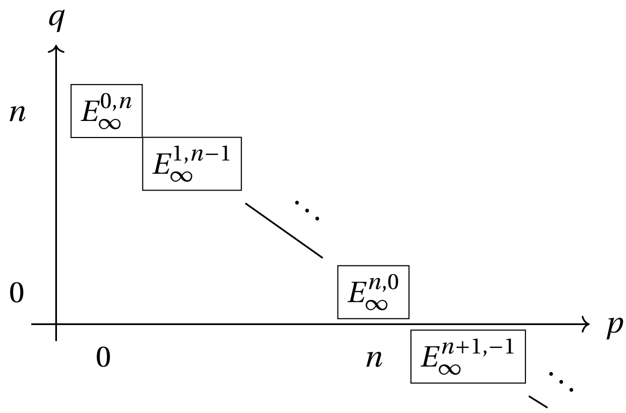


Figure 2.3: Elements of **Gr** H^n are given by **Gr** $\mathrm{H}^n = \bigoplus_{p+q=n} E_\infty^{p,q} = \bigoplus_p \frac{F^p\mathrm{H}^n}{F^{p+1}\mathrm{H}^n}$, i.e. they are obtained by taking the sum over the nth anti-diagonal.

and $\delta F^p C^{p+q}(\mathscr{L},M) \subseteq F^p C^{p+q+1}(\mathscr{L},M)$, where δ is the differential of the complex $C(\mathscr{L},M)$, hence $\{F^p\}$ is a filtration of $C(\mathscr{L},M)$. By taking the cohomology of the zeroth page $E_0^{pq} = \frac{F^p C^{p+q}}{F^{p+1}C^{p+q}}$ with respect of the differential d_0^{pq} of the zeroth page, one can prove that the first page of the spectral sequence is given by,

$$E_1^{pq} = \mathrm{H}^q\left(\mathscr{H}, C^p\left(\frac{\mathscr{L}}{\mathscr{H}}, M\right)\right).$$

Taking again the cohomology with respect to the differential d_1^{pq} of the first page, one can prove that the second page of the Hochschild-Serre spectral sequence is given by,

$$E_2^{pq} = \mathrm{H}^p\left(\frac{\mathscr{L}}{\mathscr{H}}, \mathrm{H}^q(\mathscr{H}, M)\right).$$

For details of these computations, we refer the reader to Theorem 1.5.1 p. 40 of [43]. In the cases treated in this thesis, the spectral sequence starts with the second page and will already converge on the third page. For later convenience, we summarize the results for the Hochschild-Serre spectral sequence in Theorem 2.2.6.

Theorem [Hochschild-Serre] 2.2.6. *Let $\mathscr{H}$ be an ideal of a Lie algebra $\mathscr{L}$. Then there is a first quadrant convergent spectral sequence,*

$$E_2^{pq} = \mathrm{H}^p\left(\frac{\mathscr{L}}{\mathscr{H}}, \mathrm{H}^q(\mathscr{H}, M)\right) \Rightarrow \mathrm{H}^{p+q}(\mathscr{L}, M),$$

where M is a $\mathscr{L}$-module and via $\mathscr{H} \hookrightarrow \mathscr{L}$ also a $\mathscr{H}$-module.

Proof. A very concise proof of this theorem can be found for example in Theorem 7.5.2 p. 232 of [131]. The original literature consists of the articles [60, 61] by Hochschild and Serre, see also Lyndon [80]. □

In general, explicit expressions for the differentials of the Hochschild-Serre spectral sequence are not known. However, in the case of the first few pages, results are known see e.g. the articles by André [7], Hilton and Stammbach [56], Charlap and Vasquez [19] and Huebschmann [62–64]. We will come back to this later in the thesis, in Section 3.3.

2.2.6 Results for the algebraic cohomology

In order to give the reader an overview, we summarize in this section the results known in the case of the algebraic cohomology of the Witt, the Virasoro and the Krichever-Novikov vector field algebra. The summary includes the results obtained in this thesis as well as previously obtained results. To highlight the work done in this thesis, and to distinguish the new results from old results, we underlined the results obtained in the present thesis. We first discuss the Witt and the Virasoro algebra, then we will discuss the Krichever-Novikov vector field algebra.
For the zeroth cohomology characterizing invariants, we immediately obtain from (2.42) the following results for the Witt and the Virasoro algebra:

$$\mathrm{H}^0(\mathscr{W},\mathbb{K}) = \mathbb{K}\,,\quad \mathrm{H}^0(\mathscr{W},\mathscr{W}) = \{0\}\,,\quad \dim\,\mathrm{H}^0(\mathscr{W},\mathscr{F}^\lambda) = \begin{cases} 1 \text{ if } \lambda = 0 \\ 0 \text{ else} \end{cases}\,,$$
$$\mathrm{H}^0(\mathscr{V},\mathbb{K}) = \mathbb{K}\,,\quad \mathrm{H}^0(\mathscr{V},\mathscr{V}) = \mathbb{K}\; t\,,\quad \dim\,\mathrm{H}^0(\mathscr{V},\mathscr{F}^\lambda) = \begin{cases} 1 \text{ if } \lambda = 0 \\ 0 \text{ else} \end{cases}\,,$$

where $t = i(1)$ is the central element. For $\mathrm{H}^0(\mathscr{W},\mathscr{F}^0)$, the action of $\mathscr{W}$ on $\mathscr{F}^0$ is given by $e_n \cdot f_m^0 = m f_{n+m}^0$. Hence, the element f_0^0 is $\mathscr{W}$-invariant, and we have $\dim\,\mathrm{H}^0(\mathscr{W},\mathscr{F}^0) = 1$. For $\lambda \neq 0$, there is no $\mathscr{W}$-invariant. The same holds true for the Virasoro algebra.
Concerning the first cohomology with values in the trivial module, we also get immediate results using the fact that the Witt and the Virasoro are perfect Lie algebras, as pointed out in Section 2.1. Regarding the adjoint module, we derive the results in this thesis for the Witt and the Virasoro algebra in Chapter 3. The results concerning the general tensor-densities modules are derived in Chapter 4.

$$\mathrm{H}^1(\mathscr{W},\mathbb{K}) = \{0\}\,,\quad \underline{\mathrm{H}^1(\mathscr{W},\mathscr{W})} = \{0\}\,,\quad \underline{\dim\,\mathrm{H}^1(\mathscr{W},\mathscr{F}^\lambda)} = \begin{cases} 2 \text{ if } \lambda = 0 \\ 1 \text{ if } \lambda = 1,2 \\ 0 \text{ else} \end{cases}\,,$$
$$\mathrm{H}^1(\mathscr{V},\mathbb{K}) = \{0\}\,,\quad \underline{\mathrm{H}^1(\mathscr{V},\mathscr{V})} = \{0\}\,,\quad \underline{\dim\,\mathrm{H}^1(\mathscr{V},\mathscr{F}^\lambda)} = \begin{cases} 2 \text{ if } \lambda = 0 \\ 1 \text{ if } \lambda = 1,2 \\ 0 \text{ else} \end{cases}\,.$$

The results for the trivial module are found by using (2.45) and the fact that the Lie algebras under consideration are perfect, i.e. they satisfy $[\mathscr{L},\mathscr{L}] = \mathscr{L}$. The results for the adjoint module state that all derivations of the Witt and the Virasoro algebra into themselves are inner. This is a well-known result in the case of the Witt algebra. In fact, according to Tang [116], this was proven by Zhu and Meng in [137]. Similarly, we find that all derivations of the Witt and the Virasoro algebra into the general tensor-densities modules $\mathscr{F}^\lambda$ are inner, except for $\lambda \in \{0,1,2\}$, where we also have outer derivations.

For the second cohomology, related to extensions and deformations, we have the following:

$$\dim\,\mathrm{H}^2(\mathscr{W},\mathbb{K}) = 1\,,\quad \mathrm{H}^2(\mathscr{W},\mathscr{W}) = \{0\}\,,\quad \underline{\dim\,\mathrm{H}^2(\mathscr{W},\mathscr{F}^\lambda)} = \begin{cases} 2 \text{ if } \lambda = 0,1,2 \\ 1 \text{ if } \lambda = 5,7 \\ 0 \text{ else} \end{cases}\,,$$
$$\mathrm{H}^2(\mathscr{V},\mathbb{K}) = \{0\}\,,\qquad \mathrm{H}^2(\mathscr{V},\mathscr{V}) = \{0\}\,,\quad \underline{\dim\,\mathrm{H}^2(\mathscr{V},\mathscr{F}^\lambda)} = \begin{cases} 2 \text{ if } \lambda = 1,2 \\ 1 \text{ if } \lambda = 0,5,7 \\ 0 \text{ else} \end{cases}\,,$$

The first result in the left column is a well known result and states that the Witt algebra admits a unique non-trivial central extension, which is universal. An algebraic proof can be found for

example in [8, 67]. The second result in the left column was shown in [106], and states that the Virasoro algebra admits no non-trivial central extension. The first two results for the adjoint module were shown in [34, 106, 106], stating that the Witt and the Virasoro algebra are infinitesimally and formally rigid. For the Witt and the Virasoro algebra, we obtain the results for the general tensor densities modules in Chapter 4. These results state that the Witt algebra admits two non-trivial abelian extensions by the modules $\mathcal{F}^0, \mathcal{F}^1, \mathcal{F}^2$, and one non-trivial extension by the modules $\mathcal{F}^5, \mathcal{F}^7$, while the Virasoro algebra has two non-trivial extensions by the modules $\mathcal{F}^1, \mathcal{F}^2$ and one non-trivial extension by the modules $\mathcal{F}^0, \mathcal{F}^5, \mathcal{F}^7$, where the modules $\mathcal{F}^\lambda$ are considered as Lie algebras with trivial Lie bracket.
For the third cohomology yielding crossed modules and obstructions, we get:

$$\begin{aligned}
&\underline{\dim \mathrm{H}^3(\mathcal{W},\mathbb{K})} = 1\,, \quad \underline{\mathrm{H}^3(\mathcal{W},\mathcal{W})} = \{0\}\,, \quad \underline{\mathrm{H}^3(\mathcal{W},\mathcal{F}^\lambda)} = \{0\} \text{ if } \lambda \in I\,,\\
&\underline{\dim \mathrm{H}^3(\mathcal{V},\mathbb{K})} = 1\,, \quad \underline{\dim \mathrm{H}^3(\mathcal{V},\mathcal{V})} = 1\,, \quad \underline{\mathrm{H}^3(\mathcal{V},\mathcal{F}^\lambda)} = \{0\} \text{ if } \lambda \in I\,,
\end{aligned}$$

where $I = \{-100,\dots,-1\} \cup \{6,8,10,14,16,18,20,22,24,26\}$. These results are derived in this thesis in the chapters 3 and 4.
The results on the Witt and the Virasoro algebra obtained in this thesis were already made publicly accessible in [26–28].
Next, let us give a summary of the results known in case of the Krichever-Novikov vector field algebra. For the zeroth algebraic cohomology, we have:

$$\mathrm{H}^0(\mathcal{KN},\mathbb{K}) = \mathbb{K}\,, \qquad \mathrm{H}^0(\mathcal{KN},\mathcal{KN}) = \{0\}\,, \qquad \dim \mathrm{H}^0(\mathcal{KN},\mathcal{F}^\lambda) = \begin{cases} 1 \text{ if } \lambda = 0 \\ 0 \text{ else} \end{cases}\,.$$

The first result is straightforward to obtain. In fact, $\mathbb{K}$ is a trivial $\mathcal{KN}$-module, hence every element of $\mathbb{K}$ is invariant under the $\mathcal{KN}$-action. Concerning the second result, we know that $\mathcal{KN}$ is a simple and hence a perfect Lie algebra, see Proposition 6.99 p. 153 [108]. Simple means that there are no non-trivial proper ideals in $\mathcal{KN}$, see Definition 2.1.5. On the other hand, we know that $\mathrm{H}^0(\mathcal{KN},\mathcal{KN}) = C(\mathcal{KN})$, see (2.42), and the center $C(\mathcal{KN})$ is an ideal, $C(\mathcal{KN}) \trianglelefteq \mathcal{KN}$, see Definitions 2.1.4 and 2.1.3. Since $\mathcal{KN}$ has no non-trivial proper ideals, it means that $C(\mathcal{KN})$ is either the zero-ideal or $\mathcal{KN}$ itself. However, $\mathcal{KN}$ is non-abelian, therefore $C(\mathcal{KN})$ cannot be $\mathcal{KN}$. Thus, $C(\mathcal{KN})$ is the zero-ideal and $\mathrm{H}^0(\mathcal{KN},\mathcal{KN})$ must be zero. To obtain the third result, more work is needed. We will present the proof in Chapter 5.

For the first algebraic cohomology, we have:

$$\mathrm{H}^1(\mathcal{KN},\mathbb{K}) = \{0\}\,.$$

As $\mathcal{KN}$ is a perfect Lie algebra, this result is immediate, see (2.5) and (2.45). To the best knowledge of the author, the first algebraic cohomology with values in the other modules is unknown.
For the second algebraic cohomology, we have the following results,

$$\dim \mathrm{H}^2_l(\mathcal{KN},\mathbb{K}) = 1\,, \qquad \dim \mathrm{H}^2_b(\mathcal{KN},\mathbb{K}) = K\,, \qquad \dim \mathrm{H}^2(\mathcal{KN},\mathbb{K}) = 2g + N - 1\,,$$

where K stands for the number of in-points, see (2.21), N is the total number of punctures, and g is the genus of the surface. The first and the second results were proven by Schlichenmaier [108], the third result can be derived from results by Skryabin [115]. The subscript l stands for *local* cohomology, b stands for *bounded* cohomology, and no subscript denotes the total cohomology. We will provide more details on this in Chapter 5. To the author's best knowledge, no results related to the other modules are known.
In Chapter 5, we derive an upper bound for the dimension of the third bounded algebraic cohomology,

$$\underline{\dim \mathrm{H}^3_b(\mathcal{KN},\mathbb{K})} \leq K\,.$$

Chapter 3

The trivial and the adjoint module

In this chapter, we present the results obtained related to the trivial and the adjoint module, for the Witt and the Virasoro algebra. The trivial and the adjoint modules are related to an infinite family of modules, which will be considered in the next Chapter 4. The results in this chapter concern mainly the third cohomology, although in the case of the adjoint module, we also give the computation of the first cohomology. Section 3.1 deals with the trivial module. Inspired by the results obtained in Section 3.1, we aim to find a canonical relation between the cohomology of the Witt and the Virasoro algebra at the end of Section 3.1. This will be deepened further in Section 3.3. Section 3.2 focuses on the adjoint module.
The main tools at our disposal are the cocycle condition and the coboundary condition, meaning the condition for a cochain to be a cocycle and a coboundary, respectively. When evaluated on combinations of basis elements e_i of the Witt or the Virasoro algebra, these conditions yield actually an infinite number of linear equations with an infinite number of variables. The aim is then to find recurrence relation between the variables in order to express all of them in terms of a finite number of variables. This provides information on the dimension of the cohomology. In case of a non-zero dimension, explicit generating cocycles are provided, which are obtained by either solving recurrence relations or by inspiration from continuous cohomology.
The analysis can be simplified by considering only the degree-zero cohomology, recall Theorem 2.2.1. However, the degree-non-zero part can be derived rather easily. We will not do it for the trivial module, as it will show up later in Chapter 5 anyway when considering the Krichever-Novikov vector field algebra. We will derive the degree-non-zero cohomology in case of the adjoint module though, as it will constitute a nice illustration of the proof of Theorem 2.2.1 with simple concrete examples.
The content in Sections 3.1 and 3.2 of this chapter has been published beforehand as mostly a verbatim reproduction in [26–29]. The results in Section 3.3 have not been published before.

3.1 The trivial module

3.1.1 Analysis of $\mathrm{H}^3(\mathcal{W},\mathbb{K})$ and $\mathrm{H}^3(\mathcal{V},\mathbb{K})$

In this section, we prove that the third algebraic cohomology of the Witt and the Virasoro algebra with coefficients in the trivial module is one-dimensional. We will start by writing down the appropriate cocycle and coboundary conditions. Subsequently, we will express cochains and cocycles in terms of their coefficients and derive the cocycle and coboundary conditions on the coefficients. These coefficients constitute an infinite number of variables. The proofs

for the Witt and the Virasoro algebra are very similar, the ones for the Virasoro algebra being only slightly longer than the ones for the Witt algebra. Here, we will write down the analysis for the Virasoro algebra. The corresponding analysis for the Witt algebra can be obtained in a straightforward manner by dropping the central terms.
The condition for a 3-cochain $\psi \in C^3(\mathcal{V},\mathbb{K})$ to be a 3-cocycle with values in the trivial module can be deduced from (2.68) and is given by:

$$\begin{aligned}(\delta_3\psi)(x_1,x_2,x_3,x_4) =& \psi([x_1,x_2],x_3,x_4) - \psi([x_1,x_3],x_2,x_4) + \psi([x_1,x_4],x_2,x_3)\\ &+ \psi([x_2,x_3],x_1,x_4) - \psi([x_2,x_4],x_1,x_3) + \psi([x_3,x_4],x_1,x_2) = 0\,,\end{aligned} \tag{3.1}$$

where $x_1,x_2,x_3,x_4 \in \mathcal{V}$. The condition for a 3-cocycle $\psi \in Z^3(\mathcal{V},\mathbb{K})$ to be a coboundary with values in the trivial module can be deduced from (2.67) and is given by:

$$\psi(x_1,x_2,x_3) = (\delta_2\phi)(x_1,x_2,x_3) = \phi([x_1,x_2],x_3) + \phi([x_2,x_3],x_1) + \phi([x_3,x_1],x_2)\,, \tag{3.2}$$

where $\phi \in \mathrm{C}^2(\mathcal{V},\mathbb{K})$.
Recall from (2.19) the Lie structure for the Virasoro algebra. In order to simplify the notation, we will denote in the following the generators $\hat{e}_i$ of the Virasoro algebra simply by e_i, omitting the hat. Inserting the Lie structure (2.19) into the cocycle condition (3.1), evaluated on the basis elements e_i,e_j,e_k,e_l, we obtain:

$$\begin{aligned}(\delta_3\psi)(e_i,e_j,e_k,e_l) =& \psi\left([e_i,e_j],e_k,e_l\right) - \psi\left([e_i,e_k],e_j,e_l\right) + \psi\left([e_i,e_l],e_j,e_k\right)\\ &+ \psi\left([e_j,e_k],e_i,e_l\right) - \psi\left([e_j,e_l],e_i,e_k\right) + \psi\left([e_k,e_l],e_i,e_j\right) = 0\\ \Leftrightarrow 0 =& (j-i)\psi(e_{i+j},e_k,e_l) + \alpha(e_i,e_j)\psi(t,e_k,e_l)\\ &-(k-i)\psi(e_{i+k},e_j,e_l) - \alpha(e_i,e_k)\psi(t,e_j,e_l)\\ &+(l-i)\psi(e_{i+l},e_j,e_k) + \alpha(e_i,e_l)\psi(t,e_j,e_k)\\ &+(k-j)\psi(e_{j+k},e_i,e_l) + \alpha(e_j,e_k)\psi(t,e_i,e_l)\\ &-(l-j)\psi(e_{j+l},e_i,e_k) - \alpha(e_j,e_l)\psi(t,e_i,e_k)\\ &+(l-k)\psi(e_{l+k},e_i,e_j) + \alpha(e_k,e_l)\psi(t,e_i,e_j)\,,\end{aligned} \tag{3.3}$$

where α is the Virasoro 2-cocycle, given in (2.20). Let ψ be a 3-cochain and ϕ a 2-cochain of $C^3(\mathcal{V},\mathbb{K})$ and $C^2(\mathcal{V},\mathbb{K})$, respectively. These cochains will be given by their system of coefficients $\phi_{i,j},b_i,\psi_{i,j,k},c_{i,j} \in \mathbb{K}$ defined as follows:

$$\begin{aligned}\psi(e_i,e_j,e_k) &:= \psi_{i,j,k} \qquad \text{and} \qquad \psi(e_i,e_j,t) := c_{i,j}\,,\\ \phi(e_i,e_j) &:= \phi_{i,j} \qquad \text{and} \qquad \phi(e_i,t) := b_i\,,\end{aligned} \tag{3.4}$$

with the obvious identification coming from the alternating property of the cochains. In the case of the Witt algebra, only the coefficients on the left are present.
Replacing the cocycle ψ in (3.3) by its coefficients (3.4), as well as the Virasoro 2-cocycle α by its expression (2.20), we can immediately deduce the cocycle condition on the coefficients corresponding to the basis elements e_i,e_j,e_k,e_l:

$$\begin{aligned}&(j-i)\psi_{i+j,k,l} - (k-i)\psi_{i+k,j,l} + (l-i)\psi_{i+l,j,k}\\ &+(k-j)\psi_{j+k,i,l} - (l-j)\psi_{l+j,i,k} + (l-k)\psi_{l+k,i,j}\\ &-\frac{1}{12}(i^3-i)\delta_{i,-j}c_{k,l} + \frac{1}{12}(i^3-i)\delta_{i,-k}c_{j,l} - \frac{1}{12}(i^3-i)\delta_{i,-l}c_{j,k}\\ &-\frac{1}{12}(j^3-j)\delta_{j,-k}c_{i,l} + \frac{1}{12}(j^3-j)\delta_{j,-l}c_{i,k} - \frac{1}{12}(k^3-k)\delta_{k,-l}c_{i,j} = 0\,.\end{aligned} \tag{3.5}$$

In case of the Virasoro algebra, we obtain two types of conditions. One type corresponds to the cocycle or coboundary conditions obtained when they are evaluated on basis elements e_i, like the one we just derived. The second type appears when the conditions are evaluated on a combination of basis elements including the basis element t. Due to the alternating property of cochains, if t appears more than once in the arguments, the cochain will be zero. Hence, we include t only once in the arguments. Moreover, t can always be brought to the last position of the arguments, with appropriate sign changes.
The cocycle condition (3.1) evaluated on the basis elements e_i, e_j, e_k, t yields, after replacing the Lie bracket by (2.19):

$$\begin{aligned}(\delta_3\psi)(e_i,e_j,e_k,t) =& \psi\left(\left[e_i,e_j\right],e_k,t\right) - \psi\left(\left[e_i,e_k\right],e_j,t\right) + \psi(\underbrace{[e_i,t]}_{=0},e_j,e_k) \\ &+ \psi\left(\left[e_j,e_k\right],e_i,t\right) - \psi(\underbrace{[e_j,t]}_{=0},e_i,e_k) + \psi(\underbrace{[e_k,t]}_{=0},e_i,e_j) = 0 \\ \Leftrightarrow 0 =& (j-i)\psi(e_{i+j},e_k,t) + \alpha(e_i,e_j)\underbrace{\psi(t,e_k,t)}_{=0} \\ &- (k-i)\psi(e_{i+k},e_j,t) - \alpha(e_i,e_k)\underbrace{\psi(t,e_j,t)}_{=0} \\ &+ (k-j)\psi(e_{k+j},e_i,t) + \alpha(e_j,e_k)\underbrace{\psi(t,e_i,t)}_{=0}.\end{aligned}$$

From this expression we can immediately deduce the cocycle condition on the coefficients for the generators e_i, e_j, e_k, t, by using (3.4):

$$(j-i)c_{i+j,k} - (k-i)c_{i+k,j} + (k-j)c_{j+k,i} = 0. \tag{3.6}$$

In the case of the Witt algebra, we do not have the condition (3.6). Only the condition (3.5) shows up in the case of the Witt algebra, but without the central terms appearing in the last two lines of (3.5). Next, we do the same for the coboundary conditions.
Replacing the Lie bracket in (3.2) by its expression (2.19), the coboundary condition evaluated on the basis elements e_i, e_j, e_k becomes:

$$\begin{aligned}\psi(e_i,e_j,e_k) &= \phi\left(\left[e_i,e_j\right],e_k\right) + \phi\left(\left[e_j,e_k\right],e_i\right) + \phi\left(\left[e_k,e_i\right],e_j\right) \\ &= \phi\left((j-i)e_{i+j} + \alpha(e_i,e_j)t,e_k\right) + \phi\left((k-j)e_{j+k} + \alpha(e_j,e_k)t,e_i\right) \\ &+ \phi\left((i-k)e_{i+k} + \alpha(e_k,e_i)t,e_j\right) \\ &= (j-i)\phi(e_{i+j},e_k) + \alpha(e_i,e_j)\phi(t,e_k) + (k-j)\phi(e_{j+k},e_i) + \alpha(e_j,e_k)\phi(t,e_i) \\ &+ (i-k)\phi(e_{i+k},e_j) + \alpha(e_k,e_i)\phi(t,e_j).\end{aligned}$$

Replacing ϕ in terms of its coefficients (3.4), and the Virasoro 2-cocycle by its expression (2.20), we immediately deduce the coboundary condition on the coefficients for e_i, e_j, e_k:

$$\psi_{i,j,k} = (j-i)\phi_{i+j,k} - (k-i)\phi_{i+k,j} + (k-j)\phi_{j+k,i} \tag{3.7}$$

$$+\frac{1}{12}(i^3-i)\delta_{i,-j}b_k - \frac{1}{12}(i^3-i)\delta_{i,-k}b_j + \frac{1}{12}(j^3-j)\delta_{j,-k}b_i. \tag{3.8}$$

Next, we compute the coboundary condition (3.2) evaluated on the basis elements e_i, e_j, t, which yields after replacing the Lie bracket (2.19):

$$\begin{aligned}\psi(e_i, e_j, t) &= \phi\big([e_i, e_j], t\big) + \phi(\underbrace{[e_j, t]}_{=0}, e_i) + \phi(\underbrace{[t, e_i]}_{=0}, e_j) \\ &= (j-i)\phi(e_{i+j}, t) + \alpha(e_i, e_j)\underbrace{\phi(t,t)}_{=0}.\end{aligned}$$

Replacing ϕ in terms of its coefficients (3.4), we immediately deduce the coboundary condition on the coefficients evaluated on the basis elements e_i, e_j, t:

$$c_{i,j} = (j-i)b_{i+j}. \tag{3.9}$$

As we only need to consider cochains of degree zero due to Theorem 2.2.1, and since our trivial module $\mathbb{K}$ has only degree zero elements, non-zero coefficients are only possible if the indices of the said coefficients add up to zero, hence:

$$\begin{aligned} \psi_{i,j,k} &= 0 \qquad i+j+k \neq 0, \\ c_{i,j} &= 0 \qquad i+j \neq 0, \\ \phi_{i,j} &= 0 \qquad i+j \neq 0, \\ b_i &= 0 \qquad i \neq 0. \end{aligned} \tag{3.10}$$

Finally, we gathered the necessary ingredients to prove the main result of this section.

Theorem 3.1.1. *The third algebraic cohomology of the Witt and the Virasoro algebra over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in the trivial module $\mathbb{K}$ is one-dimensional, i.e.:*

$$\dim(\mathrm{H}^3(\mathcal{W}, \mathbb{K})) = \dim(\mathrm{H}^3(\mathcal{V}, \mathbb{K})) = 1.$$

The proof proceeds in two steps. We will first prove that the dimension is at most one and subsequently we prove that the dimension is at least one.

Proposition 3.1.1. *The third algebraic cohomology of the Witt and the Virasoro algebra over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in the trivial module $\mathbb{K}$ is at most one-dimensional, i.e.:*

$$\dim(\mathrm{H}^3(\mathcal{W}, \mathbb{K})) \leq 1 \qquad \textit{and} \qquad \dim(\mathrm{H}^3(\mathcal{V}, \mathbb{K})) \leq 1.$$

We will treat both the Witt algebra and the Virasoro algebra simultaneously. The proof consists of two lemmata. The first lemma mostly involves a cohomological change. The second lemma involves the cocycle condition (3.5).

Lemma 3.1.1. *Every 3-cocycle $\psi' \in \mathrm{H}^3(\mathcal{V}, \mathbb{K})$ is cohomologous to a 3-cocycle $\psi \in \mathrm{H}^3(\mathcal{V}, \mathbb{K})$ with coefficients $c_{i,j}, \psi_{i,j,k} \in \mathbb{K}$ fulfilling:*

$$\begin{aligned} c_{i,-i} &= \frac{1}{6}(-i+1)(-i)(-i-1)c_{-2,2} \qquad \forall\, i,j \in \mathbb{Z} \\ \textit{and}\ \ \psi_{i,j,1} &= 0 \qquad \forall\, i+j+1 = 0,\ \textit{except possibly for}\ \psi_{-1,0,1}. \end{aligned} \tag{3.11}$$

Every 3-cocycle $\psi' \in \mathrm{H}^3(\mathcal{W}, \mathbb{K})$ is cohomologous to a 3-cocycle $\psi \in \mathrm{H}^3(\mathcal{W}, \mathbb{K})$ with coefficients $\psi_{i,j,k} \in \mathbb{K}$ fulfilling:

$$\psi_{i,j,1} = 0 \qquad \forall\, i+j+1 = 0,\ \textit{except possibly for}\ \psi_{-1,0,1}. \tag{3.12}$$

Proof. Let ψ' be a 3-cocycle for $\mathcal{W}$ or $\mathcal{V}$. We will perform a cohomological change $\psi = \psi' - \delta_2\phi$ with a suitable coboundary $\delta_2\phi$ such that for the cocycle ψ we have for $\mathcal{V}$ and $\mathcal{W}$ the relations (3.11) and (3.12), respectively. Note that the coboundary is obtained by constructing a 2-cochain ϕ such that $\delta_2\phi$ has the desired properties.
Recall from (3.4) and (3.10) that ψ is given by coefficients $\psi_{i,j,-i-j}$ and $c_{i,-i}$, and that ϕ is given by the system of coefficients b_0 and $\phi_{i,-i}$. The first part of the proof concerns only the coefficients $c_{i,-i}$ and b_0 and is irrelevant for the Witt algebra.
Part one:
First, we consider $b_0 = \phi(e_0, t)$. By setting:

$$\phi(e_0, t) = b_0 := \frac{c'_{-1,1}}{2},$$

we obtain after the cohomological change:

$$\psi(e_{-1}, e_1, t) = \psi'(e_{-1}, e_1, t) - \delta_2\phi(e_{-1}, e_1, t) = c'_{-1,1} - 2b_0 = 0 = c_{-1,1}.$$

Next, let us determine the relation between a coefficient of the form $c_{i,-i}$ and the generator $c_{-2,2}$. Consider the cocycle condition (3.6) for (e_i, e_{-i-1}, e_1, t), and take $i < -2$:

$$\begin{aligned}
&(-1-2i)\cancel{c_{-1,1}} - (1-i)c_{i+1,-i-1} + (i+2)c_{-i,i} = 0 \Leftrightarrow c_{i,-i} = \frac{(1-i)}{(-i-2)}c_{i+1,-i-1} \\
&\Leftrightarrow c_{i,-i} = \frac{1}{3!}\frac{(1-i)!}{(-i-2)!}c_{-2,2} \Leftrightarrow c_{i,-i} = \frac{1}{6}(1-i)(-i)(-i-1)c_{-2,2}.
\end{aligned}$$

Hence we obtained the first equation in (3.11).
Part two:
The second part focuses on the coefficients of the form $\psi_{i,j,-i-j}$. We will explicitly prove the statement concerning the Virasoro algebra, but the proof concerning the Witt algebra can be treated simultaneously. In fact, in the equations which we will consider, the central terms drop out. Hence, the conclusions obtained are valid for both the Witt algebra and the Virasoro algebra.
We will start with a cohomological change $\psi = \psi' - (\delta_2\phi)$ in order to annihilate as many coefficients $\psi_{i,j,k}$ with $i+j+k=0$ as possible. To do this, we will use the coefficients $\phi_{i,j}$ with $i+j=0$ of the 2-cochain ϕ. Note that we cannot use the coefficient b_0 of the 2-cochain $\phi(e_0, t) = b_0$ as this one has already been used in the previous part to annihilate the coefficient $c_{-1,1}$. We start by defining $\phi_{-1,1} = \phi_{-2,2} = 0$ in order to simplify the notation. In fact, the structure of the equations is such that these coefficients cannot be consistently associated to some coefficient $\psi_{i,j,-i-j}$. Consider the coboundary condition (3.7)-(3.8) for the generators (e_i, e_{-1-i}, e_1), which will yield a suggestion for the definition of ϕ:

$$\begin{aligned}
&\psi'_{i,-1-i,1} = (-1-2i)\cancel{\phi_{-1,1}} - (1-i)\phi_{i+1,-1-i} + (2+i)\phi_{-i,i} \\
&+ \frac{1}{12}(i^3-i)\cancel{\delta_{i,1+i}}b_1 - \frac{1}{12}\cancel{(i^3-i)\delta_{i,-1}}b_{-1-i} + \frac{1}{12}\cancel{((-1-i)^3+1+i)\delta_{-1-i,-1}}b_i \\
&\Rightarrow \phi_{i+1,-1-i} := -\frac{\psi'_{i,-1-i,1}}{(1-i)} - \frac{(2+i)}{(1-i)}\phi_{i,-i}.
\end{aligned}$$

The coefficient $\phi_{-1,1}$ is zero due to our normalization, the other three slashed terms are obviously zero due to the Kronecker delta's and their pre-factors. Starting with $i = 2$, i increasing, recalling that $\phi_{2,-2} = 0$ due to our normalization, we obtain a definition for $\phi_{i,j}$ with $i + j = 0$,

$i > 2$ and we have after the cohomological change $\psi_{i,j,1} = 0 \; \forall \; i + j + 1 = 0$ and $i \geq 2$. The coefficient for $i = 1$ is obviously zero due to the alternating property, i.e. $\psi_{1,-2,1} = 0$. Hence, we have $\psi_{i,j,1} = 0 \; \forall \; i + j + 1 = 0$ except for the coefficient $\psi_{-1,0,1}$. □

In the second lemma, we use the fact that we are dealing with cocycles fulfilling (3.5). The proof proceeds as follows: we start by fixing one index of the coefficients, which we will refer to as *level*, and derive results for this particular level. For the coefficients $\psi_{i,j,k}$ for example, we will first analyze $\psi_{i,-i,0}$, $\psi_{i,-i-1,1}$, $\psi_{i,-i+1,-1}$, $\psi_{i,-i-2,2}$, and $\psi_{i,-i+2,-2}$, $\forall \; i \in \mathbb{Z}$, or coefficients with some permutation of these indices, and refer to these as coefficients of level zero, one, minus one, two and minus two respectively. Subsequently, we use induction to consider general coefficients $\psi_{i,-i-k,k} \; \forall \; i, k \in \mathbb{Z}$.

Lemma 3.1.2. *Let* $\psi \in \mathrm{H}^3(\mathcal{V}, \mathbb{K})$ *be a* 3*-cocycle such that:*

$$c_{i,-i} = \frac{1}{6}(-i+1)(-i)(-i-1)c_{-2,2} \qquad \forall \; i, j \in \mathbb{Z}$$
$$\textit{and } \; \psi_{i,j,1} = 0 \qquad \forall \; i + j + 1 = 0, \; \textit{except possibly for } \psi_{-1,0,1}\,.$$

Then :

$c_{i,j} = 0 \qquad \forall \; i, j \in \mathbb{Z} \qquad$ *and*
The coefficients $\psi_{i,j,k}$ *are linearly generated by the coefficient* $\psi_{-1,1,0}$, $\forall \; i, j, k \in \mathbb{Z}$.

Let $\psi \in \mathrm{H}^3(\mathcal{W}, \mathbb{K})$ *be a* 3*-cocycle such that:*

$$\psi_{i,j,1} = 0 \qquad \forall \; i + j + 1 = 0, \; \textit{except possibly for } \psi_{-1,0,1}\,.$$

Then :

The coefficients $\psi_{i,j,k}$ *are linearly generated by the coefficient* $\psi_{-1,1,0}$, $\forall \; i, j, k \in \mathbb{Z}$.

Proof. Once again, we write down the proof in the setting of the Virasoro algebra. However, the central terms cancel in most of the cocycle conditions which we will consider in the following. Hence, the conclusions obtained are valid for both the Witt algebra and the Virasoro algebra. The central terms only appear towards the end of the proof, where we will explicitly point out the differences between the proof for the Witt algebra and for the Virasoro algebra.
The coefficients to consider are of the form $\psi_{i,j,k}$ with $i + j + k = 0$, and can be written as $\psi_{-i-k,i,k}$. We will see that they can a priori be expressed in terms of two generating coefficients $\psi_{-1,1,0}$ and $\psi_{-2,2,0}$, although we will show in the end that they are related by a non-trivial relation, hence we will have only one linearly independent generating coefficient in the end.
We will proceed by constructing recurrence relations. In order to have recurrence relations, at least one of the generators e_i appearing in the cocycle condition (3.5) is taken to be of degree plus or minus one, $e_i = e_{\pm 1}$. This means that no coefficient $c_{i,j}$ will appear in the recurrence relations, because either the coefficient $c_{i,j}$ is equal to a coefficient of the form $c_{i,\pm 1}$, which is zero due to our assumption inspired by the previous lemma, or its pre-factor of the form $(i^3 - i)\delta_{i,j}$ is zero because $((\pm 1)^3 \mp 1) = 0$. Therefore, we can once again treat the Virasoro algebra and the Witt algebra simultaneously.
We will start with level zero, $k = 0$. The cocycle condition (3.5) on the generators $(e_{-i-1}, e_i, e_0, e_1)$

yields:

$$
\begin{aligned}
&(1+2i)\psi_{-1,0,1} + \cancel{\psi_{1,-1-i,i}} - \cancel{(1+i)\psi_{-1-i,i,1}} \\
&+ (2+i)\psi_{-i,i,0} - \cancel{i\psi_{i,-1-i,1}} + (-1+i)\psi_{1+i,-1-i,0} = 0 \\
&\Leftrightarrow \psi_{-1-i,1+i,0} = \frac{1}{(-1+i)}\left((1+2i)\psi_{-1,0,1} + (2+i)\psi_{-i,i,0}\right). \qquad (3.13)
\end{aligned}
$$

The slashed terms cancel each other, and are zero anyway for $i \neq -1, 0$ because of our assumption. Starting with $i = 2$, increasing i, we see that level zero is generated by two generators, $\psi_{-1,1,0}$ and $\psi_{-2,2,0}$. Negative values of i do not yield any new information due to the alternating property $\psi_{i,-i} = -\psi_{-i,i}$.
Let us continue with level minus one, $k = -1$. A recurrence relation for level minus one is obtained by considering the cocycle condition (3.5) on the generators $(e_{-i}, e_i, e_{-1}, e_1)$:

$$
\begin{aligned}
&2i\psi_{0,-1,1} + 2\psi_{0,-i,i} - (-1+i)\cancel{\psi_{-1-i,i,1}} + (1+i)\psi_{1-i,i,-1} \\
&- (1+i)\cancel{\psi_{-1+i,-i,1}} + (-1+i)\psi_{1+i,-i,-1} = 0 \\
&\Leftrightarrow \psi_{1-i,i,-1} = \frac{1}{(1+i)}(-2i\psi_{0,-1,1} - 2\psi_{0,-i,i} - (-1+i)\psi_{1+i,-i,-1}). \qquad (3.14)
\end{aligned}
$$

The slashed terms are zero for $i \notin \{-1, 0\}$ due to our assumption. We do not need to consider $i \in \{-1, 0\}$, since for these values of i the equation above is trivially satisfied and does not yield any information. Starting with $i = -2$, decreasing i, we see that level minus one is generated by the same generators as level zero, namely $\psi_{-1,1,0}$ and $\psi_{-2,2,0}$. Values of i bigger than minus two $i > -2$ do not lead to any new information due to the alternating property.
We can proceed similarly with level minus two $k = -2$. The cocycle condition (3.5) for the generators $(e_{-i+1}, e_i, e_{-2}, e_1)$ gives us a recurrence relation:

$$
\begin{aligned}
&3\psi_{-1,1-i,i} - (-3+i)\cancel{\psi_{-1-i,i,1}} + i\psi_{2-i,i,-2} \\
&\qquad - (2+i)\cancel{\psi_{-2+i,1-i,1}} + (-1+i)\psi_{1+i,1-i,-2} = 0 \\
&\Leftrightarrow \psi_{2-i,i,-2} = \frac{1}{i}(-3\psi_{-1,1-i,i} - (-1+i)\psi_{1+i,1-i,-2}). \qquad (3.15)
\end{aligned}
$$

The slashed terms are zero for $i \notin \{-1, 0\}$ due to our assumption. Since we consider $i \leq -3$ in the following, we can omit these terms. Starting with $i = -3$, i decreasing, we see that also level minus two is generated by $\psi_{-1,1,0}$ and $\psi_{-2,2,0}$. Again, taking $i > -3$ does not lead to any new information.
The same can be done for level plus two $k = 2$, by considering the cocycle condition (3.5) for the generators $(e_{-i-1}, e_i, e_2, e_{-1})$:

$$
\begin{aligned}
&-3\cancel{\psi_{1,-1-i,i}} + i\psi_{-2-i,i,2} - (3+i)\psi_{1-i,i,-1} \\
&\qquad + (1+i)\psi_{-1+i,-1-i,2} - (-2+i)\psi_{2+i,-1-i,-1} = 0 \\
&\Leftrightarrow \psi_{-2-i,i,2} = \frac{1}{i}((3+i)\psi_{1-i,i,-1} - (1+i)\psi_{-1+i,-1-i,2} + (-2+i)\psi_{2+i,-1-i,-1}). \qquad (3.16)
\end{aligned}
$$

The slashed term is zero for $i \notin \{-1, 0\}$ by assumption. Since we consider $i \geq 3$ in the following, we can omit this term. Starting with $i = 3$, i increasing, it is clear that also level plus two is solely generated by $\psi_{-1,1,0}$ and $\psi_{-2,2,0}$. Once more, taking $i < 3$ does not yield any new information.
Finally, we can produce recurrence relations for generic level k. Starting with positive k, the

cocycle condition (3.5) for the generators $(e_{-i-k-1}, e_i, e_k, e_1)$ yields:

$$
\begin{aligned}
&-(1+i+2k)\cancel{\psi_{-1-i,i,1}} + (-1+i)\psi_{1+i,-1-i-k,k} + (1+2i+k)\cancel{\psi_{-1-k,k,1}} \\
&+(2+i+k)\psi_{-i-k,i,k} - (-1+k)\psi_{1+k,-1-i-k,i} + (-i+k)\cancel{\psi_{i+k,-1-i-k,1}} = 0 \\
&\Leftrightarrow \psi_{-1-i-k,i,1+k} = \frac{1}{1-k}\left(-(-1+i)\psi_{1+i,-1-i-k,k} - (2+i+k)\psi_{-i-k,i,k}\right). \qquad (3.17)
\end{aligned}
$$

For the indices i and k under consideration, the slashed terms are zero due to our assumptions. Starting with $k=2$ and $i=k+2$, increasing k and i, we see that a generic positive level k is build from the generators $\psi_{-1,1,0}$ and $\psi_{-2,2,0}$. Considering $i<k+2$ does not yield new information due to the alternating property.
The same can be done for negative k by considering the cocycle condition (3.5) for the generators $(e_{-i-k+1}, e_i, e_k, e_{-1})$:

$$
\begin{aligned}
&-(-1+i+2k)\psi_{1-i,i,-1} + (1+i)\psi_{-1+i,1-i-k,k} + (-1+2i+k)\psi_{1-k,k,-1} \\
&+(-2+i+k)\psi_{-i-k,i,k} - (1+k)\psi_{-1+k,1-i-k,i} + (-i+k)\psi_{i+k,1-i-k,-1} = 0 \\
&\Leftrightarrow \psi_{1-i-k,i,-1+k} = \frac{1}{-1-k}\big((-1+i+2k)\psi_{1-i,i,-1} - (1+i)\psi_{-1+i,1-i-k,k} \\
&-(-1+2i+k)\psi_{1-k,k,-1} - (-2+i+k)\psi_{-i-k,i,k} - (-i+k)\psi_{i+k,1-i-k,-1}\big). \qquad (3.18)
\end{aligned}
$$

Starting with $k=-2$ and $i=-2+k$, decreasing i and k, we see that also a generic negative level k is build from the generators $\psi_{-1,1,0}$ and $\psi_{-2,2,0}$. Again, taking $i>-2+k$ does not yield new information.

Let us summarize the results we obtained so far. We showed that the coefficients $\psi_{i,j,k}$ are solely determined by two generating coefficients $\psi_{-1,1,0}$ and $\psi_{-2,2,0}$. In the analysis above, no central terms appeared and thus, the conclusion is valid for both the Witt and the Virasoro algebra. Moreover, in the case of the Virasoro algebra, we showed in the previous lemma that the coefficients $c_{i,j}$ are generated by a single coefficient, namely $c_{-2,2}$. This means that up to now, the dimension of $\mathrm{H}^3(\mathcal{W},\mathbb{K})$ is at most two, and the dimension of $\mathrm{H}^3(\mathcal{V},\mathbb{K})$ is at most three. In the last step of the proof, we have to check whether there are non-trivial relations between the three generators $c_{-2,2}$, $\psi_{-1,1,0}$ and $\psi_{-2,2,0}$. In order to prove that the dimension is at most one, we need to find at least two non-trivial relations in case of the Virasoro algebra, and one non-trivial relation in case of the Witt algebra. We will consider the cocycle condition (3.5) for the generators $(e_{-4}, e_{-3}, e_2, e_5)$ and $(e_{-3}, e_{-2}, e_2, e_3)$. The cocycle condition for $(e_{-4}, e_{-3}, e_2, e_5)$ yields:

$$
\begin{aligned}
&\psi_{-7,2,5} - 6\psi_{-2,-3,5} + 5\psi_{-1,-4,5} + 9\underbrace{\psi_{1,-3,2}}_{=0} + 3\psi_{7,-4,-3} = 0 \\
&\Leftrightarrow -\psi_{-7,5,2} + 6\psi_{5,-3,-2} - 5\psi_{5,-4,-1} + 3\psi_{7,-4,-3} = 0, \qquad (3.19)
\end{aligned}
$$

whereas the one for $(e_{-3}, e_{-2}, e_2, e_3)$ yields:

$$
\begin{aligned}
&\frac{1}{2}c_{-3,3} + 2c_{-2,2} + \psi_{-5,2,3} - 5\psi_{-1,-2,3} + 4\psi_{0,-3,3} + 6\psi_{0,-2,2} - 5\underbrace{\psi_{1,-3,2}}_{=0} + \psi_{5,-3,-2} = 0 \\
&\Leftrightarrow \frac{1}{2}c_{-3,3} + 2c_{-2,2} - \psi_{-5,3,2} + 5\psi_{3,-2,-1} + 4\psi_{0,-3,3} + 6\psi_{0,-2,2} + \psi_{5,-3,-2} = 0. \qquad (3.20)
\end{aligned}
$$

The terms of level plus one are zero due to our assumption. We will use the recurrence relations (3.11), (3.13), (3.14), (3.15), (3.16), (3.17) and (3.18) to express all the coefficients $\psi_{i,j,k}$ and $c_{i,j}$ appearing in the conditions above in terms of the generators $c_{-2,2}$, $\psi_{-1,1,0}$ and $\psi_{-2,2,0}$. We will write down all the coefficients needed implicitly and explicitly, in order to expose the structure of the recurrence relations and their entanglement. We will see that the cocycle condition (3.19) for $(e_{-4}, e_{-3}, e_2, e_5)$ yields a non-trivial relation between $\psi_{-1,1,0}$ and $\psi_{-2,2,0}$. As no central terms appear, the cocycle condition (3.19) is valid both for the Witt and the Virasoro algebra. In case of the Witt algebra, the cocycle condition (3.19) is sufficient to conclude. In case of the Virasoro algebra, the cocycle condition (3.19), together with the second cocycle condition (3.20) for $(e_{-3}, e_{-2}, e_2, e_3)$, yields $c_{i,j} = 0 \ \forall \ i, j \in \mathbb{Z}$, which allows to conclude.
Let us begin with the coefficients of level zero. The recurrence relation (3.13) yields for $i = 2$ the following expression for $\psi_{-3,3,0}$:

$$\psi_{-3,3,0} = -5\psi_{-1,1,0} + 4\psi_{-2,2,0} \tag{3.21}$$

Continuing with $i = 3, 4, 5$ we obtain respectively:

$$\psi_{-4,4,0} = \frac{1}{2}(7\psi_{-1,0,1} + 5\psi_{-3,3,0}) \overset{(3.21)}{\Leftrightarrow} \psi_{-4,4,0} = -16\psi_{-1,1,0} + 10\psi_{-2,2,0}\,, \tag{3.22}$$

and

$$\psi_{-5,5,0} = \frac{1}{3}(9\psi_{-1,0,1} + 6\psi_{-4,4,0}) \overset{(3.22)}{\Leftrightarrow} \psi_{-5,5,0} = -35\psi_{-1,1,0} + 20\psi_{-2,2,0}\,, \tag{3.23}$$

and

$$\psi_{-6,6,0} = \frac{1}{4}(11\psi_{-1,0,1} + 7\psi_{-5,5,0}) \overset{(3.23)}{\Leftrightarrow} \psi_{-6,6,0} = -64\psi_{-1,1,0} + 35\psi_{-2,2,0}\,. \tag{3.24}$$

More coefficients of level zero will not be needed. Hence, let us consider the coefficients of level minus one.
The recurrence relation (3.14) yields for $i = -2, -3, -4, -5, -6$ the following coefficients:

$$\psi_{3,-2,-1} = -(4\psi_{0,-1,1} - 2\psi_{0,2,-2} + 3\cancel{\psi_{-1,2,-1}}) \Leftrightarrow \psi_{3,-2,-1} = -4\psi_{-1,1,0} - 2\psi_{-2,2,0}\,, \tag{3.25}$$

and for $i = -3$

$$\psi_{4,-3,-1} = -\frac{1}{2}(6\psi_{0,-1,1} - 2\psi_{0,3,-3} + 4\psi_{-2,3,-1}) \overset{(3.25),(3.21)}{\Leftrightarrow} \psi_{4,-3,-1} = -6\psi_{-1,1,0} - 8\psi_{-2,2,0}\,, \tag{3.26}$$

for $i = -4$,

$$\psi_{5,-4,-1} = -\frac{1}{3}(8\psi_{0,-1,1} - 2\psi_{0,4,-4} + 5\psi_{-3,4,-1}) \overset{(3.26),(3.22)}{\Leftrightarrow} \psi_{5,-4,-1} = -2\psi_{-1,1,0} - 20\psi_{-2,2,0}\,, \tag{3.27}$$

for $i = -5$,

$$\psi_{6,-5,-1} = -\frac{1}{4}(10\psi_{0,-1,1} - 2\psi_{0,5,-5} + 6\psi_{-4,5,-1}) \overset{(3.27),(3.23)}{\Leftrightarrow} \psi_{6,-5,-1} = 12\psi_{-1,1,0} - 40\psi_{-2,2,0}\,, \tag{3.28}$$

and finally for $i = -6$,

$$\psi_{7,-6,-1} = -\frac{1}{5}(12\psi_{0,-1,1} - 2\psi_{0,6,-6} + 7\psi_{-5,6,-1}) \overset{(3.28),(3.24)}{\Leftrightarrow} \psi_{7,-6,-1} = 40\psi_{-1,1,0} - 70\psi_{-2,2,0}\,. \tag{3.29}$$

More coefficients of level minus one will not be needed. We continue with the coefficients of level plus two.
The recurrence relation (3.16) yields for $i = 3, 4, 5$ the following coefficients:

$$\psi_{-5,3,2} = \frac{1}{3}(6\psi_{-2,3,-1} - 4\cancel{\psi_{2,-4,2}} + \psi_{5,-4,-1}) \overset{(3.25),(3.27)}{\Leftrightarrow} \psi_{-5,3,2} = \frac{22}{3}\psi_{-1,1,0} - \frac{8}{3}\psi_{-2,2,0}\,, \tag{3.30}$$

for $i = 4$,

$$\psi_{-6,4,2} = \frac{1}{4}(7\psi_{-3,4,-1} - 5\psi_{3,-5,2} + 2\psi_{6,-5,-1}) \overset{(3.26),(3.28),(3.30)}{\Leftrightarrow} \psi_{-6,4,2} = \frac{77}{3}\psi_{-1,1,0} - \frac{28}{3}\psi_{-2,2,0}\,, \quad (3.31)$$

and finally for $i = 5$,

$$\psi_{-7,5,2} = \frac{1}{5}(8\psi_{-4,5,-1} - 6\psi_{4,-6,2} + 3\psi_{7,-6,-1}) \overset{(3.27),(3.29),(3.31)}{\Leftrightarrow} \psi_{-7,5,2} = 58\psi_{-1,1,0} - \frac{106}{5}\psi_{-2,2,0}\,. \quad (3.32)$$

More coefficients of level plus two are not needed. We will continue with the coefficients of level minus two.
The recurrence relation (3.15) yields for $i = -3, -4, -5$ the following coefficients:

$$\psi_{5,-3,-2} = -\frac{1}{3}(-3\psi_{-1,4,-3} + 4\cancel{\psi_{-2,4,-2}}) \overset{(3.26)}{\Leftrightarrow} \psi_{5,-3,-2} = -6\psi_{-1,1,0} - 8\psi_{-2,2,0}\,, \quad (3.33)$$

for $i = -4$ we obtain

$$\psi_{6,-4,-2} = -\frac{1}{4}(-3\psi_{-1,5,-4} + 5\psi_{-3,5,-2}) \overset{(3.27),(3.33)}{\Leftrightarrow} \psi_{6,-4,-2} = -9\psi_{-1,1,0} - 25\psi_{-2,2,0}\,, \quad (3.34)$$

and for $i = -5$

$$\psi_{7,-5,-2} = -\frac{1}{5}(-3\psi_{-1,6,-5} + 6\psi_{-4,6,-2}) \overset{(3.28),(3.34)}{\Leftrightarrow} \psi_{7,-5,-2} = -\frac{18}{5}\psi_{-1,1,0} - 54\psi_{-2,2,0}\,. \quad (3.35)$$

These are all the coefficients needed for level minus two. Next, we need some coefficients for level minus three.
Putting $k = -2$ in the recurrence relation (3.18), we obtain a recurrence relation for level minus three:

$$\begin{aligned} \psi_{3-i,i,-3} =& (-5+i)\psi_{1-i,i,-1} - (1+i)\psi_{-1+i,3-i,-2} - (-3+2i)\psi_{3,-2,-1} \\ & - (-4+i)\psi_{2-i,i,-2} - (-i-2)\psi_{i-2,3-i,-1}\,. \end{aligned} \quad (3.36)$$

We need only one coefficient of level minus three, namely the coefficient obtained by taking $i = -4$ in (3.36):

$$\begin{aligned} \psi_{7,-4,-3} &= -9\psi_{5,-4,-1} + 3\psi_{-5,7,-2} + 11\psi_{3,-2,-1} + 8\psi_{6,-4,-2} - 2\psi_{-6,7,-1} \\ & \overset{(3.27),(3.35),(3.25),(3.34),(3.29)}{\Leftrightarrow} \psi_{7,-4,-3} = -\frac{36}{5}\psi_{-1,1,0} - 20\psi_{-2,2,0}\,. \end{aligned} \quad (3.37)$$

These recurrence relations are valid both for the Witt algebra and the Virasoro algebra, since no central terms appear.
At last, we will need the coefficient $c_{-3,3}$ in the case of the Virasoro algebra. Taking the relation (3.11) and putting $i = -3$, $j = 3$, we obtain:

$$c_{-3,3} = 4\, c_{-2,2} \quad (3.38)$$

Finally, we obtained all the coefficients needed. Inserting the coefficients (3.32), (3.33), (3.27) and (3.37) into the cocycle condition (3.19), we obtain:

$$\begin{aligned} & -58\psi_{-1,1,0} + \frac{106}{5}\psi_{-2,2,0} - 36\psi_{-1,1,0} - 48\psi_{-2,2,0} + 10\psi_{-1,1,0} \\ & + 100\psi_{-2,2,0} - \frac{108}{5}\psi_{-1,1,0} - 60\psi_{-2,2,0} = 0 \\ & \Leftrightarrow 8\psi_{-1,1,0} - \psi_{-2,2,0} = 0 \end{aligned}$$

This already allows to conclude for the Witt algebra. Similarly, inserting the coefficients (3.30), (3.25), (3.21), (3.33) and (3.38) into the cocycle condition (3.20), we obtain:

$$\begin{aligned}&-\frac{22}{3}\psi_{-1,1,0}+\frac{8}{3}\psi_{-2,2,0}-20\psi_{-1,1,0}-10\psi_{-2,2,0}-20\psi_{-1,1,0}\\&+16\psi_{-2,2,0}+6\psi_{-2,2,0}-6\psi_{-1,1,0}-8\psi_{-2,2,0}+4c_{-2,2}=0\\&\Leftrightarrow 3c_{-2,2}-5(8\psi_{-1,1,0}-\psi_{-2,2,0})=0\end{aligned}$$

Hence we obtain $c_{-2,2}=0$ and a non-trivial relation between the remaining two generators, meaning that we end up with only one generator, e.g. $\psi_{-1,1,0}$, for both the Witt algebra and the Virasoro algebra. This concludes the proof of Lemma 3.1.2. □

The Lemmata 3.1.1 and 3.1.2 allow to prove Proposition 3.1.1.

Proof of Proposition 3.1.1. Starting with a 3-cocycle $\psi\in\mathrm{H}^3(\mathcal{V},\mathbb{K})$ or $\psi\in\mathrm{H}^3(\mathcal{W},\mathbb{K})$, by Lemma 3.1.1, we can perform a cohomological change to obtain an equivalent 3-cocycle ψ', such that the hypotheses of Lemma 3.1.2 are fulfilled. Using Lemma 3.1.2, we obtain that all the coefficients $\psi'_{i,j,k}$ are uniquely determined by a single coefficient. Hence, the dimension of $\mathrm{H}^3(\mathcal{V},\mathbb{K})$ and $\mathrm{H}^3(\mathcal{W},\mathbb{K})$ is at most one. □

Next, we prove that the dimension of the spaces $\mathrm{H}^3(\mathcal{V},\mathbb{K})$ and $\mathrm{H}^3(\mathcal{W},\mathbb{K})$ is at least one. To show this, we will construct an explicit degree-zero 3-cocycle of $\mathrm{H}^3(\mathcal{V},\mathbb{K})$ and $\mathrm{H}^3(\mathcal{W},\mathbb{K})$, that is not trivial, i.e. not a coboundary.

Proposition 3.1.2. *The third algebraic cohomology of the Witt and the Virasoro algebra over a field* $\mathbb{K}$ *with* $char(\mathbb{K})=0$ *and values in the trivial module* $\mathbb{K}$ *is at least one-dimensional, i.e.:*

$$\dim(\mathrm{H}^3(\mathcal{W},\mathbb{K}))\geq 1 \textit{ and } \dim(\mathrm{H}^3(\mathcal{V},\mathbb{K}))\geq 1$$

Proof. Consider the following trilinear map:

$$\Psi:\mathcal{W}\times\mathcal{W}\times\mathcal{W}\to\mathbb{K},$$

defined on the basis elements via:

$$\Psi(e_i,e_j,e_k)=(i-j)(j-k)(i-k)\delta_{i+j+k,0}, \tag{3.39}$$

which we extend trivially to:

$$\hat{\Psi}:\mathcal{V}\times\mathcal{V}\times\mathcal{V}\to\mathbb{K},$$

by setting $\hat{\Psi}(x_1,x_2,x_3)=0$ whenever one of the elements x_1, x_2 or x_3 is a multiple of the central element t. By their very definition, Ψ and $\hat{\Psi}$ are alternating, hence $\Psi\in C^3(\mathcal{W},\mathbb{K})$ and $\hat{\Psi}\in C^3(\mathcal{V},\mathbb{K})$. Moreover, due to the Kronecker Delta in the definition, these cochains are clearly of degree zero.
A straight-forward calculation of (3.1) for a quadruplet of basis elements e_i,e_j,e_k,e_l yields $\delta_3\Psi=0$. Hence Ψ is a three-cocycle of $\mathcal{W}$. Concerning the Virasoro algebra, we have $\delta_3\hat{\Psi}(x_1,x_2,x_3,x_4)=0$ if one of the arguments is central. If all the arguments are coming from $\mathcal{W}$, we obtain[1]: $\delta_3\hat{\Psi}(x_1,x_2,x_3,x_4)=\delta_3\Psi(x_1,x_2,x_3,x_4)=0$. Thus, $\hat{\Psi}$ is a 3-cocycle for $\mathcal{V}$. It remains to be shown that these cocycles are not trivial.
Let us assume that Ψ and $\hat{\Psi}$ are coboundaries, which will lead us to a contradiction. So, let

[1] In abuse of notation, we use the same symbol x to refer both to $x\in\mathcal{V}$ and its projection $\pi(x)\in\mathcal{W}$.

$\Phi : \mathcal{W} \times \mathcal{W} \to \mathbb{K}$ be a 2-cochain with $\Psi = \delta_2\Phi$. On the one hand, evaluating Ψ on the triple e_{-1}, e_1, e_0, we obtain by the very definition (3.39) of Ψ:

$$\Psi(e_{-1}, e_1, e_0) = 2\,. \tag{3.40}$$

On the other hand, Ψ being a coboundary we obtain using (3.2):

$$\begin{aligned}\Psi(e_{-1}, e_1, e_0) &= \Phi([e_{-1}, e_1], e_0) + \Phi([e_1, e_0], e_{-1}) + \Phi([e_0, e_{-1}], e_1) \\ &= 2\Phi(e_0, e_0) - \Phi(e_1, e_{-1}) - \Phi(e_{-1}, e_1) = 0\,.\end{aligned} \tag{3.41}$$

Therefore, Ψ cannot be a coboundary. Similarly, assume $\hat{\Psi} = \delta_2\hat{\Phi}$, with $\hat{\Phi} : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$. Again, we obtain $\hat{\Psi}(e_{-1}, e_1, e_0) = \Psi(e_{-1}, e_1, e_0) = 2$ as well as $\delta_2\hat{\Phi}(e_{-1}, e_1, e_0) = 0$, hence $\hat{\Psi}$ cannot be a coboundary. Note that for any two elements out of the three elements e_{-1}, e_1, e_0, the defining cocycle (2.20) for the central extension of $\mathcal{W}$ vanishes, hence exactly the same expression (3.41) will also appear for $\hat{\Psi}$. □

Remark 3.1.1. The Godbillon-Vey cocycle is known in the context of the continuous cohomology $\mathrm{H}^3_c(Vect(S^1), \mathbb{R})$. For the interested reader, we exhibit the relation in the following.
Let t be the coordinate along S^1. The elements of $Vect(S^1)$ can be represented by functions on S^1. Assigning to the vector field $f(t)\frac{d}{dt}$ the function $f(t)$, it was shown in [45] that the continuous cohomology $\mathrm{H}^*_c(Vect(S^1), \mathbb{R})$ of $Vect(S^1)$ with values in $\mathbb{R}$ is the free graded-commutative algebra generated by an element ω of cohomological dimension two and an element θ of cohomological dimension three. The generator of dimension two, called *Gelfand-Fuks* cocycle, is given by:

$$\omega : \left(f\frac{d}{dt}, g\frac{d}{dt}\right) \mapsto \int_{S^1} \det\begin{pmatrix} f' & g' \\ f'' & g'' \end{pmatrix} dt, \tag{3.42}$$

Note that the Gelfand-Fuks cocycle can be related to the Virasoro 2-cocycle (2.20). The generator of dimension three is given by:

$$\theta : \left(f\frac{d}{dt}, g\frac{d}{dt}, h\frac{d}{dt}\right) \mapsto \int_{S^1} \det\begin{pmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{pmatrix} dt, \tag{3.43}$$

with $f, g, h \in C^\infty(S^1)$ and the prime denoting the derivative with respect to t. The generator θ in (3.43) is commonly called the *Godbillon-Vey* cocycle.
If one considers the complexified vector field $\tilde{e}_n = ie^{int}\frac{d}{dt}$ then

$$\begin{aligned}\theta(\tilde{e}_n, \tilde{e}_m, \tilde{e}_k) &= -\int_{S^1} \det\begin{pmatrix} 1 & 1 & 1 \\ n & m & k \\ n^2 & m^2 & k^2 \end{pmatrix} e^{i(n+m+k)t} dt \\ &= (n-m)(n-k)(m-k)\int_{S^1} e^{i(n+m+k)t} dt\,.\end{aligned} \tag{3.44}$$

The integral evaluates to zero if $n + m + k \neq 0$, otherwise it yields the value 1. The expression (3.44) makes perfect sense for our algebraic generators e_n of $\mathcal{W}$ for every field $\mathbb{K}$ with char $(\mathbb{K}) = 0$, and we obtain the expression (3.39).

Proof of Theorem 3.1.1. Proposition 3.1.1 and Proposition 3.1.2 clearly prove Theorem 3.1.1. □

Remark 3.1.2. Note that the cocycle condition (3.6) is the same as the cocycle condition in $\mathrm{H}^2(\mathcal{W},\mathbb{K})$. Similarly, the coboundary condition (3.9) is the same as the coboundary condition in $\mathrm{H}^2(\mathcal{W},\mathbb{K})$. This leads to the fact that the 2-cocycle in (3.11) actually corresponds to the Virasoro 2-cocycle, see (2.20). This is no coincidence but a general fact valid for every cohomological dimension. This will be shown in Theorem 3.3.2 in Section 3.3.

We have seen that the proofs for the Witt and the Virasoro algebra are very similar for the trivial module, and we obtained that the third cohomology has the same dimension for both Lie algebras. This leads to the sensible question whether the result is a mere coincidence or whether there is a deeper, more fundamental link between the two cohomology spaces. This can best be investigated with spectral sequences, which we will do in the next section.

3.1.2 Relation between $\mathrm{H}^k(\mathcal{W},M)$ and $\mathrm{H}^k(\mathcal{V},M)$

The aim in this section is to find a first relation between $\mathrm{H}^k(\mathcal{W},M)$ and $\mathrm{H}^k(\mathcal{V},M)$, by using the Hochschild-Serre spectral sequence introduced in Section 2.2.5. The result we will prove in this section is the formula in Proposition 3.1.3 below. We will prove the result for a general module M. It will be used in Section 3.2 for $M=\mathcal{W}$ and deepened further in Section 3.3 both for $M=\mathcal{W}$ and $M=\mathcal{F}^\lambda$.

Proposition 3.1.3. *Let $\varphi_p := d_2^{p,1} : E_2^{p,1} \to E_2^{p+2,0}$ and M a $\mathcal{W}$-module. Then we have the following relation between the algebraic cohomology of the Witt $\mathcal{W}$ and the Virasoro $\mathcal{V}$ algebra:*

$$\mathrm{H}^k(\mathcal{V},M) \cong \frac{\mathrm{H}^k(\mathcal{W},M)}{im\,\varphi_{k-2}} \oplus ker\,(\varphi_{k-1} : \mathrm{H}^{k-1}(\mathcal{W},M) \to \mathrm{H}^{k+1}(\mathcal{W},M)). \tag{3.45}$$

Proof. We will prove this via the Hochschild-Serre spectral sequence. Take $\mathcal{E}=\mathcal{V}$ and $\mathcal{H}=\mathbb{K}$ in Theorem 2.2.6, then $\mathcal{E}/\mathcal{H}=\mathcal{W}$. Hence, the second stage spectral sequence E_2^{pq} becomes in our case:

$$E_2^{p,q} = \mathrm{H}^p(\mathcal{W},\mathrm{H}^q(\mathbb{K},M)) \Rightarrow \mathrm{H}^{p+q}(\mathcal{V},M)\,.$$

Due to the alternating property of the cochains, we have $\mathrm{H}^k(\mathbb{K},M)=0$ for $k>1$. This is true for any module M.

Consequently, the Hochschild-Serre spectral sequence has only two lines in our case:

$$\begin{array}{c|cccc} & 0 & 0 & 0 & \dots \\ 1 & \mathrm{H}^0(\mathcal{W},\mathrm{H}^1(\mathbb{K},M)) & \mathrm{H}^1(\mathcal{W},\mathrm{H}^1(\mathbb{K},M)) & \mathrm{H}^2(\mathcal{W},\mathrm{H}^1(\mathbb{K},M)) & \dots \\ 0 & \mathrm{H}^0(\mathcal{W},\mathrm{H}^0(\mathbb{K},M)) & \mathrm{H}^1(\mathcal{W},\mathrm{H}^0(\mathbb{K},M)) & \mathrm{H}^2(\mathcal{W},\mathrm{H}^0(\mathbb{K},M)) & \dots \\ \hline & 0 & 1 & 2 & \end{array} \tag{3.46}$$

The entries with $p<0$ or $q<0$ are zero. In addition, the entries for $q\geq 2$ are zero because of $\mathrm{H}^k(\mathbb{K},M)=0$ for $k>1$.

The second stage spectral sequence comes with the differentials:

$$d_2^{p,q} : E_2^{p,q} \longrightarrow E_2^{p+2,q-1}\,.$$

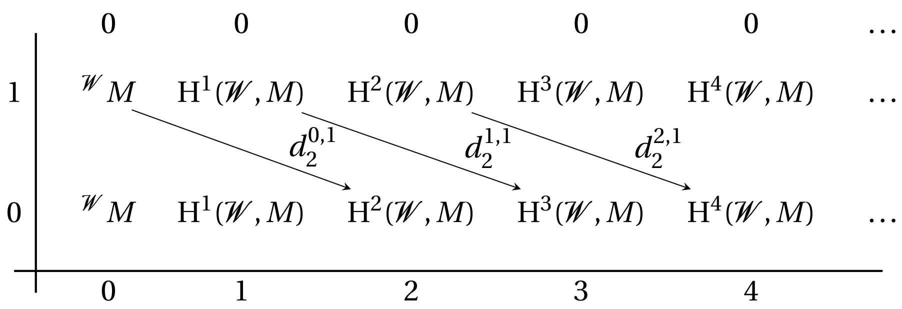
With these maps, we can take the cohomology $E_3^{p,q}$ of $E_2^{p,q}$, which gives the third page spectral sequence:

$$E_3^{p,q} = \frac{\ker d_2^{p,q}}{\operatorname{im} d_2^{p-2,q+1}}\,.$$

The third page spectral sequence has the same shape as the second page spectral sequence, meaning it too has only two lines different from zero. Again, the third page spectral sequence comes with maps:

$$d_3^{p,q} : E_3^{p,q} \longrightarrow E_3^{p+3,q-2} .$$

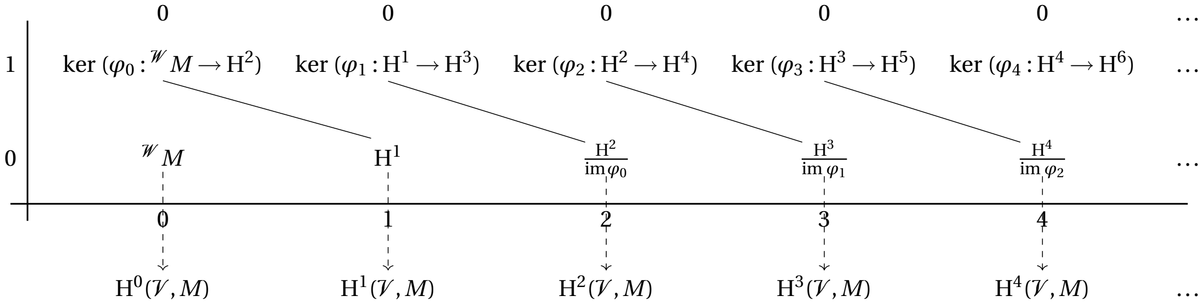
However, the operator $d_3^{p,q}$ corresponds to going three entries to the right and two entries to the bottom. Since we only have two lines different from zero , we always obtain im $d_3^{p,q} = 0$. Moreover, the kernel of $d_3^{p,q}$ then corresponds to $E_3^{p,q}$. Therefore, we obtain $E_3^{p,q} = E_4^{p,q} = \cdots = E_\infty^{p,q}$, meaning in our case, the Hochschild-Serre spectral sequence converges already on the third page.
We consider a $\mathscr{W}$-module M in (3.46), which is a $\mathscr{V}$-module as a quotient module and a trivial $\mathbb{K}$-module, see the Remark 2.1.1. The latter implies $\mathrm{H}^0(\mathbb{K}, M) = {}^{\mathbb{K}}M = M$, where ${}^{\mathbb{K}}M$ denotes the space of $\mathbb{K}$-invariants of M, see (2.41). Moreover, we have $\mathrm{H}^1(\mathbb{K}, M) = M$. In fact every linear map $\phi \in C^1(\mathbb{K}, M)$ is a cocycle since $(\delta_1\phi)(t_1, t_2) = 0 \ \forall \ t_1, t_2 \in \mathbb{K}$ due to the alternating property. Therefore, $\mathrm{H}^1(\mathbb{K}, M)$ just corresponds one-to-one to all linear maps $1 \mapsto \omega \in M$, i.e. to all elements of M. The $(0,0)$ and $(0,1)$ entries in (3.46) thus become: $\mathrm{H}^0(\mathscr{W}, M) = {}^{\mathscr{W}}M$.
Our second stage spectral sequence thus becomes:



In order to simplify the notation, we define:

$$\varphi_p := d_2^{p,1} : E_2^{p,1} \longrightarrow E_2^{p+2,0} . \tag{3.47}$$

Next, we take the cohomology of the sequence with respect to φ_p, which gives us the third page spectral sequence $E_3^{p,q} = E_\infty^{p,q}$. We will abbreviate $\mathrm{H}^i(\mathscr{W}, M)$ by H^i:



The elements $E_2^{p,q}$ converge to $\mathrm{H}^{p+q}(\mathscr{E}, M)$, i.e. $\mathrm{H}^{p+q}(\mathscr{V}, M)$ in our case. This means that in the case under consideration, we can write the elements $\mathrm{H}^{p+q}(\mathscr{V}, M)$ of degree $n = p + q$ as a direct sum of the elements $E_3^{p,q}$ of degree n, i.e. the elements $E_3^{p,q}$ with $p + q = n$ lying on the n-th diagonal:

$$\mathrm{H}^k(\mathscr{V}, M) \cong E_3^{k,0} \oplus E_3^{k-1,1} .$$

Clearly, the diagram above gives us formula (3.45). For the low-dimensional cohomology, we thus obtain,

- ${}^{\mathcal{V}}M = {}^{\mathcal{W}}M$
- $\mathrm{H}^1(\mathcal{V}, M) \cong \mathrm{H}^1(\mathcal{W}, M) \oplus \ker(\varphi_0 : {}^{\mathcal{W}}M \to \mathrm{H}^2(\mathcal{W}, M))$.
- $\mathrm{H}^2(\mathcal{V}, M) \cong \frac{\mathrm{H}^2(\mathcal{W},M)}{\mathrm{im}\,\varphi_0} \oplus \ker(\varphi_1 : \mathrm{H}^1(\mathcal{W}, M) \to \mathrm{H}^3(\mathcal{W}, M))$.
- $\mathrm{H}^3(\mathcal{V}, M) \cong \frac{\mathrm{H}^3(\mathcal{W},M)}{\mathrm{im}\,\varphi_1} \oplus \ker(\varphi_2 : \mathrm{H}^2(\mathcal{W}, M) \to \mathrm{H}^4(\mathcal{W}, M))$.
- $\mathrm{H}^4(\mathcal{V}, M) \cong \frac{\mathrm{H}^4(\mathcal{W},M)}{\mathrm{im}\,\varphi_2} \oplus \ker(\varphi_3 : \mathrm{H}^3(\mathcal{W}, M) \to \mathrm{H}^5(\mathcal{W}, M))$.
- ...

The maps φ_{-1} and φ_{-2} are zero, hence the first two lines. □

From (3.45), we see that we have $\mathrm{H}^3(\mathcal{V},\mathbb{K}) = \mathrm{H}^3(\mathcal{W},\mathbb{K})$ only if φ_2 is injective. In that case, $\ker\varphi_2 = 0$, and since $\mathrm{H}^1(\mathcal{W},\mathbb{K}) = 0$, we have $\mathrm{im}\,\varphi_1 = 0$, hence the result. In order to prove that $\mathrm{H}^3(\mathcal{V},\mathbb{K}) = \mathrm{H}^3(\mathcal{W},\mathbb{K})$ in a canonical way, we need a better understanding of the differentials of the second page of the Hochschild-Serre spectral sequence. We will analyze these in Section 3.3 only, as the results are not necessary for the proof of the next main results given in the next Section 3.2.

3.2 The adjoint module

This section focuses on the first and the third algebraic cohomology of the Witt and the Virasoro algebra with values in the adjoint module. We start by deriving the cohomologies for the Witt algebra, then we will derive the cohomologies for the Virasoro algebra by using the results from the previous Section 3.1.

3.2.1 Analysis of $\mathrm{H}^1(\mathcal{W},\mathcal{W})$ and $\mathrm{H}^3(\mathcal{W},\mathcal{W})$

The first cohomology $\mathrm{H}^1(\mathcal{W},\mathcal{W})$

In this section, we prove $\mathrm{H}^1(\mathcal{W},\mathcal{W}) = \{0\}$ in the Theorem 3.2.1 below. Recall from (2.48) that $\mathrm{H}^1(\mathcal{W},\mathcal{W}) = \dfrac{\mathrm{Der}\,\mathcal{W}}{\mathrm{ad}_{\mathcal{W}}} = \mathrm{Out}(\mathcal{W})$. Therefore, Theorem 3.2.1 gives a simple proof of the statement that for the Witt algebra, all derivations are inner derivations.

Theorem 3.2.1. *The first algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in the adjoint module vanishes, i.e.*

$$\mathrm{H}^1(\mathcal{W},\mathcal{W}) = \{0\}\,.$$

Proof. The proof follows in two steps, the first step concentrating on the non-zero degree cohomology of the Witt algebra, the second step focusing on the degree zero cohomology.
Recall from (2.47) that the condition for a 1-cochain ψ to be a 1-cocycle with values in the adjoint module is:

$$\delta_1\psi(x_1,x_2) = 0 = \psi([x_1,x_2]) - [x_1,\psi(x_2)] - [\psi(x_1),x_2]\ ,$$

with $x_1, x_2 \in \mathscr{W}$. The condition for a 1-cocycle $\psi \in \mathrm{H}^1(\mathscr{W},\mathscr{W})$ to be a coboundary with values in the adjoint module is given by (2.46):

$$\psi(x) = (\delta_0\phi)(x) = -x\cdot\phi = \left[\phi, x\right] ,$$

with $x \in \mathscr{W}$ and $\phi \in \mathrm{C}^0(\mathscr{W},\mathscr{W}) = \mathscr{W}$. As the values are taken in the adjoint module, we have $\cdot = [.,.]$.
We will start by analyzing $\mathrm{H}^1_{(d)}(\mathscr{W},\mathscr{W})$ with $d \neq 0$ in order to exhibit the proof of Theorem 2.2.1 on a concrete and simple example. The result shall be the following:

$$\mathrm{H}^1_{(d)}(\mathscr{W},\mathscr{W}) = \{0\} \text{ for } d \neq 0 \quad \text{and} \quad \mathrm{H}^1(\mathscr{W},\mathscr{W}) = \mathrm{H}^1_{(0)}(\mathscr{W},\mathscr{W}) .$$

Let $\psi \in \mathrm{H}^1_{(d\neq 0)}(\mathscr{W},\mathscr{W})$.
Let us perform a cohomological change $\psi' = \psi - \delta_0\phi$ with the following 0-cochain ϕ:

$$\phi = -\frac{1}{d}\psi(e_0) \in \mathscr{W} \Rightarrow (\delta_0\phi)(x) = \frac{1}{d}\left[x, \psi(e_0)\right] ,$$

which gives us:

$$\psi'(x) = \psi(x) - (\delta_0\phi)(x) = \psi(x) + \frac{1}{d}\left[\psi(e_0), x\right] .$$

Hence,

$$\psi'(e_0) = \psi(e_0) + \frac{1}{d}\left[\psi(e_0), e_0\right] = \psi(e_0) - \frac{1}{d}\,\mathrm{deg}(\psi(e_0))\,\psi(e_0) = \psi(e_0) - \frac{1}{d}\,d\,\psi(e_0) = 0 .$$

We thus have $\boxed{\psi'(e_0) = 0}$.
Next, let us write down the cocycle condition for ψ' on the doublet (x, e_0) for $x \in \mathscr{W}$:

$$\begin{aligned} 0 = \psi'\left([x, e_0]\right) - \underbrace{\left[x, \psi'(e_0)\right]}_{=0} - \left[\psi'(x), e_0\right] &\Leftrightarrow\ 0 = \psi'(-\mathrm{deg}(x)x) + \mathrm{deg}(\psi'(x))\psi'(x) \\ \Leftrightarrow\ 0 = -\mathrm{deg}(x)\psi'(x) + (\mathrm{deg}(x) + d)\psi'(x) &\Leftrightarrow\ 0 = d\,\psi'(x) . \end{aligned}$$

As $d \neq 0$, we get $\psi'(x) = 0\ \forall x \in \mathscr{W}$, meaning that ψ is a coboundary on $\mathscr{W}$. We conclude that the first cohomology of the Witt algebra reduces to the degree zero cohomology, in accordance with Theorem 2.2.1.

Next, we focus on the a priori non-trivial degree-zero cohomology $\mathrm{H}^1_{(0)}(\mathscr{W},\mathscr{W})$. We will show the following:

$$\mathrm{H}^1_{(0)}(\mathscr{W},\mathscr{W}) = \{0\} .$$

Let ψ be a degree zero 1-cocycle, i.e. we can write it as $\psi(e_i) = \psi_i e_i$ with suitable coefficients $\psi_i \in \mathbb{K}$. Consider the following 0-cochain $\phi = \psi_1 e_0$. The coboundary condition for ϕ gives:

$$(\delta_0\phi)(e_i) = \left[\phi, e_i\right] = i\psi_1 e_i .$$

The cohomological change $\psi' = \psi - \delta_0\phi$ leads to $\psi'_1 = 0$. In the following, we will work with a 1-cocycle normalized to $\psi'_1 = 0$, although we will drop the apostrophe in order to augment readability.
The 1-cocycle condition for ψ on the doublet (e_i, e_j) becomes:

$$\begin{aligned} 0 &= \psi\left(\left[e_i, e_j\right]\right) - \left[e_i, \psi(e_j)\right] - \left[\psi(e_i), e_j\right] \\ \Leftrightarrow\ 0 &= (j-i)\left(\psi_{i+j} - \psi_j - \psi_i\right) . \end{aligned}$$

For $j = 1$ and $i = 0$, we obtain from the 1-cocycle condition: $\psi_0 = 0$.
For $j = 1$ and $i < 0$ decreasing, we obtain from the 1-cocycle condition: $\psi_i = \psi_{i+1} = 0$.
For $j = 1$ and $i > 1$ increasing, we obtain from the 1-cocycle condition: $\psi_{i+1} = \psi_i = \psi_2$, where the value of ψ_2 is unknown for the moment.
Next, taking $j = 2$ and for example $i = 3$, we obtain:

$$\begin{aligned}
&\psi_5 - \psi_2 - \psi_3 = 0\\
\Leftrightarrow\ &\psi_2 - \psi_2 - \psi_2 = 0 \text{ as we have } \psi_i = \psi_2\ \forall i > 1\\
\Leftrightarrow\ &\psi_2 = 0\,.
\end{aligned}$$

All in all, we conclude $\boxed{\psi_i = 0\ \forall i \in \mathbb{Z}}$.

