On idempotent *n*-ary semigroups

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Summary of the talk

Part I:

Characterizations of

- single-peakedness and other related properties
- the class of totally ordered quasitrivial semigroups
- the class of totally ordered commutative idempotent semigroups

$\rightarrow$ Surprising link between social choice theory and semigroup theory

Enumeration of the class of totally ordered commutative idempotent semigroups

$\rightarrow$ New definition of the Catalan numbers

Summary of the talk

Part II:

Characterizations of

- the class of quasitrivial n-ary semigroups
- hierarchical classes of idempotent n-ary semigroups
- the class of symmetric idempotent n-ary semigroups

$\rightarrow$ Constructive descriptions

Reducibility criteria for symmetric idempotent n-ary semigroups

Part I: Single-peakedness and idempotent semigroups

Order

X: non-empty set

A *preorder* on X is a binary relation $\precsim$ on X that is reflexive and transitive

$\precsim$ total $\Rightarrow$ $\precsim$ is a *weak order*

$\precsim$ antisymmetric $\Rightarrow$ $\precsim$ is a *partial order* and we denote it by $\preceq$

$\preceq$ total $\Rightarrow$ $\preceq$ is a *total order* and we denote it by $\leq$

$(X, \preceq)$ *partially ordered set* (or *poset*)

$(X, \preceq)$ is a called a *semilattice* if every pair $\{x, y\} \subseteq X$ has a supremum $x \curlyvee y$

Ideal

$(X, \precsim)$ preordered set

I nonempty subset of $(X, \precsim)$

I is an *ideal* if it is a directed lower set, i.e., if

- $\forall\, x \in X$ and $\forall\, y \in I$ such that $x \precsim y$ we have $x \in I$
- $\forall\, y, z \in I \; \exists\, u \in I$ such that $y \precsim u$ and $z \precsim u$

In a poset $(X, \preceq)$ we define $[a, b]_{\preceq} = \{x \in X \colon a \preceq x \preceq b\}$

A subset C of $(X, \preceq)$ is *convex (for $\preceq$)* if it contains $[a, b]_{\preceq}$ $\forall\, a \preceq b \in C$

Single-peakedness

Definition. (Black, 1948)

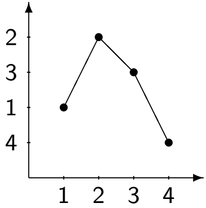
$\leq'$ is *single-peaked for* $\leq$ if $\forall\, a, b, c \in X$,

$$a < b < c \quad \Rightarrow \quad b <' a \quad \text{or} \quad b <' c$$

Example. On $X = \{1, 2, 3, 4\}$ consider $\leq$ and $\leq'$ defined by

$$1 < 2 < 3 < 4 \qquad \text{and} \qquad 2 <' 3 <' 1 <' 4$$

$$\mathrm{id}\colon X \to X$$



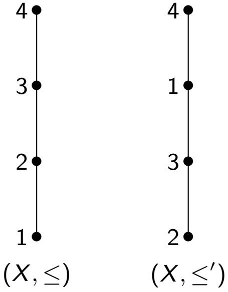
Single-peakedness

Proposition

The following assertions are equivalent.

(i) $\leq'$ is single-peaked for $\leq$

(ii) Every ideal of $(X, \leq')$ is a convex subset of $(X, \leq)$



$\leq'$ is single-peaked for $\leq$

Generalizations

Two generalizations of single-peakedness for weak orders:

- *single-plateauedness* (Black, 1987)
- existential single-peakedness (Fitzsimmons, 2015)

Single-plateauedness

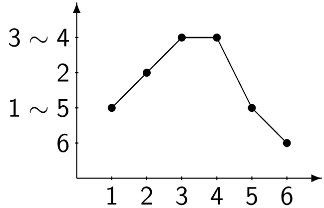
$\precsim$ weak order on X

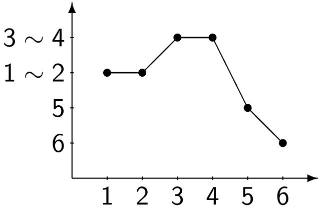
Definition.

