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ESSAYS ON MONETARY ECONOMICS AND ASSET PRICING

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Abstract

This dissertation consists of three chapters based on three applied theory papers, which all use microfoundations to study mechanisms behind asset prices in the context of monetary policy and financial stability.

Market Fragility and the Paradox of the Recent Stock-Bond Dissonance. The objective of this study is to jointly explain stock prices and bond prices. After the Lehman-Brothers collapse, the stock index has exceeded its pre-Lehman-Brothers peak by 36% in real terms. Seemingly, markets have been demanding more stocks instead of bonds. Yet, instead of observing higher bond rates, paradoxically, bond rates have been persistently negative after the Lehman-Brothers collapse. To explain this paradox, we suggest that, in the post-Lehman-Brothers period, investors changed their perceptions on disasters, thinking that disasters occur once every 30 years on average, instead of disasters occurring once every 60 years. In our asset-pricing calibration exercise, this rise in perceived market fragility alone can explain the drop in both bond rates and price-dividend ratios observed after the Lehman-Brothers collapse, which indicates that markets mostly demanded bonds instead of stocks.

Time-Consistent Welfare-Maximizing Monetary Rules. The objective of this study is to jointly explain capital prices, bond prices and money supply/demand. We analyze monetary policy from the perspective that a Central Bank conducts monetary policy serving the ultimate goal of maximizing social welfare, as dictated by a country's constitution. Given recent empirical findings that many households are hand-to-mouth, we study time-consistent welfare-maximizing monetary-policy rules within a neoclassical framework of a cash-in-advance economy with a liquidity-constrained good. The Central Bank performs open-market operations buying government bonds in order to respond to fiscal shocks

and to productivity shocks. We formulate the optimal policy as a dynamic Stackelberg game between the Central Bank and private markets. A key goal of optimal monetary policy is to improve the mixture between liquidity constrained and non-liquidity constrained goods. Optimal monetary responses to fiscal shocks aim at stabilizing aggregate consumption fluctuations, while optimal monetary responses to productivity shocks allow aggregate consumption fluctuations to be more volatile.

Jump Shocks, Endogenous Investment Leverage and Asset Prices: Analytical Results. The objective of this study is to jointly model leveraging and stock prices in an environment with rare stock-market disaster shocks. Financial intermediaries invest in the stock market using household savings. This investment leveraging, and its extent, affects stock price movements and, in turn, stock-price movements affect investment leveraging. If the price mechanism is unable to absorb a rare stock-market disaster, then with leverage ratios of 20 or more, financial intermediaries can go bankrupt. We model the interplay between leverage ratios and stock prices in an environment with rare stock-market disaster shocks. First we introduce dividend shocks that follow a Poisson jump process to an endowment economy with pure exchange between two types of agents: (i) shareholders of financial intermediaries that invest in the stock market ("experts"), and (ii) savers, who deposit their savings to financial intermediaries (households). Under the assumption that the households and the so called "experts" both have logarithmic utility, we obtain a closed-form solution for the endowment economy. This closed-form solution serves as a guide for numerically solving the model with recursive Epstein-Zin preferences in continuous-time settings. In our extension we introduce production based on capital investments, but with adjustment costs for investment changes. Jump shocks directly hit the productive capital stock, but the way they influence stock returns of productive firms passes through the leveraging channel, which is endogenous. The production economy also has endogenous growth, and investment adjustment costs partly influence the model's stability properties. Importantly, risk has an endogenous component due to leveraging, and this endogenous-risk component influences growth opportunities, bridging endogenous cycles with endogenous growth. This chapter is part of a broader project on financial stability. Future extensions will include an evaluation of the Basel II-III regulatory framework in order to assess their effectiveness and their impact on growth performance.

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INTRODUCTION

This dissertation consists of three chapters based on three applied theory papers, which all use microfoundations to study mechanisms behind asset prices in the context of monetary policy and financial stability. For the design of optimal monetary policy and macroprudential policy, it is essential to understand the mechanisms behind investment decisions in all asset markets. The following three chapters intend to shed light on the underlying mechanisms.

The first chapter is titled "Market fragility and the paradox of the recent stock-bond dissonance". It is joint work with Christos Koulovatianos and Jian Li. It appeared as a Center for Financial Studies (CFS) Working Paper, No. 589 in 2017 and a shorter version of this chapter has been published in *Economics Letters*, Volume 162, in January 2018. We ask whether, based on current market observations, stock and bond markets are dissonant. Asset pricing theory teaches us that when markets demand more stocks, asset prices increase and bond prices decrease (bond rate increases) and vice versa, when markets demand more bonds, bond prices increase (bond rate decreases) and asset prices decrease. Currently, we observe that in the post-Lehman-Brothers period, the real US stock market index has grown above and beyond its pre-Lehman-Brothers peak at an annual rate exceeding real-GDP growth. On the other hand, in the same period, we observe that the real 6-month bond rate has decreased significantly. These two observations seem as a stock-bond dissonance according to standard asset pricing theory. To explain this paradox of the persistent growth of stock prices and the persistent drop in bond rates, we use the Lucas (1978) asset pricing model with rare disasters, as in Barro (2006). The seeming stock-bond dissonance can be explained by a post-crisis shift in investors' expectations. Specifically, an increase in investors' perceived frequency of a rare-disaster shock can explain the paradox. Because shifts in expectations are unobservable, we have employed a calibrated asset-pricing model that addresses the impact of expectations on observed asset prices. Our research urges to not misinterpret seemingly good market trends as market robustness at times of underlying market fragility.

The second chapter is titled "Time-consistent welfare-maximizing monetary rules" and is joint work with Christos Koulovatianos. We are interested in how a Central Bank should optimally respond to fiscal and productivity shocks, taking into account that households face liquidity constraints. Empirical research shows that large fractions of households are hand-to-mouth, hence they face liquidity constrains. As consumption choices are influenced by bond markets and Central Banks do open-market operations, how should the presence of liquidity-constrained households affect monetary policy? In order to answer this question, we employ a cash-in-advance model with two types of consumption goods: a credit-constrained and a non-credit constrained good. The Central Bank performs open-market operations in order to manage money supply. Its objective is to maximize social welfare. To derive the optimal time-consistent monetary policy, we use concepts and algorithms from the literature on optimal time-consistent fiscal policy. We find that optimal time-consistent monetary policy has real effects along the transition to the steady state from a shock. Along the transition, monetary policy improves the distorted ratio between the two consumption-good types. Fiscal shocks are offset and productivity shocks are accommodated by optimal time-consistent monetary policy.