This concludes the proof of Theorem 3.2.1. □

The third cohomology $\mathrm{H}^3(\mathcal{W}, \mathcal{W})$

Theorem 3.2.2. *The third algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in the adjoint module vanishes, i.e.*

$$\mathrm{H}^3(\mathcal{W}, \mathcal{W}) = \{0\}\,.$$

Due to Theorem 2.2.1, we already know that the non-zero degree cohomology of the Witt algebra is zero. However, for reasons of completeness, we will prove this result again for the third cohomology, since the proof is short and simple. The proof is an application of the proof of Theorem 2.2.1 to the particular case of the third cohomology with values in the adjoint module. Thus, we proceed in two steps, the first step concentrating on the non-zero degree cohomology of the Witt algebra, the second step focusing on the degree zero part.
Recall from (2.70) that the condition for a 3-cochain ψ to be a 3-cocycle with values in the adjoint module is given by:

$$\begin{aligned}
&(\delta_3\psi)(x_1, x_2, x_3, x_4)\\
=\ &\psi([x_1, x_2], x_3, x_4) - \psi([x_1, x_3], x_2, x_4) + \psi([x_1, x_4], x_2, x_3)\\
&+ \psi([x_2, x_3], x_1, x_4) - \psi([x_2, x_4], x_1, x_3) + \psi([x_3, x_4], x_1, x_2)\\
&- \left[x_1, \psi(x_2, x_3, x_4)\right] + \left[x_2, \psi(x_1, x_3, x_4)\right] - \left[x_3, \psi(x_1, x_2, x_4)\right] + \left[x_4, \psi(x_1, x_2, x_3)\right] = 0\,,
\end{aligned}$$

with $x_1, x_2, x_3, x_4 \in \mathcal{W}$.
The condition for a 3-cocycle $\psi \in \mathrm{H}^3(\mathcal{W}, \mathcal{W})$ to be a coboundary with values in the adjoint module is given by (2.69):

$$\begin{aligned}
\psi(x_1, x_2, x_3) = (\delta_2\phi)(x_1, x_2, x_3) =\ &\phi([x_1, x_2], x_3) + \phi([x_2, x_3], x_1) + \phi([x_3, x_1], x_2)\\
&- \left[x_1, \phi(x_2, x_3)\right] + \left[x_2, \phi(x_1, x_3)\right] - \left[x_3, \phi(x_1, x_2)\right]\,,
\end{aligned}$$

where $x_1, x_2, x_3 \in \mathcal{W}$ and $\phi \in \mathrm{C}^2(\mathcal{W}, \mathcal{W})$.
We start by showing the following,

$$\mathrm{H}^3_{(d)}(\mathcal{W}, \mathcal{W}) = \{0\} \text{ for } d \neq 0 \quad \text{and} \quad \mathrm{H}^3(\mathcal{W}, \mathcal{W}) = \mathrm{H}^3_{(0)}(\mathcal{W}, \mathcal{W})\,. \tag{3.48}$$

Let $\psi \in \mathrm{H}^3_{(d\neq 0)}(\mathcal{W}, \mathcal{W})$. Let us perform a cohomological change $\psi' = \psi - \delta_2\phi$ with the following 2-cochain ϕ:

$$\phi(x_1, x_2) = -\frac{1}{d}\psi(x_1, x_2, e_0)\,,$$

which gives us, taking into account that $\phi(e_0,\cdot) = \phi(\cdot,e_0) = 0$:

$$\begin{aligned}
&\psi'(x_1,x_2,e_0) = \psi(x_1,x_2,e_0) - (\delta_2\phi)(x_1,x_2,e_0)\\
&= \psi(x_1,x_2,e_0) - \underbrace{\phi([x_1,x_2],e_0)}_{=0} - \phi([x_2,e_0],x_1) - \phi([e_0,x_1],x_2)\\
&+ [x_1,\underbrace{\phi(x_2,e_0)}_{=0}] - [x_2,\underbrace{\phi(x_1,e_0)}_{=0}] + \big[e_0,\phi(x_1,x_2)\big]\\
&= \psi(x_1,x_2,e_0) + \deg(x_2)\underbrace{\phi(x_2,x_1)}_{=-\phi(x_1,x_2)} - \deg(x_1)\phi(x_1,x_2) + \big(\deg(x_1)+\deg(x_2)+d\big)\phi(x_1,x_2)\\
&= -d\,\phi(x_1,x_2) + d\,\phi(x_1,x_2) = 0\,.
\end{aligned}$$

We thus have $\boxed{\psi'(x_1,x_2,e_0) = 0}$. Next, let us write down the cocycle condition for ψ' on the quadruplet (x_1,x_2,x_3,e_0) of homogeneous elements:

$$\begin{aligned}
&(\delta_3\psi')(x_1,x_2,x_3,e_0) = 0\\
\Leftrightarrow\ &\underbrace{\psi'([x_1,x_2],x_3,e_0)}_{=0} - \underbrace{\psi'([x_1,x_3],x_2,e_0)}_{=0} + \psi'([x_1,e_0],x_2,x_3)\\
&+ \underbrace{\psi'([x_2,x_3],x_1,e_0)}_{=0} - \psi'([x_2,e_0],x_1,x_3) + \psi'([x_3,e_0],x_1,x_2)\\
&- [x_1,\underbrace{\psi'(x_2,x_3,e_0)}_{=0}] + [x_2,\underbrace{\psi'(x_1,x_3,e_0)}_{=0}] - [x_3,\underbrace{\psi'(x_1,x_2,e_0)}_{=0}] + \big[e_0,\psi'(x_1,x_2,x_3)\big] = 0\\
\Leftrightarrow\ &-\deg(x_1)\psi(x_1,x_2,x_3) + \deg(x_2)\underbrace{\psi(x_2,x_1,x_3)}_{=-\psi(x_1,x_2,x_3)} - \deg(x_3)\underbrace{\psi(x_3,x_1,x_2)}_{=\psi(x_1,x_2,x_3)}\\
&+ \big(\deg(x_1)+\deg(x_2)+\deg(x_3)+d\big)\psi(x_1,x_2,x_3) = 0\\
\Leftrightarrow\ &d\,\psi(x_1,x_2,x_3) = 0 \Leftrightarrow \psi(x_1,x_2,x_3) = 0 \text{ as } d \neq 0\,.
\end{aligned}$$

We conclude that the third cohomology of the Witt algebra reduces to the degree zero cohomology, in agreement with the result of Theorem 2.2.1.

Next, we focus on the hard part of the proof, which aims to prove the following,

$$\mathrm{H}^3_{(0)}(\mathcal{W},\mathcal{W}) = \{0\}\,. \tag{3.49}$$

Clearly, the results (3.49) and (3.48) prove Theorem 3.2.2. The proof of (3.49) is accomplished in six steps and is similar to the proof performed for $\mathrm{H}^2_{(0)}(\mathcal{W},\mathcal{W})$ in [105, 106], albeit the proof here is quite a bit more complicated.
Let ψ be a degree zero 3-cocycle, i.e. we can write it as $\psi(e_i,e_j,e_k) = \psi_{i,j,k}e_{i+j+k}$ with suitable coefficients $\psi_{i,j,k} \in \mathbb{K}$. We say that $\psi_{\cdot,\cdot,\cdot}$ is of *level* $l \in \mathbb{Z}$ if one of its indices is equal to l, i.e. $\psi_{\cdot,\cdot,\cdot} = \psi_{\cdot,\cdot,l}$ or some permutation thereof.
Consequently, five steps of the proof correspond to the analysis of the levels plus one, minus one, zero, plus two and minus two. The final step consists in the analysis of generic levels, which is obtained by induction. In each step, there are always three cases to consider depending on the signs of the indices. One of the three indices corresponds to the level and is fixed. In that case, the three cases to consider correspond to both remaining indices being negative, both being positive, or one being negative and one being positive. It does not matter which of the indices are chosen to be positive or negative, nor does it matter which one of the three indices is chosen to be fixed, because of the alternating property of the cochains. In the following, we provide a brief and superficial summary of the proof:

- **Level plus one / minus one:** There is a cohomological change $\psi' = \psi - \delta_2\phi$, $\phi \in C^2_{(0)}(\mathcal{W},\mathcal{W})$ which allows to normalize to zero either the coefficients of level plus one or the coefficients of level minus one, depending on the signs of the two remaining indices. More precisely, we normalize ψ' to $\psi'_{i,j,-1} = 0$ if i and j are both positive and $\psi'_{i,j,1} = 0$ else.
The aim is to use the coboundary condition to produce recurrence relations which provide a consistent definition of ϕ, i.e. of all the $\phi_{i,j}$ $\forall\ i,j \in \mathbb{Z}$. Each degree of freedom given by some $\phi_{\cdot,\cdot}$ should be used to cancel some coefficient of the form $\psi_{\cdot,\cdot,1}$ or $\psi_{\cdot,\cdot,-1}$. In the case where both indices of $\phi_{i,j}$ have the same sign, the definition of the $\phi_{i,j}$'s can be obtained in a straightforward manner from the recurrence relations. In the case where the two indices are of opposite sign, poles occur in the recurrence relations, and the definition of the $\phi_{i,j}$'s has to be obtained in a somewhat roundabout manner.

- **Level zero:** For a cocycle ψ normalized as described in the previous bullet point, the cocycle conditions imply $\psi_{i,j,0} = 0$ $\forall\ i,j \in \mathbb{Z}$.
The cocycle conditions provide recurrence relations which allow to deduce the result immediately for i and j of the same sign. For i and j of different signs, the proof is an (almost) straightforward generalization of the proof of $\mathrm{H}^2(\mathcal{W},\mathcal{W})_{(0)} = \{0\}$ given in [105, 106].

- **Level minus one / plus one:** The cocycle conditions imply $\psi_{i,j,1} = 0$ if i and j are both positive and $\psi_{i,j,-1} = 0$ else. Together with the result of the first bullet point, we have $\psi_{i,j,1} = \psi_{i,j,-1} = 0$ $\forall\ i,j \in \mathbb{Z}$.
This step is the simplest one of the entire proof. The cocycle conditions provide again recurrence relations which allow to deduce the results directly.

- **Levels plus two and minus two / Generic Level k:** The cocycle conditions imply $\psi_{i,j,-2} = 0$ and $\psi_{i,j,2} = 0$ $\forall\ i,j \in \mathbb{Z}$. Induction on k subsequently implies $\psi_{i,j,k} = 0$ $\forall\ i,j,k \in \mathbb{Z}$.
For both indices i and j negative, the first step consists in proving that level minus two is zero, i.e. $\psi_{i,j,-2} = 0$. Induction on the third index allows to conclude that the coefficients $\psi_{i,j,k}$ are zero for all negative indices $i,j,k \leq 0$. These results can be obtained directly from the recurrence relations given by the cocycle conditions.
In the case of one positive and one negative index, the first step consists in proving that both levels plus two and minus two are zero, $\psi_{i,j,2} = \psi_{i,j,-2} = 0$. This has to be done by using induction on either i or j depending on the level under consideration. Note that in the proof of $\mathrm{H}^2(\mathcal{W},\mathcal{W})_{(0)} = \{0\}$ in [105, 106], the vanishing of the levels plus two and minus two could be proved directly without using induction. Obviously, the number of times induction has to be used increases with the number of indices. Due to poles and zeros in the recurrence relations, the proof again follows a somewhat roundabout way.
The second and final step consists in using induction on the third index in order to prove $\psi_{i,j,k} = 0$ for mixed indices, i.e. two indices positive and one index negative or two indices negative and one index positive.
The final case with both indices i and j positive starts with the proof that level plus two is zero, i.e. $\psi_{i,j,2} = 0$. Induction on the third index allows to conclude that the coefficients $\psi_{i,j,k}$ are zero for all positive indices $i,j,k \geq 0$. These results follow directly from the recurrence relations.

We now come to the detailed proof. Let us write down the coboundary and cocycle conditions for later use. If ϕ is a degree zero 2-cochain, i.e. $\phi(e_i,e_j) = \phi_{i,j}e_{i+j}$, the coboundary condition for ψ on the triplet (e_i, e_j, e_k) becomes:

$$\psi_{i,j,k} = (\delta_2\phi)_{i,j,k} = (j-i)\phi_{i+j,k} + (k-j)\phi_{k+j,i} + (i-k)\phi_{i+k,j} - (j+k-i)\phi_{j,k} + (i+k-j)\phi_{i,k} - (i+j-k)\phi_{i,j}\,.$$

The cocycle condition for ψ on the quadruplet (e_i, e_j, e_k, e_l) becomes:

$$\begin{aligned}(\delta_3\psi)_{i,j,k,l} =& (j-i)\psi_{i+j,k,l} - (k-i)\psi_{i+k,j,l} + (l-i)\psi_{i+l,j,k} \\ &+ (k-j)\psi_{k+j,i,l} - (l-j)\psi_{l+j,i,k} + (l-k)\psi_{l+k,i,j} \\ &- (j+k+l-i)\psi_{j,k,l} + (i+k+l-j)\psi_{i,k,l} \\ &- (i+j+l-k)\psi_{i,j,l} + (i+j+k-l)\psi_{i,j,k} = 0\,.\end{aligned}$$

In the proofs below, when we evaluate the cocycle and coboundary conditions on quadruplets (e_i, e_j, e_k, e_l) and triplets (e_i, e_j, e_k) respectively, we will refer to these quadruplets and triplets simply by the indices of the basis elements, and write them as (i, j, k, l) and (i, j, k), respectively. The first step of the proof is achieved with a cohomological change:

Lemma 3.2.1. *Every 3-cocycle ψ of degree zero is cohomologous to a degree zero 3-cocycle ψ' with:*

$$\begin{aligned} &\psi'_{i,j,1} = 0 && \forall\, i \le 0,\ \forall\, j \in \mathbb{Z}, \\ \textit{and}\quad &\psi'_{i,j,-1} = 0 && \forall\, i, j > 0, \\ \textit{and}\quad &\psi'_{i,-1,2} = 0 && \forall\, i \in \mathbb{Z}, \\ \textit{and}\quad &\psi'_{-4,2,-2} = 0\,. \end{aligned} \tag{3.50}$$

Proof. The aim is to define consistently a 2-cochain ϕ that leads to the results (3.50) after the cohomological change. Writing ϕ in terms of its coefficients, we start by defining $\phi_{i,1} = 0\ \forall\, i \in \mathbb{Z}$ and $\phi_{-1,2} = 0$. Hence, we will perform a cohomological change $\psi' = \psi - \delta_2\phi$ with ϕ normalized to $\boxed{\phi_{i,1} = 0\ \forall\, i \in \mathbb{Z}}$ and $\boxed{\phi_{-1,2} = 0}$. This simplifies the notations considerably.
To increase the readability of the proof, we will separate the analysis depending on the signs of the indices i, j. Let us start with the case i and j both being negative.

Case 1: $i, j \le 0$

Our aim is to show that we can find coefficients $\phi_{i,j}$ such that $\psi'_{i,j,1} = 0$. Writing down the coboundary condition for $(i, j, 1)$ and dropping the terms of the form $\phi_{.,1}$, we need:

$$\psi_{i,j,1} = -(i+j-1)\phi_{i,j} + (i-1)\phi_{i+1,j} - (j-1)\phi_{j+1,i}\,.$$

This is the case if we define ϕ:

$$\phi_{i,j} := \frac{i-1}{i+j-1}\phi_{i+1,j} - \frac{j-1}{i+j-1}\phi_{j+1,i} - \frac{\psi_{i,j,1}}{i+j-1}\,.$$

Starting with $i = 0$, $j = -1$, j decreasing and using $\phi_{.,1} = 0$, this recurrence relation defines in a first step $\phi_{0,j}$ for $j \le -1$. In a second step, $\phi_{-1,j}$ with $j \le -2$ can be obtained, and so on for all $i \le -2$ with $j < i$. It is sufficient to consider $j < i$ due to the alternating character of the cochains. Thus, this recurrence relation defines $\phi_{i,j}$ for $i, j \le 0$. It follows that we can perform a cohomological change such that $\psi'_{i,j,1} = 0\ \forall\, i, j \le 0$.

Case 2: $i \le 0$ and $j > 0$

We will start by proving that we can obtain $\psi'_{i,2,-1} = 0\ \forall\, i \le 0$ for a suitable choice of the coefficients $\phi_{i,j}$.

Let us consider the coboundary condition for $(-3,2,-1)$. Taking into account the normalization $\phi_{2,-1}=0$ we obtain:

$$-2\phi_{-4,2}-6\phi_{-3,-1}=\psi_{-3,2,-1}\,.$$

The quantity $\phi_{-3,-1}$ has been defined in the previous case $i,j\leq 0$. Thus, we obtain a definition for $\phi_{-4,2}$. From there, we can obtain $\phi_{i,2}$ $i\leq -5$ by using the coboundary condition for $(i,2,-1)$ and $\phi_{2,-1}=0$, which gives us:

$$\phi_{i-1,2}=\frac{3+i}{i+1}\phi_{i,2}+\frac{\psi_{i,2,-1}}{i+1}-\frac{i-3}{i+1}\phi_{i,-1}+\frac{i-2}{i+1}\phi_{i+2,-1}\,. \tag{3.51}$$

The last two terms have been defined in the previous case $i,j\leq 0$. Thus, this defines $\phi_{i,2}$ $i\leq -4$ such that we have $\psi'_{i,2,-1}=0\ \forall\ i\leq -3$. Next, let us consider the coboundary condition for $(-4,2,-2)$:

$$\psi_{-4,2,-2}=-2\phi_{-6,2}-8\phi_{-4,-2}-4\phi_{0,-4}-4\phi_{2,-2}\,.$$

The coefficients $\phi_{-4,-2}$ and $\phi_{0,-4}$ have been defined in the previous case $i,j\leq 0$. The coefficient $\phi_{-6,2}$ has been defined in (3.51) for $i\leq -4$. Therefore, we obtain a definition for $\phi_{2,-2}$, which annihilates $\psi'_{-4,2,-2}$, $\boxed{\psi'_{-4,2,-2}=0}$.
As $\phi_{2,-2}$ is now defined, we can come back to Equation (3.51), insert $i=-2$ and obtain a definition for $\phi_{-3,2}$, annihilating $\psi'_{-2,2,-1}$. Since $\phi_{-1,2}=0$ due to our normalization, the only remaining $\phi_{i,2}$ $i\leq 0$ to define is $\phi_{0,2}$.
Let us write down the coboundary condition for $(0,2,-1)$:

$$\begin{aligned}&-(3\phi_{0,2}+3\phi_{0,-1})=\psi_{0,2,-1}\\ \Leftrightarrow\ &\phi_{0,2}=-\phi_{0,-1}-\frac{1}{3}\psi_{0,2,-1}\,.\end{aligned}$$

This defines $\phi_{0,2}$ and consequently, $\psi'_{0,2,-1}=0$. Since $\psi'_{-1,2,-1}=0$ due to the alternating property, we obtain all in all that $\psi'_{i,2,-1}=0\ \forall\ i\leq 0$.
Next, let us prove that we can obtain $\psi'_{i,j,1}=0\ \forall\ i\leq 0\ \forall\ j>0$. It suffices to write down the coboundary condition for $(i,j,1)$ in the following way:

$$\phi_{i,j+1}:=\frac{i+j-1}{j-1}\phi_{i,j}-\frac{i-1}{j-1}\phi_{i+1,j}+\frac{\psi_{i,j,1}}{j-1}\,.$$

Fixing $i=0$, and starting with $j=2$ (recall that $\phi_{i,1}=0$ and that we have just defined all $\phi_{i,2}$ $i\leq 0$), j increasing, we obtain $\phi_{0,j}\ \forall\ j>2$ and $\psi'_{0,j,1}=0\ \forall\ j\geq 2$. Similarly, fixing $i=-1$, and starting with $j=2$, j increasing, we obtain $\phi_{-1,j}\ \forall\ j>2$ and $\psi'_{-1,j,1}=0\ \forall\ j\geq 2$. Continuing along the same lines, we obtain $\phi_{i,j}\ \forall\ i\leq 0,\ j>0$ and $\psi'_{i,j,1}=0\ \forall\ i\leq 0,\ j>0$. Together with the result $\psi'_{i,j,1}=0\ \forall\ i,j\leq 0$ obtained from the previous case with $i,j\leq 0$, we get $\boxed{\psi'_{i,j,1}=0\ \forall\ i\leq 0,\ \forall\ j\in\mathbb{Z}}$

Case 3: $i>0$ and $j>0$

Let us write down the coboundary condition for $(i,j,-1)$:

$$\begin{aligned}\psi_{i,j,-1}=&(i+1)\phi_{i-1,j}+(i-j-1)\phi_{i,-1}-(1+i+j)\phi_{i,j}\\ &+(j+1)\phi_{i,j-1}+(1+i-j)\phi_{j,-1}+(j-i)\phi_{i+j,-1}\,.\end{aligned}$$

From there, we can define ϕ via recurrence as follows:

$$\phi_{i,j} = \frac{(i+1)}{(1+i+j)}\phi_{i-1,j} + \frac{(j+1)}{(1+i+j)}\phi_{i,j-1} - \frac{\psi_{i,j,-1}}{(1+i+j)} + \frac{(i-j-1)}{(1+i+j)}\phi_{i,-1} + \frac{(1+i-j)}{(1+i+j)}\phi_{j,-1} + \frac{(j-i)}{(1+i+j)}\phi_{i+j,-1}\,.$$

Note that $\phi_{\cdot,-1}$ have been defined in the previous case for $i \le 0,\ j > 0$. Starting with $i = 2$, $j = 3$ and j increasing, we obtain in a first step $\phi_{2,j},\ \forall\ j \ge 3$ and $\psi'_{2,j,-1} = 0\ \forall\ j \ge 3$. Next, fixing $i = 3$, starting with $j = 4$ and j increasing, we obtain in a second step $\phi_{3,j},\ \forall\ j \ge 4$ and $\psi'_{3,j,-1} = 0\ \forall\ j \ge 4$. Continuing similarly with i increasing, we finally obtain all $\phi_{i,j},\ \forall\ i, j > 0$, and $\boxed{\psi'_{i,j,-1} = 0\ \forall\ i, j > 0}$. Note that we already have $\psi'_{1,j,-1} = 0\ \forall\ j > 0$ due to the previous case, which yielded $\psi'_{i,j,1} = 0\ \forall\ i \le 0,\ \forall\ j \in \mathbb{Z}$. Combining the result $\psi'_{i,j,-1} = 0\ \forall\ i, j > 0$ with the result $\psi'_{i,2,-1} = 0\ \forall\ i \le 0$ from the previous case $i \le 0,\ j > 0$, we also obtain $\boxed{\psi'_{i,2,-1} = 0\ \forall\ i \in \mathbb{Z}}$. □

Lemma 3.2.2. *Let ψ be a degree zero 3-cocycle such that:*

$$\begin{aligned} &\psi_{i,j,1} = 0 \quad \forall\ i \le 0,\ \forall\ j \in \mathbb{Z}, \\ \textit{and}\quad &\psi_{i,j,-1} = 0 \quad \forall\ i, j > 0, \\ \textit{and}\quad &\psi_{i,-1,2} = 0 \quad \forall\ i \in \mathbb{Z}, \end{aligned}$$

then

$$\psi_{i,j,0} = 0 \quad \forall\ i, j \in \mathbb{Z}. \tag{3.52}$$

Proof. Again, we split the proof into the three cases depending on the signs of i and j.

Case 1: $i, j \le 0$

Let us write down the cocycle condition for $(i, j, 0, 1)$, neglecting the terms of the form $\psi_{i,j,1}\ i, j \le 0$:

$$(i+j-1)\psi_{i,j,0} - (i-1)\psi_{i+1,j,0} + (j-1)\psi_{j+1,i,0} = 0\,.$$

We can define the following recurrence relation for i and j decreasing:

$$\psi_{i,j,0} = \frac{(i-1)}{(i+j-1)}\psi_{i+1,j,0} - \frac{(j-1)}{(i+j-1)}\psi_{j+1,i,0}\,.$$

Fixing $i = -1$, starting with $j = -2$ and j decreasing, we obtain $\psi_{-1,j,0} = 0\ \forall\ j \le -2$. Repeating the same procedure with decreasing values for i and $j < i$, we obtain $\psi_{i,j,0} = 0\ \forall\ i, j \le 0$.

Case 2: $i \le 0,\ j > 0$

Let us write down the cocycle condition for $(i, 2, 0, -1)$:

$$-\cancel{\psi_{-1,i,2}} + 3\psi_{1,i,0} + (-1+i)\psi_{2,0,-1} - 2\cancel{\psi_{2,i,-1}} - (1+i)\psi_{-1+i,2,0} + (-3+i)\psi_{i,0,-1} - \cancel{\psi_{i,2,-1}} + (3+i)\psi_{i,2,0} - (-2+i)\psi_{2+i,0,-1} = 0\,.$$

The slashed terms cancel each other, although they are zero anyway as we have $\psi_{i,2,-1} = 0\ \forall\ i \in \mathbb{Z}$. The term $\psi_{1,i,0}$ is zero as we have $\psi_{i,j,1} = 0\ \forall\ i, j \le 0$. The term $\psi_{2,0,-1}$ is zero due to $\psi_{i,2,-1} = 0\ \forall\ i \in \mathbb{Z}$. The terms $\psi_{i,0,-1}$ and $\psi_{2+i,0,-1}$ (for $i \le -2$) are zero because of Case 1, i.e. they are of the form $\psi_{i,j,0} = 0\ \forall\ i, j \le 0$. Therefore, we are left with:

$$\psi_{i-1,2,0} = \frac{i+3}{i+1}\psi_{i,2,0}\,. \tag{3.53}$$

Putting $i=-3$ in the equation above, this recurrence relation implies $\psi_{-4,2,0}=0$ and by recursion $\psi_{i,2,0}=0\ \forall i\leq -4$. Next, consider the cocycle condition for $(i,2,-2,0)$:

$$\begin{aligned}&2\cancel{\psi_{-2,i,2}}+i\psi_{2,-2,0}+2\cancel{\psi_{2,i,-2}}+(2+i)\psi_{-2+i,2,0}\\&+(-4+i)\psi_{i,-2,0}-(4+i)\psi_{i,2,0}-(-2+i)\psi_{2+i,-2,0}=0\,.\end{aligned}$$

The slashed terms cancel each other, the terms $\psi_{i,-2,0}$ and $\psi_{2+i,-2,0}$ (for $i\leq -2$) are zero because of $\psi_{i,j,0}=0\ \forall\ i,j\leq 0$. As we have $\psi_{i,2,0}=0\ \forall i\leq -4$, we can put for example $i=-4$ in the equation above and obtain $\psi_{2,-2,0}=0$. Inserting this value in Equation (3.53) with $i=-2$, we obtain $\psi_{-3,2,0}=0$. Recall that we also have $\psi_{-1,2,0}=0$ due to $\psi_{i,-1,2}=0\ \forall\ i\in\mathbb{Z}$. All in all, we have $\psi_{i,2,0}\ \forall\ i\leq 0$.
This result is needed to write down a well-defined recurrence relation. Writing down the cocycle condition for $(i,j,0,1)$ and neglecting the terms of the form $\psi_{i,j,1}$ with $i,j\leq 0$ and $i\leq 0\,,j>0$, we obtain the following recurrence relation:

$$\psi_{i,j+1,0}=\frac{i+j-1}{j-1}\psi_{i,j,0}-\frac{i-1}{j-1}\psi_{i+1,j,0}\,.$$

Fixing $i=-1$, one starts with $j=2$ (since we already have $\psi_{i,j,1}=0\ i,j\leq 0$ and $\psi_{i,2,0}=0\ i\leq 0$), which gives, with increasing j, $\psi_{-1,j,0}=0\ j\geq 3$. Continuing with fixing $i=-2$, starting again with $j=2$ and increasing j, we obtain $\psi_{-2,j,0}=0\ j\geq 3$. Doing this for all $i\leq 0$, we finally obtain $\psi_{i,j,0}=0\ \forall\ i\leq 0,\ j>0$.

Case 3: $i,j>0$

Writing down the cocycle condition for $(i,j,0,-1)$, we obtain:

$$\begin{aligned}&-\cancel{\psi_{-1,i,j}}-(1+i)\psi_{-1+i,j,0}+(-1+i-j)\psi_{i,0,-1}+i\cancel{\psi_{i,j,-1}}\\&-\cancel{(-1+i+j)\psi_{i,j,-1}}+(1+i+j)\psi_{i,j,0}+(1+j)\psi_{-1+j,i,0}\\&+(1+i-j)\psi_{j,0,-1}-j\cancel{\psi_{j,i,-1}}+(-i+j)\psi_{i+j,0,-1}=0\,.\end{aligned}$$

The slashed terms cancel each other, though they are zero anyway due to $\psi_{i,j,-1}=0\ i,j>0$. The terms $\psi_{i,0,-1}$, $\psi_{j,0,-1}$ and $\psi_{i+j,0,-1}$ are zero due to the previous case, $\psi_{i,j,0}=0\ i\leq 0,\ j>0$. Thus, we obtain the following recurrence relation:

$$\psi_{i,j,0}=\frac{(1+i)}{(1+i+j)}\psi_{-1+i,j,0}-\frac{(1+j)}{(1+i+j)}\psi_{-1+j,i,0}\,.$$

Fixing $i=1$, starting with $j=2$, j increasing, we obtain $\psi_{1,j,0}=0\ j\geq 2$. Fixing $i=2$, starting with $j=3$, we get $\psi_{2,j,0}=0\ j\geq 3$. Continuing with increasing i and keeping $j>i$ due to skew-symmetry, we finally obtain $\psi_{i,j,0}=0\ \forall\ i,j>0$.
Taking all three cases together, we obtain the announced result, $\psi_{i,j,0}=0\ \forall\ i,j\in\mathbb{Z}$. □

Lemma 3.2.3. *Let ψ be a degree zero 3-cocycle such that:*

$$\begin{aligned}&\psi_{i,j,1}=0 &&\forall\ i\leq 0,\ \forall\ j\in\mathbb{Z}\,,\\ \textit{and}\quad&\psi_{i,j,-1}=0 &&\forall\ i,j>0\,,\\ \textit{and}\quad&\psi_{i,-1,2}=0 &&\forall\ i\in\mathbb{Z}\,,\\ \textit{and}\quad&\psi_{i,j,0}=0 &&\forall\ i,j\in\mathbb{Z}\,,\end{aligned}$$

then

$$\psi_{i,j,1}=\psi_{i,j,-1}=0\quad\forall\ i,j\in\mathbb{Z}\,.$$

Proof. Again, the proof is split into the three cases depending on the signs of i, j.

Case 1: $i, j \leq 0$

Writing down the cocycle condition for $(i, j, 1, -1)$ and neglecting $\psi_{i,j,1}$ $i, j \leq 0$ as well as $\psi_{i,j,0}$ $i, j \leq 0$, we obtain:

$$\begin{aligned} &-(-2+i+j)\psi_{i,j,-1} + (-1+i)\psi_{1+i,j,-1} - (-1+j)\psi_{1+j,i,-1} = 0 \\ \Leftrightarrow\ &\psi_{i,j,-1} = \frac{(-1+i)}{(-2+i+j)}\psi_{1+i,j,-1} - \frac{(-1+j)}{(-2+i+j)}\psi_{1+j,i,-1}\,. \end{aligned}$$

Fixing $i = -2$ (since level zero $\psi_{0,j,-1}$ $j \leq 0$ is already done and $\psi_{-1,j,-1} = 0$), starting with $j = -3$ and j decreasing, we obtain $\psi_{-2,j,-1} = 0$ $j \leq -3$. Fixing $i = -3$, starting with $j = -4$ and j decreasing, we get $\psi_{-3,j,-1} = 0$ $j \leq -4$. Continuing along the same lines, we obtain $\psi_{i,j,-1} = 0$ $i, j \leq 0$.

Case 2: $i \leq 0,\ j > 0$

Writing down the cocycle condition for $(i, j, 1, -1)$ and neglecting $\psi_{i,j,1}$ $i \leq 0,\ j > 0$, $\psi_{i,j,1}$ $i, j \leq 0$ as well as $\psi_{i,j,0}$ $i \leq 0,\ j > 0$, we obtain:

$$\begin{aligned} &-(-2+i+j)\psi_{i,j,-1} + (-1+i)\psi_{1+i,j,-1} - (-1+j)\psi_{1+j,i,-1} = 0 \\ \Leftrightarrow\ &\psi_{i,1+j,-1} = \frac{(-2+i+j)}{(-1+j)}\psi_{i,j,-1} - \frac{(-1+i)}{(-1+j)}\psi_{1+i,j,-1}\,. \end{aligned}$$

Fixing $i = -2$ (since level zero $\psi_{0,j,-1}$ $j > 0$ is already done and $\psi_{-1,j,-1} = 0$) and starting with $j = 2$ (since $\psi_{i,2,-1} = 0$ $i \leq 0$), increasing j, we obtain $\psi_{-2,j,-1} = 0$ $j \geq 3$. Fixing $i = -3$, starting again with $j = 2$, j increasing, we get $\psi_{-3,j,-1} = 0$ $j \geq 3$. Continuing with i decreasing, we get $\psi_{i,j,-1} = 0$ $i \leq 0,\ j > 0$.

Case 3: $i, j > 0$

Writing again down the cocycle condition for $(i, j, 1, -1)$, this time neglecting the terms $\psi_{i,j,-1}$ $i, j > 0$ and $\psi_{i,j,0}$ $i, j > 0$, we obtain:

$$\begin{aligned} &-(1+i)\psi_{-1+i,j,1} + (2+i+j)\psi_{i,j,1} + (1+j)\psi_{-1+j,i,1} = 0 \\ \Leftrightarrow\ &\psi_{i,j,1} = \frac{(1+i)}{(2+i+j)}\psi_{-1+i,j,1} - \frac{(1+j)}{(2+i+j)}\psi_{-1+j,i,1}\,. \end{aligned}$$

Fixing $i = 2$ (since $\psi_{1,j,1} = 0$) and starting with $j = 3$, increasing j, we obtain $\psi_{2,j,1} = 0$ $j \geq 3$. Increasing i and keeping $j > i$ we finally obtain $\psi_{i,j,1} = 0\ \forall\ i, j > 0$.

Taking all three cases together, we have proven that $\psi_{i,j,1} = 0\ \forall\ i, j \in \mathbb{Z}$ and $\psi_{i,j,-1} = 0\ \forall\ i, j \in \mathbb{Z}$. □

Lemma 3.2.4. *Let ψ be a degree zero 3-cocycle such that:*

$$\psi_{i,j,1} = \psi_{i,j,-1} = \psi_{i,j,0} = 0 \quad \forall\ i, j \in \mathbb{Z} \quad \textit{and} \quad \psi_{-4,2,-2} = 0\,,$$

then

$$\psi_{i,j,k} = 0 \quad \forall\ i, j, k \in \mathbb{Z}\,.$$

Proof. Again, the proof is split in the three cases depending on the signs of i and j.

Case 1: $i, j \leq 0$

In a first step, we shall prove the following statement: $\boxed{\psi_{i,j,-2} = 0\ \forall\ i, j \leq 0}$.

Writing down the cocycle condition for $(i, j, -2, 1)$ and neglecting the terms of level one $\psi_{i,j,1}$ and level minus one $\psi_{i,j,-1}$, we obtain the following:

$$(-3+i+j)\psi_{i,j,-2} - (-1+i)\psi_{1+i,j,-2} + (-1+j)\psi_{1+j,i,-2} = 0$$
$$\Leftrightarrow \ \psi_{i,j,-2} = \frac{(-1+i)}{(-3+i+j)}\psi_{1+i,j,-2} - \frac{(-1+j)}{(-3+i+j)}\psi_{1+j,i,-2}\,.$$

Fixing $i = -3$ (since the levels zero $\psi_{i,j,0}$ and minus one $\psi_{i,j,-1}$ are already done and $\psi_{-2,j,-2} = 0$), starting with $j = -4$ and decreasing j, we obtain $\psi_{-3,j,-2} = 0 \ j \leq -4$. Continuing along the same lines with decreasing i and keeping $j < i$, we obtain $\psi_{i,j,-2} = 0 \ \forall \ i, j \leq 0$ as a first step.
In a second step, we shall prove $\boxed{\psi_{i,j,k} = 0 \ \forall \ i, j, k \leq 0}$. This can be done by induction. We know the result is true for $k = 0, -1, -2$. Hence, we will assume it is true for some $k \leq -2$ and check whether it remains true for $k-1$. The cocycle condition for $(i, j, k, -1)$ is given by, after omitting terms of level minus one $\psi_{i,j,-1}$:

$$-(1+i)\psi_{-1+i,j,k} + (1+i+j+k)\psi_{i,j,k} + (1+j)\psi_{-1+j,i,k} - (1+k)\psi_{-1+k,i,j} = 0$$
$$\Leftrightarrow \ -(1+k)\psi_{-1+k,i,j} = 0 \Leftrightarrow \ \psi_{-1+k,i,j} = 0 \text{ as } k \leq -2\,.$$

The terms $\psi_{-1+i,j,k}$, $\psi_{i,j,k}$ and $\psi_{-1+j,i,k}$ are zero since they are of level k and thus zero by induction hypothesis. It follows $\psi_{i,j,k} = 0 \ \forall \ i, j, k \leq 0$.

Case 2: $i \leq 0, \ j > 0$

In a first step, we shall prove the following two statements for levels minus two and plus two: $\boxed{\psi_{i,j,-2} = 0 \ \forall \ i \leq 0, \ j > 0}$ as well as $\boxed{\psi_{i,j,2} = 0 \ \forall \ i \leq 0, \ j > 0}$, respectively.
The cocycle condition for $(-3, 2, -2, -1)$ reads, after dropping the terms of level one, minus one and zero:

$$2\psi_{-4,2,-2} - 2\psi_{-3,2,-2} = 0\,.$$

Since we have $\psi_{-4,2,-2} = 0$, the equation above implies $\psi_{-3,2,-2} = 0$. Next, let us write down the cocycle condition for $(-3, j, -2, 1)$, which gives after dropping terms of level one and of level minus one:

$$(-6+j)\psi_{-3,j,-2} + (-1+j)\psi_{1+j,-3,-2} = 0$$
$$\Leftrightarrow \ \psi_{1+j,-3,-2} = \frac{(-6+j)}{(-1+j)}\psi_{j,-3,-2}\,.$$

Starting with $j = 2$, we obtain $\psi_{j,-3,-2} = 0 \ \forall \ j \geq 3$ since the starting point is zero: $\psi_{2,-3,-2} = 0$. Adding the level one, we obtain $\psi_{j,-3,-2} = 0 \ \forall \ j > 0$.
Next, let us write down the cocycle condition for $(i, 3, 2, -1)$ after dropping terms of level one and level minus one:

$$-(1+i)\psi_{-1+i,3,2} + (6+i)\psi_{i,3,2} = 0$$
$$\Leftrightarrow \ \psi_{-1+i,3,2} = \frac{(6+i)}{(1+i)}\psi_{i,3,2}\,. \tag{3.54}$$

This gives us for $i = -2$: $\psi_{-3,3,2} = -4\psi_{-2,3,2}$.
For $i = -3$: $\psi_{-4,3,2} = \frac{3}{-2}\psi_{-3,3,2} = 6\psi_{-2,3,2}$.
For $i = -4$: $\psi_{-5,3,2} = \frac{2}{-3}\psi_{-4,3,2} = -4\psi_{-2,3,2}$.

Now, let us write down the cocycle condition for $(-3,3,2,-2)$ and drop the terms of level one, level minus one and level zero, as well as the terms of the form $\psi_{j,-3,-2}$ $j > 0$:

$$\begin{aligned}
&\psi_{-5,3,2} + 4\psi_{-3,3,2} - 6\psi_{3,2,-2} = 0\\
&\Leftrightarrow\ -4\psi_{-2,3,2} - 16\psi_{-2,3,2} - 6\psi_{-2,3,2} = 0\\
&\Leftrightarrow\ \psi_{-2,3,2} = 0\,.
\end{aligned}$$

To obtain the second line, the first two terms were simply replaced by their expressions computed above. Putting $i = -2$ and $\psi_{-2,3,2} = 0$ in the recurrence relation (3.54), we obtain $\psi_{i,3,2} = 0\ \forall\ i \le -3$. Together with the levels minus one and zero, we obtain $\psi_{i,3,2} = 0\ \forall\ i \le 0$.

To prove $\psi_{i,j,-2} = 0\ \forall\ i \le 0,\ j > 0$, we will use induction on i. Indeed, we have proven that $\psi_{i,j,-2} = 0\ \forall\ j > 0$, for $i = 0,-1,-2,-3$ (recall that we have $\psi_{j,-3,-2} = 0$ $j > 0$). Suppose the statement holds true down to $i+1$, $i \le -4$, and let us see what happens for i. The cocycle condition for $(i, j,-2,1)$ gives, after dropping terms of level one and level minus one:

$$\begin{aligned}
&(-3+i+j)\psi_{i,j,-2} - (-1+i)\underbrace{\psi_{1+i,j,-2}}_{=0} + (-1+j)\psi_{1+j,i,-2} = 0\\
&\Leftrightarrow\ \psi_{i,j+1,-2} = \frac{(-3+i+j)}{(-1+j)}\psi_{i,j,-2}\,.
\end{aligned} \tag{3.55}$$

The term in the middle is zero due to the induction hypothesis.

This gives, for $j = 2$: $\psi_{i,3,-2} = (-1+i)\psi_{i,2,-2}$.

For $j = 3$: $\psi_{i,4,-2} = \frac{i}{2}\psi_{i,3,-2} = \frac{(-1+i)(i)}{2}\psi_{i,2,-2}$.

For $j = 4$: $\psi_{i,5,-2} = \frac{(1+i)}{3}\psi_{i,4,-2} = \frac{(-1+i)(i)(1+i)}{6}\psi_{i,2,-2}$.

Next, we will insert these values into the cocycle condition for $(i,3,-2,2)$, after dropping terms of level zero and level one:

$$\begin{aligned}
&(-3+i)\psi_{3,-2,2} + \psi_{5,i,-2} + (2+i)\psi_{-2+i,3,2} + (-3+i)\psi_{i,-2,2}\\
&+ (-1+i)\psi_{i,3,-2} - (7+i)\psi_{i,3,2} - (-2+i)\psi_{2+i,3,-2} - (-3+i)\psi_{3+i,-2,2} = 0\\
&\Leftrightarrow\ -\frac{(-1+i)(i)(1+i)}{6}\psi_{i,2,-2} - (-3+i)\psi_{i,2,-2} + (-1+i)(-1+i)\psi_{i,2,-2} = 0\\
&\Leftrightarrow\ (i-3)(i^2-3i+8)\psi_{i,2,-2} = 0\,.
\end{aligned}$$

The terms $\psi_{3,-2,2}$, $\psi_{-2+i,3,2}$ and $\psi_{i,3,2}$ are zero due to what was proved before, $\psi_{i,3,2} = 0\ \forall\ i \le 0$. The terms $\psi_{2+i,3,-2}$ and $\psi_{3+i,-2,2}$ are zero as a consequence of the induction hypothesis. In the last line, we have $(i-3) \ne 0$, since $i \le -4$, and also $(i^2-3i+8) \ne 0$ since its discriminant is negative. It follows $\psi_{i,2,-2} = 0$. Reinserting this into (3.55) and taking into account that level one is zero, we obtain that the induction holds true for i, and thus: $\psi_{i,j,-2} = 0\ \forall\ i \le 0, j > 0$.

Next, we proceed similarly, but with induction on j, to prove $\psi_{i,j,2} = 0\ \forall\ i \le 0, j > 0$. We already know that the statement holds true for $j = 1,2,3$. Let us suppose it is true up to $j-1$, $j \ge 4$, and show that it remains true for j. Let us write down the cocycle condition for $(i, j,2,-1)$, after dropping terms of level one and level minus one:

$$\begin{aligned}
&-(1+i)\psi_{-1+i,j,2} + (3+i+j)\psi_{i,j,2} + (1+j)\underbrace{\psi_{-1+j,i,2}}_{=0} = 0\\
&\Leftrightarrow\ \psi_{-1+i,j,2} = \frac{(3+i+j)}{(1+i)}\psi_{i,j,2}\,.
\end{aligned} \tag{3.56}$$

The third term is zero due to the induction hypothesis. From the recurrence relation above, we obtain for $i = -2$: $\psi_{-3,j,2} = -(1+j)\psi_{-2,j,2}$.
For $i = -3$: $\psi_{-4,j,2} = \frac{j}{(-2)}\psi_{-3,j,2} = \frac{j(1+j)}{2}\psi_{-2,j,2}$
For $i = -4$: $\psi_{-5,j,2} = \frac{(-1+j)}{(-3)}\psi_{-4,j,2} = -\frac{(-1+j)j(1+j)}{6}\psi_{-2,j,2}$.
Next, we insert these values into the cocycle condition for $(-3, j, 2, -2)$ after dropping terms of level zero and level minus one:

$$\begin{aligned}
&\psi_{-5,j,2} - (3+j)\psi_{-3,2,-2} - (-7+j)\psi_{-3,j,-2} + (1+j)\psi_{-3,j,2} \\
&+ (3+j)\psi_{-3+j,2,-2} + (2+j)\psi_{-2+j,-3,2} - (3+j)\psi_{j,2,-2} - (-2+j)\psi_{2+j,-3,-2} = 0 \\
&\Leftrightarrow\ -\frac{(-1+j)j(1+j)}{6}\psi_{-2,j,2} - (1+j)(1+j)\psi_{-2,j,2} - (3+j)\psi_{-2,j,2} = 0 \\
&\Leftrightarrow\ (j+3)(8+3j+j^2)\psi_{-2,j,2} = 0\,.
\end{aligned}$$

The terms $\psi_{-3,2,-2}$, $\psi_{-3,j,-2}$ and $\psi_{2+j,-3,-2}$ are zero due to what was shown before in this proof, $\psi_{j,-3,-2} = 0\ \forall\ j > 0$. The terms $\psi_{-3+j,2,-2}$ and $\psi_{-2+j,-3,2}$ are zero due to the induction hypothesis. In the last line, we have $(j+3) \neq 0$, since $j \geq 4$, and also $(j^2+3j+8) \neq 0$ since its discriminant is negative. It follows $\psi_{-2,j,2} = 0$. Reinserting this into (3.56) and taking into account that level minus one is zero, we obtain that the induction holds true for j, and thus: $\psi_{i,j,2} = 0\ \forall\ i \leq 0, j > 0$. Now that the terms of level two and of level minus two are zero for the case $i \leq 0, j > 0$, we can use induction on k to first prove $\boxed{\psi_{i,j,k} = 0,\ \forall\ i \leq 0, j > 0, k \geq 0}$ and then $\boxed{\psi_{i,j,k} = 0,\ \forall\ i \leq 0, j > 0, k \leq 0}$.
The result is true for $k = 0, 1, 2$. Let us assume the result is true for k, $k \geq 2$ and show that it remains true for $k+1$. The cocycle condition for $(i, j, k, 1)$ gives, after dropping terms of level one,

$$\begin{aligned}
&(-1+i+j+k)\psi_{i,j,k} - (-1+i)\psi_{1+i,j,k} + (-1+j)\psi_{1+j,i,k} - (-1+k)\psi_{1+k,i,j} = 0 \\
&\Leftrightarrow\ (-1+k)\psi_{1+k,i,j} = 0 \Leftrightarrow\ \psi_{1+k,i,j} = 0 \text{ as } k \geq 2\,.
\end{aligned}$$

The terms $\psi_{i,j,k}$, $\psi_{1+i,j,k}$ and $\psi_{1+j,i,k}$ are zero because of the induction hypothesis (if $1+i = 1$, the term $\psi_{1+i,j,k}$ is still zero because the level plus one is zero for all $j, k \in \mathbb{Z}$). It follows that the result holds true for $k+1$.
All the same, the result is true for $k = 0, -1, -2$. Let us assume it is true for k, $k \leq -2$, and show that it holds true for $k-1$. The cocycle condition for $(i, j, k, -1)$ yields, after dropping terms of level minus one,

$$\begin{aligned}
&-(1+i)\psi_{-1+i,j,k} + (1+i+j+k)\psi_{i,j,k} + (1+j)\psi_{-1+j,i,k} - (1+k)\psi_{-1+k,i,j} = 0 \\
&\Leftrightarrow\ (1+k)\psi_{-1+k,i,j} = 0 \Leftrightarrow\ \psi_{-1+k,i,j} = 0 \text{ as } k \leq -2\,.
\end{aligned}$$

The terms $\psi_{-1+i,j,k}$, $\psi_{i,j,k}$ and $\psi_{-1+j,i,k}$ are zero because of the induction hypothesis (if $-1+j = 0$, the term $\psi_{-1+j,i,k}$ is still zero because the level zero vanishes for all $i, k \in \mathbb{Z}$). It follows that the result holds true for $k-1$. Thus, we have obtained the desired result for the case $i \leq 0, j > 0$.

Case 3: $i > 0, j > 0$

In a first step, we shall prove the following statement: $\boxed{\psi_{i,j,2} = 0\ \forall\ i, j > 0}$. The cocycle condition for $(i, j, 2, -1)$ yields, after dropping the terms of level one and of level minus one:

$$\begin{aligned}
&-(1+i)\psi_{-1+i,j,2} + (3+i+j)\psi_{i,j,2} + (1+j)\psi_{-1+j,i,2} = 0 \\
&\Leftrightarrow\ \psi_{i,j,2} = \frac{(1+i)}{(3+i+j)}\psi_{-1+i,j,2} + \frac{(1+j)}{(3+i+j)}\psi_{i,-1+j,2}\,.
\end{aligned}$$

Fixing $i = 3$ and starting with $j = 4$, j ascending, we obtain $\psi_{3,j,2} = 0\ \forall j \geq 4$. Fixing $i = 4$ and starting with $j = 5$, j ascending, we get $\psi_{4,j,2} = 0\ \forall j \geq 5$. Continuing with ascending i, keeping $j > i$, we finally obtain $\psi_{i,j,2} = 0\ \forall\ i, j > 0$.
Finally, we want to prove $\boxed{\psi_{i,j,k} = 0\ \forall\ i, j > 0, k \geq 0}$. This can be done with induction on k. Indeed, the result is true for level zero, level one and level two, i.e. $k = 0,1,2$. Thus, let us assume the result is true for k, $k \geq 2$ and show that it holds true for $k+1$. The cocycle condition for $(i, j, k, 1)$ gives, after dropping the terms of level one:

$$\begin{aligned} &(-1+i+j+k)\psi_{i,j,k} - (-1+i)\psi_{1+i,j,k} + (-1+j)\psi_{1+j,i,k} - (-1+k)\psi_{1+k,i,j} = 0 \\ &\Leftrightarrow\ (-1+k)\psi_{1+k,i,j} = 0 \Leftrightarrow\ \psi_{1+k,i,j} = 0 \text{ as } k \geq 2\,. \end{aligned}$$

The terms $\psi_{i,j,k}$, $\psi_{1+i,j,k}$ and $\psi_{1+j,i,k}$ are zero because of the induction hypothesis. It follows that the statement holds true for $k+1$.
Taking all three cases together, we find the announced result $\boxed{\psi_{i,j,k} = 0\ \forall\ i, j, k \in \mathbb{Z}}$ □

Proof of the Theorem 3.2.2. Let us collect the statements of the four lemmata. Let ψ be a degree-zero 3-cocycle of $\mathcal{W}$ with values in $\mathcal{W}$. By Lemma 3.2.1 we can perform a cohomological change such that we obtain a cohomologous degree-zero 3-cocycle with coefficients fulfilling (3.50). Hence, the assumptions of Lemma 3.2.2 are satisfied and we obtain (3.52). Together with Lemma 3.2.3, the assumptions of Lemma 3.2.4 are fulfilled and Lemma 3.2.4 shows $\boxed{\psi_{i,j,k} = 0\ \forall\ i, j, k \in \mathbb{Z}}$, which proves the result (3.49). The results (3.49) and (3.48) prove Theorem 3.2.2 □

3.2.2 Analysis of $\mathrm{H}^1(\mathcal{V},\mathcal{V})$ and $\mathrm{H}^3(\mathcal{V},\mathcal{V})$

In this Section, we will compute the first and the third cohomology for the Virasoro algebra, with values in the adjoint module. Of course, one could compute these directly as we did it for the Witt algebra in the previous section. However, this would be lengthy and rather uninteresting. A better way to proceed is to use the results for the Witt algebra we already have. In order to achieve this, we will use a long exact sequence as in Theorem 2.2.2. In fact, the short exact sequence (2.18) giving the central extension $\mathcal{V}$ of $\mathcal{W}$ by $\mathbb{K}$ can also be viewed as a short exact sequence of $\mathcal{V}$-modules. Actually, $\mathcal{V}$ is a $\mathcal{V}$-module as the adjoint module, and $\mathcal{W}$ is a quotient $\mathcal{V}$-module while $\mathbb{K}$ is the trivial $\mathcal{V}$-module, see the Remark 2.1.1. Therefore, this short exact sequence of $\mathcal{V}$-modules gives rise to a long exact sequence in cohomology, accordingly to Theorem 2.2.2:

$$\begin{array}{l} 0 \longrightarrow \mathrm{H}^0(\mathcal{V},\mathbb{K}) \longrightarrow \mathrm{H}^0(\mathcal{V},\mathcal{V}) \longrightarrow \mathrm{H}^0(\mathcal{V},\mathcal{W}) \\ \quad\longrightarrow \mathrm{H}^1(\mathcal{V},\mathbb{K}) \longrightarrow \mathrm{H}^1(\mathcal{V},\mathcal{V}) \longrightarrow \mathrm{H}^1(\mathcal{V},\mathcal{W}) \\ \quad\longrightarrow \mathrm{H}^2(\mathcal{V},\mathbb{K}) \longrightarrow \mathrm{H}^2(\mathcal{V},\mathcal{V}) \longrightarrow \mathrm{H}^2(\mathcal{V},\mathcal{W}) \\ \quad\longrightarrow \mathrm{H}^3(\mathcal{V},\mathbb{K}) \longrightarrow \mathrm{H}^3(\mathcal{V},\mathcal{V}) \longrightarrow \mathrm{H}^3(\mathcal{V},\mathcal{W}) \\ \quad\longrightarrow \mathrm{H}^4(\mathcal{V},\mathbb{K}) \longrightarrow \cdots \end{array} \tag{3.57}$$

The relevant part for us is the part involving the first and the third cohomology,

$$\ldots \longrightarrow \mathrm{H}^0(\mathcal{V},\mathcal{W}) \longrightarrow \mathrm{H}^1(\mathcal{V},\mathbb{K}) \longrightarrow \mathrm{H}^1(\mathcal{V},\mathcal{V}) \longrightarrow \mathrm{H}^1(\mathcal{V},\mathcal{W}) \longrightarrow \ldots, \tag{3.58}$$

and

$$\ldots \longrightarrow \mathrm{H}^2(\mathcal{V},\mathcal{W}) \longrightarrow \mathrm{H}^3(\mathcal{V},\mathbb{K}) \longrightarrow \mathrm{H}^3(\mathcal{V},\mathcal{V}) \longrightarrow \mathrm{H}^3(\mathcal{V},\mathcal{W}) \longrightarrow \ldots, \tag{3.59}$$

We will start with the first cohomology.
We already showed $\mathrm{H}^1(\mathcal{W},\mathcal{W}) = \{0\}$, and we also have $\mathrm{H}^1(\mathcal{V},\mathbb{K}) = \{0\}$, see Section 2.2.6. These results will be used in the proof of $\mathrm{H}^1(\mathcal{V},\mathcal{V}) = \{0\}$ based on long exact sequences. This time we will focus only on the degree zero cohomology accordingly to the result of Theorem 2.2.1.

Theorem 3.2.3. *The first algebraic cohomology of the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)$=0$ and values in the adjoint module vanishes, i.e.*

$$\mathrm{H}^1(\mathcal{V},\mathcal{V}) = \{0\}.$$

Proof. Since we already have $\mathrm{H}^1(\mathcal{V},\mathbb{K}) = \{0\}$ and also $\mathrm{H}^1(\mathcal{W},\mathcal{W}) = \{0\}$, it suffices to prove $\mathrm{H}^1(\mathcal{V},\mathcal{W}) \cong \mathrm{H}^1(\mathcal{W},\mathcal{W})$ in order to conclude via (3.58). However, this is an immediate consequence from Proposition 3.1.3. In fact, taking $k=1$ in Formula (3.45), $M=\mathcal{W}$, we obtain $\mathrm{H}^0(\mathcal{W},\mathcal{W}) = {}^{\mathcal{W}}\mathcal{W} = 0$ since $C(\mathcal{W}) = 0$. Thus, $\ker\varphi_0 = 0$ and since $\varphi_{-1} = 0$, we obtain $\mathrm{H}^1(\mathcal{V},\mathcal{W}) \cong \mathrm{H}^1(\mathcal{W},\mathcal{W})$. □

In the following, we would like to present an alternative proof of $\mathrm{H}^1(\mathcal{V},\mathcal{W}) \cong \mathrm{H}^1(\mathcal{W},\mathcal{W})$, using direct computations, as was done to prove $\mathrm{H}^2(\mathcal{V},\mathcal{W}) \cong \mathrm{H}^2(\mathcal{W},\mathcal{W})$ in [106]. Actually, we will have similar considerations in the proof of Theorem 3.3.2 in Section 3.3, but for general cohomological dimension. That proof will be easier to understand after reading through a similar proof, but for cohomological dimension one, as the one given below.

Alternative proof. The proof consists of two steps. First, we will compare the cocycles of $Z^1(\mathcal{V},\mathcal{W})$ to the cocycles of $Z^1(\mathcal{W},\mathcal{W})$. In the second step, we will compare the coboundaries of $B^1(\mathcal{V},\mathcal{W})$ to the ones of $B^1(\mathcal{W},\mathcal{W})$.
Let $\hat{\psi} : \mathcal{V} \to \mathcal{W}$ be a cocycle of $Z^1(\mathcal{V},\mathcal{W})$. Our aim is to show that the restriction of this cocycle to $\mathcal{W}$, i.e. $\psi := \hat{\psi}|_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}$, is a cocycle of $Z^1(\mathcal{W},\mathcal{W})$. Let $x_1, x_2 \in \mathcal{V}$. We will use the same symbols to denote the projections $\pi(x_1), \pi(x_2) \in \mathcal{W}$. Writing the Virasoro bracket $[\cdot,\cdot]_{\mathcal{V}}$ in terms of the Witt bracket $[\cdot,\cdot]_{\mathcal{W}}$ and the 2-cocycle $\alpha(\cdot,\cdot)$ giving the central extension, i.e. $[\cdot,\cdot]_{\mathcal{V}} = [\cdot,\cdot]_{\mathcal{W}} + \alpha(\cdot,\cdot)\cdot t$, the cocycle condition for $\hat{\psi}$ becomes:

$$\begin{aligned}
&0 = (\delta_1^{\mathcal{V}}\hat{\psi})(x_1,x_2) = \hat{\psi}([x_1,x_2]_{\mathcal{V}}) - x_1\cdot\hat{\psi}(x_2) + x_2\cdot\hat{\psi}(x_1) \\
&\Leftrightarrow 0 = (\delta_1^{\mathcal{V}}\hat{\psi})(x_1,x_2) = \hat{\psi}([x_1,x_2]_{\mathcal{W}}) + \alpha(x_1,x_2)\,\hat{\psi}(t) - [x_1,\hat{\psi}(x_2)]_{\mathcal{W}} + [x_2,\hat{\psi}(x_1)]_{\mathcal{W}} \\
&\Leftrightarrow 0 = (\delta_1^{\mathcal{V}}\hat{\psi})(x_1,x_2) = (\delta_1^{\mathcal{W}}\hat{\psi})(x_1,x_2) + \alpha(x_1,x_2)\,\hat{\psi}(t).
\end{aligned} \tag{3.60}$$

Since we are considering degree-zero cocycles, the cocycle $\hat{\psi}$ evaluated on the central element reads as follows:

$$\hat{\psi}(t) = c\, e_0,$$

for suitable $c \in \mathbb{K}$. Next, let us insert this expression into the cocycle condition for (e_1, t), which yields:

$$\begin{aligned}
&(\delta_1^{\mathcal{V}}\hat{\psi})(e_1,t) = \hat{\psi}(\cancel{[e_1,t]_{\mathcal{V}}}) - e_1\cdot\hat{\psi}(t) + \cancel{t\cdot\hat{\psi}(e_1)} = 0 \\
&\Leftrightarrow -[e_1,\hat{\psi}(t)]_{\mathcal{W}} = -c\,[e_1,e_0]^{\mathcal{W}} = c\, e_1 = 0 \\
&\Leftrightarrow c = 0.
\end{aligned}$$

Inserting $\hat{\psi}(t) = 0$ into (3.60), we obtain

$$0 = (\delta_1^{\mathcal{V}}\hat{\psi})(x_1, x_2) = (\delta_1^{\mathcal{W}}\psi)(x_1, x_2). \tag{3.61}$$

This means that a cocycle $\hat{\psi} \in Z^1(\mathcal{V},\mathcal{W})$ corresponds to a cocycle $\psi \in Z^1(\mathcal{W},\mathcal{W})$ when projected to $\mathcal{W}$. Moreover, a cocycle ψ of $Z^1(\mathcal{W},\mathcal{W})$ can also be lifted to a cocycle $\hat{\psi} := \psi \circ \nu$ in $Z^1(\mathcal{V},\mathcal{W})$. By definition, we thus have $\hat{\psi}(t) = 0$ and the relation (3.61) holds true. Hence, a cocycle $\psi \in Z^1(\mathcal{W},\mathcal{W})$ yields a cocycle $\hat{\psi} \in Z^1(\mathcal{V},\mathcal{W})$ and we have $Z^1(\mathcal{V},\mathcal{W}) \cong Z^1(\mathcal{W},\mathcal{W})$ in a canonical way.
The second step of the proof consists in comparing the coboundaries of $B^1(\mathcal{V},\mathcal{W})$ and those of $B^1(\mathcal{W},\mathcal{W})$. However, this is trivial. In fact, the coboundary condition applied on a 0-cochain $\phi \in C^0(\mathcal{V},\mathcal{W})$ is the same as the one applied on a 0-cochain $\phi \in C^0(\mathcal{W},\mathcal{W})$, yielding in both cases:

$$(\delta_0\phi)(x) = -x\cdot\phi \text{ with } \phi \in \mathcal{W}.$$

Since the central element of $\mathcal{V}$ acts trivially on $\mathcal{W}$, we have $B^1(\mathcal{V},\mathcal{W}) \cong B^1(\mathcal{W},\mathcal{W})$. All in all, we conclude $\mathrm{H}^1(\mathcal{V},\mathcal{W}) \cong \mathrm{H}^1(\mathcal{W},\mathcal{W})$ in a canonical way. □

Next, we will consider the third cohomology.

Theorem 3.2.4. *The third algebraic cohomology of the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in the adjoint module is one-dimensional, i.e.*

$$\dim \mathrm{H}^3(\mathcal{V},\mathcal{V}) = 1.$$

Proof. In [106], it was proven that $\mathrm{H}^2(\mathcal{V},\mathcal{W}) \cong \mathrm{H}^2(\mathcal{W},\mathcal{W})$ and also $\mathrm{H}^2(\mathcal{W},\mathcal{W}) = 0$. Consider Formula (3.45) for $k = 3$ and $M = \mathcal{W}$. Since we already computed $\mathrm{H}^1(\mathcal{W},\mathcal{W}) = 0$ we obtain $\operatorname{im}\varphi_1 = 0$, and due to $\mathrm{H}^2(\mathcal{W},\mathcal{W}) = 0$, $\ker\varphi_2 = 0$. Therefore, we obtain $\mathrm{H}^3(\mathcal{V},\mathcal{W}) \cong \mathrm{H}^3(\mathcal{W},\mathcal{W})$, which is zero as we proved it in the previous section. Consequently, our long exact sequence (3.59) reduces to the short exact sequence,

$$0 \longrightarrow \mathrm{H}^3(\mathcal{W},\mathbb{K}) \longrightarrow \mathrm{H}^3(\mathcal{V},\mathcal{V}) \longrightarrow 0.$$

Since we already proved $\dim \mathrm{H}^3(\mathcal{W},\mathbb{K}) = 1$, we obtain by exactness and the Rank-Nullity Theorem that $\dim \mathrm{H}^3(\mathcal{V},\mathcal{V}) = 1$. □

3.3 The differential $d_2^{p,1}$

In this section, we aim to render the formula (3.45) more explicit. To do this, we need a better understanding of the differentials $\varphi_p = d_2^{p,1}$. In fact, we will first show that the differentials φ_p correspond to the cup product of a p-cocycle with the extension class, i.e. with the class of the Virasoro 2-cocycle (2.20). In a second step, we will then prove by explicit considerations that φ_p is injective for low p, for the modules $M = \mathbb{K}$ and $M = \mathcal{F}^\lambda$. Thus, we obtain a direct link between the low-dimensional algebraic cohomology of the Witt and the Virasoro algebra for these modules.
Let us recall the definition of the cup product as given in [60, 61].

Definition 3.3.1. Let $\mathcal{L}$ be a Lie algebra and M, N be $\mathcal{L}$-modules. Let $\psi \in C^p(\mathcal{L}, M)$ and $\phi \in C^q(\mathcal{L}, N)$ be p-cochains and q-cochains with values in M and N, respectively. The *cup product* $\smile: C^p(\mathcal{L}, M) \times C^q(\mathcal{L}, N) \to C^{p+q}(\mathcal{L}, P)$ with $P = M \otimes N$ is given by, for $x_1, \dots x_{p+q} \in \mathcal{L}$:

$$(\psi \smile \phi)(x_1, \dots x_{p+q}) := \sum_S (-1)^{\nu(S)} \psi(x_{s_1}, \dots x_{s_p}) \otimes \phi(x_{t_1}, \dots, x_{t_q}), \tag{3.62}$$

where the sum goes over all the ordered subsets $S = (s_1, \dots s_p)$ of the set $(1, 2, \dots, p+q)$, and $T = (t_1, \dots, t_q)$ is the ordered complement of S. The quantity $(-1)^{\nu(S)}$ denotes the sign of the permutation $(s_1, \dots s_p, t_1, \dots, t_q)$, and can be defined as follows: For each $1 \le j \le q$, denoting by $S(j)$ the number of indices i for which s_i is bigger than t_j, we define $\nu(S) := \sum_{j=1}^{q} S(j)$. The pairing $\otimes : M \times N \to P$, $(m, n) \mapsto m \otimes n$ satisfies $x \cdot (m \otimes n) = (x \cdot m) \otimes n + m \otimes (x \cdot n)$.
One can show [60, 61] that $\delta(\psi \smile \phi) = (\delta\psi) \smile \phi + (-1)^p \psi \smile (\delta\phi)$, and that the cup product (3.62) can be lifted to the level of cohomology, yielding a pairing $\smile$: $\mathrm{H}^p(\mathscr{L}, M) \times \mathrm{H}^q(\mathscr{L}, N) \to \mathrm{H}^{p+q}(\mathscr{L}, P)$ with $P = M \otimes N$.

3.3.1 Differential and cup product

In this section, we want to prove that the differential $d_2^{p,1}$ is given by $[\alpha] \smile$, where α is the Virasoro 2-cocycle. This result can be found in [61], Theorem 8. The Theorem 3.3.2 below can then be deduced immediately from (3.45) and Theorem 8 in [61]. This was communicated to the author by Wagemann [130]. Still, we provide a direct proof of Theorem 3.3.2 below. Subsequently, the Theorem 3.3.2, Formula (3.45) and a Theorem by André [7] introduced below immediately imply $d_2^{p,1} = [\alpha] \smile$.

Consider the following abelian Lie algebra extension,

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{l} \to 0\,, \tag{3.63}$$

with a non-trivial extension class $[\alpha] \in \mathrm{H}^2(\mathfrak{l}, \mathfrak{h})$. Let M be a $\mathfrak{g}$-module. Let $E_2^{p,q} = \mathrm{H}^p(\mathfrak{l}, \mathrm{H}^q(\mathfrak{h}, M))$ be the second page of the associated Hochschild-Serre spectral sequence, with the differential $d_2^{(\alpha)} : E_2^{p,q} \to E_2^{p+2,q-1}$. The differential $d_2^{(\alpha)}$ can be written as,

$$d_2^{(\alpha)} = d_2^{(0)} + \Delta(\alpha)\,, \tag{3.64}$$

for all p, q, where $[\alpha] \in \mathrm{H}^2(\mathfrak{l}, \mathfrak{h})$ is the extension class and $d_2^{(0)}$ is the differential in the case of the trivial extension, $\Delta(0) = 0$. In fact, the Hochschild-Serre spectral sequence associated to (3.63) has the same structure in the case of the trivial extension and the extension $[\alpha]$. In particular, the differentials $d_2^{(\alpha)}$ and $d_2^{(0)}$ start and end for all p, q, at the same spaces $E_2^{p,q}$ and $E_2^{p+2,q-1}$, respectively. The difference $d_2^{(\alpha)} - d_2^{(0)}$ thus gives a map depending on $[\alpha]$, denoted by $\Delta(\alpha)$.

Theorem [André] 3.3.1. *Consider an abelian Lie algebra extension as in* (3.63)*, with differentials of the associated Hochschild-Serre spectral sequence as in* (3.64)*. Then the map Δ (for all p, q) in* (3.64)

$$\mathrm{H}^2(\mathfrak{l}, \mathfrak{h}) \xrightarrow{\Delta} \mathrm{Hom}\left(\mathrm{H}^p(\mathfrak{l}, \mathrm{H}^q(\mathfrak{h}, M)), \mathrm{H}^{p+2}(\mathfrak{l}, \mathrm{H}^{q-1}(\mathfrak{h}, M))\right),$$

gives a homomorphism $\Delta(\alpha)$ corresponding to $[\alpha] \smile$, where $\smile$ is the cup product (3.62).

We see that we can obtain $d_2^{p,1} = [\alpha] \smile$ by proving that $\varphi_p^{(0)} := d_2^{(0),p,1}$ is zero for all p.
Let us denote by $\mathcal{V}'$ the trivial central extension of $\mathcal{W}$ by $\mathbb{K}$,

$$0 \longrightarrow \mathbb{K} \xrightarrow{i} \mathcal{V}' \xrightarrow{\pi} \mathcal{W} \longrightarrow 0\,.$$

We shall prove that $\mathrm{H}^k(\mathcal{V}', M) \cong \mathrm{H}^k(\mathcal{W}, M) \oplus \mathrm{H}^{k-1}(\mathcal{W}, M)$. By comparing this result to Formula (3.45) and replacing $\mathcal{V}$ by $\mathcal{V}'$, φ_p by $\varphi_p^{(0)}$, we conclude immediately that $\varphi_p^{(0)} = 0$ for all p.
Since $\mathcal{V}'$ is the trivial central extension, we have $\mathcal{V}' = \mathbb{K} \oplus \mathcal{W}$ on the level of Lie algebras. In the following, we analyze how $\mathcal{V}'$ can be split into direct sums on the level of cohomology with coefficients in a $\mathcal{W}$-module M.

Theorem 3.3.2. *Let $\mathcal{V}'$ be the trivial central extension of the Witt algebra by the trivial base field $\mathbb{K}$. Then the following holds,*

$$\mathrm{H}^k(\mathcal{V}', M) = \mathrm{H}^k(\mathcal{W}, M) \oplus \mathrm{H}^{k-1}(\mathcal{W}, M),$$

where M is a $\mathcal{W}$-module and hence also a $\mathcal{V}'$-module.

Proof. Let us consider general k-cocycles of $\mathrm{H}^k(\mathcal{V}', M)$. We denote by v'_q, homogeneous elements of $\mathcal{V}'$, by w_q, homogeneous elements of $\mathcal{W}$, and 1 is the generator of $\mathbb{K}$. We will use the same symbols w_q to denote elements $s_0(w_q) = (0, w_q)$ of $\mathcal{W}$ seen as elements of $\mathcal{V}'$, but we use $t = i(1) = (1,0)$ to denote the generator 1 of $\mathbb{K}$ seen as element of $\mathcal{V}'$.
Because $\mathcal{V}' = \mathbb{K} \oplus \mathcal{W}$ as Lie algebras, k-cochains ψ of $C^k(\mathcal{V}', M)$ can be decomposed as:

$$\begin{aligned}
\psi(v'_1, \dots, v'_k) =& \psi((1, w_1), \dots, (1, w_k)) \\
=& \psi(w_1, \dots, w_k) + \sum_{i=1}^{k} (-1)^{k-i} \psi(w_1, \dots, \hat{w}_i, \dots, w_k, t) \\
&+ \sum_{1 \le i < j \le k} (-1)^{i+j+1} \psi(w_1, \dots, \hat{w}_i, \dots, \hat{w}_j, \dots, w_k, \underbrace{t, t}_{=0}) \\
&+ \dots + \psi(\underbrace{t, \dots, t}_{=0}),
\end{aligned}$$

where the hat denotes omitted entries. Due to the alternating property of the cochains, and since we consider a one-dimensional central extension with only one generator t, only two terms remain in the decomposition,

$$\psi(v'_1, \dots, v'_k) = \psi(\underbrace{w_1, \dots, w_k}_{=:A}) + \sum_{i=1}^{k} (-1)^{k-i} \psi(\underbrace{w_1, \dots, \hat{w}_i, \dots, w_k, t}_{=:B}). \tag{3.65}$$

Defining an element $\psi(w_1, \dots, w_{k-1}, t)$ of $C^k(\mathcal{V}', M)$ as an element of $C^{k-1}(\mathcal{W}, M)$ by the following, $\psi(w_1, \dots, w_{k-1}, t) =: \psi(w_1, \dots, w_{k-1})$, with appropriate signs when t is in other positions, we obtain $C^k(\mathcal{V}', M) \cong C^k(\mathcal{W}, M) \oplus C^{k-1}(\mathcal{W}, M)$ on the level of cochains. Next, we need to check whether this isomorphism holds on the cohomological level.
We will first concentrate on the decomposition on the level of cocycles, then on the level of coboundaries. The first step consists in showing that if ψ is a cocycle in $Z^k(\mathcal{V}', M)$, then the restriction $\psi|_A$ will be a cocycle in $Z^k(\mathcal{W}, M)$, whereas the restriction $\psi|_B$ will be a cocycle in $Z^{k-1}(\mathcal{W}, M)$. Conversely, we need to show that if ψ_1 and ψ_2 are cocycles in $Z^k(\mathcal{W}, M)$ and $Z^{k-1}(\mathcal{W}, M)$, respectively, then they can be lifted to cocycles of $Z^k(\mathcal{V}', M)$. In a second step, the same reasoning has to be done for coboundaries.

$\boxed{Z^k(\mathcal{V}', M) \cong Z^k(\mathcal{W}, M) \oplus Z^{k-1}(\mathcal{W}, M)}$

$\boxed{Z^k(\mathcal{V}', M) \to Z^k(\mathcal{W}, M)}$ We want to show that every cocycle ψ of $Z^k(\mathcal{V}', M)$ yields a cocycle $\psi|_A$ in $Z^k(\mathcal{W}, M)$ when restricted to A. The cocycle condition in $Z^k|_A(\mathcal{V}', M)$ reads,

$$\begin{aligned}
(\delta_k \psi)(w_1, \dots w_{k+1}) =& \sum_{1 \le i < j \le k+1} (-1)^{i+j+1} \, \psi([w_i, w_j]_{\mathcal{V}'}, w_1, \dots, \hat{w}_i, \dots, \hat{w}_j, \dots, w_{k+1}) \\
&+ \sum_{i=1}^{k+1} (-1)^i \, w_i \cdot \psi(w_1, \dots, \hat{w}_i, \dots, w_{k+1}) = 0.
\end{aligned}$$

However, this corresponds exactly to the cocycle condition in $Z^k(\mathcal{W},M)$. In fact, since $\mathcal{V}' = \mathbb{K}\oplus\mathcal{W}$ as Lie algebras, we have $\left[w_i,w_j\right]_{\mathcal{V}'} = \left[w_i,w_j\right]_{\mathcal{W}}$, meaning we do not have the additional term $\alpha(w_i,w_j)\ t$ that we obtain for non-trivial extensions $\alpha\neq 0$. The action terms are also the same, see the Remark 2.1.1.

$\boxed{Z^k(\mathcal{V}',M)\leftarrow Z^k(\mathcal{W},M)}$ Let ψ_1 be a cocycle in $Z^k(\mathcal{W},M)$. We want to define a lift $\hat{\psi}_1$ such that $\hat{\psi}_1$ is a cocycle in $Z^k(\mathcal{V}',M)$. We define the lift in the following canonical way, $\hat{\psi}_1 := \psi_1\circ\pi$, hence we obtain by definition $\hat{\psi}_1(w_1,\ldots,w_{k-1},t)=0$. The cocycle condition in $Z^k(\mathcal{V}',M)$ reads,

$$\begin{aligned}(\delta_k\hat{\psi}_1)(v'_1,\ldots v'_{k+1}) &= \sum_{1\le i<j\le k+1}(-1)^{i+j+1}\ \hat{\psi}_1(\left[v'_i,v'_j\right]_{\mathcal{V}'},v'_1,\ldots,\hat{v}'_i,\ldots,\hat{v}'_j,\ldots,v'_{k+1})\\ &+\sum_{i=1}^{k+1}(-1)^i\ v'_i\cdot\hat{\psi}_1(v'_1,\ldots,\hat{v}'_i,\ldots,v'_{k+1})\\ &\overset{(3.65)}{=}\sum_{1\le i<j\le k+1}(-1)^{i+j+1}\ \hat{\psi}_1(\left[w_i,w_j\right]_{\mathcal{V}'},w_1,\ldots,\hat{w}_i,\ldots,\hat{w}_j,\ldots,w_{k+1})\\ &+\sum_{i=1}^{k+1}(-1)^i\ w_i\cdot\hat{\psi}_1(w_1,\ldots,\hat{w}_i,\ldots,w_{k+1})\\ &+\sum\hat{\psi}_1(\underbrace{[\cdot,t]_{\mathcal{V}'}}_{=0},\ldots)+\sum\underbrace{t\cdot\hat{\psi}_1(\ldots)}_{=0}+\sum\underbrace{\hat{\psi}_1(\ldots,t)}_{=0}\\ &=(\delta_k\psi_1)(w_1,\ldots w_{k+1})\,.\end{aligned}$$

To obtain the last line, we use the definition $\hat{\psi}_1 := \psi_1\circ\pi$ as well as $\left[w_i,w_j\right]_{\mathcal{V}'} = \left[w_i,w_j\right]_{\mathcal{W}}$. Moreover, the three sums represented symbolically are zero because $\mathcal{V}'$ is a central extension, because M is a trivial $\mathbb{K}$-module (see Remark 2.1.1), and because $\hat{\psi}_1(\ldots,t)=0$ by definition of the lift, respectively.

As we have $(\delta_k\psi_1)(w_1,\ldots w_{k+1})=0$, we obtain $(\delta_k\hat{\psi}_1)(v'_1,\ldots v'_{k+1})=0$, hence every cocycle in $Z^k(\mathcal{W},M)$ gives rise to a cocycle in $Z^k(\mathcal{V}',M)$.

$\boxed{Z^k(\mathcal{V}',M)\rightarrow Z^{k-1}(\mathcal{W},M)}$ We want to prove that if ψ is a cocycle of $Z^k(\mathcal{V}',M)$, then the restriction $\psi|_B$ yields a cocycle of $Z^{k-1}(\mathcal{W},M)$. Defining the element $\psi(w_1,\ldots,w_{k-1},t)$ of $C^k|_B(\mathcal{V}',M)$ as an element $\tilde{\psi}$ of $C^{k-1}(\mathcal{W},M)$ by $\tilde{\psi}(w_1,\ldots,w_{k-1}) := \psi(w_1,\ldots,w_{k-1},t)$, the cocycle condition in $Z^k|_B(\mathcal{V}',M)$ becomes,

$$\begin{aligned}(\delta_k\psi)(w_1,\ldots w_k,t) &= \sum_{1\le i<k+1}(-1)^{i+k+2}\psi(\underbrace{[w_i,t]_{\mathcal{V}'}}_{=0},w_1,\ldots\hat{w}_i,\ldots,w_k)\\ &+(-1)^{k+1}\ \underbrace{t\cdot\psi(w_1,\ldots,\hat{w}_i,\ldots,w_k)}_{=0}\\ &+\sum_{1\le i<j\le k}(-1)^{i+j+1}\ \psi(\left[w_i,w_j\right]_{\mathcal{V}'},w_1,\ldots,\hat{w}_i,\ldots,\hat{w}_j,\ldots,w_k,t)\\ &+\sum_{i=1}^{k}(-1)^i\ w_i\cdot\psi(w_1,\ldots,\hat{w}_i,\ldots,w_k,t)\\ =(\delta_{k-1}\tilde{\psi})(w_1,\ldots w_k)\,.\end{aligned}$$

The first two terms correspond to the entries $j=k+1$ and $i=k+1$ of the sums. They are zero because $\mathcal{V}'$ is a central extension and M is a trivial $\mathbb{K}$-module, respectively. The last line is obtained by definition of $\tilde{\psi}$ and due to the fact that the extension is trivial, $\left[w_i,w_j\right]_{\mathcal{V}'} = \left[w_i,w_j\right]_{\mathcal{W}}$. Since $(\delta_k\psi)(w_1,\ldots w_k,t)=0$ as $\psi\in Z^k(\mathcal{V}',M)$, we obtain $(\delta_{k-1}\tilde{\psi})(w_1,\ldots w_k)=0$, hence ψ yields indeed a cocycle $\tilde{\psi}$ in $Z^{k-1}(\mathcal{W},M)$.