$\precsim$ is *single-plateaued for* $\leq$ if $\forall\, a, b, c \in X$,

$$a < b < c \quad \Rightarrow \quad b \prec a \quad \text{or} \quad b \prec c \quad \text{or} \quad a \sim b \sim c$$

Examples. On $X = \{1 < 2 < 3 < 4 < 5 < 6\}$





Quasitriviality

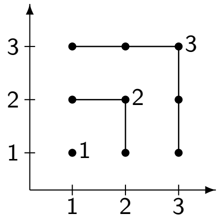
(X, G) groupoid : X nonempty set and $G\colon X^2 \to X$

G associative $\Rightarrow$ (X, G) semigroup

$G\colon X^2 \to X$ is said to be *quasitrivial* (or *conservative*) if

$$G(x, y) \in \{x, y\} \qquad x, y \in X$$

Example. $G = \max_{\leq}$ on $X = \{1, 2, 3\}$ endowed with the usual $\leq$



Ordinal sum

$(Y, \leq)$ totally ordered set

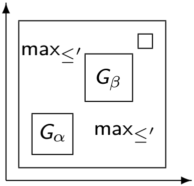
$\{(X_\alpha, G_\alpha) \colon \alpha \in Y\}$ set of semigroups such that

$$X_\alpha \cap X_\beta = \varnothing, \qquad \alpha \neq \beta$$

Definition. (Clifford, 1954)
(X, G) is an *ordinal sum of semigroups* (X_α, G_α) if

- $X = \bigcup_{\alpha \in Y} X_\alpha$
- $G|_{X_\alpha^2} = G_\alpha \ \forall \alpha \in Y$
- $\forall (x, y) \in X_\alpha \times X_\beta$ such that $\alpha < \beta$, we have $G(x, y) = G(y, x) = y$

$\rightarrow (X, G)$ is a semigroup



Projections

The *projection operations* $\pi_1 \colon X^2 \to X$ and $\pi_2 \colon X^2 \to X$ are respectively defined by

$$\begin{aligned} \pi_1(x,y) &= x, \qquad x,y \in X \\ \pi_2(x,y) &= y, \qquad x,y \in X \end{aligned}$$

(X, π_1): *left zero semigroup*

(X, π_2): *right zero semigroup*

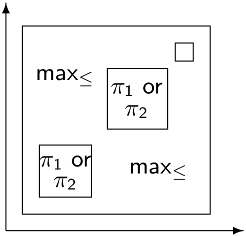
(X, G) is a *singular band* if $G \in \{\pi_1, \pi_2\}$

Quasitrivial semigroups

Theorem (Länger, 1980)

G is associative and quasitrivial if and only if

$$\exists \precsim \ : G|_{A\times B} = \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \quad \forall A, B \in X/\sim$$



Corollary

G is associative, quasitrivial, and commutative $\Leftrightarrow$ $\exists \leq \ : G = \max_{\leq}$

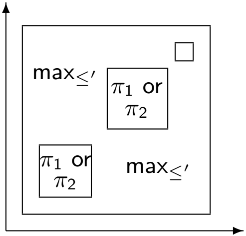
Totally ordered semigroups

$G: X^2 \to X$ is *$\leq$-preserving* for some $\leq$ on X if

$$\left.\begin{array}{l} x \leq x' \\ y \leq y' \end{array}\right\} \quad \Rightarrow \quad G(x,y) \leq G(x',y')$$

$\to$ (X, G) totally ordered

Totally ordered quasitrivial semigroups

(∗) 

single-plateauedness : $a < b < c \quad \Rightarrow \quad b \prec a$ or $b \prec c$ or $a \sim b \sim c$

Proposition

Let $\leq$ on X and $G\colon X^2 \to X$. The following assertions are equivalent.

(i) G is associative, quasitrivial, and $\leq$-preserving

(ii) $\exists \precsim$: G is of the form $(\ast)$ and $\precsim$ is single-plateaued for $\leq$

Moreover, if G is commutative, then (i)-(ii) are equivalent to

(iii) $\exists \leq'$: G is of the form $(\ast)$ and $\leq'$ is single-peaked for $\leq$

Semilattices

$G\colon X^2 \to X$ is *idempotent* if $G(x,x) = x \ \forall x \in X$

Proposition (folklore)

(X, G) commutative idempotent semigroup $\Leftrightarrow \ \exists \preceq$ on X such that $G = \curlyvee$

$$(X, \preceq) \quad \Leftrightarrow \quad (X, \curlyvee)$$

Definition.

$\preceq$ has the *convex-ideal property* (*CI-property* for short) *for* $\leq$ if for all $a, b, c \in X$,

$$a < b < c \quad \Rightarrow \quad b \preceq a \curlyvee c$$

Definition.