The third chapter is titled "Jump shocks, endogenous investment leveraging and asset prices: analytical results" and is part of a bigger project with Christos Koulovatianos and Jian Li. The question of the broader project is, how capital requirements can influence balance sheets of banks and asset prices in general? One could also rephrase the question to, how does leverage impact investments into the real economy and what is the optimal level of leverage for financial intermediaries? An answer to this could shed light on the mechanics of financial stability and the design of optimal macroprudential policy. Insights about the optimal leverage ratio provided by a model can guide the design of capital requirements. A key question is whether financial intermediaries stay too close to the boundary of optimal capital requirements. For example, banks may stay above that boundary foreseeing future requirements in order to avoid future pressures to rebalance.

This bigger project is motivated by the 2008 financial crisis, which revealed the need to better understand the source of financial instability. Reinhart and Rogoff (2009) documented that "we have been there before". Similar persistent declines and long-lasting changes in asset prices, output or investment were observed after financial crisis, which motivates the need to make the financial system more resilient.

In order to achieve this, one needs to develop analytical results, that can serve as guides for calibration and computation. The third chapter takes this step, focusing on theory and offering some key analytical results on this class of models of financial stability with disaster risk. A crucial gap in the literature is the understanding of asset allocation in a world of rare disaster shocks. The goal is to jointly analyze both the leverage ratio of intermediaries and asset prices, in a micro-founded continuous-time framework with endogenous growth and exogenous jump shocks, in order to study how the anticipation of rare disasters affects the mechanism behind financial stability.

In all three chapters we use microfoundations and focus on asset pricing. The optimal design of monetary and macroprudential policy relies on understanding the mechanisms behind asset pricing, behind investments in all asset markets. The driver of asset pricing is utility maximization of all agents. Endogenous asset pricing makes allocations of wealth in the economy sensitive to future-earnings expectations. In the first chapter we jointly explain stock prices and bond prices. We show how in an extended Lucas' asset pricing tree model (Lucas (1978)) small changes in expectations, triggered by extreme-tail events, impact asset allocations of economic agents and therefore asset pricing. In the second chapter we model the Central Bank as a benevolent social planner, who maximizes social welfare by recognizing the ultimate means through which households achieve utility (consumption and leisure). The Central Bank makes open market operations, which has an impact on money, capital price and bond price in our model economy. In the third chapter, asset prices are driven through utility maximization of unproductive households and intermediaries which help channelling resources from unproductive households to productive investments. Those intermediaries use household savings to invest in the stock market. We model leveraging and stock prices in an environment with rare stock-market disaster shocks, both in an endowment economy and a production economy with logarithmic utility and Epstein-Zin preferences.

1. CHAPTER

Market Fragility and the Paradox of the Recent Stock-Bond Dissonance

1.1 Introduction

Since the first oil crisis of 1973, the US stock exchange has been marked by two major setback episodes of its *aggregate dividend index*: the dot-com bust and the Lehman-Brothers collapse (see Figure 1.1). These two disaster episodes mark two subperiods, as depicted by Figure 1.1: the pre- and post-Lehman-Brothers regimes.

In the post-Lehman-Brothers period (subperiod 2), by the end of July 2017, the stock market has grown above and beyond its pre-Lehman-Brothers peak by more than 36% in real terms (58% in nominal terms). On the other hand, we see that the real 6-month bond rate has decreased significantly from a mean bond rate of 0.33% in subperiod 1 to -1.29% in subperiod 2. This high increase in stock prices together with persistently negative bond rates after the Lehman-Brothers collapse, looks like a stock-bond dissonance according to standard asset pricing theory. We further observe that the price-dividend ratio has significantly fallen from subperiod 1 to subperiod 2 from 60.24 to 49.26 as depicted in Table 1.1. These two empirical observations are essential for explaining the current stock-bond dissonance. Later in the subsection 1.4.1 these are our four targets which we want to match.