$\boxed{Z^k(\mathcal{V}',M) \leftarrow Z^{k-1}(\mathcal{W},M)}$ Let ψ_2 be a cocycle in $Z^{k-1}(\mathcal{W},M)$. We are going to define a lift $\hat{\psi}_2$ such that $\hat{\psi}_2$ is a cocycle in $Z^k(\mathcal{V}',M)$. We define the lift $\hat{\psi}_2$ as follows, $\hat{\psi}_2(w_1,\dots,w_k) := 0$ and $\hat{\psi}_2(w_1,\dots,w_{k-1},t) := \psi_2(w_1,\dots,w_{k-1})$ with appropriate signs when t is in other positions. The cocycle condition in $Z^k(\mathcal{V}',M)$ reads,

$$
\begin{aligned}
&(\delta_k\hat{\psi}_2)(v_1',\dots v_{k+1}') \overset{(3.65)}{=} (\delta_k\hat{\psi}_2)(w_1,\dots w_{k+1}) + \sum_{l=1}^{k+1}(\delta_k\hat{\psi}_2)(w_1,\dots,\hat{w}_l,t,w_{l+1},\dots,w_{k+1})\\
&= \underbrace{(\delta_k\hat{\psi}_2)(w_1,\dots w_{k+1})}_{=0} + \sum_{l=1}^{k+1}(-1)^{k-l+1}(\delta_k\hat{\psi}_2)(w_1,\dots,\hat{w}_l,\dots,w_{k+1},t)\\
&= \sum \hat{\psi}_2(\underbrace{[\cdot,t]_{\mathcal{V}'}}_{=0},\dots) + \sum \underbrace{t\cdot\hat{\psi}_2(\dots)}_{=0}\\
&+ \sum_{l=1}^{k+1}(-1)^{k-l+1}\sum_{1\le i<j\le k+1,i,j\neq l}(-1)^{i+j+1}\;\hat{\psi}_2([w_i,w_j]_{\mathcal{V}'},w_1,\dots,\hat{w}_l,\dots,\hat{w}_i,\dots,\hat{w}_j,\dots,w_{k+1},t)\\
&+ \sum_{l=1}^{k+1}(-1)^{k-l+1}\sum_{i=1,i\neq l}^{k+1}(-1)^i\; w_i\cdot\hat{\psi}_2(w_1,\dots,\hat{w}_l,\dots,\hat{w}_i,\dots,w_{k+1},t)\\
&= \sum_{l=1}^{k+1}(-1)^{k-l+1}\sum_{1\le i<j\le k+1,i,j\neq l}(-1)^{i+j+1}\;\psi_2([w_i,w_j]_{\mathcal{V}'},w_1,\dots,\hat{w}_l,\dots,\hat{w}_i,\dots,\hat{w}_j,\dots,w_{k+1})\\
&+ \sum_{l=1}^{k+1}(-1)^{k-l+1}\sum_{i=1,i\neq l}^{k+1}(-1)^i\; w_i\cdot\psi_2(w_1,\dots,\hat{w}_l,\dots,\hat{w}_i,\dots,w_{k+1})\\
&= \sum_{l=1}^{k+1}(-1)^{k-l+1}(\delta_{k-1}\psi_2)(w_1,\dots,\hat{w}_l,\dots w_{k+1})\,.
\end{aligned}
$$

The first term in the second line is zero by definition of our lift, the first and second sums in the third line are zero due to the fact that we consider a central extension and that M is a trivial $\mathbb{K}$-module, respectively. Since $(\delta_{k-1}\psi_2) = 0$, we obtain $(\delta_k\hat{\psi}_2) = 0$. Hence the conclusion. Altogether, we obtain $Z^k(\mathcal{V}',M) \cong Z^k(\mathcal{W},M) \oplus Z^{k-1}(\mathcal{W},M)$.

$\boxed{B^k(\mathcal{V}',M) \cong B^k(\mathcal{W},M) \oplus B^{k-1}(\mathcal{W},M)}$

Next, we need to perform the same analysis on the level of the coboundaries. As it is similar to the previous one, we will keep it short.

$\boxed{B^k(\mathcal{V}',M) \to B^k(\mathcal{W},M)}$ We want to show that every coboundary ψ of $B^k(\mathcal{V}',M)$ yields a coboundary $\psi|_A$ in $B^k(\mathcal{W},M)$ when restricted to A. The coboundary condition in $B^k|_A(\mathcal{V}',M)$ states that $\psi \in B^k|_A(\mathcal{V}',M)$ if there exists a $k-1$-cochain $\phi \in C^{k-1}|_A(\mathcal{V}',M)$ such that:

$$
\begin{aligned}
\psi(w_1,\dots w_k) = (\delta_{k-1}\phi)(w_1,\dots w_k) = &\sum_{1\le i<j\le k}(-1)^{i+j+1}\;\phi([w_i,w_j]_{\mathcal{V}'},w_1,\dots,\hat{w}_i,\dots,\hat{w}_j,\dots,w_k)\\
&+\sum_{i=1}^{k}(-1)^i\; w_i\cdot\phi(w_1,\dots,\hat{w}_i,\dots,w_k)\,.
\end{aligned}
$$

Since we have $[w_i,w_j]_{\mathcal{V}'} = [w_i,w_j]_{\mathcal{W}}$, this corresponds exactly to the coboundary condition in $B^k(\mathcal{W},M)$.

$\boxed{B^k(\mathcal{V}',M) \leftarrow B^k(\mathcal{W},M)}$ Let ψ_1 be a coboundary in $B^k(\mathcal{W},M)$, i.e. $\psi_1 = \delta_{k-1}\phi_1$ for some $\phi_1 \in$

$C^{k-1}(\mathcal{W}, M)$. We define a lift $\hat{\phi}_1$ of ϕ_1 from $\mathcal{W}$ to $\mathcal{V}'$ in the following canonical way, $\hat{\phi} := \phi \circ \pi$, hence we obtain by definition $\hat{\phi}_1(w_1, \ldots, w_{k-2}, t) = 0$. Thus, $\hat{\phi}_1$ is a $k-1$-cochain from $\mathcal{V}'$ to M. Applying the coboundary operator to $\hat{\phi}_1$, we thus obtain a coboundary $\delta_{k-1}\hat{\phi}_1 \in B^k(\mathcal{V}', M)$,

$$
\begin{aligned}
(\delta_{k-1}\hat{\phi}_1)(v'_1, \ldots v'_k) &= \sum_{1 \le i < j \le k} (-1)^{i+j+1}\, \hat{\phi}_1\left(\left[v'_i, v'_j\right]_{\mathcal{V}'}, v'_1, \ldots, \hat{v}'_i, \ldots, \hat{v}'_j, \ldots, v'_k\right) \\
&+ \sum_{i=1}^{k} (-1)^i\, v'_i \cdot \hat{\phi}_1(v'_1, \ldots, \hat{v}'_i, \ldots, v'_k) \\
&\overset{(3.65)}{=} \sum_{1 \le i < j \le k} (-1)^{i+j+1}\, \hat{\phi}_1([w_i, w_j]_{\mathcal{V}'}, w_1, \ldots, \hat{w}_i, \ldots, \hat{w}_j, \ldots, w_k) \\
&+ \sum_{i=1}^{k} (-1)^i\, w_i \cdot \hat{\phi}_1(w_1, \ldots, \hat{w}_i, \ldots, w_k) \\
&+ \sum \hat{\phi}_1(\underbrace{[\cdot, t]_{\mathcal{V}'}}_{=0}, \ldots) + \sum \underbrace{t \cdot \hat{\phi}_1(\ldots)}_{=0} + \sum \underbrace{\hat{\phi}_1(\ldots, t)}_{=0} \\
&= (\delta_{k-1}\phi_1)(w_1, \ldots w_k) = \psi_1(w_1, \ldots w_k)\,.
\end{aligned}
$$

Reading the development above backwards, we see that every coboundary ψ_1 of $B^k(\mathcal{W}, M)$ can be seen as a coboundary of $B^k(\mathcal{V}', M)$.

$\boxed{B^k(\mathcal{V}', M) \to B^{k-1}(\mathcal{W}, M)}$ We want to show that if ψ is a coboundary of $B^k(\mathcal{V}', M)$, i.e. $\psi = \delta_{k-1}\phi$ for some $\phi \in C^{k-1}(\mathcal{V}', M)$, then the restriction $\psi|_B$ yields a coboundary in $B^{k-1}(\mathcal{W}, M)$. As before, we define an element $\phi(w_1, \ldots w_{k-2}, t)$ of $C^{k-1}|_B(\mathcal{V}', M)$ as an element $\tilde{\phi}$ of $C^{k-2}(\mathcal{W}, M)$ by $\tilde{\phi}(w_1, \ldots w_{k-2}) := \phi(w_1, \ldots w_{k-2}, t)$. A coboundary in $B^k|_B(\mathcal{V}', M)$ is given by,

$$
\begin{aligned}
(\delta_{k-1}\phi)(w_1, \ldots w_{k-1}, t) &= \sum_{1 \le i < k} (-1)^{i+k+1} \phi(\underbrace{[w_i, t]_{\mathcal{V}'}}_{=0}, w_1, \ldots \hat{w}_i, \ldots, w_{k-1}) \\
&+ (-1)^k \underbrace{t \cdot \phi(w_1, \ldots, \hat{w}_i, \ldots, w_{k-1})}_{=0} \\
&+ \sum_{1 \le i < j \le k-1} (-1)^{i+j+1}\, \phi([w_i, w_j]_{\mathcal{V}'}, w_1, \ldots, \hat{w}_i, \ldots, \hat{w}_j, \ldots, w_{k-1}, t) \\
&+ \sum_{i=1}^{k-1} (-1)^i\, w_i \cdot \phi(w_1, \ldots, \hat{w}_i, \ldots, w_{k-1}, t) \\
= (\delta_{k-2}\tilde{\phi})(w_1, \ldots w_{k-1})\,.
\end{aligned}
$$

Therefore, the coboundary $\psi = \delta_{k-1}\phi$ in $B^k(\mathcal{V}', M)$ can be viewed as a coboundary $(\delta_{k-2}\tilde{\phi})$ in $B^{k-1}(\mathcal{W}, M)$.

$\boxed{B^k(\mathcal{V}', M) \leftarrow B^{k-1}(\mathcal{W}, M)}$ Let ψ_2 be a coboundary in $B^{k-1}(\mathcal{W}, M)$, i.e. $\psi_2 = \delta_{k-2}\phi_2$ for some $\phi_2 \in C^{k-2}(\mathcal{W}, M)$. We want to lift this coboundary to a coboundary of $B^k(\mathcal{V}', M)$. We define the following lift of the cochain ϕ_2 to a cochain $\hat{\phi}_2$ of $C^{k-1}(\mathcal{V}', M)$, $\hat{\phi}_2(w_1, \ldots w_{k-1}) := 0$ and

$\hat{\phi}_2(w_1, \dots w_{k-2}, t) := \phi_2(w_1, \dots w_{k-2})$. A coboundary in $B^k(\mathcal{V}', M)$ reads,

$$(\delta_{k-1}\hat{\phi}_2)(v'_1, \dots v'_k) \overset{(3.65)}{=} (\delta_{k-1}\hat{\phi}_2)(w_1, \dots w_k) + \sum_{l=1}^{k} (\delta_{k-1}\hat{\phi}_2)(w_1, \dots, \hat{w}_l, t, w_{l+1}, \dots, w_k) \tag{3.66}$$

$$= \underbrace{(\delta_{k-1}\hat{\phi}_2)(w_1, \dots w_k)}_{=0} + \sum_{l=1}^{k} (-1)^{k-l} (\delta_{k-1}\hat{\phi}_2)(w_1, \dots, \hat{w}_l, \dots, w_k, t)$$

$$= \sum \hat{\phi}_2(\underbrace{[\cdot, t]_{\mathcal{V}'}}_{=0}, \dots) + \sum \underbrace{t \cdot \hat{\phi}_2(\dots)}_{=0} \tag{3.67}$$

$$+ \sum_{l=1}^{k} (-1)^{k-l} \sum_{1 \le i < j \le k, i,j \ne l} (-1)^{i+j+1} \hat{\phi}_2([w_i, w_j]_{\mathcal{V}'}, w_1, \dots, \hat{w}_l, \dots, \hat{w}_i, \dots, \hat{w}_j, \dots, w_k, t)$$

$$+ \sum_{l=1}^{k} (-1)^{k-l} \sum_{i=1, i \ne l}^{k} (-1)^i \, w_i \cdot \hat{\phi}_2(w_1, \dots, \hat{w}_l, \dots, \hat{w}_i, \dots, w_k, t)$$

$$= \sum_{l=1}^{k} (-1)^{k-l} \sum_{1 \le i < j \le k, i,j \ne l} (-1)^{i+j+1} \phi_2([w_i, w_j]_{\mathcal{V}'}, w_1, \dots, \hat{w}_l, \dots, \hat{w}_i, \dots, \hat{w}_j, \dots, w_k)$$

$$+ \sum_{l=1}^{k} (-1)^{k-l} \sum_{i=1, i \ne l}^{k} (-1)^i \, w_i \cdot \phi_2(w_1, \dots, \hat{w}_l, \dots, \hat{w}_i, \dots, w_k)$$

$$= \sum_{l=1}^{k} (-1)^{k-l} (\delta_{k-2}\phi_2)(w_1, \dots, \hat{w}_l, \dots w_k). \tag{3.68}$$

The computation above needs to be read backwards. Starting with the coboundary ψ_2, we use ϕ_2 to define the cochain $\hat{\phi}_2 \in C^{k-1}(\mathcal{V}', M)$. Applying the coboundary operator to ϕ_2, evaluating on k different combinations of $k-1$ elements of k elements of $\mathcal{W}$, and combining them as in (3.68), we can do the computation backwards, adding zero terms (3.67), to finally obtain the coboundary $(\delta_{k-1}\hat{\phi}_2)$ in $B^k(\mathcal{V}', M)$ in (3.66). Hence, every coboundary in $B^{k-1}(\mathcal{W}, M)$ can be interpreted as a coboundary of $B^k(\mathcal{V}', M)$. □

Comparison of the result of Theorem 3.3.2 to Formula (3.45) yields that the differential $d_2^{(0),p,1} = \varphi_p^{(0)}$ for the trivial central extension must be zero for all p. Consequently, we obtain that the differential $d_2^{p,1} = \varphi_1$ for the non-trivial central extension $\mathcal{V}$ is given by the cup product with the extension class $[\alpha]$.

Remark 3.3.1. The result of Theorem 3.3.2, as well as $d_2^{(0),p,1} = 0$, is not only valid for the trivial central extension of the Witt algebra $\mathcal{W}$ by the trivial module $\mathbb{K}$. It is valid for any extension $0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{e} \longrightarrow 0$ of a Lie algebra $\mathfrak{e}$ by a Lie algebra $\mathfrak{h}$ satisfying the following properties,

- the extension must be central,
- the extension must be trivial,
- the extension must be one-dimensional,
- M must be a trivial $\mathfrak{h}$-module.

See also Theorem 8 of [61].

In the following two sections, the goal is to prove the injectivity of the differentials φ_p for low p, for $M = \mathbb{K}$ and $M = \mathscr{F}^\lambda$. Using (3.45), we thus obtain a canonical relation between $\mathrm{H}^k(\mathscr{V}, M)$ and $\mathrm{H}^k(\mathscr{W}, M)$ for the low-dimensional cohomology. The adjoint case for the Witt algebra $M = \mathscr{W}$ is included in the analysis of $M = \mathscr{F}^\lambda$, corresponding to $\lambda = -1$. The adjoint case for the Virasoro algebra $M = \mathscr{V}$ is not included. However, using the long exact sequence (3.57) and the results for $M = \mathbb{K}$ and $M = \mathscr{F}^{-1}$, a relation between $\mathrm{H}^k(\mathscr{V},\mathscr{V})$, $\mathrm{H}^k(\mathscr{W},\mathscr{W})$ and $\mathrm{H}^k(\mathscr{W},\mathbb{K})$ can be found, just as we did in Section 3.2 in case of the third cohomology. The results will be used in the next Chapter 4.

3.3.2 The trivial module

Although results are already known for the low-dimensional algebraic cohomology, we will write down the proofs explicitly also for the known cases, in order to see how the complexity of the proofs increases with the cohomological dimension. For example, although we already know that $\mathrm{H}^1(\mathscr{W},\mathbb{K}) = 0$ or that every cocycle class of $\mathrm{H}^2(\mathscr{W},\mathbb{K})$ and $\mathrm{H}^3(\mathscr{W},\mathbb{K})$ are multiples of the cocycle classes of α in (2.20) or of Ψ in (3.39), respectively, we still write down the cup product with general degree-zero cocycle classes $[\psi]$, to exhibit possible patterns in the proofs when increasing the cohomological dimension. If patterns appear, the proofs could possibly be generalized to higher cohomological dimension. Since only the degree-zero cohomology is a a priori non-trivial, we will stick to degree-zero cocycles, though.
The aim in this section is to prove the injectivity of $\varphi_p = [\alpha] \smile \, \forall p$ where $[\alpha] \in \mathrm{H}^2(\mathscr{W},\mathbb{K})$ is the Virasoro 2-cocycle cohomology class, and φ_p applies to p-cocycles classes $[\psi^{(p)}] \in \mathrm{H}^p(\mathscr{W},\mathbb{K})$. The product $[\alpha] \smile [\psi^{(p)}]$ yields in accordance with Definition 3.3.1 an element of $\mathrm{H}^{p+2}(\mathscr{W},\mathbb{K}\otimes\mathbb{K}) = \mathrm{H}^{p+2}(\mathscr{W},\mathbb{K})$.
To show the injectivity of the φ_p, we need to show the injectivity on the cohomological level, i.e. $\ker\varphi_p = 0$ up to coboundaries. Concretely, we need to prove,

$$\Big(\big([\alpha] \smile \big[\psi^{(p)}\big]\big)(e_{i_1}, e_{i_2}, \dots, e_{i_p}, e_{i_{p+1}}, e_{i_{p+2}}) = [\mathbf{0}] \;\; \forall i_1, \dots, i_{p+2} \in \mathbb{Z}\Big) \Leftrightarrow \psi^{(p)} \in [\mathbf{0}] \,,$$

meaning that the kernel of φ_p has to be the zero-cocycle up to coboundaries. For reasons of simplicity, we will work in the proofs with cohomological choices such that the right-hand side on the left of the formula above is zero exactly. As we do not know the precise form of the cohomological choice under question, we work with general elements of $[\alpha]$ and $[\psi^{(p)}]$. A general degree-zero cocycle in the cohomology class of the Virasoro 2-cocycle is of the form,

$$\alpha(e_i, e_j) = (i^3 - \beta i)\delta_{i+j,0}\,, \tag{3.69}$$

where $\beta \in \mathbb{K}$. Actually, a generic degree-zero 1-coboundary with values in the trivial module is of the form $\phi([e_i, e_{-i}]) = -2i\phi(e_0) := -\beta i$.

Theorem 3.3.3. *For $n = 0,1,2,3,4$, the following relation between the algebraic cohomology of the Witt and the Virasoro algebra over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in the trivial module holds true:*

$$\mathrm{H}^0(\mathscr{V},\mathbb{K}) = \mathrm{H}^0(\mathscr{W},\mathbb{K})\,, \qquad \mathrm{H}^1(\mathscr{V},\mathbb{K}) = \mathrm{H}^1(\mathscr{W},\mathbb{K})\,,$$
$$\mathrm{H}^n(\mathscr{V},\mathbb{K}) = \frac{\mathrm{H}^n(\mathscr{W},\mathbb{K})}{\mathrm{H}^{n-2}(\mathscr{W},\mathbb{K})}\,, \quad n = 2,3,4\,.$$

Proof. $\boxed{n=0}$: For $n=0$ in (3.45), we have $\mathrm{im}\,\varphi_{n-2}=0$ and $\ker\varphi_{n-1}=0$ since $\mathrm{H}^{-2}(\mathcal{W},\mathbb{K})$ and $\mathrm{H}^{-1}(\mathcal{V},\mathbb{W})=0$ by definition.

$\boxed{n=1}$: Let ψ be in $\mathrm{H}^0(\mathcal{W},\mathbb{K})$. Recall that the 0-cochains are the elements of the module, hence ψ is an element of $\mathbb{K}$. The general form of the cup product $(\alpha\smile\psi)$ with $\psi\in\mathrm{H}^0(\mathcal{W},\mathbb{K})$ evaluated on the basis elements (e_i,e_j) has the following form:

$$(\alpha\smile\psi)(e_i,e_j)=\alpha(e_i,e_j)\psi=(i^3-\beta i)\delta_{i+j,0}\psi\,.$$

Choosing $i+j=0$ and $i\notin\{0,\pm\sqrt{\beta}\}$, we immediately obtain that $(\alpha\smile\psi)(e_i,e_j)=0\ \forall i,j\in\mathbb{Z}$ if and only if $\psi=0$. Hence, $\ker\varphi_0=0$ and φ_0 is injective. Together with the fact that $\mathrm{im}\,\varphi_{-1}=0$ because $\mathrm{H}^{-1}(\mathcal{W},\mathbb{K})=0$ by definition, we obtain the result for $n=1$.

$\boxed{n=2}$: Let ψ be a degree zero 1-cocycle of $\mathrm{H}^1(\mathcal{W},\mathbb{K})$. Such a degree zero 1-cocycle can be written as $\psi(e_i)=\psi_i\delta_{i,0}$. The cup product of α and ψ is of the form:

$$\begin{aligned}(\alpha\smile\psi)(e_i,e_j,e_k)&=\alpha(e_i,e_j)\psi(e_k)-\alpha(e_i,e_k)\psi(e_j)+\alpha(e_j,e_k)\psi(e_i)\\&=(i^3-\beta i)\delta_{i+j,0}\psi_k\delta_{k,0}-(i^3-\beta i)\delta_{i+k,0}\psi_j\delta_{j,0}+(j^3-\beta j)\delta_{k+j,0}\psi_i\delta_{i,0}\,.\end{aligned}$$

Choosing $i+j=0$, $i\notin\{0,\pm\sqrt{\beta}\}$, $k=0$, we obtain $\psi_0=0$. Hence $\psi(e_i)=0\ \forall i\in\mathbb{Z}$ and ψ is the zero map. Thus, we obtain that $(\alpha\smile\psi)(e_i,e_j,e_k)=0\ \forall i,j,k\in\mathbb{Z}$ if and only if ψ is the zero map, meaning that $\ker\varphi_1=0$ and φ_1 is injective. Together with the previous result that φ_0 is injective, we obtain the result for $n=2$.

$\boxed{n=3}$: Let ψ be a degree zero 2-cocycle of $\mathrm{H}^2(\mathcal{W},\mathbb{K})$. In terms of coefficients, ψ can be written as $\psi(e_i,e_j):=\psi_{i,j}\delta_{i+j,0}$.
The general form of the cup product $(\alpha\smile\psi)$ with $\psi\in\mathrm{H}^2(\mathcal{W},\mathbb{K})$ evaluated on the basis elements (e_i,e_j,e_k,e_l) has the following form:

$$\begin{aligned}(\alpha\smile\psi)(e_i,e_j,e_k,e_l)&=\alpha(e_i,e_j)\psi(e_k,e_l)-\alpha(e_i,e_k)\psi(e_j,e_l)+\alpha(e_i,e_l)\psi(e_j,e_k)\\&\quad+\alpha(e_j,e_k)\psi(e_i,e_l)-\alpha(e_j,e_l)\psi(e_i,e_k)+\alpha(e_k,e_l)\psi(e_i,e_j)\,.\end{aligned}\tag{3.70}$$

In order to compute the kernel of this cup product, we have to solve in ψ the equation $(\alpha\smile\psi)(e_i,e_j,e_k,e_l)=0\ \forall\ i,j,k,l\in\mathbb{Z}$. We want to show that this equation implies $\psi(e_i,e_j)=0\ \forall\ i,j\in\mathbb{Z}$ or $\psi(e_i,e_j)=\delta_2\phi(e_i,e_j)\ \forall\ i,j\in\mathbb{Z}$ for some 2-cochain ϕ. Inserting our notation and the definition (3.69) for the Virasoro 2-cocycle, we obtain:

$$\begin{aligned}&(\alpha\smile\psi)(e_i,e_j,e_k,e_l)=0\\&\Leftrightarrow(i^3-\beta i)\delta_{i+j,0}\delta_{k+l,0}\psi_{k,l}-(i^3-\beta i)\delta_{i+k,0}\delta_{j+l,0}\psi_{j,l}+(i^3-\beta i)\delta_{i+l,0}\delta_{j+k,0}\psi_{j,k}\\&+(j^3-\beta j)\delta_{j+k,0}\delta_{i+l,0}\psi_{i,l}-(j^3-\beta j)\delta_{j+l,0}\delta_{i+k,0}\psi_{i,k}+(k^3-\beta k)\delta_{k+l,0}\delta_{i+j,0}\psi_{i,j}=0\,.\end{aligned}$$

The cup product $(\alpha\smile\psi)$ evaluated on $(e_{-k-1},e_{k+1},e_k,e_{-k})$ yields:

$$\begin{aligned}&(\alpha\smile\psi)(e_{-k-1},e_{k+1},e_k,e_{-k})=0\\&\Leftrightarrow(k^3-\beta k)\psi_{-1-k,1+k}+((-1-k)^3+\beta(1+k))\psi_{k,-k}=0\\&\Leftrightarrow\psi_{-1-k,1+k}=\frac{(-1-k)^3+\beta(1+k)}{k^3-\beta k}\psi_{-k,k}\,.\end{aligned}\tag{3.71}$$

Clearly, there are poles appearing in the recurrence relation (3.71). Poles appear for $k = 0$ and $k = \pm\sqrt{\beta}$. Due to the poles, the values $\psi_{-1,1}$ and $\psi_{-1-\sqrt{\beta},1+\sqrt{\beta}}$ have to be analyzed separately. Of course, $\beta \in \mathbb{K}$ is not necessarily integer, nor is $\sqrt{\beta}$. If it is not integer, no poles appear and the analysis below related to the poles $k = \pm\sqrt{\beta}$ can be skipped.
We will start with proving $\psi_{-1-\sqrt{\beta},1+\sqrt{\beta}} = 0$. Moreover, we suppose $\beta \neq 0$ as the case of the pole $k = 0$ will by analyzed later. First of all, we need to express the coefficients $\psi_{-2-\sqrt{\beta},2+\sqrt{\beta}}$ and $\psi_{-3-\sqrt{\beta},3+\sqrt{\beta}}$ in terms of $\psi_{-1-\sqrt{\beta},1+\sqrt{\beta}}$. Due to antisymmetry, we can work either with k positive or k negative. In the following, we will work with k positive. Using (3.71), we find,

$$\begin{aligned}
k = 1+\sqrt{\beta}: \qquad \psi_{-2-\sqrt{\beta},2+\sqrt{\beta}} &= \frac{(-2-\sqrt{\beta})^3 + \beta(2+\sqrt{\beta})}{(1+\sqrt{\beta})^3 - \beta(1+\sqrt{\beta})}\psi_{-1-\sqrt{\beta},1+\sqrt{\beta}},\\
k = 2+\sqrt{\beta}: \qquad \psi_{-3-\sqrt{\beta},3+\sqrt{\beta}} &= \frac{(-3-\sqrt{\beta})^3 + \beta(3+\sqrt{\beta})}{(2+\sqrt{\beta})^3 - \beta(2+\sqrt{\beta})}\psi_{-2-\sqrt{\beta},2+\sqrt{\beta}}\\
&= -\frac{(-3-\sqrt{\beta})^3 + \beta(3+\sqrt{\beta})}{(1+\sqrt{\beta})^3 - \beta(1+\sqrt{\beta})}\psi_{-1-\sqrt{\beta},1+\sqrt{\beta}}.
\end{aligned}$$

No poles appear in the denominators above. Let us insert these values into the cup product for $(e_{-1-\sqrt{\beta}}, e_{1+\sqrt{\beta}}, e_{-3-\sqrt{\beta}}, e_{3+\sqrt{\beta}})$, which yields:

$$\begin{aligned}
&(\alpha \smile \psi)(e_{-1-\sqrt{\beta}}, e_{1+\sqrt{\beta}}, e_{-3-\sqrt{\beta}}, e_{3+\sqrt{\beta}}) = 0 \qquad (3.72)\\
\Leftrightarrow &\left(\left(-1-\sqrt{\beta}\right)^3 - \beta\left(-1-\sqrt{\beta}\right)\right)\psi_{-3-\sqrt{\beta},3+\sqrt{\beta}} + \left(\left(-3-\sqrt{\beta}\right)^3 - \beta\left(-3-\sqrt{\beta}\right)\right)\psi_{-1-\sqrt{\beta},1+\sqrt{\beta}} = 0\\
\Leftrightarrow &\left(\left(-3-\sqrt{\beta}\right)^3 + \beta\left(3+\sqrt{\beta}\right)\right)\psi_{-1-\sqrt{\beta},1+\sqrt{\beta}} = 0\,.
\end{aligned}$$

There are no zeros of $(-3-\sqrt{\beta})^3 + \beta(3+\sqrt{\beta})$. Thus, we obtain the desired result $\psi_{-1-\sqrt{\beta},1+\sqrt{\beta}} = 0$.

Next, we consider the value $\psi_{-1,1}$, which is problematic due to the pole at $k = 0$ in (3.71). The analysis of this pole also includes the case $\sqrt{\beta} = 0$. We proceed similarly to the reasoning above. The reasoning below is valid for $\beta \notin \{1,4,9\}$. The values $\beta \in \{1,4,9\}$ will be analyzed separately later. First, we express the coefficients $\psi_{-2,2}$ and $\psi_{-3,3}$ in terms of $\psi_{-1,1}$, by using (3.71),

$$\begin{aligned}
k = 1: \qquad \psi_{-2,2} &= \frac{-8+2\beta}{1-\beta}\psi_{-1,1} \quad \beta \neq 1\,,\\
k = 2: \qquad \psi_{-3,3} &= \frac{-27+3\beta}{8-2\beta}\psi_{-2,2} = \frac{27-3\beta}{1-\beta}\psi_{-1,1} \quad \beta \notin \{1,4\}\,,
\end{aligned}$$

Subsequently, we insert these into the cup product evaluated on $(e_1, e_{-1}, e_3, e_{-3})$,

$$\begin{aligned}
&(\alpha \smile \psi)(e_1, e_{-1}, e_3, e_{-3}) = 0\\
&\Leftrightarrow (1-\beta)\psi_{3,-3} + (27-3\beta)\psi_{-1,1} = 0 \Leftrightarrow -2(27-3\beta)\psi_{-1,1} = 0\,.
\end{aligned}$$

The zero of $27-3\beta$ is $\beta = 9$. Therefore, we obtain $\psi_{-1,1} = 0$ except for $\beta \in \{1,4,9\}$.

We are now able to conclude for $\beta \notin \{1,4,9\}$. Inserting $\psi_{-1,1} = 0$ into (3.71) with $k = 1$, increasing k until $k = \sqrt{\beta} - 1$, we obtain $\psi_{-i,i} = 0$ for $0 \leq i \leq \sqrt{\beta}$. At $k = \sqrt{\beta}$, there is a pole in the recurrence relation (3.71). However, we already proved at the beginning of the reasoning for the case $n=3$ that $\psi_{-1-\sqrt{\beta},1+\sqrt{\beta}}=0$. Thus, inserting $\psi_{-1-\sqrt{\beta},1+\sqrt{\beta}}=0$ into (3.71) with $k=1+\sqrt{\beta}$, increasing k, we obtain $\psi_{-i,i}=0\ \forall\ i\in\mathbb{Z}$, for $\beta\notin\{1,4,9\}$.

Let us come back to $\beta\in\{1,4,9\}$. For these values, the crucial point consists in proving $\psi_{-1,1}=0$. Let us consider $\beta=1$. For $\beta=1$, the conclusion is immediate. In fact, $k=1$ in (3.71) yields $(1-\beta)\psi_{-2,2}=-8+2\beta\psi_{-1,1}$ and hence $\psi_{-1,1}=0$ for $\beta=1$. Note that for $\beta=1$, the poles in (3.71) occur for $k=\pm 1$. By the reasoning we did before for general poles of the form $k=\pm\sqrt{\beta}$, we also obtain $\psi_{-2,2}=0$. Using induction in (3.71) as before, we obtain $\psi_{-i,i}=0\ \forall\ i\in\mathbb{Z}$ also for $\beta=1$.
Next, consider $\beta=4$. From (3.71) for $k=1$, we immediately obtain $\psi_{-2,2}=0$ for $\beta=4$. Starting with $k=2$ in (3.71), using $\psi_{-2,2}=0$ and induction, we obtain with the same reasoning as before, $\psi_{-i,i}=0\ \forall\ i\in\mathbb{Z}$ except for $\psi_{-1,1}$. However, recall that ψ is a degree-zero 2-cocycle of $\mathrm{H}^2(\mathscr{W},\mathbb{K})$, hence it needs to fulfill the cocycle condition, which reads in terms of coefficients, evaluated on (e_i,e_j,e_k),

$$(j-i)\psi_{i+j,k}-(k-i)\psi_{i+k,j}+(k-j)\psi_{k+j,i}=0\,.$$

Consider the cocycle condition evaluated on (e_{-3},e_2,e_1),

$$5\psi_{-1,1}-4\psi_{-2,2}-\psi_{3,-3}=0\,. \tag{3.73}$$

Since $\psi_{-2,2}=\psi_{-3,3}=0$, we also obtain $\psi_{-1,1}=0$ and thus, $\psi_{-i,i}=0\ \forall\ i\in\mathbb{Z}$.
It remains $\beta=9$. The relation (3.71) for $k=2$ yields immediately $\psi_{3,-3}=0$ for $\beta=9$. Moreover, for $k=1$, the relation (3.71) yields $-8\psi_{-2,2}=10\psi_{-1,1}$ for $\beta=9$. Comparing this to the result from the cocycle condition in (3.73), we obtain $\psi_{-1,1}=0$. Using induction in (3.71) as before, we obtain $\psi_{-i,i}=0\ \forall\ i\in\mathbb{Z}$ also for $\beta=9$.

As we consider degree-zero cocycles, we obtain $\psi(e_i,e_j)=0\ \forall\ i,j\in\mathbb{Z}$. Thus, we have $(\alpha\smile\psi)(e_i,e_j,e_k,e_l)=0\ \forall i,j,k,l\in\mathbb{Z}$ if and only if ψ is the zero-map, and $\ker\varphi_2=0$. Next, we consider the situation for 3-cocycles.

$\boxed{n=4}$: Let ψ be a degree-zero 3-cocycle of $\mathrm{H}^3(\mathscr{W},\mathbb{K})$, $\psi(e_i,e_j,e_k)=\psi_{i,j,k}\delta_{i+j+k,0}$. The cup product evaluated on the basis elements (e_i,e_j,e_k,e_l,e_m) yields:

$$\begin{aligned}
&(\alpha\smile\psi)(e_i,e_j,e_k,e_l,e_m)=0\\
&\Leftrightarrow \alpha(e_i,e_j)\psi(e_k,e_l,e_m)-\alpha(e_i,e_k)\psi(e_j,e_l,e_m)+\alpha(e_i,e_l)\psi(e_j,e_k,e_m)\\
&-\alpha(e_i,e_m)\psi(e_j,e_k,e_l)+\alpha(e_j,e_k)\psi(e_i,e_l,e_m)-\alpha(e_j,e_l)\psi(e_i,e_k,e_m)\\
&+\alpha(e_j,e_m)\psi(e_i,e_k,e_l)+\alpha(e_k,e_l)\psi(e_i,e_j,e_m)-\alpha(e_k,e_m)\psi(e_i,e_j,e_l)\\
&+\alpha(e_l,e_m)\psi(e_i,e_j,e_k)=0\,.
\end{aligned}$$

Inserting our notation and the definition of the Virasoro 2-cocycle (3.69), we obtain:

$$\begin{aligned}
&(\alpha\smile\psi)(e_i,e_j,e_k,e_l,e_m)=0\\
&\Leftrightarrow -\left(i^3-\beta i\right)\delta_{0,i+m}\delta_{0,j+k+l}\psi_{j,k,l}+\left(i^3-\beta i\right)\delta_{0,i+l}\delta_{0,j+k+m}\psi_{j,k,m}-\left(i^3-\beta i\right)\delta_{0,i+k}\delta_{0,j+l+m}\psi_{j,l,m}\\
&+\left(i^3-\beta i\right)\delta_{0,i+j}\delta_{0,k+l+m}\psi_{k,l,m}+\left(j^3-\beta j\right)\delta_{0,j+m}\delta_{0,i+k+l}\psi_{i,k,l}-\left(j^3-\beta j\right)\delta_{0,j+l}\delta_{0,i+k+m}\psi_{i,k,m}\\
&+\left(j^3-\beta j\right)\delta_{0,j+k}\delta_{0,i+l+m}\psi_{i,l,m}-\left(k^3-\beta k\right)\delta_{0,k+m}\delta_{0,i+j+l}\psi_{i,j,l}+\left(k^3-\beta k\right)\delta_{0,k+l}\delta_{0,i+j+m}\psi_{i,j,m}\\
&+\left(l^3-\beta l\right)\delta_{0,l+m}\delta_{0,i+j+k}\psi_{i,j,k}=0\,.
\end{aligned}$$

Next, consider the cup product evaluated on $(e_i, e_{-i}, e_k, e_l, e_{-k-l})$. We will assume $i \neq \pm l$, $i \neq \pm k$ and $i \neq \pm(k+l)$, because for these values the cup product is zero anyway due to its alternating property. In that case, only four terms remain:

$$\begin{aligned}&(\alpha \smile \psi)(e_i, e_{-i}, e_k, e_l, e_{-k-l}) = 0, \qquad i \neq \pm l, i \neq \pm k, i \neq \pm(k+l)\\ &\Leftrightarrow \left(i^3 - \beta i\right)\psi_{k,l,-k-l} + \left(k^3 - \beta k\right)\delta_{0,-k-l}\delta_{0,k+l}\psi_{i,-i,-k-l}\\ &- \left(k^3 - \beta k\right)\delta_{0,-l}\delta_{0,l}\psi_{i,-i,l} + \left(l^3 - \beta l\right)\delta_{0,-k}\delta_{0,k}\psi_{i,-i,k} = 0.\end{aligned} \tag{3.74}$$

Case 1: Let us choose $k \neq 0$, $l \neq 0$, $k+l \neq 0$ and $i \notin \{0, \pm\sqrt{\beta}\}$. In that case, we obtain:

$$\psi_{k,l,-k-l} = 0 \text{ except for entries of the form } \psi_{k,-k,0}. \tag{3.75}$$

Case 2: In order to analyze the remaining entries, we consider $k = 0$ and $l \neq 0$ in Equation (3.74):

$$\begin{aligned}&(\alpha \smile \psi)(e_i, e_{-i}, e_0, e_l, e_{-l}) = 0\\ &\Leftrightarrow (-\beta l + l^3)\psi_{i,-i,0} + (-\beta i + i^3)\psi_{0,l,-l} = 0, \qquad i \neq \pm l, i \neq 0, l \neq 0\end{aligned}$$

Choosing furthermore $l = i+1$, we obtain the following recurrence relation for $i \notin \{0, -1\}$:

$$\begin{aligned}&(\alpha \smile \psi)(e_i, e_{-i}, e_0, e_{i+1}, e_{-i-1}) = 0\\ &\Leftrightarrow \psi_{0,-i-1,i+1} = -\frac{(i+1)^3 - \beta(i+1)}{i^3 - \beta i}\psi_{0,-i,i}.\end{aligned} \tag{3.76}$$

Comparing the expression (3.76) to (3.71), we see that they are the same. Moreover, consider the cup product evaluated on $(e_{-1-\sqrt{\beta}}, e_{1+\sqrt{\beta}}, e_0, e_{-3-\sqrt{\beta}}, e_{3+\sqrt{\beta}})$, which yields:

$$\begin{aligned}&(\alpha \smile \psi)(e_{-1-\sqrt{\beta}}, e_{1+\sqrt{\beta}}, e_0, e_{-3-\sqrt{\beta}}, e_{3+\sqrt{\beta}}) = 0\\ &\Leftrightarrow \left(\left(-1-\sqrt{\beta}\right)^3 - \beta\left(-1-\sqrt{\beta}\right)\right)\psi_{-3-\sqrt{\beta},3+\sqrt{\beta},0}\\ &+ \left(\left(-3-\sqrt{\beta}\right)^3 - \beta\left(-3-\sqrt{\beta}\right)\right)\psi_{-1-\sqrt{\beta},1+\sqrt{\beta},0} = 0.\end{aligned}$$

Also this expression is exactly the same as in (3.72). Therefore, we can redo exactly the same reasoning as before for the case $n = 3$ to obtain $\psi_{-i,i,0} = 0 \; \forall \; i \in \mathbb{Z}$, except for $\beta = 4$ and $\beta = 9$. For $\beta = 4$ and $\beta = 9$, we can nevertheless already obtain $\psi_{-i,i,0} = 0 \; \forall \; i \in \mathbb{Z}$ except for $\psi_{-1,1,0}$ in the case of $\beta = 4$ and except for $\psi_{-1,1,0}$, $\psi_{-2,2,0}$ in the case of $\beta = 4$, by the same reasoning as the one for the case $n = 3$. The annihilation of the values $\psi_{-1,1,0}$, $\psi_{-2,2,0}$ requires the cocycle condition, which are different for $\mathrm{H}^2(\mathcal{W}, \mathbb{K})$ and $\mathrm{H}^3(\mathcal{W}, \mathbb{K})$, therefore we will treat these cases explicitly. The cocycle condition of $\mathrm{H}^3(\mathcal{W}, \mathbb{K})$ expressed in terms of coefficients reads,

$$\begin{aligned}&(j-i)\psi_{i+j,k,l} - (k-i)\psi_{i+k,j,l} + (l-i)\psi_{i+l,j,k}\\ &+ (k-j)\psi_{j+k,i,l} - (l-j)\psi_{l+j,i,k} + (l-k)\psi_{l+k,i,j},\end{aligned}$$

and yields, when evaluated on (e_{-3}, e_2, e_1, e_0),

$$5\psi_{-1,1,0} - 4\psi_{-2,2,0} + \cancel{3\psi_{-3,2,1}} - \psi_{3,-3,0} + \cancel{2\psi_{2,-3,1}} - \cancel{\psi_{1,-3,2}} = 0.$$

The slashed terms cancel each other and are zero anyway due to Case 1. The remaining equation is exactly the same as in (3.73). The same reasoning as the one done for $n = 3$ allows to conclude $\psi_{-i,i,0} = 0 \ \forall \ i \in \mathbb{Z}$.

The results (3.75) and $\psi_{-i,i,0} = 0 \ \forall \ i \in \mathbb{Z}$ yield $\psi(e_i, e_j, e_k) = 0 \ \forall \ i, j, k \in \mathbb{Z}$ as we consider degree zero 3-cocycles. Thus, we have $(\alpha \smile \psi)(e_i, e_j, e_k, e_l, e_m) = 0 \ \forall i, j, k, l, m \in \mathbb{Z}$ if and only if ψ is the zero-map, and $\ker \varphi_3 = 0$. Together with the previous result that φ_2 is also injective, we obtain the statement for $n = 4$.

All in all, we proved Theorem 3.3.3. □

Remark 3.3.2. The author tried to go beyond cohomological dimension four. However, the complexity of the analysis increased in such a manner that the methods used in the proof above are insufficient to obtain the results. Although the complexity of the proofs does not seem to increase with the cohomological dimension as dramatically as when trying to compute $\mathrm{H}^k(\mathcal{V}, \mathbb{K})$ directly, still there is no obvious pattern apparent, i.e. there is no straightforward generalization of the proofs to higher dimension.

Remark 3.3.3. We see that in the case of the third cohomology, we have: $\mathrm{H}^3(\mathcal{V}, \mathbb{K}) = \dfrac{\mathrm{H}^3(\mathcal{W}, \mathbb{K})}{\mathrm{H}^1(\mathcal{W}, \mathbb{K})}$. Since we have $\mathrm{H}^1(\mathcal{W}, \mathbb{K}) = \{0\}$, we obtain $\mathrm{H}^3(\mathcal{V}, \mathbb{K}) = \mathrm{H}^3(\mathcal{W}, \mathbb{K})$ in a canonical way, corroborating our previous result from Section 3.1.
In the case of the second cohomology, we obtain $\mathrm{H}^2(\mathcal{V}, \mathbb{K}) = \dfrac{\mathrm{H}^2(\mathcal{W}, \mathbb{K})}{\mathcal{W}\mathbb{K}} = \dfrac{\mathrm{H}^2(\mathcal{W}, \mathbb{K})}{\mathbb{K}}$. Since $\dim \mathrm{H}^2(\mathcal{W}, \mathbb{K}) = 1$, we obtain $\mathrm{H}^2(\mathcal{V}, \mathbb{K}) = \{0\}$, meaning that the Virasoro algebra $\mathcal{V}$ admits no non-trivial central extension. This was already computed explicitly in [106].

3.3.3 The general tensor densities modules

In this section, we perform the same analysis but with the $\mathcal{F}^\lambda$ modules instead of the trivial module $\mathbb{K}$. Thus, the aim in this section is to prove the injectivity of $\varphi_p = \alpha \smile \forall p$ where $\alpha \in \mathrm{H}^2(\mathcal{W}, \mathbb{K})$ is the Virasoro 2-cocycle, and φ_p applies to p-cocycles $\psi^{(p)} \in \mathrm{H}^p(\mathcal{W}, \mathcal{F}^\lambda)$. The product $\alpha \smile \psi^{(p)}$ yields in accordance with Definition 3.3.1 an element of $\mathrm{H}^{p+2}(\mathcal{W}, \mathbb{K} \otimes \mathcal{F}^\lambda) = \mathrm{H}^{p+2}(\mathcal{W}, \mathcal{F}^\lambda)$, as the tensor product is $\mathbb{K} \otimes \mathcal{F}^\lambda = \mathcal{F}^\lambda$.

Theorem 3.3.4. *For $n = 0, 1, 2, 3$, the following relations between the algebraic cohomology of the Witt and the Virasoro algebra over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in $\mathcal{F}^\lambda$ hold true:*

$$\mathrm{H}^0(\mathcal{V}, \mathcal{F}^\lambda) = \mathrm{H}^0(\mathcal{W}, \mathcal{F}^\lambda), \qquad \mathrm{H}^1(\mathcal{V}, \mathcal{F}^\lambda) = \mathrm{H}^1(\mathcal{W}, \mathcal{F}^\lambda),$$
$$\mathrm{H}^n(\mathcal{V}, \mathcal{F}^\lambda) = \frac{\mathrm{H}^n(\mathcal{W}, \mathcal{F}^\lambda)}{\mathrm{H}^{n-2}(\mathcal{W}, \mathcal{F}^\lambda)}, \quad n = 2, 3, \quad \forall \lambda \in \mathbb{C}.$$

Proof. $\boxed{n = 0}$: For $n = 0$ in (3.45), we have $\mathrm{im}\,\varphi_{n-2} = 0$ and $\ker \varphi_{n-1} = 0$, because by definition, we have $\mathrm{H}^{-2}(\mathcal{W}, \mathcal{F}^\lambda) = 0$ and $\mathrm{H}^{-1}(\mathcal{W}, \mathcal{F}^\lambda) = 0$, respectively.
To prove the other results, we show the injectivity of $\alpha \smile$ on the cohomological level.

$\boxed{n = 1}$: Let ψ be a degree zero 0-cocycle, i.e. ψ is of the form $\psi = \psi_0 f_0^\lambda$, where $\psi_0 \in \mathbb{K}$ and $f_0^\lambda \in \mathcal{F}^\lambda$. The cup product with the Virasoro 2-cocycle α yields,

$$(\alpha \smile \psi)(e_i, e_j) = (i^3 - \beta i)\delta_{i+j,0}\psi_0 f_0^\lambda .$$

Choosing $i+j=0$ and $i \notin \{0, \pm\sqrt{\beta}\}$, we obtain that $(i^3-\beta i)\delta_{i+j,0}\psi_0 f_0^\lambda$ is zero if and only if $\psi_0 = 0$. Hence, $(\alpha \smile \psi)(e_i, e_j) = 0\ \forall i, j \in \mathbb{Z}$ if and only if ψ is the zero map and thus $\ker \varphi_0 = 0$. Together with the fact that $\operatorname{im}\varphi_{-1} = 0$ because $\mathrm{H}^{-1}(\mathcal{W}, \mathcal{F}^\lambda) = 0$, we obtain the result for $n = 1$.

$\boxed{n=2}$: Let ψ be a degree zero 1-cocycle, i.e. ψ is of the form $\psi(e_i) = \psi_i f_i^\lambda$, with $\psi_i \in \mathbb{K}$. The cup product $(\alpha \smile \psi)$ is zero if and only if,

$$
\begin{aligned}
&(\alpha \smile \psi)(e_i, e_j, e_k) = 0 \Leftrightarrow \alpha(e_i, e_j)\psi(e_k) - \alpha(e_i, e_k)\psi(e_j) + \alpha(e_j, e_k)\psi(e_i) = 0 \\
&\Leftrightarrow (i^3-\beta i)\delta_{i+j,0}\psi_k f_k^\lambda - (i^3-\beta i)\delta_{i+k,0}\psi_j f_j^\lambda + (j^3-\beta j)\delta_{j+k,0}\psi_i f_i^\lambda = 0\,.
\end{aligned}
$$

Let us choose $i+j=0$ and $i \notin \{0, \pm\sqrt{\beta}\}$, as well as $k \notin \{-i, i\}$. Then we obtain immediately $\psi_k = 0$. Different choices of i lead to $\psi_k = 0\ \forall k \in \mathbb{Z}$. Hence φ_1 is injective as its kernel is zero, and together with the previous result that φ_0 is injective, we obtain the result for $n = 2$.

$\boxed{n=3}$: Let ψ be a degree zero 2-cocycle, i.e. ψ is of the form $\psi(e_i, e_j) = \psi_{i,j} f_{i+j}^\lambda$, with $\psi_{i,j} \in \mathbb{K}$. The cup product with the Virasoro 2-cocycle $\alpha \smile \psi$ can be found in (3.70), and yields after writing α and ψ in terms of their coefficients,

$$
\begin{aligned}
&(\alpha \smile \psi)(e_i, e_j, e_k, e_l) = 0 \\
&\Leftrightarrow i\left(i^2-\beta\right) f_{j+k}^\lambda \delta_{0,i+l}\psi_{j,k} - i\left(i^2-\beta\right) f_{j+l}^\lambda \delta_{0,i+k}\psi_{j,l} + i\left(i^2-\beta\right) f_{k+l}^\lambda \delta_{0,i+j}\psi_{k,l} \\
&- j\left(j^2-\beta\right) f_{i+k}^\lambda \psi_{i,k}\delta_{0,j+l} + j\left(j^2-\beta\right) f_{i+l}^\lambda \psi_{i,l}\delta_{0,j+k} + k\left(k^2-\beta\right) f_{i+j}^\lambda \psi_{i,j}\delta_{0,k+l} = 0\,. \qquad (3.77)
\end{aligned}
$$

In the following, we will choose the indices i, j, k, l in such a manner that only one of the Kronecker Deltas will be different from zero, then two, then three and so on.
Case 1: Let us choose the indices such that $\delta_{0,i+j}$ will be different from zero and all other Kronecker Deltas zero, i.e. we choose $i+j=0$, $k \neq \pm i$, $l \neq \pm i$, and $k+l \neq 0$. Furthermore, we choose $i \notin \{0, \pm\sqrt{\beta}\}$. Then we immediately obtain from the equation (3.77) above,

$$
\psi_{k,l} = 0 \qquad \forall\ k+l \neq 0\,,
$$

meaning all coefficients $\psi_{k,l}$ are zero except those of the form $\psi_{k,-k}$. The latter will be considered below.
Case 2: We will now choose the indices in (3.77) such that two of the Kronecker Deltas will be different from zero. We choose $i+j=0$, $k+l=0$, $k \neq \pm i$ in (3.77), which yields:

$$
(\alpha \smile \psi)(e_i, e_{-i}, e_k, e_{-k}) = 0 \Leftrightarrow (i^3-\beta i)\psi_{k,-k} f_0^\lambda + (k^3-\beta k)\psi_{i,-i} f_0^\lambda = 0\,, \qquad k \neq \pm i\,. \qquad (3.78)
$$

Let us choose $i = 1$ in the equation above, and $k \notin \{0, \pm\sqrt{\beta}\}$. Then we immediately obtain $\psi_{1,-1} = 0$. Next, we take $i = -k-1$ in (3.78), which is compatible with our assumptions and yields,

$$
\begin{aligned}
&(\alpha \smile \psi)(e_{-k-1}, e_{k+1}, e_k, e_{-k}) = 0 \\
&\Leftrightarrow \left((k^3-\beta k)\psi_{-1-k,1+k} + ((-1-k)^3 + \beta(1+k))\psi_{k,-k}\right) f_0^\lambda = 0\,. \qquad (3.79)
\end{aligned}
$$

This expression is exactly the same as (3.71). Moreover, the cup product evaluated on the basis elements $(e_{-1-\sqrt{\beta}}, e_{1+\sqrt{\beta}}, e_{-3-\sqrt{\beta}}, e_{3+\sqrt{\beta}})$ yields:

$$
\begin{aligned}
&(\alpha \smile \psi)(e_{-1-\sqrt{\beta}}, e_{1+\sqrt{\beta}}, e_{-3-\sqrt{\beta}}, e_{3+\sqrt{\beta}}) = 0 \\
&\Leftrightarrow \left(\left(-1-\sqrt{\beta}\right)^3 - \beta\left(-1-\sqrt{\beta}\right)\right)\psi_{-3-\sqrt{\beta},3+\sqrt{\beta}}\, f_0^\lambda \\
&+ \left(\left(-3-\sqrt{\beta}\right)^3 - \beta\left(-3-\sqrt{\beta}\right)\right)\psi_{-1-\sqrt{\beta},1+\sqrt{\beta}}\, f_0^\lambda = 0\,.
\end{aligned}
$$

This expression is the same as (3.72). We can thus do exactly the same reasoning as for $n=3$ in the case of the trivial module, see the proof of Theorem 3.3.3, to obtain $\psi_{i,-i}=0\ \forall\ i\in\mathbb{Z}$ except for $\beta\in\{4,9\}$, $\psi_{i,-i}=0\ \forall\ i\in\mathbb{Z}\setminus\{\pm1\}$ for $\beta=4$, and $\psi_{i,-i}=0\ \forall\ i\in\mathbb{Z}\setminus\{\pm1,\pm2\}$ for $\beta=9$.
For $\beta=4$ and $\beta=9$, we need to consider the fact that ψ is a cocycle of $\mathrm{H}^2(\mathscr{W},\mathscr{F}^\lambda)$ and that it has to satisfy the cocycle condition. The cocycle condition of $\mathrm{H}^2(\mathscr{W},\mathscr{F}^\lambda)$ can be found in (4.14) and yields when evaluated on (e_{-3},e_2,e_1),

$$5\psi_{-1,1}-4\psi_{-2,2}-\psi_{3,-3}-(3-3\lambda)\underline{\psi_{2,1}}+(-2+2\lambda)\underline{\psi_{-3,1}}-(-1+\lambda)\underline{\psi_{-3,2}}=0\,.$$

The underlined terms are of the form $\psi_{k,l}$ with $k+l\neq0$ and thus zero due to Case 1. The remaining equation is the same as in (3.73). By the same reasoning as the one performed in the proof of Theorem 3.3.3, we obtain $\psi_{i,-i}=0\ \forall\ i\in\mathbb{Z}$.

Together with the coefficients analyzed in Case 1, we obtain that $\ker\varphi_2=0$, hence φ_2 is injective, and since φ_1 is also injective, the result is shown for $n=3$.

All in all, we proved Theorem 3.3.4.

□

Chapter 4

General tensor densities modules

The analysis of the previous chapter focused on the trivial and the adjoint module. However, these two modules are related to a larger, actually infinite family of modules, called the general tensor densities modules $\mathcal{F}^\lambda$, $\lambda \in \mathbb{C}$. As we already mentioned in Section 2.1.5, the adjoint module for the Witt algebra corresponds to the module $\mathcal{F}^{-1}$. Hence, the proof presented for Theorem 3.2.2 is only an example of the family of proofs we will present in Section 4.3. Nevertheless, we presented the proof for $\lambda = -1$ explicitly for Theorem 3.2.2, because it allows a better understanding of the general proof given in Section 4.3. Note that in case of the Virasoro algebra, the adjoint module does not correspond to $\mathcal{F}^{-1}$.

The trivial module $\mathbb{K}$ is included in $\mathcal{F}^0$. In fact, recall from Section 2.1.5 that the action of $\mathcal{W}$ or $\mathcal{V}$ on $\mathcal{F}^0$ is given by $e_n \cdot f_m^0 = m f_{n+m}^0$. The trivial module $\mathbb{K}$ can thus be associated to the element f_0^0, on which $\mathcal{W}$ and $\mathcal{V}$ indeed act trivially, $e_n \cdot f_0^0 = 0 f_n^0 = 0$. When interpreting the modules $\mathcal{F}^0$ as meromorphic functions, $f_m^0 = z^m$, the trivial module $\mathbb{K}$ associated to f_0^0 corresponds to constant functions.

In Section 4.1, we concentrate on the first algebraic cohomology. We have complete results for the Witt algebra for all $\lambda \in \mathbb{C}$. In Section 4.2, we analyze the second algebraic cohomology. In Section 4.3, we present results for the third algebraic cohomology. Here, we only have results for a finite number of values of λ, due to the complexity of the proofs. In Sections 4.1, 4.2 and 4.3, we only consider the Witt algebra. The results for the Virasoro algebra can be deduced immediately from the results for the Witt algebra, thanks to the results given in Theorem 3.3.4; they will be given in Section 4.4. Most results of this chapter have already been announced in [25, 29].

4.1 The first algebraic cohomology

We will start by deriving results for all $\lambda \in \mathbb{C}$ except for $\lambda = 0, 1, 2$. These values will be considered in a separate analysis later on.

We focus only on the degree zero cohomology, as the non-zero degree cohomology is zero, see Theorem 2.2.1. As we consider only the degree zero cohomology of the Witt algebra, we can write a 1-cochain of $C^1(\mathcal{W}, \mathcal{F}^\lambda)$ as follows: $\psi^\lambda(e_i) = \psi_i^\lambda f_i^\lambda$, with the $\psi_i^\lambda \in \mathbb{K}$, the $e_i \in \mathcal{W}$ and the $f_i^\lambda \in \mathcal{F}^\lambda$. In that case, the condition for a 1-cochain $\psi^\lambda \in C^1(\mathcal{W}, \mathcal{F}^\lambda)$ to be a 1-cocycle is given by, when evaluated on basis elements $(e_i, e_j) =: (i, j)$, see (2.44):

$$(\delta_1 \psi^\lambda)(e_i, e_j) = (j - i)\psi_{i+j}^\lambda - (j + \lambda i)\psi_j^\lambda + (i + \lambda j)\psi_i^\lambda = 0. \qquad (4.1)$$

Next, let ϕ^λ be a zero-cochain, i.e $\phi^\lambda \in \mathcal{F}^\lambda$. As we consider the degree zero cohomology, we have $\phi^\lambda = \phi_0 f_0^\lambda$, with $\phi_0 \in \mathbb{K}$. The condition for a 1-cocycle ψ^λ to be a coboundary reads, when evaluated on a basis element e_i, see (2.43):

$$\psi^\lambda(e_i) = (\delta_0\phi^\lambda)(e_i) = -\phi_0 e_i \cdot f_0^\lambda = -\phi_0 \lambda i f_i^\lambda \,. \tag{4.2}$$

4.1.1 General analysis

The aim of this section is to prove the Theorem 4.1.1 below.

Theorem 4.1.1. *The first algebraic cohomology of the Witt algebra $\mathscr{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathscr{F}^\lambda$ with $\lambda \in \mathbb{C}\setminus\{0,1,2\}$ vanishes, i.e.*

$$\forall \lambda \in \mathbb{C}\setminus\{0,1,2\}: \ \mathrm{H}^1(\mathscr{W},\mathscr{F}^\lambda) = \{0\}\,. \tag{4.3}$$

Proof. Let ψ^λ be a 1-cocycle of $\mathrm{H}^1(\mathscr{W},\mathscr{F}^\lambda)$. The first step consists in performing a cohomological change such that $\psi_1^\lambda = 0$. Let us chose the following zero-cochain: $\phi^\lambda = \frac{1}{\lambda}\psi_1^\lambda f_0^\lambda \in \mathscr{F}^\lambda$. Note that $\lambda \neq 0$ by assumption. The coboundary condition (4.2) gives us:

$$(\delta_0\phi^\lambda)(e_i) = -e_i \cdot \frac{1}{\lambda}\psi_1^\lambda f_0^\lambda = -(0+\lambda i)\frac{1}{\lambda}\psi_1^\lambda f_i^\lambda = -i\psi_1^\lambda f_i^\lambda \,.$$

The cohomological change $(\psi^\lambda)' = \psi^\lambda + (\delta_0\phi^\lambda)$ then leads to $(\psi_1^\lambda)' = 0$. In the following, we will work with $(\psi^\lambda)'$, although we will drop the apostrophe to augment readability.
Next, the cocycle condition (4.1) evaluated on $(i,1)$ yields:

$$\begin{aligned}(\delta_1\psi^\lambda)(e_i,e_1) &= -(1+i\lambda)\cancel{\psi_1^\lambda} + (i+\lambda)\psi_i^\lambda + (1-i)\psi_{i+1}^\lambda = 0\\ &\Leftrightarrow \psi_{i+1}^\lambda = \frac{i+\lambda}{i-1}\psi_i^\lambda \,.\end{aligned} \tag{4.4}$$

From this relation, we obtain:
For $i=2$: $\psi_3^\lambda = (2+\lambda)\psi_2^\lambda$.
For $i=3$: $\psi_4^\lambda = \frac{(3+\lambda)}{2}\psi_3^\lambda = \frac{(3+\lambda)(2+\lambda)}{2}\psi_2^\lambda$.
For $i=4$: $\psi_5^\lambda = \frac{(4+\lambda)}{3}\psi_4^\lambda = \frac{(4+\lambda)(3+\lambda)(2+\lambda)}{6}\psi_2^\lambda$.
Writing down the cocycle condition (4.1) for (3,2) and inserting the values of ψ_3^λ and ψ_5^λ gives us:

$$\begin{aligned}(\delta_1\psi^\lambda)(e_3,e_2) &= -(2+3\lambda)\psi_2^\lambda + (3+2\lambda)\psi_3^\lambda - \psi_5^\lambda = 0\\ &\Leftrightarrow -(2+3\lambda)\psi_2^\lambda + (3+2\lambda)(2+\lambda)\psi_2^\lambda - \frac{(4+\lambda)(3+\lambda)(2+\lambda)}{6}\psi_2^\lambda = 0\\ &\Leftrightarrow -\frac{1}{6}(\lambda-2)(\lambda-1)\lambda\,\psi_2^\lambda = 0 \Leftrightarrow \psi_2^\lambda = 0 \text{ as } \lambda \notin \{0,1,2\}\,.\end{aligned}$$

Inserting $\psi_2^\lambda = 0$ into the recurrence relation defined in (4.4), we obtain $\psi_i^\lambda = 0\ \forall\ i \geq 3$. Inserting $i=0$ in (4.4) gives $\psi_0^\lambda = 0$ since $\lambda \neq 0$. Thus, we obtain $\boxed{\psi_i^\lambda = 0\ \forall\ i \geq 0}$.
Finally, consider the cocycle condition for $(-i,i)$, $i>0$, which yields:

$$\begin{aligned}(\delta_1\psi^\lambda)(e_{-i},e_i) &= 2i\cancel{\psi_0^\lambda} + (-i+i\lambda)\psi_{-i}^\lambda - (i-i\lambda)\underbrace{\psi_i^\lambda}_{=0 \text{ for } i>0} = 0\\ &\Leftrightarrow (-i+i\lambda)\psi_{-i}^\lambda = 0 \Leftrightarrow \psi_{-i}^\lambda = 0 \text{ as } \lambda \neq 1\,.\end{aligned}$$

As $i>0$, we obtain $\boxed{\psi_i^\lambda = 0\ \forall\ i < 0}$. All in all, the statement is proved. □

4.1.2 Exceptional cases

In this section, we focus on the remaining values of λ, namely $\lambda = 0, 1, 2$. We will see that the cohomology does not vanish for these cases. We will proceed similarly to the analysis for $\mathrm{H}^3(\mathcal{W}, \mathbb{K})$ given in Section 3.1, by first deriving an upper bound for the dimension, then a lower bound. The lower bound is obtained by constructing explicit generating cocycles. These will be obtained by solving recurrence relations.
The first pathological value of λ we will analyze is $\lambda = 0$. The main result we obtain is the Theorem 4.1.2 below.

Theorem 4.1.2. *The first algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda = 0$ has a dimension of two, i.e.*

$$\dim \mathrm{H}^1(\mathcal{W}, \mathcal{F}^0) = 2 . \tag{4.5}$$

Proof. In a first step, we will derive an upper bound for the dimension of the cohomology space, i.e. $\boxed{\dim \mathrm{H}^1(\mathcal{W}, \mathcal{F}^0) \leq 2}$.
Let $\psi \in \mathrm{H}^1(\mathcal{W}, \mathcal{F}^0)$. First of all, let us point out that a cohomological change is not possible in the case $\lambda = 0$. In fact, we see from the formula (4.2) that in the case $\lambda = 0$, every cohomological change is trivial. Therefore, we cannot normalize $\psi_1 = 0$. In order to determine an upper bound for the dimension, we have to count the generating coefficients of a 1-cocycle fulfilling the 1-cocycle condition (4.1), which for $\lambda = 0$ yields:

$$(j-i)\psi_{i+j} - j\psi_j + i\psi_i = 0 .$$

Putting $j = 1$ gives a recurrence relation:

$$\begin{aligned} &(1-i)\psi_{i+1} - \psi_1 + i\psi_i = 0 \\ \Leftrightarrow\ &\psi_{i+1} = \frac{1}{1-i}\psi_1 + \frac{i}{i-1}\psi_i \text{ for } i \text{ increasing} \\ \Leftrightarrow\ &\psi_i = \frac{i-1}{i}\psi_{i+1} + \frac{1}{i}\psi_1 \text{ for } i \text{ decreasing} . \end{aligned} \tag{4.6}$$

Starting with $i = 3$ in the first recurrence relation above, and increasing i, we see that all elements ψ_i $i \geq 3$ are generated by two generators ψ_1 and ψ_2. Starting with $i = -1$ in the second recurrence relation above and decreasing i, we see that all elements ψ_i $i \leq -1$ are generated by two elements, ψ_0 and ψ_1. Hence, three generating coefficients appear a priori. However, there is a relation between them. Taking for example the cocycle condition for $(-2, 2)$ and inserting the recurrence relations yields:

$$-2\psi_{-2} + 4\psi_0 - 2\psi_2 = 0 \Leftrightarrow -2(\psi_0 - 2\psi_1 + \psi_2) = 0 \Leftrightarrow \psi_2 = 2\psi_1 - \psi_0 . \tag{4.7}$$

Thus, there are only two independent generating coefficients and the dimension of the first algebraic cohomology is at most two.

Next, we prove that the dimension is at least two, $\boxed{\dim \mathrm{H}^1(\mathcal{W}, \mathcal{F}^0) \geq 2}$. We need to find two generating cocycles, which are not coboundaries, and not cohomologously equivalent, i.e. their difference must not be a coboundary. These can be found by solving the recurrence relation (4.6), and are given by:

$$\begin{aligned} \psi^{(1)}(e_i) &= \psi_i^{(1)} f_i^0 := i\psi_1 f_i^0 , \\ \psi^{(2)}(e_i) &= \psi_i^{(2)} f_i^0 := -(i-1)\psi_0 f_i^0 . \end{aligned}$$

The choice of signs and the generating coefficients is not important. A straightforward verification shows that both cochains above are cocycles. Moreover, they are not coboundaries. Actually, as mentioned before, all the degree zero coboundaries are trivial for $\lambda = 0$, while ψ_0 and ψ_1 need not be trivial. Similarly, their difference needs not be zero, hence it cannot be a coboundary so that $\psi^{(1)}$ and $\psi^{(2)}$ are not equivalent. Also, the two cocycles are clearly linearly independent. Hence, $\mathrm{H}^1(\mathscr{W},\mathscr{F}^0)$ is generated by two non-trivial cocycles, meaning that the dimension of $\mathrm{H}^1(\mathscr{W},\mathscr{F}^1)$ is at least two. Together with the first part of the proof, we obtain that the dimension is exactly two. □

The next theorem concerns the critical value $\lambda = 1$.

Theorem 4.1.3. *The first algebraic cohomology of the Witt algebra $\mathscr{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathscr{F}^\lambda$ with $\lambda = 1$ has a dimension of one, i.e.*

$$\dim \mathrm{H}^1(\mathscr{W},\mathscr{F}^1) = 1\,. \tag{4.8}$$

Proof. As before, we will start by determining an upper bound for the dimension, meaning we start by showing $\boxed{\dim \mathrm{H}^1(\mathscr{W},\mathscr{F}^1) \le 1}$.
Let $\psi \in \mathrm{H}^1(\mathscr{W},\mathscr{F}^1)$. By choosing a degree zero zero-cochain ϕ equal to $\phi = -\psi_1 f_0^1$, we can normalize our 1-cocycle ψ to $\psi_1 = 0$ by a cohomological change. The cocycle condition (4.1) for $\lambda = 1$ is given by:

$$(j-i)\psi_{i+j} - (i+j)\psi_j + (i+j)\psi_i = 0\,.$$

Putting $j = 1$ gives us the recurrence relations:

$$-(1+i)\cancel{\psi_1} + (1+i)\psi_i + (1-i)\psi_{i+1} = 0$$
$$\Leftrightarrow \psi_i = \frac{i-1}{i+1}\psi_{i+1} \text{ for } i \text{ decreasing} \tag{4.9}$$
$$\Leftrightarrow \psi_{i+1} = \frac{i+1}{i-1}\psi_i \text{ for } i \text{ increasing}. \tag{4.10}$$

For $i = 0$ the first recurrence relation (4.9) immediately gives $\psi_0 = 0$. Starting with $i = -2$ and i decreasing in the first relation, we see that ψ_{-1} generates all ψ_i $i \le -2$. Starting with $i = 2$ and increasing i in the second recurrence relation (4.10), we obtain that ψ_2 generates all ψ_i $i \ge 3$. Thus, there are two generating coefficients a priori. However, the cocycle condition for $(i,j) = (-1,2)$ yields:

$$\psi_{-1} - \psi_2 + 3\cancel{\psi_1} = 0 \Leftrightarrow \psi_{-1} = \psi_2\,.$$

Hence, there is only one generating coefficient and the dimension of the first cohomology is at most one.

Next, we will show that the dimension is at least one, $\boxed{\dim \mathrm{H}^1(\mathscr{W},\mathscr{F}^1) \ge 1}$. The second recurrence relation (4.10) can be rewritten as follows, for $i \ge 3$:

$$\psi_i = \frac{i}{i-2}\psi_{i-1} = \frac{1}{2}\frac{i!}{(i-2)!}\psi_2 = \frac{1}{2}i(i-1)\psi_2\,.$$

The first recurrence relation (4.9) for negative i can be rewritten as follows, for $i \le -2$:

$$\psi_i = \frac{i-1}{i+1}\psi_{i+1} = \frac{1}{2}\frac{(-i+1)!}{(-i-1)!}\psi_2 = \frac{1}{2}(-i+1)(-i)\psi_2 = \frac{1}{2}i(i-1)\psi_2\,.$$

Putting $i = 0, 1, 2, -1$ into the formula $\psi_i = \frac{1}{2}i(i-1)\psi_2$, we obtain $\psi_0 = 0$, $\psi_1 = 0$, $\psi_2 = \psi_2$ and $\psi_{-1} = \psi_2$, respectively. These values are all compatible with our previous results, hence the formula $\psi_i = \frac{1}{2}i(i-1)\psi_2$ is valid for all i. Next, we can insert this expression into the cocycle condition for generic i and j, and direct verification shows that the cocycle condition is satisfied. Hence, we showed above that the cochain with coefficients $\psi_i = \frac{1}{2}i(i-1)\psi_2$ is a generating cocycle. We can check explicitly that it is non-trivial, i.e. that it is no coboundary. Suppose the contrary, i.e. suppose $\psi_i = \frac{1}{2}i(i-1)\psi_2$ is a coboundary. Then it would fulfill (4.2) for $\lambda = 1$ for some 0-cochain ϕ_0, $\psi_i = -\phi_0 i \Leftrightarrow \frac{1}{2}i(i-1)\psi_2 = -\phi_0 i$. For $i = 1$, the relation implies $\phi_0 = 0$. For $i = 2$, the relation implies $\phi_0 = -\frac{1}{2}\psi_2$. Since ψ_2 is not necessary zero, we obtain that the cocycle $\psi_i = \frac{1}{2}i(i-1)\psi_2$ cannot be a coboundary. Hence, we found an explicit, non-trivial, generating cocycle of $\mathrm{H}^1(\mathscr{W}, \mathscr{F}^1)$, meaning that the dimension of $\mathrm{H}^1(\mathscr{W}, \mathscr{F}^1)$ is at least one. Together with the first part of the proof, we obtain that the dimension is exactly one. □

The procedure for $\lambda = 2$ is very similar to the one for $\lambda = 1$. We will give it anyway for reasons of completeness. The statement to prove is given below in Theorem 4.1.4.

Theorem 4.1.4. *The first algebraic cohomology of the Witt algebra $\mathscr{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathscr{F}^\lambda$ with $\lambda = 2$ has a dimension of one, i.e.*

$$\dim \mathrm{H}^1(\mathscr{W}, \mathscr{F}^2) = 1\,. \tag{4.11}$$

Proof. Once again, we will start by deriving an upper bound for the dimension, in order to obtain $\boxed{\dim \mathrm{H}^1(\mathscr{W}, \mathscr{F}^2) \leq 1}$. Let $\psi \in \mathrm{H}^1(\mathscr{W}, \mathscr{F}^2)$. By choosing a degree zero zero-cochain ϕ equal to $\phi = -\frac{1}{2}\psi_1 f_0^2$, we can normalize our 1-cocycle ψ to $\psi_1 = 0$. The cocycle condition (4.1) for $\lambda = 2$ is given by:

$$(j-i)\psi_{i+j} - (2i+j)\psi_j + (i+2j)\psi_i = 0\,.$$

Putting $j = 1$ gives us the recurrence relations:

$$-(1+2i)\cancel{\psi_1} + (2+i)\psi_i + (1-i)\psi_{i+1} = 0$$

$$\Leftrightarrow \psi_i = \frac{i-1}{i+2}\psi_{i+1} \text{ for } i \text{ decreasing} \tag{4.12}$$

$$\Leftrightarrow \psi_{i+1} = \frac{i+2}{i-1}\psi_i \text{ for } i \text{ increasing}. \tag{4.13}$$

For $i = 0$ and $i = -1$ the first recurrence relation (4.12) immediately gives $\psi_0 = 0$ and $\psi_{-1} = 0$, respectively. Starting with $i = -3$ and i decreasing in the first relation (4.12), we see that ψ_{-2} generates all ψ_i $i \leq -3$. Starting with $i = 2$ and increasing i in the second recurrence relation (4.13), we obtain that ψ_2 generates all ψ_i $i \geq 3$. Thus, there are two generating coefficients a priori. However, the cocycle condition for $(i, j) = (-2, 2)$ yields:

$$2\psi_{-2} + 2\psi_2 + 4\cancel{\psi_0} = 0 \Leftrightarrow \psi_{-2} = -\psi_2\,.$$

Hence, there is only one generating coefficient and the dimension of the first cohomology is at most one.

As before, we next prove that the dimension is at least one, $\boxed{\dim \mathrm{H}^1(\mathscr{W}, \mathscr{F}^2)) \geq 1}$. We start by solving the recurrence relations (4.12) and (4.13), the relation (4.13) yielding for $i \geq 3$,

$$\psi_i = \frac{i+1}{i-2}\psi_{i-1} = \frac{1}{6}\frac{(i+1)!}{(i-2)!}\psi_2 = \frac{1}{6}i(i+1)(i-1)\psi_2\,,$$

and (4.12) yields for $i \le -3$:

$$\psi_i = \frac{i-1}{i+2}\psi_{i+1} = \frac{1}{6}\frac{(-i+1)!}{(-i-2)!}\psi_2 = \frac{1}{6}i(i+1)(i-1)\psi_2 .$$

Putting $i = 0,1,2,-1,-2$ into the relation $\psi_i = \frac{1}{6}i(i+1)(i-1)\psi_2$, we obtain $\psi_0 = 0$, $\psi_1 = 0$, $\psi_2 = \psi_2$, $\psi_{-1} = 0$ and $\psi_{-2} = -\psi_2$. This is consistent with the results obtained before, thus the relation $\psi_i = \frac{1}{6}i(i+1)(i-1)\psi_2$ is valid for all i. Inserting this relation into the cocycle condition for generic i and j, we obtain that the cocycle condition is satisfied. Hence, we showed that the cochain with coefficients $\psi_i = \frac{1}{6}i(i+1)(i-1)\psi_2$ is a generating cocycle. We can verify explicitly that it is not a coboundary. Suppose it was a coboundary, then it would need to satisfy (4.2) for $\lambda = 2$ for some 0-cochain ϕ_0, $\psi_i = -2\phi_0 i \Leftrightarrow \frac{1}{6}i(i+1)(i-1)\psi_2 = -2\phi_0 i$. Inserting $i = 0$ implies $\phi_0 = 0$. Inserting $i = 2$ implies $-\frac{1}{4}\psi_2 = \phi_0$. Since ψ_2 is not necessarily zero, we obtain that ψ with coefficients ψ_i cannot be a coboundary. Hence, we obtained a non-trivial generating cocycle, meaning that the dimension is at least one. Together with the first part of the proof, we find that the dimension is exactly one. □

4.2 The second algebraic cohomology

As we focus on the degree zero cohomology of the Witt algebra, we can write a degree zero 2-cochain $\psi^\lambda \in C^2(\mathcal{W},\mathcal{F}^\lambda)$ and a degree zero 1-cochain $\phi^\lambda \in C^1(\mathcal{W},\mathcal{F}^\lambda)$ with values in $\mathcal{F}^\lambda$ as follows: $\psi^\lambda(e_i,e_j) = \psi^\lambda_{i,j} f^\lambda_{i+j}$ and $\phi^\lambda(e_i) = \phi^\lambda_i f^\lambda_i$, respectively, with the $\psi^\lambda_{i,j}, \phi^\lambda_i \in \mathbb{K}$ and $\psi^\lambda_{i,j} = -\psi^\lambda_{j,i}$, the $e_i \in \mathcal{W}$ and the $f^\lambda_i \in \mathcal{F}^\lambda$. In that case, the condition for ψ^λ to be a 2-cocycle expressed in terms of its coefficients is given by, see (2.51),

$$\begin{aligned}(\delta_2\psi^\lambda)(e_i,e_j,e_k) = (j-i)\psi^\lambda_{i+j,k} - (k-i)\psi^\lambda_{i+k,j} + (k-j)\psi^\lambda_{k+j,i} \\ - (j+k+\lambda i)\psi^\lambda_{j,k} + (i+k+\lambda j)\psi^\lambda_{i,k} - (i+j+\lambda k)\psi^\lambda_{i,j} = 0 .\end{aligned} \tag{4.14}$$

Similarly, the condition for the 2-cocycle ψ to be a coboundary expressed in terms of the coefficients $\psi^\lambda_{i,j}, \phi^\lambda_i$ is given by, see (2.50):

$$(\delta_1\phi^\lambda)(e_i,e_j) = \psi^\lambda_{i,j} = (j-i)\phi^\lambda_{i+j} - (j+\lambda i)\phi^\lambda_j + (i+\lambda j)\phi^\lambda_i . \tag{4.15}$$

In the following, we will drop the superscript λ in order to augment readability. In the proofs, we generally start by fixing one of the two indices in $\psi_{i,j}$, and we will refer to the fixed index as level.

As in the previous Section 4.1, we start by performing a general analysis for all $\lambda \in \mathbb{C}$ except for a few exceptional values of λ. In a second step, we analyze these exceptional values of λ.

4.2.1 General analysis

The aim of this section is to prove Theorem 4.2.1 below.

Theorem 4.2.1. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda \in \mathbb{C} \setminus \{0,1,2,5,7\}$ vanishes, i.e.*

$$\forall \lambda \in \mathbb{C} \setminus \{0,1,2,5,7\} : \mathrm{H}^2(\mathcal{W},\mathcal{F}^\lambda) = \{0\} .$$

The proof is done in six steps. The first step consists in finding a cohomological change such that level plus one vanishes. Subsequently, the cocycle conditions are used to show that level zero, level minus one, level plus two and level minus two vanish. In the final step, induction is used to show that the 2-cocycles identically vanish.
The proofs consist of two cases. The first case corresponds to $\lambda \in \mathbb{C} \setminus \mathbb{N}$ and the proof of that case is similar to the proof of $\mathrm{H}^2(\mathscr{W},\mathscr{W}) = \{0\}$ given in [106]. The second case concentrates on λ being a positive integer, except for $\lambda \in \{0,1,2,5,7\}$. The proofs are more complicated in the second case, because for positive integer $\lambda \in \mathbb{N} \setminus \{0,1,2,5,7\}$, poles appear in the recurrence relations, creating gaps in the recurrence relations. The coefficients corresponding to gaps in the recurrence relations have to be annihilated in a rather roundabout way.

Lemma 4.2.1. *Every 2-cocycle of degree zero is cohomologous to a degree zero 2-cocycle ψ' with:*

$$\text{For } \lambda \in \mathbb{C} \setminus \mathbb{N}: \; \psi'_{i,1} = 0 \;\forall\; i \in \mathbb{Z} \text{ and } \psi'_{-1,2} = 0.$$
$$\text{For } \lambda \in \mathbb{N} \setminus \{0,1,2\}: \; \psi'_{i,1} = 0 \;\forall\; i \in \mathbb{Z} \setminus \{-\lambda\} \text{ and } \psi'_{-\lambda+1,-1} = 0 \text{ and } \psi'_{-1,2} = 0.$$

Proof. Let ψ be a 2-cocycle and let ϕ be the 1-cochain used to perform the cohomological change $\psi' = \psi - (\delta_1 \phi)$. We start by defining $\boxed{\phi_1 = 0}$ in order to simplify notation. Actually, the structure of the coboundary condition is such that ϕ_1 cannot be consistently used to annihilate some coefficient $\psi_{i,j}$ of ψ. The coboundary condition (4.15) on $(i,1)$ suggests to define the remaining coefficients of ϕ in the following way,

$$\begin{aligned} \psi_{i,1} &= -(1+i\lambda)\cancel{\phi_1} + (i+\lambda)\phi_i + (1-i)\phi_{1+i} \\ \Leftrightarrow \phi_{1+i} &= \frac{(i+\lambda)}{i-1}\phi_i - \frac{\psi_{i,1}}{i-1} \text{ for } i \text{ increasing} \qquad (4.16)\\ \Leftrightarrow \phi_i &= \frac{(i-1)}{(i+\lambda)}\phi_{1+i} + \frac{\psi_{i,1}}{(i+\lambda)} \text{ for } i \text{ decreasing}. \qquad (4.17) \end{aligned}$$

$\boxed{\textbf{Case 1}: \lambda \in \mathbb{C} \setminus \mathbb{N}}$
From relation (4.17), starting with $i = 0$ (recall $\lambda \notin \mathbb{N}$) and i decreasing, we immediately obtain a definition for $\phi_i \forall\; i \le 0$ and so $\psi'_{i,1} = 0 \;\forall\; i \le 0$ after the cohomological change. In fact, since $\lambda \notin \mathbb{N}$ and $i \le 0$, the recurrence relation (4.17) is valid for all $i \le 0$ since no zeros appear in the denominator. Next, let us write down the coboundary condition for $(-1,2)$, which suggests the following definition for ϕ_2:

$$\begin{aligned} (-1+2\lambda)\phi_{-1} + 3\cancel{\phi_1} - (2-\lambda)\phi_2 &= \psi_{-1,2} \\ \Leftrightarrow \phi_2 &= \frac{(-1+2\lambda)}{(2-\lambda)}\phi_{-1} - \frac{\psi_{-1,2}}{(2-\lambda)}. \qquad (4.18) \end{aligned}$$

This expression for ϕ_2 is well-defined, since ϕ_{-1} has been defined in the previous step and $\lambda \neq 2$. So we obtain $\psi'_{-1,2} = 0$ after the cohomological change. Starting with $i = 2$ in (4.16) and inserting ϕ_2, increasing i, we obtain a definition for $\phi_i \;\forall\; i \ge 3$ and $\psi'_{i,1} = 0 \;\forall\; i \ge 2$. Due to the alternating property, we have $\psi'_{1,1} = 0$.
All in all, we obtain $\psi'_{i,1} = 0 \;\forall\; i \in \mathbb{Z}$.