$\preceq$ is *internal for* $\leq$ if for all $a, b, c \in X$,

$$a < b < c \quad \Rightarrow \quad (a \neq b \curlyvee c \quad \text{and} \quad c \neq a \curlyvee b)$$

Order-preserving semilattice operations

Proposition

Let $\leq$ on X and $G\colon X^2 \to X$. The following assertions are equivalent.

(i) G is associative, idempotent, commutative, and $\leq$-preserving

(ii) $\exists \preceq \ : G = \curlyvee$ and $\preceq$ satisfies

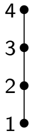
  (a) CI-property for $\leq$

  (b) internality for $\leq$

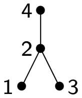
(iii) $\exists \preceq \ : G = \curlyvee$ and $\preceq$ satisfies

  (a) Every ideal of $(X, \preceq)$ is a convex subset of $(X, \leq)$

  (b) $\forall\, a \leq b \in X$, we have $a \curlyvee b \in [a, b]_{\leq}$



$(X, \leq)$



$(X, \preceq)$

Enumeration of order-preserving semilattice operations

Assume that $X = \{1, \ldots, n\}$, is endowed with the usual $\leq$ defined by

$$1 < \ldots < n$$

$$\{\leq \text{-preserving semilattice operations on X}\}$$
$$\updownarrow$$
$$\{\text{ordered rooted binary trees with } n \text{ vertices}\}$$

Proposition

The number of $\leq$-preserving semilattice operations on X is the nth Catalan number.

Part II: Idempotent n-ary semigroups

n-ary semigroups

(X, F) n-ary groupoid : X nonempty set and $F\colon X^n \to X$

Definition. (Dörnte, 1928)
$F\colon X^n \to X$ is *associative* if

$$\begin{aligned} & F(x_1, \ldots, x_{i-1}, F(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{2n-1}) \\ = \quad & F(x_1, \ldots, x_i, F(x_{i+1}, \ldots, x_{i+n}), x_{i+n+1}, \ldots, x_{2n-1}) \end{aligned}$$

for all $x_1, \ldots, x_{2n-1} \in X$ and all $1 \leq i \leq n-1$.

$\Longrightarrow$ (X, F) is an *n-ary semigroup*

$F\colon X^n \to X$ is

- *quasitrivial* if $F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$ $\forall\, x_1, \ldots, x_n \in X$
- *idempotent* if $F(x, \ldots, x) = x$ $\forall\, x \in X$
- *symmetric* if F is invariant under the action of permutations

n-ary extensions

(X, G) semigroup

Define a sequence $(G^m)_{m\geq 1}$ of $(m+1)$-ary operation inductively by the following rules : $G^1 = G$ and

$$G^m(x_1, \ldots, x_{m+1}) \;=\; G^{m-1}(x_1, \ldots, x_{m-1}, G(x_m, x_{m+1})), \qquad m \geq 2.$$

Setting $F = G^{n-1}$, the pair (X, F) is the *n-ary extension* of (X, G).

G is a *binary reduction* of F

$\rightarrow (X, F)$ is an n-ary semigroup

Not every n-ary semigroup is obtained like this

Example.

$(\mathbb{R}, F)$ where $F\colon \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$F(x, y, z) = x - y + z, \qquad x, y, z \in \mathbb{R}$$

Quasitrivial n-ary semigroups

Combining a result of Ackerman (2011) with a result of Dudek and Mukhin (2006) we conclude the following result.

Theorem

Every quasitrivial n-ary semigroup is the n-ary extension of a semigroup

But the binary reduction is in general neither quasitrivial nor unique.

Example

$$F(x,y,z) = x+y+z \pmod 2$$

$$G(x,y) = x+y \pmod 2 \qquad G'(x,y) = x+y+1 \pmod 2$$

$G'\colon Y^2 \to Y$ is said to be *conjugate* to $G\colon X^2 \to X$ if $\exists\, \varphi\colon X \to Y$ bijection such that

$$\varphi(G(x,y)) = G'(\varphi(x),\varphi(y)), \qquad x,y \in X$$

Neutral elements

$e \in X$ is said to be a *neutral element for F* if

$$F(x, e, \ldots, e) = F(e, x, e, \ldots, e) = \cdots = F(e, \ldots, e, x) = x, \qquad x \in X$$

E_F : set of neutral elements of F

Example. $F(x, y, z) = x + y + z \pmod 2$

If $E_F \neq \varnothing$, then

$$E_F \longleftrightarrow \{\text{binary reductions of } F\}$$

Proposition

If $F\colon X^n \to X$ is associative and quasitrivial, then $|E_F| \leq 2$

Quasitrivial n-ary semigroups

Theorem

Let $F\colon X^n \to X$ be associative and quasitrivial. The following assertions are equivalent.