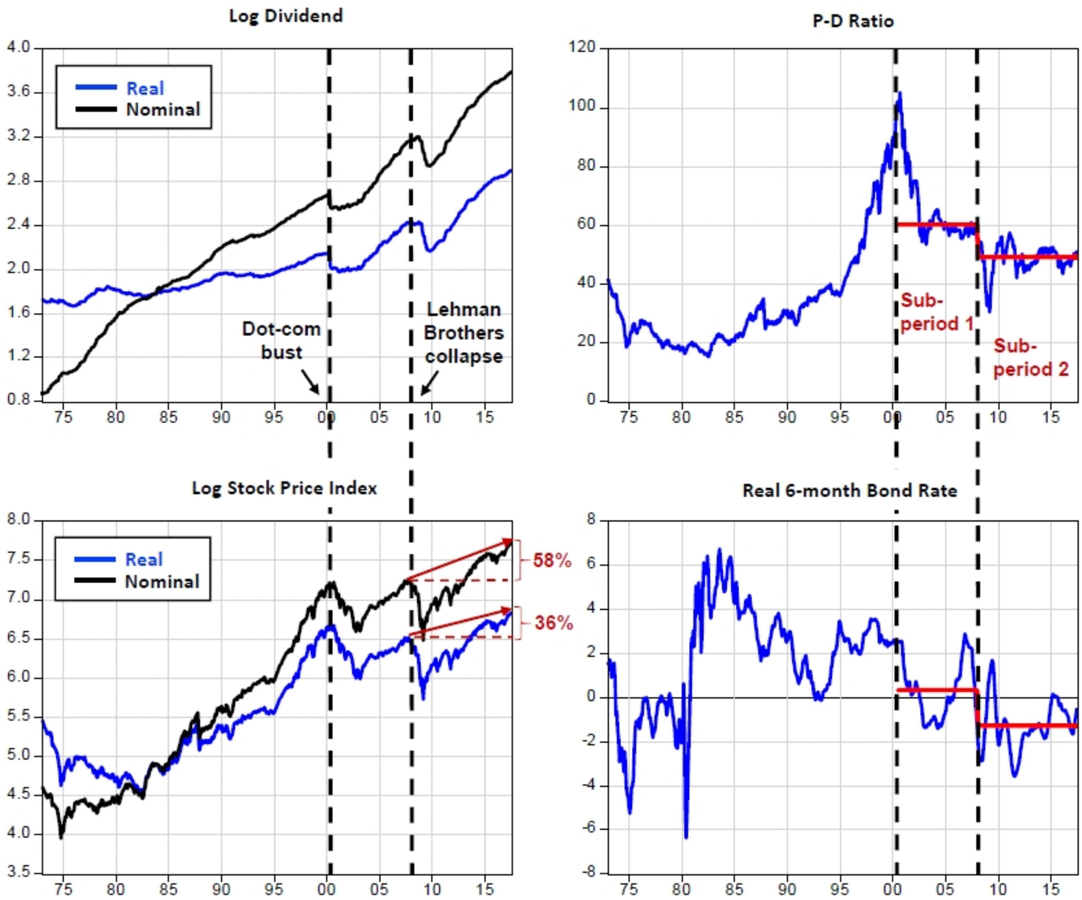

Figure 1.1 - US Data. Flat lines are statistics reported in Table 1.1. Sources: Datastream (TOTMKUS) and Board of Governors of the Federal Reserve System (US), 6-Month Treasury Bill: Secondary Market Rate (TB6MS).

According to standard asset-pricing theory, it is reasonable to think that investors have rebalanced their portfolios, demanding more stocks instead of bonds. If this was true, then the bond price would have fallen, leading to an increase in the bond rate. However, in the post-Lehman-Brothers period, the bond rate has significantly decreased and persistently stayed in a negative regime. *This stock-bond dissonance, (a) the persistent drop in bond rates, and (b) the persistent growth of stock prices, is a paradox.*

subperiods	1	2
	2000/7–2008/1	**2008/2–2017/7**
mean real interest rate	0.33% (1.32%)	−1.29% (1.12%) [b]
median P-D ratio [a]	60.24 (3.05)	49.26 (2.31) [c]

Table 1.1 - Descriptive statistics. Bond rate, and P-D-ratio statistics that appear in Figure 1.1.

[a] Medians are reported when normality tests fail. Standard errors are reported in parentheses for means and median absolute deviations for medians.

[b] Difference-of-means t-test for difference from previous subperiod's statistic is 9.53 (p-value is 0).

[c] Wilcoxon signed-ranks test for difference from previous subperiod's median is 12.13 (p-value is 0).

To explain this stock-bond dissonance we use the Lucas (1978) asset pricing model with rare disasters as in Barro (2006). We focus on two key observations in the data, which are the significant drop in the bond rate and the significant drop in the price-dividend ratio, which we try to match with the model. We show that a change in investors' expectations about the frequency of rare disasters can explain the observed dissonance. More precisely, after the Lehman-Brothers collapse, the investors have the perception of higher market fragility, i.e. higher disaster risk hitting the real economy, such as a sudden drop in dividends. In our calibration exercise we show that without changing the investors' preference parameters, nor market fundamentals, in both subperiods, the model can match the data only by allowing for an increase in investors' perceived frequency of a rare disaster.

Our model builds upon investors' increasing fear for more frequent market disruptions. As in Barro (2006) a rare disaster can be any low-probability event that triggers a sharp drop in per capita GDP or consumption. An economic disaster can be triggered by economic events that affect the business sector and specifically the aggregate dividend index (Great Depression in 1929, the 2008-2009 Global Financial Crisis), by natural disasters, or by wartime destruction (World War I, World War II, nuclear conflicts). As in the asset-pricing literature with rare disaster risks, e.g., Barro (2006, 2009), Gabaix (2012), Gourio (2012), and Wachter (2013), we assume that rare disasters are exogenous events. Although bonds are not a perfect hedge against disaster risks, investors substitute bonds for stocks in case of higher market fragility.

According to our model, an explanation to the paradox is based on investors' perceptions about disaster risk: investors think that disasters occur once every 30 years on average compared to once every 60 years on average before the Lehman-Brothers collapse. Our argument is based on the fact that there is no decrease in the dividend growth rate nor an increase in the dividend volatility (outside crashes) observed after the Lehman-Brothers collapse. So, without changing perceptions about disaster risk (market fragility), the drop in the price-dividend ratio or the drop in the bond rate cannot be explained.

Our sensitivity analysis supports our market-fragility explanation. Being aware that disaster risk is considered to be "dark matter", in our sensitivity analysis, we use a range of initial disaster probabilities from 1.7% to 2.5% in subperiod 1 and doubling the frequency for subperiod 2. Our model still performs relatively well which reconfirms our working hypothesis of an increase in market fragility.

We also calibrate our model allowing for the possibility of a partial default in government bonds, making them not completely risk-free. We show that the sovereign-default risk is less important quantitatively to explain the observed stock-bond dissonance. Indeed sovereign-default risk raises the government bond rate as markets require a default premium. Hence, it is market fragility alone that can explain the persistently negative government bond rates.

Our market-fragility explanation is in line with a number of studies focusing on rare disaster risks in asset pricing. First, an influential body of literature suggests that disaster risk is variable. More specifically we refer to Gabaix (2012), Gourio (2012), and Wachter (2013), who demonstrate that this variability can explain many asset-pricing puzzles. In addition, Marfe and Penasse (2017) find empirical evidence for disaster-risk variability. Another body of literature assumes imperfect information about rare disaster risk and argues that parameter learning implies more pessimistic disaster-risk beliefs after a rare disaster (Collin-Dufresne et al., 2016, Koulovatianos and Wieland, 2017, and Kozlowski et al., 2017). All these studies agree that after the Lehman-Brothers collapse, beliefs about rare disaster risk should be more pessimistic, backing up the working hypothesis examined in this paper. Yet, for the sake of simplicity, here we employ only rational expectations and an unexpected post-disaster structural break.

Due to challenges in observing disaster risk, John Campbell in his 2008 Princeton Finance lectures called disaster risk the "dark matter for economists". But ever since much progress has been made regarding disaster-risk estimation and its role in calibration. Chen, Joslin and Tran (2012) demonstrate that small changes in the distribution of heterogeneous beliefs can have substantial impact on the aggregate-market implications of disaster risk. Chen, Dou and Kogan (2017) also stress that disaster risk is difficult to infer, and offer a comprehensive robustness measure for estimating asset-pricing models with disaster risk.