$\boxed{\textbf{Case 2}: \lambda \in \mathbb{N} \setminus \{0,1,2\}}$
Inserting $i = 0$ into (4.17), we obtain a definition for ϕ_0, since we assumed $\lambda \neq 0$ and by definition $\phi_1 = 0$. Thus, $\psi'_{0,1} = 0$ after the cohomological change. Inserting $i = -1$ into (4.17), we obtain ϕ_{-1}, since we assumed $\lambda \neq 1$. Thus, $\psi'_{-1,1} = 0$. Since we assume that $\lambda \neq 2$, we obtain

from (4.18) ϕ_2 and thus $\psi'_{-1,2} = 0$. Starting with $i = 2$ in (4.16) and inserting ϕ_2, we obtain $\phi_i\ \forall\ i \geq 3$ and $\psi'_{i,1} = 0\ \forall\ i \geq 2$. All in all, $\phi_i\ \forall\ i \geq -1$ are defined and we have $\psi'_{i,1} = 0\ \forall\ i \geq -1$, as well as $\psi'_{-1,2} = 0$.
From (4.17), we see that all the coefficients ϕ_i are defined for i negative from $i = -1$ down to $i = -\lambda + 1$ included, so we have $\psi'_{i,1} = 0\ ,\ -\lambda + 1 \leq i\ \leq 0$. At $i = -\lambda$, we obtain a pole.
Let us write down the coboundary condition (4.15) for $(-\lambda+1,-1)$, which suggests the following definition for $\phi_{-\lambda}$:

$$\begin{aligned}
&-(-1+(1-\lambda)\lambda)\phi_{-1} + (1-2\lambda)\phi_{1-\lambda} + (-2+\lambda)\phi_{-\lambda} = \psi_{-\lambda+1,-1}\\
&\Leftrightarrow \phi_{-\lambda} = \frac{(-1+(1-\lambda)\lambda)}{(-2+\lambda)}\phi_{-1} - \frac{(1-2\lambda)}{(-2+\lambda)}\phi_{1-\lambda} + \frac{\psi_{-\lambda+1,-1}}{(-2+\lambda)}\,.
\end{aligned}$$

Since ϕ_{-1} and $\phi_{-\lambda+1}$ are already defined, and since by assumption $\lambda \neq 2$, we obtain a definition for $\phi_{-\lambda}$, and also $\psi'_{-\lambda+1,-1} = 0$.
Starting with $i = -\lambda-1$ in (4.17), decreasing i, we obtain $\phi_i\ \forall\ i \leq -\lambda-1$ and $\psi'_{i,1} = 0\ \forall\ i \leq -\lambda-1$. This concludes the proof. □

Lemma 4.2.2. *Let ψ be a 2-cocycle such that:*

For $\lambda \in \mathbb{C}\setminus\mathbb{N}$: $\psi_{i,1} = 0\ \forall\ i \in \mathbb{Z}$ and $\psi_{-1,2} = 0$.
For $\lambda \in \mathbb{N}\setminus\{0,1,2\}$: $\psi_{i,1} = 0\ \forall\ i \in \mathbb{Z}\setminus\{-\lambda\}$ and $\psi_{-\lambda+1,-1} = 0$ and $\psi_{-1,2} = 0$.

Then:

$$\forall \lambda \in \mathbb{C}\setminus\{0,1,2\}: \qquad \psi_{i,0} = 0 \quad \forall\ i \in \mathbb{Z}.$$

Proof. The cocycle condition (4.14) for $(i,1,0)$ gives:

$$-(1+i\lambda)\underline{\psi_{1,0}} - \cancel{\psi_{1,i}} + (i+\lambda)\psi_{i,0} + i\cancel{\psi_{i,1}} - (1+i)\cancel{\psi_{i,1}} + (1-i)\psi_{1+i,0} = 0$$

$$\Leftrightarrow \psi_{1+i,0} = \frac{(i+\lambda)}{i-1}\psi_{i,0} \text{ for } i \text{ increasing} \tag{4.19}$$

$$\Leftrightarrow \psi_{i,0} = \frac{i-1}{(i+\lambda)}\psi_{1+i,0} \text{ for } i \text{ decreasing}. \tag{4.20}$$

The slashed terms cancel each other. The underlined term $\psi_{1,0}$ is zero by assumption, for both cases $\lambda \in \mathbb{C}\setminus\mathbb{N}$ and $\lambda \in \mathbb{N}\setminus\{0,1,2,5,7\}$. Indeed, in the latter case, $\psi_{1,0}$ does not correspond to the pathological $\psi_{-\lambda,1}$, as we have $\lambda \neq 0$ by assumption. Thus, the formulas above are valid for both cases.

Case 1: $\lambda \in \mathbb{C}\setminus\mathbb{N}$

Starting with $i = -1$ in (4.20), decreasing i, we immediately obtain $\psi_{i,0} = 0\ \forall\ i \leq -1$.
Next, let us write down the cocycle condition for $(0,-1,2)$:

$$-2\cancel{\psi_{-1,2}} - (-1+2\lambda)\cancel{\psi_{0,-1}} + (2-\lambda)\psi_{0,2} + 3\cancel{\psi_{1,0}} - 2\cancel{\psi_{2,-1}} = 0$$

$$\Leftrightarrow (2-\lambda)\psi_{0,2} = 0 \Leftrightarrow \psi_{0,2} = 0 \text{ as } \lambda \neq 2\,. \tag{4.21}$$

The slashed terms are zero either by assumption or the previous result $\psi_{i,0} = 0\ \forall\ i \leq -1$. Starting with $i = 2$ in (4.19) and inserting $\psi_{0,2} = 0$, increasing i, we obtain $\psi_{i,0} = 0\ \forall\ i \geq 3$. All in all, we obtain $\psi_{i,0} = 0\ \forall\ i \in \mathbb{Z}$.

Case 2: $\lambda \in \mathbb{N}\setminus\{0,1,2\}$

Inserting $i = -1$ into (4.20), we obtain $\psi_{-1,0} = 0$ since $\lambda \neq 1$ by assumption. In that case, all slashed terms in (4.21) are zero and we get $\psi_{0,2} = 0$ since $\lambda \neq 2$ by assumption. Starting with $i = 2$ in (4.19) and inserting $\psi_{0,2} = 0$, increasing i, we obtain $\psi_{i,0} = 0\ \forall\ i \geq 3$. All in all, we have $\psi_{i,0} = 0\ \forall\ i \geq -1$.

From (4.20), starting with $i = -1$, decreasing i, we obtain $\psi_{i,0} = 0\ , -\lambda+1 \leq i \leq 0$. At $i = -\lambda$, we have a pole. The cocycle condition (4.14) for $(-\lambda, \lambda, 0)$ yields:

$$\begin{aligned}&(-\lambda+\lambda^2)\psi_{-\lambda,0} - \lambda\cancel{\psi_{-\lambda,\lambda}} - (\lambda-\lambda^2)\underline{\psi_{\lambda,0}} - \lambda\cancel{\psi_{\lambda,-\lambda}} = 0\\ \Leftrightarrow\ &(-\lambda+\lambda^2)\psi_{-\lambda,0} = 0 \Leftrightarrow \psi_{-\lambda,0} = 0 \text{ as } \lambda \notin \{0,1\}\,.\end{aligned}$$

The slashed terms cancel each other, the underlined term is zero as we showed $\psi_{i,0} = 0\ \forall\ i \geq -1$ in the previous step. Continuing with $i = -\lambda - 1$ in (4.20) and inserting $\psi_{-\lambda,0} = 0$, decreasing i, we obtain $\psi_{i,0} = 0\ \forall\ i \leq -\lambda-1$. Therefore, we have $\psi_{i,0} = 0\ \forall\ i \leq 0$. This concludes the proof. □

Lemma 4.2.3. *Let ψ be a 2-cocycle such that:*

For $\lambda \in \mathbb{C}\setminus\mathbb{N}$: $\psi_{i,1} = \psi_{i,0} = 0\ \forall\ i \in \mathbb{Z}$ and $\psi_{-1,2} = 0$.
For $\lambda \in \mathbb{N}\setminus\{0,1,2\}$: $\psi_{i,1} = 0\ \forall\ i \in \mathbb{Z}\setminus\{-\lambda\}$ and $\psi_{i,0} = 0\ \forall\ i \in \mathbb{Z}$ and $\psi_{-\lambda+1,-1} = 0$ and $\psi_{-1,2} = 0$.

Then:

$$\forall \lambda \in \mathbb{C}\setminus\{0,1,2\}: \qquad \psi_{i,-1} = 0\ \forall\ i \in \mathbb{Z} \text{ and } \psi_{i,1} = 0\ \forall\ i \in \mathbb{Z}.$$

Proof. **Case 1**: $\lambda \in \mathbb{C}\setminus\mathbb{N}$

The cocycle condition (4.14) for $(i, 1, -1)$ yields:

$$-2\underline{\psi_{0,i}} - i\lambda\underline{\psi_{1,-1}} - (-1-i)\underline{\psi_{-1+i,1}} \tag{4.22}$$
$$+(-1+i+\lambda)\psi_{i,-1} - (1+i-\lambda)\underline{\psi_{i,1}} + (1-i)\psi_{1+i,-1} = 0 \tag{4.23}$$
$$\Leftrightarrow \psi_{1+i,-1} = \frac{(-1+i+\lambda)}{i-1}\psi_{i,-1} \text{ for } i \text{ increasing} \tag{4.24}$$
$$\Leftrightarrow \psi_{i,-1} = \frac{i-1}{(-1+i+\lambda)}\psi_{1+i,-1} \text{ for } i \text{ decreasing}. \tag{4.25}$$

The underlined terms are of level zero and level plus one, which are zero $\forall\ i$ for $\lambda \in \mathbb{C}\setminus\mathbb{N}$. Starting with $i = 0$ in (4.25), decreasing i, we immediately obtain $\psi_{i,-1} = 0\ \forall\ i \leq 0$. Starting with $i = 2$ in (4.24), increasing i, we obtain $\psi_{i,-1} = 0\ \forall\ i \geq 3$. All in all, we have $\psi_{i,-1} = 0\ \forall\ i \in \mathbb{Z}$.

Case 2: $\lambda \in \mathbb{N}\setminus\{0,1,2\}$

For $i \geq 0$, the underlined terms in (4.22)-(4.23) also disappear for $\lambda \in \mathbb{N}\setminus\{0,1,2,5,7\}$, as the pathological case $\psi_{-\lambda,1}$ does not appear in $\psi_{i,1}$, $i \geq 0$ as we have $\lambda \in \mathbb{N}\setminus\{0,1,2,5,7\}$. Also, $\psi_{-1,1}$ is zero since it does not correspond to the pathological case as $\lambda \neq 1$ by assumption. Therefore, the recurrence relations (4.24) and (4.25) are valid also for the case $\lambda \in \mathbb{N}\setminus\{0,1,2,5,7\}$.

Starting with $i = 2$ in (4.24), increasing i, we obtain $\psi_{i,-1} = 0\ \forall\ i \geq 3$. All in all, together with the assumptions, this gives $\psi_{i,-1} = 0\ \forall\ i \geq 0$.

Next, consider i lying in $-\lambda+2 \leq i \leq 0$. In that case, the underlined terms in (4.22)-(4.23) vanish, as they are of level zero and of level plus one and the pathological case $\psi_{-\lambda,1}$ is excluded. Starting with $i = 0$ in (4.25), decreasing i, we obtain $\psi_{i,-1} = 0$ for $-\lambda+2 \leq i \leq 0$. Next, let us put $i = -\lambda+1$ in (4.22)-(4.23), which gives:

$$\begin{aligned}&-2\underline{\psi_{0,-\lambda+1}} - (-\lambda+1)\lambda\underline{\psi_{1,-1}} - (\lambda-2)\psi_{-\lambda,1} - (-2\lambda+2)\underline{\psi_{-\lambda+1,1}} + \lambda\underline{\underline{\psi_{-\lambda+2,-1}}} = 0\\ \Leftrightarrow\ &(\lambda-2)\psi_{-\lambda,1} = 0 \Leftrightarrow \psi_{-\lambda,1} = 0 \text{ as } \lambda \neq 2\,.\end{aligned}$$

The terms underlined once disappear as they are of level zero or plus one and they do not correspond to the pathological $\psi_{-\lambda,1}$. The term underlined twice is zero due to the previous step, i.e. $\psi_{i,-1} = 0$ for $-\lambda + 2 \le i \le 0$. Therefore, $\psi_{-\lambda,1} = 0$, the underlined terms in (4.22)-(4.23) are zero for all i and the relation (4.25) is valid for all i. Continuing with $i = -\lambda$ in (4.25), recalling that we have $\psi_{-\lambda+1,-1} = 0$ by assumption, decreasing i, we obtain $\psi_{i,-1} = 0 \ \forall \ i \le -\lambda$. All in all, we have $\psi_{i,-1} = 0 \ \forall \ i \le 0$. This concludes the proof. □

Lemma 4.2.4. *Let ψ be a 2-cocycle such that:*

$$\forall \lambda \in \mathbb{C} \setminus \{0,1,2\}: \qquad \psi_{i,1} = \psi_{i,0} = \psi_{i,-1} = 0 \ \forall \ i \in \mathbb{Z}.$$

Then:

$$\forall \lambda \in \mathbb{C} \setminus \{0,1,2,5,7\}: \qquad \psi_{i,j} = 0 \ \forall \ i, j \in \mathbb{Z}.$$

Proof. We will start by proving $\boxed{\psi_{i,2} = \psi_{i,-2} = 0 \ \forall \ i \in \mathbb{Z}}$.
The cocycle condition (4.14) for $(i,-2,1)$ yields:

$$\begin{aligned}
&-(-1+i\lambda)\cancel{\psi_{-2,1}} + 3\cancel{\psi_{-1,i}} + (-2-i)\cancel{\psi_{-2+i,1}} \\
&-(-2+i+\lambda)\psi_{i,-2} + (1+i-2\lambda)\cancel{\psi_{i,1}} - (1-i)\psi_{1+i,-2} = 0 \\
&\Leftrightarrow \psi_{i,-2} = \frac{(i-1)}{(-2+i+\lambda)}\psi_{1+i,-2} \text{ for } i \text{ decreasing} && (4.26)\\
&\Leftrightarrow \psi_{1+i,-2} = \frac{(-2+i+\lambda)}{(i-1)}\psi_{i,-2} \text{ for } i \text{ increasing}. && (4.27)
\end{aligned}$$

The cocycle condition for $(i,2,-1)$ yields:

$$\begin{aligned}
&-3\cancel{\psi_{1,i}} - (1+i\lambda)\cancel{\psi_{2,-1}} - (-1-i)\psi_{-1+i,2} \\
&+(-1+i+2\lambda)\cancel{\psi_{i,-1}} - (2+i-\lambda)\psi_{i,2} + (2-i)\cancel{\psi_{2+i,-1}} = 0 \\
&\Leftrightarrow \psi_{i,2} = \frac{(1+i)}{(2+i-\lambda)}\psi_{-1+i,2} \text{ for } i \text{ increasing} && (4.28)\\
&\Leftrightarrow \psi_{-1+i,2} = \frac{(2+i-\lambda)}{(1+i)}\psi_{i,2} \text{ for } i \text{ decreasing}. && (4.29)
\end{aligned}$$

Case 1: $\lambda \in \mathbb{C} \setminus \mathbb{N}$
Starting with $i = 0$ in (4.26), decreasing i, we immediately obtain $\psi_{i,-2} = 0 \ \forall \ i \le 0$. Starting with $i = 0$ in (4.28), increasing i, we immediately obtain $\psi_{i,2} = 0 \ \forall \ i \ge 0$.
From (4.27), we obtain the following:
For $i = 2$: $\psi_{3,-2} = \lambda\psi_{2,-2}$
For $i = 3$: $\psi_{4,-2} = \frac{(\lambda+1)}{2}\psi_{3,-2} = \frac{(\lambda+1)\lambda}{2}\psi_{2,-2}$
For $i = 4$: $\psi_{5,-2} = \frac{(\lambda+2)}{3}\psi_{4,-2} = \frac{(\lambda+2)(\lambda+1)\lambda}{6}\psi_{2,-2}$
Inserting these values into the cocycle condition for $(2,-2,3)$ yields:

$$\begin{aligned}
&-(1+2\lambda)\psi_{-2,3} - 4\cancel{\psi_{0,3}} + 5\cancel{\psi_{1,2}} - 3\lambda\psi_{2,-2} + (5-2\lambda)\cancel{\psi_{2,3}} - \psi_{5,-2} = 0 && (4.30)\\
&\Leftrightarrow (1+2\lambda)\lambda\psi_{2,-2} - 3\lambda\psi_{2,-2} - \frac{(\lambda+2)(\lambda+1)\lambda}{6}\psi_{2,-2} = 0 \\
&\Leftrightarrow (-7+\lambda)(-2+\lambda)\lambda\psi_{2,-2} = 0 \Leftrightarrow \psi_{2,-2} = 0 \text{ as } \lambda \notin \{0,2,7\}. && (4.31)
\end{aligned}$$

The slashed terms are zero either by assumption or by the previous result $\psi_{i,2} = 0 \ \forall \ i \ge 0$. Starting with $i = -2$ in (4.29) and inserting $\psi_{2,-2} = 0$, decreasing i, we obtain $\psi_{i,2} = 0 \ \forall \ i \le -3$.

Starting with $i = 2$ in (4.27) and inserting $\psi_{2,-2} = 0$, we obtain $\psi_{i,-2} = 0 \ \forall \ i \geq 3$. All in all, we have $\psi_{i,2} = \psi_{i,-2} = 0 \ \forall \ i \in \mathbb{Z}$.

Case 2: $\lambda \in \mathbb{N} \setminus \{0,1,2,5,7\}$

Starting with $i = 0$ in (4.26), we obtain $\psi_{i,-2} = 0$ for $-\lambda+3 \leq i \leq 0$. At $i = -\lambda+2$, we have a pole. Starting with $i = 0$ in (4.28), we obtain $\psi_{i,2} = 0$ for $0 \leq i \leq \lambda-3$. At $i = \lambda-2$, we have a pole. In particular, inserting $i = 3$ in (4.28) yields $\psi_{3,2} = 0$ since $\lambda \neq 5$ by assumption. The latter implies that the slashed terms in Equation (4.30) are also zero for the case $\lambda \in \mathbb{N} \setminus \{0,1,2,5,7\}$. Using (4.27), we can again express the coefficients $\psi_{3,-2}$ and $\psi_{5,-2}$ in terms of the coefficient $\psi_{2,-2}$ as we did before, and insert them into (4.30) to obtain (4.31).
Starting with $i = -2$ in (4.29) and inserting $\psi_{2,-2} = 0$, decreasing i, we obtain $\psi_{i,2} = 0 \ \forall \ i \leq -3$. Together with the results of level minus one and zero, we have $\psi_{i,2} = 0 \ \forall \ i \leq 0$.
Starting with $i = 2$ in (4.27) and inserting $\psi_{2,-2} = 0$, we obtain $\psi_{i,-2} = 0 \ \forall \ i \geq 3$. Together with the results of level plus one and zero, we have $\psi_{i,-2} = 0 \ \forall \ i \geq 0$.
Next, let us write down the cocycle condition for $(-\lambda+2,-2,2)$, which yields:

$$-(2-\lambda)\lambda\cancel{\psi_{-2,2}} + 4\cancel{\psi_{0,2-\lambda}} - \lambda\psi_{2-\lambda,-2} + (4-3\lambda)\underline{\psi_{2-\lambda,2}} - \lambda\underline{\underline{\psi_{4-\lambda,-2}}} + (-4+\lambda)\underline{\psi_{-\lambda,2}} = 0$$

$$\Leftrightarrow \lambda\psi_{2-\lambda,-2} = 0 \Leftrightarrow \psi_{2-\lambda,-2} = 0 \text{ as } \lambda \neq 0\,.$$

The slashed terms are zero by assumption and because $\phi_{-2,2} = 0$ as proven earlier. The terms underlined once are zero since we already showed that $\psi_{i,2} = 0 \ \forall \ i \leq 0$ (Recall that $\lambda \notin \{0,1,2\}$). The term underlined twice is zero because we already argued that $\psi_{i,-2} = 0$ for $-\lambda+3 \leq i \leq 0$, and we also have $\psi_{1,-2} = 0$ if $\lambda = 3$.
Starting from $i = -\lambda+1$ in (4.26) and inserting $\psi_{2-\lambda,-2} = 0$, decreasing i, we obtain $\psi_{i,-2} = 0 \ \forall \ i \leq -\lambda+1$. We thus obtain $\psi_{i,-2} = 0 \ \forall \ i \leq 0$ and all together we have $\psi_{i,-2} = 0 \ \forall \ i \in \mathbb{Z}$.
Finally, let us write down the cocycle condition for $(\lambda-2,-2,2)$, which gives:

$$-(-2+\lambda)\lambda\cancel{\psi_{-2,2}} + 4\cancel{\psi_{0,-2+\lambda}} - \lambda\underline{\underline{\psi_{-4+\lambda,2}}} - (-4+3\lambda)\underline{\psi_{-2+\lambda,-2}} - \lambda\psi_{-2+\lambda,2} - (4-\lambda)\underline{\psi_{\lambda,-2}} = 0$$

$$\Leftrightarrow \lambda\psi_{-2+\lambda,2} = 0 \Leftrightarrow \psi_{-2+\lambda,2} = 0 \text{ as } \lambda \neq 0\,.$$

The slashed terms are zero by assumption and because $\phi_{-2,2} = 0$ as proven earlier. The terms underlined once are zero as they are of level minus two, and we showed $\psi_{i,-2} = 0 \ \forall \ i \in \mathbb{Z}$ in the previous step. The term underlined twice is zero as we showed before that $\psi_{i,2} = 0$ for $0 \leq i \leq \lambda-3$, and we also have $\psi_{-1,2} = 0$ if $\lambda = 3$.
Starting from $i = \lambda-1$ in (4.28) and inserting $\psi_{-2+\lambda,2} = 0$, increasing i, we obtain $\psi_{i,2} = 0 \ \forall \ i \geq \lambda-1$. We thus obtain $\psi_{i,2} = 0 \ \forall \ i \geq 0$ and all together we have $\psi_{i,2} = 0 \ \forall \ i \in \mathbb{Z}$.

The final step consists in proving $\boxed{\psi_{i,j} = 0 \ \forall \ i,j \in \mathbb{Z}}$.
This can be proved for both cases of λ simultaneously. In order to do this, we use induction. The result is valid for the levels zero, plus one and plus two. We suppose it is valid for level $k \geq 2$ and check whether it remains valid for level $k+1$. The cocycle condition for $(i,1,k)$ produces the following:

$$\begin{aligned} &-(1+k+i\lambda)\cancel{\psi_{1,k}} - (1+i+k\lambda)\cancel{\psi_{i,1}} + (i+k+\lambda)\underline{\psi_{i,k}} \\ &+ (1-i)\underline{\psi_{1+i,k}} + (-1+k)\psi_{1+k,i} - (-i+k)\cancel{\psi_{i+k,1}} = 0 \\ &\Leftrightarrow (-1+k)\psi_{1+k,i} = 0 \Leftrightarrow \psi_{1+k,i} = 0 \text{ as } k \geq 2\,. \end{aligned}$$

The underlined terms are of level k and are zero by induction. The statement remains true for level $k+1$.

We also know that the statement is true for the levels zero, minus one and minus two. Let us suppose that the statement is valid for level $k \leq -2$ and check that it remains true for level $k-1$. The cocycle condition for $(i,-1,k)$ yields:

$$\begin{aligned}
&-(-1+k+i\lambda)\cancel{\psi_{-1,k}} + (-1-i)\underline{\psi_{-1+i,k}} - (-1+i+k\lambda)\cancel{\psi_{i,-1}} \\
&+(i+k-\lambda)\underline{\psi_{i,k}} + (1+k)\psi_{-1+k,i} - (-i+k)\cancel{\psi_{i+k,-1}} = 0 \\
&\Leftrightarrow (1+k)\psi_{-1+k,i} = 0 \Leftrightarrow \psi_{-1+k,i} = 0 \text{ as } k \leq -2\,.
\end{aligned}$$

The underlined terms are of level k and are zero by induction. The statement remains true for level $k-1$. This concludes the proof. □

Proof of Theorem 4.2.1. We use Lemma 4.2.1 to perform a cohomological change such that the assumptions of Lemma 4.2.2 are fulfilled. By Lemma 4.2.2, we obtain results such that the assumptions of Lemma 4.2.3 are fulfilled. Lemma 4.2.3 then gives the results necessary to fulfill the assumptions of Lemma 4.2.4, which allows to prove Theorem 4.2.1 immediately. □

4.2.2 Exceptional cases

In this section, we will analyze the dimension of the second cohomology for the critical values of λ. We will proceed with the proofs in two steps, similarly to the proofs we gave in the previous section for the first algebraic cohomology. We first derive an upper bound for the dimension, then a lower bound. Unfortunately, the methods used to determine the lower bound of the dimension for the first cohomology do not work so well for the second cohomology. In fact, for $\lambda = 5$ and $\lambda = 7$, the recurrence relations can be solved and provide generating cocycles for the second cohomology. However, for $\lambda \in \{0,1,2\}$, the recurrence relations are hard to solve explicitly. An easier way to proceed consists in making guesses for candidate generators, based on the results of continuous cohomology for $\mathrm{H}^2_{\mathrm{cont}}(Vect(S^1),\mathcal{F}^\lambda)$, and subsequently in proving that the candidate generators are indeed generating cocycles of $\mathrm{H}^2(\mathcal{W},\mathcal{F}^\lambda)$ with $\lambda \in \{0,1,2,5,7\}$. This is the same strategy as the one used in the proof of Theorem 3.1.1, see the Remark 3.1.1. Explicit expressions for the generating cocycles of the continuous cohomology $\mathrm{H}^2_{\mathrm{cont}}(Vect(S^1),\mathcal{F}^\lambda)$ on the circle for $\lambda \in \{0,1,2,5,7\}$ can be found in [90].

Results for $\lambda = 0$

We will start with the pathological value $\lambda = 0$. The statement to prove is Theorem 4.2.2 below.

Theorem 4.2.2. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda = 0$ has dimension two, i.e.*

$$\dim \mathrm{H}^2(\mathcal{W},\mathcal{F}^0) = 2\,.$$

We prove the Theorem 4.2.2 in two steps given by Propositions 4.2.1 and 4.2.2 below.

Proposition 4.2.1. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda = 0$ has maximally a dimension of two, i.e.*

$$\dim \mathrm{H}^2(\mathcal{W},\mathcal{F}^0) \leq 2\,.$$

Proof. We shall start by deriving the upper bound on the dimension, i.e. $\boxed{\dim \mathrm{H}^2(\mathscr{W},\mathscr{F}^0) \le 2}$. Let ψ be a degree-zero 2-cocycle of $\mathrm{H}^2(\mathscr{W},\mathscr{F}^0)$. The first step consists in performing a cohomological change $\psi' = \psi - \delta_1\phi$ with a 1-cochain ϕ in order to obtain the following condition on ψ', $\boxed{\psi'_{i,1} = 0\ \forall\ i \in \mathbb{Z}\setminus\{0\} \text{ and } \psi'_{-1,2} = 0}$. We start by defining $\phi_1 := 0$ and $\phi_0 := 0$ in order to simplify the notation. In fact, the structure of the coboundary condition for $\lambda = 0$ is such that these cannot be put consistently in a non-trivial relation with some coefficient $\psi_{i,j}$. The coboundary condition (4.15) for $\lambda = 0$ is as follows:

$$i\phi_i - j\phi_j + (j-i)\phi_{i+j} = \psi_{i,j}\,. \tag{4.32}$$

Putting $j = 1$, we obtain a recurrence relation we can use to define ϕ:

$$
\begin{aligned}
&-\cancel{\phi_1} + i\phi_i + (1-i)\phi_{i+1} = \psi_{i,1} \\
&\Leftrightarrow \phi_i := \frac{i-1}{i}\phi_{i+1} + \frac{1}{i}\psi_{i,1} \text{ for } i \text{ decreasing} && (4.33)\\
&\Leftrightarrow \phi_{i+1} := \frac{i}{i-1}\phi_i - \frac{1}{i-1}\psi_{i,1} \text{ for } i \text{ increasing}. && (4.34)
\end{aligned}
$$

Starting with $i = -1$ in the first recurrence relation (4.33), decreasing i, we obtain a definition for ϕ_i $i \le -1$ and we get $\psi'_{i,1} = 0\ \forall\ i \le -1$ after the cohomological change. Next, the coboundary condition (4.32) for $(i,j) = (2,-1)$ yields a definition for ϕ_2:

$$\phi_{-1} - 3\cancel{\phi_1} + 2\phi_2 = \psi_{2,-1} \Leftrightarrow \phi_2 := -\frac{1}{2}\phi_{-1} + \frac{1}{2}\psi_{2,-1}\,,$$

and we have $\psi'_{-1,2} = 0$. Starting with $i = 2$ in the second recurrence relation (4.34), increasing i, we obtain $\phi_i\ \forall\ i \ge 3$ and $\psi'_{i,1} = 0\ \forall\ i \ge 2$.
In the following, we will drop the prime and work with a 2-cocycle $\psi_{i,j}$ which has been cohomologically normalized to $\psi_{i,1} = 0\ \forall\ i \in \mathbb{Z}\setminus\{0\}$ and $\psi_{-1,2} = 0$.
As usual, we will start with level zero and count the generating coefficients necessary to generate $\boxed{\psi_{i,0}\ \forall\ i \in \mathbb{Z}}$. The cocycle condition (4.14) for $(i,j,k) = (i,0,1)$ produces the recurrence relations:

$$
\begin{aligned}
&-\psi_{0,1} + \cancel{\psi_{1,i}} - i\psi_{i,0} - i\cancel{\psi_{i,1}} + (1+i)\cancel{\psi_{i,1}} - (1-i)\psi_{1+i,0} = 0 \\
&\Leftrightarrow \psi_{i,0} = -\frac{\psi_{0,1}}{i} - \frac{(1-i)}{i}\psi_{1+i,0} \text{ for } i \text{ decreasing} && (4.35)\\
&\Leftrightarrow \psi_{1+i,0} = -\frac{i}{1-i}\psi_{i,0} - \frac{\psi_{0,1}}{1-i} \text{ for } i \text{ increasing}. && (4.36)
\end{aligned}
$$

The slashed terms cancel each other. Starting with $i = -1$ and i decreasing in the first recurrence relation (4.35), we see that $\psi_{0,1}$ generates all $\psi_{i,0}$ $i \le -1$. Starting with $i = 2$ and i increasing in the second recurrence relation (4.36), we obtain that $\psi_{0,1}$ and $\psi_{2,0}$ generate all $\psi_{i,0}$ $i \ge 3$. Writing down the cocycle condition for $(0,-1,2)$ provides a non-trivial relation between the two generating coefficients:

$$
\begin{aligned}
&-2\cancel{\psi_{-1,2}} + \psi_{0,-1} + 2\psi_{0,2} + 3\psi_{1,0} - 2\cancel{\psi_{2,-1}} = 0 \\
&\Leftrightarrow -\psi_{0,1} + 2\psi_{0,2} + 3\psi_{1,0} = 0 \Leftrightarrow \psi_{0,2} = -2\psi_{1,0}\,.
\end{aligned}
$$

Hence, $\psi_{i,0}\ \forall\ i \in \mathbb{Z}$ is generated by a single generating coefficient, $\psi_{1,0}$.
Next, we will focus on level minus one and count the generating coefficients of $\boxed{\psi_{i,-1}\ \forall\ i \in \mathbb{Z}}$.

The cocycle condition (4.14) on $(i,-1,1)$ for $\lambda = 0$ yields:

$$
\begin{aligned}
&2\psi_{0,i} + (-1-i)\psi_{-1+i,1} - (-1+i)\psi_{i,-1} + (1+i)\psi_{i,1} - (1-i)\psi_{1+i,-1} = 0 \\
&\Leftrightarrow \psi_{i,-1} = \frac{2}{i-1}\psi_{0,i} + \frac{(-1-i)}{i-1}\psi_{-1+i,1} + \frac{(1+i)}{i-1}\psi_{i,1} + \psi_{1+i,-1} \text{ for } i \text{ decreasing} \qquad (4.37) \\
&\Leftrightarrow \psi_{1+i,-1} = \frac{2}{1-i}\psi_{0,i} + \frac{(-1-i)}{1-i}\psi_{-1+i,1} + \psi_{i,-1} + \frac{(1+i)}{1-i}\psi_{i,1} \text{ for } i \text{ increasing}. \qquad (4.38)
\end{aligned}
$$

Starting with $i = 0$, i decreasing in the first recurrence relation (4.37), we immediately obtain $\psi_{i,-1}$ $i \le 0$ without a new generating coefficient appearing. Similarly, starting with $i = 2$ and i increasing in the second recurrence relation (4.38), recalling that $\psi_{-1,2} = 0$, we obtain $\psi_{i,-1}$ $i \ge 3$ without any new generating coefficient becoming apparent. Since $\psi_{-1,1} = 0$ due to the cohomological change, we see that $\psi_{1,0}$ is also sufficient to generate all $\psi_{i,-1}$ $\forall\ i \in \mathbb{Z}$.
Next, we will count the generating coefficients necessary to generate level two and level minus two, i.e. $\boxed{\psi_{i,2} \text{ and } \psi_{i,-2}\ \forall\ i \in \mathbb{Z}}$. The cocycle condition (4.14) for $(i,2,-1)$ gives:

$$
\begin{aligned}
&-3\psi_{1,i} - \cancel{\psi_{2,-1}} - (-1-i)\psi_{-1+i,2} + (-1+i)\psi_{i,-1} - (2+i)\psi_{i,2} + (2-i)\psi_{2+i,-1} = 0 \\
&\Leftrightarrow \psi_{-1+i,2} = -\frac{3}{-1-i}\psi_{1,i} + \frac{(-1+i)}{-1-i}\psi_{i,-1} - \frac{(2+i)}{-1-i}\psi_{i,2} + \frac{(2-i)}{-1-i}\psi_{2+i,-1} \text{ for } i \text{ decreasing} \qquad (4.39) \\
&\Leftrightarrow \psi_{i,2} = -\frac{3}{i+2}\psi_{1,i} + \frac{(-1+i)}{i+2}\psi_{i,-1} + \frac{(2-i)}{i+2}\psi_{2+i,-1} - \frac{(-1-i)}{i+2}\psi_{-1+i,2} \text{ for } i \text{ increasing}. \qquad (4.40)
\end{aligned}
$$

Starting with $i = 0$ and i increasing in the second recurrence relation (4.40), we obtain $\psi_{i,2}$ $i \ge 0$ without a new generating coefficient appearing. Starting with $i = -2$, i decreasing in the first recurrence relation (4.39), we obtain $\psi_{i,2}$ $i \le -3$ with no new generating coefficient appearing. However, the value $\psi_{2,-2}$ is missing and is not related to the generating coefficient $\psi_{1,0}$. Thus, $\psi_{2,-2}$ corresponds to a new generating coefficient. Similarly, the cocycle condition (4.14) $(i,-2,1)$ gives the recurrence relations for level minus two:

$$
\begin{aligned}
&\cancel{\psi_{-2,1}} + 3\psi_{-1,i} + (-2-i)\psi_{-2+i,1} - (-2+i)\psi_{i,-2} + (1+i)\psi_{i,1} - (1-i)\psi_{1+i,-2} = 0 \\
&\Leftrightarrow \psi_{i,-2} = \frac{3}{i-2}\psi_{-1,i} + \frac{(-2-i)}{i-2}\psi_{-2+i,1} + \frac{(1+i)}{i-2}\psi_{i,1} - \frac{(1-i)}{i-2}\psi_{1+i,-2} \text{ for } i \text{ decreasing} \qquad (4.41) \\
&\Leftrightarrow \psi_{1+i,-2} = \frac{3}{1-i}\psi_{-1,i} + \frac{(-2-i)}{1-i}\psi_{-2+i,1} - \frac{(-2+i)}{1-i}\psi_{i,-2} + \frac{(1+i)}{1-i}\psi_{i,1} \text{ for } i \text{ increasing}. \qquad (4.42)
\end{aligned}
$$

Starting with $i = 0$ in the first recurrence relation (4.41), i decreasing, we obtain $\psi_{i,-2}$ $i \le 0$ without any new generating coefficient appearing. Starting with $i = 2$ in the second recurrence relation (4.42), i increasing, we get $\psi_{i,-2}$ $i \ge 3$. Again, the value $\psi_{-2,2}$ is missing. Thus, the levels plus one, minus one, zero, plus two and minus two are generated by at most two generating coefficients. However, there might be non-trivial relations appearing between $\psi_{1,0}$ and $\psi_{2,-2}$ for high values of i, j and k.
In the last step, we can use induction on k. We want to show that $\psi_{i,j}$ is generated at most by $\psi_{1,0}$ and $\psi_{-2,2}$. The statement is true for levels zero, plus one and plus two. Thus, let us suppose the statement is true for level $k \ge 2$, and let us prove that it holds true for level $k+1$. The cocycle condition for $(i,1,k)$ gives:

$$
\begin{aligned}
&-(1+k)\psi_{1,k} - (1+i)\psi_{i,1} + (i+k)\psi_{i,k} + (1-i)\psi_{1+i,k} + (-1+k)\psi_{1+k,i} - (-i+k)\psi_{i+k,1} = 0 \\
&\Leftrightarrow (1-k)\psi_{1+k,i} = -(1+k)\psi_{1,k} - (1+i)\psi_{i,1} + (i+k)\psi_{i,k} + (1-i)\psi_{1+i,k} - (-i+k)\psi_{i+k,1}\,.
\end{aligned}
$$

The terms on the right are of level plus one or level k, which are generated by $\psi_{1,0}$ and $\psi_{-2,2}$ due to the induction hypothesis. We see that the term of level $k+1$ on the left can be expressed, for all i, in terms of level-k-terms without any new generating coefficient becoming apparent. Hence, level $k+1$ is also generated by at most two generating coefficients.
The same can be done for decreasing k. The statement is true for the levels zero, minus one and minus two. Let us suppose the statement is true for level $k \le -2$, and let us prove that it holds true for level $k-1$. The cocycle condition for $(i,-1,k)$ gives:

$$\begin{aligned} &-(-1+k)\psi_{-1,k} + (-1-i)\psi_{-1+i,k} - (-1+i)\psi_{i,-1} \\ &\quad +(i+k)\psi_{i,k} + (1+k)\psi_{-1+k,i} - (-i+k)\psi_{i+k,-1} = 0 \\ \Leftrightarrow &-(1+k)\psi_{-1+k,i} = -(-1+k)\psi_{-1,k} + (-1-i)\psi_{-1+i,k} - (-1+i)\psi_{i,-1} \\ &\qquad + (i+k)\psi_{i,k} - (-i+k)\psi_{i+k,-1}\,. \end{aligned}$$

A similar reasoning as before leads to the fact that also level $k-1$ is generated by at most two generating coefficients. Therefore, the dimension of the second cohomology for $\lambda = 0$ is at most two. □

Proposition 4.2.2. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda = 0$ has minimally a dimension of two, i.e.*

$$2 \le \dim \mathrm{H}^2(\mathcal{W}, \mathcal{F}^0))\ .$$

Proof. To prove that the dimension is at least two, we will give two explicit cochains which fulfill the cocycle conditions and which are not coboundaries.
The cocycles we found are inspired from continuous cohomology. Continuous cohomology yields two cocycles, one of them corresponds to the Virasoro 2-cocycle (2.20) with values in the trivial module $\mathbb{C} \subset \mathcal{F}^0$. Inspired from these cocycles, we can define algebraic cochains suited for our purpose:

$$\begin{aligned} \psi^{(1)}(e_i, e_j) &= \psi^{(1)}_{i,j}\, f_{i+j} := (j-i)\, f_{i+j}\,, \\ \psi^{(2)}(e_i, e_j) &= \psi^{(2)}_{i,j}\, f_{i+j} := (ij^2 - i^2 j)\delta_{i+j,0}\, f_{i+j}\,. \end{aligned}$$

Clearly, the expressions above are antisymmetric in i and j. We will start by proving that these two cochains are cocycles. The proof is immediate for the generator $\psi^{(1)}(e_i, e_j)$. It suffices to insert the expression for its coefficients $\psi^{(1)}_{i,j} = (j-i)$ into the cocycle condition (4.14) for $\lambda = 0$, and a direct computation shows that the cocycle condition is fulfilled. Thus, $\psi^{(1)}(e_i, e_j)$ is a cocycle. In case of the second generator $\psi^{(2)}(e_i, e_j)$, we need to consider several case differentiations because of the Kronecker Delta appearing in its expression.

Case 1: $i+j+k \neq 0$

In this case, the cocycle condition for $\lambda = 0$ reduces to:

$$-(i+j)\psi^{(2)}_{i,j} + (i+k)\psi^{(2)}_{i,k} - (j+k)\psi^{(2)}_{j,k} = 0\,, \tag{4.43}$$

where $\psi^{(2)}_{i,j} = (ij^2 - i^2 j)\delta_{i+j,0}$. Next, we can distinguish further three sub-cases according to whether zero, one, two or three terms in (4.43) remain:

- $j+k \neq 0$ and $i+k \neq 0$ and $i+j \neq 0$: In this case, all the coefficients $\psi^{(2)}_{i,j}$ appearing in the cocycle condition (4.43) above are zero due to the Kronecker Delta's appearing in the expression of the coefficients $\psi^{(2)}_{i,j}$, meaning that the cocycle condition is trivially satisfied.

- $j+k=0$ and $i+k\neq 0$ and $i+j\neq 0$: In this case, only the coefficient $\psi_{j,k}$ remains in the cocycle condition (4.43) given above:

$$-\underbrace{(j+k)}_{=0}\psi^{(2)}_{j,k}=0\,. \qquad \checkmark$$

The factor $(j+k)$ is zero by assumption, and hence the cocycle condition is fulfilled in this case.

- $j+k=0$ and $i+k=0$ and $i+j\neq 0$: The first condition implies $j=-k$ and the second one implies $i=-k$. There are two non-zero coefficients $\psi^{(2)}_{j,k}$ and $\psi^{(2)}_{i,k}$ in the cocycle condition (4.43), which becomes:

$$\begin{aligned}-(j+k)\psi^{(2)}_{j,k}+(i+k)\psi^{(2)}_{i,k} &= k(-i^3+j^3+(i-j)k^2)\\ &= k(-(-k)^3+(-k)^3+((-k)-(-k))k^2)=0\,. \qquad \checkmark\end{aligned}$$

- $j+k=0$ and $i+k=0$ and $i+j=0$: This case is not possible under the assumption $i+j+k\neq 0$. Indeed, summing the three conditions above yields $2(i+j+k)=0$, which is contradictory with our assumption.

Case 2: $i+j+k=0$

Once again, we have to consider several subcases:

- $j+k\neq 0$ and $i+k\neq 0$ and $i+j\neq 0$: In this case, the cocycle condition (4.14) for $\lambda=0$ reduces to:

$$\begin{aligned}&(j-i)\psi^{(2)}_{i+j,k}-(k-i)\psi^{(2)}_{i+k,j}+(k-j)\psi^{(2)}_{j+k,i}\\ &=-(i-j)(i-k)(j-k)\underbrace{(i+j+k)}_{=0}=0\,. \qquad \checkmark\end{aligned}$$

- $j+k=0$ and $i+k\neq 0$ and $i+j\neq 0$: In this case, the cocycle condition (4.14) contains four non-zero coefficients.

$$\begin{aligned}&(j-i)\psi^{(2)}_{i+j,k}-(k-i)\psi^{(2)}_{i+k,j}+(k-j)\psi^{(2)}_{j+k,i}-(j+k+)\psi^{(2)}_{j,k}\\ &=\underbrace{i}_{=0}(j^3-k^3+i^2(-j+k))=0\,. \qquad \checkmark\end{aligned}$$

The index i is zero because our assumptions $i+j+k=0$ and $j+k=0$ imply $i=0$.

- $j+k=0$ and $i+k=0$ and $i+j\neq 0$: This case is not possible because of our assumption $i+j+k=0$. In fact, the conditions $i+j+k=0$ and $j+k=0$ imply $i=0$, while the conditions $i+j+k=0$ and $i+k=0$ imply $j=0$, and hence $i+j=0$ which is incompatible with the assumption $i+j\neq 0$.

- $j+k=0$ and $i+k=0$ and $i+j=0$: In this case, all the coefficients in the cocycle condition (4.14) remain:

$$\begin{aligned}&-\underline{(i+j)}\psi^{(2)}_{i,j}+\underline{(i+k)}\psi^{(2)}_{i,k}-\underline{(j+k)}\psi^{(2)}_{j,k}\\ &+(-i+j)\psi^{(2)}_{i+j,k}-(-i+k)\psi^{(2)}_{i+k,j}+(-j+k)\psi^{(2)}_{j+k,i}\\ &=(-i+j)\left(-\underline{(i+j)}^2k+\underline{(i+j)}k^2\right)-(-i+k)\left(j^2\underline{(i+k)}-j\underline{(i+k)}^2\right)\\ &+(-j+k)\left(i^2\underline{(j+k)}-i\underline{(j+k)}^2\right)=0\,. \qquad \checkmark\end{aligned}$$

The underlined factors are zero by assumption.

We proved that our two generating cochains $\psi^{(1)}(e_i,e_j)$ and $\psi^{(2)}(e_i,e_j)$ are cocycles. Next, we have to prove that they are not coboundaries, and that they are not cohomologically equivalent.
Let us start with $\psi^{(1)}(e_i,e_j)$. The verification is immediate. On the one hand, the coboundary condition (4.32) for $\lambda=0$ evaluated on the generators e_1 and e_0 yields $\psi_{0,1}=0$. On the other hand, the generating cocycle $\psi^{(1)}$ evaluated on e_1 and e_0 yields in general a non-zero coefficient: $\psi^{(1)}_{0,1}=1-0=1\neq 0$. Consequently, the cocycle $\psi^{(1)}$ cannot be a coboundary.
Let us proceed with the second generating cocycle $\psi^{(2)}$. We consider the coboundary condition (4.32) evaluated on several combinations of the generators $e_{-3},e_{-2},e_{-1},e_0,e_1,e_2$ and we compare them to the corresponding coefficients $\psi^{(2)}_{i,j}$ of the generating cocycle:

$$
\begin{aligned}
\psi_{2,-2} &= 2\phi_{-2}-4\phi_0+2\phi_2 & \psi^{(2)}_{2,-2} &= 16\,,\\
\psi_{1,-1} &= \phi_{-1}-2\phi_0+\phi_1 & \psi^{(2)}_{1,-1} &= 2\,,\\
\psi_{1,-2} &= 2\phi_{-2}-3\phi_{-1}+\phi_1 & \psi^{(2)}_{1,-2} &= 0\,,\\
\psi_{-1,2} &= -\phi_{-1}+3\phi_1-2\phi_2 & \psi^{(2)}_{-1,2} &= 0\,,\\
\psi_{-3,1} &= -3\phi_{-3}+4\phi_{-2}-\phi_1 & \psi^{(2)}_{-3,1} &= 0\,.
\end{aligned}
$$

Forcing equality between the coboundary condition and the coefficients $\psi^{(2)}_{i,j}$ yields a linear system which is incompatible:

$$
\operatorname{rk}\begin{pmatrix} 0 & 2 & 0 & -4 & 0 & 2\\ 0 & 0 & 1 & -2 & 1 & 0\\ 0 & 2 & -3 & 0 & 1 & 0\\ 0 & 0 & -1 & 0 & 3 & -2\\ -3 & 4 & 0 & 0 & -1 & 0 \end{pmatrix} = 4 \neq \operatorname{rk}\left(\begin{array}{cccccc|c} 0 & 2 & 0 & -4 & 0 & 2 & 16\\ 0 & 0 & 1 & -2 & 1 & 0 & 2\\ 0 & 2 & -3 & 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 3 & -2 & 0\\ -3 & 4 & 0 & 0 & -1 & 0 & 0 \end{array}\right) = 5\,. \qquad (4.44)
$$

where the columns from left to right are taken to correspond to the variables $\phi_{-3},\phi_{-2},\ldots\phi_2$. Hence, our cocycle $\psi^{(2)}$ cannot be a coboundary.
Finally, let us consider the difference $\psi^{(1)}-\psi^{(2)}$. Taking the same system as before, but replacing the independent terms by the differences, we obtain for the outermost right column in (4.44), $(-20,-4,-3,3,4)$. The corresponding matrix has still rank five, hence the system is still incompatible. Thus, $\psi^{(1)}-\psi^{(2)}$ cannot be a coboundary, meaning $\psi^{(1)}$ and $\psi^{(2)}$ are not equivalent.
All in all, we found two non-trivial and non-equivalent 2-cocycles, meaning that the dimension of $\mathrm{H}^2(\mathcal{W},\mathcal{F}^0)$ is at least two. □

Proof of Theorem 4.2.2. Proposition 4.2.1 and Proposition 4.2.2 together clearly prove Theorem 4.2.2. □

Results for $\lambda=1$

We will continue with the critical value $\lambda=1$. The analysis is very similar to the case $\lambda=0$. Still, putting both analysis's together into a single proof is rather awkward, making the resulting proof hard to read. We prefer giving explicit separate proofs for all the exceptional values of λ. The statement to prove is Theorem 4.2.3 below.

Theorem 4.2.3. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda=1$ has dimension two, i.e.*

$$\dim \mathrm{H}^2(\mathcal{W},\mathcal{F}^1)=2\,.$$

The Theorem 4.2.3 will be proven in two steps corresponding to the Propositions 4.2.3 and 4.2.4 below.

Proposition 4.2.3. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda = 1$ has maximally a dimension of two, i.e.*

$$\dim \mathrm{H}^2(\mathcal{W}, \mathcal{F}^1) \leq 2\,.$$

Proof. Let ψ be a degree-zero 2-cocycle of $\mathrm{H}^2(\mathcal{W}, \mathcal{F}^1)$. We will start by performing a cohomological change $\psi' = \psi - \delta_1\phi$ with a one-cochain ϕ in order to obtain the following condition, $\boxed{\psi'_{i,1} = 0\ \forall\ i \in \mathbb{Z}\setminus\{-1\} \text{ and } \psi'_{-1,2} = 0}$. We start by defining $\phi_1 = 0$ and $\phi_2 = 0$. The coboundary condition (4.15) for $\lambda = 1$ is as follows:

$$(i+j)\phi_i - (i+j)\phi_j + (-i+j)\phi_{i+j} = \psi_{i,j}\,. \tag{4.45}$$

Putting $j = 1$, the coboundary condition suggests to define ϕ in the following recursive way.:

$$-(1+i)\cancel{\phi_1} + (1+i)\phi_i + (1-i)\phi_{1+i} = \psi_{i,1}$$
$$\Leftrightarrow \phi_i := \frac{i-1}{i+1}\phi_{i+1} + \frac{1}{i+1}\psi_{i,1} \text{ for } i \text{ decreasing} \tag{4.46}$$
$$\Leftrightarrow \phi_{i+1} := \frac{i+1}{i-1}\phi_i - \frac{1}{i-1}\psi_{i,1} \text{ for } i \text{ increasing}. \tag{4.47}$$

Starting with $i = 2$ in the second recurrence relation (4.47), increasing i, we obtain a definition for ϕ_i $i \geq 3$ and we get $\psi'_{i,1} = 0\ \forall\ i \geq 2$ after the cohomological change. Putting $i = 0$ in the first recurrence relation (4.46), we obtain a definition for ϕ_0 and we get $\psi'_{0,1} = 0$. Next, the coboundary condition (4.45) for $(i,j) = (2,-1)$ yields a definition for ϕ_{-1}:

$$-\phi_{-1} - 3\cancel{\phi_1} + \cancel{\phi_2} = \psi_{2,-1} \Leftrightarrow \phi_{-1} := -\psi_{2,-1}\,,$$

and we have $\psi'_{-1,2} = 0$ after the cohomological change. Starting with $i = -2$ in the first recurrence relation (4.46), decreasing i, we obtain $\phi_i\ \forall\ i \leq -2$ and $\psi'_{i,1} = 0\ \forall\ i \leq -2$. We note that the value $\psi_{-1,1}$ is missing.
In the following, we will drop the prime and work with a 2-cocycle $\psi_{i,j}$ which has been cohomologically normalized to $\psi_{i,1} = 0\ \forall\ i \in \mathbb{Z}\setminus\{-1\}$ and $\psi_{-1,2} = 0$.
As usual, we will start with level zero and count the generating coefficients necessary to generate $\boxed{\psi_{i,0}\ \forall\ i \in \mathbb{Z}}$. The cocycle condition (4.14) for $(i,j,k) = (i,0,1)$ produces the recurrence relations:

$$-(1+i)\cancel{\psi_{0,1}} + \underline{\psi_{1,i}} - (1+i)\psi_{i,0} - i\underline{\psi_{i,1}} + (1+i)\underline{\psi_{i,1}} - (1-i)\psi_{1+i,0} = 0$$
$$\Leftrightarrow \psi_{i,0} = \frac{i-1}{i+1}\psi_{1+i,0} \text{ for } i \text{ decreasing} \tag{4.48}$$
$$\Leftrightarrow \psi_{1+i,0} = \frac{i+1}{i-1}\psi_{i,0} \text{ for } i \text{ increasing}. \tag{4.49}$$

The underlined terms cancel each other. Starting with $i = -2$ and i decreasing in the first recurrence relation (4.48), we see that $\psi_{-1,0}$ generates all $\psi_{i,0}$ $i \leq -2$. Starting with $i = 2$ and i increasing in the second recurrence relation (4.49), we obtain that $\psi_{2,0}$ generates all $\psi_{i,0}$ $i \geq 3$. Writing down the cocycle condition (4.14) for $(0,-1,2)$ provides a non-trivial relation between the two generating coefficients $\psi_{-1,0}$ and $\psi_{2,0}$:

$$-2\cancel{\psi_{-1,2}} - \psi_{0,-1} + \psi_{0,2} + 3\cancel{\psi_{1,0}} - 2\cancel{\psi_{2,-1}} = 0 \Leftrightarrow \psi_{0,-1} = \psi_{0,2}\,.$$

Hence, $\psi_{i,0}\ \forall\ i \in \mathbb{Z}$ is generated by a single generating coefficient, say $\psi_{2,0}$.
Next, we will focus on level minus one and count the generating coefficients of $\boxed{\psi_{i,-1}\ \forall\ i \in \mathbb{Z}}$.
The cocycle condition (4.14) on $(i,-1,1)$ for $\lambda = 1$ yields:

$$-i\psi_{-1,1} + 2\psi_{0,i} + (-1-i)\psi_{-1+i,1} - i\psi_{i,-1} + i\psi_{i,1} - (1-i)\psi_{1+i,-1} = 0$$

$$\Leftrightarrow \psi_{i,-1} = -\psi_{-1,1} + \frac{2}{i}\psi_{0,i} + \frac{(-1-i)}{i}\psi_{-1+i,1} + \psi_{i,1} - \frac{(1-i)}{i}\psi_{1+i,-1} \text{ for } i \text{ decreasing} \tag{4.50}$$

$$\Leftrightarrow \psi_{1+i,-1} = -\frac{i}{1-i}\psi_{-1,1} + \frac{2}{1-i}\psi_{0,i} + \frac{(-1-i)}{1-i}\psi_{-1+i,1} - \frac{i}{1-i}\psi_{i,-1} + \frac{i}{1-i}\psi_{i,1} \text{ for } i \text{ increasing}. \tag{4.51}$$

Starting with $i = -2$, i decreasing in the first recurrence relation (4.50), we see that $\psi_{i,-1}\ i \leq -2$ is solely generated by $\psi_{-1,1}$ and $\psi_{0,2}$. Similarly, starting with $i = 2$ and i increasing in the second recurrence relation (4.51), we obtain $\psi_{i,-1}\ i \geq 3$ without any new generating coefficient becoming apparent. Hence, $\psi_{2,0}$ and $\psi_{-1,1}$ are sufficient to generate all $\psi_{i,-1}\ \forall\ i \in \mathbb{Z}$.
Next, we will count the generating coefficients necessary to generate level two and level minus two, i.e. $\boxed{\psi_{i,2} \text{ and } \psi_{i,-2}\ \forall\ i \in \mathbb{Z}}$. The cocycle condition (4.14) for $(i,2,-1)$ gives:

$$-3\psi_{1,i} - (1+i)\cancel{\psi_{2,-1}} - (-1-i)\psi_{-1+i,2} + (1+i)\psi_{i,-1} - (1+i)\psi_{i,2} + (2-i)\psi_{2+i,-1} = 0$$

$$\Leftrightarrow \psi_{-1+i,2} = -\frac{3}{-1-i}\psi_{1,i} - \psi_{i,-1} + \psi_{i,2} + \frac{(2-i)}{-1-i}\psi_{2+i,-1} \text{ for } i \text{ decreasing} \tag{4.52}$$

$$\Leftrightarrow \psi_{i,2} = -\frac{3}{i+1}\psi_{1,i} + \psi_{-1+i,2} + \psi_{i,-1} + \frac{(2-i)}{i+1}\psi_{2+i,-1} \text{ for } i \text{ increasing}. \tag{4.53}$$

Starting with $i = 0$ and i increasing in the second recurrence relation (4.53), we obtain $\psi_{i,2}\ i \geq 0$ without introducing a new generating coefficient. Starting with $i = -2$, i decreasing in the first recurrence relation (4.52), we obtain $\psi_{i,2}\ i \leq -3$ with a new generating coefficient appearing, $\psi_{-2,2}$. Similarly, the cocycle condition (4.14) for $(i,-2,1)$ gives the recurrence relations for level minus two:

$$-(-1+i)\cancel{\psi_{-2,1}} + 3\psi_{-1,i} + (-2-i)\psi_{-2+i,1} - (-1+i)\psi_{i,-2} + (-1+i)\psi_{i,1} - (1-i)\psi_{1+i,-2} = 0$$

$$\Leftrightarrow \psi_{i,-2} = \frac{3}{i-1}\psi_{-1,i} + \frac{(-2-i)}{i-1}\psi_{-2+i,1} + \psi_{i,1} + \psi_{1+i,-2} \text{ for } i \text{ decreasing} \tag{4.54}$$

$$\Leftrightarrow \psi_{1+i,-2} = \frac{3}{1-i}\psi_{-1,i} + \frac{(-2-i)}{1-i}\psi_{-2+i,1} + \psi_{i,-2} - \psi_{i,1} \text{ for } i \text{ increasing}. \tag{4.55}$$

Starting with $i = 0$ in the first recurrence relation (4.54), i decreasing, we obtain $\psi_{i,-2}\ i \leq 0$ without any new generating coefficient appearing. Starting with $i = 2$ in the second recurrence relation (4.55), i increasing, we get $\psi_{i,-2}\ i \geq 3$. Again, the generating coefficient $\psi_{-2,2}$ appears. Thus, the levels plus one, minus one, zero, plus two and minus two are generated by at most three generating coefficients, $\psi_{-1,1}$, $\psi_{0,2}$ and $\psi_{-2,2}$. However, there is a non-trivial relation between these three generating coefficients. Actually, the cocycle condition (4.14) for $(2,-2,3)$ yields, after replacing the various expressions of $\psi_{i,j}$ by their corresponding recurrence relation:

$$-3\psi_{-2,3} - 4\psi_{0,3} + 5\cancel{\psi_{1,2}} - 3\psi_{2,-2} + 3\psi_{2,3} - \psi_{5,-2} = 0$$
$$\Leftrightarrow -\psi_{2,-2} + 6\psi_{2,0} - 8\psi_{-1,1} = 0 \Leftrightarrow \psi_{2,-2} = 6\psi_{2,0} - 8\psi_{-1,1}\,.$$

The slashed term is zero due to the cohomological change. Thus, there are only two generating coefficients $\psi_{2,0}$ and $\psi_{-1,1}$.

In the last step, we can use induction on k. We want to show that $\psi_{i,j}$ is generated at most by $\psi_{2,0}$ and $\psi_{-1,1}$. The statement is true for levels zero, plus one and plus two. Thus, let us suppose the statement is true for level $k \geq 2$, and let us prove that it holds true for level $k+1$. The cocycle condition (4.14) for $(i,1,k)$ gives:

$$\begin{aligned}
&-(1+i+k)\psi_{1,k} - (1+i+k)\psi_{i,1} + (1+i+k)\psi_{i,k} \\
&\quad +(1-i)\psi_{1+i,k} + (-1+k)\psi_{1+k,i} - (-i+k)\psi_{i+k,1} = 0 \\
&\Leftrightarrow (1-k)\psi_{1+k,i} = -(1+i+k)\psi_{1,k} - (1+i+k)\psi_{i,1} + (1+i+k)\psi_{i,k} \\
&\qquad\qquad + (1-i)\psi_{1+i,k} - (-i+k)\psi_{i+k,1}\,.
\end{aligned}$$

The terms on the right are of level plus one or level k, which are generated by $\psi_{2,0}$ and $\psi_{-1,1}$ due to the induction hypothesis. We see that the term of level $k+1$ on the left can be expressed, for all i, in terms of level-k-terms without having to introduce any new generating coefficient. Hence, level $k+1$ is also generated by at most two generating coefficients.
The same can be done for decreasing k. The statement is true for the levels zero, minus one and minus two. Let us suppose the statement is true for level $k \leq -2$, and let us prove that it holds true for level $k-1$. The cocycle condition (4.14) for $(i,-1,k)$ gives:

$$\begin{aligned}
&-(-1+i+k)\psi_{-1,k} + (-1-i)\psi_{-1+i,k} - (-1+i+k)\psi_{i,-1} \\
&\quad +(-1+i+k)\psi_{i,k} + (1+k)\psi_{-1+k,i} - (-i+k)\psi_{i+k,-1} = 0 \\
&\Leftrightarrow -(1+k)\psi_{-1+k,i} = -(-1+i+k)\psi_{-1,k} + (-1-i)\psi_{-1+i,k} - (-1+i+k)\psi_{i,-1} \\
&\qquad\qquad + (-1+i+k)\psi_{i,k} - (-i+k)\psi_{i+k,-1}\,.
\end{aligned}$$

A similar reasoning as before leads to the fact that also level $k-1$ is generated by at most two generating coefficients. Therefore, the dimension of the second cohomology for $\lambda = 1$ is at most two. □

Next, we will prove that the dimension of $\mathrm{H}^2(\mathcal{W},\mathcal{F}^1)$ is at least two. The proof is more straightforward as in the case of $\lambda = 0$, as no Kronecker Delta's are needed in our definitions of the generating 2-cocycles, which renders case differentiations unnecessary.

Proposition 4.2.4. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda = 1$ has minimally a dimension of two, i.e.*

$$2 \leq \dim\, \mathrm{H}^2(\mathcal{W},\mathcal{F}^1)\,.$$

Proof. Similarly to the case $\lambda = 0$, our definitions of the generating 2-cocycles are inspired from continuous cohomology, which can be found in [90]. We will define our generating degree-zero cochains in the following manner:

$$\begin{aligned}
\psi^{(1)}(e_i,e_j) &= \psi^{(1)}_{i,j}\, f_{i+j} := (i j^2 - i^2 j)\, f_{i+j}\,, \\
\psi^{(2)}(e_i,e_j) &= \psi^{(2)}_{i,j}\, f_{i+j} := (j^2 - i^2)\, f_{i+j}\,.
\end{aligned}$$

Clearly, the expressions are antisymmetric in i and j, thus they are cochains. The first cochain is again similar to the Virasoro 2-cocycle (2.20). The verification that $\psi^{(1)}$ is a cocycle is straightforward. It suffices to insert the expression for the coefficients $\psi^{(1)}_{i,j} = (i j^2 - i^2 j)$ into the cocycle condition (4.14) for $\lambda = 1$, and a direct computation yields that the cocycle condition is indeed fulfilled. The same can be done for the second cochain $\psi^{(2)} = (j^2 - i^2)$, yielding also immediately that $\psi^{(2)}$ is a cocycle. Consequently, our two generating 2-cochains are cocycles. Next, let

us check that they are not coboundaries, and not equivalent.
We will consider the coboundary conditions 4.45 evaluated on several combinations of the elements e_{-3}, e_{-2}, e_{-1}, e_1, e_2, e_3, and compare them to the coefficients $\psi^{(1)}_{i,j}$, which yields:

$$\begin{aligned}
\psi_{-1,2} &= \phi_{-1} + 3\phi_1 - \phi_2 & \psi^{(1)}_{-1,2} &= -6\,,\\
\psi_{1,-2} &= \phi_{-2} - 3\phi_{-1} - \phi_1 & \psi^{(1)}_{1,-2} &= 6\,,\\
\psi_{-3,1} &= -2\phi_{-3} + 4\phi_{-2} + 2\phi_1 & \psi^{(1)}_{-3,1} &= -12\,,\\
\psi_{-3,2} &= -\phi_{-3} + 5\phi_{-1} + \phi_2 & \psi^{(1)}_{-3,2} &= -30\,,\\
\psi_{-1,3} &= 2\phi_{-1} + 4\phi_2 - 2\phi_3 & \psi^{(1)}_{-1,3} &= -12\,.
\end{aligned}$$

Forcing equality between the coboundary conditions and the coefficients $\psi^{(1)}_{i,j}$ yields a system which is incompatible:

$$\operatorname{rk}\begin{pmatrix} 0 & 0 & 1 & 3 & -1 & 0\\ 0 & 1 & -3 & -1 & 0 & 0\\ -2 & 4 & 0 & 2 & 0 & 0\\ -1 & 0 & 5 & 0 & 1 & 0\\ 0 & 0 & 2 & 0 & 4 & -2 \end{pmatrix} = 4 \neq \operatorname{rk}\left(\begin{array}{cccccc|c} 0 & 0 & 1 & 3 & -1 & 0 & -6\\ 0 & 1 & -3 & -1 & 0 & 0 & 6\\ -2 & 4 & 0 & 2 & 0 & 0 & -12\\ -1 & 0 & 5 & 0 & 1 & 0 & -30\\ 0 & 0 & 2 & 0 & 4 & -2 & -12 \end{array}\right) = 5\,.$$

The columns correspond, from left to right, to the coefficients ϕ_{-3}, ϕ_{-2}, ϕ_{-1}, ϕ_1, ϕ_2 and ϕ_3. The system is incompatible, and therefore, $\psi^{(1)}$ cannot be a coboundary. Next, we will proceed the same for $\psi^{(2)}$. We can use the same system of coboundary conditions:

$$\begin{aligned}
\psi_{-1,2} &= \phi_{-1} + 3\phi_1 - \phi_2 & \psi^{(2)}_{-1,2} &= 3\,,\\
\psi_{1,-2} &= \phi_{-2} - 3\phi_{-1} - \phi_1 & \psi^{(2)}_{1,-2} &= 3\,,\\
\psi_{-3,1} &= -2\phi_{-3} + 4\phi_{-2} + 2\phi_1 & \psi^{(2)}_{-3,1} &= -8\,,\\
\psi_{-3,2} &= -\phi_{-3} + 5\phi_{-1} + \phi_2 & \psi^{(2)}_{-3,2} &= -5\,,\\
\psi_{-1,3} &= 2\phi_{-1} + 4\phi_2 - 2\phi_3 & \psi^{(2)}_{-1,3} &= 8\,.
\end{aligned}$$

which is once again incompatible:

$$\operatorname{rk}\begin{pmatrix} 0 & 0 & 1 & 3 & -1 & 0\\ 0 & 1 & -3 & -1 & 0 & 0\\ -2 & 4 & 0 & 2 & 0 & 0\\ -1 & 0 & 5 & 0 & 1 & 0\\ 0 & 0 & 2 & 0 & 4 & -2 \end{pmatrix} = 4 \neq \operatorname{rk}\left(\begin{array}{cccccc|c} 0 & 0 & 1 & 3 & -1 & 0 & 3\\ 0 & 1 & -3 & -1 & 0 & 0 & 3\\ -2 & 4 & 0 & 2 & 0 & 0 & -8\\ -1 & 0 & 5 & 0 & 1 & 0 & -5\\ 0 & 0 & 2 & 0 & 4 & -2 & 8 \end{array}\right) = 5\,.$$

Therefore, $\psi^{(2)}$ cannot be a coboundary.
Finally, let us check whether the difference $\psi^{(1)} - \psi^{(2)}$ can be a coboundary. Taking once again the same system, replacing the ϕ_i-independent terms by the differences, we obtain for the outermost right column in the matrix on the right-hand side above, $(-9, 3, -4, -25, -20)$. The rank of the matrix thus obtained is still five, hence the system is still incompatible, so that $\psi^{(1)} - \psi^{(2)}$ cannot be a coboundary, meaning $\psi^{(1)}$ and $\psi^{(2)}$ are not equivalent.
We proved that $\mathrm{H}^2(\mathscr{W}, \mathscr{F}^1)$ is generated by two non-trivial non-equivalent cocycles $\psi^{(1)}$ and $\psi^{(2)}$. This concludes the proof. □

Proof of Theorem 4.2.3. Clearly, Propositions 4.2.3 and 4.2.4 prove Theorem 4.2.3. □

Results for $\lambda = 2$

The analysis for $\lambda = 2$ is almost identical to the previous one. Still, I write it down for reasons of completeness. The statement to prove is Theorem 4.2.4 given below.

Theorem 4.2.4. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda = 2$ has a dimension of two, i.e.*

$$\dim \mathrm{H}^2(\mathcal{W}, \mathcal{F}^2) = 2\,.$$

Again, the proof follows in two steps.

Proposition 4.2.5. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda = 2$ has maximally a dimension of two, i.e.*

$$\dim \mathrm{H}^2(\mathcal{W}, \mathcal{F}^2) \leq 2\,.$$

Proof. Let ψ be a degree-zero 2-cocycle of $\mathrm{H}^2(\mathcal{W}, \mathcal{F}^2)$. We will start by performing a cohomological change $\psi' = \psi - \delta_1\phi$ with a 1-cochain $\phi \in C^1(\mathcal{W}, \mathcal{F}^2)$ in order to obtain the following: $\boxed{\psi'_{i,1} = 0\ \forall\ i \in \mathbb{Z}\setminus\{-2\} \text{ and } \psi'_{-2,2} = 0}$. We start by defining $\phi_1 := 0$ and $\phi_2 := 0$. The coboundary condition (4.15) for $\lambda = 2$ is as follows:

$$(i+2j)\phi_i - (2i+j)\phi_j + (-i+j)\phi_{i+j} = \psi_{i,j}\,. \tag{4.56}$$

Putting $j = 1$, the coboundary condition (4.56) suggests the following recursive definition for ϕ:

$$\begin{aligned} &-(1+2i)\cancel{\phi_1} + (2+i)\phi_i + (1-i)\phi_{1+i} = \psi_{i,1} \\ &\Leftrightarrow \phi_i = \frac{i-1}{i+2}\phi_{i+1} + \frac{1}{i+2}\psi_{i,1} \text{ for } i \text{ decreasing} \qquad (4.57)\\ &\Leftrightarrow \phi_{i+1} = \frac{i+2}{i-1}\phi_i - \frac{1}{i-1}\psi_{i,1} \text{ for } i \text{ increasing}. \qquad (4.58)\end{aligned}$$

Starting with $i = 2$ in the second recurrence relation (4.58), increasing i, we obtain a definition for ϕ_i $i \geq 3$ and we get $\psi'_{i,1} = 0\ \forall\ i \geq 2$ after the cohomological change. Putting $i = 0$ in the first recurrence relation (4.57), we obtain a definition for ϕ_0 and $\psi'_{0,1} = 0$. Next, the coboundary condition (4.56) for $(i, j) = (2, -2)$ yields a definition for ϕ_{-2}:

$$-2\phi_{-2} - 4\phi_0 - 2\cancel{\phi_2} = \psi_{2,-2} \Leftrightarrow \phi_{-2} = -2\phi_0 - \frac{1}{2}\psi_{2,-2}\,,$$

and we get $\psi'_{-2,2} = 0$. Putting $i = -1$ in the first recurrence relation (4.57), we obtain a definition for ϕ_{-1} and also $\psi'_{-1,1} = 0$. Starting with $i = -3$ in the first recurrence relation (4.57), i decreasing, we obtain $\phi_i\ \forall\ i \leq -3$ and $\psi'_{i,1} = 0\ \forall\ i \leq -3$. We note that the value $\psi_{-2,1}$ is missing.
In the following, we will drop the prime and work with a 2-cocycle $\psi_{i,j}$ which has been cohomologically normalized to $\psi_{i,1} = 0\ \forall\ i \in \mathbb{Z}\setminus\{-2\}$ and $\psi_{-2,2} = 0$.
As usual, we will start with level zero and count the generating coefficients necessary to generate $\boxed{\psi_{i,0}\ \forall\ i \in \mathbb{Z}}$. The cocycle condition (4.14) for $(i, j, k) = (i, 0, 1)$ produces the recurrence relations:

$$\begin{aligned} &-(1+2i)\cancel{\psi_{0,1}} + \underline{\psi_{1,i}} - (2+i)\psi_{i,0} - i\underline{\psi_{i,1}} + (1+i)\underline{\psi_{i,1}} - (1-i)\psi_{1+i,0} = 0 \\ &\Leftrightarrow \psi_{i,0} = \frac{i-1}{i+2}\psi_{1+i,0} \text{ for } i \text{ decreasing} \qquad (4.59)\\ &\Leftrightarrow \psi_{1+i,0} = \frac{i+2}{i-1}\psi_{i,0} \text{ for } i \text{ increasing}. \qquad (4.60)\end{aligned}$$

The underlined terms cancel each other. Inserting $i = -1$ in the first recurrence relation (4.59), we obtain $\psi_{-1,0} = -2\psi_{0,0} = 0$. Continuing with $i = -3$ and i decreasing in the first recurrence relation (4.59), we see that $\psi_{-2,0}$ generates all $\psi_{i,0}$ $i \le -3$. Starting with $i = 2$ and i increasing in the second recurrence relation (4.60), we obtain that $\psi_{2,0}$ generates all $\psi_{i,0}$ $i \ge 3$. Writing down the cocycle condition (4.14) for $(-2,2,0)$ provides a non-trivial relation between the two generating coefficients $\psi_{-2,0}$ and $\psi_{2,0}$:

$$2\psi_{-2,0} - 2\cancel{\psi_{-2,2}} - 2\cancel{\psi_{2,-2}} + 2\psi_{2,0} = 0 \Leftrightarrow \psi_{-2,0} = -\psi_{2,0}\,.$$

Hence, $\psi_{i,0}$ $\forall\ i \in \mathbb{Z}$ is generated by a single generating coefficient, say $\psi_{2,0}$.
Next, we will focus on level minus one and count the generating coefficients of $\boxed{\psi_{i,-1}\ \forall\ i \in \mathbb{Z}}$. The cocycle condition (4.14) on $(i,-1,1)$ for $\lambda = 2$ yields:

$$\begin{aligned}
&-2i\cancel{\psi_{-1,1}} + 2\psi_{0,i} + (-1-i)\psi_{-1+i,1} - (i+1)\psi_{i,-1} + (i-1)\psi_{i,1} - (1-i)\psi_{1+i,-1} = 0\\
&\Leftrightarrow \psi_{i,-1} = \frac{2}{i+1}\psi_{0,i} - \psi_{-1+i,1} + \frac{i-1}{i+1}\psi_{i,1} - \frac{(1-i)}{i+1}\psi_{1+i,-1} \text{ for } i \text{ decreasing} \qquad (4.61)\\
&\Leftrightarrow \psi_{1+i,-1} = \frac{2}{1-i}\psi_{0,i} + \frac{(-1-i)}{1-i}\psi_{-1+i,1} - \frac{i+1}{1-i}\psi_{i,-1} - \psi_{i,1} \text{ for } i \text{ increasing}. \qquad (4.62)
\end{aligned}$$

Starting with $i = -2$, i decreasing in the first recurrence relation (4.61), we see that $\psi_{i,-1}$ $i \le -2$ is solely generated by $\psi_{-2,1}$ and $\psi_{0,2}$. Starting with $i = 2$ and i increasing in the second recurrence relation (4.62), we obtain $\psi_{i,-1}$ $i \ge 3$ with a new generating coefficient becoming apparent, $\psi_{2,-1}$. Hence, we have á priori three generating coefficients $\psi_{-2,1}$, $\psi_{0,2}$ and $\psi_{2,-1}$, to generate all $\psi_{i,-1}$ $\forall\ i \in \mathbb{Z}$.
Next, we will count the generating coefficients necessary to generate level two and level minus two, i.e. $\boxed{\psi_{i,2} \text{ and } \psi_{i,-2}\ \forall\ i \in \mathbb{Z}}$. The cocycle condition (4.14) for $(i,2,-1)$ gives:

$$\begin{aligned}
&-3\psi_{1,i} - (1+2i)\psi_{2,-1} - (-1-i)\psi_{-1+i,2} + (3+i)\psi_{i,-1} - i\psi_{i,2} + (2-i)\psi_{2+i,-1} = 0\\
&\Leftrightarrow \psi_{-1+i,2} = -\frac{(1+2i)}{-1-i}\psi_{2,-1} - \frac{3}{-1-i}\psi_{1,i} + \frac{3+i}{-1-i}\psi_{i,-1} - \frac{i}{-1-i}\psi_{i,2} + \frac{(2-i)}{-1-i}\psi_{2+i,-1} \qquad (4.63)\\
&\Leftrightarrow \psi_{i,2} = -\frac{(1+2i)}{i}\psi_{2,-1} - \frac{3}{i}\psi_{1,i} + \frac{i+1}{i}\psi_{-1+i,2} + \frac{3+i}{i}\psi_{i,-1} + \frac{(2-i)}{i}\psi_{2+i,-1}\,. \qquad (4.64)
\end{aligned}$$

Starting with $i = 3$ and i increasing in the second recurrence relation (4.64), we obtain $\psi_{i,2}$ $i \ge 3$ without introducing a new generating coefficient. Starting with $i = -2$, i decreasing in the first recurrence relation (4.63), we obtain $\psi_{i,2}$ $i \le -3$ with no new generating coefficient appearing. Similarly, the cocycle condition (4.14) for $(i,-2,1)$ gives the recurrence relations for level minus two:

$$\begin{aligned}
&-(-1+2i)\psi_{-2,1} + 3\psi_{-1,i} + (-2-i)\psi_{-2+i,1} - i\psi_{i,-2} + (-3+i)\psi_{i,1} - (1-i)\psi_{1+i,-2} = 0\\
&\Leftrightarrow \psi_{i,-2} = -\frac{(-1+2i)}{i}\psi_{-2,1} + \frac{3}{i}\psi_{-1,i} + \frac{(-2-i)}{i}\psi_{-2+i,1} + \frac{i-3}{i}\psi_{i,1} - \frac{1-i}{i}\psi_{1+i,-2} \qquad (4.65)\\
&\Leftrightarrow \psi_{1+i,-2} = -\frac{(-1+2i)}{1-i}\psi_{-2,1} + \frac{3}{1-i}\psi_{-1,i} + \frac{(-2-i)}{1-i}\psi_{-2+i,1} - \frac{i}{1-i}\psi_{i,-2} + \frac{i-3}{1-i}\psi_{i,1}\,. \qquad (4.66)
\end{aligned}$$

Starting with $i = -3$ in the first recurrence relation (4.65), i decreasing, we obtain $\psi_{i,-2}$ $i \le -3$ without any new generating coefficient appearing. Starting with $i = 2$ in the second recurrence relation (4.66), i increasing, we get $\psi_{i,-2}$ $i \ge 3$. Thus, the levels plus one, minus one, zero, plus two and minus two are generated by at most three generating coefficients, $\psi_{-1,2}$, $\psi_{0,2}$ and

$\psi_{-2,1}$. However, there is a non-trivial relation between these three coefficients. Actually, the cocycle condition (4.14) for $(2,-2,3)$ yields, after replacing the various expressions of $\psi_{i,j}$ by their recurrence relation:

$$\begin{aligned}&-5\psi_{-2,3}-4\psi_{0,3}+5\cancel{\psi_{1,2}}-6\cancel{\psi_{2,-2}}+\psi_{2,3}-\psi_{5,-2}=0\\&\Leftrightarrow \psi_{-2,1}+2\psi_{2,0}-\psi_{2,-1}=0\Leftrightarrow \psi_{2,-1}=\psi_{-2,1}+2\psi_{2,0}\,.\end{aligned}$$

Thus, there are only two generating coefficients, say $\psi_{2,0}$ and $\psi_{-2,1}$.
In the last step, we can use induction on k. We want to show that $\psi_{i,j}$ is generated at most by $\psi_{2,0}$ and $\psi_{-2,1}$. The statement is true for levels zero, plus one and plus two. Thus, let us suppose the statement is true for level $k \geq 2$, and let us prove that it holds true for level $k+1$. The cocycle condition (4.14) for $(i,1,k)$ gives:

$$\begin{aligned}&-(1+2i+k)\psi_{1,k}-(1+i+2k)\psi_{i,1}+(2+i+k)\psi_{i,k}\\&+(1-i)\psi_{1+i,k}+(-1+k)\psi_{1+k,i}-(-i+k)\psi_{i+k,1}=0\\&\Leftrightarrow (1-k)\psi_{1+k,i}=-(1+2i+k)\psi_{1,k}-(1+i+2k)\psi_{i,1}\\&+(2+i+k)\psi_{i,k}+(1-i)\psi_{1+i,k}-(-i+k)\psi_{i+k,1}\,.\end{aligned}$$

The terms on the right are of level plus one or level k, which are generated by $\psi_{2,0}$ and $\psi_{-2,1}$ due to the induction hypothesis. We see that the term of level $k+1$ on the left can be expressed, for all i, in terms of level-k-terms without having to introduce any new generating coefficient. Hence, level $k+1$ is also generated by at most two generating coefficients.
The same can be done for decreasing k. The statement is true for the levels zero, minus one and minus two. Let us suppose the statement is true for level $k \leq -2$, and let us prove that it holds true for level $k-1$. The cocycle condition (4.14) for $(i,-1,k)$ gives:

$$\begin{aligned}&-(-1+2i+k)\psi_{-1,k}+(-1-i)\psi_{-1+i,k}-(-1+i+2k)\psi_{i,-1}\\&+(-2+i+k)\psi_{i,k}+(1+k)\psi_{-1+k,i}-(-i+k)\psi_{i+k,-1}=0\\&\Leftrightarrow -(1+k)\psi_{-1+k,i}=-(-1+2i+k)\psi_{-1,k}+(-1-i)\psi_{-1+i,k}\\&-(-1+i+2k)\psi_{i,-1}+(-2+i+k)\psi_{i,k}-(-i+k)\psi_{i+k,-1}\,.\end{aligned}$$

A similar reasoning as before leads to the fact that also level $k-1$ is generated by at most two generating coefficients. Therefore, the dimension of the second cohomology group for $\lambda=2$ is at most two. □

Next, we prove that the dimension of $\mathrm{H}^2(\mathcal{W},\mathcal{F}^2)$ is at least two.