(i) $|E_F| \leq 1$

(ii) The associative and quasitrivial $G\colon X^2 \to X$ defined by

$$G(x,y) \quad = \quad F(x,\ldots,x,y) \quad = \quad F(x,y,\ldots,y), \qquad x,y \in X$$

is the unique binary reduction of F

Quasitrivial n-ary semigroups

Theorem

Let $F\colon X^n \to X$ be associative and quasitrivial and let $e_1 \neq e_2 \in X$. The following assertions are equivalent.

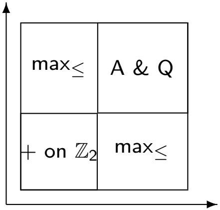
(i) $E_F = \{e_1, e_2\}$

(ii) n is odd and the associative $G_{e_1}, G_{e_2}\colon X^2 \to X$ defined by

$$G_{e_1}(x,y) = F(x, e_1, \ldots, e_1, y) \text{ and } G_{e_2}(x,y) = F(x, e_2, \ldots, e_2, y), \quad x, y \in X$$

are the only binary reductions of F

Moreover, if any of the assertions $(i) - (ii)$ is satisfied, then $G_{e_1} \neq G_{e_2}$ and neither of them is quasitrivial



Towards idempotent n-ary semigroups

$\forall k \in \{1, \ldots, n\}$, let

$$D_k^n = \bigcup_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| \geq k}} \{(x_1, \ldots, x_n) \in X^n : \forall i, j \in S,\ x_i = x_j\}$$

$\mathcal{F}_k^n$: class of associative n-ary operations $F\colon X^n \to X$ that satisfy

$$F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}, \qquad (x_1, \ldots, x_n) \in D_k^n$$

$\to \mathcal{F}_1^n$: class of associative and quasitrivial n-ary operations $F\colon X^n \to X$

$\to \mathcal{F}_n^n$: class of associative and idempotent n-ary operations $F\colon X^n \to X$

$$\mathcal{F}_1^n \subseteq \mathcal{F}_2^n \subseteq \cdots \subseteq \mathcal{F}_n^n$$

Proposition

$$\mathcal{F}_1^n = \mathcal{F}_2^n = \cdots = \mathcal{F}_{n-2}^n \subseteq \mathcal{F}_{n-1}^n \subseteq \mathcal{F}_n^n, \qquad n \geq 3$$

Towards idempotent n-ary semigroups

A group $(X, *)$ with neutral element e has *bounded exponent* if $\exists\, m \geq 1$ such that

$$\underbrace{x * \cdots * x}_{m \text{ times}} \quad = \quad e, \qquad x \in X$$

An element $z \in X$ is an *annihilator* of $F\colon X^n \to X$ if

$$F(x_1, \ldots, x_n) \quad = \quad z, \qquad \text{whenever} \quad z \in \{x_1, \ldots, x_n\}$$

Towards idempotent n-ary semigroups

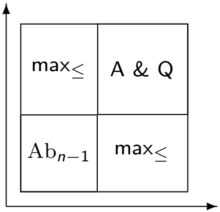
Definition

$\mathcal{H}$: class of operations $G\colon X^2 \to X$ such that $\exists Y \subseteq X$ with $|Y| \geq 3$ for which

(a) $(Y, G|_{Y^2})$ is an Abelian group whose exponent divides $n-1$

(b) $G|_{(X\setminus Y)^2}$ is associative and quasitrivial

(c) Any $x \in X \setminus Y$ is an annihilator for $G|_{(\{x\}\cup Y)^2}$



Theorem

$$F \in \mathcal{F}^n_{n-1} \setminus \mathcal{F}^n_1 \quad \Leftrightarrow \quad F \text{ is reducible to } G \in \mathcal{H}$$

Towards idempotent n-ary semigroups

Theorem

$$F \in \mathcal{F}^n_{n-1} \setminus \mathcal{F}^n_1 \quad \Leftrightarrow \quad F \text{ is reducible to } G \in \mathcal{H}$$