1.2 More empirical details on the paradox

To describe the aggregate dividend process, depicted in Figure 1.1, we assume that, as in Barro (2006, 2009), dividends, D_t, follow the process,

$$\ln\left(D_{t+1}\right) = \mu - \frac{\sigma^2}{2} + \ln\left(D_t\right) + \sigma\varepsilon_{t+1} + \nu_{t+1}\ln\left(1-\zeta_{t+1}\right) \ , \tag{1.1}$$

in which the random term $\varepsilon_{t+1} \sim N(0,1)$, is i.i.d. normal with mean 0 and variance 1. The random term, ν_{t+1},

$$\nu_{t+1} = \begin{cases} 1 \quad , \quad \text{with Prob. } \lambda \\ 0 \quad , \quad \text{with Prob. } 1-\lambda \end{cases} , \tag{1.2}$$

introduced low-probability, rare disasters to the dividend process, i.e., with probability $\lambda \in (0,1)$ dividends are hit by a negative rare disaster shock of size ζ_{t+1}. Variable $\zeta_{t+1} \in (0,1)$ is a random variable with given time-invariant distribution and compact support, $\mathcal{Z} \subset (0,1)$.

An interesting feature of the post-Lehman-Brothers stock prices is that the P-D ratio has fallen significantly (see Figure 1.1 and Table 1.1). For explaining the persistent drop in the P-D ratio, it would be reasonable to focus on changes in fundamentals. The three parameters involved in equations (1.1) and (1.2) are μ, σ, and λ. We first examine whether the transition from subperiod 1 to subperiod 2 has been marked by any changes in the trend, μ, and in the component of volatility that is not related to disasters, namely parameter σ. *Interestingly, neither μ, nor σ have changed across subperiods 1 and 2.*

1.2.1 What remained constant across subperiods 1 and 2: the dividend trend and the non-crash dividend volatility

In order to see that neither μ, nor σ changed across subperiods 1 and 2, we first need to obtain an estimate for σ, the non-disaster-shock dividend volatility, by excluding crash-episode periods. The criterion for determining non-crash periods is explained by Figure 1.2.

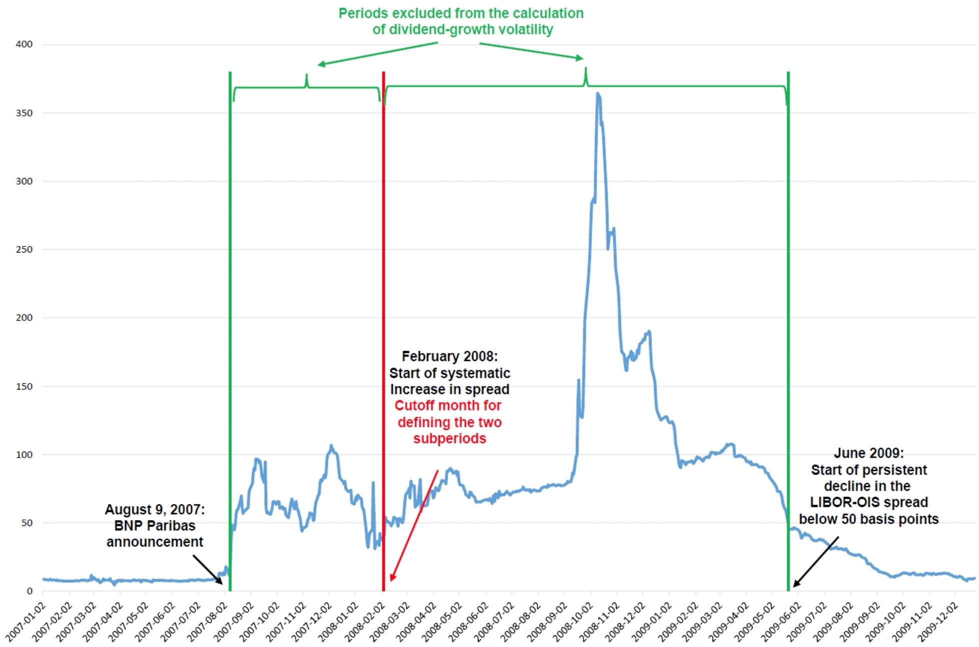

Figure 1.2 - Three-month LIBOR-OIS Spread (US daily data). Source for London Interbank Offered Rate (LIBOR): Federal Reserve Bank of St. Louis, Economic Research Division. Source for Overnight Indexed Swap (OIS): Datastream.

Figure 1.2 depicts the spread between the 3-month London Interbank Offered Rate (LIBOR) and the 3-month Overnight Indexed Swap (OIS). According to Thornton (2009, p. 1), the LIBOR-OIS spread is "*a measure of the health of banks because it reflects what banks believe is the risk of default associated with lending to other banks.*" This interpretation of the LIBOR-OIS spread, and the overall pattern revealed by Figure 1.2, motivate that a systematic rise of the LIBOR-OIS spread above 50 basis points indicates times of problems in the banking sector. The first green vertical line indicates the date at which the LIBOR-OIS spread suddenly increased beyond the 50-basis-points threshold. That date was August 9, 2007, when BNP Paribas, France's largest bank, announced that it would halt redemptions on three investment funds (see St. Louis Fed, 2007). Certainly, this date marked the start of a broader period of uneasiness regarding the solvency of the banking sector. Nevertheless, the 50-basis-points threshold of the LIBOR-OIS spread has been exceeded systematically only since February 2008. Therefore, February 2008, marked by the red vertical line in Figure 1.2, is the cutoff month separating subperiod 1 (the pre-subprime-crisis phase / Lehman-Brothers collapse) and subperiod 2 (the post-subprime-crisis phase / Lehman-Brothers collapse) in our sample.