Proposition 4.2.6. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda=2$ has minimally a dimension of two, i.e.*

$$2 \leq \dim\ \mathrm{H}^2(\mathcal{W},\mathcal{F}^2)\,.$$

Proof. Our expressions for the generating cocycles are again inspired from continuous cohomology. Let us define our generating degree-zero cochains in the following way:

$$\begin{aligned}\psi^{(1)}(e_i,e_j)&=\psi^{(1)}_{i,j}\,f_{i+j}:=(i\,j^3-i^3 j)\,f_{i+j}\,,\\ \psi^{(2)}(e_i,e_j)&=\psi^{(2)}_{i,j}\,f_{i+j}:=(j^3-i^3)\,f_{i+j}\,.\end{aligned}$$

The antisymmetry in i and j is again plain to see. The verification that these two cochains are cocycles is also straightforward. It suffices to insert the expression of the coefficients $\psi^{(1)}_{i,j} = (ij^3 - i^3 j)$ and coefficients $\psi^{(2)}_{i,j} = (j^3 - i^3)$ into the cocycle condition (4.14) for $\lambda = 2$, a direct calculation shows that they fulfill the cocycle condition. Hence, the two cochains are cocycles. Let us check that they are not coboundaries, and not equivalent. For the coefficients $\psi^{(1)}_{i,j}$, this is straightforward. Consider the following coboundary conditions:

$$\begin{aligned} \psi_{-1,2} &= 3\phi_{-1} + 3\phi_1 & \psi^{(1)}_{-1,2} &= -6\,, \\ \psi_{1,-2} &= -3\phi_{-1} - 3\phi_1 & \psi^{(1)}_{1,-2} &= -6\,. \end{aligned} \tag{4.67}$$

Clearly, the coefficients ϕ_{-1} and ϕ_1 cannot fulfill the equations obtained by equaling the coboundary conditions and the coefficients $\psi^{(1)}_{i,j}$. Hence, $\psi^{(1)}$ cannot be a coboundary. Next, we consider $\psi^{(2)}$. We will consider the coboundary condition for several combinations of the elements $e_{-3},\dots,e_3$ and compare them to the coefficients $\psi^{(2)}_{i,j}$:

$$\begin{aligned} \psi_{2,-2} &= -2\phi_{-2} - 4\phi_0 - 2\phi_2 & \psi^{(2)}_{2,-2} &= -16\,, \\ \psi_{-1,2} &= 3\phi_{-1} + 3\phi_1 & \psi^{(2)}_{-1,2} &= 9\,, \\ \psi_{-3,2} &= \phi_{-3} + 5\phi_{-1} + 4\phi_2 & \psi^{(2)}_{-3,2} &= 35\,, \\ \psi_{-1,3} &= 5\phi_{-1} + 4\phi_2 - \phi_3 & \psi^{(2)}_{-1,3} &= 28\,, \\ \psi_{-2,3} &= 4\phi_{-2} + 5\phi_1 + \phi_3 & \psi^{(2)}_{-2,3} &= 35\,, \\ \psi_{-3,3} &= 3\phi_{-3} + 6\phi_0 + 3\phi_3 & \psi^{(2)}_{-3,3} &= 54\,. \end{aligned}$$

Putting the coboundary conditions equal to the coefficients $\psi^{(2)}_{i,j}$ yields a linear system which is incompatible:

$$\mathrm{rk}\begin{pmatrix} 0 & -2 & 0 & -4 & 0 & -2 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & 0 & 0 & 4 & -1 \\ 0 & 4 & 0 & 0 & 5 & 0 & 1 \\ 3 & 0 & 0 & 6 & 0 & 0 & 3 \end{pmatrix} = 5 \neq \mathrm{rk}\begin{pmatrix} 0 & -2 & 0 & -4 & 0 & -2 & 0 & -16 \\ 0 & 0 & 3 & 0 & 3 & 0 & 0 & 9 \\ 1 & 0 & 5 & 0 & 0 & 4 & 0 & 35 \\ 0 & 0 & 5 & 0 & 0 & 4 & -1 & 28 \\ 0 & 4 & 0 & 0 & 5 & 0 & 1 & 35 \\ 3 & 0 & 0 & 6 & 0 & 0 & 3 & 54 \end{pmatrix} = 6\,.$$

From left to right, the columns correspond to the variables $\phi_{-3},\dots,\phi_3$. The system is incompatible, meaning that $\psi^{(2)}$ cannot be a coboundary.
Finally, let us check whether $\psi^{(1)} - \psi^{(2)}$ can be a coboundary. We consider the system (4.67). Doing the difference, we obtain for the ϕ_i-independent terms $\psi^{(1)}_{-1,2} - \psi^{(2)}_{-1,2} = -15$ and $\psi^{(1)}_{1,-2} - \psi^{(2)}_{1,-2} = 3$. The rank of the associated homogeneous system is one, whereas it is two for the entire system. Thus, the system is not compatible and $\psi^{(1)} - \psi^{(2)}$ cannot be a coboundary, meaning that $\psi^{(1)}$ and $\psi^{(2)}$ are not equivalent.
In conclusion, we found two non-trivial non-equivalent cocycles $\psi^{(1)}$ and $\psi^{(2)}$, hence $\mathrm{H}^2(\mathscr{W},\mathscr{F}^2)$ is at least two-dimensional. □

Proof of Theorem 4.2.4. Clearly, the Propositions 4.2.5 and 4.2.6 together prove Theorem 4.2.4. □

Results for $\lambda = 5$

Next, we will turn our attention to the critical value $\lambda = 5$. The statement to prove is given in Theorem 4.2.5 below.

Theorem 4.2.5. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda = 5$ is one-dimensional, i.e.*

$$\dim \mathrm{H}^2(\mathcal{W},\mathcal{F}^5)) = 1 \,.$$

As usual, we will start by deriving an upper limit for the dimension.

Proposition 4.2.7. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda = 5$ has maximally a dimension of one, i.e.*

$$\dim \mathrm{H}^2(\mathcal{W},\mathcal{F}^5)) \leq 1 \,.$$

Proof. In the proof of Theorem 4.2.1, we see that the assumption $\lambda \neq 5$ is relevant only quite late in the proof, at the analysis of levels plus and minus two in Lemma 4.2.4. Thus, the proof of $\psi_{i,1} = \psi_{i,0} = \psi_{i,-1} = 0$ given by the Lemmata 4.2.1, 4.2.2 and 4.2.3, is valid for the case $\lambda = 5$. Consequently, we can immediately start with the analysis of levels plus and minus two. The cocycle condition (4.14) for $(i,-2,1)$ and $\lambda = 5$ gives the recurrence relations for level minus two:

$$\begin{aligned}
&-(-1+5i)\cancel{\psi_{-2,1}} + 3\cancel{\psi_{-1,i}} + (-2-i)\cancel{\psi_{-2+i,1}} \\
&-(3+i)\psi_{i,-2} + (-9+i)\cancel{\psi_{i,1}} - (1-i)\psi_{1+i,-2} = 0 \\
&\Leftrightarrow \psi_{i,-2} = -\frac{(1-i)}{i+3}\psi_{1+i,-2} \text{ for } i \text{ decreasing} && (4.68)\\
&\Leftrightarrow \psi_{1+i,-2} = -\frac{(i+3)}{1-i}\psi_{i,-2} \text{ for } i \text{ increasing}. && (4.69)
\end{aligned}$$

Starting with $i = -4$ in the first recurrence relation (4.68), decreasing i, we see that all $\psi_{i,-2}$ $i \leq -4$ are generated by $\psi_{-3,-2}$. Starting with $i = 2$ in the second recurrence relation (4.69), increasing i, we see that all $\psi_{i,-2}$ $i \geq 3$ are generated by $\psi_{2,-2}$. A priori, level minus two is generated by two generating coefficients.
The cocycle condition (4.14) for $(i,2,-1)$ gives the recurrence relations for level plus two:

$$\begin{aligned}
&-3\cancel{\psi_{1,i}} - (1+5i)\cancel{\psi_{2,-1}} - (-1-i)\psi_{-1+i,2} \\
&+(9+i)\cancel{\psi_{i,-1}} - (-3+i)\psi_{i,2} + (2-i)\cancel{\psi_{2+i,-1}} = 0 \\
&\Leftrightarrow \psi_{i,2} = -\frac{(-1-i)}{i-3}\psi_{-1+i,2} \text{ for } i \text{ increasing} && (4.70)\\
&\Leftrightarrow \psi_{-1+i,2} = -\frac{i-3}{(-1-i)}\psi_{i,2} \text{ for } i \text{ decreasing}. && (4.71)
\end{aligned}$$

Starting with $i = 4$ in the first recurrence relation (4.70), i increasing, we see that all $\psi_{i,2}$ $i \geq 4$ are generated by $\psi_{3,2}$. Starting with $i = -2$ in the second recurrence relation (4.71), i decreasing, we see that all $\psi_{i,2}$ $i \leq -3$ are generated by $\psi_{-2,2}$. Thus we have three generating coefficients in total for the time being. However, the cocycle condition (4.14) for $(2,-2,3)$ produces a non-trivial relation between two generating coefficients:

$$-11\psi_{-2,3} - 4\cancel{\psi_{0,3}} + 5\cancel{\psi_{1,2}} - 15\psi_{2,-2} - 5\psi_{2,3} - \psi_{5,-2} = 0 \Leftrightarrow \psi_{2,-2} = \psi_{2,3}\,.$$

Similarly, the cocycle condition (4.14) for $(2,-2,-3)$ gives another non-trivial relation:

$$-\psi_{-5,2} - 5\psi_{-2,-3} + 5\cancel{\psi_{-1,-2}} - 4\cancel{\psi_{0,-3}} - 11\psi_{2,-3} + 15\psi_{2,-2} = 0 \Leftrightarrow \psi_{-3,-2} = \psi_{2,-2}\,.$$

Therefore, only one generating coefficient remains.
As usual, we can use induction on k for the last step. We want to show that $\psi_{i,j}$ is generated by at most one generating coefficient, say $\psi_{2,-2}$. The statement holds true for $k = 0,1,2$. Suppose it is true for level $k \geq 2$, and let us see what happens for level $k+1$. The cocycle condition (4.14) for $(i,1,k)$ yields:

$$\begin{aligned}
&-(1+5i+k)\cancel{\psi_{1,k}} - (1+i+5k)\cancel{\psi_{i,1}} + (5+i+k)\psi_{i,k} \\
&+(1-i)\psi_{1+i,k} + (-1+k)\psi_{1+k,i} - (-i+k)\cancel{\psi_{i+k,1}} = 0 \\
&\Leftrightarrow (1-k)\psi_{1+k,i} = (5+i+k)\psi_{i,k} + (1-i)\psi_{1+i,k} . \qquad (4.72)
\end{aligned}$$

The terms on the right side are of level k and generated at most by $\psi_{2,-2}$ due to the induction hypothesis. Consequently, the term on the left of level $k+1$ is also generated by at most one generating coefficient.
The statement is also true for $k = 0,-1,-2$. Suppose it is true for level $k \leq -2$, and let us see what happens for level $k-1$. The cocycle condition (4.14) for $(i,-1,k)$ yields:

$$\begin{aligned}
&-(-1+5i+k)\cancel{\psi_{-1,k}} + (-1-i)\psi_{-1+i,k} - (-1+i+5k)\cancel{\psi_{i,-1}} \\
&+(-5+i+k)\psi_{i,k} + (1+k)\psi_{-1+k,i} - (-i+k)\cancel{\psi_{i+k,-1}} = 0 \\
&\Leftrightarrow -(1+k)\psi_{-1+k,i} = +(-1-i)\psi_{-1+i,k} + (-5+i+k)\psi_{i,k} . \qquad (4.73)
\end{aligned}$$

The terms on the right side are of level k and generated at most by $\psi_{2,-2}$ due to the induction hypothesis. Consequently, the term on the left of level $k-1$ is also generated by at most one generating coefficient.
All in all, we can conclude that the second cohomology with $\lambda = 5$ is at most one-dimensional. □

Proposition 4.2.8. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda = 5$ has minimally a dimension of one, i.e.*

$$1 \leq \dim \mathrm{H}^2(\mathcal{W}, \mathcal{F}^5) .$$

Proof. To prove that the dimension is at least one, we have to find a non-trivial cochain fulfilling the cocycle condition for $\lambda = 5$. We can obtain a generating, non-trivial cocycle by solving the recurrence relations appearing in the previous proof. An alternative way of obtaining a non-trivial cocycle is given by guessing a candidate generator of $\mathrm{H}^2(\mathcal{W}, \mathcal{F}^5)$ based on the continuous cohomology [90], and then prove that it is indeed a generator. We will define our candidate generating degree-zero cochain the following way:

$$\psi(e_i, e_j) = \psi_{i,j}\ f_{i+j} := (i^3 j^4 - i^4 j^3)\ f_{i+j} . \qquad (4.74)$$

Clearly, this expression is antisymmetric in i and j. Inserting the coefficients $\psi_{i,j} = (i^3 j^4 - i^4 j^3)$ into the cocycle condition for $\lambda = 5$, expanding everything, we can verify straightaway that ψ is a cocycle.
Let us check that ψ is not a coboundary. We will write down the coboundary condition (4.15) for $\lambda = 5$ for several combinations of the elements $e_{-3}, \ldots, e_3$ and compare them to the coefficients

$\psi_{i,j}$:

$$\begin{array}{ll}
-8\phi_{-2}-4\phi_0-8\phi_2 & \psi_{2,-2}=256\,,\\
9\phi_{-1}+3\phi_1+3\phi_2 & \psi_{-1,2}=-24\,,\\
-3\phi_{-2}-3\phi_{-1}-9\phi_1 & \psi_{1,-2}=24\,,\\
7\phi_{-3}+5\phi_{-1}+13\phi_2 & \psi_{-3,2}=-1080\,,\\
2\phi_{-3}+4\phi_{-2}+14\phi_1 & \psi_{-3,1}=-108\,,\\
13\phi_{-2}+5\phi_1+7\phi_3 & \psi_{-2,3}=-1080\,,\\
12\phi_{-3}+6\phi_0+12\phi_3 & \psi_{-3,3}=-4374\,.
\end{array}$$

Equaling the coboundary conditions with the coefficients $\psi_{i,j}$ yields a system which is incompatible:

$$\mathrm{rk}\begin{pmatrix} 0 & -8 & 0 & -4 & 0 & -8 & 0\\ 0 & 0 & 9 & 0 & 3 & 3 & 0\\ 0 & -3 & -3 & 0 & -9 & 0 & 0\\ 7 & 0 & 5 & 0 & 0 & 13 & 0\\ 2 & 4 & 0 & 0 & 14 & 0 & 0\\ 0 & 13 & 0 & 0 & 5 & 0 & 7\\ 12 & 0 & 0 & 6 & 0 & 0 & 12 \end{pmatrix}=6$$

$$\neq \mathrm{rk}\left(\begin{array}{ccccccc|c} 0 & -8 & 0 & -4 & 0 & -8 & 0 & 256\\ 0 & 0 & 9 & 0 & 3 & 3 & 0 & -24\\ 0 & -3 & -3 & 0 & -9 & 0 & 0 & 24\\ 7 & 0 & 5 & 0 & 0 & 13 & 0 & -1080\\ 2 & 4 & 0 & 0 & 14 & 0 & 0 & -108\\ 0 & 13 & 0 & 0 & 5 & 0 & 7 & -1080\\ 12 & 0 & 0 & 6 & 0 & 0 & 12 & -4374 \end{array}\right)=7\,.$$

Hence, our cocycle ψ cannot be a coboundary, meaning that $\mathrm{H}^2(\mathcal{W},\mathcal{F}^5)$ is at least one-dimensional. □

Proof of Theorem 4.2.5. Propositions 4.2.7 and 4.2.8 prove Theorem 4.2.5. □

Remark 4.2.1. Recall that for $\lambda = 5$ and $\lambda = 7$, the recurrence relations are much easier to solve than for $\lambda \in \{0,1,2\}$. For $\lambda = 5$, the solution of the recurrence relations reads,

$$\psi_{i,k} = -\frac{1}{144}(i-1)i(i+1)(i-k)k(k-1)(k+1)\psi_{2,-2}\,. \tag{4.75}$$

In comparison, the expression for the cocycle we found inspired by continuous cohomology (4.74), i.e. $\psi_{i,k} = i^3k^4 - i^4k^3$, is much simpler, meaning that the cocycle in (4.75) has many coboundary terms.

Results for $\lambda = 7$

The last critical value corresponds to $\lambda = 7$.

Theorem 4.2.6. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda = 7$ is one-dimensional, i.e.*

$$\dim \mathrm{H}^2(\mathcal{W},\mathcal{F}^7) = 1\,.$$

Proposition 4.2.9. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda = 7$ has a maximal dimension of one, i.e.*

$$\dim \mathrm{H}^2(\mathcal{W}, \mathcal{F}^7) \leq 1\,. \tag{4.76}$$

Proof. In the proof of Theorem 4.2.1, we see that just like for $\lambda = 5$, the assumption $\lambda \neq 7$ appears quite late in the proof, at levels plus and minus two, analyzed in Lemma 4.2.4. Thus, the proof of $\psi_{i,1} = \psi_{i,0} = \psi_{i,-1} = 0$, given by Lemmata 4.2.1, 4.2.2 and 4.2.3, is valid for the case $\lambda = 7$ too. Consequently, we can immediately start with the levels plus and minus two. The cocycle condition (4.14) for $(i,-2,1)$ and $\lambda = 7$ gives the recurrence relations for level minus two:

$$\begin{aligned}
&-(-1+7i)\cancel{\psi_{-2,1}} + 3\cancel{\psi_{-1,i}} + (-2-i)\cancel{\psi_{-2+i,1}} \\
&-(5+i)\psi_{i,-2} + (-13+i)\cancel{\psi_{i,1}} - (1-i)\psi_{1+i,-2} = 0 \\
&\Leftrightarrow \psi_{i,-2} = -\frac{(1-i)}{i+5}\psi_{1+i,-2} \text{ for } i \text{ decreasing} &(4.77)\\
&\Leftrightarrow \psi_{1+i,-2} = -\frac{(i+5)}{1-i}\psi_{i,-2} \text{ for } i \text{ increasing}. &(4.78)
\end{aligned}$$

From the first recurrence relation 4.77, we see that $\psi_{-4,-2} = \psi_{-3,-2} = 0$, by inserting successively $i = -3$ and $i = -4$. Starting with $i = -6$ in the first recurrence relation 4.77, decreasing i, we see that all $\psi_{i,-2}$ $i \leq -6$ are generated by $\psi_{-5,-2}$. Starting with $i = 2$ in the second recurrence relation 4.78, increasing i, we see that all $\psi_{i,-2}$ $i \geq 3$ are generated by $\psi_{2,-2}$. A priori, level minus two is generated by two generating coefficients.

The cocycle condition (4.14) for $(i,2,-1)$ gives the recurrence relations for level plus two:

$$\begin{aligned}
&-3\cancel{\psi_{1,i}} - (1+7i)\cancel{\psi_{2,-1}} - (-1-i)\psi_{-1+i,2} \\
&+(13+i)\cancel{\psi_{i,-1}} - (-5+i)\psi_{i,2} + (2-i)\cancel{\psi_{2+i,-1}} = 0 \\
&\Leftrightarrow \psi_{i,2} = -\frac{(-1-i)}{i-5}\psi_{-1+i,2} \text{ for } i \text{ increasing} &(4.79)\\
&\Leftrightarrow \psi_{-1+i,2} = -\frac{i-5}{(-1-i)}\psi_{i,2} \text{ for } i \text{ decreasing}. &(4.80)
\end{aligned}$$

From the first recurrence relation (4.79), we see that $\psi_{4,2} = \psi_{3,2} = 0$, by inserting successively $i = 3$ and $i = 4$. Starting with $i = 6$ in the first recurrence relation (4.79), i increasing, we see that all $\psi_{i,2}$ $i \geq 6$ are generated by $\psi_{5,2}$. Starting with $i = -2$ in the second recurrence relation (4.80), i decreasing, we see that all $\psi_{i,2}$ $i \leq -3$ are generated by $\psi_{-2,2}$. Thus, we currently have three generating coefficients in total. However, the cocycle condition (4.14) for $(2,-2,5)$ produces a non-trivial relation between two generating coefficients:

$$-17\psi_{-2,5} - 4\cancel{\psi_{0,5}} - 35\psi_{2,-2} - 7\psi_{2,5} + 7\cancel{\psi_{3,2}} - 3\psi_{7,-2} = 0 \Leftrightarrow \psi_{2,-2} = -\psi_{5,2}\,.$$

Similarly, the cocycle condition (4.14) for $(2,-2,-5)$ gives another non-trivial relation:

$$-3\psi_{-7,2} + 7\cancel{\psi_{-3,-2}} - 7\psi_{-2,-5} - 4\cancel{\psi_{0,-5}} - 17\psi_{2,-5} + 35\psi_{2,-2} = 0 \Leftrightarrow \psi_{-5,-2} = \psi_{2,-2}\,.$$

Therefore, only one generating coefficient remains.

We use induction on k for the last step. We want to show that $\psi_{i,j}$ is generated by at most one generating coefficient $\psi_{2,-2}$. The statement holds true for $k = 0,1,2$. Suppose it is true for level $k \geq 2$, and let us see what happens for level $k+1$. The cocycle condition (4.14) for $(i,1,k)$ yields:

$$\begin{aligned}
&-(1+7i+k)\cancel{\psi_{1,k}} - (1+i+7k)\cancel{\psi_{i,1}} + (7+i+k)\psi_{i,k} \\
&+(1-i)\psi_{1+i,k} + (-1+k)\psi_{1+k,i} - (-i+k)\cancel{\psi_{i+k,1}} = 0 \\
&\Leftrightarrow (1-k)\psi_{1+k,i} = (7+i+k)\psi_{i,k} + (1-i)\psi_{1+i,k}\,. &(4.81)
\end{aligned}$$

The terms on the right side are of level k and generated at most by $\psi_{2,-2}$ due to the induction hypothesis. Consequently, the term on the left of level $k+1$ is also generated by at most one generating coefficient.
The statement is also true for $k = 0, -1, -2$. Suppose it is true for level $k \leq -2$, and let us see what happens for level $k-1$. The cocycle condition (4.14) for $(i, -1, k)$ yields:

$$\begin{aligned} &-(-1+7i+k)\cancel{\psi_{-1,k}} + (-1-i)\psi_{-1+i,k} - (-1+i+7k)\cancel{\psi_{i,-1}} \\ &+(-7+i+k)\psi_{i,k} + (1+k)\psi_{-1+k,i} - (-i+k)\cancel{\psi_{i+k,-1}} = 0 \\ &\Leftrightarrow -(1+k)\psi_{-1+k,i} = (-1-i)\psi_{-1+i,k} + (-7+i+k)\psi_{i,k}\,. \end{aligned} \tag{4.82}$$

The terms on the right side are of level k and generated at most by $\psi_{2,-2}$ due to the induction hypothesis. Consequently, the term on the left of level $k-1$ is also generated by at most one generating coefficient.
All in all, we can conclude that the second cohomology with $\lambda = 7$ is at most one-dimensional.
□

Proposition 4.2.10. *The second algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda = 7$ has minimally a dimension of one, i.e.*

$$1 \leq \dim \mathrm{H}^2(\mathcal{W}, \mathcal{F}^7)\,.$$

Proof. Once again, we will define our generating degree-zero cocycle inspired by the continuous cohomology:

$$\psi(e_i, e_j) = \psi_{i,j}\, f_{i+j} := 2(i^3 j^6 - i^6 j^3) - 9(i^4 j^5 - i^5 j^4)\, f_{i+j}\,. \tag{4.83}$$

The expression is antisymmetric in i and j. To verify that this cochain is a cocycle, it suffices to insert its coefficients $\psi_{i,j} = 2(i^3 j^6 - i^6 j^3) - 9(i^4 j^5 - i^5 j^4)$ into the cocycle condition (4.14) for $\lambda = 7$, a direct computation shows that the cocycle condition is fulfilled.
Next, we have to check whether our generating cocycle is non-trivial, i.e. whether it is not a coboundary. Let us consider the coboundary condition (4.15) for $\lambda = 7$ for the following combinations of elements $e_{-3}, \ldots, e_3$:

$$\begin{aligned} &-6\phi_{-1} - 2\phi_0 - 6\phi_1 && \psi_{1,-1} = 22\,,\\ &-12\phi_{-2} - 4\phi_0 - 12\phi_2 && \psi_{2,-2} = 11264\,,\\ &-5\phi_{-2} - 3\phi_{-1} - 13\phi_1 && \psi_{1,-2} = 576\,,\\ &11\phi_{-3} + 5\phi_{-1} + 19\phi_2 && \psi_{-3,2} = -73440\,,\\ &4\phi_{-3} + 4\phi_{-2} + 20\phi_1 && \psi_{-3,1} = -4428\,,\\ &19\phi_{-2} + 5\phi_1 + 11\phi_3 && \psi_{-2,3} = -73440\,,\\ &18\phi_{-3} + 6\phi_0 + 18\phi_3 && \psi_{-3,3} = -433026\,. \end{aligned}$$

Forcing equality between the coboundary conditions and the coefficients $\psi_{i,j}$ yields a system which is incompatible:

$$\mathrm{rk}\begin{pmatrix} 0 & 0 & -6 & -2 & -6 & 0 & 0 \\ 0 & -12 & 0 & -4 & 0 & -12 & 0 \\ 0 & -5 & -3 & 0 & -13 & 0 & 0 \\ 11 & 0 & 5 & 0 & 0 & 19 & 0 \\ 4 & 4 & 0 & 0 & 20 & 0 & 0 \\ 0 & 19 & 0 & 0 & 5 & 0 & 11 \\ 18 & 0 & 0 & 6 & 0 & 0 & 18 \end{pmatrix} = 6$$

$$\neq \mathrm{rk}\left(\begin{array}{ccccccc|c} 0 & 0 & -6 & -2 & -6 & 0 & 0 & 22 \\ 0 & -12 & 0 & -4 & 0 & -12 & 0 & 11264 \\ 0 & -5 & -3 & 0 & -13 & 0 & 0 & 576 \\ 11 & 0 & 5 & 0 & 0 & 19 & 0 & -73440 \\ 4 & 4 & 0 & 0 & 20 & 0 & 0 & -4428 \\ 0 & 19 & 0 & 0 & 5 & 0 & 11 & -73440 \\ 18 & 0 & 0 & 6 & 0 & 0 & 18 & -433026 \end{array}\right) = 7\,.$$

The incompatibility of this system implies that our generating cocycle ψ is not a coboundary. Hence, the dimension of $\mathrm{H}^2(\mathscr{W},\mathscr{F}^7)$ is at least two. □

Proof of Theorem 4.2.6. Propositions 4.2.9 and 4.2.10 prove Theorem 4.2.6. □

Remark 4.2.2. Just as for $\lambda = 5$, the recurrence relations for $\lambda = 7$ are not too hard to solve, and yield:

$$\psi_{i,k} = -\frac{1}{8640} i(i^2-1)(i-k)k(k^2-1)(2i^2-7ik+16+2k^2)\psi_{2,-2}\,. \tag{4.84}$$

Clearly, the expression found by inspiring from continuous cohomology (4.83), i.e. $\psi_{i,j} = 2(i^3 j^6 - i^6 j^3) - 9(i^4 j^5 - i^5 j^4)$, is much simpler than the one (4.84) coming from solving the recurrence relation. This means that the cocycle in (4.84) contains many coboundary terms.

4.3 The third algebraic cohomology

In this section, we analyze the third algebraic cohomology of the Witt algebra with values in $\mathscr{F}^\lambda$. Unfortunately, we were able to settle with our method only the case of finitely many values for λ. In fact, the complexity of the proof concerning the third cohomology increases rather dramatically compared to the proofs involving the second cohomology. To simplify the proofs, we consider $\lambda \in \mathbb{Z}$ rather than $\lambda \in \mathbb{C}$. A main problem consists in the fact that the length of the proof increases with the absolute value $|\lambda|$ of λ. To avoid doing all the computations by hand, it is sensible to implement the proof in a programming language able to do symbolic computations. For this thesis, I used *Mathematica*. However, the running times obviously increase with the length of the proofs, hence only a finite number of values of $\lambda \in \mathbb{Z}$ with low $|\lambda|$ have been treated.

In the case of the second cohomology $\mathrm{H}^2(\mathscr{W},\mathscr{F}^\lambda)$, the proof of its vanishing given in Theorem 4.2.1 was much easier for $\lambda \in \mathbb{C}\setminus\mathbb{N}$ than for $\lambda \in \mathbb{N}$, which also contains values of λ where the cohomology does not vanish. In the former case, only minor modifications had to be implemented in the proof corresponding to $\lambda = -1$ in order to extend it to $\lambda \in \mathbb{C}\setminus\mathbb{N}$, whereas major changes had to be applied in the second case.

The situation is similar for the third cohomology $\mathrm{H}^3(\mathscr{W},\mathscr{F}^\lambda)$ in the sense that the proof for $\lambda < 0$

is more straightforward than the one for $\lambda > 0$. However, contrary to the second cohomology, minor modifications of the proof for $\lambda = -1$ are not sufficient to extend it to $\lambda < 0$, at least not in the last step of the proof.
We will only focus on the degree-zero cohomology, as the cohomology related to the non-zero degree cohomology of the Witt algebra is zero due Theorem 2.2.1. Let us start by writing down the coboundary and cocycle conditions (2.68) with generic λ, in terms of coefficients. Suppose ψ is a degree-zero 3-cochain of $C^3(\mathcal{W},\mathcal{F}^\lambda)$, then it can be written as $\psi^\lambda(e_i,e_j,e_k) = \psi^\lambda_{i,j,k} f^\lambda_{i+j+k}$ for suitable coefficients $\psi_{i,j,k} \in \mathbb{K}$. The condition for ψ to be a 3-cocycle, evaluated on basis elements (e_i,e_j,e_k,e_l) and expressed in terms of the coefficients $\psi_{i,j,k}$, is given by,

$$\begin{aligned}(\delta_3\psi^\lambda)(e_i,e_j,e_k,e_l) =&(j-i)\psi^\lambda_{i+j,k,l} - (k-i)\psi^\lambda_{i+k,j,l} + (l-i)\psi^\lambda_{i+l,j,k}\\ &+ (k-j)\psi^\lambda_{k+j,i,l} - (l-j)\psi^\lambda_{l+j,i,k} + (l-k)\psi^\lambda_{l+k,i,j}\\ &- (j+k+l+\lambda i)\psi^\lambda_{j,k,l} + (i+k+l+\lambda j)\psi^\lambda_{i,k,l}\\ &- (i+j+l+\lambda k)\psi^\lambda_{i,j,l} + (i+j+k+\lambda l)\psi^\lambda_{i,j,k} = 0. \end{aligned} \tag{4.85}$$

Let ϕ be a degree-zero 2-cochain $\phi \in C^2(\mathcal{W},\mathcal{F}^\lambda)$, i.e. $\phi^\lambda(e_i,e_j) = \phi^\lambda_{i,j} f^\lambda_{i+j}$ with adequate coefficients $\phi_{i,j} \in \mathbb{K}$. The condition for the 3-cocycle ψ to be a coboundary, evaluated on basis elements (e_i,e_j,e_k) and expressed in terms of the coefficients $\psi_{i,j,k}$ and $\phi_{i,j}$, is given by, see (2.67),

$$\begin{aligned}\psi^\lambda_{i,j,k} = (\delta_2\phi^\lambda)(e_i,e_j,e_k) =&(j-i)\phi^\lambda_{i+j,k} + (k-j)\phi^\lambda_{k+j,i} + (i-k)\phi^\lambda_{i+k,j}\\ &- (j+k+\lambda i)\phi^\lambda_{j,k} + (i+k+\lambda j)\phi^\lambda_{i,k} - (i+j+\lambda k)\phi^\lambda_{i,j}. \end{aligned} \tag{4.86}$$

In the following, we will drop the superscript λ in order to simplify the notation.

4.3.1 Negative values of λ

We start by analyzing $\mathrm{H}^3(\mathcal{W},\mathcal{F}^\lambda)$ with negative λ, i.e. $\lambda \in \mathbb{Z}\setminus\mathbb{N}$. The aim of this section is to prove Theorem 4.3.1 below.

Theorem 4.3.1. *The third algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in the tensor densities modules $\mathcal{F}^\lambda$ is zero for $\lambda \in I = \{-100,\ldots,-1\}$, i.e.*

$$\mathrm{H}^3(\mathcal{W},\mathcal{F}^\lambda) = \{0\} \textit{ if } \lambda \in I.$$

The proof follows the same structure as the proof of Theorem 3.2.2. We start by performing a cohomological change, subsequently we fix one of the three indices of $\psi_{i,j,k}$ to a certain *level* and derive results for it. In the last step, induction is used on the fixed index to get the desired result. Some parts of the proof are almost identical to the proof of Theorem 3.2.2 given for $\lambda = -1$. Hence, we will not repeat these parts here.

Lemma 4.3.1. *Every 3-cocycle $\psi \in \mathrm{H}^3(\mathcal{W},\mathcal{F}^\lambda)$ with $\lambda \in \mathbb{Z}\setminus\mathbb{N}$ of degree zero is cohomologous to a degree zero 3-cocycle ψ' with:*

$$\begin{aligned}\forall \lambda \in \mathbb{Z}\setminus\mathbb{N}: \quad &\psi'_{i,j,1} = 0 \;\forall\, i \le 0,\; \forall\, j \in \mathbb{Z} \textit{ and } \psi'_{i,j,-1} = 0 \;\forall\, i,j > 0,\\ &\textit{and } \psi'_{i,2,-1} = 0 \;\forall\, i \in \mathbb{Z} \textit{ and } \psi'_{-3+\lambda,2,-2} = 0. \end{aligned}$$

Proof. Let $\psi \in \mathrm{H}^3(\mathscr{W},\mathscr{F}^\lambda)$ and $\phi \in C^2(\mathscr{W},\mathscr{F}^\lambda)$ both be of degree zero with $\lambda \in \mathbb{Z}\setminus\mathbb{N}$. We will begin with a cohomological change $\psi' = \psi - (\delta_2\phi)$ in order to annihilate as many coefficients of ψ as possible. We define: $\phi_{i,1} = 0\ \forall\ i \in \mathbb{Z}$ and $\phi_{2,-1} = 0$. This choice allows one to simplify the notation to a maximum. As we did in the case of $\lambda = -1$ in Lemma 3.2.1, the proof will be separated into three cases depending on the signs of i, j.

Case 1: $i, j \le 0$

The proof for this case is identical to the one given for $\lambda = -1$, hence we will not repeat it here. Actually, there are no poles appearing in the recurrence relations when defining ϕ, because $i, j \le 0$ and also $\lambda < 0$. Thus, we refer instead to the proof of the corresponding case of Lemma 3.2.1. It leads to a definition of all $\phi_{i,j}$, $i, j \le 0$ and $\psi'_{i,j,1} = 0$, $i, j \le 0$ after the cohomological choice.

Case 2: $i \le 0,\ j > 0$

The proof is very similar to the one performed for $\lambda = -1$ given in Lemma 3.2.1, and only slight adaptions are necessary. Let us consider the coboundary condition (4.86) for $(i, j, k) = (-2+\lambda, 2, -1)$:

$$\begin{aligned}&-3\cancel{\phi_{1,-2+\lambda}} - (1+(-2+\lambda)\lambda)\cancel{\phi_{2,-1}} - (1-\lambda)\phi_{-3+\lambda,2}\\ &+(-3+3\lambda)\underline{\phi_{-2+\lambda,-1}} + (4-\lambda)\underline{\phi_{\lambda,-1}} = \psi_{-2+\lambda,2,-1}\,.\end{aligned}$$

The slashed terms are zero by definition of ϕ. The underlined terms with both indices negative have been defined already in the previous case. Since the coefficient $(1-\lambda)$ is different from zero for $\lambda < 0$, we obtain a definition for $\phi_{-3+\lambda,2}$, and consequently $\psi'_{-2+\lambda,2,-1} = 0$.
The coboundary condition (4.86) for $(i, 2, -1)$ suggests the following recursive definition for $\phi_{i,2}$:

$$\begin{aligned}&-3\cancel{\phi_{1,i}} - (1+i\lambda)\cancel{\phi_{2,-1}} - (-1-i)\phi_{-1+i,2}\\ &+(-1+i+2\lambda)\underline{\phi_{i,-1}} - (2+i-\lambda)\phi_{i,2} + (2-i)\underline{\phi_{2+i,-1}} = \psi_{i,2,-1}\\ \Leftrightarrow \phi_{-1+i,2} &= -\frac{(-1+i+2\lambda)}{i+1}\underline{\phi_{i,-1}} + \frac{(2+i-\lambda)}{i+1}\phi_{i,2} - \frac{(2-i)}{i+1}\underline{\phi_{2+i,-1}} + \frac{\psi_{i,2,-1}}{i+1}\,. \end{aligned}\tag{4.87}$$

starting with $i = -3+\lambda$ in the relation above (4.87), decreasing i, we obtain a consistent definition for $\phi_{i,2}$, $i \le -3+\lambda$ since the underlined terms have both indices negative and thus have already been defined in the previous case $i, j \le 0$. As a consequence, we have $\psi'_{i,2,-1} = 0\ \forall\ i \le -2+\lambda$. Next, consider the coboundary condition (4.86) for $(-3+\lambda, 2, -2)$:

$$\begin{aligned}&-4\underline{\phi_{0,-3+\lambda}} - (-3+\lambda)\lambda\phi_{2,-2} - (1-\lambda)\underline{\underline{\phi_{-5+\lambda,2}}}\\ &+(-5+3\lambda)\underline{\phi_{-3+\lambda,-2}} - (-1-\lambda)\underline{\underline{\phi_{-3+\lambda,2}}} + (5-\lambda)\underline{\phi_{-1+\lambda,-2}} = \psi_{-3+\lambda,2,-2}\,.\end{aligned}$$

The terms underlined once have both indices negative and have thus been defined in the previous case $i, j \le 0$. The terms underlined twice are of the form $\phi_{i,2}$, $i \le -3+\lambda$ and are thus also defined already. Finally, as the coefficient $(-3+\lambda)\lambda$ is different from zero for $\lambda < 0$, we obtain a definition for $\phi_{2,-2}$. Moreover, we get $\psi'_{-3+\lambda,2,-2} = 0$. Reinserting the definition of $\phi_{2,-2}$ into the recurrence relation (4.87), starting with $i = -2$, decreasing i, we obtain a definition for $\phi_{i,2}$, $-2 \ge i > -3+\lambda$ leading to $\psi'_{i,2,-1} = 0$, $-2 \ge i > -2+\lambda$.
Since $\phi_{-1,2} = 0$ by definition, only $\phi_{0,2}$ remains to be defined. This can be done by considering the coboundary condition (4.86) for $(0, 2, -1)$:

$$\cancel{\phi_{-1,2}} + (-1+2\lambda)\underline{\phi_{0,-1}} - (2-\lambda)\phi_{0,2} - 3\cancel{\phi_{1,0}} + \cancel{\phi_{2,-1}} = \psi_{0,2,-1}\,.$$

The underlined term has been defined in the previous case, $i, j \leq 0$. Since the coefficient $(2-\lambda)$ is not zero for $\lambda < 0$, we obtain a definition for $\phi_{0,2}$ leading to $\psi'_{0,2,-1} = 0$. Taking everything together, we get $\psi'_{i,2,-1} = 0 \; \forall \; i \leq 0$.
The final step can be performed exactly in the same way as in the proof for $\lambda = -1$ given in the corresponding case in Lemma 3.2.1, leading to a definition of all $\phi_{i,j}$, $i \leq 0, j > 0$ and $\psi'_{i,j,1} = 0$, $i \leq 0, j > 0$. We will not reproduce the reasoning here.

Case 3: $i, j > 0$

The reasoning here is exactly the same as the one from $\lambda = -1$ given in the corresponding case in Lemma 3.2.1, hence I will not reproduce it. It leads to a definition of all $\phi_{i,j}$, $i, j > 0$ and $\psi'_{i,j,-1} = 0$, $i, j > 0$. This concludes the proof. □

Lemma 4.3.2. *Let* $\psi \in \mathrm{H}^3(\mathscr{W}, \mathscr{F}^\lambda)$ *with* $\lambda \in \mathbb{Z} \setminus \mathbb{N}$ *be a cocycle such that:*

$$\forall \lambda \in \mathbb{Z} \setminus \mathbb{N}: \qquad \psi_{i,j,1} = 0 \; \forall \; i \leq 0, \; \forall \; j \in \mathbb{Z} \; and \; \psi_{i,j,-1} = 0 \; \forall \; i, j > 0, \\ and \; \psi_{i,-1,2} = 0 \; \forall \; i \in \mathbb{Z},$$

Then

$$\forall \lambda \in \mathbb{Z} \setminus \mathbb{N}: \qquad \psi_{i,j,0} = 0 \; \forall \; i, j \in \mathbb{Z}.$$

Proof. **Case 1:** $i, j \leq 0$

The corresponding proof given for $\lambda = -1$ in Lemma 3.2.2 also applies to $\lambda < 0$. It leads to the result that level zero with both indices negative is zero: $\psi_{i,j,0} = 0$, $i, j \leq 0$.

Case 2: $i \leq 0$, $j > 0$

In this case, the proof for $\lambda = -1$ in Lemma 3.2.2 has to be adapted slightly in order to make it work for $\lambda < 0$. The cocycle condition (4.85) for $(i, 2, 0, -1)$ provides a recurrence relation on i for $\psi_{i,2,0}$:

$$\begin{aligned} &-\cancel{\psi_{-1,i,2}} + 3\cancel{\psi_{1,i,0}} - (1+i\lambda)\cancel{\psi_{2,0,-1}} - 2\cancel{\psi_{2,i,-1}} - (1+i)\psi_{-1+i,2,0} \\ &+ (-1+i+2\lambda)\underline{\psi_{i,0,-1}} - \cancel{\psi_{i,2,-1}} + (2+i-\lambda)\psi_{i,2,0} - (-2+i)\underline{\psi_{2+i,0,-1}} = 0 \\ \Leftrightarrow \psi_{-1+i,2,0} &= \frac{(2+i-\lambda)}{(1+i)}\psi_{i,2,0}. \end{aligned} \tag{4.88}$$

The slashed terms are zero by assumption, the underlined terms are either zero by assumption or due to the previous case, $\psi_{i,j,0} = 0$, $i, j \leq 0$. Inserting $i = -2+\lambda$ and decreasing i in the recurrence relation (4.88) above, we obtain $\psi_{i,2,0} = 0$, $i \leq -3+\lambda$. The cocycle condition (4.85) for $(i, 2, -2, 0)$ yields:

$$\begin{aligned} &\cancel{2\psi_{-2,i,2}} - i\lambda\psi_{2,-2,0} + \cancel{2\psi_{2,i,-2}} - (-2-i)\underline{\underline{\psi_{-2+i,2,0}}} \\ &+ (-2+i+2\lambda)\underline{\psi_{i,-2,0}} - (2+i-2\lambda)\underline{\underline{\psi_{i,2,0}}} + (2-i)\underline{\psi_{2+i,-2,0}} = 0. \end{aligned}$$

The slashed terms cancel each other and the terms underlined once are zero (for $i \leq -2$) because of the previous case, $\psi_{i,j,0} = 0$, $i, j \leq 0$. If we insert for example $i = -3+\lambda$ into the expression above, the terms underlined twice are zero due to $\psi_{i,2,0} = 0$, $i \leq -3+\lambda$. We thus obtain $(-3+\lambda)\lambda\psi_{2,-2,0} = 0$ and hence $\psi_{2,-2,0} = 0$ as $\lambda < 0$. Inserting $\psi_{2,-2,0} = 0$ into the recurrence relation (4.88), decreasing i, we obtain $\psi_{i,2,0} = 0$, $i \leq -2$. As we also have $\psi_{-1,2,0} = 0$ by assumption, we get $\psi_{i,2,0} = 0$, $i \leq 0$.
The final step is again exactly the same as the one performed for $\lambda = -1$ for the corresponding case in Lemma 3.2.2, which allows one to get $\psi_{i,j,0} = 0$, $i \leq 0$, $j > 0$.

Case 3: $i,j>0$

The reasoning is exactly the same as for $\lambda=-1$, see the corresponding case in the proof of Lemma 3.2.2, which gives $\psi_{i,j,0}=0$, $i,j>0$. All in all, we obtain the announced result. □

Lemma 4.3.3. *Let $\psi\in \mathrm{H}^3(\mathcal{W},\mathcal{F}^\lambda)$ with $\lambda\in\mathbb{Z}\setminus\mathbb{N}$ be a cocycle such that:*

$$\begin{aligned}\forall\lambda\in\mathbb{Z}\setminus\mathbb{N}: \qquad &\psi_{i,j,1}=0\ \forall\ i\le 0,\ \forall\ j\in\mathbb{Z}\ \textit{and}\ \psi_{i,j,-1}=0\ \forall\ i,j>0,\\ &\textit{and}\ \psi_{i,-1,2}=0\ \forall\ i\in\mathbb{Z},\ \textit{and}\ \psi_{i,j,0}=0\ \forall\ i,j\in\mathbb{Z}.\end{aligned}$$

Then

$$\forall\lambda\in\mathbb{Z}\setminus\mathbb{N}: \qquad \psi_{i,j,1}=0\ \forall\ i,j\in\mathbb{Z}\ \textit{and}\ \psi_{i,j,-1}=0\ \forall\ i,j\in\mathbb{Z}.$$

Proof. For all three cases, $i,j\le 0$, $i\le 0$, $j>0$ and $i,j>0$, the reasoning is exactly the same as the one given in the proof of Lemma 3.2.3 for $\lambda=-1$. Therefore, we will not reproduce the proof here. □

The techniques employed in the proof below remain the same as the ones used in the proof for $\lambda=-1$ in Lemma 3.2.4, but the calculus becomes longer the lower λ is. This leads to the fact that for a fixed λ, a proof can in principle be given explicitly. Of course, if one wants to go to very low values of λ, it is sensible to implement the calculus into a symbolic programming language to speed it up. However, this method does not provide a proof working for generic negative values of λ.

Lemma 4.3.4. *Let $\psi\in \mathrm{H}^3(\mathcal{W},\mathcal{F}^\lambda)$ with $\lambda\in\mathbb{Z}\setminus\mathbb{N}$ be a cocycle such that:*

$$\forall\lambda\in\mathbb{Z}\setminus\mathbb{N}: \qquad \psi_{i,j,1}=\psi_{i,j,-1}=\psi_{i,j,0}=0\ \forall\ i,j\in\mathbb{Z}\ \textit{and}\ \psi_{-3+\lambda,2,-2}=0.$$

Then

$$\forall\lambda\in\{-100,\dots,-1\}: \qquad \psi_{i,j,k}=0\ \forall\ i,j,k\in\mathbb{Z}.$$

Proof. **Case 1:** $i,j\le 0$

The reasoning made for $\lambda=-1$ in Lemma 3.2.4 holds true for $\lambda<0$, leading to the result $\psi_{i,j,k}=0$, $i,j,k\le 0$.

Case 2: $i\le 0$, $j>0$

Again, it is for this case where the differences with the proof for $\lambda=-1$ in Lemma 3.2.4 show up. The major problem results from the fact that we were forced in Lemma 4.3.1 to put $\psi_{-3+\lambda,2,-2}$ equal to zero with a cohomological change instead of $\psi_{-4,2,-2}$ as was the case for $\lambda=-1$. The coefficient $\psi_{-4,2,-2}$ can be put immediately into a relation with $\psi_{-3,2,-2}$. Hence, if $\psi_{-4,2,-2}$ is zero it implies immediately the vanishing of $\psi_{-3,2,-2}$, which is the basis step upon which the whole proof relies. For $\lambda=-2$, the coefficient $\psi_{-5,2,-2}$ cannot be put into an immediate relation with $\psi_{-3,2,-2}$. There is an intermediate step involving $\psi_{-4,2,-2}$. For $\lambda=-3$, the coefficient $\psi_{-6,2,-2}$ has to be put into a relation with $\psi_{-3,2,-2}$, which necessitates two intermediate steps involving $\psi_{-5,2,-2}$ and $\psi_{-4,2,-2}$, and so on. We will show this explicitly in the following.

The main difficulty consists in proving $\psi_{i,j,2}=\psi_{i,j,-2}=0$, $i\le 0$, $j>0$. To do this, we first need to establish a relation between $\psi_{2,-3,-2}$ and $\psi_{-2,3,2}$, which are the basements of the proof. The cocycle condition (4.85) for $(i,3,2,-1)$ provides us with a recurrence relation on i for $\psi_{i,3,2}$:

$$\begin{aligned}&-3\cancel{\psi_{1,i,3}}-(4+i\lambda)\cancel{\psi_{3,2,-1}}-\psi_{5,i,-1}-(1+i)\psi_{-1+i,3,2}+(1+i+3\lambda)\cancel{\psi_{i,2,-1}}\\ &-(2+i+2\lambda)\cancel{\psi_{i,3,-1}}+(5+i-\lambda)\psi_{i,3,2}+(-2+i)\cancel{\psi_{2+i,3,-1}}-(-3+i)\cancel{\psi_{3+i,2,-1}}=0\\ &\Leftrightarrow \psi_{-1+i,3,2}=\frac{(5+i-\lambda)}{(1+i)}\psi_{i,3,2}\,.\end{aligned} \tag{4.89}$$

The slashed terms are of level plus one or minus one, and are thus zero by assumption. For later use, we will express the values of $\psi_{i,3,2}$ for $i = -3, -5$ in terms of $\psi_{-2,3,2}$:

$$\psi_{-3,3,2} = (-3+\lambda)\psi_{-2,3,2}\,, \tag{4.90}$$

$$\psi_{-5,3,2} = \frac{(-3+\lambda)(-2+\lambda)(-1+\lambda)}{6}\psi_{-2,3,2}\,. \tag{4.91}$$

The cocycle condition (4.85) on $(-3, j, -2, 1)$ provide us with a recurrence relation on j for $\psi_{j,-3,-2}$:

$$\begin{aligned}
&-\cancel{\psi_{-5,j,1}} + (-4+j\lambda)\cancel{\psi_{-3,-2,1}} + (-5+j+\lambda)\psi_{-3,j,-2} + (2-j+2\lambda)\cancel{\psi_{-3,j,1}} + 3\cancel{\psi_{-1,-3,j}}\\
&+(3+j)\cancel{\psi_{-3+j,-2,1}} - (2+j)\cancel{\psi_{-2+j,-3,1}} + (1-j+3\lambda)\cancel{\psi_{j,-2,1}} + (-1+j)\psi_{1+j,-3,-2} = 0\\
&\Leftrightarrow \psi_{1+j,-3,-2} = \frac{(-5+j+\lambda)}{(-1+j)}\psi_{j,-3,-2}\,.
\end{aligned} \tag{4.92}$$

The slashed terms are of level plus one or minus one and are thus zero because of our assumptions. For later use, we will express the values of $\psi_{j,-3,-2}$ for $j = 3, 5$ in terms of $\psi_{2,-3,-2}$:

$$\psi_{3,-3,-2} = (-3+\lambda)\psi_{2,-3,-2}\,, \tag{4.93}$$

$$\psi_{5,-3,-2} = \frac{(-3+\lambda)(-2+\lambda)(-1+\lambda)}{6}\psi_{2,-3,-2}\,. \tag{4.94}$$

By inserting the coefficients (4.90), (4.91), (4.93) and (4.94) into the cocycle condition (4.85) for $(-3, 3, 2, -2)$, we obtain a relation between $\psi_{2,-3,-2}$ and $\psi_{-2,3,2}$:

$$\begin{aligned}
&\psi_{-5,3,2} + 3(-1+\lambda)\psi_{-3,2,-2} - 2(-1+\lambda)\psi_{-3,3,-2} - 2(-1+\lambda)\psi_{-3,3,2} - 5\cancel{\psi_{-1,3,-2}}\\
&-4\cancel{\psi_{0,-3,3}} + 6\cancel{\psi_{0,2,-2}} + 5\cancel{\psi_{1,-3,2}} + 3(-1+\lambda)\psi_{3,2,-2} - \psi_{5,-3,-2} = 0\\
&\Leftrightarrow \frac{(-3+\lambda)(-2+\lambda)(-1+\lambda)}{6}\psi_{-2,3,2} - 3(-1+\lambda)\psi_{2,-3,-2} + 2(-1+\lambda)(-3+\lambda)\psi_{2,-3,-2}\\
&-2(-1+\lambda)(-3+\lambda)\psi_{-2,3,2} + 3(-1+\lambda)\psi_{-2,3,2} - \frac{(-3+\lambda)(-2+\lambda)(-1+\lambda)}{6}\psi_{2,-3,-2} = 0\\
&\Leftrightarrow \frac{1}{6}(-12+\lambda)(-5+\lambda)(-1+\lambda)(\psi_{-2,3,2} - \psi_{2,-3,-2}) = 0\,.
\end{aligned}$$

The slashed terms are zero by assumption, as they are of level plus one, level minus one or level zero. The zeros of the polynomial in λ in the last line are given by $\lambda = 1, 5, 12$. Hence, for $\lambda < 0$ the coefficient in front of $(\psi_{-2,3,2} - \psi_{2,-3,-2})$ is different from zero, which implies $\psi_{-2,3,2} = \psi_{2,-3,-2}$. In the next step, we want to obtain a relation between $\psi_{-2,3,2} = \psi_{2,-3,-2}$ and the value which is cohomologously set to zero, i.e. $\psi_{2,-3+\lambda,-2}$. This is done by considering the cocycle condition (4.85) for $(i, 3, 2, -2)$ with $-3+\lambda \le i < -3$, and by expressing all the coefficients $\psi_{i,j,k}$ appearing in there in terms of $\psi_{-2,3,2} = \psi_{2,-3,-2}$ and $\psi_{2,i,-2}$. The cocycle condition (4.85) for $(i, 3, 2, -2)$ appears as follows, after dropping terms of level zero and one:

$$\begin{aligned}
&-(3+i\lambda)\psi_{3,2,-2} - \psi_{5,i,-2} - (2+i)\psi_{-2+i,3,2} + (i+3\lambda)\psi_{i,2,-2}\\
&-(1+i+2\lambda)\psi_{i,3,-2} + (5+i-2\lambda)\psi_{i,3,2} + (-2+i)\psi_{2+i,3,-2} - (-3+i)\psi_{3+i,2,-2} = 0\,.
\end{aligned} \tag{4.95}$$

We need recurrence relations on i for $\psi_{3,i,-2}$, indirectly for $\psi_{4,i,-2}$, and for $\psi_{5,i,-2}$, which must be expressed in terms of $\psi_{2,i,-2}$ and $\psi_{2,-3,-2}$. The recurrence relations can be obtained by considering the cocycle conditions (4.85) for $(i,2,-2,1)$, $(i,3,-2,1)$ and $(i,4,-2,1)$, which yield respectively after dropping terms of level zero and one:

$$\psi_{3,i,-2} = (i+\lambda)\psi_{2,i,-2} - (-1+i)\psi_{2,1+i,-2}\,, \tag{4.96}$$

$$\psi_{4,i,-2} = \frac{(1+i+\lambda)}{2}\psi_{3,i,-2} - \frac{(-1+i)}{2}\psi_{3,1+i,-2}\,, \tag{4.97}$$

$$\psi_{5,i,-2} = \frac{(2+i+\lambda)}{3}\psi_{4,i,-2} - \frac{(-1+i)}{3}\psi_{4,1+i,-2}\,. \tag{4.98}$$

For later use, let us solve the recurrence relations (4.96)-(4.98) above for $i = -4,-5,-6$:

For $i=-4$:

$$\begin{aligned}
\psi_{3,-4,-2} &= (-4+\lambda)\psi_{2,-4,-2} + 5\psi_{2,-3,-2}\,,\\
\psi_{4,-4,-2} &= \frac{1}{2}(-3+\lambda)((-4+\lambda)\psi_{2,-4,-2} + 10\psi_{2,-3,-2})\,,\\
\psi_{5,-4,-2} &= \frac{1}{6}(-3+\lambda)(-2+\lambda)((-4+\lambda)\psi_{2,-4,-2} + 15\psi_{2,-3,-2})\,.
\end{aligned}$$

For $i=-5$:

$$\begin{aligned}
\psi_{3,-5,-2} &= (-5+\lambda)\psi_{2,-5,-2} + 6\psi_{2,-4,-2}\,,\\
\psi_{4,-5,-2} &= \frac{1}{2}(-4+\lambda)((-5+\lambda)\psi_{2,-5,-2} + 12\psi_{2,-4,-2}) + 15\psi_{2,-3,-2}\,,\\
\psi_{5,-5,-2} &= \frac{1}{6}(-3+\lambda)((-4+\lambda)((-5+\lambda)\psi_{2,-5,-2} + 18\psi_{2,-4,-2}) + 90\psi_{2,-3,-2})\,.
\end{aligned}$$

For $i=-6$:

$$\begin{aligned}
\psi_{3,-6,-2} &= (-6+\lambda)\psi_{2,-6,-2} + 7\psi_{2,-5,-2}\,,\\
\psi_{4,-6,-2} &= \frac{1}{2}((-5+\lambda)((-6+\lambda)\psi_{2,-6,-2} + 14\psi_{2,-5,-2}) + 42\psi_{2,-4,-2})\,,\\
\psi_{5,-6,-2} &= \frac{1}{6}(-4+\lambda)((-5+\lambda)((-6+\lambda)\psi_{2,-6,-2} + 21\psi_{2,-5,-2}) + 126\psi_{2,-4,-2})\\
&\quad + 35\psi_{2,-3,-2}\,.
\end{aligned}$$

As an example, we will write down the cocycle condition (4.95) for $i = -4,-5,-6$. Using the recurrence relations (4.96)-(4.98) and (4.89) for $i = -4,-5,-6$, as well as the fact that $\psi_{-2,3,2} = \psi_{2,-3,-2}$, the cocycle condition $(i,3,2,-2)$ for $i = -4,-5,-6$ yields respectively:

For $i=-4$:

$$\frac{1}{12}(-6+\lambda)(20+(-15+\lambda)\lambda)(-2\psi_{2,-4,-2} + (3+\lambda)\psi_{2,-3,-2}) = 0\,, \tag{4.99}$$

For $i=-5$:

$$\begin{aligned}
&-\frac{1}{6}(-15+\lambda)(-7+\lambda)(-2+\lambda)\psi_{2,-5,-2} - 3(20+(-11+\lambda)\lambda)\psi_{2,-4,-2}\\
&+\frac{1}{120}(2520+\lambda(-138+\lambda(-425+\lambda(255+\lambda(-55+3\lambda)))))\psi_{2,-3,-2} = 0\,,
\end{aligned} \tag{4.100}$$

For $i=-6$:

$$\begin{aligned}
&\frac{1}{360}(-60((-8+\lambda)(42+(-19+\lambda)\lambda)\psi_{2,-6,-2}\\
&+21(-10+\lambda)(-3+\lambda)\psi_{2,-5,-2} + 78(-4+\lambda)\psi_{2,-4,-2})\\
&+(-2520+\lambda(2226+\lambda(23+\lambda(-210+\lambda(155+2(-18+\lambda)\lambda)))))\psi_{2,-3,-2}) = 0\,.
\end{aligned} \tag{4.101}$$

The first equation (4.99) for $i = -4$ gives us a non-trivial relation between $\psi_{2,-4,-2}$ and $\psi_{2,-3,-2}$ for $\lambda \neq -3$, hence $\psi_{2,-4,-2}$ can be expressed in terms of $\psi_{2,-3,-2}$. The coefficient in front is different from zero for integer λ. For $\lambda = -3$, the first equation (4.99) immediately implies $\psi_{2,-4,-2} = 0$. This does not influence the generic reasoning. Actually, for $\lambda = -3$ the coefficient zero by assumption is $\psi_{2,-3+\lambda,-2} = \psi_{2,-6,-2}$, thus the relevant cocycle condition corresponds to

$(-6,3,2,-2)$, i.e. the third equation (4.101) above for $i=-6$. Let us check whether it provides us with a non-trivial relation between $\psi_{2,-6,-2}$ and $\psi_{2,-3,-2}$. In the second equation (4.100) for $i=-5$ the coefficient $\psi_{2,-4,-2}$ can be written in terms of $\psi_{2,-3,-2}$ by using the first equation (4.99) for $i=-4$, which gives:

$$\frac{1}{120}(-15+\lambda)(-2+\lambda)(-20(-7+\lambda)\psi_{2,-5,-2}+(-276+\lambda(-83+\lambda(-4+3\lambda)))\psi_{2,-3,-2})=0\,.$$

The coefficients involving λ are different from zero for negative integer λ. Therefore, we obtain a non-trivial relation between $\psi_{2,-5,-2}$ and $\psi_{2,-3,-2}$. Expressing the former in terms of the latter and inserting it into the cocycle condition (4.101) for $(-6,3,2,-2)$, we obtain:

$$\begin{aligned}&\frac{1}{180(-7+\lambda)}((-8+\lambda)(42+(-19+\lambda)\lambda)(-30(-7+\lambda)\psi_{2,-6,-2}\\&+(5+\lambda)(-102+\lambda(-16+(-3+\lambda)\lambda))\psi_{2,-3,-2}))=0\,.\end{aligned}$$

Except for the factor $(5+\lambda)$, which is zero for $\lambda=-5$, the other factors involving λ are different from zero for negative integer λ. In particular, for $\lambda=-3$, we see that we obtain a non-trivial relation between $\psi_{2,-6,-2}$ and $\psi_{2,-3,-2}$. The coefficient $\psi_{2,-6,-2}$ is zero by assumption, which implies that $\psi_{2,-3,-2}$ is also zero. This was the aim. Note that for $\lambda=-5$, we obtain $\psi_{2,-6,-2}=0$. Again, this is not an issue, since for $\lambda=-5$, the coefficient which is zero by assumption corresponds to $\psi_{2,-3+\lambda,-2}=\psi_{2,-8,-2}$, implying that the relevant cocycle condition corresponds to $(-8,3,2,-2)$. This condition will provide a non-trivial relation between $\psi_{2,-8,-2}$ and $\psi_{2,-3,-2}$. It continues like that. The lower λ is, the more equations one has to consider. I found no proof with a fixed length for all $\lambda\in\mathbb{Z}\setminus\mathbb{N}$. Induction on i in the cocycle condition $(i,3,2,-2)$ is hard to use as the degree of the polynomials in λ increases with decreasing i. Induction on λ is also hard to implement, as the coefficients $\psi^{\lambda}_{i,j,k}$ and $\psi^{\lambda-1}_{i,j,k}$ are not related a priori.
The procedure can be encoded in a symbolic programming language though. Basically, it can be implemented in three steps,

- In a first step, the recurrence relations (4.89) on i for $\psi_{i,3,2}$ have to be implemented as well as the recurrence relations (4.96)-(4.98) on i for $\psi_{3,i,-2}$, $\psi_{4,i,-2}$ and $\psi_{3,i,-2}$.

- In a second step, we implement our favorite cocycle condition $(i,3,2,-2)$ given in (4.95), and we write it down for $i=-4,\cdots-3+\lambda$. In each of the equations thus obtained, we replace the coefficients $\psi_{i,3,2}$, $\psi_{3,i,-2}$, $\psi_{4,i,-2}$ and $\psi_{3,i,-2}$ by the recurrence relations previously implemented, and we also take into account $\psi_{2,-3+\lambda,-2}=0$.

- In the last step, we solve the system of linear equations in the variables $\psi_{-2,2,i}$, $i=-3,\cdots-3+\lambda$, obtained in the previous bullet point. We do it for $\lambda=-1,\cdots-100$, and check whether we always obtain $\psi_{-2,2,-3}=0$ as part of the solutions. This is indeed the case.

Therefore, we obtain $\psi_{-2,-3,2}=0$ for $\lambda=-1,\cdots-100$.

The next steps consist of a direct generalization of the proof given for $\lambda=-1$ in Lemma 3.2.4. Inserting $\psi_{2,-3,-2}=\psi_{-2,3,2}=0$ into the recurrence relations (4.92) and (4.89), increasing j and decreasing i, we immediately obtain $\psi_{j,-3,-2}=0$, $j\geq 2$ and $\psi_{i,3,2}=0$, $i\leq -2$, respectively. Next, we want to prove $\psi_{j,i,-2}=0$, $\forall\ j>0$, $\forall\ i\leq 0$. We can do this by using induction on i. Indeed, we already proved that the statement $\psi_{j,i,-2}=0$, $\forall\ j>0$ is true for $i=0,-1,-2,-3$. Let us suppose it is true down to $i+1$ $i \leq -4$, and let us show that it holds true for i. Consider the cocycle condition (4.85) on $(i, j, -2, 1)$, which gives us a recurrence relation on j for $\psi_{j,i,-2}$:

$$(-2+i+j+\lambda)\psi_{i,j,-2} - (-1+i)\underbrace{\psi_{1+i,j,-2}}_{=0} + (-1+j)\psi_{1+j,i,-2} = 0$$
$$\Leftrightarrow \psi_{i,1+j,-2} = \frac{(-2+i+j+\lambda)}{(-1+j)}\psi_{i,j,-2}\,. \tag{4.102}$$

The term in the middle is zero because of the induction hypothesis. We will need the relations for $j = 3, 5$, given by:

$$\text{For } j=2\text{: } \psi_{i,3,-2} = (i+\lambda)\psi_{i,2,-2}\,,$$
$$\text{For } j=4\text{: } \psi_{i,5,-2} = \frac{(2+i+\lambda)(1+i+\lambda)(i+\lambda)}{6}\psi_{i,2,-2}\,.$$

Consider the cocycle condition $(i, 3, 2, -2)$ given by (4.95). The terms of the form $\psi_{i,3,2}$ $i \leq 0$ are zero. The terms $\psi_{i+2,3,-2}$ and $\psi_{i+3,2,-2}$ are zero because of the induction hypothesis. The terms $\psi_{i,5,-2}$ and $\psi_{i,3,-2}$ can be expressed in terms of $\psi_{i,2,-2}$ as shown above, yielding:

$$\left(-(-2+i+\lambda)\left(i^2+(-7+\lambda)\lambda+i(-1+2\lambda)\right)\right)\psi_{i,2,-2} = 0\,.$$

A simple analysis shows that there are no zeros of the polynomial with i and λ both being negative integers. Hence, we obtain $\psi_{i,2,-2} = 0$. Inserting $\psi_{i,2,-2} = 0$ into the recurrence relation on j in (4.102), starting with $j = 2$ and increasing j, we obtain $\psi_{j,i,-2} = 0\ \forall\ j > 0$. Thus our induction holds true for i, and altogether we obtain $\psi_{i,j,-2} = 0$, $j > 0$, $i \leq 0$.
Next, we want to prove $\psi_{i,j,2} = 0$, $\forall\ i \leq 0\ \forall\ j > 0$. This time, we shall use induction on j. Indeed, we have seen that the result holds true for $j = 1, 2, 3$. Let us suppose it holds true up to $j-1$ $j \geq 4$, and let us show that it holds true for j. Consider the cocycle condition (4.85) for $(i, j, 2, -1)$, which provides a recurrence relation on i for $\psi_{i,j,2}$:

$$-(1+i)\psi_{-1+i,j,2} + (2+i+j-\lambda)\psi_{i,j,2} + (1+j)\underbrace{\psi_{-1+j,i,2}}_{=0} = 0$$
$$\Leftrightarrow \psi_{-1+i,j,2} = \frac{(2+i+j-\lambda)}{(1+i)}\psi_{i,j,2}\,. \tag{4.103}$$

The third term in the first line is zero due to the induction hypothesis. We shall need $i = -3, -5$ given by:

$$\text{For } i=-2\text{: } \psi_{-3,j,2} = -(j-\lambda)\psi_{-2,j,2}\,,$$
$$\text{For } i=-4\text{: } \psi_{-5,j,2} = \frac{(-2+j-\lambda)(-1+j-\lambda)(j-\lambda)}{-6}\psi_{-2,j,2}\,.$$

Next, consider the cocycle condition(4.85) for $(-3, j, 2, -2)$ after dropping terms of level zero and level minus one, given by:

$$\psi_{-5,j,2} + (-3+j\lambda)\underline{\psi_{-3,2,-2}} - (-5+j+2\lambda)\underline{\psi_{-3,j,-2}} + (-1+j-2\lambda)\psi_{-3,j,2}$$
$$+ (3+j)\underline{\underline{\psi_{-3+j,2,-2}}} + (2+j)\underline{\underline{\psi_{-2+j,-3,2}}} - (j-3\lambda)\psi_{j,2,-2} - (-2+j)\underline{\psi_{2+j,-3,-2}} = 0$$
$$\left(-(2+j-\lambda)\left(j+j^2-2j\lambda+(-7+\lambda)\lambda\right)\right)\psi_{-2,j,2} = 0\,.$$

The terms underlined once are of the form $\psi_{j,-3,-2}$ $j>0$ and are thus zero. The terms underlined twice are zero due to the induction hypothesis. The last line is obtained by replacing the remaining terms by the expressions in terms of $\psi_{-2,j,2}$ given above. A simple analysis shows that the polynomial has no zeros for positive integer j and negative integer λ. Consequently, we have $\psi_{-2,j,2}=0$. Reinserting $\psi_{-2,j,2}=0$ into the recurrence relation on i in (4.103), starting with $i=-2$ i decreasing, we obtain $\psi_{i,j,2}=0\ \forall\ i\leq 0$. Hence, our result holds true for j, and by induction we thus have $\psi_{i,j,2}=0\ \forall\ i\leq 0,\ \forall\ j>0$.
The proof for $\psi_{i,j,k}=0,\ \forall\ i\leq 0,\ \forall\ j>0,\ \forall\ k\in\mathbb{Z}$ is exactly the same as the one done for $\lambda=-1$ given in Lemma 3.2.4, thus we will not reproduce it here.

Case 3: $i,j>0$

The proof is again the same as the one done for $\lambda=-1$ given in Lemma 3.2.4, leading to $\psi_{i,j,k}=0\ \forall\ i,j,k>0$. We will not reproduce it here.
This concludes the proof. □

Proof of Theorem 4.3.1. By using Lemma 4.3.1, we can perform a cohomological change such that the assumptions of Lemma 4.3.2 are fulfilled. Subsequently, we can use Lemma 4.3.2 such that the assumptions of Lemma 4.3.3 are fulfilled, which in turn allows to fulfill the assumptions of Lemma 4.3.4. Lemma 4.3.4 then allows to prove Theorem 4.3.1. □

Remark 4.3.1. In the preceding proofs, we were constantly working with coefficients $\psi_{i,j,k}$ at least one index of which included λ, like for instance the coefficient $\psi_{-3+\lambda,2,-2}$. Since the indices i,j,k are always integers, the parameter λ has to be an integer as well. Thus, the preceding proofs do not work with complex $\lambda\in\mathbb{C}$.

4.3.2 Positive values of λ

In this section, we will analyze $\mathrm{H}^3(\mathcal{W},\mathcal{F}^\lambda)$ with $\lambda\in\mathbb{N}$. The aim is to prove Theorem 4.3.2 below.

Theorem 4.3.2. *The third algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ with $\lambda\in I$ is zero, i.e.*

$$\forall\lambda\in I:\qquad \mathrm{H}^3(\mathcal{W},\mathcal{F}^\lambda)=\{0\},$$

where $I=\{6,8,10,14,16,18,20,22,24,26\}$.

The proof for positive λ is distinctly more complicated than the proof for λ negative given in the previous section. We see from Theorem 4.3.2 that we only have results for certain values of λ. The proof is divided into several Lemmata. The firsts three Lemmata are valid for any $\lambda\in\mathbb{N}\setminus\{0,1,2,5,7,12,15\}$. The values $\{0,1,2,5,7,12,15\}$ correspond to exceptional values of λ, for which $\mathrm{H}^3(\mathcal{W},\mathcal{F}^\lambda)$ will most probably not vanish. They are inspired from continuous cohomology. We had a similar situation in the case of the first and the second algebraic cohomology. For the proofs given in this section, we will consider the following assumption: $\boxed{\lambda\in\mathbb{N}\setminus\{0,1,2,5,7,12,15\}}$.
In the fourth Lemma, there arises again the problem that the length of the proof increases with λ. Therefore, the proof had to be implemented once again into a symbolic programming language. However, in comparison to the case of λ negative, the running time of the proof is much longer, as the algorithm is more complicated. Thus, we did not go beyond $\lambda=26$. Moreover, the algorithm does not work for λ odd. Due to the complexity of equations and the recurrence relations involved, it is impossible with our methods to determine as to why the algorithm does not

work for λ odd, nor is it possible to prove that the algorithm would work for all λ even. Actually, already in the third Lemma 4.3.7 differences show up between λ odd and λ even, the proof for λ odd being more complicated. This is due to the structure of the cocycle condition (4.85). The fifth and last Lemma 4.3.9 is again purely algebraic.

As before, we only focus on the degree zero part of the cohomology since the non-zero degree part is already zero due to Theorem 2.2.1. The cocycle condition is given by (4.85), the coboundary condition by (4.86).

The first result we want to prove is given by Lemma 4.3.5. Note that in the Lemma 4.3.5, we assume a certain ordering in the indices i, j of the coefficients $\psi_{i,j,\pm 1}$ and $\psi_{i,j,\pm 2}$, in order to augment the readability. We place the greatest index in absolute value to the left, i.e. $|i| > |j|$, for $i, j \leq 0$ and $i, j > 0$ in the expressions $\psi_{i,j,\pm 1}$ and $\psi_{i,j,\pm 2}$ below. For indices of different signs in $\psi_{i,j,\pm 1}$, we take $i \leq 0$ and $j > 0$.

Lemma 4.3.5. *Every 3-cocycle $\psi \in \mathrm{H}^3(\mathcal{W}, \mathcal{F}^\lambda)$ with $\lambda \in \mathbb{N}\backslash\{0,1,2,5,7,12,15\}$ of degree zero is cohomologous to a degree zero 3-cocycle ψ' with coefficients $\psi'_{i,j,1}$, $\psi'_{i,j,-1}$, $\psi'_{i,j,2}$ and $\psi'_{i,j,-2}$ satisfying the following conditions:*

$$
\begin{array}{lll}
\boxed{i, j \leq 0, i < j}: & \underline{j = 0}: & \begin{cases} i \neq -\lambda: & \psi'_{i,0,1} = 0 \\ \textit{and}: & \psi'_{-\lambda+1,0,-1} = 0 \end{cases}, \\
& \underline{j = -1}: & \psi'_{i,-1,1} = 0, \\
& \underline{j = -2}: & \begin{cases} i \neq -\lambda+2: & \psi'_{i,-2,1} = 0 \\ \textit{and}: & \psi'_{-\lambda+2,-2,2} = 0 \end{cases}, \\
& \underline{j < -2}: & \begin{cases} i \neq -\lambda-j: & \psi'_{i,j,1} = 0 \\ \textit{and}: & \psi'_{-\lambda-j,j+1,-1} = 0 \end{cases}, \\
\boxed{i \leq 0, j > 0}: & & \psi'_{i,j,1} = 0, \\
& & \psi'_{i,2,-1} = 0, \\
\boxed{i, j > 0, i > j}: & \underline{j = 2} & \begin{cases} i \neq \lambda-2: & \psi'_{i,2,-1} = 0 \\ \textit{and}: & \psi'_{\lambda-2,2,-2} = 0 \end{cases}, \\
& \underline{j > 2} & \begin{cases} i \neq \lambda-j: & \psi'_{i,j,-1} = 0 \\ \textit{and}: & \psi'_{\lambda-j,j-1,1} = 0 \end{cases}.
\end{array}
$$

Proof. As usual, we will begin with a cohomological change $\psi' = \psi - (\delta_2 \phi)$ in order to annihilate as many degrees of freedom of ψ as possible.

We start by defining the following coefficients of a 2-cochain ϕ,

$$
\phi_{i,1} = 0 \ \forall \ i \in \mathbb{Z} \setminus \{-\lambda\},
$$
$$
\phi_{-\lambda+1,-1} = 0 \text{ and } \phi_{2,-1} = 0.
$$

This normalization allows one to simplify the notation to a maximum, it is found by analyzing the structure of the coboundary condition. We will proof the results mostly in the same order than presented above, though it is not always possible to keep the exact ordering. In other words, we shall start with the following case:

$\boxed{i, j \leq 0, i < j}$: $\underline{j = 0}$:

The coboundary condition (4.86) for $(i, 0, 1)$ suggests to define ϕ as follows:

$$
\begin{aligned}
&-(1 + i\lambda)\phi_{0,1} + \cancel{\phi_{1,i}} - (i + \lambda)\phi_{i,0} + \cancel{\phi_{i,1}} + (-1 + i)\phi_{1+i,0} = \psi_{i,0,1} \\
&\Leftrightarrow \phi_{i,0} := \frac{(-1 + i)}{(i + \lambda)}\phi_{1+i,0} + \frac{\psi_{i,0,1}}{-(i + \lambda)}.
\end{aligned}
$$

The slashed terms cancel each other, the term $\phi_{0,1}$ is zero due to our normalization. Starting with $i=-1$, i decreasing, we obtain a definition for $\phi_{i,0}$ and get $\psi'_{i,0,1}=0$ for $-\lambda<i<0$ after the cohomological change. Next, consider the coboundary condition (4.86) for $(-\lambda+1,0,-1)$, which suggests to define,

$$-\cancel{\phi_{-1,1-\lambda}}+(1+(-1+\lambda)\lambda)\phi_{0,-1}-\cancel{\phi_{1-\lambda,-1}}+(-1+2\lambda)\phi_{1-\lambda,0}-(-2+\lambda)\phi_{-\lambda,0}=\psi_{-\lambda+1,0,-1}$$
$$\Leftrightarrow \phi_{-\lambda,0}:=\frac{\psi_{-\lambda+1,0,-1}}{-(-2+\lambda)}-\frac{(1+(-1+\lambda)\lambda)}{-(-2+\lambda)}\phi_{0,-1}-\frac{(-1+2\lambda)}{-(-2+\lambda)}\phi_{1-\lambda,0}.$$

The slashed terms cancel, although they are zero anyway due to our normalization. The coefficients $\phi_{0,-1}$ and $\phi_{-\lambda+1,0}$ are already defined due to the previous result, i.e. the $\phi_{i,0}$ are defined for $-\lambda<i<0$. Because of our assumption $\lambda\neq 2$, we thus obtain a definition for $\phi_{-\lambda,0}$, and we also obtain $\psi'_{-\lambda+1,0,-1}=0$. As we now have a definition for $\phi_{-\lambda,0}$, we can come back to the coboundary condition (4.86) for $(i,0,1)$, which yielded:

$$\phi_{i,0}:=\frac{(-1+i)}{(i+\lambda)}\phi_{1+i,0}+\frac{\psi_{i,0,1}}{-(i+\lambda)}.$$

Starting with $i=-\lambda-1$, i decreasing, we obtain a definition for $\phi_{i,0}$ and get $\psi'_{i,0,1}=0$ for $i<-\lambda$. All in all, we used up all $\phi_{i,0}$ with i negative in order to obtain the first two announced results. Let us consider the next case:
$i,j\leq 0, i<j$: $j=-1$:
Consider the coboundary condition (4.86) for $(i,-1,1)$, which suggest the following definition:

$$-i\lambda\phi_{-1,1}+2\phi_{0,i}-(1+i)\phi_{-1+i,1}-(-1+i+\lambda)\phi_{i,-1}+(1+i-\lambda)\phi_{i,1}+(-1+i)\phi_{1+i,-1}=\psi_{i,-1,1}$$
$$\Leftrightarrow \phi_{i,-1}:=\frac{2}{(-1+i+\lambda)}\phi_{0,i}+\frac{(-1+i)}{(-1+i+\lambda)}\phi_{1+i,-1}-\frac{\psi_{i,-1,1}}{(-1+i+\lambda)}\qquad\text{for}\quad -\lambda+1<i<-1.$$

The coefficients $\phi_{-1,1}$, $\phi_{-1+i,1}$ and $\phi_{i,1}$ are zero for $-\lambda+2<i<-1$ due to our normalization. The coefficients $\phi_{0,i}$ have been defined previously for $i\leq 0$. Hence, we obtain a definition for $\phi_{i,-1}$ and get $\psi'_{i,-1,1}=0$ for $-\lambda+1<i<-1$. Next, let us see what happens to the coboundary condition (4.86) for $(-\lambda+1,-1,1)$:

$$(-1+\lambda)\lambda\phi_{-1,1}+2\phi_{0,1-\lambda}-2(-1+\lambda)\phi_{1-\lambda,1}-\lambda\phi_{2-\lambda,-1}+(-2+\lambda)\phi_{-\lambda,1}=\psi_{-\lambda+1,-1,1}$$
$$\Leftrightarrow \phi_{-\lambda,1}:=-\frac{2}{(-2+\lambda)}\phi_{0,1-\lambda}+\frac{\lambda}{(-2+\lambda)}\phi_{2-\lambda,-1}+\frac{\psi_{-\lambda+1,-1,1}}{(-2+\lambda)}.$$

The coefficients $\phi_{-1,1}$ and $\phi_{1-\lambda,1}$ are zero due to our normalization. The coefficient $\phi_{0,1-\lambda}$ has been defined previously, as it is of the form $\phi_{0,i}$ with i negative. Moreover, the coefficient $\phi_{-\lambda+2,-1}$ is already defined, as we have a definition for $\phi_{i,-1}$ for $-\lambda+1<i<-1$. Thus, as we have $\lambda\neq 2$ by assumption, we obtain a definition for $\phi_{-\lambda,1}$ and in the process we get $\psi'_{-\lambda+1,-1,1}=0$. Next, let us come back to the coboundary condition (4.86) for $(i,-1,1)$ which yielded:

$$\begin{aligned}\phi_{i,-1}:=&\frac{2}{(-1+i+\lambda)}\phi_{0,i}-\frac{(1+i)}{(-1+i+\lambda)}\phi_{-1+i,1}+\frac{(1+i-\lambda)}{(-1+i+\lambda)}\phi_{i,1}\\&+\frac{(-1+i)}{(-1+i+\lambda)}\phi_{1+i,-1}-\frac{\psi_{i,-1,1}}{(-1+i+\lambda)}.\end{aligned}$$

In particular, for $i=-\lambda$, we may define,

$$\phi_{-\lambda,-1}:=-2\phi_{0,-\lambda}+\phi_{-1-\lambda,1}-\lambda\phi_{-1-\lambda,1}+\phi_{1-\lambda,-1}+\lambda\phi_{1-\lambda,-1}-\phi_{-\lambda,1}+2\lambda\phi_{-\lambda,1}+\psi_{-\lambda,-1,1}.$$

The coefficients $\phi_{-1-\lambda,1}$ and $\phi_{1-\lambda,-1}$ are zero due to our normalization, while the coefficients $\phi_{0,-\lambda}$ and $\phi_{-\lambda,1}$ have been defined previously. Consequently, the coefficient $\phi_{-\lambda,-1}$ is well-defined and we get $\psi'_{-\lambda,-1,1}=0$. Coming back to the coboundary condition (4.86) for $(i,-1,1)$ with $i<-\lambda$, we define:

$$\phi_{i,-1} := \frac{2}{(-1+i+\lambda)}\phi_{0,i} + \frac{(-1+i)}{(-1+i+\lambda)}\phi_{1+i,-1} - \frac{\psi_{i,-1,1}}{(-1+i+\lambda)} \qquad \text{for} \quad i<-\lambda\,.$$

Recall that $\phi_{0,i}$ is already defined for i negative. Starting with $i=-\lambda-1$, decreasing i, we obtain definitions for $\phi_{i,-1}$ $i<-\lambda$ and we get $\psi'_{i,-1,1}=0$ for $i<-\lambda$. This proves the third result.