Sketch of the proof

$\Leftarrow$: Check the axioms

$\Rightarrow$: Follows from the following steps

(1) $\forall\, x_1, \ldots, x_n \in X\colon\ F(x_1, \ldots, x_n) \notin \{x_1, \ldots, x_n\} \Rightarrow x_1, \ldots, x_n, F(x_1, \ldots, x_n) \in E_F$

(2) Step (1) $\Rightarrow F = G_e^{n-1}\ \forall\, e \in E_F$

(3) Check that G_e inherits the good properties from F

Symmetric idempotent n-ary semigroups

A *symmetric n-ary band* is a symmetric idempotent n-ary semigroup

Examples

- $(X, \curlyvee)$ semillatice $\Rightarrow$ $(X, \curlyvee^{n-1})$ symmetric n-ary band
- $(X, *)$ Abelian group whose exponent divides $n-1$
  $\Rightarrow$ $(X, *^{n-1})$ symmetric n-ary band

$$*^{n-1}(x, \ldots, x) \quad = \quad \underbrace{x * \cdots * x}_{=e} * x \quad = \quad x$$

n-ary semilattices of n-ary semigroups

Extend the concept of semilattices of semigroups (Clifford, Yamada,...) to n-ary semigroups

$(Y, \curlyvee^{n-1})$ n-ary semilattice

$\{(X_\alpha, F_\alpha)\colon \alpha \in Y\}$ set of n-ary semigroups such that

$$X_\alpha \cap X_\beta = \varnothing, \qquad \alpha \neq \beta$$

Definition
(X, F) is an *n-ary semilattice $(Y, \curlyvee^{n-1})$ of n-ary semigroups (X_α, F_α)* if

- $X = \bigcup_{\alpha \in Y} X_\alpha$
- $F|_{X_\alpha^n} = F_\alpha \; \forall \alpha \in Y$
- $\forall (x_1, \ldots, x_n) \in X_{\alpha_1} \times \cdots \times X_{\alpha_n}$

$$F(x_1, \ldots, x_n) \in X_{\alpha_1 \curlyvee \cdots \curlyvee \alpha_n}$$

We write $(X, F) = ((Y, \curlyvee^{n-1}), (X_\alpha, F_\alpha))$

Not an n-ary semigroup in general!

Strong n-ary semilattices of n-ary semigroups

Extend the concept of strong semilattices of semigroups (Kimura,. . .) to n-ary semigroups

Definition

Let $(X, F) = ((Y, \curlyvee^{n-1}), (X_\alpha, F_\alpha))$. Suppose that $\forall\, \alpha \preceq \beta \in Y$
$\exists\, \varphi_{\alpha,\beta} \colon X_\alpha \to X_\beta$ homomorphism such that

(a) $\varphi_{\alpha,\alpha}$ is the identity on X_α

(b) $\forall\, \alpha \preceq \beta \preceq \gamma \in Y$ we have $\varphi_{\beta,\gamma} \circ \varphi_{\alpha,\beta} = \varphi_{\alpha,\gamma}$

(c) $\forall\, (x_1, \ldots, x_n) \in X_{\alpha_1} \times \cdots \times X_{\alpha_n}$ we have

$$F(x_1, \ldots, x_n) = F_{\alpha_1 \curlyvee \cdots \curlyvee \alpha_n}(\varphi_{\alpha_1, \alpha_1 \curlyvee \cdots \curlyvee \alpha_n}(x_1), \ldots, \varphi_{\alpha_n, \alpha_1 \curlyvee \cdots \curlyvee \alpha_n}(x_n))$$

(X, F) is a *strong n-ary semilattice of n-ary semigroups*

We write $(X, F) = ((Y, \curlyvee^{n-1}), (X_\alpha, F_\alpha), \varphi_{\alpha,\beta})$

Strong n-ary semilattices of n-ary semigroups

Proposition

Every strong n-ary semilattice of n-ary semigroups is an n-ary semigroup

Description of symmetric n-ary bands

Theorem

The following assertions are equivalent.