Table 1.2 presents statistics regarding the average dividend growth rate (means and medians), and also a measure of variability of the dividend growth rate, but focusing on the non-disaster-shock dividend volatility across subperiods 1 and 2.

subperiods	**1**	**2**
	2000/7–2008/1	**2008/2–2017/7**
mean dividend growth rate	5.59 [a] (12.02)	4.89 [b] (17.27)
median dividend growth rate	4.88 (6.23)	6.66 (5.67)
standard deviation of dividend growth rate (excluding crash period)	12.22	11.57

Table 1.2 - Descriptive statistics of dividends in real terms appearing in Figure 1.1. All numbers are percentages. Standard errors are reported in parentheses for means, and median absolute deviations are reported for medians.

[a] Normality test does not fail (Jarque-Bera test statistic is 3.19 with p-value 20.3%).

[b] Normality test fails (Jarque-Bera test statistic is 564.43 with p-value 0%).

Regarding measures of the average dividend growth rate, in Table 1.2 Jarque-Bera test statistics are reported, testing normality of distributions. While the dividend growth rate does not fail a normality test in subperiod 1, normality is rejected in subperiod 2. For this reason, a test of equality of means across subperiods 1 and 2 is not appropriate. Instead, in Table 1.3 we report a number of equality tests for the medians of the dividend growth rate across subperiods 1 and 2.

Method	df	Value	Probability
Wilcoxon/Mann-Whitney		0.849519	0.3956
Wilcoxon/Mann-Whitney (tie-adj.)		0.849519	0.3956
Med. Chi-square	1	1.446670	0.2291
Adj. Med. Chi-square	1	1.128270	0.2881
Kruskal-Wallis	1	0.723697	0.3949
Kruskal-Wallis (tie-adj.)	1	0.723697	0.3949
van der Waerden	1	0.293416	0.5880

Category Statistics

Variable	Count	Median	>Overall Median	Mean Rank	Mean Score
GR_D_1	91	4.881276	41	99.05495	-0.041503
GR_D_2	114	6.663943	61	106.1491	0.033129
All	205	6.315151	102	103.0000	1.13E-16

Table 1.3 - Tests of equality of medians of dividend growth across subperiods 1 and 2. Variable "GR_D_1" is the dividend growth rate in subperiod 1 and Variable "GR_D_2" is the dividend growth rate in subperiod 2.

Since according to all median tests reported in Table 1.3 the null hypothesis of equality between the two medians cannot be rejected, *in our calibration of the model below, we use the total-sample median spanning the two subperiods, from July 2000 until July 2017 (205 months in total) which is equal to 6.32% (see also the last line in Table 1.3).*

Regarding the measure of non-disaster dividend volatility (volatility of dividend growth excluding crash periods), for subperiod 1 we include the months from 2000/7 until 2007/7 and for subperiod 2 we include the months from 2009/6 until 2017/7.

Method	df	Value	Probability
F-test	$(97, 84)$	1.117252	0.6040
Siegel-Tukey		1.282917	0.1995
Bartlett	1	0.275745	0.5995
Levene	$(1, 181)$	0.971887	0.3255
Brown-Forsythe	$(1, 181)$	0.954303	0.3299

Category Statistics

Variable	Count	Std. Dev.	Mean Abs. Mean Diff.	Mean Abs. Median Diff.	Mean Tukey-Siegel Rank
GRD1_NO	85	12.22474	8.994378	8.981747	86.60000
GRD2_NO	98	11.56549	7.768607	7.765153	96.68367
All	183	11.90850	8.337954	8.330238	92.00000

Bartlett weighted standard deviation: 11.87599

Table 1.4 - Tests of equality of variances of dividend growth across subperiods 1 and 2, excluding crash periods. Variable "GRD1_NO" is the dividend growth rate in subperiod 1 for which crash periods are excluded, and variable "GRD2_NO" is the dividend growth rate in subperiod 2, for which crash periods are excluded as well.

Based on Table 1.4, the standard deviations for these no-disaster subperiods are 12.22% and 11.57%, but all tests cannot reject the null hypothesis that these two standard deviations are equal. So, throughout the rest of the paper, in the calibration below, we use the total-sample standard deviation, spanning the two subperiods, from July 2000 until July 2017

(183 months in total, after excluding the period from August 2007 until May 2009), which is equal to 11.91% (see also the penultimate line in Table 1.4).

1.2.2 What changed across subperiods 1 and 2: perceptions about disaster risk

While neither μ, nor σ changed across subperiods 1 and 2, parameter λ, seems to have changed. Our working hypothesis is that there has been a pessimistic shift in rare-disaster beliefs about parameter λ after the Lehman-Brothers collapse, i.e. λ has increased. This working hypothesis is corroborated by an increase in the "SKEW" index, depicted by Figure 1.3.

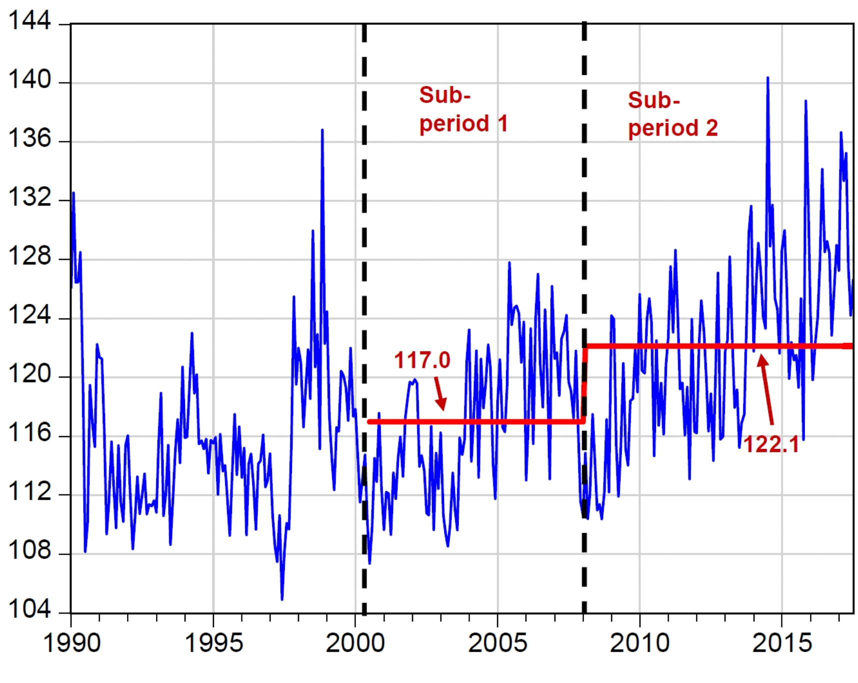

Figure 1.3 - The SKEW index, US monthly data. Source: Chicago Board Options Exchange.