$\boxed{i,j\leq 0, i<j}$: $\underline{j=-2}$:

Consider the coboundary condition (4.86) for $(i,-2,1)$, which suggests to take the following definition for ϕ:

$$\begin{aligned}
&(1-i\lambda)\cancel{\phi_{-2,1}} + 3\phi_{-1,i} - (2+i)\phi_{-2+i,1} - (-2+i+\lambda)\phi_{i,-2}\\
&\qquad + (1+i-2\lambda)\phi_{i,1} + (-1+i)\phi_{1+i,-2} = \psi_{i,-2,1}\\
&\Leftrightarrow \phi_{i,-2} := \frac{1}{-2+i+\lambda}\left(3\phi_{-1,i} - (2+i)\phi_{-2+i,1} + (1+i-2\lambda)\phi_{i,1} + (-1+i)\phi_{1+i,-2} - \psi_{i,-2,1}\right).
\end{aligned}$$

The slashed term is zero because of the normalization. The coefficient of level minus one has been defined previously, while the coefficients of level plus one are either zero due to the normalization, or have been defined previously. Starting with $i=-3$, i decreasing, we obtain a definition for the coefficients $\phi_{i,-2}$ and get $\psi'_{i,-2,1}=0$ for $-\lambda+2<i<-2$. Next, we want to obtain a definition for $\phi_{-\lambda+2,-2}$. To do this, we need to find a definition for $\phi_{i,2}$ with negative i first. Consider the following coboundary condition (4.86), for $(i,2,-1)$:

$$\begin{aligned}
&-3\phi_{1,i} - (1+i\lambda)\cancel{\phi_{2,-1}} + (1+i)\phi_{-1+i,2} + (-1+i+2\lambda)\phi_{i,-1}\\
&\qquad + (-2-i+\lambda)\phi_{i,2} - (-2+i)\phi_{2+i,-1} = \psi_{i,2,-1}\\
&\Leftrightarrow \phi_{-1+i,2} := \frac{1}{1+i}(3\phi_{1,i} - (-1+i+2\lambda)\phi_{i,-1} + (2+i-\lambda)\phi_{i,2} + (-2+i)\phi_{2+i,-1} + \psi_{i,2,-1})\,.
\end{aligned}$$

We have to leave the coefficient $\phi_{2,-2}$ arbitrary, or put it to zero. The author was not able to find a relation providing a consistent definition for $\phi_{2,-2}$. Hence, the coefficient given by $\phi_{2,-2}$ is lost in the sense that it cannot be used to cancel some coefficient $\psi_{i,j,k}$. Starting with $i=-2$ in the expression above, decreasing i, we obtain a definition for $\phi_{i,2}$ $i<-2$ and we get $\psi'_{i,2,-1}=0$ for $i<0$. Recall that the coefficients of level minus one and plus one are either zero or already defined for $i\leq -2$ in the expression above. Remember also that we have $\lambda\neq 2$ by assumption. Next, taking $i=0$ in the expression above, we obtain a definition for the coefficient $\phi_{0,2}$ defined as:

$$\phi_{0,2} := \frac{1}{-2+\lambda}(-\cancel{\phi_{-1,2}} + (1-2\lambda)\phi_{0,-1} + 3\cancel{\phi_{1,0}} - \cancel{\phi_{2,-1}} + \psi_{0,2,-1})\,.$$

The coefficient $\phi_{0,-1}$ being already defined, we obtain a consistent definition for $\phi_{0,2}$ and $\psi'_{0,2,-1}=0$. All in all, we obtain $\psi'_{i,2,-1}=0$ for $i\leq 0$, which corresponds to the second result listed in the case for $i\leq 0$, $j>0$. In total, we now proved four results out of the 13 results. Recall that our most recent aim was to find a definition for $\phi_{-\lambda+2,-2}$. This can now be achieved by considering the coboundary condition (4.86) for $(-\lambda+2,-2,2)$, yielding:

$$\phi_{2-\lambda,-2} := \frac{1}{\lambda}((-2+\lambda)\lambda\phi_{-2,2} + 4\phi_{0,2-\lambda} + (4-3\lambda)\phi_{2-\lambda,2} - \lambda\phi_{4-\lambda,-2} + (-4+\lambda)\phi_{-\lambda,2} - \psi_{2-\lambda,-2,2})\,.$$

Recall that we have $\lambda \neq 0$ by assumption. The coefficient $\phi_{-2,2}$ is left arbitrary as stated before. The coefficients of level zero and of level plus two appearing in the expression above have been defined previously. Similarly, we already have a definition for the coefficients $\phi_{i,-2}$ for $-\lambda + 2 < i < 2$, thus $\phi_{-\lambda+4,-2}$ is defined. Consequently, we obtain a definition for $\phi_{2-\lambda,-2}$ and get $\psi'_{2-\lambda,-2,2} = 0$. Having obtained the desired definition, we can come back to the coboundary condition (4.86) for $(i,-2,1)$, which yielded:

$$\phi_{i,-2} := \frac{1}{-2+i+\lambda}\left(3\phi_{-1,i} - (2+i)\phi_{-2+i,1} + (1+i-2\lambda)\phi_{i,1} + (-1+i)\phi_{1+i,-2} - \psi_{i,-2,1}\right).$$

Starting with $i = -\lambda + 1$, i decreasing, we obtain a definition for $\phi_{i,-2}$ and get $\psi'_{i,-2,1} = 0$ for $i < -\lambda + 2$. This finishes the proof of the two results listed in the present case. The next case to consider is the following:

$\boxed{i, j \leq 0, i < j}$: $\underline{j < -2}$:

The result we want to prove is the following: $\psi'_{i,j,1} = 0$ for $i \neq -\lambda - j$ and $\psi'_{-\lambda-j,j+1,-1} = 0$. This can be done by induction. The base step corresponds to $j = -3$ and must be computed explicitly. The coboundary condition (4.86) for $(i,-3,1)$ gives us a definition for $\phi_{i,-3}$:

$$\begin{aligned}\phi_{i,-3} := \frac{1}{-3+i+\lambda}&((2-i\lambda)\phi_{-3,1} + 4\phi_{-2,i} - (3+i)\phi_{-3+i,1} \\ &+ (1+i-3\lambda)\phi_{i,1} + (-1+i)\phi_{1+i,-3} - \psi_{i,-3,1}).\end{aligned} \tag{4.104}$$

The coefficients of level superior to level minus three appearing in the expression above are zero or already defined for negative i. Hence, the expression above offers a definition for the coefficients $\phi_{i,-3}$ for $-\lambda+3 < i < -3$ leading to $\psi'_{i,-3,1} = 0$ for $-\lambda+3 < i < -3$. As usual, we need to find a definition for $\phi_{-\lambda+3,-3}$ by using a different relation. In fact, the coboundary condition (4.86) for $(-\lambda+3,-2,-1)$ provides such a definition:

$$\begin{aligned}\phi_{-3,3-\lambda} :=& -(3+(-3+\lambda)\lambda)\phi_{-2,-1} - (-5+\lambda)\phi_{1-\lambda,-1} + (-4+\lambda)\phi_{2-\lambda,-2} \\ &+ (1-2\lambda)\phi_{3-\lambda,-2} - 2\phi_{3-\lambda,-1} + 3\lambda\phi_{3-\lambda,-1} + \psi_{3-\lambda,-2,-1}.\end{aligned}$$

The coefficients of levels minus one and minus two appearing in the expression above have already been defined previously. Hence, the coefficient $\phi_{-3,3-\lambda}$ is well-defined and we obtain $\psi'_{3-\lambda,-2,-1} = 0$. Next, we can come back to the coboundary condition for $(i,-3,1)$ given in (4.104). Starting with $i = -\lambda+2$ in (4.104), decreasing i, we obtain a definition for $\phi_{i,-3}$ and get $\psi'_{i,-3,1} = 0$ for $i < -\lambda+3$. This concludes the proof of the basis step.

In the following, we can use induction. Let us use induction on j. Suppose the statement holds true for $j+1$, $j < -3$, i.e. $\phi_{i,j+1}$ is defined for all $i \leq 0$ and we have $\psi'_{i,j+1,1} = 0$ for $i \neq -j-1-\lambda$ $(i \leq 0)$, as well as $\psi'_{-\lambda-j-1,j+2,-1} = 0$. Let us check whether the statement holds true for j, i.e. whether we can find a consistent definition for $\phi_{i,j}$ $i \leq 0$ leading to $\psi'_{i,j,1} = 0$ for $i \neq -j-\lambda$ $(i \leq 0)$, as well as $\psi'_{-\lambda-j,j+1,-1} = 0$. Consider the coboundary condition (4.86) for $(i,j,1)$ with $i, j \leq 0$:

$$\begin{aligned}\phi_{i,j} := \frac{1}{i+j+\lambda}&((1+i+j\lambda)\phi_{i,1} + (-1+i)\phi_{1+i,j} - (1+j+i\lambda)\phi_{j,1} \\ &+ \phi_{1+j,i} - j\phi_{1+j,i} + (-i+j)\phi_{i+j,1} - \psi_{i,j,1}).\end{aligned} \tag{4.105}$$

The coefficients of level one appearing in the expression above are either zero or already defined. The coefficient $\phi_{1+j,i}$ is defined by induction hypothesis. For fixed j, starting with

$i = j-1$, i decreasing ($j < i \le 0$ is already defined because of antisymmetry and the induction hypothesis), we obtain a definition for $\phi_{i,j}$ for $-\lambda - j < i \le 0$, leading to $\psi'_{i,j,1} = 0$ for $-\lambda - j < i \le 0$. Note that the pole $i = -\lambda - j$ with $i, j \le 0$ is not realized if $-\lambda - j > 0$. In that case, no pole occurs in the expression above, the $\phi_{i,j}$'s can be directly defined and the statement is obtained immediately. So let us assume in the following the problematic case, $-\lambda - j < 0$. Next, we need a definition for $\phi_{-\lambda-j,j}$. This can be obtained by considering the coboundary condition (4.86) for $(-\lambda - j, j+1, -1)$, which yields:

$$\phi_{j,-j-\lambda} := \frac{1}{2+j}((j(-1+\lambda)+\lambda^2)\phi_{1+j,-1} + (1+2j+\lambda)\phi_{1-\lambda,-1} - (-1+j+\lambda)\phi_{-1-j-\lambda,1+j} + (-1+j(-1+\lambda))\phi_{-j-\lambda,-1} - \phi_{-j-\lambda,1+j} + 2\lambda\phi_{-j-\lambda,1+j} - \psi_{-j-\lambda,1+j,-1}).$$

The coefficients of level minus one have been defined previously. The coefficients of level $j+1$ are defined by induction hypothesis. Since we consider $j < -3$ (i.e. in particular $j \neq -2$), we obtain a consistent definition for the coefficient $\phi_{j,-j-\lambda}$ and we get $\psi'_{-j-\lambda,1+j,-1} = 0$. Next, we can come back to the coboundary condition for $(i, j, 1)$ given in (4.105). Starting with $i = -j - \lambda - 1$ in (4.105), decreasing i, we obtain a definition for the coefficients $\phi_{i,j}$ and get $\psi'_{i,j,1} = 0$ for $i < -\lambda - j$. We conclude that the statement holds true for j. This concludes the proof of the case under consideration.

$\boxed{i \le 0, j > 0}$:

First of all, recall that the second result listed in the present case has already been obtained, i.e. $\psi'_{i,2,-1} = 0$ for $i \le 0$ via the definition of the coefficients $\phi_{i,2}$, $i \le 0$, see the case: $i, j \le 0$ $j = -2$. Recall also that $\phi_{i,1}$ is already defined or zero for all $i \in \mathbb{Z}$. The remaining coefficients to define in the present case correspond to the $\phi_{i,j}$ with $i \le 0$ and $j > 2$. This can be done straightforwardly by considering the coboundary condition (4.86) for $(i, j, 1)$, yielding:

$$\phi_{i,1+j} := \frac{1}{-1+j}(-(1+i+j\lambda)\phi_{i,1} + (i+j+\lambda)\phi_{i,j} - (-1+i)\phi_{1+i,j} + (1+j+i\lambda)\phi_{j,1} + (i-j)\phi_{i+j,1} + \psi_{i,j,1}).$$

Fixing $j = 2$, starting with $i = 0$, i decreasing, we obtain a definition for $\phi_{i,3}$ $i \le 0$ and we get $\psi'_{i,2,1} = 0$ for $i \le 0$. Doing the same for increasing j, we obtain a consistent definition for all $\phi_{i,j}$ $i \le 0$, $j > 2$ leading to $\psi'_{i,j,1} = 0$ for $i \le 0$, $j > 0$. This corresponds to the first result announced in the present case. The penultimate case to consider is the following:

$\boxed{i, j > 0, i > j}$: $\underline{j = 2}$

First of all, recall that the coefficients $\phi_{i,-1}$ have been defined for $i \le 0$ in the case $i, j \le 0, i < j : j = -1$. The coefficients $\phi_{-1,j}$ have been defined for $j > 0$ in the previous case, just above. Hence, all the coefficients of level minus one are already defined. From the coboundary condition (4.86) for $(i, 2, -1)$, we obtain the following:

$$\phi_{i,2} := -\frac{1}{2+i-\lambda}(3\phi_{1,i} + (1+i\lambda)\cancel{\phi_{2,-1}} - (1+i)\phi_{-1+i,2} - (-1+i+2\lambda)\phi_{i,-1} + (-2+i)\phi_{2+i,-1} + \psi_{i,2,-1}). \tag{4.106}$$

Starting with $i = 3$, increasing i, we obtain a definition for $\phi_{i,2}$ with $2 < i < \lambda - 2$, leading to $\psi'_{i,2,-1} = 0$ for $2 < i < \lambda - 2$. As before, we need to obtain a definition for $\phi_{\lambda-2,2}$ by using a different coboundary condition. Consider the coboundary condition (4.86) for $(-2+\lambda, -2, 2)$, which yields the following definition:

$$\phi_{-2+\lambda,2} := \frac{1}{\lambda}\big(-(-2+\lambda)\lambda\phi_{-2,2} + 4\phi_{0,-2+\lambda} - \lambda\phi_{-4+\lambda,2} + (4-3\lambda)\phi_{-2+\lambda,-2} + (-4+\lambda)\phi_{\lambda,-2} - \psi_{-2+\lambda,-2,2}\big)\,.$$

Recall that the coefficients of level zero $\phi_{i,0}$ have been defined for $i>0$ in the previous case (except for $\phi_{2,0}$, defined in the case $i,j\le 0$, $j=-2$, and $\phi_{1,0}$, which is zero by normalization). Recall also that the coefficient $\phi_{-2,2}$ is left arbitrary. Moreover, the coefficients $\phi_{-2+\lambda,-2}$ and $\phi_{\lambda,-2}$ correspond to coefficients of the form $\phi_{i,j}$ with $i\le 0$ and $j>2$, which have been defined in the previous case. Note that for $\lambda=3$ and for $\lambda=4$, $\phi_{-2+\lambda,-2}$ is of level plus one and equal to $\phi_{2,-2}$, respectively. We already abundantly commented on both. Finally, the coefficient $\phi_{-4+\lambda,2}$ is also defined because we obtained above a definition for $\phi_{i,2}$ with $2<i<\lambda-2$. Consequently, since we also have $\lambda\neq 0$ by assumption, we obtain a consistent definition for $\phi_{-2+\lambda,2}$, leading to $\psi'_{-2+\lambda,-2,2}=0$. Coming back to the coboundary condition for $(i,2,-1)$ given in (4.106), we can start with $i=\lambda-1$, i increasing, and obtain a definition for $\phi_{i,2}$ with $i>\lambda-2$ leading to $\psi'_{i,2,-1}=0$ for $i>\lambda-2$. This proves the two results listed for the present case. We have one final case left to consider:

$\boxed{i,j>0,i>j}$: $\underline{j>2}$:

We will proceed once again by induction. The basis step corresponds to $j=3$ and needs to be computed explicitly. Consider the coboundary condition (4.86) for $(i,3,-1)$ yielding:

$$\phi_{i,3} := -\frac{1}{3+i-\lambda}\big(4\phi_{2,i} + (2+i\lambda)\phi_{3,-1} - (1+i)\phi_{-1+i,3} - (-1+i+3\lambda)\phi_{i,-1} + (-3+i)\phi_{3+i,-1} + \psi_{i,3,-1}\big)\,. \tag{4.107}$$

The coefficients of levels plus two and minus one have been defined previously. Starting with $i=4$, increasing i, we obtain a definition for $\phi_{i,3}$ $3<i<\lambda-3$ leading to $\psi'_{i,3,-1}=0$ for $3<i<\lambda-3$. Note again that the pole $i=\lambda-3$ is not reached for $\lambda<7$ when i starts at $i=4$. In that case, no pole arises and the coefficients can be obtained directly. As before, we will treat the more complicated situation and assume that the pole is realized. A definition for the coefficient $\phi_{\lambda-3,3}$ can be obtained by considering the coboundary condition (4.86) for $(\lambda-3,2,1)$:

$$\phi_{-3+\lambda,3} := (3+(-3+\lambda)\lambda)\cancel{\phi_{2,1}} - (-2+3\lambda)\cancel{\phi_{-3+\lambda,1}} - (1-2\lambda)\phi_{-3+\lambda,2} - (-4+\lambda)\phi_{-2+\lambda,2} + (\lambda-5)\cancel{\phi_{-1+\lambda,1}} + \psi_{-3+\lambda,2,1}\,.$$

The coefficients of level plus two appearing in the expression above have been defined in the previous case. Hence, we obtain a consistent definition for $\phi_{-3+\lambda,3}$ and we get $\psi'_{-3+\lambda,2,1}=0$. Coming back to the coboundary condition for $(i,3,-1)$ given in (4.107), we can start with $i=\lambda-2$, i increasing, and obtain the $\phi_{i,3}$ for $i>\lambda-3$, yielding $\psi'_{i,3,-1}=0$ for $i>\lambda-3$. This concludes the proof of the basis step.

Next, let us use induction on j. Suppose the result is true for $j-1$, $j>3$, i.e. we have a definition for $\phi_{i,j-1}$ for all $i>0$ and we have $\psi'_{i,j-1,-1}=0$ for $i\neq -j+1+\lambda$ $(i>0)$, as well as $\psi'_{-j+1+\lambda,j-2,1}=0$. Let us check whether the statement holds true for j, i.e. we need to find a definition for $\phi_{i,j}$ for all $i>0$ leading to $\psi'_{i,j,-1}=0$ for $i\neq -j+\lambda$ $(i>0)$, as well as $\psi'_{-j+\lambda,j-1,1}=0$. Consider the coboundary condition (4.86) for $(i,j,-1)$, yielding:

$$\phi_{i,j} := \frac{1}{i+j-\lambda}\big((1+i)\phi_{-1+i,j} + (-1+i+j\lambda)\phi_{i,-1} - (1+j)\phi_{-1+j,i} - (j+i\lambda-1)\phi_{j,-1} + (-i+j)\phi_{i+j,-1} - \psi_{i,j,-1}\big)\,. \tag{4.108}$$

The coefficient of level $j-1$ is defined per induction hypothesis. Starting with $i=j+1$, i increasing, we obtain a definition for $\phi_{i,j}$ with $j<i<\lambda-j$, yielding $\psi'_{i,j,-1}=0$ for $j<i<\lambda-j$. We assume again that the pole $i=\lambda-j$ can be realized, the other case being immediate. In particular, this implies $\lambda-j>0$. We thus need a definition for the coefficient $\phi_{\lambda-j,j}$, which is obtained from the coboundary condition (4.86) for $(\lambda-j,j-1,1)$:

$$\begin{aligned}\phi_{j,-j+\lambda} := \frac{1}{-2+j}\big((j(-1+\lambda)-\lambda^2)\phi_{-1+j,1}+(-1+2j-\lambda)\phi_{-1+\lambda,1}+(1+j(-1+\lambda))\phi_{-j+\lambda,1}\\ +(1-2\lambda)\phi_{-j+\lambda,-1+j}+(-j+\lambda-1)\phi_{1-j+\lambda,-1+j}-\psi_{-j+\lambda,-1+j,1}\big)\,.\end{aligned}$$

The coefficients of level plus one are zero or have been defined previously. The coefficients of level $j-1$ are defined by induction hypothesis. As we consider $j>3$ and in particular $j\neq 2$, we obtain a consistent definition for $\phi_{j,-j+\lambda}$ leading to $\psi'_{-j+\lambda,-1+j,1}=0$. Next, we can come back to the coboundary condition for $(i,j,-1)$ given in (4.108). Starting with $i=\lambda-j+1$, increasing i, we obtain a definition for $\phi_{i,j}$ $i>\lambda-j$ resulting in $\psi'_{i,j,-1}=0$ for $i>\lambda-j$. Therefore, the statement holds true for j. This concludes the proof of Lemma 4.3.5. □

Note that for $\lambda=3$ and $\lambda=4$, the critical situations corresponding to $i+j=-\lambda$ and $i+j=\lambda$ are never reached except for coefficients of the form $\psi_{i,0,1}$ with $i\le 0$. Hence, for these values of λ the proof of level zero being zero immediately implies the vanishing of levels plus one and minus one.
This first result for $\lambda>0$ given by Lemma 4.3.5 already presents fundamental differences with the case $\lambda<0$. In fact, there are missing values of coefficients in level plus and minus one which could not be annihilated by a cohomological change. Instead, coefficients of level plus and minus two have been canceled. This will lead to the fact that level minus one can not be treated exclusively later on. Instead, it will have to be analyzed in parallel with levels plus and minus two. However, a real issue is due to the fact that the coefficient $\phi_{-2,2}$ is lost, meaning it could not be used to cancel a coefficient of the 3-cocycle. In the case of $\lambda<0$, the coefficient canceled by $\phi_{-2,2}$ corresponded to $\psi_{-3+\lambda,-2,2}$. In the case of $\lambda>0$, its analogue would have been the cancellation of a coefficient similar to $\psi_{3-\lambda,-2,2}$. This missing piece leads to the fact that a proof similar to the proof of Theorem 4.2.1 is very hard to obtain.

Concerning level zero, it is possible to construct a proof that all the coefficients of level zero, i.e. all $\psi_{i,j,0}$, are zero. The result is given by Lemma 4.3.6 below. The assumptions of Lemma 4.3.6 correspond to the results of Lemma 4.3.5.

Lemma 4.3.6. *Let* $\psi\in \mathrm{H}^3(\mathscr{W},\mathscr{F}^\lambda)$ *with* $\lambda\in\mathbb{N}\setminus\{0,1,2,5,7,12,15\}$ *be a 3-cocycle of degree zero with* *coefficients* $\psi_{i,j,1}$, $\psi_{i,j,-1}$, $\psi_{i,j,2}$ *and* $\psi_{i,j,-2}$ *satisfying the following conditions:*

$$
\begin{array}{lll}
\boxed{i,j\le 0, i<j}: & \underline{j=0}: & \left\{\begin{array}{ll} i\neq -\lambda: & \psi_{i,0,1}=0 \\ \textit{and}: & \psi_{-\lambda+1,0,-1}=0 \end{array}\right. , \\
 & \underline{j=-1}: & \psi_{i,-1,1}=0, \\
 & \underline{j=-2}: & \left\{\begin{array}{ll} i\neq -\lambda+2: & \psi_{i,-2,1}=0 \\ \textit{and}: & \psi_{-\lambda+2,-2,2}=0 \end{array}\right. , \\
 & \underline{j<-2}: & \left\{\begin{array}{ll} i\neq -\lambda-j: & \psi_{i,j,1}=0 \\ \textit{and}: & \psi_{-\lambda-j,j+1,-1}=0 \end{array}\right. , \\
\boxed{i\le 0, j>0}: & & \psi_{i,j,1}=0, \\
 & & \psi_{i,2,-1}=0, \\
\boxed{i,j>0, i>j}: & \underline{j=2} & \left\{\begin{array}{ll} i\neq \lambda-2: & \psi_{i,2,-1}=0 \\ \textit{and}: & \psi_{\lambda-2,2,-2}=0 \end{array}\right. , \\
 & \underline{j>2} & \left\{\begin{array}{ll} i\neq \lambda-j: & \psi_{i,j,-1}=0 \\ \textit{and}: & \psi_{\lambda-j,j-1,1}=0 \end{array}\right. ,
\end{array}
$$

Then:

$$\psi_{i,j,0}=0 \qquad \forall\ i,j\in\mathbb{Z}.$$

Proof. Let us write down the cocycle condition (4.85) for $(i,j,0,1)$, giving the following recurrence relations:

$$
\begin{aligned}
0 &= \cancel{\psi_{1,i,j}}+(1+i+j\lambda)\psi_{i,0,1}+(i+j+\lambda)\psi_{i,j,0}+\cancel{i\psi_{i,j,1}}-\cancel{(1+i+j)\psi_{i,j,1}} \\
&\quad -(-1+i)\psi_{1+i,j,0}-(1+j+i\lambda)\psi_{j,0,1}-\cancel{j\psi_{j,i,1}}+(-1+j)\psi_{1+j,i,0}+(-i+j)\psi_{i+j,0,1} \\
\Leftrightarrow \psi_{i,j,0} &= \frac{1}{i+j+\lambda}(-(1+i+j\lambda)\psi_{i,0,1}+(-1+i)\psi_{1+i,j,0}+(1+j+i\lambda)\psi_{j,0,1} \\
&\quad -(-1+j)\psi_{1+j,i,0}+(i-j)\psi_{i+j,0,1}) \text{ for } i,j \text{ decreasing} \qquad (4.109) \\
\Leftrightarrow \psi_{1+j,i,0} &= \frac{1}{-1+j}(-(1+i+j\lambda)\psi_{i,0,1}-(i+j+\lambda)\psi_{i,j,0}+(-1+i)\psi_{1+i,j,0} \\
&\quad +(1+j+i\lambda)\psi_{j,0,1}+i\psi_{i+j,0,1}-j\psi_{i+j,0,1}) \text{ for } i \text{ decreasing, } j \text{ increasing.} \qquad (4.110)
\end{aligned}
$$

The slashed terms cancel each other. We will mostly follow the order of cases given in the statement above, i.e. the first case we consider is the following:

$\boxed{i,j\le 0}: \underline{j=-1}$

Putting $j=-1$ in the recurrence relation (4.109), starting with $i=-2$, i decreasing, we see that $\psi_{i,-1,0}=0$ for $-\lambda+1<i\le 0$. Recall that we have $\psi_{i,0,1}=0$ for $i\le 0$ and $i\neq-\lambda$ and also $\psi_{-\lambda+1,-1,0}=0$ by assumption. Inserting $j=-1$ and $i=-\lambda+1$ into (4.109), we obtain:

$$
\begin{aligned}
&(-1+\lambda)\lambda\psi_{-1,0,1}+\cancel{\psi_{-1,1-\lambda,1}}+\cancel{\psi_{1,1-\lambda,-1}} \\
&-2(-1+\lambda)\psi_{1-\lambda,0,1}+\lambda\psi_{2-\lambda,-1,0}+(-2+\lambda)\psi_{-\lambda,0,1}=0.
\end{aligned}
$$

The slashed terms cancel each other. The terms $\psi_{-1,0,1}$ and $\psi_{1-\lambda,0,1}$ are zero by the assumption $\psi_{i,0,1}=0$ for $i\le 0$ and $i\neq-\lambda$. The term $\psi_{2-\lambda,-1,0}$ is zero because of the result $\psi_{i,-1,0}=0$ for $-\lambda+1<i\le 0$. Since we also have $\lambda\neq 2$ by assumption, we obtain $\psi_{-\lambda,0,1}=0$. This means that the terms of level plus one appearing in (4.109) can now be ignored for all $i\le 0$. Hence, we can put $i=-\lambda$, $j=-1$ in (4.109) and continue with i decreasing such that we obtain:

$$\psi_{i,-1,0}=0 \qquad \forall\ i\le 0. \qquad (4.111)$$

The next case to consider is the following:
$\boxed{i,j\leq 0}$: $\overline{j=-2}$
Inserting $j=-2$ into (4.109), starting with $i=-3$ and decreasing i, we obtain $\psi_{i,-2,0}=0$ for $-\lambda+2<i\leq 0$. Next, we will need to get a result for $\psi_{-\lambda+2,-2,0}$. This can only be achieved by considering the recurrence relation (4.110) with $i\leq 0$, $j>0$. More precisely, we will need a result for $\psi_{i,2,0}$ for $i\leq 0$.
Consider the cocycle condition (4.85) for $(i,2,0,-1)$:

$$
\begin{aligned}
&-\cancel{\psi_{-1,i,2}}+3\psi_{1,i,0}-(1+i\lambda)\psi_{2,0,-1}-\cancel{2\psi_{2,i,-1}}-(1+i)\psi_{-1+i,2,0}\\
&+(-1+i+2\lambda)\psi_{i,0,-1}-\cancel{\psi_{i,2,-1}}+(2+i-\lambda)\psi_{i,2,0}-(-2+i)\psi_{2+i,0,-1}=0\\
&\Leftrightarrow \psi_{i-1,2,0}=\frac{2+i-\lambda}{i+1}\psi_{i,2,0}\,. \qquad (4.112)
\end{aligned}
$$

The slashed terms cancel each other. The term $\psi_{1,i,0}$ is zero because of the result $\psi_{i,0,1}=0$ for $i\leq 0$. The term $\psi_{2,0,-1}$ is zero because of the assumption $\psi_{i,2,-1}=0$ for $i\leq 0$. Finally, the terms $\psi_{i,0,-1}$ and $\psi_{2+i,0,-1}$ (for $i\leq -2$) are zero because of our result (4.111). Clearly in the recurrence relation above (4.112), we need a result for $\psi_{-2,2,0}$ as a starting point. From the recurrence relation (4.112) above, we get the following expressions for $i=-3,-5$:

$$
\begin{aligned}
\psi_{-3,2,0}&=\lambda\psi_{-2,2,0}\,,\\
\psi_{-5,2,0}&=\frac{(\lambda+2)(\lambda+1)\lambda}{6}\psi_{-2,2,0}\,.
\end{aligned}
$$

These values can be inserted into the cocycle condition (4.85) for $(-3,-2,2,0)$:

$$
\begin{aligned}
&\psi_{-5,2,0}+(5-2\lambda)\underbrace{\psi_{-3,-2,0}}_{=0}-(1+2\lambda)\psi_{-3,2,0}-\cancel{2\psi_{-2,-3,2}}+3\lambda\psi_{-2,2,0}-5\underbrace{\psi_{-1,-2,0}}_{=0}-\cancel{2\psi_{2,-3,-2}}=0\\
&\Leftrightarrow \frac{(\lambda+2)(\lambda+1)\lambda}{6}\psi_{-2,2,0}-(1+2\lambda)\lambda\psi_{-2,2,0}+3\lambda\psi_{-2,2,0}=0\\
&\Leftrightarrow (-7+\lambda)(-2+\lambda)\lambda\psi_{-2,2,0}=0\,.
\end{aligned}
$$

As $\lambda\notin\{0,2,7\}$ by assumption, we obtain $\psi_{-2,2,0}=0$. In the first line of the expression above, the term $\psi_{-1,-2,0}$ is zero because of (4.111). The term $\psi_{-3,-2,0}$ is zero because from (4.109), we have:

$$
\psi_{-3,-2,0}=\frac{1}{-5+\lambda}(-4\underbrace{\psi_{-2,-2,0}}_{=0}+3\underbrace{\psi_{-1,-3,0}}_{=0\text{ cf. (4.111)}})\,.
$$

Since $\lambda\neq 5$ by assumption, we have $\psi_{-3,-2,0}=0$. Inserting $\psi_{-2,2,0}=0$ into our recurrence relation (4.112), starting with $i=-2$, i decreasing, we obtain $\psi_{i,2,0}=0$ for $i\leq -3$. All in all, we have:

$$
\psi_{i,2,0}=0 \qquad \forall\, i\leq 0\,. \tag{4.113}
$$

Recall that this result is needed to analyze $\psi_{-\lambda+2,-2,0}$, which can be done by considering the cocycle condition (4.85) for $(-\lambda+2,-2,0)$:

$$
\begin{aligned}
&(-2+\lambda)\lambda\psi_{-2,0,2}+\cancel{2\psi_{-2,2-\lambda,2}}+\cancel{2\psi_{2,2-\lambda,-2}}+\lambda\psi_{2-\lambda,-2,0}+(4-3\lambda)\psi_{2-\lambda,0,2}\\
&+\lambda\psi_{4-\lambda,-2,0}+(-4+\lambda)\psi_{-\lambda,0,2}=0\,.
\end{aligned}
$$

The terms $\psi_{-2,0,2}$, $\psi_{2-\lambda,0,2}$ and $\psi_{-\lambda,0,2}$ are zero because of our previous result (4.113). The coefficient $\psi_{4-\lambda,-2,0}$ is zero because we already have established that $\psi_{i,-2,0}=0$ for $-\lambda+2<i\leq 0$.

Since we have $\lambda \neq 0$ by assumption, we obtain $\psi_{2-\lambda,-2,0} = 0$. Putting $j = -2$ in (4.109), starting with $i = -\lambda + 1$, i decreasing and using $\psi_{2-\lambda,-2,0} = 0$, we obtain $\psi_{i,-2,0} = 0$ for $i < -\lambda + 2$. All in all, we obtain the following result:

$$\psi_{i,-2,0} = 0 \qquad \forall\ i \leq 0\,. \tag{4.114}$$

In the last step, we can proceed by induction to prove the result for generic negative j.

$\boxed{i, j \leq 0}$: $\underline{j \leq -3}$

We have proven that $\psi_{i,j,0} = 0\ \forall\ i \leq 0$ for $j = -1, -2$. Suppose the result holds true for $j+1$, i.e. $\psi_{i,j+1,0} = 0\ \forall\ i \leq 0,\ j \leq -3$. Let us check whether it holds true for j, i.e. $\psi_{i,j,0} = 0\ \forall\ i \leq 0$. Consider the recurrence relation (4.109) for fixed j. Note again that the terms of level plus one are zero because we have $\psi_{i,0,1} = 0$ for $i \leq 0$ by assumption and also due to one of our previous results. Starting with $i = j-1$, i decreasing, we obtain $\psi_{i,j,0} = 0$ for $-\lambda - j < i \leq 0$. In the case that $-\lambda - j$ is positive, the pole is never realized and we immediately obtain $\psi_{i,j,0} = 0\ \forall\ i \leq 0$. Let us consider the non-trivial case where the pole $i = -\lambda - j$ is realized. In that case, we need to obtain a result for the coefficient $\psi_{-\lambda-j,j,0}$. This can be achieved by considering the cocycle condition (4.85) for $(-\lambda - j, j+1, 0, -1)$, yielding:

$$\begin{aligned}
&-\cancel{\psi_{-1,-j-\lambda,1+j}} + (2+j)\psi_{j,-j-\lambda,0} + (j(-1+\lambda)+\lambda^2)\psi_{1+j,0,-1} - \cancel{(1+j)\psi_{1+j,-j-\lambda,-1}}\\
&+(1+2j+\lambda)\psi_{1-\lambda,0,-1} + (-1+j+\lambda)\psi_{-1-j-\lambda,1+j,0} + (-1+j(-1+\lambda))\psi_{-j-\lambda,0,-1}\\
&+\cancel{\lambda\psi_{-j-\lambda,1+j,-1}} - \cancel{(j+\lambda)\psi_{-j-\lambda,1+j,-1}} + (1-2\lambda)\psi_{-j-\lambda,1+j,0} = 0\,.
\end{aligned}$$

The terms $\psi_{1+j,0,-1}$, $\psi_{1-\lambda,0,-1}$ and $\psi_{-j-\lambda,0,-1}$ are zero because of our result (4.111). The terms $\psi_{-1-j-\lambda,1+j,0}$ and $\psi_{-j-\lambda,1+j,0}$ are zero by our induction hypothesis. It follows $\psi_{j,-j-\lambda,0} = 0$ as we have $j \leq -3$, i.e. $j \neq -2$ in particular. Coming back to the recurrence relation (4.109), starting with $i = -1 - j - \lambda$, i decreasing, we get $\psi_{i,j,0} = 0$ for $i < -j - \lambda$. All in all, we get $\psi_{i,j,0} = 0$ for $i \leq 0$. Thus, the induction holds true for j, and all in all we obtain:

$$\psi_{i,j,0} = 0 \qquad \forall\ i, j \leq 0\,. \tag{4.115}$$

The next case to consider is the following:

$\boxed{i \leq 0, j > 0}$

First of all, the terms of level plus one appearing in (4.110) are zero because we have $\psi_{i,0,1} = 0$ for $i \leq 0$ by assumption and also due to one of our previous results, and also because of another of our assumptions, namely $\psi_{i,j,1} = 0$ for $i \leq 0$ (in particular, $i = 0$) and $j > 0$. Next, recall that we already have the following result: $\psi_{i,2,0} = 0$ for $i \leq 0$, see (4.113). Hence, starting with $j = 2$ and $i = -1$ in (4.110), decreasing i and increasing j, we immediately obtain the following result:

$$\psi_{i,j,0} = 0 \qquad \forall\ i \leq 0,\ \forall\ j > 0\,. \tag{4.116}$$

The final main case to consider is the following:

$\boxed{i, j > 0}$

To analyze this case, we have to write down the cocycle condition (4.85) for $(i, j, 0, -1)$, giving.

$$\begin{aligned}
&-\cancel{\psi_{-1,i,j}} - (1+i)\psi_{-1+i,j,0} + (-1+i+j\lambda)\psi_{i,0,-1} + \cancel{i\psi_{i,j,-1}} - \cancel{(-1+i+j)\psi_{i,j,-1}}\\
&+(i+j-\lambda)\psi_{i,j,0} + (1+j)\psi_{-1+j,i,0} - (-1+j+i\lambda)\psi_{j,0,-1}\\
&-\cancel{j\psi_{j,i,-1}} + (-i+j)\psi_{i+j,0,-1} = 0\\
&\Leftrightarrow \psi_{i,j,0} = \frac{(1+i)}{(i+j-\lambda)}\psi_{-1+i,j,0} - \frac{(1+j)}{(i+j-\lambda)}\psi_{-1+j,i,0}\,.
\end{aligned} \tag{4.117}$$

The terms $\psi_{i,0,-1}$, $\psi_{j,0,-1}$ and $\psi_{i+j,0,-1}$ are zero due to our previous result (4.116). Note that for $j=1$, the coefficients $\psi_{i,1,0}$ $i>0$ are included in the coefficients of the form $\psi_{i,j,1}$ with $i \le 0$ (in particular $i=0$) and $j>0$, which are zero by assumption. Hence, it suffices to start with $j=2$. Putting $j=2$ in (4.117), starting with $i=3$ and increasing i, we obtain $\psi_{i,2,0}=0$ for $0<i<-2+\lambda$. Again a pole appears such that we have to find a result for $\psi_{-2+\lambda,2,0}$ by considering the cocycle condition (4.85) for $(-2+\lambda,2,0,-2)$:

$$\begin{aligned}&-\cancel{2\psi_{-2,-2+\lambda,2}}-(-2+\lambda)\lambda\psi_{2,0,-2}-\cancel{2\psi_{2,-2+\lambda,-2}}-\lambda\psi_{-4+\lambda,2,0}\\&+(-4+3\lambda)\psi_{-2+\lambda,0,-2}-\lambda\psi_{-2+\lambda,2,0}-(-4+\lambda)\psi_{\lambda,0,-2}=0\,.\end{aligned}$$

The terms $\lambda\psi_{2,0,-2}$, $\psi_{-2+\lambda,0,-2}$ and $\psi_{\lambda,0,-2}$ are zero because of our result (4.116). The term $\psi_{-4+\lambda,2,0}$ is zero because we already established $\psi_{i,2,0}=0$ for $0<i<-2+\lambda$. As in addition we have $\lambda \neq 0$ by assumption, it follows $\psi_{-2+\lambda,2,0}=0$. Coming back to our recurrence relation (4.117), putting $j=2$, starting with $i=-1+\lambda$, increasing i, we obtain $\psi_{i,2,0}=0$ for $i>-2+\lambda$. All in all, we have:

$$\psi_{i,2,0}=0 \qquad \forall\ i>0\,. \tag{4.118}$$

To prove the result for generic positive j, we can proceed once more by induction. We know already that the result is true for $j=1,2$. Suppose the result holds true for $j-1$, i.e. $\psi_{i,j-1,0}=0$ $\forall\ i>0$ for $j\ge 3$. Let us check whether it holds true for j, i.e. $\psi_{i,j,0}=0\ \forall\ i>0$. For fixed j in (4.117), starting with $i=j+1$, increasing i, we obtain $\psi_{i,j,0}=0$ for $0<i<\lambda-j$. For large j, we have $\lambda-j\le 0$ and no pole appears. We immediately obtain $\psi_{i,j,0}=0\ \forall\ i>0$. Consider the non trivial case where the pole at $i=\lambda-j$ is realized. We can obtain a result for $\psi_{\lambda-j,j,0}$ by considering the cocycle condition (4.85) for $(\lambda-j,j-1,0,1)$:

$$\begin{aligned}&\cancel{\psi_{1,-j+\lambda,-1+j}}-(j-j\lambda+\lambda^2)\psi_{-1+j,0,1}-\cancel{(-1+j)\psi_{-1+j,-j+\lambda,1}}+(-2+j)\psi_{j,-j+\lambda,0}\\&+(-1+2j-\lambda)\psi_{-1+\lambda,0,1}+(1+j(-1+\lambda))\psi_{-j+\lambda,0,1}+(-1+2\lambda)\psi_{-j+\lambda,-1+j,0}\\&-\cancel{\lambda\psi_{-j+\lambda,-1+j,1}}+\cancel{(-j+\lambda)\psi_{-j+\lambda,-1+j,1}}+(1+j-\lambda)\psi_{1-j+\lambda,-1+j,0}=0\,.\end{aligned}$$

The terms $\psi_{-1+j,0,1}$, $\psi_{-1+\lambda,0,1}$ and $\psi_{-j+\lambda,0,1}$ are zero because of our assumption $\psi_{i,j,1}=0$ for $i\le 0$ (in particular $i=0$) and $j>0$. The terms $\psi_{-j+\lambda,-1+j,0}$ and $\psi_{1-j+\lambda,-1+j,0}$ are zero by induction hypothesis. Since we have $j\ge 3$ and in particular $j\neq 2$, we obtain $\psi_{j,-j+\lambda,0}=0$. Using this result in the recurrence relation (4.117) for $i=-j+\lambda+1$, increasing i, we obtain $\psi_{i,j,0}=0$ for $i>-j+\lambda$. All in all, we have $\psi_{i,j,0}=0$ for $i>0$. Therefore, the induction hypothesis holds true for j and we conclude:

$$\psi_{i,j,0}=0 \qquad \forall\ i,j>0\,. \tag{4.119}$$

The results (4.115), (4.116) and (4.119) correspond to the announced statement of Lemma 4.3.6. □

In the next step given by Lemma 4.3.7 below, we will focus on level minus one (plus one in the case of both indices being positive). However, in the cohomological change in Lemma 4.3.5 we can already see that some coefficients of level plus two and minus two appear. Hence, it is not possible to separate the analysis of levels plus and minus one completely from levels plus and minus two. In the following, the aim is to annihilate levels plus and minus one, but the proof also needs some analysis of levels plus and minus two.

Lemma 4.3.7. *Let $\psi \in \mathrm{H}^3(\mathscr{W}, \mathscr{F}^\lambda)$ with $\lambda \in \mathbb{N} \setminus \{0,1,2,5,7,12,15\}$ be a 3-cocycle of degree zero with coefficients $\psi_{i,j,0}$, $\psi_{i,j,1}$, $\psi_{i,j,-1}$, $\psi_{i,j,2}$ and $\psi_{i,j,-2}$ satisfying the following conditions:*

$$
\begin{array}{lll}
\boxed{i,j \le 0, i<j}: & \underline{j=-1}: & \psi_{i,-1,1} = 0, \\
 & \underline{j=-2}: & \begin{cases} i \neq -\lambda+2: & \psi_{i,-2,1} = 0 \\ \text{and}: & \psi_{-\lambda+2,-2,2} = 0 \end{cases}, \\
 & \underline{j<-2}: & \begin{cases} i \neq -\lambda-j: & \psi_{i,j,1} = 0 \\ \text{and}: & \psi_{-\lambda-j,j+1,-1} = 0 \end{cases}, \\
\boxed{i \le 0, j>0}: & & \psi_{i,j,1} = 0, \\
 & & \psi_{i,2,-1} = 0, \\
\boxed{i,j>0, i>j}: & \underline{j=2} & \begin{cases} i \neq \lambda-2: & \psi_{i,2,-1} = 0 \\ \text{and}: & \psi_{\lambda-2,2,-2} = 0 \end{cases}, \\
 & \underline{j>2} & \begin{cases} i \neq \lambda-j: & \psi_{i,j,-1} = 0 \\ \text{and}: & \psi_{\lambda-j,j-1,1} = 0 \end{cases},
\end{array}
$$

and

$$\psi_{i,j,0} = 0 \qquad \forall\ i,j \in \mathbb{Z}.$$

Then:

$$\psi_{i,j,1} = \psi_{i,j,-1} = 0 \qquad \forall\ i,j \in \mathbb{Z}.$$

Proof. This time, the easiest case to examine corresponds to the one with indices of mixed signs, hence we will start by considering the following:

$\boxed{i \le 0, j > 0:}$

First, note that coefficients of the form $\psi_{n,-1,1}$ $n \in \mathbb{Z}$ are always zero. In fact, for $n \le 0$, $\psi_{n,-1,1}$ is zero due to our assumption listed in the first line of the statement above. Moreover, for $n > 0$, $\psi_{n,-1,1}$ is zero due to the assumption $\psi_{i,j,1} = 0$ for $i \le 0$, $j > 0$ listed in the sixth line of the statement.

The cocycle condition (4.85) for $(i, j, 1, -1)$ yields the following recurrence relation:

$$
\begin{aligned}
\psi_{i,1+j,-1} = &-\frac{1}{-1+j}(-2\cancel{\psi_{0,i,j}} - (1+i)\underline{\psi_{-1+i,j,1}} - (i+j\lambda)\cancel{\psi_{i,-1,1}} \\
&- (-1+i+j+\lambda)\psi_{i,j,-1} + (1+i+j-\lambda)\underline{\psi_{i,j,1}} + (i-1)\psi_{1+i,j,-1} \\
&+ (1+j)\underline{\psi_{-1+j,i,1}} + (j+i\lambda)\cancel{\psi_{j,-1,1}} + (i-j)\cancel{\psi_{i+j,-1,1}}).
\end{aligned}
$$

The slashed terms are zero because they are either of level zero or of the form $\psi_{n,-1,1}$ $n \in \mathbb{Z}$. Moreover, the underlined coefficients of level plus one are all of the form $\psi_{i,j,1}$ with $i \le 0$, $j > 0$, which are zero. The starting point of the recurrence relation above corresponds to $\psi_{i,2,-1}$ with $i \le 0$. As these are zero due to our assumption listed in the seventh line of our statement, we obtain $\boxed{\psi_{i,j,-1} = 0,\ i \le 0, j > 0}$. The next case we will consider is more complicated and corresponds to:

$\boxed{i,j \le 0:}$

In this case, on the one hand we have a few coefficients of level plus one which are not zero, namely those of the form $\psi_{i,j,1}$ with $i + j = -\lambda$. On the other hand, we only have a few coefficients of level minus one which are zero, namely the ones of the form $\psi_{i,j,-1}$ with $i + j = -\lambda + 1$. The aim is to express the coefficients of level minus one in terms of those of level plus one, which will act as temporary generating coefficients. The corresponding recurrence relation can once again be taken from the cocycle condition (4.85) for $(i, j, -1, 1)$:

$$\psi_{i,j,-1} = \frac{1}{-1+i+j+\lambda}(-(1+i)\psi_{-1+i,j,1} + (1+i+j-\lambda)\psi_{i,j,1} + (-1+i)\psi_{1+i,j,-1} + (1+j)\psi_{-1+j,i,1} + (1-j)\psi_{1+j,i,-1}). \tag{4.120}$$

In the expression above, we already dropped the coefficients of level zero and of the form $\psi_{n,-1,1}$ $n \in \mathbb{Z}$. We see that the pole is realized at $i + j = -\lambda + 1$. The terms of level plus one are non-zero at $i + j = -\lambda + 1$ ($\psi_{-1+i,j,1}$, $\psi_{-1+j,i,1}$) and $i + j = -\lambda$ ($\psi_{i,j,1}$). Therefore we conclude $\psi_{i,j,-1} = 0$ for $i + j > -\lambda + 1$, by using the usual procedure, i.e. fixing $j = -2$, starting with $i = -3$ and decreasing i, then for decreasing j starting with $i = j - 1$, and so on. Due to our assumption $\psi_{-\lambda-j+1,j,-1} = 0$ listed in the fifth line of our statement, we can extend the result to $\psi_{i,j,-1} = 0$ for $i + j \geq -\lambda + 1$. The remaining coefficients $\psi_{i,j,-1}$ with $i + j < -\lambda + 1$ can thus all be expressed in terms of the generating coefficients of the form $\psi_{-j-\lambda,j,1}$ for $j \leq 0$ and $-j - \lambda \leq 0$. We consider $-j - \lambda \leq 0$ since we analyze here $\psi_{i,j,1}$ and $\psi_{i,j,-1}$ with both indices i and j negative, the case of one positive and one negative index having been treated before. Therefore, it suffices to show that the generating coefficients of the form $\psi_{-j-\lambda,j,1}$ for $j \leq 0$ and $-j - \lambda \leq 0$ are zero in order to get $\psi_{i,j,-1} = 0 \ \forall \ i, j \leq 0$. Actually, the number of generating coefficients can be reduced. The generating coefficients are a priori $\psi_{-j-\lambda,j,1}$ for $-\lambda \leq j \leq -2$. However, due to the alternating property, it is enough to consider $\psi_{-j-\lambda,j,1}$ for $-\frac{\lambda}{2} + 1 \leq j \leq -2$. For example, for $\lambda = 10$, a priori we have $\psi_{-8,-2,1}, \psi_{-7,-3,1}, \psi_{-6,-4,1}, \psi_{-5,-5,1}, \psi_{-4,-6,1}, \psi_{-3,-7,1}, \psi_{-2,-8,1}$ as generating coefficients. However, due to the alternating property, it is sufficient to consider only the first three coefficients. Moreover, $j \in \{0, 1\}$ reduces to cases previously analyzed.
For later use, we write down a particular relation between coefficients of level plus one and minus one. Inserting $i = -j - \lambda$ in the recurrence relation (4.120) above, we obtain the following relation:

$$\begin{aligned} \psi_{-j-\lambda,j,-1} = &- (1+j)\underline{\psi_{-1+j,-j-\lambda,1}} + (-1+j)\underline{\underline{\psi_{1+j,-j-\lambda,-1}}} - (-1+j+\lambda)\underline{\psi_{-1-j-\lambda,j,1}} \\ &+ (-1+2\lambda)\psi_{-j-\lambda,j,1} + (1+j+\lambda)\underline{\underline{\psi_{1-j-\lambda,j,-1}}} \\ \Leftrightarrow \psi_{-j-\lambda,j,-1} = &(-1+2\lambda)\psi_{-j-\lambda,j,1}. \end{aligned} \tag{4.121}$$

The indices of the terms underlined once of level plus one satisfy $i + j \neq -\lambda$ and are thus zero due to our assumption given in the fourth line of the statement. The terms of level minus one underlined twice are zero because of the assumption given in the fifth line of the statement, meaning their indices satisfy $i + j = -\lambda + 1$. The relation (4.121) will be needed later on.
Next, we can obtain a recurrence relation for our generating coefficients $\psi_{i,j,1}$ with $i + j = -\lambda$. This can be done by considering the cocycle condition (4.85) for $(-j - \lambda + 1, j, -1, 1)$, yielding after dropping terms of level zero and coefficients of the form $\psi_{n,-1,1}$ $n \in \mathbb{Z}$,

$$\begin{aligned} \psi_{1-j-\lambda,-1+j,1} = &\frac{1}{1+j}(-(-1+j)\underline{\psi_{1+j,1-j-\lambda,-1}} + (-2+j+\lambda)\psi_{-j-\lambda,j,1} \\ &-2(-1+\lambda)\underline{\underline{\psi_{1-j-\lambda,j,1}}} - (j+\lambda)\underline{\psi_{2-j-\lambda,j,-1}}) \\ \Leftrightarrow \psi_{1-j-\lambda,-1+j,1} = &\frac{-2+j+\lambda}{1+j}\psi_{-j-\lambda,j,1}. \end{aligned} \tag{4.122}$$

The indices of the terms underlined once of level minus one satisfy $i + j \geq -\lambda + 1$ and are thus zero due to one of our previous results. The indices of the coefficient of level plus one underlined twice fulfill $i + j \neq -\lambda$ and are thus zero due to our assumption given in the fourth line

of our statement. Therefore, the generating coefficients of the form $\psi_{i,j,1}$ with $i+j=-\lambda$ are reduced to a single generating coefficient, namely $\psi_{2-\lambda,-2,1}$.
Next, we consider two cases corresponding to λ being even and to λ being odd. The proof for λ being even is the shortest one, hence we will start with λ even. As mentioned after the first Lemma 4.3.5, $\lambda=4$ automatically already has levels plus one and minus one equal to zero. Hence, the following proof should be considered for $\lambda \geq 6$. Consider the relation (4.122) and take a diagonal term, i.e. suppose for example $-j-\lambda=j \Leftrightarrow j=-\frac{\lambda}{2}\neq -1$. This implies $\psi_{-\frac{\lambda}{2}+1,-\frac{\lambda}{2}-1,1}=0$, since the right-hand side is zero as the coefficient $\psi_{-\frac{\lambda}{2},-\frac{\lambda}{2},1}$ is zero due to the alternating property. We can use this as a starting point for the recurrence if we invert the relation (4.122), yielding:

$$\psi_{-j-\lambda,j,1}=\frac{1+j}{-2+j+\lambda}\psi_{1-j-\lambda,-1+j,1}\,.$$

Starting with $j=-\frac{\lambda}{2}+2$, increasing j until obtaining $j=-2$, we get $\psi_{-j-\lambda,j,1}=0$ for $-\frac{\lambda}{2}+1\leq j\leq -2$. These are all the generating coefficients of level plus one for fixed λ, which comes with $\frac{\lambda}{2}-2$ generating coefficients when taking into account the alternating property. Note that the pole at $j=-\lambda+2$ is not realized for $-\frac{\lambda}{2}+1\leq j\leq -2$. This concludes the proof for λ even, i.e. we have $\psi_{i,j,1}=\psi_{i,j,-1}=0\ \forall\ i,j\leq 0$ for λ even.
For λ odd, the proof is more complicated because it needs to include a discussion about the coefficients of level minus two. Let us assume λ is odd. As remarked after the Lemma 4.3.5, $\lambda=3$ has automatically vanishing levels plus one and minus one. Since we do not consider $\lambda=5,7$ by assumption, the following proof for λ odd should be considered for $\lambda\geq 9$. A recurrence relation for level minus two is given by the cocycle condition (4.85) for $(i,j,-2,1)$, yielding:

$$\begin{aligned}\psi_{i,j,-2}=\frac{1}{-2+i+j+\lambda}&(-3\psi_{-1,i,j}-(2+i)\psi_{-2+i,j,1}-(-1+i+j\lambda)\psi_{i,-2,1}\\&+(1+i+j-2\lambda)\psi_{i,j,1}+(-1+i)\psi_{1+i,j,-2}+(2+j)\psi_{-2+j,i,1}\\&+(-1+j+i\lambda)\psi_{j,-2,1}+(1-j)\psi_{1+j,i,-2}+(i-j)\psi_{i+j,-2,1})\,.\end{aligned}\tag{4.123}$$

Consider the situation $i+j>-\lambda+2$. No issues due to poles arise if this condition is satisfied. Moreover, all the coefficients of level plus one $\psi_{m,n,1}$ are zero if $i+j>-\lambda+2$, because in that case their indices satisfy $m+n\neq-\lambda$. Finally, also the coefficient of level minus one $\psi_{-1,i,j}$ is zero because we have $i+j\geq-\lambda+1$ due to the condition $i+j>-\lambda+2$. Starting with $j=-3$, $i=-4$, decreasing i and j, we obtain $\psi_{i,j,-2}=0$ for $i+j>-\lambda+2$. At $i+j=-\lambda+2$, poles appear in the recurrence relation (4.123). We thus have new generating coefficients appearing of the form $\psi_{i,j,-2}$ with $i+j-2=-\lambda$ for $\lambda\geq 9$, which will be discussed later in Lemma 4.3.8.
Next, we compare the cocycle conditions for $(-\lambda+6,-3,-2,-1)$ and $(-\lambda+5,-3,-2,-1)$. The first one yields:

$$\begin{aligned}&\underline{\psi_{-5,6-\lambda,-1}}-2\psi_{-4,6-\lambda,-2}+(6+(-6+\lambda)\lambda)\underline{\psi_{-3,-2,-1}}\\&+(-9+\lambda)\underline{\psi_{3-\lambda,-2,-1}}-(-8+\lambda)\underline{\psi_{4-\lambda,-3,-1}}+(-7+\lambda)\psi_{5-\lambda,-3,-2}\\&+(1-2\lambda)\underline{\underline{\psi_{6-\lambda,-3,-2}}}+(-2+3\lambda)\underline{\psi_{6-\lambda,-3,-1}}+(3-4\lambda)\underline{\psi_{6-\lambda,-2,-1}}=0\,,\end{aligned}\tag{4.124}$$

and the second one:

$$\begin{aligned}&\psi_{-5,5-\lambda,-1}-2\psi_{-4,5-\lambda,-2}+(6+(-5+\lambda)\lambda)\underline{\psi_{-3,-2,-1}}\\&+(-8+\lambda)\psi_{2-\lambda,-2,-1}-(-7+\lambda)\psi_{3-\lambda,-3,-1}+(-6+\lambda)\psi_{4-\lambda,-3,-2}\\&-2\lambda\psi_{5-\lambda,-3,-2}+(-1+3\lambda)\underline{\psi_{5-\lambda,-3,-1}}+(2-4\lambda)\underline{\psi_{5-\lambda,-2,-1}}=0\,.\end{aligned}\tag{4.125}$$

The indices of the coefficients underlined once of level minus one satisfy $i+j \geq -\lambda+1$, such that the corresponding coefficients are zero. Concerning the coefficient of level minus two underlined twice, its indices satisfy $i+j > -\lambda+2$, hence it is zero. Note that for $\lambda = 9$, only the second equation will be relevant.
First of all, let us have a look at the non-zero coefficients of level minus one appearing in the second equation (4.125). Using relation (4.121), we can express these coefficients of level minus one in terms of the generating coefficients of level plus one. In a second step, we can use relation (4.122) to express all the generating coefficients of level plus one in terms of the coefficient $\psi_{-\lambda+2,-2,1}$:

$$\begin{aligned}
&-\psi_{5-\lambda,-5,-1}+(-8+\lambda)\psi_{2-\lambda,-2,-1}-(-7+\lambda)\psi_{3-\lambda,-3,-1}\\
&=(1-2\lambda)\psi_{5-\lambda,-5,1}-(1-2\lambda)(-8+\lambda)\psi_{2-\lambda,-2,1}+(1-2\lambda)(-7+\lambda)\psi_{3-\lambda,-3,1}\\
&=-\frac{1-2\lambda}{6}(-7+\lambda)(-2+\lambda)\lambda\psi_{2-\lambda,-2,1}\,.
\end{aligned}$$

Next, let us have a look at the coefficients of level minus two appearing in (4.125). Using the recurrence relation (4.123), we can express $\psi_{-\lambda+5,-4,-2}$ in terms of $\psi_{-\lambda+6,-4,-2}$ and $\psi_{-\lambda+5,-3,-2}$, as well as $\psi_{-\lambda+4,-3,-2}$ in terms of $\psi_{-\lambda+5,-3,-2}$:

$$\begin{aligned}
\psi_{5-\lambda,-4,-2} &= 2\underline{\psi_{-6,5-\lambda,1}}+(5+(-5+\lambda)\lambda)\underline{\psi_{-4,-2,1}}-5\psi_{-3,5-\lambda,-2}\\
&\quad+3\underline{\underline{\psi_{-1,5-\lambda,-4}}}+(-9+\lambda)\underline{\psi_{1-\lambda,-2,1}}-(-7+\lambda)\underline{\psi_{3-\lambda,-4,1}}\\
&\quad+(-2+3\lambda)\underline{\psi_{5-\lambda,-4,1}}+(4-5\lambda)\underline{\psi_{5-\lambda,-2,1}}+(\lambda-4)\psi_{6-\lambda,-4,-2}\\
\Leftrightarrow\ \psi_{5-\lambda,-4,-2} &= 5\psi_{5-\lambda,-3,-2}+(\lambda-4)\psi_{6-\lambda,-4,-2}\,,\\
&\text{and}\\
\psi_{4-\lambda,-3,-2} &= \underline{\psi_{-5,4-\lambda,1}}+(-2+\lambda)^2\underline{\psi_{-3,-2,1}}+3\underline{\underline{\psi_{-1,4-\lambda,-3}}}\\
&\quad+(-7+\lambda)\underline{\psi_{1-\lambda,-2,1}}-(-6+\lambda)\underline{\psi_{2-\lambda,-3,1}}+(-2+3\lambda)\underline{\psi_{4-\lambda,-3,1}}\\
&\quad+(3-4\lambda)\underline{\psi_{4-\lambda,-2,1}}+(\lambda-3)\psi_{5-\lambda,-3,-2}\\
\Leftrightarrow\ \psi_{4-\lambda,-3,-2} &= (\lambda-3)\psi_{5-\lambda,-3,-2}\,.
\end{aligned}$$

The coefficients underlined once of level plus one are zero by assumption, because their indices fulfill $i+j \neq -\lambda$. The coefficients of level minus one underlined twice are also zero because their indices satisfy $i+j \geq -\lambda+1$. Inserting everything into relation (4.125) yields:

$$-\frac{1-2\lambda}{6}(-7+\lambda)(-2+\lambda)\lambda\psi_{2-\lambda,-2,1}+(-7+\lambda)(-4+\lambda)\psi_{5-\lambda,-3,-2}-2(4-\lambda)\psi_{6-\lambda,-4,-2}=0\,.$$

Comparing this equation to the relation (4.124), and since $\lambda \neq 4$ because λ is odd, we conclude $\psi_{2-\lambda,-2,1}=0$ for $\lambda \notin \{0,2,7,(\frac{1}{2})\}$. Due to the relation (4.122), we obtain that all the generating coefficients of level plus one are zero and hence, we obtain the desired result, $\psi_{i,j,1}=\psi_{i,j,-1}=0\ \forall\ i,j \leq 0$. It remains to show the same result for $i,j>0$.

$\boxed{i,j>0}$

The proof is very similar to the one given for $i,j \leq 0$. In this case, most coefficients of level plus one are non-zero, whereas only a few coefficients of level minus one are non-zero. Hence this time, we will take the coefficients of level minus one $\psi_{i,j,-1}$ with $i+j=\lambda$ as generating coefficients. First of all, we will express the coefficients of level plus one in terms of the coefficients of level minus one. This can be done by considering the coycle condition (4.85) for $(i, j, -1, 1)$ which yields:

$$\psi_{i,j,1} = \frac{1}{1+i+j-\lambda}((1+i)\psi_{-1+i,j,1} + (-1+i+j+\lambda)\psi_{i,j,-1} + (1-i)\psi_{1+i,j,-1} + (-1-j)\psi_{-1+j,i,1} + (-1+j)\psi_{1+j,i,-1}).$$

In the expression above, we already dropped the coefficients of level zero and of the form $\psi_{n,-1,1}$ $n \in \mathbb{Z}$. The pole is realized at $i + j = \lambda - 1$. The generating coefficients of level minus one are non-zero at $i+j = \lambda-1$ ($\psi_{1+i,j,-1}, \psi_{1+j,i,-1}$) and $i+j = \lambda$ ($\psi_{i,j,-1}$). Hence, as long as $i+j < \lambda-1$ is satisfied, we have $\psi_{i,j,1} = 0$, by the usual procedure, taking $j = 2$, $i = 3$, increasing i, then continuing with increasing j, always starting with $i = j+1$. Due to our assumption given in the last line of the statement, we can extend the strict inequality to equality, i.e. $\psi_{i,j,1} = 0$ for $i+j \le \lambda-1$. The remaining coefficients $\psi_{i,j,1}$ with $i+j > \lambda-1$ thus can all be expressed in terms of the generating coefficients $\psi_{i,j,-1}$ with $i+j = \lambda$. Therefore, it suffices to show $\psi_{i,j,-1} = 0$ with $i + j = \lambda$. In particular, we will need the following relation between coefficients of levels plus one and minus one, obtained by considering the cocycle condition (4.85) for $(-j+\lambda, j, -1, 1)$:

$$\begin{aligned} \psi_{-j+\lambda,j,1} &= -(1+j)\underline{\underline{\psi_{-1+j,-j+\lambda,1}}} + (-1+j)\underline{\psi_{1+j,-j+\lambda,-1}} + (1-j+\lambda)\underline{\underline{\psi_{-1-j+\lambda,j,1}}} \\ &\quad + (-1+2\lambda)\psi_{-j+\lambda,j,-1} + (1+j-\lambda)\underline{\psi_{1-j+\lambda,j,-1}} \\ \Leftrightarrow \psi_{-j+\lambda,j,1} &= (-1+2\lambda)\psi_{-j+\lambda,j,-1}\,. \end{aligned} \tag{4.126}$$

In the expression above, we already dropped the coefficients of level zero and of the form $\psi_{n,-1,1}$ $n \in \mathbb{Z}$. The terms underlined once of level plus one are zero due to the assumption given in the last line of our statement, since the indices of the coefficients satisfy $i+j = \lambda-1$. The coefficients of level minus one underlined twice are zero due to our assumption given in the second-to-last line, because their indices satisfy $i + j \ne \lambda$. Next, we will need a recurrence relation on our generating coefficients $\psi_{i,j,-1}$ with $i + j = \lambda$, which can be obtained from the cocycle condition (4.85) for $(-j+\lambda-1, j, -1, 1)$:

$$\begin{aligned} \psi_{1+j,-1-j+\lambda,-1} &= -\frac{1}{-1+j}(-(1+j)\underline{\psi_{-1+j,-1-j+\lambda,1}} + (-j+\lambda)\underline{\psi_{-2-j+\lambda,j,1}} \\ &\quad + 2(-1+\lambda)\underline{\underline{\psi_{-1-j+\lambda,j,-1}}} + (2+j-\lambda)\psi_{-j+\lambda,j,-1}) \\ \Leftrightarrow \psi_{1+j,-1-j+\lambda,-1} &= \frac{2+j-\lambda}{-1+j}\psi_{-j+\lambda,j,-1}\,. \end{aligned} \tag{4.127}$$

The coefficients underlined once of level plus one are zero due to our previous result, i.e. $\psi_{i,j,1} = 0$ for $i+j \le \lambda-1$. The coefficient of level minus one underlined twice is zero due to our assumption given in the second-to-last line of our statement, i.e. its indices satisfy $i + j \ne \lambda$. As before, we will make a case differentiation between λ even and λ odd, and start again with λ even.
The remark made for $\lambda = 4$ in the case $i, j \le 0$ holds true for the present case $i, j > 0$, hence the following proof should be considered for $\lambda \ge 6$. As before, we can make appear a diagonal term in the recurrence relation (4.127) by choosing $j = \frac{\lambda}{2} \ne 1$. This implies $\psi_{\frac{\lambda}{2}+1,\frac{\lambda}{2}-1,-1} = 0$, which can be used as a starting point when inverting the recurrence relation (4.127):

$$\psi_{-j+\lambda,j,-1} = \frac{-1+j}{2+j-\lambda}\psi_{1+j,-1-j+\lambda,-1}\,.$$

Starting with $j = \frac{\lambda}{2} - 2$, decreasing j until $j = 2$, we obtain $\psi_{-j+\lambda,j,-1} = 0$ for $2 \le j \le \frac{\lambda}{2} - 1$. This range includes all the generating coefficients existing for given λ, when taking into account the alternating property. The pole at $j = -2 + \lambda$ is not realized for this range. This concludes the proof for λ even. Next, we consider λ odd.
The remark concerning $\lambda = 3$ given in the case $i, j \le 0$ holds true in the present case $i, j > 0$. Since we do not consider $\lambda = 5$ nor $\lambda = 7$ by assumption, the following proof should be considered for $\lambda \ge 9$. As before, we need to go to level two to complete the proof. A recurrence relation for coefficients of level plus two can be obtained from the cocycle condition (4.85) for $(i, j, 2, -1)$:

$$\begin{aligned}\psi_{i,j,2} = \frac{1}{2+i+j-\lambda}\big(&3\psi_{1,i,j} + (1+i)\psi_{-1+i,j,2} - (1+i+j\lambda)\psi_{i,2,-1}\\ &+ (-1+i+j+2\lambda)\psi_{i,j,-1} - (-2+i)\psi_{2+i,j,-1} + (-1-j)\psi_{-1+j,i,2}\\ &+ (1+j+i\lambda)\psi_{j,2,-1} + (j-2)\psi_{2+j,i,-1} + (i-j)\psi_{i+j,2,-1}\big)\,. \end{aligned} \tag{4.128}$$

Consider the situation $i + j < \lambda - 2$. No issues due to poles arise if this condition is satisfied. Moreover, the coefficient of level plus one is zero because its indices satisfy $i + j \le \lambda - 1$ if $i + j < \lambda - 2$. Furthermore, the coefficients of level minus one are zero due to our assumption given in the second-to-last line of our statement, as their indices satisfy $i + j \ne \lambda$ in the situation $i + j < \lambda - 2$. Starting with $j = 3$, $i = 4$, increasing i and j, we obtain $\psi_{i,j,2} = 0$ for $i + j < \lambda - 2$. We obtain poles at $i + j = \lambda - 2$. We thus have new generating coefficients appearing of the form $\psi_{i,j,2}$ with $i + j + 2 = \lambda$ for $\lambda \ge 9$, which will be discussed later in 4.3.8.
Similarly to what we did before, the next step consists in comparing the cocycle condition (4.85) for $(\lambda - 6, 3, 2, 1)$ and $(\lambda - 5, 3, 2, 1)$. For $\lambda = 9$, only the second equation will be relevant. The first one yields:

$$\begin{aligned}&-(6+(-6+\lambda)\lambda)\underline{\psi_{3,2,1}} + 2\psi_{4,-6+\lambda,2} - \underline{\psi_{5,-6+\lambda,1}}\\ &+(-3+4\lambda)\underline{\psi_{-6+\lambda,2,1}} + (2-3\lambda)\underline{\psi_{-6+\lambda,3,1}} + (-1+2\lambda)\underline{\underline{\psi_{-6+\lambda,3,2}}}\\ &-(-7+\lambda)\psi_{-5+\lambda,3,2} + (-8+\lambda)\underline{\psi_{-4+\lambda,3,1}} - (-9+\lambda)\underline{\psi_{-3+\lambda,2,1}} = 0\,,\end{aligned} \tag{4.129}$$

and the second one:

$$\begin{aligned}&-(6+(-5+\lambda)\lambda)\underline{\psi_{3,2,1}} + 2\psi_{4,-5+\lambda,2} - \psi_{5,-5+\lambda,1}\\ &+(-2+4\lambda)\underline{\psi_{-5+\lambda,2,1}} + (1-3\lambda)\underline{\psi_{-5+\lambda,3,1}} + 2\lambda\psi_{-5+\lambda,3,2}\\ &-(-6+\lambda)\psi_{-4+\lambda,3,2} + (-7+\lambda)\psi_{-3+\lambda,3,1} - (-8+\lambda)\psi_{-2+\lambda,2,1} = 0\,.\end{aligned} \tag{4.130}$$

The terms underlined once of level plus one are zero because the indices of the coefficients satisfy $i + j \le \lambda - 1$. Besides, the coefficient of level plus two underlined twice is zero because its indices fulfill $i + j < \lambda - 2$.
Next, let us have a look at the non-vanishing coefficients of level plus one appearing in the second equation (4.130). We can express these in terms of the generating coefficients of level minus one by using expression (4.126). In a second step, we can express the generating coefficients of level minus one in terms of the coefficient $\psi_{\lambda-2,2,-1}$ by using (4.127). All in all, we obtain:

$$\begin{aligned}&-\psi_{5,-5+\lambda,1} + (-7+\lambda)\psi_{-3+\lambda,3,1} - (-8+\lambda)\psi_{-2+\lambda,2,1}\\ &= (-1+2\lambda)\left(\psi_{-5+\lambda,5,-1} + (-7+\lambda)\psi_{-3+\lambda,3,-1} - (-8+\lambda)\psi_{-2+\lambda,2,-1}\right)\\ &= -\frac{(-1+2\lambda)}{6}(-7+\lambda)(-2+\lambda)\lambda\psi_{\lambda-2,2,-1}\,.\end{aligned}$$

Concerning the coefficients of level plus two, we can use the recurrence relation (4.128) to express the coefficient $\psi_{\lambda-5,4,2}$ in terms of $\psi_{\lambda-6,4,2}$ and $\psi_{\lambda-5,3,2}$ as well as the coefficient $\psi_{\lambda-4,3,2}$ in terms of $\psi_{\lambda-5,3,2}$:

$$\begin{aligned}
\psi_{-5+\lambda,4,2} &= 3\underline{\underline{\psi_{1,-5+\lambda,4}}} - 5\psi_{3,-5+\lambda,2} + (5+(-5+\lambda)\lambda)\underline{\psi_{4,2,-1}} \\
&\quad + 2\underline{\psi_{6,-5+\lambda,-1}} + (-4+\lambda)\psi_{-6+\lambda,4,2} + (4-5\lambda)\underline{\psi_{-5+\lambda,2,-1}} \\
&\quad + (-2+3\lambda)\underline{\psi_{-5+\lambda,4,-1}} + (7-\lambda)\underline{\psi_{-3+\lambda,4,-1}} + (-9+\lambda)\underline{\psi_{-1+\lambda,2,-1}} \\
\Leftrightarrow \psi_{-5+\lambda,4,2} &= 5\psi_{-5+\lambda,3,2} + (-4+\lambda)\psi_{-6+\lambda,4,2}\,, \\
&\text{and} \\
\psi_{-4+\lambda,3,2} &= 3\underline{\underline{\psi_{1,-4+\lambda,3}}} + (-2+\lambda)^2\underline{\psi_{3,2,-1}} + \underline{\psi_{5,-4+\lambda,-1}} \\
&\quad + (-3+\lambda)\psi_{-5+\lambda,3,2} + (3-4\lambda)\underline{\psi_{-4+\lambda,2,-1}} + (-2+3\lambda)\underline{\psi_{-4+\lambda,3,-1}} \\
&\quad + (6-\lambda)\underline{\psi_{-2+\lambda,3,-1}} + (-7+\lambda)\underline{\psi_{-1+\lambda,2,-1}} \\
\Leftrightarrow \psi_{-4+\lambda,3,2} &= (-3+\lambda)\psi_{-5+\lambda,3,2}\,.
\end{aligned}$$

The coefficients of level plus one underlined twice are zero because their indices satisfy $i+j \leq \lambda-1$. The coefficients underlined once of level minus one are zero due to our assumption given in the second-to-last line of our statement, i.e. their indices fulfill $i+j \neq \lambda$. Replacing the coefficients of level plus one and plus two in (4.130) by the obtained expressions, we get:

$$\frac{1-2\lambda}{6}(-7+\lambda)(-2+\lambda)\lambda\psi_{\lambda-2,2,-1} - 2(-4+\lambda)\psi_{\lambda-6,4,2} - (-7+\lambda)(-4+\lambda)\psi_{\lambda-5,3,2} = 0\,.$$

Comparing this equation to (4.129), since $\lambda \neq 4$, we see that we get $\psi_{\lambda-2,2,-1} = 0$ for $\lambda \notin \{0,2,7,(\frac{1}{2})\}$. By (4.127), we thus obtain that all our generating coefficients of level minus one are zero, and hence we obtain the result $\psi_{i,j,1} = \psi_{i,j,-1} = 0 \;\forall\; i,j > 0$. All in all, we obtain the announced result of Lemma 4.3.7. □

Up to now, the proofs were completely algebraic. However, to prove the vanishing of levels plus two and minus two, we face the same problem than in the case for λ negative, namely that the length of the proof increases with λ. In the following Lemma 4.3.8, we present an algorithm to prove the vanishing of the generating coefficients of level plus two and minus two, for certain values of λ even. The algorithm was run for $\lambda \in \{6,8,10,14,16,18,20,22,24,26\}$, for which it worked and provided the desired proof. It failed to work for $\lambda = 4$. Also, it does not work for λ odd. Due to high processing times, the algorithm was not run beyond values of $\lambda = 26$. Although the entire algorithm was run up to values of $\lambda = 26$ only, parts of the algorithm can easily be run up to $\lambda = 100$. The algorithm was found by proceeding empirically, hence there is no guarantee that it will work for arbitrary high λ. In other words, it could be possible that already at $\lambda = 28$, the algorithm could fail to provide the desired proof.
Once the vanishing of the generating coefficients of levels plus two and minus two is obtained, the rest of the proof for coefficients of higher levels is algebraic, i.e. the length of the proof is independent of λ. Concretely, the results we need to obtain by the algorithm in Lemma 4.3.8 involve the generating coefficients of level plus two and minus two, $\psi_{i,j,2} = 0$ with $i+j+2 = \lambda$ and $i,j > 0$, $\psi_{i,j,-2} = 0$ with $i+j-2 = -\lambda$ and $i,j < 0$, and $\psi_{-3,-2,2} = 0$. Basically, we aim to achieve this by finding a system of linear equations with the variables being given by the generating coefficients and with trivial solution, which forces the generating coefficients to zero. The size of the analyzed system of linear equations is limited by processing time. The most of

the processing time is due not to the solution of the linear system, but to the solution of recurrence relations, in order to express all coefficients in terms of generating coefficients, to reduce the number of variables. In the case of λ odd and $\lambda = 4$, we could not find a suitable linear system with trivial solution in a reasonable processing time. Bigger linear systems need to be analyzed to possibly provide an adequate system, which involves more processing power. In the Appendix A, we provide some methods to speed up processing time.

Lemma 4.3.8. *Let $\psi \in \mathrm{H}^3(\mathcal{W}, \mathcal{F}^\lambda)$ with $\lambda \in \{6,8,10,14,16,18,20,22,24,26\}$ be a 3-cocycle of degree zero with coefficients satisfying the following conditions:*

$$\psi_{i,j,1} = \psi_{i,j,-1} = \psi_{i,j,0} = 0 \qquad \forall\ i,j \in \mathbb{Z} \qquad \textit{and} \qquad \psi_{-\lambda+2,-2,2} = \psi_{\lambda-2,-2,2} = 0\,.$$

Then

$$\psi_{i,j,2} = 0 \textit{ with } i+j+2 = \lambda \textit{ and } i,j>0, \qquad \psi_{i,j,-2} = 0 \textit{ with } i+j-2 = -\lambda \textit{ and } i,j<0,$$
$$\textit{and} \qquad \psi_{3,-2,2} = \psi_{-3,-2,2} = 0\,.$$

Proof. We will need the recurrence relations for generic level k. The recurrence relation for generic decreasing negative k is given by:

$$\psi_{i,j,k-1} = -\frac{1}{1+k}((1+i)\psi_{-1+i,j,k} - (i+j+k-\lambda)\psi_{i,j,k} - (1+j)\psi_{-1+j,i,k})\,. \tag{4.131}$$

We see that for $k < -1$, no pole is appearing and hence no new generating coefficients of level $k < -2$ appear. Similarly, the recurrence relation for generic increasing positive k is given by:

$$\psi_{i,j,k+1} = \frac{1}{1-k}(-(i+j+k+\lambda)\psi_{i,j,k} + (-1+i)\psi_{1+i,j,k} - (-1+j)\psi_{1+j,i,k})\,. \tag{4.132}$$

We see that for $k > 1$, no pole is appearing and hence no new generating coefficients of level $k > 2$ appear. In other words, the proof of the vanishing of the coefficients of level plus two and minus two is sufficient to imply the vanishing of all the remaining coefficients.

First of all, let us identify the generating coefficients of levels plus two and minus two. If we manage to force the generating coefficients equal to zero, the remaining coefficients of levels plus two and minus two will become zero, too, as we will show in the next lemma. The aim is thus to find a system of equations obtained from the cocycle condition which admits as a unique solution the trivial solution. We have finitely many generating coefficients of "pure" type with the three indices being either all positive or all negative, and we have a priori infinitely many generating coefficients of "mixed" type, with indices of both signs. However, the latter can be reduced to a single generating coefficient, $\psi_{-3,-2,2}$ or $\psi_{3,-2,2}$.