(i) (X, F) is a symmetric n-ary band

(ii) (X, F) is a strong n-ary semilattice of n-ary extensions of Abelian groups whose exponents divide $n-1$

Sketch of the proof

(ii) $\Rightarrow$ (i): Check the axioms

(i) $\Rightarrow$ (ii): Follows from the following steps

(1) $\forall\, x \in X$, define $\ell_x \colon X \to X$ by $\ell_x(y) = F(x, \ldots, x, y)\ \forall\, y \in X$

(2) The binary relation $\sim$ on X defined by

$$x \sim y \quad \Leftrightarrow \quad \ell_x = \ell_y, \qquad x, y \in X,$$

is a congruence on (X, F) such that $(X/{\sim}, \tilde{F})$ is an n-ary semilattice

(3) $\forall\, x \in X$, $([x]_\sim, F|_{[x]^n_\sim})$ is the n-ary extension of an Abelian group whose exponent divides $n-1$

(4) $\forall\, [x]_\sim \preceq_{\tilde{F}} [y]_\sim \in X/{\sim}$, the map $\ell_y|_{[x]_\sim} \colon [x]_\sim \to [y]_\sim$ is a homomorphism

(5) Check that $(X, F) = ((X/{\sim}, \tilde{F}), ([x]_\sim, F|_{[x]^n_\sim}), \ell_y|_{[x]_\sim})$

Reducibility of symmetric n-ary bands

Theorem

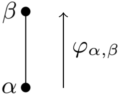
$(X, F) = ((Y, \curlyvee^{n-1}), (X_\alpha, F_\alpha), \varphi_{\alpha,\beta})$ is the n-ary extension of a semigroup (X, G) if and only if $\exists\, e \colon Y \to X$ such that

(a) $\forall\, \alpha \in Y$, $e(\alpha) = e_\alpha \in X_\alpha$

(b) $\forall\, \alpha \preceq \beta \in Y$, we have $\varphi_{\alpha,\beta}(e_\alpha) = e_\beta$

Moreover, $(X, G) = ((Y, \curlyvee), (X_\alpha, G_\alpha), \varphi_{\alpha,\beta})$ where $G_\alpha^{n-1} = F_\alpha \ \forall \alpha \in Y$

Example of symmetric ternary band



$\alpha \curlyvee \beta = \beta$

$X = \{1, 2, 3\}$, $X_\alpha = \{1\}$, $X_\beta = \{2, 3\}$

(X_α, F_α) ternary extension of the trivial group on $\{1\}$

(X_β, F_β) isomorphic to the ternary extension of $(\mathbb{Z}_2, +)$

$\varphi_{\alpha,\beta} \colon X_\alpha \to X_\beta$ defined by $\varphi_{\alpha,\beta}(1) = 2$

$\varphi_{\alpha,\alpha} = \mathrm{id}|_{X_\alpha} \colon X_\alpha \to X_\alpha$

$\varphi_{\beta,\beta} = \mathrm{id}|_{X_\beta} \colon X_\beta \to X_\beta$

Example of a symmetric ternary band

$F\colon X^3 \to X$ defined by

- $F|_{\{1\}^3} = F_\alpha$, $F|_{\{2,3\}^3} = F_\beta$
- $F(1,1,2) = F_\beta(\varphi_{\alpha,\beta}(1), \varphi_{\alpha,\beta}(1), \varphi_{\beta,\beta}(2)) = F_\beta(2,2,2) = 2 = F(1,2,1) = F(2,1,1)$
- $F(1,1,3) = F_\beta(\varphi_{\alpha,\beta}(1), \varphi_{\alpha,\beta}(1), \varphi_{\beta,\beta}(3)) = F_\beta(2,2,3) = 3 = F(1,3,1) = F(3,1,1)$
- $F(1,2,2) = F(2,1,2) = F(2,2,1) = 2$
- $F(1,3,3) = F(3,1,3) = F(3,3,1) = 2$
- $F(1,2,3) = F(1,3,2) = F(2,1,3) = F(3,1,2) = F(2,3,1) = F(3,2,1) = 2$

$\to$ (X, F) is a strong ternary semilattice of ternary extensions of Abelian groups whose exponents divide 2

Reducible to a semigroup (take $e\colon \{\alpha, \beta\} \to X$ defined by $e(\alpha) = 1$ and $e(\beta) = 2$)

Top 5 results

1. Characterizations of classes of totally ordered idempotent semigroups by means of single-peakedness
2. nth Catalan number $=$ number of totally ordered commutative idempotent semigroups on $\{1, \ldots, n\}$
3. Characterization of the class of quasitrivial n-ary semigroups by means of binary reductions that are constructed in terms of ordinal sums
4. Characterization of the class of symmetric idempotent n-ary semigroups by means of strong n-ary semilattices
5. Necessary and sufficient condition for any symmetric idempotent n-ary semigroup to be reducible to a semigroup

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