The SKEW index partially reveals the investors' beliefs on market fragility, despite that it is not a perfect proxy for the rare disaster risk hitting the dividend index. Figure 1.3 plots the Chicago Board Options Exchange (CBOE) Skew index, commonly known as "SKEW". According to Chicago Board Options Exchange CBOE (2010), the SKEW is an indicator based on options, measuring the perceived tail risk of the distribution of Standard and Poor's (S&P) 500 log returns at a 30-day horizon. The SKEW measures tail risk, and specifically the risk related to an increase in the probability of extreme negative outlier returns, two or more standard deviations below the mean. Details on the formal definition of SKEW are provided by the Chicago Board Options Exchange CBOE (2010, p. 5).

The main point made by Figure 1.3 is that the mean level of the SKEW index has increased in subperiod 2. Interestingly, the SKEW index is well approximated by a normal distribution in both subperiods (the Jarque-Bera statistic is 3.71, implying a p-value of 0.16 in subperiod 1, while for subperiod 2 the Jarque-Bera statistic is 2.51, implying a p-value of 0.29).

subperiods	**1**	**2**
	2000/7–2008/1	**2008/2–2017/7**
mean SKEW	117.0 (4.99)	122.1 (6.28)[a]

Table 1.5 - Descriptive statistics of the SKEW index appearing in Figure 1.3.

Standard errors in parentheses.

[a] Difference-of-means t-test for difference from previous subperiod's statistic is -6.37 (p-value is 0).

Table 1.5 presents a formal statistical test revealing that the mean SKEW has increased significantly in subperiod 2. This evidence supports our working hypothesis that, after the Lehman-Brothers collapse, beliefs about rare disasters have become more pessimistic. The risk interpretation of the changes reported by Figure 1.3 and Table 1.5 is given by the Chicago

Board Options Exchange CBOE (2010, p. 8). Specifically, the estimated risk-adjusted probability that the S&P 500 may experience a sudden drop of two standard deviations in the next 30 days has increased from 6.89% in subperiod 1 to 8.27% in subperiod 2 on average. Similarly, the estimated risk-adjusted probability of a sudden drop of three standard deviations has increased from 1.16% in subperiod 1 to 1.46% in subperiod 2 on average. Although these estimates are not a perfect proxy of the risk of a rare disaster hitting the dividend index, they reveal that beliefs about tail risks and market fragility have been elevated after the Lehman-Brothers collapse. In the rest of the paper *we use an asset-pricing model in order to investigate whether a change in λ alone across subperiods 1 and 2 is capable of replicating key asset-pricing features summarized by Table 1.1*, *and we obtain a model-based sense of the increase in parameter λ.*

1.3 Model

In this section we present our model with disaster risk. We follow the classic Lucas-tree setup (Lucas, 1978). There is a risky asset, the stock composite index (the market portfolio), and a one-year zero-coupon bond. Our stylized asset-pricing model that uses i.i.d. disaster shocks hitting the dividend process, and summarized by equations (1.1) and (1.2), implies a flat term structure on bond rates in equilibrium, so there is no need to introduce bonds with different maturity. The one-year zero-coupon bond is not entirely risk-free. In the case of a rare disaster hitting the dividend process, the probability of a partial default on government bonds exists. So we do not only have market fragility in our model but also a sovereign-default risk.

The budget constraint of an investor is,

$$\underbrace{S_{t-1}D_t}_{\substack{\shortparallel \\ \text{Income}}} = \underbrace{P_t\left(S_t - S_{t-1}\right)}_{\substack{\shortparallel \\ \text{Investment in Stocks}}} + \underbrace{Q_t B_t - (1-\delta)^{\nu_t \nu_t^B} B_{t-1}}_{\substack{\shortparallel \\ \text{Investment in Bonds}}} + \underbrace{C_t}_{\substack{\shortparallel \\ \text{Consumption}}} , \tag{1.3}$$

in which S_{t-1} and B_{t-1} is the number of stocks and bonds held by the investor in the beginning of period t, while P_t and Q_t are the stock and bond prices in period t. The term $(1-\delta)^{\nu_t \nu_t^B}$ multiplying B_{t-1} in (1.3) states that if there is no dividend disaster ($v_t = 0$), then the zero-coupon bond pays 1 unit of the consumable good at the maturity date; in periods that a dividend disaster occurs ($v_t = 1$), then a probabilistic sovereign-default process is triggered, governed by ν_t^B,

$$\nu_t^B = \begin{cases} 1 & , \quad \text{with Prob. } \pi \\ 0 & , \quad \text{with Prob. } 1-\pi \end{cases} , \tag{1.4}$$

with $\pi \in [0, 1)$. If both a dividend-disaster and a default occur ($\nu_t = \nu_t^B = 1$), then the zero-coupon bond pays $1 - \delta$ units of the consumable good, i.e., it defaults by the fraction $\delta \in [0, 1]$.[1] Variables ε_{t+1}, ν_{t+1}, ν_{t+1}^B, and ζ_{t+1} are independent among each other and also independent and identically distributed (i.i.d.) over time.

Preferences are recursive, of the form of Epstein-Zin-Weil (EZW), with utility in period t, denoted by J_t, given by the recursion,

$$J_t = \left\{ (1-\beta)\, C_t^{1-\frac{1}{\eta}} + \beta \left[E_t\left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1-\frac{1}{\eta}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\eta}}} , \tag{1.5}$$

in which $\eta > 0$ is the intertemporal elasticity of substitution (IES), $\gamma > 0$ is the coefficient of relative risk aversion, and $\beta \in (0, 1)$ is the utility discount factor that is inversely related to the rate of time preference, $\rho = (1-\beta)/\beta$.

[1] The concept of sovereign default follows Barro (2006, p. 836) who observes that in periods of rare market disasters the probability of a sovereign default increases. We thank an anonymous referee for raising this point.