The generators of mixed type are the same as the ones appearing already for $\lambda = -1$ and λ negative. The algorithm we use is thus quite similar to the one we used for λ negative, see the proof of Lemma 4.3.4. The cocycle conditions to consider for these are of the form $(i,3,-2,2)$ $i \le 0$, meaning we have three indices fixed, and one index only is varying. This means that the level of the coefficients involved is reduced, which makes it easy to compute them. The resulting algorithm is thus very fast. It was tested for values of λ up to $\lambda = 100$, which took only seconds.

The generators of pure type are specific for λ positive and were not present in our previous work for λ negative. Their number increases with λ. The algorithm was found by proceeding empirically. Possibly, variants might be found which are faster. The cocycle conditions to consider are of the form $(\pm\lambda - 2k - l \mp 1, k \pm 1, k, l)$. The levels of the coefficients involved in these equations

are high, and increase with λ. This makes the algorithm very slow. It was tested up to values $\lambda = 26$, which took more or less 16 hours. We will first focus on the generators of pure type.
Considering the recurrence relation of level plus two $\psi_{i,j,2}$ for $i, j > 0$ given in (4.128), we see that a pole appears at $i + j + 2 = \lambda$. This means that the coefficients $\psi_{i,j,2}$ with $i + j + 2 = \lambda$ are generating coefficients. The recurrence relation of level minus two with $i, j \leq 0$ is given in (4.123). The pole appears at $i + j - 2 = -\lambda$, meaning that our generating coefficients are of the form $\psi_{i,j,-2}$ with $i + j - 2 = -\lambda$.
The generating coefficients of pure type do not exist for small λ even. In fact, for $\lambda = 6$ and $\lambda = 8$, we see that we have no poles appearing since for these values $i + j + 2 = \lambda$ with $i > j > 2$ and $i + j - 2 = -\lambda$ with $i < j < -2$ is not possible. Hence, for $\lambda = 6$ and $\lambda = 8$, the levels plus two and minus two can be linked to levels of plus one and minus one and are thus zero for indices of pure type.
Let us continue with $\lambda = 10$. Here we have two generators of pure type given by $\psi_{-5,-3,-2}$ and $\psi_{5,3,2}$. Considering the cocycle condition (4.85) for $(-4, -3, -2, -1)$ and $(4, 3, 2, 1)$, we immediately obtain $\psi_{-5,-3,-2} = 0$ and $\psi_{5,3,2} = 0$, respectively. Indeed, after eliminating the coefficients of level minus one, the cocycle condition for e.g. $(-4, -3, -2, -1)$ yields:

$$3\psi_{-5,-3,-2} - 19\cancel{\psi_{-4,-3,-2}} = 0\,.$$

The coefficient $\psi_{-4,-3,-2}$ is zero because it satisfies $i + j > -\lambda + 2$, see the remark below (4.123). Hence the conclusion. A similar reasoning holds true for the generating coefficients with positive indices.
The value $\lambda = 12$ corresponds to a critical value, thus we will ignore it and continue with $\lambda = 14$. For this value of λ, we have three generating coefficients for each pure type, i.e. $\psi_{-9,-3,-2}$, $\psi_{-8,-4,-2}$, $\psi_{-7,-5,-2}$ and $\psi_{9,3,2}$, $\psi_{8,4,2}$, $\psi_{7,5,2}$. The cocycle conditions to consider are given by $(-8, -3, -2, -1)$, $(-6, -4, -3, -1)$, $(-5, -4, -3, -2)$ and $(8, 3, 2, 1)$, $(6, 4, 3, 1)$, $(5, 4, 3, 2)$. Let us illustrate this on the first set of equations. After dropping the coefficients of level minus one, the cocycle condition (4.85) for $(-8, -3, -2, -1)$ yields:

$$7\psi_{-9,-3,-2} - 27\cancel{\psi_{-8,-3,-2}} - 2\psi_{-4,-8,-2} = 0\,. \tag{4.133}$$

Similarly, the cocycle condition (4.85) for $(-6, -4, -3, -1)$ gives:

$$5\psi_{-7,-4,-3} - 27\psi_{-6,-4,-3} - 3\psi_{-5,-6,-3} = 0\,. \tag{4.134}$$

and the one for $(-5, -4, -3, -2)$:

$$\begin{aligned} &\psi_{-9,-3,-2} - 2\psi_{-8,-4,-2} + \psi_{-7,-5,-2} + 3\psi_{-7,-4,-3} - 2\psi_{-6,-5,-3} \\ &- 40\psi_{-5,-4,-3} + 53\cancel{\psi_{-5,-4,-2}} - 66\cancel{\psi_{-5,-3,-2}} + 79\cancel{\psi_{-4,-3,-2}} = 0\,. \end{aligned} \tag{4.135}$$

The slashed coefficients of level minus two are zero because they satisfy $i + j > -\lambda + 2$. We see that in the last two equations (4.134) and (4.135), we have coefficients of level minus three appearing. We can express these in terms of the generating coefficients of pure type of level minus two by using (4.131) with $k = -2$ and (4.123). Doing this with *Mathematica*, the last two equations (4.134) and (4.135) above reduce to:

$$-30(\psi_{-8,-4,-2} + \psi_{-7,-5,-2}) = 0 \qquad \text{and} \qquad \psi_{-9,-3,-2} - 20\psi_{-8,-4,-2} + 2\psi_{-7,-5,-2} = 0\,.$$

Together with (4.133), these equations lead to the unique trivial solution $\psi_{-9,-3,-2} = 0$, $\psi_{-8,-4,-2} = 0$ and $\psi_{-7,-5,-2} = 0$. Proceeding similarly with the other set of three equations, we also obtain

$\psi_{9,3,2} = 0$, $\psi_{8,4,2} = 0$ and $\psi_{7,5,2} = 0$. In the following, we will not give all the details anymore, since the procedure is always the same.
We continue with $\lambda = 16$. We have four generators of each pure type appearing, namely $\psi_{-11,-3,-2}$, $\psi_{-10,-4,-2}$, $\psi_{-9,-5,-2}$, $\psi_{-8,-6,-2}$ and $\psi_{11,3,2}$, $\psi_{10,4,2}$, $\psi_{9,5,2}$, $\psi_{8,6,2}$. The eight cocycle conditions we consider are the following:

$$
\begin{aligned}
&(-10,-3,-2,-1)\,, && (10,3,2,1)\,,\\
&(-8,-4,-3,-1)\,, && (8,4,3,1)\,,\\
&(-6,-5,-4,-1)\,, && (6,5,4,1)\,,\\
&(-7,-4,-3,-2)\,, && (7,4,3,2)\,.
\end{aligned}
$$

Using the recurrence relations (4.131), (4.132), (4.123) and (4.128) to express all the coefficients appearing in the cocycle conditions above in terms of the generating coefficients, we obtain as the unique solution the trivial solution $\psi_{-11,-3,-2} = 0$, $\psi_{-10,-4,-2} = 0$, $\psi_{-9,-5,-2} = 0$, $\psi_{-8,-6,-2} = 0$ and $\psi_{11,3,2} = 0$, $\psi_{10,4,2} = 0$, $\psi_{9,5,2} = 0$, $\psi_{8,6,2} = 0$. In the following, we will provide the cocycle conditions to use also for $\lambda = 18$ and $\lambda = 20$, in order to render the pattern more explicit.
For $\lambda = 18$, we obtain five generating coefficients of each pure type, $\psi_{-13,-3,-2}$, $\psi_{-12,-4,-2}$, $\psi_{-11,-5,-2}$, $\psi_{-10,-6,-2}$, $\psi_{-9,-7,-2}$ and $\psi_{13,3,2}$, $\psi_{12,4,2}$, $\psi_{11,5,2}$, $\psi_{10,6,2}$, $\psi_{9,7,2}$. The ten cocycle conditions to use are:

$$
\begin{aligned}
&(-12,-3,-2,-1)\,, && (12,3,2,1)\,,\\
&(-10,-4,-3,-1)\,, && (10,4,3,1)\,,\\
&(-8,-5,-4,-1)\,, && (8,5,4,1)\,,\\
&(-9,-4,-3,-2)\,, && (9,4,3,2)\,,\\
&(-7,-5,-4,-2)\,, && (7,5,4,2)\,.
\end{aligned}
$$

A similar reasoning to the previous one leads to the result that all generating coefficients of pure type have to be zero. As a last example, we will consider $\lambda = 20$. We have six generators of each pure type, $\psi_{-15,-3,-2}$, $\psi_{-14,-4,-2}$, $\psi_{-13,-5,-2}$, $\psi_{-12,-6,-2}$, $\psi_{-11,-7,-2}$, $\psi_{-10,-8,-2}$ and $\psi_{15,3,2}$, $\psi_{14,4,2}$, $\psi_{13,5,2}$, $\psi_{12,6,2}$, $\psi_{11,7,2}$, $\psi_{10,8,2}$. The twelve cocycle conditions to consider are:

$$
\begin{aligned}
&(-14,-3,-2,-1)\,, && (14,3,2,1)\,,\\
&(-12,-4,-3,-1)\,, && (12,4,3,1)\,,\\
&(-10,-5,-4,-1)\,, && (10,5,4,1)\,,\\
&(-8,-6,-5,-1)\,, && (8,6,5,1)\,,\\
&(-11,-4,-3,-2)\,, && (11,4,3,2)\,,\\
&(-9,-5,-4,-2)\,, && (9,5,4,2)\,.
\end{aligned}
$$

Again, a similar reasoning to the previous ones leads to the result that all generating coefficients of pure type have to be zero.
We can generalize our procedure now to arbitrary λ. The generating coefficients are of the following form:

$$
\begin{aligned}
&\psi_{-\lambda+5,-3,-2}, \psi_{-\lambda+6,-4,-2}, \psi_{-\lambda+7,-5,-2}, \ldots \psi_{-\frac{\lambda}{2},-\frac{\lambda}{2}+2,-2}\,,\\
\text{and}\qquad &\psi_{\lambda-5,3,2}, \psi_{\lambda-6,4,2}, \psi_{\lambda-7,5,2}, \ldots \psi_{\frac{\lambda}{2},\frac{\lambda}{2}-2,2}\,.
\end{aligned}
$$

Hence, there are in total $\frac{\lambda}{2}-4$ generating coefficients with negative indices $\psi_{-\lambda-j+2,j,-2}$ $j=-3,\dots,-\frac{\lambda}{2}+2$ and $\frac{\lambda}{2}-4$ generating coefficients with positive indices $\psi_{\lambda-j-2,j,2}$ $j=3,\dots,\frac{\lambda}{2}-2$. The cocycle conditions to consider are of the form $(-\lambda-2k-l+1,k-1,k,l)$ $k<l<0$ for the coefficients with negative indices and of the form $(\lambda-2k-l-1,k+1,k,l)$ $k>l>0$ for the coefficients with positive indices. Let us illustrate the procedure on the coefficients with negative indices. The steps for the algorithm are the following:

1. Fix $l=-1$.
2. Take $k=l-1$ and decrease k as long as the condition $k-1>-\lambda-2k-l+1 \Leftrightarrow k>-\frac{\lambda}{3}-\frac{l}{3}+\frac{2}{3}$ is satisfied.
3. Take $l=-2$, repeat step 2), continue decreasing l and repeating 2) until the number of equations obtained equals the number of generating coefficients.
4. Check that the system of equations obtained has a unique solution, which is trivial.

The algorithm for the coefficients with positive indices is similar. In that case, the condition in point 2) writes $k<\frac{\lambda}{3}-\frac{l}{3}-\frac{2}{3}$. This condition is necessary in order to avoid getting equivalent conditions due to the alternating property. In practice, the algorithm the author encoded in *Mathematica* for the coefficients with negative indices is given by Algorithm 1.

Algorithm 1 Algorithm for the generating coefficients with negative indices. The algorithm for the generators with positive indices is similar. The algorithm was employed up to $\lambda=26$. Duration: about 16 hours.

```
Do[ i = 1;
l = -1;
compt = 0;

While [compt < λ/2 - 4,
k = l - 1;
While [k > -λ/3 - l/3 + 2/3 && compt < λ/2 - 4,
compt++;
liste1[i] =
FullSimplify[f[-λ - 2*k - l + 1, k - 1, k, l, λ]];          ▷ f is the cocycle condition
i++;
k- -]
;
l- -
];

Do[liste2[j] =
Psi[-λ - j + 2, j, -2, λ], {j, -3, -λ/2 + 2, -1}];          ▷ Psi are the generating coefficients

Solve[Table[liste1[j], {j, 1, compt}] == 0,Table[liste2[j], {j, -3, -λ/2 + 2, -1}]] ,{λ ,14,26,2}]
```

The algorithm starts with $\lambda=14$ in order to avoid the exceptional value $\lambda=12$. The value $\lambda=10$ was treated separately. The algorithm for the generating coefficients with positive indices is similar.

Next, we will consider the generating coefficients with indices of both signs. They appear when we consider the recurrence relations of level plus two and minus two with one index negative and one index positive, say, $i \le 0$, $j > 0$. The recurrence relation of level minus two for mixed indices is obtained from the cocycle condition (4.85) for $(i, j, -2, 1)$:

$$\psi_{i,j+1,-2} = \frac{-1}{-1+j}(-(-2+i+j+\lambda)\psi_{i,j,-2} + (-1+i)\psi_{1+i,j,-2}) \,. \tag{4.136}$$

We see that we have a pole at the starting point $j = 1$. Therefore, we have to start with $j = 2$, meaning that for each i, we obtain generating coefficients of the form $\psi_{i,2,-2}$. The same problem appears for level plus two, the recurrence relation for which can be obtained from the cocycle condition (4.85) for $(i, j, 2, -1)$:

$$\psi_{-1+i,j,2} = \frac{1}{1+i}((2+i+j-\lambda)\psi_{i,j,2} + (1+j)\psi_{-1+j,i,2}) \,. \tag{4.137}$$

This time, we have a pole at $i = -1$. Therefore, for each j we obtain generating coefficient of the form $\psi_{-2,j,2}$. Consequently, in the case of coefficients with mixed indices, we obtain a priori an infinite number of generating coefficients of the form $\psi_{i,-2,2}$ with $i \in \mathbb{Z}\setminus\{-\lambda+2, -2, -1, 0, 1, 2, \lambda-2\}$. As in the case of λ negative, the crucial point consists in managing to put the coefficients $\psi_{\pm 3,-2,2}$ equal to zero. A subsequent induction procedure then allows to put the remaining generating coefficients equal to zero.
The values of $\lambda = 6$ and $\lambda = 8$ will be treated separately as they have no generators of pure type, and hence the method used for these values is different from the one used for $\lambda \ge 10$. We will start with these two values of λ and then provide the generic algorithm for $\lambda \ge 10$. For both values of λ, we can use the same set of five equations given by $(-4, -3, -2, 2)$ and $(i, 3, -2, 2)$ with $i = -7, \ldots, -4$. For $\lambda = 6$ these conditions yield:

$$\begin{aligned}
(-4,-3,-2,2): &\quad 7\psi_{-7,-2,2} - 24\psi_{-6,-2,2} + 5\psi_{-5,-2,2} + 9\psi_{-3,-2,2} = 0\,,\\
(-7,3,-2,2): &\quad 17\psi_{-7,-2,2} - 48\psi_{-6,-2,2} + 19\psi_{-5,-2,2} - 75\psi_{-2,2,3} = 0\,,\\
(-6,3,-2,2): &\quad 12\psi_{-6,-2,2} - 42\psi_{-5,-2,2} + 4\psi_{-3,-2,2} - 50\psi_{-2,2,3} = 0\,,\\
(-5,3,-2,2): &\quad \psi_{-5,-2,2} + 4\psi_{-3,-2,2} - 5\psi_{-2,2,3} = 0\,,\\
(-4,3,-2,2): &\quad \psi_{-3,-2,2} + \psi_{-2,2,3} = 0\,.
\end{aligned}$$

The coefficient $\psi_{-4,-2,2}$ does not appear as it is of the form $\psi_{-\lambda+2,-2,2}$ and hence zero by assumption. This system of equations has as unique solution, which is the trivial solution $\psi_{i,-2,2} = 0$ for $i = -7, \ldots, -3$ and $\psi_{3,-2,2} = 0$. Similarly, for $\lambda = 8$ we obtain:

$$\begin{aligned}
(-4,-3,-2,2): &\quad 3\psi_{-7,-2,2} + 4\psi_{-5,-2,2} + 32\psi_{-4,-2,2} + 5\psi_{-3,-2,2} = 0\,,\\
(-7,3,-2,2): &\quad 8\psi_{-7,-2,2} + 57\psi_{-5,-2,2} + 12\psi_{-4,-2,2} - 565\psi_{-2,2,3} = 0\,,\\
(-6,3,-2,2): &\quad -35\psi_{-5,-2,2} + 52\psi_{-4,-2,2} + 4\psi_{-3,-2,2} - 305\psi_{-2,2,3} = 0\,,\\
(-5,3,-2,2): &\quad -7\psi_{-5,-2,2} - 12\psi_{-4,-2,2} + 40\psi_{-3,-2,2} - 145\psi_{-2,2,3} = 0\,,\\
(-4,3,-2,2): &\quad 6\psi_{-4,-2,2} - 5\psi_{-3,-2,2} + 28\psi_{-2,2,3} = 0\,.
\end{aligned}$$

This time, the coefficient $\psi_{-\lambda+2,-2,2} = \psi_{-6,-2,2} = 0$ does not appear. Again, the system has a unique solution, which is the trivial one, $\psi_{i,-2,2} = 0$ for $i = -7, \ldots, -3$ and $\psi_{3,-2,2} = 0$.
Next, we will continue with the more generic case starting from $\lambda = 10$ on, with the exception of the critical value $\lambda = 12$. The aim is the same as before: finding a system of equations which has a unique solution, which is trivial, yielding in particular $\psi_{-3,-2,2} = \psi_{3,-2,2} = 0$. In order to do this, we use the fact that the generators of pure type are zero by the previous Algorithm 1, and also $\psi_{-\lambda+2,-2,2=0} = 0$. The steps to consider are:

1. Put $\psi_{-\lambda+5,-3,-2}$ equal to zero, assuming that the Algorithm 1 for the generators of pure type did its job.
2. Use two "linking" equations $(-\lambda+5,-3,-2,2)$ and $(-\lambda+4,-3,-2,2)$, which link the generator $\psi_{-\lambda+5,-3,-2}$ of pure type to generators of mixed type, the one with lowest index i being $\psi_{-\lambda+1,-2,2}$, and which also involve $\psi_{-\lambda+2,-2,2=0} = 0$.
3. Use the usual conditions of the form $(i,3,-2,2)$ with i going from $i = -\lambda+1$ to $i = -3$ to link the generators of mixed type appearing in the linking equations to the generators $\psi_{\pm 3,-2,2}$.
4. Check that the system of equations obtained has a unique solution, which is trivial.

The number of equations of the form $(i,3,-2,2)$ with i going from $i = -\lambda+1$ to $i = -3$ is $\lambda-3$. Together with the two linking equations, we obtain a total of $\lambda-1$ equations. The non-zero generators involved range from $\psi_{-\lambda+1,-2,2}$ to $\psi_{-3,-2,2}$, with the exception of $\psi_{-\lambda+2,-2,2} = 0$, and there is also $\psi_{3,-2,2}$. In total there are thus $\lambda-3$ generators. This means that we have two superfluous equations which are a linear combinations of the others. In fact, the equation $(-\lambda+2,3,-2,2)$ is in general a combination of the other ones, and there is a second one which changes depending on λ. The author could not predict for a given λ which equations would be a combination of others, thus all of them were included in the algorithm.
Let us illustrate the procedure on the example $\lambda = 22$. The linking equations are given by $(-17,-3,-2,2)$ and $(-18,-3,-2,2)$. In these equations $\psi_{-17,-3,-2}$ would appear if it were not zero. The coefficients $\psi_{i,-2,2}$ with most negative i appearing in the two linking equations are $\psi_{-19,-2,2}$ and $\psi_{-21,-2,2}$, respectively. The coefficient $\psi_{-20,-2,2}$ does not appear because it is zero for $\lambda = 22$. The remaining equations to consider are:

$$
\begin{aligned}
&(-21,3,-2,2):\ \psi_{-21,-2,2} \text{ appears and some other } \psi_{i,-2,2} \text{ with } i > -21, \text{ as well as } \psi_{3,-2,2},\\
&\underline{(-20,3,-2,2)}:\ \psi_{-19,-2,2} \text{ appears and some other } \psi_{i,-2,2} \text{ with } i > -19, \text{ as well as } \psi_{3,-2,2},\\
&(-19,3,-2,2):\ \psi_{-19,-2,2} \text{ appears and some other } \psi_{i,-2,2} \text{ with } i > -19, \text{ as well as } \psi_{3,-2,2},\\
&(-18,3,-2,2):\ \psi_{-18,-2,2} \text{ appears and some other } \psi_{i,-2,2} \text{ with } i > -18, \text{ as well as } \psi_{3,-2,2},\\
&\vdots\\
&(-11,3,-2,2):\ \psi_{-11,-2,2} \text{ appears and some other } \psi_{i,-2,2} \text{ with } i > -11, \text{ as well as } \psi_{3,-2,2},\\
&\underline{(-10,3,-2,2)}:\ \psi_{-9,-2,2} \text{ appears and some other } \psi_{i,-2,2} \text{ with } i > -9, \text{ as well as } \psi_{3,-2,2},\\
&(-9,3,-2,2):\ \psi_{-9,-2,2} \text{ appears and some other } \psi_{i,-2,2} \text{ with } i > -9, \text{ as well as } \psi_{3,-2,2},\\
&\vdots\\
&(-3,3,-2,2):\psi_{-3,-2,2} \text{ and } \psi_{3,-2,2} \text{ appear.}
\end{aligned}
$$

We see two irregularities appearing, given by the underlined equations. In fact, normally equations of the form $(m,3,-2,2)$ yield the coefficient $\psi_{m,-2,2}$ and some coefficients $\psi_{i,-2,2}$ with $i > m$ for i,m negative. We see that for $(-20,3,-2,2)$, $\psi_{-20,3,-2,2}$ does not appear, but this is normal since $\psi_{-20,-2,2} = 0$ by assumption. A genuine anomaly appears for $(-10,3,-2,2)$ which fails to produce $\psi_{-10,-2,2}$. This leads to the fact that the equation $(-10,3,-2,2)$ is not linearly independent from the others. This anomaly occurs only for isolated values of λ, where some equation $(i,3,-2,2)$ fails to produce the corresponding coefficient $\psi_{i,-2,2}$. The author could not find a link between λ and the anomalous equation $(i,3,-2,2)$, hence it is not possible to tell which equations will become linearly dependent. Nor did the author manage to predict for which values of λ this anomaly appears. Without this anomaly, the algorithm could have been simplified by dropping the equations $(-\lambda+4,-3,-2,2)$, $(-\lambda+1,3,-2,2)$ and $(-\lambda+2,3,-2,2)$, as well as the generator $\psi_{-\lambda+1,-2,2}$. The anomalous values of λ up to $\lambda=100$ are given by: $\lambda=22$, $\lambda=26$, $\lambda=40$, $\lambda=70$, $\lambda=92$ and $\lambda=100$. However, the algorithm as presented above works for all of the values of λ up to $\lambda=100$, only by assuming that the generating coefficients of pure type are zero for the corresponding values of λ. In practice, the vanishing of generating coefficients of pure type was not proven by Algorithm 1 for values of λ beyond $\lambda=26$. The algorithm the author encoded in *Mathematica* for the generators of mixed type is given by Algorithm 2. The coefficient $\psi_{-\lambda+2,-2,2}$ is already put to zero in the definition of $psi[i,j,k,\lambda]$ via the recur-

Algorithm 2 Algorithm for the generating coefficients of mixed type. The algorithm was employed for values up to $\lambda=100$. Duration: a few seconds.

```
Do[
Clear[sol];

                                                        ▷ Define:
Do[liste3[j]=Psi[j,-2,2,λ],{j,-3,-λ+3,-1}];             ▷ Generators ψ_{i,-2,2} for i = -3,··· -λ+3
Do[liste4[i]=FullSimplify[f[i,3,-2,2,λ]],{i,-3,-λ+1,-1}];  ▷ Cocycle conditions (i,3,-2,2)

                                                        ▷ Solve system of equations:
sol=Solve[ Psi[-λ+5,-3,-2,λ]==0 &&                      ▷ Assume ψ_{-λ+5,-3,-2} = 0 by Algorithm 1
FullSimplify[f[-λ+4,-3,-2,2,λ]]==0 &&                   ▷ The first linking equation
FullSimplify[f[-λ+5,-3,-2,2,λ]]==0 &&                   ▷ The second linking equation
Table[liste4[i],{i,-3,-λ+1,-1}]==0,                     ▷ Cocycle conditions (i,3,-2,2)

                                                        ▷ with respect to:
Flatten[{Table[liste3[j],{j,-3,-λ+3,-1}],               ▷ Generators ψ_{i,-2,2} for i = -3,··· -λ+3
Psi[-2,2,3,λ],                                          ▷ Generator ψ_{3,-2,2}
Psi[-λ+5,-3,-2,λ],                                      ▷ Generator ψ_{-λ+5,-3,-2} = 0 by the first equation
Psi[-λ+1,-2,2,λ]}]];                                    ▷ Generator ψ_{-λ+1,-2,2} not included in liste3

Print[sol]                                              ▷ Print the solution

,{λ,14,100,2}]
```

rence relations, which in turn appear in the cocycle condition $f[i,j,k,l,\lambda]$. The code starts with $\lambda=14$ in order to avoid the exceptional value $\lambda=12$. A successful separate run for $\lambda=10$ was done. □

Remark 4.3.2. The Algorithm 1 is capable of producing an infinite number of equations by letting l vary. There is a counter which counts the generated equations and stops the algorithm once the number of equations obtained equals the number of generating coefficients of pure type. In the explicit examples we gave, the equations produced by $l=-1$ and $l=-2$ were sufficient and we did not need to go further $l\le-3$. A sensible question to ask is whether this will always be the case for all values of λ. In order to answer this question, we have to count the number of equations $(-\lambda-2k-l+1,k-1,k,l)$ $l,k\le0$ generated by varying k from $k=l-1$ down to k satisfying $k>-\frac{\lambda}{3}-\frac{l}{3}+\frac{2}{3}$ for fixed l. In order to obtain an exact integer number, we need to consider several case separations. We will not give details of the reasoning, but the final results are:

- $\underline{\lambda \bmod 3 = 0}$:
  - ⋆ $\underline{-l \bmod 3 = 0}$: number of equations: $\frac{\lambda}{3} + \frac{4l}{3} - 1$.
  - ⋆ $\underline{-l \bmod 3 = 1}$: number of equations: $\frac{\lambda}{3} + \frac{4l}{3} - \frac{5}{3}$.
  - ⋆ $\underline{-l \bmod 3 = 2}$: number of equations: $\frac{\lambda}{3} + \frac{4l}{3} - \frac{4}{3}$.
- $\underline{\lambda \bmod 3 = 1}$:
  - ⋆ $\underline{-l \bmod 3 = 0}$: number of equations: $\frac{\lambda}{3} + \frac{4l}{3} - \frac{4}{3}$.
  - ⋆ $\underline{-l \bmod 3 = 1}$: number of equations: $\frac{\lambda}{3} + \frac{4l}{3} - 1$.
  - ⋆ $\underline{-l \bmod 3 = 2}$: number of equations: $\frac{\lambda}{3} + \frac{4l}{3} - \frac{5}{3}$.
- $\underline{\lambda \bmod 3 = 2}$:
  - ⋆ $\underline{-l \bmod 3 = 0}$: number of equations: $\frac{\lambda}{3} + \frac{4l}{3} - \frac{5}{3}$.
  - ⋆ $\underline{-l \bmod 3 = 1}$: number of equations: $\frac{\lambda}{3} + \frac{4l}{3} - \frac{4}{3}$.
  - ⋆ $\underline{-l \bmod 3 = 2}$: number of equations: $\frac{\lambda}{3} + \frac{4l}{3} - 1$.

Let us compare these numbers to the number $\lambda 2 - 4$ of equations needed, i.e. the number of generators of pure negative type.
We will start with $l = -1$ and check explicitly what happens for $\lambda \bmod 3 = 1$. If $l = -1$, then we have $-l \bmod 3 = 1$. The number of equations obtained for $l = -1$ is given by $\frac{\lambda}{3} + \frac{4l}{3} - 1 = \frac{\lambda}{3} - \frac{7}{3}$. Let us see for which values of λ this number is equal to or bigger than $\frac{\lambda}{2} - 4$:

$$\frac{\lambda}{3} - \frac{7}{3} \overset{!}{\geq} \frac{\lambda}{2} - 4 \Leftrightarrow 10 \geq \lambda\,.$$

For $\lambda \bmod 3 = 2$, the condition is $8 \geq \lambda$, and for $\lambda \bmod 3 = 0$ the condition is $6 \geq \lambda$. Since we only have generating coefficients of pure type for $\lambda \geq 10$, it means that $\lambda = 10$, which satisfies indeed $\lambda \bmod 3 = 1$, is the only value of λ where $l = -1$ produces enough equations so that we do not need $l = -2$. This is exactly what we saw explicitly in the examples.
Next, let us take both $l = -1$ and $l = -2$ into account for each of the three cases. Let us start with the case $\lambda \bmod 3 = 2$. In this case, the number of equations obtained from $l = -1$ is given by: $\frac{\lambda}{3} + \frac{4l}{3} - \frac{4}{3} = \frac{\lambda}{3} - \frac{8}{3}$. The number of equations from $l = -2$ corresponds to: $\frac{\lambda}{3} + \frac{4l}{3} - 1 = \frac{\lambda}{3} - \frac{11}{3}$. All equations together should be more than $\frac{\lambda}{2} - 4$:

$$\frac{\lambda}{3} - \frac{8}{3} + \frac{\lambda}{3} - \frac{11}{3} \overset{!}{\geq} \frac{\lambda}{2} - 4 \Leftrightarrow \lambda \geq 14\,.$$

The value $\lambda = 14$ is indeed the first value satisfying $\lambda \bmod 3 = 2$. This means that for all values of λ satisfying $\lambda \bmod 3 = 2$, $l = -1$ and $l = -2$ will always provide enough equations so that we do not need to consider $l < -2$.
Let us continue with $\lambda \bmod 3 = 1$. For $l = -1$, the number of equations is given by: $\frac{\lambda}{3} + \frac{4l}{3} - 1 = \frac{\lambda}{3} - \frac{7}{3}$. For $l = -2$, we obtain: $\frac{\lambda}{3} + \frac{4l}{3} - \frac{5}{3} = \frac{\lambda}{3} - \frac{13}{3}$. Both together should provide more than $\frac{\lambda}{2} - 4$ equations:

$$\frac{\lambda}{3} - \frac{7}{3} + \frac{\lambda}{3} - \frac{13}{3} \overset{!}{\geq} \frac{\lambda}{2} - 4 \Leftrightarrow \lambda \geq 16\,.$$

The value $\lambda = 16$ is the first value of λ satisfying $\lambda \bmod 3 = 1$ we consider. Hence, for all values of λ satisfying $\lambda \bmod 3 = 1$, $l = -1$ and $l = -2$ will always provide enough equations so that we do not need to consider $l < -2$.
Finally, let us consider the case $\lambda \bmod 3 = 0$. For $l = -1$ the number of equations in this case is given by: $\frac{\lambda}{3} + \frac{4l}{3} - \frac{5}{3} = \frac{\lambda}{3} - 3$. For $l = -2$, the number of equations obtained is given by: $\frac{\lambda}{3} + \frac{4l}{3} - \frac{4}{3} = \frac{\lambda}{3} - 4$. Both together should be bigger than $\frac{\lambda}{2} - 4$:

$$\frac{\lambda}{3} - 3 + \frac{\lambda}{3} - 4 \overset{!}{\geq} \frac{\lambda}{2} - 4 \Leftrightarrow \lambda \geq 18\,.$$

Again, the value $\lambda = 18$ is the first value of λ satisfying $\lambda \bmod 3 = 0$ which we consider. Hence, for all values of λ satisfying $\lambda \bmod 3 = 0$, $l = -1$ and $l = -2$ will always provide enough equations so that we do not need to consider $l < -2$. All in all, it means that we could actually limit Algorithm 1 to $l = -2$. This does not render the algorithm faster, but it augments its readability. As a remark, note that the first value of λ satisfying $\lambda \bmod 3 = 0$ and having generating coefficients of pure type would have been $\lambda = 12$, an exceptional value. In fact, we expect this value of λ to be exceptional because of the results from continuous cohomology, and thus we excluded this value right from the start. However, now we have a first clue that this value is also exceptional in the algebraic setting. Because we obtained $\lambda \geq 18$, it means that for $\lambda = 12$, the values $l = -1$ and $l = -2$ will not provide enough equations. We could consider $l = -3$, which might be trivial, so that there might be a left-over generator of pure type. Clearly, the exceptional values would need further investigation.

Next, we will use the results obtained algebraically previously for the levels plus one, zero and minus one, as well as the results obtained by the algorithms in Lemma 4.3.8, namely the generators of pure type are zero, as well as $\psi_{3,-2,2} = \psi_{-3,-2,2} = 0$. The proof given in Lemma 4.3.9 below is again purely algebraic and works in principle both for λ even and λ odd. However, some of the assumptions of Lemma 4.3.9 only hold true for particular values of λ even, starting with $\lambda = 6$ and ending with $\lambda = 26$, and $\lambda \neq 12$. Thus in the end, the final result holds true only for these values of λ.

Lemma 4.3.9. *Let $\psi \in \mathrm{H}^3(\mathscr{W}, \mathscr{F}^\lambda)$ with $\lambda \in \{6, 8, 10, 14, 16, 18, 20, 22, 24, 26\}$ be a 3-cocycle of degree zero with coefficients $\psi_{i,j,k}$, satisfying the following conditions:*

$$\begin{aligned}
&\psi_{i,j,1} = \psi_{i,j,-1} = \psi_{i,j,0} = 0 \qquad \forall\ i,j \in \mathbb{Z} \qquad \textit{and} \qquad \psi_{-\lambda+2,-2,2} = \psi_{\lambda-2,-2,2} = 0\,,\\
&\psi_{i,j,2} = 0 \textit{ with } i+j+2 = \lambda \textit{ and } i,j > 0\,, \qquad \psi_{i,j,-2} = 0 \textit{ with } i+j-2 = -\lambda \textit{ and } i,j < 0\,,\\
&\textit{and} \qquad \psi_{3,-2,2} = \psi_{-3,-2,2} = 0\,.
\end{aligned}$$

Then:

$$\psi_{i,j,k} = 0 \qquad \forall\ i,j,k \in \mathbb{Z}\,.$$

Proof. We will start by using our results obtained for the generators of pure type and hence focus on the coefficients of the form $\psi_{i,j,k}$ with all $i,j,k \leq 0$ or all $i,j,k \geq 0$. After dropping the coefficients of levels minus one and plus one, the cocycle condition (4.85) for $(i,j,-2,1)$ yields a recurrence relation for the coefficients $\psi_{i,j,-2}$ with $i,j \leq 0$:

$$\psi_{i,j,-2} = \frac{1}{(-2+i+j+\lambda)}\left((-1+i)\psi_{1+i,j,-2} - (-1+j)\psi_{1+j,i,-2}\right). \tag{4.138}$$

We have poles at $i+j-2 = -\lambda$. However, the coefficients of the form $\psi_{i,j,-2}$ with $i+j-2 = -\lambda$ are generators of pure type and thus zero by the assumption based on Algorithm 1. The recurrence relation above then implies $\psi_{i,j,-2} = 0$ for all $i, j \leq 0$. Similarly, after dropping the coefficients of levels minus one and plus one, the cocycle condition (4.85) for $(i, j, 2, -1)$ yields a recurrence relation for the coefficients $\psi_{i,j,2}$ with $i, j > 0$:

$$\psi_{i,j,2} = \frac{1}{(2+i+j-\lambda)}\left((1+i)\psi_{-1+i,j,2} - (1+j)\psi_{-1+j,i,2}\right).$$

We have a pole appearing at $i + j + 2 = \lambda$. However, the coefficients of the form $\psi_{i,j,2}$ with $i + j + 2 = \lambda$ are generators of pure type and thus zero by the assumption base on Algorithm 1. The recurrence relation above then implies $\psi_{i,j,2} = 0$ for all $i, j > 0$.
We can now use induction on the third index k. Let us start with generic negative k. So we proved $\psi_{i,j,k} = 0$ for all $i, j \leq 0$ for $k = 0, -1, -2$. Let us assume the result holds true for k ($k \leq -2$), and let us check whether the hypothesis remains true for induction step $k-1$. A look at Equation (4.131) immediately reveals that this is correct. In fact, at the right-hand side of (4.131), all the coefficients are of level k with $i, j \leq 0$ and thus, zero. Hence $\psi_{i,j,k-1}$ is also zero as there is no pole for $k \leq -2$. By induction, we thus obtain $\psi_{i,j,k} = 0$ for all $i, j, k \leq 0$.
We proceed similarly for generic positive k. We proved $\psi_{i,j,k} = 0$ for all $i, j > 0$ for $k = 0, 1, 2$. Let us assume the result holds true for k ($k \geq 2$), and let us check whether the hypothesis remains true for induction step $k+1$. A look at equation (4.132) immediately reveals that this is correct. In fact, at the right-hand side of (4.132), all the coefficients are of level k with $i, j > 0$ and thus, zero. Hence $\psi_{i,j,k+1}$ is also zero as there is no pole for $k \geq 2$. By induction, we thus obtain $\psi_{i,j,k} = 0$ for all $i, j, k > 0$.
The analysis of coefficients of mixed type is more involved, though the procedure per se is very similar to the one used for $\lambda < 0$. The cocycle condition (4.85) for $(i, 3, 2, -1)$ provides us with a recurrence relation on i for $\psi_{i,3,2}$:

$$\begin{aligned}
&-3\cancel{\psi_{1,i,3}} - (4+i\lambda)\cancel{\psi_{3,2,-1}} - \psi_{5,i,-1} - (1+i)\psi_{-1+i,3,2} + (1+i+3\lambda)\cancel{\psi_{i,2,-1}} \\
&-(2+i+2\lambda)\cancel{\psi_{i,3,-1}} + (5+i-\lambda)\psi_{i,3,2} + (-2+i)\cancel{\psi_{2+i,3,-1}} - (-3+i)\cancel{\psi_{3+i,2,-1}} = 0 \\
&\Leftrightarrow \psi_{-1+i,3,2} = \frac{(5+i-\lambda)}{(1+i)}\psi_{i,3,2}\,.
\end{aligned} \tag{4.139}$$

The slashed terms are of level plus one, zero or minus one, and are thus zero by assumption. Also by assumption, $\psi_{3,-2,2} = 0$. Hence, starting with $i = -2$, decreasing i, we obtain $\psi_{i,3,2} = 0$ for all $i \leq -3$. Similarly, the cocycle condition (4.85) on $(-3, j, -2, 1)$ provides us with a recurrence relation on j for $\psi_{j,-3,-2}$:

$$\begin{aligned}
&-\cancel{\psi_{-5,j,1}} + (-4+j\lambda)\cancel{\psi_{-3,-2,1}} + (-5+j+\lambda)\psi_{-3,j,-2} + (2-j+2\lambda)\cancel{\psi_{-3,j,1}} + 3\cancel{\psi_{-1,-3,j}} \\
&+(3+j)\cancel{\psi_{-3+j,-2,1}} - (2+j)\cancel{\psi_{-2+j,-3,1}} + (1-j+3\lambda)\cancel{\psi_{j,-2,1}} + (-1+j)\psi_{1+j,-3,-2} = 0 \\
&\Leftrightarrow \psi_{1+j,-3,-2} = \frac{(-5+j+\lambda)}{(-1+j)}\psi_{j,-3,-2}\,.
\end{aligned} \tag{4.140}$$

The slashed terms are of level plus one, zero or minus one and are thus zero by assumption. Moreover, by assumption, $\psi_{2,-3-,2} = 0$. Hence, starting with $j = 2$, increasing j, we obtain $\psi_{j,-3,-2} = 0$ for all $j \geq 3$.
In the next step, we will focus on coefficients of level plus two and minus two. We will start with level minus two. More precisely, we want to prove $\psi_{j,i,-2} = 0$ for all $j > 0$ and $i \leq 0$. We will do this by induction on i. Indeed, we already showed that the statement $\psi_{j,i,-2} = 0$ for all $j > 0$ is true for $i = 0, -1, -2, -3$. Let us suppose the statement holds true down to $i + 1$, $i \leq -4$, and let

us show that it remains true for i. Consider the cocycle condition (4.85) on $(i, j, -2, 1)$, which gives us a recurrence relation on j for $\psi_{j,i,-2}$:

$$
\begin{aligned}
&(-2+i+j+\lambda)\psi_{i,j,-2} - (-1+i)\underbrace{\psi_{1+i,j,-2}}_{=0} + (-1+j)\psi_{1+j,i,-2} = 0 \\
&\Leftrightarrow \psi_{i,1+j,-2} = \frac{(-2+i+j+\lambda)}{(-1+j)}\psi_{i,j,-2}\,. \qquad (4.141)
\end{aligned}
$$

The term in the middle is zero because of the induction hypothesis. We will need the relations for $j = 3,5$, given by:

$$
\begin{aligned}
&\text{For } j=2\text{: } \psi_{i,3,-2} = (i+\lambda)\psi_{i,2,-2}\,, \\
&\text{For } j=4\text{: } \psi_{i,5,-2} = \frac{(2+i+\lambda)(1+i+\lambda)(i+\lambda)}{6}\psi_{i,2,-2}\,.
\end{aligned}
$$

Consider the cocycle condition (4.85) for $(i,3,2,-2)$ yielding:

$$
\begin{aligned}
0 = &-\cancel{4\psi_{0,i,3}} + \cancel{5\psi_{1,i,2}} - (3+i\lambda)\underline{\psi_{3,2,-2}} - \psi_{5,i,-2} - (2+i)\underline{\psi_{-2+i,3,2}} + (i+3\lambda)\psi_{i,2,-2} \\
&- (1+i+2\lambda)\psi_{i,3,-2} + (5+i-2\lambda)\underline{\psi_{i,3,2}} + (-2+i)\psi_{2+i,3,-2} - (-3+i)\psi_{3+i,2,-2}\,.
\end{aligned}
$$

The underlined terms are of the form $\psi_{i,3,2}$ $i \le 0$ and are thus zero. The terms $\psi_{i+2,3,-2}$ and $\psi_{i+3,2,-2}$ are zero because of the induction hypothesis. The terms $\psi_{i,5,-2}$ and $\psi_{i,3,-2}$ can be expressed in terms of $\psi_{i,2,-2}$ as shown above, yielding:

$$
\left(-(-2+i+\lambda)\left(i^2+(-7+\lambda)\lambda+i(-1+2\lambda)\right)\right)\psi_{i,2,-2} = 0\,. \qquad (4.142)
$$

We have to check the zeros of the polynomial in i above to see for which values of i the relation will be trivial. The zeros of the polynomial are given by:

$$
i_0 = -\lambda + 2 \qquad \text{and} \qquad i_0^{\pm} = \frac{1}{2}\left(1 - 2\lambda \pm \sqrt{24\lambda+1}\right).
$$

The first zero $i_0 = -\lambda + 2$ gives a trivial relation for the coefficient $\psi_{-\lambda+2,-2,2}$. This is not a problem, as this coefficient is already zero by assumption. The second and third solution will create a problem when they yield integer negative values for i. A quick verification shows that for our λ of interest, both solutions $i_0^{\pm}$ are always negative. In order to get an idea when they yield integers, a quick scan reveals values of λ and indices i for which the coefficients $\psi_{i,-2,2}$ are not put to zero by the relation (4.142) above:

λ	i_0^-	i_0^+
0	0	1
1	−3	2
2	−5	2
5	−10	1
7	−13	0
12	−20	−3
15	−24	−5
$\boxed{22}$	−33	−10
$\boxed{26}$	−38	−13
35	−49	−20
$\boxed{40}$	−55	−24
51	−68	−33
57	−75	−38
$\boxed{70}$	−90	−49
77	−98	−55
$\boxed{92}$	−115	−68
$\boxed{100}$	−124	−75

We see that all of the exceptional values of λ show up, given in bold, but since we do not consider them, they do not cause trouble. The boxed values correspond to values of λ for which the simplified version of Algorithm 2 did not work. Without knowing all the values of λ explicitly, we will refer to the values of λ causing trouble as pathological λ.
For all non-pathological λ, the relation (4.142) yields immediately $\psi_{i,-2,2} = 0$ for fixed i in the induction procedure. For the values of λ given in the table above, more work needs to be done in order to obtain the same result, which we will do in the following.
Let us suppose we are dealing with a pathological λ, and that we are at an induction step i where i corresponds to either i_0^+ or i_0^-. The proof is the same for both values i_0^+ and i_0^-, hence we will treat them simultaneously. Note that by induction hypothesis, we already have $\psi_{i,-2,2} = 0$ for all $0 \geq i > i_0^\pm$. The cocycle condition (4.85) on $(i_0^\pm + 3, -3, -2, 2)$ yields:

$$\begin{aligned}0 = {} & \psi_{-5,3+i_0^\pm,2} - (-3 + (3+i_0^\pm)\lambda)\underline{\psi_{-3,-2,2}} - 5\cancel{\psi_{-1,3+i_0^\pm,-2}} + 4\cancel{\psi_{0,3+i_0^\pm,-3}} \\ & - (6+i_0^\pm)\psi_{i_0^\pm,-2,2} + (5+i_0^\pm)\psi_{1+i_0^\pm,-3,2} + (-2+i_0^\pm+2\lambda)\underline{\underline{\psi_{3+i_0^\pm,-3,-2}}} \\ & - (2+i_0^\pm-2\lambda)\psi_{3+i_0^\pm,-3,2} + (3+i_0^\pm-3\lambda)\underline{\psi_{3+i_0^\pm,-2,2}} - (1+i_0^\pm)\underline{\underline{\psi_{5+i_0^\pm,-3,-2}}}.\end{aligned} \tag{4.143}$$

The slashed terms are of level minus one and zero and are thus zero by assumption. The terms underlined twice $\psi_{5+i_0^\pm,-3,-2}$ and $\psi_{3+i_0^\pm,-3,-2}$ are coefficients of the form $\psi_{i,j,k}$ with $i,j,k \leq 0$ and are thus zero by our previous results. Indeed, a look at the table above reveals that for non-exceptional λ, $i_0^\pm < -5$, hence $5 + i_0^\pm$ and $3 + i_0^\pm$ are negative. Note that even if they had been positive, the coefficients would have been zero anyway as we showed before that coefficients of the form $\psi_{j,-3,-2}$ with $j > 0$ are zero. The coefficients underlined once are zero by our assumption and our induction hypothesis. The next aim is to express via recurrence relations the coefficients $\psi_{-5,3+i_0^\pm,2}$, $\psi_{3+i_0^\pm,-3,2}$ and $\psi_{1+i_0^\pm,-3,2}$ in terms of $\psi_{i_0^\pm,-2,2}$, and to see whether we obtain a non-trivial relation for $\psi_{i_0^\pm,-2,2}$. In order to obtain these recurrence relations, we need to consider cocycle conditions (4.85) of the form $(i,j,-1,k)$. More precisely, in this case we take

$k=-2$ and $j=2$, i.e. $(i,2,-1,-2)$, yielding after dropping coefficients of level minus one and plus one:

$$\psi_{i,2,-3}=(1+i)\psi_{-1+i,2,-2}-(i-\lambda)\psi_{i,2,-2}\,. \tag{4.144}$$

The coefficients we need correspond to $i=3+i_0^\pm$ and $i=1+i_0^\pm$, yielding respectively:

$$\psi_{3+i_0^\pm,2,-3}=(4+i_0^\pm)\psi_{2+i_0^\pm,2,-2}-(3+i_0^\pm-\lambda)\psi_{3+i_0^\pm,2,-2}=0\,, \tag{4.145}$$

$$\psi_{1+i_0^\pm,2,-3}=(2+i_0^\pm)\psi_{i_0^\pm,2,-2}-(1+i_0^\pm-\lambda)\psi_{1+i_0^\pm,2,-2}=(2+i_0^\pm)\psi_{i_0^\pm,2,-2}\,. \tag{4.146}$$

The coefficients of the form $\psi_{i,-2,2}$ with $0\geq i>i_0^\pm$ are zero by the induction hypothesis. The remaining coefficient in (4.143) to express in terms of $\psi_{i_0^\pm,2,-2}$ is of the form $\psi_{i,2,-5}$. It is obtained from the cocycle condition (4.85) for $(i,j,-1,k)$ with $k=-4$ and $j=2$, yielding after omitting terms of level plus and minus one:

$$\psi_{i,2,-5}=\frac{1}{3}((1+i)\psi_{-1+i,2,-4}-(-2+i-\lambda)\psi_{i,2,-4})\,. \tag{4.147}$$

We see that we need coefficients of the form $\psi_{i,2,-4}$, which can be obtained from the cocycle condition (4.85) for $(i,j,-1,k)$ with $k=-3$ and $j=2$:

$$\psi_{i,2,-4}=\frac{1}{2}((1+i)\psi_{-1+i,2,-3}-(-1+i-\lambda)\psi_{i,2,-3})$$

$$\overset{(4.144)}{\Leftrightarrow}\quad \psi_{i,2,-4}=\frac{i(i+1)}{2}\psi_{i-2,2,-2}-(i-1-\lambda)(i+1)\psi_{i-1,2,-2}+\frac{(i-1-\lambda)(i-\lambda)}{2}\psi_{i,2,-2}\,.$$

More precisely, in Equation (4.143), we will need the coefficient $\psi_{-5,3+i_0^\pm,2}$. From (4.147), we see that in this case, we need the coefficients $\psi_{2+i_0^\pm,2,-4}$ and $\psi_{3+i_0^\pm,2,-4}$, given by:

$$\psi_{2+i_0^\pm,2,-4}=\frac{(2+i_0^\pm)(3+i_0^\pm)}{2}\psi_{i_0^\pm,2,-2}-(\ldots)\underbrace{\psi_{1+i_0^\pm,2,-2}}_{=0\text{ induction}}+(\ldots)\underbrace{\psi_{2+i_0^\pm,2,-2}}_{=0\text{ induction}}$$

$$\psi_{3+i_0^\pm,2,-4}=(\ldots)\psi_{1+i_0^\pm,2,-2}-(\ldots)\psi_{2+i_0^\pm,2,-2}+(\ldots)\psi_{3+i_0^\pm,2,-2}=0$$

The coefficients of the form $\psi_{i,-2,2}$ with $0\geq i>i_0^\pm$ are zero by the induction hypothesis. Inserting these values into (4.147) with $i=3+i_0^\pm$ yields:

$$\psi_{3+i_0^\pm,2,-5}=\frac{1}{3}((4+i_0^\pm)\psi_{2+i_0^\pm,2,-4}-(1+i_0^\pm-\lambda)\psi_{3+i_0^\pm,2,-4})$$

$$\Leftrightarrow\ \psi_{3+i_0^\pm,2,-5}=\frac{(4+i_0^\pm)(3+i_0^\pm)(2+i_0^\pm)}{6}\psi_{i_0^\pm,2,-2}\,. \tag{4.148}$$

Finally, we can insert the values (4.145), (4.146) and (4.148) into (4.143), which yields:

$$(-2+i_0^\pm)(5+i_0^\pm)i_0^\pm\ \psi_{i_0^\pm,2,-2}=0\,. \tag{4.149}$$

The zeros of the polynomial in $i_0^\pm$ correspond to $i_0^\pm=0,2,-5$. The coefficients $\psi_{0,2,-2}$ and $\psi_{2,2,-2}$ are zero, hence it remains to check whether $i_0^\pm=-5$ has a solution in λ:

$$-5=i_0^\pm=\frac{1}{2}\left(1-2\lambda\pm\sqrt{24\lambda+1}\right)$$

$$\Leftrightarrow\lambda=2\qquad\text{or}\qquad\lambda=15\,.$$

These are two exceptional values of λ we do not consider. Therefore, the relations (4.142) and (4.149) show that we obtain $\psi_{i,-2,2} = 0$, where i is the fixed i in our induction step i, and it holds true even in the case when the induction step i corresponds to a critical $i = i_0^{\pm}$.
Now we can come back to (4.141) for fixed i. Starting with $j = 2$, increasing j, we obtain $\psi_{i,j,-2} = 0$ for all $j > 0$ for the induction step i, even if $i = i_0^{\pm}$. Hence, our induction remains true for step i. By induction, we thus obtain $\psi_{i,j,-2} = 0$ for all $i \leq 0$ and all $j > 0$.

Note that we could have proceeded the other way round. Instead of starting with the cocycle condition $(i,3,2,-2)$ resulting in (4.142), we could have started with the cocycle condition (4.143) given by $(i+3,-3,-2,2)$. In fact, starting with the cocycle condition $(i+3,-3,-2,2)$, redoing the same reasoning as before in the case $i = i_0^{\pm}$, we end up with the equation $(-2+i)(5+i)i\psi_{i,2,-2} = 0$, which is independent of λ. The zeros $i = 0$ and $i = 2$ obviously do not make trouble, but the zero $i = -5$ corresponding to $\psi_{-5,2,-2}$ poses difficulties for all λ under consideration, except for $\lambda = 7$, for which $\psi_{-5,2,-2}$ corresponds precisely to $\psi_{-\lambda+2,2,-2} = 0$. Hence the induction step $i = -5$ now leads to trouble, and a separate non-trivial relation for $\psi_{-5,2,-2}$ has to be found. This can be done by considering the cocycle condition $(i,3,2,-2)$ with $i = -5$, which by an identical reasoning to the one done in the first method leads to (4.142) with $i = -5$:

$$(\lambda-7)((\lambda-7)\lambda - 5(2\lambda-1) + 25)\psi_{-5,2,-2} = 0\,,$$

which has zeros in λ given by $\lambda = 2$, $\lambda = 7$ and $\lambda = 15$. However, for $\lambda = 7$, the coefficient $\psi_{-5,2,-2}$ precisely corresponds to $\psi_{-\lambda+2,2,-2}$, which is zero by assumption. Hence, we end up again with the same exceptional values of λ we found before. We see that with this method, the critical induction step $i = -5$ is of a simpler form than before $i = i_0^{\pm}$, which simplifies the expressions involved. However, the general procedure and length of the proof remains unchanged. Next, we will prove the same result for level plus two $\psi_{i,j,2} = 0$ for $i \leq 0$ and $j > 0$, which we will do by using positive induction on j. We will do this by using the second method to change a bit.

Let us briefly summarize our assumptions and results obtained so far. We already have $\psi_{i,j,k} = 0$ for all $i,j,k \geq 0$, and also $\psi_{-3,2,-2} = \psi_{3,2,-2} = 0$ and consequently $\psi_{i,3,2} = 0$ for $i \leq 0$ and $\psi_{j,-3,-2} = 0$ for $j > 0$ due to (4.139) and (4.140), respectively. We have $\psi_{i,j,2} = 0$ for all $i \leq 0$ for $j = 1,2$ and $j = 3$. Let us suppose the result $\psi_{i,j,2} = 0$ for all $i \leq 0$ holds true for induction step $j-1$ $(j \geq 4)$, and let us check whether it remains true for induction step j. The cocycle condition (4.85) for $(3, j-3, -2, 2)$ yields:

$$\begin{aligned}
&4\cancel{\psi_{0,3,-3+j}} + 5\cancel{\psi_{1,-3+j,2}} + (3+(-3+j)\lambda)\underline{\psi_{3,-2,2}} + (-2+j+2\lambda)\psi_{3,-3+j,-2} \\
&-(2+j-2\lambda)\underline{\underline{\psi_{3,-3+j,2}}} - \psi_{5,-3+j,-2} - (-1+j)\underline{\underline{\psi_{-5+j,3,2}}} \\
&-(-3+j+3\lambda)\underline{\psi_{-3+j,-2,2}} + (-5+j)\psi_{-1+j,3,-2} + (-6+j)\psi_{j,-2,2} = 0\,.
\end{aligned} \tag{4.150}$$

The twice underlined terms are zero because for $j \geq 4$ either they are of the form $\psi_{i,j,k}$ with $i,j,k \geq 0$ or of level plus one, minus one or zero. The terms underlined once are zero by induction assumption, as they are of the form $\psi_{l,i,2}$ with $i \leq 0$ and $0 < l < j$. Next, we will express the coefficients $\psi_{3,-3+j,-2}$, $\psi_{5,-3+j,-2}$ and $\psi_{-1+j,3,-2}$ in terms of $\psi_{j,-2,2}$. These coefficients are of generic level $k = 3$ and $k = 5$, hence we will start by considering the cocycle condition $(i,j,1,k)$ for $k = 2$ and $i = -2$, i.e. $(-2,j,1,2)$ yielding after dropping coefficients of levels minus one, zero and plus one:

$$\psi_{-2,j,3} = (j+\lambda)\psi_{-2,j,2} + (-1+j)\psi_{1+j,-2,2}\,. \tag{4.151}$$

In (4.150), we see that we need the coefficients with $j-1$ and $j-3$. Replacing j by $j-1$ and $j-3$ in the expression above leads to:

$$\psi_{-2,j-1,3} = (j-1+\lambda)\underbrace{\psi_{-2,j-1,2}}_{=0 \text{ induction}} + (-2+j)\psi_{j,-2,2}, \tag{4.152}$$

$$\psi_{-2,j-3,3} = (j-3+\lambda)\underbrace{\psi_{-2,j-3,2}}_{=0 \text{ induction}} + (-4+j)\underbrace{\psi_{-2+j,-2,2}}_{=0 \text{ induction}}. \tag{4.153}$$

Several coefficients are of the form $\psi_{l,i,2}$ with $i \leq 0$ and $0 < l < j$ and thus zero by induction assumption. Next, we focus on the coefficient $\psi_{5,-3+j,-2}$ of level $k=5$. We first need to derive an expression for coefficients of level $k=4$. We consider the cocycle condition $(i, j, 1, k)$ with $i=-2$ and $k=3$, which yields after dropping coefficients of level plus one and minus one:

$$\begin{aligned}
\psi_{-2,j,4} &= \frac{1}{2}((1+j+\lambda)\psi_{-2,j,3} + (-1+j)\psi_{1+j,-2,3}) \\
\overset{(4.151)}{\Leftrightarrow} \psi_{-2,j,4} &= \frac{(1+j+\lambda)(j+\lambda)}{2}\psi_{-2,j,2} + (1+j+\lambda)(-1+j)\psi_{j+1,-2,2} \\
&\quad - \frac{j(-1+j)}{2}\psi_{j+2,-2,2}.
\end{aligned} \tag{4.154}$$

Next, we can write down the coefficient of level $k=5$. Considering the cocycle condition (4.85) for $(i, j, 1, k)$ with $i=-2$ and $k=4$, we obtain after dropping coefficients of levels plus one and minus one:

$$\psi_{-2,j,5} = \frac{1}{3}((2+j+\lambda)\psi_{-2,j,4} + (-1+j)\psi_{1+j,-2,4}). \tag{4.155}$$

More precisely, from (4.150) we see that we need $\psi_{-2,j-3,5}$ and hence $\psi_{-2,j-3,4}$ and $\psi_{-2+j,-2,4}$. These are obtained from (4.154) by replacing j with $j-3$ and $j-2$ respectively, yielding:

$$\begin{aligned}
\psi_{-2,j-3,4} &= (\dots)\underbrace{\psi_{-2,j-3,2}}_{=0 \text{ induction}} + (\dots)\underbrace{\psi_{j-2,-2,2}}_{=0 \text{ induction}} - (\dots)\underbrace{\psi_{j-1,-2,2}}_{=0 \text{ induction}} = 0, \\
\psi_{-2,j-2,4} &= (\dots)\underbrace{\psi_{-2,j-2,2}}_{=0 \text{ induction}} + (\dots)\underbrace{\psi_{j-1,-2,2}}_{=0 \text{ induction}} - \frac{(j-2)(-3+j)}{2}\psi_{j,-2,2}.
\end{aligned} \tag{4.156}$$

Several coefficients are of the form $\psi_{l,i,2}$ with $i \leq 0$ and $0 < l < j$ and thus zero by induction assumption. Inserting the expressions of the coefficients of level four (4.156) into (4.155) with j replaced by $j-3$, we obtain:

$$\begin{aligned}
\psi_{-2,j-3,5} &= \frac{1}{3}((-1+j+\lambda)\underbrace{\psi_{-2,j-3,4}}_{=0} + (-4+j)\psi_{-2+j,-2,4}) \\
\Leftrightarrow \psi_{-2,j-3,5} &= \frac{(j-2)(j-3)(j-4)}{6}\psi_{j,-2,2}.
\end{aligned} \tag{4.157}$$

Finally, inserting the coefficients (4.152), (4.153) and (4.157) into (4.150), we obtain:

$$(-5+j)(2+j)j\psi_{j,2,-2} = 0.$$

We obtain a trivial relation for $j = 0, -2$ and $j = 5$. Since we consider positive j, only the induction step $j_0 = 5$ leads to trouble. Therefore, we need another relation to force the coefficient

$\psi_{5,2,-2}$ to zero, which is true for all λ except for $\lambda = 7$, in which case the coefficient $\psi_{5,2,-2}$ corresponds to $\psi_{\lambda-2,2,-2} = 0$ zero by assumption. We start by considering the cocycle condition (4.85) for $(i, j, 2, -1)$, which yields, after dropping zero coefficients, a recurrence relation on i for fixed j for $\psi_{i,j,2}$:

$$\begin{aligned} &-(1+i)\psi_{-1+i,j,2} + (2+i+j-\lambda)\psi_{i,j,2} + (1+j)\underbrace{\psi_{-1+j,i,2}}_{=0} = 0 \\ &\Leftrightarrow \psi_{-1+i,j,2} = \frac{(2+i+j-\lambda)}{(1+i)}\psi_{i,j,2}\,. \end{aligned} \tag{4.158}$$

The third term in the first line is zero due to the induction hypothesis. This is true for any induction step j, independent of whether $j = j_0$ or not. We shall need $i = -3, -5$ given by:

$$\begin{aligned} &\text{For } i = -2\text{: } \psi_{-3,j,2} = -(j-\lambda)\psi_{-2,j,2}\,, \\ &\text{For } i = -4\text{: } \psi_{-5,j,2} = \frac{(-2+j-\lambda)(-1+j-\lambda)(j-\lambda)}{-6}\psi_{-2,j,2}\,. \end{aligned}$$

Next, consider the cocycle condition (4.85) for $(-3, j, 2, -2)$, which is, after dropping terms of level zero and level minus one, given by:

$$\begin{aligned} &\psi_{-5,j,2} + (-3+j\lambda)\underline{\psi_{-3,2,-2}} - (-5+j+2\lambda)\underline{\psi_{-3,j,-2}} + (-1+j-2\lambda)\psi_{-3,j,2} \\ &+ (3+j)\underline{\underline{\psi_{-3+j,2,-2}}} + (2+j)\underline{\underline{\psi_{-2+j,-3,2}}} - (j-3\lambda)\psi_{j,2,-2} - (-2+j)\underline{\psi_{2+j,-3,-2}} = 0 \\ &\Leftrightarrow \left(-(2+j-\lambda)\left(j+j^2-2j\lambda+(-7+\lambda)\lambda\right)\right)\psi_{-2,j,2} = 0\,. \end{aligned}$$

The terms underlined once are of the form $\psi_{j,-3,-2}$ $j > 0$ and are thus zero due to previous results. The terms underlined twice are of the form $\psi_{i,l,2}$ with $i \le 0$ and $0 < l < j$ and are thus zero due to the induction hypothesis. The last line is obtained by replacing the remaining terms by the expressions in terms of $\psi_{-2,j,2}$ given above. Next, remember that there is only one induction step causing trouble, namely $j = 5$. Considering $j = 5$ in the expression above, we obtain a new relation involving $\psi_{5,-2,2}$:

$$(7-\lambda)((\lambda-7)\lambda - 10\lambda + 30)\psi_{-2,5,2} = 0\,.$$

The relation is trivial for $\lambda = 2, 7, 15$. The value $\lambda = 7$ is not troublesome, as for this value of λ, $\psi_{-2,5,2} = \psi_{-2,\lambda-2,2}$ is zero by assumption. The other two values are exceptional values of λ which we do not consider anyway. Hence, we obtain $\psi_{-2,j,2} = 0$ for the induction step j even if $j = 5$. Next, we go back to equation (4.158), and starting with $i = -2$, decreasing i, we obtain $\psi_{i,j,2} = 0$ for all $i \le 0$ for the fixed induction step j, even if $j = 5$. Hence our induction hypothesis holds true for the induction step j. By induction, we thus obtain $\psi_{i,j,2} = 0$ for all $i \le 0$ and all $j > 0$.

Finally, we come to generic level k. Let us start with negative k. So we proved $\psi_{i,j,k} = 0$ for all $i \le 0$ and all $j > 0$ and for $k = 0, -1, -2$. Let us assume the result holds true for k ($k \le -2$), and let us check whether the hypothesis remains true for induction step $k-1$. A look at equation (4.131) immediately reveals that this is correct. In fact, at the right-hand side of (4.131), all the coefficients are of level k with $i \le 0$ and $j \ge 0$ and thus, zero. Hence $\psi_{i,j,k-1}$ is also zero as there is no pole for $k \le -2$. By induction, we thus obtain $\psi_{i,j,k} = 0$ for all $i, k \le 0$ and all $j > 0$. We proceed similarly for generic positive k. So we proved $\psi_{i,j,k} = 0$ for all $i \le 0$ and all $j > 0$ and for $k = 0,1,2$. Let us assume the result holds true for k ($k \geq 2$), and let us check whether the hypothesis remains true for induction step $k+1$. A look at equation (4.132) immediately reveals that this is correct. In fact, at the right-hand side of (4.132), all the coefficients are of level k with $i \leq 1$ and $j > 0$ and thus, zero. Hence $\psi_{i,j,k+1}$ is also zero as there is no pole for $k \geq 2$. By induction, we thus obtain $\psi_{i,j,k} = 0$ for all $j, k > 0$ and all $i \leq 0$.

All in all, we obtain the result stated in Lemma 4.3.9. □

Proof of Theorem 4.3.2. Starting with a degree-zero 3-cocycle $\psi \in \mathrm{H}^3(\mathscr{W}, \mathscr{F}^\lambda)$ which has $\lambda \in \{6,8,10,14,16,18,20,22,24,26\}$, the Lemma 4.3.5 allows to fulfill the assumptions of Lemma 4.3.6, which in turn, together with Lemma 4.3.5, allows to fulfill the assumptions of Lemma 4.3.7. Lemma 4.3.7, together with the previous lemmata, then allows to get the assumptions of Lemma 4.3.8. The algorithms proposed in Lemma 4.3.8, together with the previous lemmata, yield the assumptions of 4.3.9, which allows to prove Theorem 4.3.2. □

Remark 4.3.3. In the Algorithms 1 and 2, the most time-consuming part is the process of expressing an arbitrary coefficient $\psi_{i,j,k}$ in terms of the generating coefficients, i.e. the resolution of the recurrence relations. The bigger $|\lambda|$ is, the higher is the level of the coefficients that must be considered, and thus, the more steps are needed to solve the recurrence relations. In Algorithm 2, only one index of the $\psi_{i,j,k}$ involved increases with $|\lambda|$, while in Algorithm 1, all three indices of the coefficients increase with $|\lambda|$. Hence, in Algorithm 2, non-linear recurrence relations in only one variable need to be solved, whereas in Algorithm 1, non-linear recurrence relations in three variables need to be solved, which takes much more time. In order to speed up processing time, one could try to solve the recurrence relations explicitly by hand and derive explicit formulas to express arbitrary coefficients $\psi_{i,j,k}$ in terms of their generating coefficients directly. This was attempted, and the results are given in the Appendix A. However, the approach is empirical, meaning we guessed the solution of the recurrence relations and tested it for some $\psi_{i,j,k}$, but we did not prove that it is indeed the correct solution of the recurrence relations. Thus we did not get genuine proofs.
Apart from the recurrence relations, another possibility to shorten the running times would be to avoid solving the systems of linear equations, which consist of several thousand equations. Instead, considerations about the rank are already sufficient to analyze the triviality of the solutions. The gain in speed would be slight, though. However, the speed could be increased drastically by parallelizing these computations. Numerous algorithms and packages already exist, see for instance Mulmuley [85] or Dekker, Hoffmann and Potma [22] for early examples.

4.4 The Virasoro algebra

In this section, we summarize the results obtained for the Virasoro algebra. They can be deduced immediately from the results for the Witt algebra by using Theorem 3.3.4.

Theorem 4.4.1. *The first algebraic cohomology of the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathscr{F}^\lambda$ is given by:*

$$\mathrm{H}^1(\mathcal{V}, \mathscr{F}^\lambda) = \{0\} \qquad \forall \lambda \in \mathbb{C} \setminus \{0,1,2\},$$

$$\dim \mathrm{H}^1(\mathcal{V}, \mathscr{F}^0) = 2 \quad \textit{and} \quad \dim \mathrm{H}^1(\mathcal{V}, \mathscr{F}^1) = \dim \mathrm{H}^1(\mathcal{V}, \mathscr{F}^2) = 1\,.$$

Proof. Recall from Theorem 3.3.4 that we have $\mathrm{H}^1(\mathcal{V},\mathcal{F}^\lambda) = \mathrm{H}^1(\mathcal{W},\mathcal{F}^\lambda)$. Therefore, Theorem 4.4.1 is obtained immediately from the Theorems 4.1.1, 4.1.2, 4.1.3 and 4.1.4. □

Theorem 4.4.2. *The second algebraic cohomology of the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ is given by:*

$$\mathrm{H}^2(\mathcal{V},\mathcal{F}^\lambda) = \{0\} \qquad \forall \lambda \in \mathbb{C} \setminus \{0,1,2,5,7\},$$

$$\dim\, \mathrm{H}^2(\mathcal{V},\mathcal{F}^1) = \dim\, \mathrm{H}^2(\mathcal{V},\mathcal{F}^2) = 2,$$

$$\dim\, \mathrm{H}^2(\mathcal{V},\mathcal{F}^0) = \dim\, \mathrm{H}^2(\mathcal{V},\mathcal{F}^5) = \dim\, \mathrm{H}^2(\mathcal{V},\mathcal{F}^7) = 1.$$

Proof. Recall from Theorem 3.3.4 that we have for the second algebraic cohomology, $\mathrm{H}^2(\mathcal{V},\mathcal{F}^\lambda) = \dfrac{\mathrm{H}^2(\mathcal{W},\mathcal{F}^\lambda)}{\mathrm{H}^0(\mathcal{W},\mathcal{F}^\lambda)}$. Recall also that $\mathrm{H}^0(\mathcal{W},\mathcal{F}^\lambda)$ is the space of $\mathcal{W}$-invariants, ${}^{\mathcal{W}}\mathcal{F}^\lambda$. At the beginning of this chapter, we mentioned that the trivial module $\mathbb{K}$ is included in $\mathcal{F}^0$, corresponding to f_0^0, as we have $e_i \cdot f_0^0 = 0$. This is the only trivial action for all λ, meaning we have ${}^{\mathcal{W}}\mathcal{F}^\lambda = \{0\}\ \forall \lambda \in \mathbb{C}\setminus\{0\}$ and $\dim\, {}^{\mathcal{W}}\mathcal{F}^0 = 1$. Together with the Theorems 4.2.1, 4.2.2, 4.2.3, 4.2.4, 4.2.5 and 4.2.6, we obtain Theorem 4.4.2. □

Theorem 4.4.3. *The third algebraic cohomology of the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with char($\mathbb{K}$)= 0 and values in $\mathcal{F}^\lambda$ is given by:*

$$\mathrm{H}^3(\mathcal{V},\mathcal{F}^\lambda) = \{0\} \qquad \forall \lambda \in I,$$

where $I = \{-100,\ldots,-1\} \cup \{6,8,10,14,16,18,20,22,24,26\}$.

Proof. Theorem 3.3.4 yields $\mathrm{H}^3(\mathcal{V},\mathcal{F}^\lambda) = \dfrac{\mathrm{H}^3(\mathcal{W},\mathcal{F}^\lambda)}{\mathrm{H}^1(\mathcal{W},\mathcal{F}^\lambda)}$. Thus, Theorems 4.1.1, 4.3.1 and 4.3.2 immediately yield Theorem 4.4.3. □

Chapter 5

The Krichever-Novikov vector field algebra

In this chapter, we derive the zeroth algebraic cohomology Krichever-Novikov vector field algebra with values in general tensor densities modules as well as an upper bound for the dimension of the third algebraic bounded cohomology of the Krichever-Novikov vector field algebra with values in the trivial module $\mathbb{K}$. The methods used to derive the latter result are very close to the ones used to derive the upper bound for the second algebraic bounded cohomology, see [102] and [108]. So far, the author was not able to derive a minimal bound for the dimension of the third bounded cohomology.

5.1 Analysis of $\mathrm{H}^0(\mathscr{K}\mathscr{N}, \mathscr{F}^\lambda)$

In this section, we derive the zeroth algebraic cohomology $\mathrm{H}^0(\mathscr{K}\mathscr{N}, \mathscr{F}^\lambda)$ with values in general tensor-densities modules. The proof was given to the author by Schlichenmaier in a private communication [109].

Theorem 5.1.1. *The zeroth algebraic cohomology of the N-point-Krichever-Novikov vector field algebra $\mathscr{K}\mathscr{N}$ with higher genus over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in $\mathscr{F}^\lambda$ is given by,*

$$\forall \lambda \in \frac{1}{2}\mathbb{Z}: \qquad \dim\ \mathrm{H}^0(\mathscr{K}\mathscr{N}, \mathscr{F}^\lambda) = \begin{cases} 1 & if\, \lambda = 0 \\ 0 & else \end{cases} .$$

Proof. Recall from Section 2.1.5 that elements of $\mathscr{F}^\lambda$ correspond to meromorphic differential forms of weight λ. The action of elements of $\mathscr{K}\mathscr{N}$ on elements of $\mathscr{F}^\lambda$ is given by the Lie derivative,

$$e \cdot f = e\frac{df}{dz} + \lambda f \frac{de}{dz}, \tag{5.1}$$

where we use e to denote an element of $\mathscr{K}\mathscr{N}$ as well as the representing function with respect to the coordinate z, and similarly for f an element of $\mathscr{F}^\lambda$. This action is described in terms of the basis elements of $\mathscr{K}\mathscr{N}$ as,

$$e_{n,p} \cdot f^\lambda_{m,r} = (m + \lambda n)\delta_{p,r} f^\lambda_{n+m,r} + \sum \text{h.d.}, \tag{5.2}$$

where h.d. stands for higher degree terms, i.e. elements $f^\lambda_{i,r}$ with $R + n + m > i > n + m$, see Section 2.1.5.

Let us start with $\lambda = 0$. The Lie derivative action (5.1) on 1 for $\lambda = 0$ yields,

$$e \cdot 1 = e\frac{d1}{dz} = 0 \qquad \forall\ e \in \mathscr{K}\mathscr{N},$$

thus 1 is an invariant of $\mathscr{F}^0$ under $\mathscr{K}\mathscr{N}$. The vector space generated by 1 over $\mathbb{K}$ thus gives constant functions $f(z) = c$, $c \in \mathbb{K}$, satisfying

$$e \cdot f = e\frac{df}{dz} = 0 \qquad \forall\ e \in \mathscr{K}\mathscr{N}. \tag{5.3}$$

Conversely, Equation (5.3) is only true if f is constant. Hence, we obtain that the space of $\mathscr{K}\mathscr{N}$-invariants of $\mathscr{F}^0$ is one-dimensional, dim ${}^{\mathscr{K}\mathscr{N}}\mathscr{F}^0 = 1$. Consequently, we obtain the result for $\lambda = 0$, as we have $\mathrm{H}^0(\mathscr{K}\mathscr{N}, \mathscr{F}^0) = {}^{\mathscr{K}\mathscr{N}}\mathscr{F}^0$, see Section 2.2.4.

Next, we consider $\lambda \neq 0$. We want to show that there are no $\mathscr{K}\mathscr{N}$-invariants of $\mathscr{F}^\lambda$ for $\lambda \neq 0$. We proceed by argument to absurdity and suppose that there is a non-trivial element $\psi \in {}^{\mathscr{K}\mathscr{N}}\mathscr{F}^\lambda$ for $\lambda \neq 0$, i.e. there is $\psi \neq 0$ such that $\mathscr{K}\mathscr{N} \cdot \psi = 0$. We can write ψ as a combination of elements of $\mathscr{F}^\lambda$,

$$\psi = \sum_{n=n_0}^{n_0+R} \sum_p \alpha_{n,p} f^\lambda_{n,p},$$

with at least one coefficient $\alpha_{n_0,p} \neq 0$ as we assume $\psi \neq 0$. Applying a general basis element $e_{m,r}$ on ψ, we obtain from (5.2),

$$e_{m,r} \cdot \psi = \sum_{p,r} \alpha_{n_0,p} \delta_{r,p} (n_0 + \lambda m) f^\lambda_{n_0+m,p} + \sum_{n,r,p} \text{h.d.}. \tag{5.4}$$

In particular, we obtain,

$$\left(\sum_r e_{0,r}\right) \cdot \psi = \sum_{r,p} \alpha_{n_0,p} \delta_{r,p} n_0 f^\lambda_{n_0,p} + \sum \text{h.d.} = \sum_p \alpha_{n_0,p} n_0 f^\lambda_{n_0,p} + \sum \text{h.d.}.$$

As we assume that ψ is a $\mathscr{K}\mathscr{N}$-invariant, we have $\left(\sum_r e_{0,r}\right) \cdot \psi = 0$. By the previous formula, this implies $\alpha_{n_0,p} n_0 = 0\ \forall p$, since the $f^\lambda_{n_0,p}$ are a basis of degree n_0-elements. Since there is by assumption at least one p such that $\alpha_{n_0,p} \neq 0$, we obtain $n_0 = 0$. Next, considering $\left(\sum_r e_{1,r}\right) \cdot \psi = 0$, we similarly obtain from (5.4),

$$\sum_p \alpha_{n_0,p} (n_0 + \lambda) f^\lambda_{n_0+1,p} + \sum \text{h.d.} = 0 \Rightarrow \alpha_{n_0,p} (\cancel{n_0} + \lambda) = 0\ \forall p \Rightarrow \alpha_{n_0,p} \lambda = 0\ \forall p.$$

As we consider $\lambda \neq 0$, we obtain $\alpha_{n_0,p} = 0\ \forall p$, which is in contradiction with the assumption that at least one coefficient $\alpha_{n_0,p}$ is non-zero. Therefore, for $\lambda \neq 0$, there are no $\mathscr{K}\mathscr{N}$-invariants of $\mathscr{F}^\lambda$. □

5.2 Analysis of $\mathrm{H}^3_b(\mathscr{K}\mathscr{N}, \mathbb{K})$

In this section, we will work with bounded cohomology, and more precisely, with cohomology bounded from above. The definitions of boundedness, as well as the definition of local cohomology, which we provide for reasons of completeness, can be found in Definition 5.2.1 below.

Definition 5.2.1. Consider the complex $\mathrm{H}^q(\mathcal{KN},\mathbb{K})$ of the Krichever-Novikov vector field algebra over a field $\mathbb{K}$ with $char(\mathbb{K})=0$ with values in the trivial module $\mathbb{K}$. Let $\psi \in \mathrm{H}^q(\mathcal{KN},\mathbb{K})$.

- The cocycle ψ is *bounded from above* if and only if there exists $L \in \mathbb{Z}$ such that we have $\psi(e_{i_1,p_1},\dots,e_{i_q,p_q})=0$ for all $i_1,\dots,i_q$ with $i_1+\cdots+i_q \geq L$.
- The cocycle ψ is *bounded from below* if and only if there exists $L \in \mathbb{Z}$ such that we have $\psi(e_{i_1,p_1},\dots,e_{i_q,p_q})=0$ for all $i_1,\dots,i_q$ with $i_1+\cdots+i_q \leq L$.
- A *local* cocycle is a cocycle that is bounded from below and from above.
- A *bounded from above* cohomology class is a class $[\psi]$ of $\mathrm{H}^q(\mathcal{KN},\mathbb{K})$ which admits at least one cocycle ψ which is bounded from above.
- A *bounded from below* cohomology class is a class $[\psi]$ of $\mathrm{H}^q(\mathcal{KN},\mathbb{K})$ which admits at least one cocycle ψ which is bounded from below.
- A *local* cohomology class is a class $[\psi]$ of $\mathrm{H}^q(\mathcal{KN},\mathbb{K})$ which admits in each cocycle class $[\psi]$ at least one cocycle ψ which is local.

In this thesis, we consider the third cohomology bounded from above, which we simply refer to as bounded cohomology, and denote by $\mathrm{H}^q_b(\mathcal{KN},\mathbb{K})$. This space is the subspace formed by all cohomology classes that are bounded from above.
The methods used to derive an upper bound of $\mathrm{H}^3_b(\mathcal{KN},\mathbb{K})$ are similar to those used in the previous chapters. We do not need to work with particular realizations of the basis elements $e_{n,p}$. An important difference to the previous analyses consists in the fact that the Theorem 2.2.1 is not valid for the Krichever-Novikov vector field algebra, since Krichever-Novikov algebras are not graded Lie algebras, but only almost graded. Therefore, we need to consider cochains and cocycles of arbitrary degree, and not only those of degree zero.
The condition for a 3-cochain ψ to be a 3-cocycle with values in the trivial module (3.1) evaluated on basis elements $(e_{i,r},e_{j,s},e_{k,t},e_{l,u})$ is given by:

$$\begin{aligned}&(\delta_3\psi)(e_{i,r},e_{j,s},e_{k,t},e_{l,u})=0\\&\Leftrightarrow \psi([e_{i,r},e_{j,s}],e_{k,t},e_{l,u})-\psi([e_{i,r},e_{k,t}],e_{j,s},e_{l,u})+\psi([e_{i,r},e_{l,u}],e_{j,s},e_{k,t})\\&+\psi([e_{j,s},e_{k,t}],e_{i,r},e_{l,u})-\psi([e_{j,s},e_{l,u}],e_{i,r},e_{k,t})+\psi([e_{k,t},e_{l,u}],e_{i,r},e_{j,s})=0\,,\end{aligned}\tag{5.5}$$

where $e_{i,r},e_{j,s},e_{k,t},e_{l,u} \in \mathcal{KN}$. We will refer to the sum $i+j+k+l$ of the indices of the basis elements the cocycle condition is evaluated on as *level.* For the convenience of the reader, we reproduce the Lie structure equation for the Krichever-Novikov vector field algebra (2.23) here:

$$[e_{k,r},e_{n,s}]=\delta_{r,s}(n-k)e_{k+n,r}+\sum_{h=n+k+1}^{n+k+R}\sum_t c^{(h,t)}_{(k,r),(n,s)}e_{h,t}\,,$$

with $c^{(h,t)}_{(k,r),(n,s)} \in \mathbb{K}$ and R a constant, the second sum goes from $t=1$ to $t=K$, where K is the number of in-points, see Section 2.1.4. Inserting this expression into the cocycle condition 5.5, we obtain:

$$\begin{aligned}&(\delta_3\psi)(e_{i,r},e_{j,s},e_{k,t},e_{l,u})=0\\&\Leftrightarrow \delta_{r,s}(j-i)\psi(e_{i+j,r},e_{k,t},e_{l,u})-\delta_{r,t}(k-i)\psi(e_{i+k,r},e_{j,s},e_{l,u})\\&+\delta_{r,u}(l-i)\psi(e_{i+l,r},e_{j,s},e_{k,t})+\delta_{s,t}(k-j)\psi(e_{j+k,s},e_{i,r},e_{l,u})\\&-\delta_{s,u}(l-j)\psi(e_{j+l,s},e_{i,r},e_{k,t})+\delta_{t,u}(l-k)\psi(e_{k+l,t},e_{i,r},e_{j,s})+\text{h.d.}=0\,,\end{aligned}\tag{5.6}$$

where h.d. denotes terms of degree higher than the level given by $i+j+k+l$, i.e. terms of the form $\psi(e_{m,q}, e_{n,v}, e_{p,w})$ with $m+n+p > i+j+k+l$. The condition for ψ to be a coboundary is given by:

$$\begin{aligned} \psi(e_{i,r}, e_{j,s}, e_{k,t}) &= (\delta_2\phi)(e_{i,r}, e_{j,s}, e_{k,t}) \\ &= \phi([e_{i,r}, e_{j,s}], e_{k,t}) - \phi([e_{i,r}, e_{k,t}], e_{j,s}) + \phi([e_{j,s}, e_{k,t}], e_{i,r}), \end{aligned} \tag{5.7}$$

where ϕ is a 2-cochain with values in the trivial module.
In order to derive an upper bound for the dimension of $\mathrm{H}^3_b(\mathcal{KN}, \mathbb{K})$, we will need the result given in Lemma 5.2.1 below.