1.3.1 Asset prices

Equation (1.1) implies that dividend growth is random, following i.i.d. shocks over time. In Appendices 1.6.A through 1.6.C we prove that these i.i.d. shocks imply a constant price-dividend (P-D) ratio over time, denoted by x,

$$\frac{P_t}{D_t} = x = \frac{\omega}{1-\omega} \quad \text{with} \quad \omega = \beta e^{\left(1-\frac{1}{\eta}\right)\left(\mu-\gamma\frac{\sigma^2}{2}\right)} \left(1-\lambda\xi\right)^{\frac{1-\frac{1}{\eta}}{1-\gamma}} \ , \ \ t = 0, 1, ..., \tag{1.6}$$

in which $\xi = 1 - E_\zeta\left[(1-\zeta)^{1-\gamma}\right]$, with $E_\zeta(\cdot)$ denoting expectation with respect to variable ζ only. The expected bond rate, denoted by r^B, is

$$E\left(r^B\right) = \frac{1}{\beta} e^{\frac{1}{\eta}\mu-\gamma\left(1+\frac{1}{\eta}\right)\frac{\sigma^2}{2}} \frac{\left(1-\lambda\xi\right)^{\frac{\frac{1}{\eta}-\gamma}{1-\gamma}}\left(1-\lambda\pi\delta\right)}{1-\lambda\left\{1-E_\zeta\left[(1-\zeta)^{-\gamma}\right](1-\pi\delta)\right\}} - 1 \ , \ \ t = 0, 1, ... \tag{1.7}$$

1.3.2 Empirical implications and tests of the model

The flat P-D ratio implied by equation (1.6) is not a bad approximation of the P-D ratio dynamics in both subperiods 1 and 2. As Figure 1.1 indicates, after the P-D ratio overreactions to the disaster episodes calmed down, P-D ratios remained almost constant throughout subperiods 1 and 2, but at different levels.

subperiods	**1**	**2**
	2000/7–2008/1	**2008/2–2017/7**
estimator α_1 in equation (1.9)	1.05 (0.37) [a]	0.88 (0.07) [a]
ADF statistic for unit root of $\ln(P_t)$	−2.09 [b]	−0.52 [b]
ADF statistic for unit root of $\ln(D_t)$	1.69 [b]	1.39 [b]

Table 1.6 - Cointegration coefficients (standard errors in parentheses) and ADF unit-root tests. [a] Max eigenvalue test indicates one cointegrating equation at the 5% level. [b] ADF test cannot reject a unit root (1% critical value is -3.50, 5% critical value is -2.89).

For empirical evidence on the validity of (1.6), notice that another way of writing (1.6) is,

$$\ln(P_t) = \ln(x) + \ln(D_t) \ . \tag{1.8}$$

In Table 1.6 we report estimates of α_1 in the cointegrating equation,

$$\ln(P_t) = \alpha_0 + \alpha_1 \ln(D_t) + u_t \ , \tag{1.9}$$

whenever cointegration is applicable, based on Augmented Dickey-Fuller (ADF) unit-root tests.

Table 1.6 provides evidence that, in subperiods 1 and 2, $\ln(P_t)$ and $\ln(D_t)$ are both integrated of order 1, and that the estimates of α_1 do not differ much from 1.[2] In subperiod 1, coefficient α_1 is not significantly different from 1. Although in subperiod 2 coefficient α_1 differs from 1, the value of 0.88 supports that equation (1.8) is not a bad big-picture approximation of financial markets in the US after the Lehman-Brothers collapse. This evidence validates using the Barro (2006, 2009) model for asset-pricing purposes during subperiods 1 and 2, despite that these subperiods have relatively short length of about 8 years each.

1.4 Calibration

1.4.1 Benchmark calibration and key targets

We summarize our calibrating parameter values in Table 1.7, focusing on the benchmark case of no sovereign fragility ($\pi = \delta = 0$).

[2] ADF tests showing that $\ln(P_t)$ and $\ln(D_t)$ are not integrated of order 2 or above can be provided by the authors upon request.

subperiods	**1**	**2**
	2000/7–2008/1	**2008/2–2017/7**
$\boldsymbol{\eta}$	1.85	same
$\boldsymbol{\rho}$	2.91%	same
$\boldsymbol{\gamma}$	3.92	same
$\boldsymbol{\mu}$	6.32% (data)	same
$\boldsymbol{\sigma}$	11.91% (data)	same
$\boldsymbol{\alpha}$	7.08 (data)	same
$\boldsymbol{\lambda}$	1.7% (benchmark)	**3.5%**

Table 1.7 - Calibrating parameter values. $E_{\zeta}(\zeta) = 23.28\%$.

Parameter α is a newly introduced parameter. It is based on the finding by Barro and Jin (2011) that, after transforming disaster sizes using the formula $z = 1/(1-\zeta)$, empirically, variable z is Pareto distributed with density $f(z) = \alpha z_0^{\alpha}/z^{\alpha+1}$, in which z_0 is the minimum value of z. Our chosen value for ζ_0 (obeying $z_0 = 1/(1-\zeta_0)$, the minimum cutoff disaster size) is 12.5%. We independently estimate α, and choose calibrating values of α from the 95% confidence interval of its estimated value.[3] For estimating α, we use the database by Barro and Jin (2011) that refers to GDP disasters, which is downloadable from,

https://scholar.harvard.edu/barro/publications/size-distribution-macroeconomic-disasters-data

Barro and Jin (2011) present a sensitivity analysis of all their results considering that the lower bound for the disasters, ζ_0, ranges from 9.5% to 14.5%, while Barro (2006, 2009)

[3] We use this estimated Pareto distribution in order to compute all expressions involving the expectation $E_{\zeta}(\cdot)$. Barro and Jin (2011) demonstrate that the goodness of fit to disaster-size data increases if one uses two Pareto distributions, each being effective for a different interval of the support of z. Yet, a single Pareto distribution also gives a good approximation, so we use this for simplicity.

also work with $\zeta_0 = 15\%$. We pick a value somewhere in the middle of this range, setting $\zeta_0 = 12.5\%$, which leaves 110 disasters out of 157 in the Barro and Jin (2011) sample. Using the maximum-likelihood estimation of the "shape" parameter α in the Pareto distribution, we obtain an estimate for α equal to 5.986 (standard error 0.57), and a 95% confidence interval implying that $\hat{\alpha} \in [4.87, 7.11]$.[4]