Lemma 5.2.1. *Every bounded 3-cocycle ψ of $\mathrm{H}^3_b(\mathcal{KN}, \mathbb{K})$ is cohomologous to a bounded 3-cocycle ψ' fulfilling:*

$$\psi'(e_{i,r}, e_{0,r}, e_{k,s}) = 0 \quad \forall\ i \in \mathbb{Z}, \forall\ k \neq 0 \ \textit{and}\ r < s, \tag{5.8}$$
$$\psi'(e_{i,r}, e_{0,r}, e_{k,s}) = 0 \ \textit{or h.d.} \quad \forall\ i \in \mathbb{Z}, \forall\ k \neq 0 \ \textit{and}\ r > s, \tag{5.9}$$
$$\psi'(e_{i,r}, e_{0,r}, e_{0,s}) = 0 \quad \forall\ i \in \mathbb{Z}, \tag{5.10}$$
$$\psi'(e_{-1,r}, e_{1,r}, e_{0,s}) = 0 \quad r < s, \tag{5.11}$$
$$\psi'(e_{-1,r}, e_{1,r}, e_{0,s}) = h.d. \quad r > s, \tag{5.12}$$
$$\psi'(e_{i,r}, e_{j,r}, e_{0,r}) = 0 \quad \forall\ i+j \neq 0, \tag{5.13}$$
$$\psi'(e_{i,r}, e_{j,r}, e_{1,r}) = 0 \quad \forall\ i+j+1 = 0 \ \textit{except possibly for}\ \psi'(e_{-1,r}, e_{0,r}, e_{1,r}). \tag{5.14}$$

Proof. Let $\psi \in \mathrm{H}^3_b(\mathcal{KN}, \mathbb{K})$ be a bounded cocycle. We need to consistently define a 2-cochain ϕ in such a way that the cohomological change $\psi' = \psi - \delta\phi$ yields a 3-cocycle ψ' fulfilling the conditions (5.8)-(5.14) given in the statement. As we consider bounded cohomology, we can assume that ψ vanishes from level L upwards, i.e. $\psi(e_{i,r}, e_{j,s}, e_{k,t}) = 0\ \forall\ i+j+k \geq L$. Accordingly, we define $\phi(e_{i,r}, e_{j,s}) = 0\ \forall\ i+j \geq L$. This implies that ψ' will also be bounded, even at the same level as ψ, i.e. $\psi'(e_{i,r}, e_{j,s}, e_{k,t}) = 0\ \forall\ i+j+k \geq L$. We continue to define ϕ consistently by decreasing induction on the level, and by juggling between different coboundary conditions. Note that it is not possible to first consider all levels for $r \neq s$ in $\phi(e_{i,r}, e_{j,s})$ and then all levels for $r = s$, because in the coboundary condition, the higher degree terms of ϕ mix both cases $r \neq s$ and $r = s$. Instead, one has to proceed level by level and consider at each level both cases $r \neq s$ and $r = s$.
For $L = 0$ or $L < 0$, one can jump immediately to the case of level zero or of level $\ell \leq -1$, respectively, treated towards the end of this proof. We assume $L > 0$ in the following, which is the more interesting case.
Level $L-1$: Consider the coboundary condition (5.7) on the basis elements $(e_{i,r}, e_{0,r}, e_{k,s})$ with $r < s$ and $i, k \neq 0$, from which we can get an expression for $\phi(e_{i,r}, e_{k,s})$ with $r < s$ and $i, k \neq 0$:

$$\phi(e_{i,r}, e_{k,s}) := -\frac{1}{i}\psi(e_{i,r}, e_{0,r}, e_{k,s}), \quad i+k = L-1. \tag{5.15}$$

Note that $\psi'(e_{i,r}, e_{0,r}, e_{k,s})$ for $i = 0$ is trivially zero due to the alternating property. Another possible definition for $\phi(e_{i,r}, e_{k,s})$ with $r < s$ and $i, k \neq 0$ could be given by the coboundary condition (5.7) on the basis elements $(e_{k,s}, e_{0,s}, e_{i,r})$ with $r < s$ and $i, k \neq 0$:

$$\phi(e_{i,r}, e_{k,s}) := -\frac{1}{k}\psi(e_{i,r}, e_{0,s}, e_{k,s}), \quad i+k = L-1. \tag{5.16}$$

In other words, we can choose whether we want to have an $e_{0,r}$ or an $e_{0,s}$ in the ψ's. If we compare the two possible definitions (5.15) and (5.16), we see that they are not equivalent as we do not necessarily have $-\frac{1}{i}\psi(e_{i,r},e_{0,r},e_{k,s}) = -\frac{1}{k}\psi(e_{i,r},e_{0,s},e_{k,s})$. Hence, it is only possible to put one set of ψ''s equal to zero, either $\psi'(e_{i,r},e_{0,r},e_{k,s})$ with $r<s$ or $\psi'(e_{i,r},e_{0,r},e_{k,s})$ with $r>s$. We will choose the definition (5.15) and obtain the statement (5.8) for level $L-1$ after the cohomological change. For example, in the case $N=3$, with definition (5.15) we could put to zero the entries $\psi'(e_{k,1},e_{0,1},e_{i,2})$, $\psi'(e_{k,1},e_{0,1},e_{i,3})$ and $\psi'(e_{k,2},e_{0,2},e_{i,3})$ for $i\neq 0$, but not the entries $\psi'(e_{k,2},e_{0,2},e_{i,1})$, $\psi'(e_{k,3},e_{0,3},e_{i,1})$ nor $\psi'(e_{k,3},e_{0,3},e_{i,2})$. However, let us see whether it is possible to find some relation between $\psi'(e_{i,r},e_{0,r},e_{k,s})$ and $\psi'(e_{i,s},e_{0,s},e_{k,r})$, with $r<s$. Consider the cocycle condition (5.6) on ψ' and the basis elements $(e_{i,r},e_{0,r},e_{k,s},e_{0,s})$ with $i,k\neq 0$ and $r<s$, which yields:

$$i\psi'(e_{i,r},e_{0,s},e_{k,s}) - k\psi'(e_{i,r},e_{0,r},e_{k,s}) = \text{h.d.}\,, \quad i+k=L-1\,, \tag{5.17}$$

where h.d. denote higher degree terms of ψ'. As we consider level $L-1$, i.e. $i+k=L-1$, we have h.d. $=0$ as these are at least of level L. Recall that due to our definition of ϕ, ψ' and ψ vanish at the same level. As we already have after the cohomological change $\psi'(e_{i,r},e_{0,r},e_{k,s})=0$ for $r<s$ and $k\neq 0$, Equation (5.17) gives $\psi'(e_{i,r},e_{0,s},e_{k,s})=0$ for $r<s$ and $i\neq 0$. This means that even with definition (5.15), we can annihilate the entries $\psi'(e_{i,r},e_{0,s},e_{k,s})$ with $r<s$ and $i\neq 0$. Therefore we obtain the statements (5.8) and (5.9) for level $L-1$.
The next step to consider is whether we can find a definition for $\phi(e_{i,r},e_{k,s})$ when $i=0$ or $k=0$. Consider the definition (5.15) with $i\neq 0$ and $k=0$, and $r\neq s$:

$$\phi(e_{i,r},e_{0,s}) := -\frac{1}{i}\psi(e_{i,r},e_{0,r},e_{0,s})\,, \quad i=L-1\,. \tag{5.18}$$

As we consider $i\neq 0$, the $\phi(e_{i,r},e_{0,s})$ with $r<s$ are not related to the $\phi(e_{i,r},e_{0,s})$ with $r>s$ in any way, i.e. they constitute independent coefficients. Therefore, we can consider both $r<s$ and $r>s$ in the definition (5.18) above and annihilate $\psi'(e_{i,r},e_{0,r},e_{0,s})$ both for $r<s$ and $r>s$ immediately. In the case $r=s$, $\psi'(e_{i,r},e_{0,r},e_{0,s})$ is trivially zero due to the alternating property. Hence, we obtain the statement (5.10) for level $L-1$. The statements (5.11) and (5.12) concern level zero and will be obtained with $\phi(e_{0,r},e_{0,s})$. However, for now we are considering level $L-1$, hence we will skip these two statements for now and come back to them later.
Let us see what happens if $r=s$ and continue with statement (5.13). First, we define $\phi(e_{i,r},e_{0,r}) := 0$ for $i=L-1$. Next, using the coboundary condition (5.7) on the basis elements $(e_{i,r},e_{j,r},e_{0,r})$, we can define $\phi(e_{i,r},e_{j,r})$ as follows:

$$\phi(e_{i,r},e_{j,r}) = \frac{\psi(e_{i,r},e_{j,r},e_{0,r})}{i+j}\,, \quad i+j=L-1\neq 0\,. \tag{5.19}$$

Hence we obtain $\psi'(e_{i,r},e_{j,r},e_{0,r})=0$ for $i+j=L-1\neq 0$, which corresponds to statement (5.13) for level $L-1$. Note that if $L-1=0$ in the Definition (5.19) above, it means that $\psi(e_{i,r},e_{j,p},e_{k,q}) = 0\ \forall\ i+j+k\geq 1$, and one can jump immediately to the step of level zero $\ell=0$ treated further below in this proof. The last statement (5.14) is again a statement for level zero, which we will consider later. We now continue with level $\ell=L-2$.
Level $\ell=L-2$: The definitions we use and the reasoning will be the same as for level $L-1$, except that terms of higher degrees of ϕ and ψ will appear. However, since all ϕ of degree higher than ℓ have already been defined, everything will be consistent. Still, we give the definitions explicitly. First, consider the coboundary condition (5.7) on the basis elements $(e_{i,r},e_{0,r},e_{k,s})$ with $r<s$ and $i,k\neq 0$, from which we can get an expression for $\phi(e_{i,r},e_{k,s})$ with $r<s$ and

$i, k \neq 0$:

$$\begin{aligned}\phi(e_{i,r}, e_{k,s}) &:= -\frac{1}{i}\psi(e_{i,r}, e_{0,r}, e_{k,s}) + \frac{1}{i}\phi\left(\sum_{h=i+1}^{i+R}\sum_{t} c_{(i,r),(0,r)}^{h,t} e_{h,t}, e_{k,s}\right) \quad i, k \neq 0, i+k=\ell \\ &- \frac{1}{i}\phi\left(\sum_{h=i+k+1}^{i+k+R}\sum_{t} c_{(i,r),(k,s)}^{h,t} e_{h,t}, e_{0,r}\right) + \frac{1}{i}\phi\left(\sum_{h=k+1}^{k+R}\sum_{t} c_{(0,r),(k,s)}^{h,t} e_{h,t}, e_{i,r}\right) \quad r < s.\end{aligned} \tag{5.20}$$

We see in the Definition (5.20) that in the higher degree terms, we have entries $\phi(e_{i,r}, e_{k,s})$ of the form $\phi(e_{i,r}, e_{0,s})$, $\phi(e_{i,r}, e_{k,r})$ and others appearing, that are not included in the Definition (5.20) for degree $i + k = \ell$. However, all of these entries have already been defined at the previous levels, hence the Definition (5.20) is consistent. This shows why one has to consider at each level all cases (e.g. $r \neq s$, $r = s$) separately, instead of defining all levels at once for each case. In the Definition (5.20), we consider $r < s$, hence we obtain $\psi'(e_{i,r}, e_{0,r}, e_{k,s}) = 0$ for $r < s$ and $k \neq 0$. As for the previous level, the entries of the type $\psi'(e_{i,r}, e_{0,r}, e_{k,s})$ with $r > s$ and $k \neq 0$ are not necessarily annihilated. An additional investigation is of order. The Equation (5.17) holds for any level $i + k = \ell$. However, contrary to the previous level $L-1$, the higher degree terms h.d. appearing on the right-hand side are now at least of level $L-1$ and not necessarily zero. Inserting $\psi'(e_{i,r}, e_{0,r}, e_{k,s}) = 0$ for $r < s$ and $k \neq 0$ into Equation (5.17), we obtain $\psi'(e_{k,s}, e_{0,s}, e_{i,r}) =$ h.d. for $r < s$ and $i \neq 0$. Hence, we obtain no longer zero but higher degree terms in ψ', which is nonetheless sufficient for our purpose. We thus obtain the statements (5.8) and (5.9) for level ℓ. Next, the coboundary condition (5.7) on the basis elements $(e_{i,r}, e_{0,r}, e_{0,s})$ with $r \neq s$ and $i \neq 0$ gives an expression for $\phi(e_{i,r}, e_{0,s})$ with $r \neq s$ and $i \neq 0$:

$$\begin{aligned}\phi(e_{i,r}, e_{0,s}) &:= -\frac{1}{i}\psi(e_{i,r}, e_{0,r}, e_{0,s}) + \frac{1}{i}\phi\left(\sum_{h=i+1}^{i+R}\sum_{t} c_{(i,r),(0,r)}^{h,t} e_{h,t}, e_{0,s}\right) \quad i = \ell \neq 0 \\ &- \frac{1}{i}\phi\left(\sum_{h=i+1}^{i+R}\sum_{t} c_{(i,r),(0,s)}^{h,t} e_{h,t}, e_{i,r}\right) + \frac{1}{i}\phi\left(\sum_{h=1}^{R}\sum_{t} c_{(0,r),(0,s)}^{h,t} e_{h,t}, e_{i,r}\right) \quad r \neq s.\end{aligned} \tag{5.21}$$

The higher degree terms in ϕ have been defined previously for all cases. Moreover, just as for the previous level, the $\phi(e_{i,r}, e_{0,s})$ with $r < s$ and $r > s$ are not related for $i \neq 0$. Thus, we immediately obtain $\psi'(e_{i,r}, e_{0,r}, e_{0,s}) = 0$ both for $r < s$ and $r > s$, yielding the statement (5.10) for level ℓ, the case $r = s$ being trivially true. The statements (5.11) and (5.12) correspond to level zero, which we analyze later. We continue with the case $r = s$.
We define $\phi(e_{i,r}, e_{0,r}) := 0$ for $i = \ell$. Moreover, consider the coboundary condition (5.7) on the basis elements $(e_{i,r}, e_{j,r}, e_{0,r})$. The term $\phi([e_{i,r}, e_{j,r}], e_{0,r})$ with $i + j = \ell$ is zero, as it yields terms of the form $\phi(e_{n,r}, e_{0,r})$ with $n \geq \ell$, which are zero by definition. We define $\phi(e_{i,r}, e_{j,r})$ as follows:

$$\begin{aligned}\phi(e_{i,r}, e_{j,r}) = \frac{\psi(e_{i,r}, e_{j,r}, e_{0,r})}{i+j} &- \frac{i}{i+j}\phi\left(\sum_{h=i+1}^{i+R}\sum_{t} c_{(i,r),(0,r)}^{h,t} e_{h,t}, e_{j,r}\right) \\ &+ \frac{j}{i+j}\phi\left(\sum_{h=j+1}^{j+R}\sum_{t} c_{(j,r),(0,r)}^{h,t} e_{h,t}, e_{i,r}\right), \quad i + j = \ell \neq 0.\end{aligned} \tag{5.22}$$

Hence we obtain $\psi'(e_{i,r}, e_{j,r}, e_{0,r}) = 0$ for $i + j = \ell \neq 0$, which corresponds to statement (5.13) for level ℓ. The last statement (5.14) is again a statement for level zero, which we will consider later.
Level $1 \leq \ell \leq L-2$: The reasoning and the definitions used are exactly the same as for level $\ell = L-2$. Thus, we obtain immediately the statements (5.8)-(5.10) and (5.13) for these levels. Next, let us consider level zero.

Level $\ell = 0$: The definition (5.20) is valid also for level zero, $i+k=0$. Therefore, we obtain the statement (5.8) for level zero. Besides, Equation (5.17) is also valid for $i+k=0$. Equation (5.17) together with the statement (5.8) yield the statement (5.9) for level zero. The third statement (5.10) is trivial for level zero $i=0$ due to the alternating property. Let us consider the fourth statement (5.11). The aim is to find a definition for $\phi(e_{0,r},e_{0,s})$, which can be deduced from the coboundary condition (5.7) evaluated on the basis elements $(e_{-1,r},e_{1,r},e_{0,s})$ with $r<s$:

$$\begin{aligned}\phi(e_{0,r},e_{0,s}) &= \frac{1}{2}\psi(e_{-1,r},e_{1,r},e_{0,s}) - \frac{1}{2}\phi\left(\sum_{h=1}^{R}\sum_t c^{h,t}_{(-1,r),(1,r)}e_{h,t},e_{0,s}\right)\\ &+\frac{1}{2}\phi\left(\sum_{h=0}^{-1+R}\sum_t c^{h,t}_{(-1,r),(0,s)}e_{h,t},e_{1,r}\right)-\frac{1}{2}\phi\left(\sum_{h=2}^{1+R}\sum_t c^{h,t}_{(1,r),(0,s)}e_{h,t},e_{-1,r}\right).\end{aligned}\tag{5.23}$$

The higher terms are all of level $\ell\geq 1$ and thus already defined for all cases. We have again a subtlety concerning antisymmetry. The quantities $\phi(e_{0,r},e_{0,s})$ and $\phi(e_{0,s},e_{0,r})$ are related by antisymmetry, but not necessarily so $\psi(e_{-1,r},e_{1,r},e_{0,s})$ and $\psi(e_{-1,s},e_{1,s},e_{0,r})$. Hence, only one set of ψ''s can a priori be annihilated. We choose the Definition (5.23) for $r<s$, thus we obtain the statement (5.11), $\psi'(e_{-1,r},e_{1,r},e_{0,s})=0$ for $r<s$. The Equation (5.17) is in this case of no use, as the zero-mode in (5.17) is sitting on the puncture-index which is double, i.e. $\psi'(e_{\cdot,r},e_{0,r},e_{\cdot,s})$, while we need the zero-mode to sit on the single puncture-index, $\psi'(e_{\cdot,r},e_{\cdot,r},e_{0,s})$. Instead of using (5.17), let us have a look at the cocycle condition (5.6) for ψ' evaluated on $(e_{-1,r},e_{1,r},e_{-1,s},e_{1,s})$ with $r<s$, which yields:

$$2\psi'(e_{0,r},e_{-1,s},e_{1,s})+2\psi'(e_{0,s},e_{-1,r},e_{1,r})=\text{h.d.},\tag{5.24}$$

where h.d. denotes terms of higher degree, i.e. terms of the form $\psi'(e_{m,u},e_{n,v},e_{p,w})$ with $m+n+p\geq 1$. As we already obtained the statement (5.11), the Equation (5.24) immediately yields the statement (5.12), $\psi'(e_{-1,s},e_{1,s},e_{0,r})=\text{h.d.}$ for $r<s$. The statement (5.13) does not concern level zero, hence we continue with statement (5.14). In the case of $r=s$, we need to define ϕ at level zero. First, we define $\phi(e_{1,r},e_{-1,r}):=0$ and $\phi(e_{2,r},e_{-2,r}):=0$. Moreover, the coboundary condition (5.7) evaluated on $(e_{i,r},e_{-1-i,r},e_{1,r})$ suggests to take the following definition for ϕ at level zero:

$$\begin{aligned}\phi(e_{i+1,r},e_{-i-1,r}) &:= -\frac{2+i}{1-i}\phi(e_{i,r},e_{-i,r})-\frac{\psi'(e_{i,r},e_{-1-i,r},e_{1,r})}{1-i}\\ &-\frac{1}{1-i}\phi\left(\sum_{h=0}^{-1+R}\sum_t c^{h,t}_{(i,r),(-1-i,r)}e_{h,t},e_{1,r}\right)+\frac{1}{1-i}\phi\left(\sum_{h=i+2}^{i+1+R}\sum_t c^{h,t}_{(i,r),(1,r)}e_{h,t},e_{-1-i,r}\right)\\ &-\frac{1}{1-i}\phi\left(\sum_{h=-i+1}^{-i+R}\sum_t c^{h,t}_{(-1-i,r),(1,r)}e_{h,t},e_{i,r}\right).\end{aligned}\tag{5.25}$$

The higher degree terms have already been defined before for all cases. Starting with $i=2$ in (5.25), using increasing induction on i, we obtain a consistent definition for ϕ at level zero and also the statement (5.14).

Level $\ell\leq -1$: The reasoning and the definitions used are exactly the same as for level $\ell = L-2$. Thus, we obtain immediately the statements (5.8)-(5.10) and (5.13) for these levels.

This concludes the proof of Lemma 5.2.1. □

Theorem 5.2.1. *The third algebraic bounded (from above) cohomology of the N-point-Krichever-Novikov vector field algebra $\mathcal{KN}$ with higher genus over a field $\mathbb{K}$ with $char(\mathbb{K})=0$ and values* *in the trivial module is at most K-dimensional,*

$$\dim \mathrm{H}^3_b(\mathcal{KN},\mathbb{K}) \leq K\,,$$

where K is the number of in-points.

Proof. Let $\tilde{\psi} \in \mathrm{H}^3_b(\mathcal{KN},\mathbb{K})$. The proof is separated into three cases, depending on whether we have two identical second indices in the cocycle condition (5.6), three identical ones or four identical ones. The aim is to express $\psi(e_{i,r}, e_{j,s}, e_{k,t})$ of a given level $i+j+k$ in terms of higher degree terms $\psi(e_{m,q}, e_{n,v}, e_{p,w})$ with $m+n+p > i+j+k$. At the end, we conclude by using boundedness and decreasing induction on the level. The level zero of the third case with four identical puncture-indices corresponds to the Witt algebra. To analyze this level, we will use the analysis of the upper bound of dim $\mathrm{H}^3(\mathcal{W},\mathbb{K})$, given in the proof of Proposition 3.1.1, without reproducing it here.
Starting with a 3-cocycle $\tilde{\psi}$ bounded at level L, we perform a cohomological change as described in Lemma 5.2.1 to obtain a 3-cocycle ψ bounded at level L and satisfying the conditions (5.8)-(5.14).
Case 1: $t = u$: Consider the cocycle condition (5.6) on the basis elements $(e_{i,r}, e_{j,s}, e_{k,t}, e_{l,t})$, with $s \neq t$, $r \neq t$ and $r \neq s$. We obtain:

$$(l-k)\psi(e_{k+l,t}, e_{i,r}, e_{j,s}) = \text{h.d.}.$$

Putting $l = (m+1)/2$ and $k = (m-1)/2$ if m is odd and $l = (m+2)/2$ and $k = (m-2)/2$ if m is even, we obtain,

$$\psi(e_{m,t}, e_{i,r}, e_{j,s}) = \text{h.d.} \qquad \forall\ m,i,j \in \mathbb{Z} \quad \text{and} \quad \forall\ t \neq r \neq s \neq t \in \{1,\ldots,K\}. \tag{5.26}$$

Case 2: $r = s$ and $t = u$: The cocycle condition (5.6) on the basis elements $(e_{i,r}, e_{j,r}, e_{k,s}, e_{l,s})$ with $r \neq s$ yields:

$$(j-i)\psi(e_{i+j,r}, e_{k,s}, e_{l,s}) + (l-k)\psi(e_{k+l,s}, e_{i,r}, e_{j,r}) = \text{h.d.}, \tag{5.27}$$

where h.d. denotes higher degree terms of level $\ell > i+j+k+l$. Taking $l = 0$, we obtain:

$$(j-i)\psi(e_{i+j,r}, e_{k,s}, e_{0,s}) - k\psi(e_{k,s}, e_{i,r}, e_{j,r}) = \text{h.d.}. \tag{5.28}$$

Because of (5.8)-(5.10), the term $\psi(e_{i+j,r}, e_{k,s}, e_{0,s})$ is either zero or a higher degree term. Hence, for $k \neq 0$, we obtain:

$$\psi(e_{k,s}, e_{i,r}, e_{j,r}) = \text{h.d.}, \quad k \neq 0. \tag{5.29}$$

Next, we choose $j = -i$ in Equation (5.27), and we consider $k + l \neq 0$:

$$-2i\psi(e_{0,r}, e_{k,s}, e_{l,s}) + (l-k)\psi(e_{k+l,s}, e_{i,r}, e_{-i,r}) = \text{h.d.}. \tag{5.30}$$

Due to the result (5.29) and since $k + l \neq 0$, the term $\psi(e_{k+l,s}, e_{i,r}, e_{-i,r})$ is a higher degree term. Taking $i \neq 0$ yields:

$$\psi(e_{0,r}, e_{k,s}, e_{l,s}) = \text{h.d.}, \quad k+l \neq 0. \tag{5.31}$$

Finally, we choose $i + j = 0$ and $k = 1$, $l = -1$ in Equation (5.27), yielding:

$$-2i\psi(e_{0,r}, e_{1,s}, e_{-1,s}) - 2\psi(e_{0,s}, e_{i,r}, e_{-i,r}) = \text{h.d.}. \tag{5.32}$$

Due to the conditions (5.11) and (5.12), the term $\psi(e_{0,r}, e_{1,s}, e_{-1,s})$ is either zero or a higher degree term. Hence, we obtain:

$$\psi(e_{0,s}, e_{i,r}, e_{-i,r}) = \text{h.d.}. \tag{5.33}$$

The three results (5.29), (5.31) and (5.33) together yield that,

$$\psi(e_{i,s}, e_{j,r}, e_{k,r}) = \text{h.d.} \qquad \forall\ i,j,k \in \mathbb{Z} \quad \text{and} \quad \forall\ r \neq s \in \{1,\dots,K\}. \tag{5.34}$$

Case 3: $r = s = t = u$: As all puncture-indices are equal, we will drop the second index to increase readability. Consider the cocycle condition (5.6) on the basis elements (e_i, e_j, e_k, e_0), yielding:

$$(j-i)\psi(e_{i+j}, e_k, e_0) - (k-i)\psi(e_{i+k}, e_j, e_0) - (i+j+k)\psi(e_i, e_j, e_k) + (k-j)\psi(e_{j+k}, e_i, e_0) = \text{h.d.}.$$

Considering the restriction $i + j + k \neq 0$, the terms of the form $\psi(e_{i+j}, e_k, e_0)$ are zero because of condition (5.13), and the equation above reduces to:

$$(i+j+k)\psi(e_i, e_j, e_k) = \text{h.d.} \Leftrightarrow \psi(e_i, e_j, e_k) = \text{h.d.} \qquad i+j+k \neq 0. \tag{5.35}$$

This means that all entries $\psi(e_i, e_j, e_k)$ can be expressed in terms of higher degree entries, except for level zero.

Conclusion:

Let us now conclude by using decreasing induction on the level. Taking $i + j + k = L-1 \neq 0$ in (5.26), (5.34) and (5.35), we obtain $\psi(e_{i,r}, e_{j,s}, e_{k,p}) = 0$ for $i+j+k = L-1$ and $\forall r,s,p \in \{1,\dots,K\}$, because the higher degree terms are of level L and higher and thus zero by boundedness. Therefore, terms of level $L-1$ are zero, for all puncture-indices, i.e. for all three cases considered above. Next, taking $i+j+k = L-2$ in (5.26), (5.34) and (5.35), we obtain $\psi(e_{i,r}, e_{j,s}, e_{k,p}) = 0$ for $i+j+k = L-2$ and $\forall r,s,p \in \{1,\dots,K\}$, because the higher degree terms are of level $L-1$ and higher and are zero because of the previous induction step, for all three cases. Continuing with decreasing level until taking $i+j+k = 1$ in (5.26), (5.34) and (5.35), we obtain $\psi(e_{i,r}, e_{j,s}, e_{k,p}) = 0$ for $i+j+k = 1$ and $\forall r,s,p \in \{1,\dots,K\}$ by induction. Next, take $i+j+k = 0$ in (5.26) and (5.34). Since the higher degree terms on the right-hand side are of level one and higher, they are zero, and thus, we obtain $\psi(e_{k,t}, e_{i,r}, e_{j,s}) = 0\ \forall\ k,i,j \in \mathbb{Z}$ and $\forall\ t \neq r \neq s \neq t \in \{1,\dots,K\}$ and $\psi(e_{i,s}, e_{j,r}, e_{k,r}) = 0\ \forall\ i,j,k \in \mathbb{Z}$ and $\forall\ r \neq s \in \{1,\dots,K\}$. For the third case listed above, we see from (5.35) that the level zero $i+j+k = 0$ cannot be expressed in terms of higher degree entries. However, at level zero, the analysis of Case 3 reduces to the analysis of the Witt algebra. In fact, the condition (5.14) corresponds exactly to the condition for the Witt algebra of Lemma 3.1.2. Moreover, as we already proved $\dim \mathrm{H}^3(\mathscr{W},\mathbb{K}) = 1$, we know that level zero is completely fixed by the non-trivial entry $\psi(e_{-1}, e_1, e_0)$, which corresponds in the present case to K entries $\psi(e_{-1,r}, e_{1,r}, e_{0,r})$ with $r \in \{1,\dots,K\}$, see the proof of Lemma 3.1.2. The reasoning in the proof of Lemma 3.1.2 exactly also applies to the present case.

Continuing with decreasing induction on the level, we obtain by (5.26), (5.34) and (5.35) that level minus one is completely determined by level zero and higher terms, i.e. it is completely determined by the entries $\psi(e_{-1,r}, e_{1,r}, e_{0,r})$ with $r \in \{1,\dots,K\}$, for all three cases. The same holds true for levels minus two, minus three and so on. Therefore, all $\psi(e_{k,t}, e_{i,r}, e_{j,s})$, $\forall\ i,j,k \in \mathbb{Z}$, $\forall\ t,r,s \in \{1,\dots,K\}$ are completely determined by the entries $\psi(e_{-1,r}, e_{1,r}, e_{0,r})$ with $r \in \{1,\dots,K\}$. This concludes the proof of Theorem 5.2.1. □

Remark 5.2.1. In the introduction, Chapter 1, we already mentioned the Feigin-Novikov conjecture, which has been proven for the continuous cohomology, but not for the algebraic cohomology. The Feigin-Novikov conjecture states that the full cohomology of the multipoint

Krichever-Novikov vector field algebra is given by the free graded commutative algebra generated by elements c_k of cohomological dimension two and one element θ of cohomological dimension 3,

$$\mathrm{H}(\mathcal{KN},\mathbb{C}) \cong \wedge(c_1,\ldots,c_{2g+N-1},\theta)\,.$$

If the conjecture holds true in the algebraic case, then $\mathrm{H}^3(\mathcal{KN},\mathbb{K})$ should be generated by θ and thus, it should be one-dimensional. Since the space of bounded cohomology $\mathrm{H}^3_b(\mathcal{KN},\mathbb{K})$ is a subspace of $\mathrm{H}^3(\mathcal{KN},\mathbb{K})$, the dimension of $\mathrm{H}^3_b(\mathcal{KN},\mathbb{K})$ should be at most one. Therefore, it should be possible in the proof above to find relations between the coefficients $\psi(e_{-1,r}, e_{1,r}, e_{0,r})$, $r \in \{1,\ldots K\}$. The author did not further investigate this possibility, though.

Chapter 6

Conclusion and Outlook

6.1 Conclusion

The aim of this thesis was to compute the low-dimensional algebraic cohomology with values in natural modules of infinite dimensional Lie algebras of Virasoro-type, including the Witt algebra, the Virasoro algebra and the multipoint Krichever-Novikov vector field algebra. Natural modules included the trivial module, the adjoint module, and general tensor densities modules. Most of the explicit computations focussed on the Witt algebra, as the results for the Virasoro algebra were deduced from the analysis of the Witt algebra using a relation we exhibited via the Hochschild-Serre spectral sequence. Results for the multipoint Krichever-Novikov vector field algebra are limited and need further investigation.
The focus in this thesis lied on algebraic cohomology. An interesting question is whether the algebraic cohomology we derived in this thesis agrees with the continuous cohomology. Let us recall the results from continuous cohomology. In case of the trivial module, it was shown [43, 45] that the continuous cohomology $\mathrm{H}^k(Vect(S^1),\mathbb{R})$ is the free graded-commutative algebra generated by an element ω of cohomological dimension two and an element θ of cohomological dimension three. More precisely, it is given by the tensor product of the algebra of polynomials generated by ω with an external algebra generated by θ. Thus, the dimensions of the continuous cohomology are given by,

$\mathrm{H}^0(Vect(S^1),\mathbb{R})$	$\mathrm{H}^1(Vect(S^1),\mathbb{R})$	$\mathrm{H}^2(Vect(S^1),\mathbb{R})$	$\mathrm{H}^3(Vect(S^1),\mathbb{R})$	$\dots$	$\mathrm{H}^k(Vect(S^1),\mathbb{R})$	$\dots$
1	0	1	1	$\dots$	1	$\dots$

We see that these results agree with our algebraic computation of $\dim\,\mathrm{H}^3(\mathscr{W},\mathbb{K})=1$.

Under the assumption that the entire cohomology agrees in the algebraic and the continuous cases, and that the maps φ_k from (3.45) are injective for all k, such that we have $\mathrm{H}^k(\mathscr{V},\mathbb{K}) = \dfrac{\mathrm{H}^k(\mathscr{W},\mathbb{K})}{\mathrm{H}^{k-2}(\mathscr{W},\mathbb{K})}$ for all $k \geq 0$, as we already have for k up to four, see Theorem 3.3.3, then the dimensions of the cohomology of the Virasoro algebra are,

$\mathrm{H}^0(\mathscr{V},\mathbb{K})$	$\mathrm{H}^1(\mathscr{V},\mathbb{K})$	$\mathrm{H}^2(\mathscr{V},\mathbb{K})$	$\mathrm{H}^3(\mathscr{V},\mathbb{K})$	$\mathrm{H}^4(\mathscr{V},\mathbb{K})$	$\dots$	$\mathrm{H}^k(\mathscr{V},\mathbb{K})$	$\dots$
1	0	0	1	0	$\dots$	0	$\dots$

Therefore, only $\mathrm{H}^0(\mathscr{V},\mathbb{K})$ and $\mathrm{H}^3(\mathscr{V},\mathbb{K})$ is non-zero. We proved the assumption for the low-dimensional cohomology in this thesis, but it is not proven for all k.

The continuous cohomology of $Vect(S^1)$ with values in tensor densities modules, including the adjoint module, is given in [36]. The result is more complicated than the result for the trivial module, hence we will not present it in detail here. We only give the results for the first and the second cohomology spaces, which are:

$$
\begin{aligned}
&\mathrm{H}^1(Vect(S^1), C^\infty(S^1)d\varphi^\lambda) = \{0\} \quad \forall \lambda \in \mathbb{Z} \setminus \{0,1,2\}, \\
&\dim \mathrm{H}^1(Vect(S^1), C^\infty(S^1)d\varphi^0) = 2, \\
&\dim \mathrm{H}^1(Vect(S^1), C^\infty(S^1)d\varphi^1) = \dim \mathrm{H}^1(Vect(S^1), C^\infty(S^1)d\varphi^2) = 1, \\
&\mathrm{H}^2(Vect(S^1), C^\infty(S^1)d\varphi^\lambda) = \{0\} \quad \forall \lambda \in \mathbb{Z} \setminus \{0,1,2,5,7\}, \\
&\dim \mathrm{H}^2(Vect(S^1), C^\infty(S^1)d\varphi^0) = \dim \mathrm{H}^2(Vect(S^1), C^\infty(S^1)d\varphi^1) = 2, \\
&\dim \mathrm{H}^2(Vect(S^1), C^\infty(S^1)d\varphi^2) = 2, \\
&\dim \mathrm{H}^2(Vect(S^1), C^\infty(S^1)d\varphi^5) = \dim \mathrm{H}^2(Vect(S^1), C^\infty(S^1)d\varphi^7) = 1, \\
&\mathrm{H}^3(Vect(S^1), C^\infty(S^1)d\varphi^\lambda) = \{0\} \quad \forall \lambda \in \mathbb{Z} \setminus \{0,1,2,5,7,12,15\}, \\
&\dim \mathrm{H}^3(Vect(S^1), C^\infty(S^1)d\varphi^0) = \dim \mathrm{H}^3(Vect(S^1), C^\infty(S^1)d\varphi^1) = 2, \\
&\dim \mathrm{H}^3(Vect(S^1), C^\infty(S^1)d\varphi^2) = \dim \mathrm{H}^3(Vect(S^1), C^\infty(S^1)d\varphi^5) = 2, \\
&\dim \mathrm{H}^3(Vect(S^1), C^\infty(S^1)d\varphi^7) = 2, \\
&\dim \mathrm{H}^3(Vect(S^1), C^\infty(S^1)d\varphi^{12}) = \dim \mathrm{H}^3(Vect(S^1), C^\infty(S^1)d\varphi^{15}) = 1,
\end{aligned}
$$

where φ is the angular coordinate on the circle.
We see that they are in agreement with the algebraic results we obtained in this thesis. Also for the third cohomology, continuous cohomology is $\mathrm{H}^3(Vect(S^1), C^\infty(S^1)d\varphi^\lambda) = \{0\}$ $\forall \lambda \in \mathbb{Z} \setminus \{0,1,2,5,7,12,15\}$, and it is non-vanishing for $\lambda \in \{0,1,2,5,7,12,15\}$. This is compatible with the results we obtained so far for the third algebraic cohomology, namely that it is zero for negative λ and also for large positive λ.

Under the assumption that the entire cohomology agrees in the algebraic and the continuous cases, and that the maps φ_k from (3.45) are injective for all k, such that we have $\mathrm{H}^k(\mathcal{V}, \mathcal{F}^\lambda) = \dfrac{\mathrm{H}^k(\mathcal{W}, \mathcal{F}^\lambda)}{\mathrm{H}^{k-2}(\mathcal{W}, \mathcal{F}^\lambda)}$ for all $k \geq 0$, as we already have for k up to three, see Theorem 3.3.4, then the dimensions of the cohomology of the Virasoro algebra are for $\mathcal{F}^0$,

H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8	H^9	...
1	2	1	0	1	2	1	0	1	2	...

where we abbreviated $\mathrm{H}^k := \mathrm{H}^k(\mathcal{V}, \mathcal{F}^0)$. For $\mathcal{F}^1$, we obtain under the assumption above,

H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8	H^9	...
0	1	2	1	0	1	2	1	0	1	...

where we abbreviated $\mathrm{H}^k := \mathrm{H}^k(\mathcal{V}, \mathcal{F}^1)$. The same is obtained for $\mathcal{F}^2$. For $\mathcal{F}^5$ and $\mathcal{F}^7$, we obtain,

H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8	H^9	...
0	0	1	2	1	0	1	2	1	0	...

For all non-exceptional λ, we obtain that $\mathrm{H}^k(\mathcal{V}, \mathcal{F}^\lambda)$ is zero, under the assumption given above.

In case of the Witt algebra, for the adjoint module in particular, continuous cohomology is zero, $\mathrm{H}^k(Vect(S^1), Vect(S^1)) = \{0\}\ \forall k \in \mathbb{N}$. This result also agrees with our algebraic result $\mathrm{H}^3(\mathcal{W},\mathcal{W}) = \{0\}$. Concerning the Virasoro algebra, remember its cohomology with values in the adjoint module can be derived from the long exact sequence,

$$\ldots \longrightarrow \mathrm{H}^{k-1}(\mathcal{V},\mathcal{W}) \longrightarrow \mathrm{H}^k(\mathcal{V},\mathbb{K}) \longrightarrow \mathrm{H}^k(\mathcal{V},\mathcal{V}) \longrightarrow \mathrm{H}^k(\mathcal{V},\mathcal{W}) \longrightarrow \ldots .$$

Under the assumption that algebraic and continuous cohomology coincide and that $\mathrm{H}^k(\mathcal{V},\mathcal{W}) \cong \mathrm{H}^k(\mathcal{W},\mathcal{W})$ for all k, we obtain $\dim\ \mathrm{H}^k(\mathcal{V},\mathbb{K}) = \dim\ \mathrm{H}^k(\mathcal{V},\mathcal{V})$ for all k.

6.2 Outlook

On the short term, there are some obvious analyses that should be completed. In fact, the analysis of $\mathrm{H}^3(\mathcal{W},\mathcal{F}^\lambda)$ should be performed for all $\lambda \in \mathbb{C}$ or at least for all $\lambda \in \mathbb{Z}$. In particular, it would be interesting to obtain a proof for some λ odd. Also, the exceptional values of λ, namely $\lambda \in \{0,1,2,5,7,12,15\}$, should be investigated in detail.
In case of the multipoint Krichever-Novikov vector field algebra, more results need to be completed. First of all, the dimension of the third bounded cohomology $\mathrm{H}^3_b(\mathcal{KN},\mathbb{K})$ needs to be determined exactly. Also, the third local cohomology $\mathrm{H}^3_{loc}(\mathcal{KN},\mathbb{K})$ should be investigated. Concerning the tensor densities modules, including the adjoint module, almost all of the low-dimensional cohomology is still unknown.

On the long term, higher cohomology could be explored, starting with the analysis of the fourth cohomology $\mathrm{H}^4(\mathcal{W},\mathcal{W})$. In fact, comparing the analyses of $\mathrm{H}^1(\mathcal{W},\mathcal{W})$, $\mathrm{H}^2(\mathcal{W},\mathcal{W})$ and $\mathrm{H}^3(\mathcal{W},\mathcal{W})$, no pattern becomes apparent. The increase of complexity between these three analyses is substantial. Also, the complications that arise when going to higher cohomology are hard to predict. For example, in the proof of $\mathrm{H}^3(\mathcal{W},\mathcal{W}) = \{0\}$, subtleties arose that were hard to anticipate when regarding the proof of $\mathrm{H}^2(\mathcal{W},\mathcal{W}) = \{0\}$. It is thus clear that the analysis of $\mathrm{H}^4(\mathcal{W},\mathcal{W})$ could be much more complicated than the one of $\mathrm{H}^3(\mathcal{W},\mathcal{W})$ when using our elementary algebraic methods, and that a generalization of our proofs to higher cohomology is maybe not possible. Clearly, the computation of $\mathrm{H}^k(\mathcal{W},\mathcal{F}^\lambda)$ for higher k is even more complicated, as the analysis of $\mathrm{H}^3(\mathcal{W},\mathcal{F}^\lambda)$ clearly demonstrated. One could also attempt to extend the Theorems 3.3.3 and 3.3.4 to higher cohomology, especially since the increase in complexity with the cohomological dimension is not as strong as in the case of the computation of $\mathrm{H}^k(\mathcal{W},\mathcal{W})$. Possibly, a pattern may become apparent after computations for several k.
In case of the trivial module, another approach may stem from the analysis of the cup product. Similarly to the continuous cohomology, one could suspect that the entire cohomology $\mathrm{H}^k(\mathcal{W},\mathbb{K})$ is generated by elements of $\mathrm{H}^2(\mathcal{W},\mathbb{K})$ and $\mathrm{H}^3(\mathcal{W},\mathbb{K})$, via the cup product in our algebraic case. In a first step, one could try to prove that the cup product is injective, which would provide a minimal dimension for $\mathrm{H}^k(\mathcal{W},\mathbb{K})$. In a second step, one would need to prove that all k-cocycles stem from the cup product between two cocycles. Using this approach, one could also eventually aim to prove that algebraic and continuous cohomology are the same, and obtain the algebraic cohomology from the continuous cohomology. However, is is not clear how to proceed concretely to get a proof of these hypotheses.
Another approach to relate continuous cohomology to algebraic cohomology could be given by the one used by Hennion and Kapranov in [55], where a relation was established in the case of the trivial module. The author also tried a different method using algebraic discrete Morse

theory, in collaboration with Viktor Lopatkin. However, the analysis boiled down to heavy computations similar to those used in the main text of this thesis, not allowing to get new insights. An example where the algebraic discrete Morse theory was applied successively to compute cohomology is given by the article of Lopatkin [78]. Algebraic discrete Morse theory was introduced independently by Jöllenbeck and Welker [66] and Sköldberg [114], and is an algebraic version of Forman's discrete Morse theory [40, 41].

The topic considered in this thesis can be extended in various directions. For instance, the same analysis could be performed for the superalgebraic versions of the Lie algebras considered here, which come with an extra $\mathbb{Z}_2$-grading. In [119], $\mathrm{H}^2(k(1), k(1))$ and $\mathrm{H}^2(k(1)^+, k(1)^+)$ were computed by using elementary algebra, which correspond to the two supersymmetric versions of $\mathrm{H}^2(\mathcal{W}, \mathcal{W})$, and the same was done for the second cohomology of the Neveu-Schwarz and the Ramond superalgebras, which are the two supersymmetric versions of the Virasoro algebra. The same could be done for the third algebraic cohomology. The analysis could also be extended to the tensor densities modules.
In this thesis, we focussed on infinite-dimensional Lie algebras, thus we always considered Lie algebras over a base field $\mathbb{K}$ with characteristic zero, $char(\mathbb{K}) = 0$. Another direction that could be investigated would be to consider $char(\mathbb{K}) \neq 0$. The case $char(\mathbb{K}) = 2$ is of particular interest, as it has led to the so-called *commutative cohomology*, see [79]. Recently, also a Hochschild-Serre spectral sequence has been constructed in this setting, see [129].
Another very popular subject is given by the so-called *Hom-Lie* algebras. A Hom-Lie algebra $(\mathcal{L}, \zeta)$ is a pair of a non-associative algebra $\mathcal{L}$ together with an algebra homomorphism ζ such that the product $[\cdot,\cdot]_{\mathcal{L}}$ on $\mathcal{L}$ is antisymmetric, $[x, y]_{\mathcal{L}} = -[y, x]_{\mathcal{L}}$, and satisfies a ζ-twisted Jacobi identity, $[(\mathrm{id}+\zeta)(x), [y, z]_{\mathcal{L}}]_{\mathcal{L}} + [(\mathrm{id}+\zeta)(z), [x, y]_{\mathcal{L}}]_{\mathcal{L}} + [(\mathrm{id}+\zeta)(y), [z, x]_{\mathcal{L}}]_{\mathcal{L}} = 0, \forall x, y, z \in \mathcal{L}$. The Hom-Lie algebras were first introduced by Hartwig, Larsson and Silvestrov [54, 75, 76], though similar structures already played an important role in physics and mathematics before, see e.g. [2, 16–18, 91, 95–97]. Since their introduction, they have been studied intensively, including their cohomology and representations, a very non-exhaustive list being given by [1, 4–6, 9, 15, 113, 136]. It would be interesting to explicitly compute the low-dimensional algebraic cohomology with values in various modules of the Witt Hom-Lie algebra and the Virasoro Hom-Lie algebra, and their supersymmetric analogues.

Appendix A

Solutions of recurrence relations

In this appendix, we write down how solutions of the recurrence relations, which we established during the analyses of $\mathrm{H}^2(\mathscr{W},\mathscr{F}^\lambda$ and $\mathrm{H}^3(\mathscr{W},\mathscr{F}^\lambda$, can be found. We will start with the second cohomology, as it allows for a warm-up example to the third cohomology.

A.1 Solving the recurrence relations of $\mathrm{H}^2(\mathscr{W},\mathscr{F}^5)$ and $\mathrm{H}^2(\mathscr{W},\mathscr{F}^7)$

In this section, we give details about the solving of the recurrence relations established during the analysis of $\mathrm{H}^2(\mathscr{W},\mathscr{F}^5)$ and $\mathrm{H}^2(\mathscr{W},\mathscr{F}^7)$, see the Remarks 4.2.1 and 4.2.2, respectively. The strategy is to write down the first terms of the recurrence relations and to guess a solution. Subsequently, by induction it is proven that the guess of the solution is indeed a correct solution of the recurrence relations.

A.1.1 The case of $\lambda = 5$

We already proved that $\mathrm{H}^2(\mathscr{W},\mathscr{F}^5)$ is one-dimensional, hence we have one generating coefficient. In the following, we seek to express a generic 2-cocycle in terms of the generating coefficient $\psi_{2,-2}$. We already know that the levels zero, one and minus one are trivial, so we can start with levels minus two and plus two. For positive i, the recurrence relation for level minus two yields, see (4.69):

$$\psi_{i+1,-2} = -\frac{i+3}{1-i}\psi_{i,-2} \Leftrightarrow \psi_{i,-2} = \frac{i+2}{i-2}\psi_{i-1,-2}$$
$$\Leftrightarrow \psi_{i,-2} = \frac{1}{4!}\frac{(i+2)!}{(i-2)!}\psi_{2,-2} \Leftrightarrow \psi_{i,-2} = \frac{1}{24}(i+2)(i+1)i(i-1)\psi_{2,-2}. \tag{A.1}$$

Doing the same reasoning for negative i using (4.68) leads to an identical formula. Similarly, level plus two can be expressed as follows:

$$\psi_{i,2} = -\frac{1}{24}(i-2)(i-1)i(i+1)\psi_{2,-2}. \tag{A.2}$$

Comparing the two formulas, we see that the factor $(i-1)i(i+1)$ remains the same. As a guess, we may assume that for generic k, the formula may become:

$$\psi_{i,k} = \frac{1}{24}(a(k)i + b(k))(i-1)i(i+1)\psi_{2,-2}, \tag{A.3}$$

where $a(k)$ and $b(k)$ are some functions of k, which have to be determined. Encoding the formulas for level minus two (A.1), level plus two (A.2) and the recurrence relation for generic level k (4.73,4.72) in some symbolic programming language, we can construct a table giving the values of a and b for various values of k:

k	$b(k)$	$a(k)$
−10	1650	165
−9	1080	120
−8	672	84
−7	392	56
−6	210	35
−5	100	20
−4	40	10
−3	12	4
−2	2	1
−1	0	0

k	$b(k)$	$a(k)$
0	0	0
1	0	0
2	2	−1
3	12	−4
4	40	−10
5	100	−20
6	210	−35
7	392	−56
8	672	−84
9	1080	−120
10	1650	−165

From the table above, we see that both functions a and b look polynomial. The function a has three roots in $k = 0, 1, -1$ and is antisymmetric in k, thus it has minimal degree 3. The polynomial a is of the following form:

$$a(k) = -\frac{1}{6}k(k-1)(k+1)\,.$$

The function b also has three roots, but it is symmetric in k, thus it is at least of degree 4. The polynomial b is of the following form:

$$b(k) = \frac{1}{6}k^2(k+1)(k-1)\,.$$

Replacing the expressions of a and b into (A.3), we obtain a generic formula for the two-cocycle:

$$\begin{aligned} \psi_{i,k} &= -\frac{1}{144}\left(k(k^2-1)i - k^2(k^2-1)\right)(i+1)(i-1)\ i\ \psi_{2,-2} \\ \Leftrightarrow \psi_{i,k} &= -\frac{1}{144}(i-1)i(i+1)(i-k)k(k-1)(k+1)\psi_{2,-2}\,. \end{aligned} \tag{A.4}$$

The expression above is clearly antisymmetric in i and k, and its roots are consistent with the triviality of level zero, level plus one and level minus one. In order to prove that the recurrence relations of level plus two, (4.71,4.70), level minus two (4.68,4.69) and generic level k (4.72,4.73) are equal to (A.4) for every k, we can use induction on k, by assuming formula (A.4) valid for $k \geq 2$ and checking whether it stays valid for $k+1$ by using (4.72). Inserting the generic formula (A.4) for $\psi_{i,k}$, $\psi_{i+1,k}$ and $\psi_{i,k+1}$ into the recurrence relation (4.72), a direct computation leads to agreement. Hence, the cochain (A.4) is a solution of the recurrence relations of $\mathrm{H}^2(\mathcal{W},\mathcal{F}^5)$. Inserting the expression (A.4) into the cocycle condition (4.14) of $\mathrm{H}^2(\mathcal{W},\mathcal{F}^5)$, a direct verification shows that (A.4) is a cocycle. The same can be done for k decreasing.

A.1.2 The case of $\lambda = 7$

We already proved that the dimension of $\mathrm{H}^2(\mathcal{W},\mathcal{F}^7)$ is one, hence all coefficients can be expressed in terms of a single generating coefficient. From the recurrence relation for level minus two (4.78, 4.77) we can immediately deduce the following generic expression:

$$\psi_{i,-2} = \frac{1}{720}(i+4)(i+3)(i+2)(i+1)i(i-1)\psi_{2,-2}\,. \tag{A.5}$$

The same can be done for level plus two using (4.79, 4.80):

$$\psi_{i,2} = -\frac{1}{720}(i-4)(i-3)(i-2)(i+1)i(i-1)\psi_{2,-2}\,. \tag{A.6}$$

Comparing both formulas, we can make a guess concerning the formula for generic k:

$$\psi_{i,k} = \frac{1}{720}\left(a(k)i^3 + b(k)i^2 + c(k)i + d(k)\right)(i+1)i(i-1)\psi_{2,-2}\,, \tag{A.7}$$

where $a(k)$, $b(k)$, $c(k)$ and $d(k)$ are some functions of k. Again, implementing formulas (A.6) and (A.5), as well as the recurrence relation for generic k (4.81,4.82) into some symbolic programming language, we obtain:

k	$d(k)$	$c(k)$	$b(k)$	$a(k)$
0	0	0	0	0
1	0	0	0	0
2	24	−26	9	−1
3	204	−194	54	−4
4	960	−800	180	−10
5	3300	−2410	450	−20
6	9240	−5950	945	−35
7	22344	−12796	1764	−56
8	48384	−24864	3024	−84
9	96120	−44700	4860	−120
10	178200	−75570	7425	−165

k	$d(k)$	$c(k)$	$b(k)$	$a(k)$
−10	178200	75570	7425	165
−9	96120	44700	4860	120
−8	48384	24864	3024	84
−7	22344	12796	1764	56
−6	9240	5950	945	35
−5	3300	2410	450	20
−4	960	800	180	10
−3	204	194	54	4
−2	24	26	9	1
−1	0	0	0	0

Having a sharp look at the values for $a(k)$ and $b(k)$, we see that they are similar to the ones encountered for $\lambda = 5$:

$$a(k) = -\frac{1}{6}k(k-1)(k+1) \quad \text{and} \quad b(k) = \frac{3}{4}k^2(k+1)(k-1)\,.$$

The remaining functions $c(k)$ and $d(k)$ can be determined by using some spreadsheet software, which yields:

$$c(k) = -\frac{1}{12}(k-1)k(k+1)(16+9k^2)\,, \qquad d(k) = \frac{1}{6}(k-1)k^2(k+1)(8+k^2)\,.$$

Inserting the definitions of a, b, c and d into (A.7), simplifying, we obtain a generic formula for a 2-cocycle:

$$\psi_{i,k} = -\frac{1}{8640}i(i^2-1)(i-k)k(k^2-1)(2i^2-7ik+16+2k^2)\psi_{2,-2}\,. \tag{A.8}$$

In this last expression, we can see that it is antisymmetric in i and k and its roots are consistent with the triviality of the levels zero, plus one and minus one. Once again, in order to prove that the recurrence relations (4.77,4.78) and (4.82,4.81) are equal to the generic formula (A.8) for every k, we can use induction on k, by assuming formula (A.8) valid for $k \geq 2$ and checking whether it stays valid for $k+1$ by using (4.81). Inserting the generic formula (A.8) for $\psi_{i,k}$, $\psi_{i+1,k}$ and $\psi_{i,k+1}$ into the recurrence relation (4.81), a direct computation yields agreement. The same can be done for k decreasing. Hence, (A.8) is genuinely a solution of the recurrence relations we found for $\mathrm{H}^2(\mathscr{W},\mathscr{F}^7)$. Inserting the expression (A.8) into the cocycle condition (4.14), direct computation shows that it is fulfilled.

A.2 Solving the recurrence relations of $\mathrm{H}^3(\mathscr{W},\mathscr{F}^\lambda)$

The main aim in Section 4.3 and in particular in Section 4.3.2 was to find a system of equations for the coefficients $\psi_{i,j,k}$ with a trivial solution, implying in particular $\psi_{-3,2,-2} = 0$. The problem is that a system with that property was not found for λ odd. Such a system can be searched by numerical means, simply by testing systematically the equations resulting from the cocycle condition when all the coefficients $\psi_{i,j,k}$ have been expressed in terms of the generating coefficients of pure and mixed type. The problem is that expressing the coefficients in terms of the pure type generators takes a lot of processing time, which limits the number of analyzed equations very much. A solution to this problem is given by resolving the recurrence relations by hand and by obtaining an explicit expression for the coefficients $\psi_{i,j,k}$ in terms of the generating coefficients of pure and mixed type. Below, we solve the recurrence relations in an empirical manner, which allows to increase the number of tested equations considerably. For example, for $\lambda = 19$, a system of 4152 equations with 45 variables (=generating coefficients of mixed and pure type) was tested. Unfortunately, a non-trivial solution still could be found. Although solving the recurrence relations algebraically did not help, we will in the following show how the recurrence relations can be solved in a semi-empirical manner, since this allows to speed up the processing times also for λ even, and to go beyond $\lambda = 26$. We do not prove that the algebraic solution holds true for any $\psi_{i,j,k}$, but we compared the algebraic solutions with the numerical solutions found by *Mathematica* for low-level coefficients. The aim is only to gain processing time. Since this does not help with the generic problem of λ odd, the author did not do the effort of providing a rigorous proof for the resolution of the recurrence relations.

The strategy is simple. We will write down the first terms of the recurrence relation and then guess a generic law. We will start with the mixed type generators, which arise for the coefficients $\psi_{i,j,k}$ which have two positive indices and one negative index or vice versa. We will start with level minus two, i.e. we consider $\psi_{i,j,-2}$ with $j \geq 2$ and $i < -2$. Using (4.136), we try for example $j = 2$, $j = 3$ and leave i arbitrary, which gives respectively:

$$\psi_{i,3,-2} = (i+\lambda)\psi_{i,2,-2} + (1-i)\psi_{i+1,2,-2}\,,$$
$$\psi_{i,4,-2} = \frac{1}{2}(i+\lambda)(i+1+\lambda)\psi_{i,2,-2} + (1-i)(1+i+\lambda)\psi_{i+1,2,-2} + \frac{1}{2}(i-1)i\psi_{i+2,2,-2}\,.$$

The bigger j is, the more terms there will be. Let us write down the terms from $j = 3,\ldots,9$, they can be found in Tables A.1-A.3. Note that in *Mathematica*, $\prod_{c=m}^{j-3}(i+\lambda+c)$ gives one for $j < m+3$. From these tables, we can now guesstimate a generic rule for the coefficients $\psi_{i,j,-2}$ with $i < -2$ and $j > 2$, given by:

$$\psi_{i,j,-2} = \sum_{l=0}^{j-2} \frac{1}{l!(j-l-2)!} \prod_{b=-1}^{l-2}(-i-b) \prod_{c=l}^{j-3}(i+\lambda+c)\psi_{i+l,2,-2}\,. \tag{A.9}$$

The coefficients with $i > 2$, $j < -2$ are obtained from this expression by antisymmetry: $\psi_{i,j,-2} := -\psi_{j,i,-2}$.

	$\psi_{i,2,-2}$	$\psi_{i+1,2,-2}$
$\psi_{i,3,-2}$	$(i+\lambda)$	$-i+1$
$\psi_{i,4,-2}$	$\frac{1}{2}(i+\lambda)(i+1+\lambda)$	$(-i+1)(i+1+\lambda)$
$\psi_{i,5,-2}$	$\frac{1}{6}(i+\lambda)(i+1+\lambda)(i+2+\lambda)$	$\frac{1}{2}(-i+1)(i+1+\lambda)(i+2+\lambda)$
$\psi_{i,6,-2}$	$\frac{1}{24}(i+\lambda)(i+1+\lambda)(i+2+\lambda)(i+3+\lambda)$	$\frac{1}{6}(-i+1)(i+1+\lambda)(i+2+\lambda)(i+3+\lambda)$
$\psi_{i,7,-2}$	$\frac{1}{120}(i+\lambda)(i+1+\lambda)(i+2+\lambda)(i+3+\lambda)(i+4+\lambda)$	$\frac{1}{24}(-i+1)(i+1+\lambda)(i+2+\lambda)(i+3+\lambda)(i+4+\lambda)$
$\cdots$	$\cdots$	$\cdots$
	$\frac{1}{(j-2)!}\prod_{c=0}^{j-3}(i+\lambda+c)$	$\frac{1}{(j-3)!}(-i+1)\prod_{c=1}^{j-3}(i+\lambda+c)$

Table A.1: The table gives the coefficients of the generating terms $\psi_{i,2,-2}$ and $\psi_{i+1,2,-2}$ appearing in the expressions of the terms $\psi_{i,j,-2}$ with $j=3,\ldots,7$. Derived from these coefficients, the last line contains an ansatz for the expressions of these for generic j.

	$\psi_{i+2,2,-2}$	$\psi_{i+3,2,-2}$
$\psi_{i,3,-2}$	0	0
$\psi_{i,4,-2}$	$\frac{1}{2}(i-1)i$	0
$\psi_{i,5,-2}$	$\frac{1}{2}(i-1)i(i+2+\lambda)$	$\frac{1}{6}(-i+1)(-i)(-i-1)$
$\psi_{i,6,-2}$	$\frac{1}{4}(i-1)i(i+2+\lambda)(i+3+\lambda)$	$\frac{1}{6}(-i+1)(-i)(-i-1)(i+3+\lambda)$
$\psi_{i,7,-2}$	$\frac{1}{12}(i-1)i(i+2+\lambda)(i+3+\lambda)(i+4+\lambda)$	$\frac{1}{12}(-i+1)(-i)(-i-1)(i+3+\lambda)(i+4+\lambda)$
$\cdots$	$\cdots$	$\cdots$
	$\frac{1}{2(j-4)!}(-i+1)(-i)\prod_{c=2}^{j-3}(i+\lambda+c)$	$\frac{1}{6(j-5)!}\prod_{b=-1}^{1}(-i-b)\prod_{c=3}^{j-3}(i+\lambda+c)$

Table A.2: The table gives the coefficients of the generating terms $\psi_{i+2,2,-2}$ and $\psi_{i+3,2,-2}$ appearing in the expressions of the terms $\psi_{i,j,-2}$ with $j=3,\ldots,7$. Derived from the non-zero coefficients, the last line contains an ansatz for the expressions of these for generic j.

	$\psi_{i+4,2,-2}$	$\psi_{i+5,2,-2}$
$\psi_{i,3,-2}$	0	0
$\psi_{i,4,-2}$	0	0
$\psi_{i,5,-2}$	0	0
$\psi_{i,6,-2}$	$\frac{1}{24}(-i+1)(-i)(-i-1)(-i-2)$	0
$\psi_{i,7,-2}$	$\frac{1}{24}(-i+1)(-i)(-i-1)(-i-2)(i+4+\lambda)$	$\frac{1}{120}(-i+1)(-i)(-i-1)(-i-2)(-i-3)$
$\psi_{i,8,-2}$	$\frac{1}{48}(-i+1)(-i)(-i-1)(-i-2)(i+4+\lambda)(i+5+\lambda)$	$\frac{1}{120}(-i+1)(-i)(-i-1)(-i-2)(-i-3)(i+5+\lambda)$
$\psi_{i,9,-2}$	$\frac{1}{144}(-i+1)(-i)(-i-1)(-i-2)(i+4+\lambda)(i+5+\lambda)(i+6+\lambda)$	$\frac{1}{240}(-i+1)(-i)(-i-1)(-i-2)(-i-3)(i+5+\lambda)(i+6+\lambda)$
	$\frac{1}{24(j-6)!}\prod_{b=-1}^{2}(-i-b)\prod_{c=4}^{j-3}(i+\lambda+c)$	$\frac{1}{120(j-7)!}\prod_{b=-1}^{3}(-i-b)\prod_{c=5}^{j-3}(i+\lambda+c)$

Table A.3: The table gives the coefficients of the generating terms $\psi_{i+4,2,-2}$ and $\psi_{i+5,2,-2}$ appearing in the expressions of the terms $\psi_{i,j,-2}$ with $j=3,\ldots,9$. Derived from the non-zero coefficients, the last line contains an ansatz for the expressions of these for generic j.

Next, we will consider $\psi_{i,j,k}$ of generic level k, with $k < -2$. The reasoning is valid for both situations $i < -2$, $j > 2$ and $i, j < -2$, but the last situation gives us generating coefficients of mixed type, which we will consider later. Let us abbreviate the coefficients $\psi_{i,j,-2}$ in (A.9) by $\psi_{i,j,-2} := F(i,j)$. We will proceed as before. Considering the recurrence relation for generic k with $k \leq -2$ given in (4.131), we write down $\psi_{i,j,k}$ with $k = -1, \ldots, -5$ expressed in terms of $F(i,j)$, in Table A.4. From Table A.4, we can guesstimate a generic expression:

$$\psi_{i,j,k} = \sum_{x=k+2}^{0} \sum_{y=k+2-x}^{0} \frac{(-1)^{x+y+k}}{(-k-2+x+y)!(-x)!(-y)!} \prod_{m=x+2}^{1} (i+m) \prod_{n=y+2}^{1} (j+n) \prod_{r=k+1}^{x+y-2} (i+j-\lambda+r) F(i+x, j+y), \tag{A.10}$$

where $F(i,j)$ is as in (A.9). This expression gives us the coefficients $\psi_{i,j,k}$ with two indices negative and one index positive.
To conclude the analysis of mixed type generators, we also need to find an expression of the coefficients $\psi_{i,j,k}$ with two indices positive and one index negative. To do this, we need to guesstimate an expression for $\psi_{i,j,2}$ with $i < -2$, $j > 2$, which can be done using (A.9) and symmetry, yielding:

$$\psi_{i,j,2} = \sum_{l=0}^{-i-2} \frac{1}{l!(-i-l-2)!} \prod_{b=-1}^{l-2} (j-b) \prod_{c=l}^{-i-3} (-j+\lambda+c) \psi_{-2,j-l,2}. \tag{A.11}$$

The coefficients with $i > 2$, $j < -2$ are obtained from this expression by antisymmetry: $\psi_{i,j,-2} := -\psi_{j,i,-2}$. Let us denote this expression for $\psi_{i,j,2}$ by $\widetilde{F}(i,j)$. The expression of the coefficients $\psi_{i,j,k}$ with $k > 2$ can be derived semi-empirically from (A.10) using symmetry:

$$\psi_{i,j,k} = \sum_{x=-k+2}^{0} \sum_{y=-k+2-x}^{0} \frac{(-1)^{x+y}}{(k-2+x+y)!(-x)!(-y)!} \prod_{m=x+2}^{1} (i-m) \prod_{n=y+2}^{1} (j-n) \prod_{r=-k+1}^{x+y-2} (i+j+\lambda-r) \widetilde{F}(i-x, j-y),$$

where $\widetilde{F}(i,j)$ is as in (A.11).This expression gives us the coefficients $\psi_{i,j,k}$ with two indices positive and one index negative.

	$\psi_{i,j,-3}$	$\psi_{i,j,-4}$	$\psi_{i,j,-5}$	$\psi_{i,j,-6}$
$F(i-4,j)$	0	0	0	$\frac{1}{24}(i-2)(i-1)i(i+1)$
$F(i-3,j)$	0	0	$\frac{1}{6}(i)(i-1)(i+1)$	$-\frac{1}{6}(i)(i-1)(i+1)(i+j-5-\lambda)$
$F(i-2,j)$	0	$\frac{1}{2}(i)(i+1)$	$-\frac{1}{2}(i)(i+1)(i+j-4-\lambda)$	$\frac{1}{4}(i)(i+1)(i+j-5-\lambda)(i+j-4-\lambda)$
$F(i-1,j)$	$(i+1)$	$-(i+1)(i+j-3-\lambda)$	$\frac{1}{2}(i+1)(i+j-4-\lambda)(i+j-3-\lambda)$	$-\frac{1}{6}(i+1)(i+j-5-\lambda)(i+j-4-\lambda)(i+j-3-\lambda)$
$F(i,j)$	$-(i+j-2-\lambda)$	$\frac{1}{2}(i+j-3-\lambda)(i+j-2-\lambda)$	$-\frac{1}{6}(i+j-4-\lambda)(i+j-3-\lambda)(i+j-2-\lambda)$	$\frac{1}{24}(i+j-5-\lambda)(i+j-4-\lambda)(i+j-3-\lambda)(i+j-2-\lambda)$
$F(i-3,j-1)$	0	0	0	$\frac{1}{6}(i)(i-1)(i+1)(j+1)$
$F(i-2,j-1)$	0	0	$\frac{1}{2}(i)(i+1)(j+1)$	$-\frac{1}{2}(i)(i+1)(j+1)(i+j-5-\lambda)$
$F(i-1,j-1)$	0	$(i+1)(j+1)$	$-(i+1)(j+1)(i+j-4-\lambda)$	$\frac{1}{2}(i+1)(j+1)(i+j-5-\lambda)(i+j-4-\lambda)$
$F(i,j-1)$	$(j+1)$	$-(j+1)(i+j-3-\lambda)$	$\frac{1}{2}(j+1)(i+j-4-\lambda)(i+j-3-\lambda)$	$-\frac{1}{6}(j+1)(i+j-5-\lambda)(i+j-4-\lambda)(i+j-3-\lambda)$
$F(i-2,j-2)$	0	0	0	$\frac{1}{4}(i)(i+1)(j+1)(j)$
$F(i-1,j-2)$	0	0	$\frac{1}{2}(i+1)(j+1)(j)$	$-\frac{1}{2}(i+1)(j+1)(j)(i+j-5-\lambda)$
$F(i,j-2)$	0	$\frac{1}{2}(j)(j+1)$	$-\frac{1}{2}(j)(j+1)(i+j-4-\lambda)$	$\frac{1}{4}(j)(j+1)(i+j-5-\lambda)(i+j-4-\lambda)$
$F(i-1,j-3)$	0	0	0	$\frac{1}{6}(j)(j-1)(j+1)(i+1)$
$F(i,j-3)$	0	0	$\frac{1}{6}(j)(j-1)(j+1)$	$-\frac{1}{6}(j)(j-1)(j+1)(i+j-5-\lambda)$
$F(i,j-4)$	0	0	0	$\frac{1}{24}(j-2)(j-1)j(j+1)$

Table A.4: The table gives the coefficients of the functions $F(i,j)$ appearing in the expressions of the terms $\psi_{i,j,k}$ with $k=-3,\dots,-6$. The coefficients are both valid for $i<-2,\ j>2$ and for $i,j<-2$ by considering $F(i,j)$ as in (A.9) and as in (A.12), respectively.

Now we will turn our attention to the generating coefficients of pure type, which appear when solving the recurrence relations for coefficients of the form $\psi_{i,j,k}$ with all indices negative or all indices positive. Here we concentrate on the former, i.e. $\psi_{i,j,k}$ with $i, j, k \leq 0$. The situation with all indices positive is symmetric and should not produce new insights, hence we do not consider it here. In case of the pure type generators, it is harder to guesstimate a solution. We will start by searching a solution for level minus two. The generic solution is then derived from (A.10). The recurrence relation to consider for level minus two is given in (4.138). We will proceed as before and write down the first terms of the recurrence relation in order to guess the solution. However, contrary to before, we cannot leave the one index arbitrary. Both indices need to be fixed. We start with $j = -3$.

$\boxed{j=-3}$

If $j = -3$ in (4.138) there is one generating coefficient appearing given by $\psi_{-\lambda+5,-3,-2}$. For $i > -\lambda+5$, we have $\psi_{i,-3,-2} = 0$. Hence, the interesting cases to consider are given by $i < -\lambda+5$. We have:

$$\begin{aligned}
\psi_{-\lambda+4,-3,-2} &= -(-\lambda+3)\psi_{-\lambda+5,-3,-2}\,,\\
\psi_{-\lambda+3,-3,-2} &= \frac{1}{2}(-\lambda+2)(-\lambda+3)\psi_{-\lambda+5,-3,-2}\,,\\
\psi_{-\lambda+2,-3,-2} &= -\frac{1}{6}(-\lambda+1)(-\lambda+2)(-\lambda+3)\psi_{-\lambda+5,-3,-2}\,,\\
\psi_{-\lambda+1,-3,-2} &= \frac{1}{24}(-\lambda)(-\lambda+1)(-\lambda+2)(-\lambda+3)\psi_{-\lambda+5,-3,-2}\,,\\
\psi_{-\lambda,-3,-2} &= -\frac{1}{120}(-\lambda-1)(-\lambda+1)(-\lambda+2)(-\lambda+3)\psi_{-\lambda+5,-3,-2}\,.
\end{aligned}$$

Next, let us continue with $j = -4$.

$\boxed{j=-4}$

We have: $\psi_{i,-4,-2} = 0$ for $i > -\lambda+6$. The interesting cases to consider are thus given by $i < -\lambda+6$. We obtain two generating coefficients, namely $\psi_{-\lambda+5,-3,-2}$ and $\psi_{-\lambda+6,-4,-2}$:

i	$\psi_{-\lambda+6,-4,-2}$	$\psi_{-\lambda+5,-3,-2}$
$-\lambda+5$	$(\lambda-4)$	5
$-\lambda+4$	$\frac{1}{2}(\lambda-4)(\lambda-3)$	$5(\lambda-3)$
$-\lambda+3$	$\frac{1}{6}(\lambda-4)(\lambda-3)(\lambda-2)$	$\frac{5}{2}(\lambda-3)(\lambda-2)$
$-\lambda+2$	$\frac{1}{24}(\lambda-4)(\lambda-3)(\lambda-2)(\lambda-1)$	$\frac{5}{6}(\lambda-3)(\lambda-2)(\lambda-1)$

$\boxed{j=-5}$

The interesting cases to consider are given by $i < -\lambda+7$. We obtain three generating coefficients, namely $\psi_{-\lambda+5,-3,-2}$, $\psi_{-\lambda+6,-4,-2}$ and $\psi_{-\lambda+7,-5,-2}$:

i	$\psi_{-\lambda+7,-5,-2}$	$\psi_{-\lambda+6,-4,-2}$	$\psi_{-\lambda+5,-3,-2}$
$-\lambda+6$	$(\lambda-5)$	6	0
$-\lambda+5$	$\frac{1}{2}(\lambda-5)(\lambda-4)$	$6(\lambda-4)$	15
$-\lambda+4$	$\frac{1}{6}(\lambda-5)(\lambda-4)(\lambda-3)$	$3(\lambda-4)(\lambda-3)$	$15(\lambda-3)$
$-\lambda+3$	$\frac{1}{24}(\lambda-5)(\ldots)(\lambda-2)$	$(\lambda-4)(\lambda-3)(\lambda-2)$	$\frac{15}{2}(\lambda-3)(\lambda-2)$
$-\lambda+2$	$\frac{1}{120}(\lambda-5)(\ldots)(\lambda-1)$	$\frac{1}{4}(\lambda-4)(\ldots)(\lambda-1)$	$\frac{5}{2}(\lambda-3)(\lambda-2)(\lambda-1)$
$-\lambda+1$	$\frac{1}{720}(\lambda-5)(\ldots)\lambda$	$\frac{1}{20}(\lambda-4)(\ldots)\lambda$	$\frac{5}{8}(\lambda-3)(\lambda-2)(\lambda-1)\lambda$

Obviously, the lower j is, the more generating coefficients appear. However, for fixed λ, the number of generating coefficients of pure type is finite, due to the alternating property. For example, for $\lambda = 9$, the generator $\psi_{-\lambda+7,-5,-2}$ above would be zero, and the generators $\psi_{-\lambda+6,-4,-2}$

and $\psi_{-\lambda+5,-3,-2}$ would be identical. However, for now, we will not take this into account, but consider λ very big, and solve the recurrence relations in that case. Later on in *Mathematica*, a manual cut-off can be implemented into the solution very easily.

$j=-6$

The interesting cases to consider are given by $i<-\lambda+8$. We obtain:

i	$\psi_{-\lambda+8,-6,-2}$	$\psi_{-\lambda+7,-5,-2}$	$\psi_{-\lambda+6,-4,-2}$	$\psi_{-\lambda+5,-3,-2}$
$-\lambda+7$	$(\lambda-6)$	7	0	0
$-\lambda+6$	$\frac{1}{2}(\lambda-6)(\lambda-5)$	$7(\lambda-5)$	21	0
$-\lambda+5$	$\frac{1}{6}(\lambda-6)(\lambda-5)(\lambda-4)$	$\frac{7}{2}(\lambda-5)(\lambda-4)$	$21(\lambda-4)$	35
$-\lambda+4$	$\frac{1}{24}(\lambda-6)(\ldots)(\lambda-3)$	$\frac{7}{6}(\lambda-5)(\ldots)(\lambda-3)$	$\frac{21}{2}(\lambda-4)(\lambda-3)$	$35(\lambda-3)$
$-\lambda+3$	$\frac{1}{120}(\lambda-6)(\ldots)(\lambda-2)$	$\frac{7}{24}(\lambda-5)(\ldots)(\lambda-2)$	$\frac{7}{2}(\lambda-4)(\ldots)(\lambda-2)$	$\frac{35}{2}(\lambda-3)(\lambda-2)$

The rule of the factors involving λ is easy to guesstimate. The prefactors are a bit more complicated to guess. Let us write down the prefactors for $j=-7$ and $j=-8$ without the factors involving λ.

$j=-7$

i	$\psi_{-\lambda+9,-7,-2}$	$\psi_{-\lambda+8,-6,-2}$	$\psi_{-\lambda+7,-5,-2}$	$\psi_{-\lambda+6,-4,-2}$	$\psi_{-\lambda+5,-3,-2}$
$-\lambda+8$	$\mathbf{1}=\frac{1}{1!}$	$\mathit{8}$	0	0	0
$-\lambda+7$	$\frac{1}{2}=\frac{1}{2!}$	$8=\frac{8}{1!}$	$\mathbf{28}=\frac{8\cdot7}{2}$	0	0
$-\lambda+6$	$\frac{1}{6}=\frac{1}{3!}$	$4=\frac{8}{2!}$	$28=\frac{28}{1!}$	$\mathbf{56}=\frac{8\cdot7\cdot6}{2\cdot3}$	0
$-\lambda+5$	$\frac{1}{24}=\frac{1}{4!}$	$\frac{4}{3}=\frac{8}{3!}$	$14=\frac{28}{2!}$	$56=\frac{56}{1!}$	$\mathbf{70}=\frac{8\cdot7\cdot6\cdot5}{2\cdot3\cdot4}$

$j=-8$

i	$\psi_{-\lambda+10,-8,-2}$	$\psi_{-\lambda+9,-7,-2}$	$\psi_{-\lambda+8,-6,-2}$	$\psi_{-\lambda+7,-5,-2}$	$\psi_{-\lambda+6,-4,-2}$	$\psi_{-\lambda+5,-3,-2}$
$-\lambda+9$	$\mathbf{1}=\frac{1}{1!}$	$\mathit{9}$	0	0	0	0
$-\lambda+8$	$\frac{1}{2}=\frac{1}{2!}$	$9=\frac{9}{1!}$	$\mathbf{36}=\frac{9\cdot8}{2}$	0	0	0
$-\lambda+7$	$\frac{1}{6}=\frac{1}{3!}$	$\frac{9}{2}=\frac{9}{2!}$	$36=\frac{36}{1!}$	$\mathbf{84}=\frac{9\cdot8\cdot7}{2\cdot3}$	0	0
$-\lambda+6$	$\frac{1}{24}=\frac{1}{4!}$	$\frac{3}{2}=\frac{9}{3!}$	$18=\frac{36}{2!}$	$84=\frac{84}{1!}$	$\mathbf{126}=\frac{9\cdot8\cdot7\cdot6}{2\cdot3\cdot4}$	0
$-\lambda+5$	$\frac{1}{24}=\frac{1}{4!}$	$\frac{3}{8}=\frac{9}{4!}$	$6=\frac{36}{3!}$	$42=\frac{84}{2!}$	$126=\frac{126}{1!}$	$\mathbf{126}=\frac{9\cdot8\cdot7\cdot6\cdot5}{2\cdot3\cdot4\cdot5}$

We are now able to guesstimate a solution. If we write the generating coefficients appearing in the tables above under the form $\psi_{-\lambda-j+2-l,j+l,-2}$, the prefactors written in bold and italic are given by $\frac{1}{l!}\prod_{m=2-l}^{1}(-j+m)$. Moving down vertically, these entries are divided by some factorial. All in all, the prefactors are given by $\frac{1}{l!}\frac{1}{(-i-\lambda-j+2-l)!}\prod_{m=2-l}^{1}(-j+m)$. Including the factors with λ, we can guesstimate the final solution for coefficients $\psi_{i,j,-2}$ with $i,j<-2$, $i<j$, $i+j<-\lambda+2$ and λ big:

$$\psi_{i,j,-2}=\sum_{l=0}^{-\lambda+2-j-i}\frac{1}{l!(-i-\lambda-j+2-l)!}\prod_{m=2-l}^{1}(-j+m)\prod_{n=j+l}^{-(i+\lambda)+1}(\lambda+n)\psi_{-\lambda-j+2-l,j+l,-2}. \tag{A.12}$$

As mentioned before, to find the generic $\psi_{i,j,k}$, with $i,j,k<0$, we can use the formula (A.10) and replace $F(i,j)$ with the coefficients $\psi_{i,j,-2}$ as given in (A.12).

This completes the resolution of the recurrence relations. The solutions were implemented into *Mathematica*. Some properties such as antisymmetry, the fact that there are only finitely many generators of pure type, and the fact that $\psi_{i,j,k}=0$ for $i+j+k>-\lambda$ and $i,j,k<0$, were added manually into the implementation. This rendered the computation of the coefficients $\psi_{i,j,k}$ much faster. However, there is now a new time limit appearing in the resolution of the linear system.

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