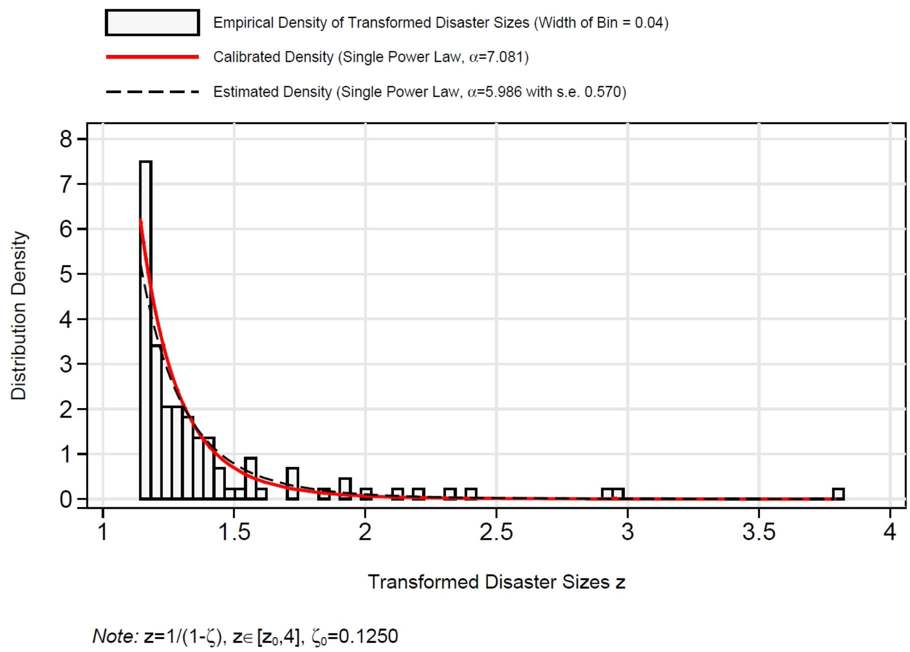

Figure 1.4 - Goodness of fit of transformed disaster-size data above the 12.5% threshold. Source: Barro and Jin (2011).

In our calibration exercise we compute all expectations involving ζ, using $E_\zeta(\cdot)$ based on a Pareto distribution for the transformed variable $z = 1/(1-\zeta)$ with a calibrating parameter

[4] Specifically, we use the package 'PARETOFIT', a module to fit a Type 1 Pareto distribution by Stephen P. Jenkins, which is implementable using Stata and downloadable from,

http://fmwww.bc.edu/RePEc/bocode/p

α^*, taken from this 95% confidence interval, i.e., $\alpha^* \in [4.87, 7.11]$. As Figure 1.4 reveals, our calibrating value, $\alpha^* = 7.081$ fits the disaster data very well, doing approximately the same good job in fitting the disaster distribution as the point estimate $\hat{\alpha} = 5.986$.

We have four targets: r_1^B, r_2^B, $(P/D)_1$, and $(P/D)_2$, denoting the bond rate and the P-D ratio in the two subperiods. Parameter values μ, σ, and α are directly inferred from data. We use parameter $\lambda_1 = 1.7\%$ as a benchmark value. Then our calibration exercise is to match the four targets using four parameter values, the three preference parameters, η, ρ, and γ, which are constant across the two subperiods, and also λ_2, which is the fourth parameter value.

	$\mathbf{r}^B$	$\mathbf{r}^B$	**P-D ratio**	**P-D ratio**
subperiods	**Model**	**Data**	**Model**	**Data**
2000/7–2008/1	0.33%	0.33%	60.61	60.24
2008/2–2017/7	−1.29%	−1.29%	48.34	49.26

Table 1.8 - Model vs. data. Case with no sovereign fragility: $\pi = \delta = 0$.

The key element of our calibration exercise is that, in subperiod 2, after the Lehman-Brothers collapse, the disaster-risk parameter, λ, has more than doubled, reflecting that disasters occur in slightly less than 30 years ($1/3.5\% \simeq 29$) on average.[5] *In Table 1.8 we can see that this simple modification in perceived market fragility is capable of replicating the persistent changes in the bond rate and the P-D ratio that occurred after the Lehman-Brothers collapse.*

[5] Barro's (2009) benchmark suggests disasters occurring once every 60 years ($1/1.7\% = 59$). This change is consistent with models of rational learning about disaster risk implying that perceived disaster risk increases after a disaster episode and then remains high for a long period afterwards (see Koulovatianos and Wieland, 2017).

1.4.2 Sensitivity analysis: varying the disaster probability in the first subperiod

In this subsubsection we provide a sensitivity analysis using a different benchmark for λ_1: instead of fixing λ_1 to 1.7%. We vary its values in a range from 1.5% to 2.5%, doubling λ_2 in each calibration exercise. The key message here is that none of the matching preference parameters η, ρ, and γ, change drastically.

		Benchmark	Sensitivity analysis changing λ										
								Parameters					
λ_1		0.017	0.015	0.016	0.017	0.018	0.019	0.020	0.021	0.022	0.023	0.024	0.025
λ_2		0.035	0.030	0.032	0.034	0.036	0.038	0.040	0.042	0.044	0.046	0.048	0.050
ρ		0.029	0.026	0.028	0.027	0.028	0.028	0.028	0.029	0.030	0.032	0.030	0.033
γ		3.92	4.06	3.98	3.93	3.88	3.82	3.77	3.73	3.70	3.68	3.58	3.60
η		1.85	1.44	1.68	1.59	1.69	1.77	1.84	1.95	2.08	2.29	2.16	2.55
α		7.08	7.08	7.08	7.08	7.08	7.08	7.08	7.08	7.08	7.08	7.08	7.08
	Data							Model					
r_1	**0.33**	0.33	0.33	0.33	0.33	0.33	0.33	0.33	0.33	0.33	0.33	0.33	0.33
r_2	**-1.29**	-1.29	-1.29	-1.23	-1.29	-1.29	-1.29	-1.29	-1.29	-1.30	-1.29	-1.29	-1.29
$PDratio_1$	**60.24**	60.61	55.99	59.01	59.50	60.22	61.56	62.69	63.03	62.23	60.39	66.93	60.35
$PDratio_2$	**49.26**	48.34	49.26	49.32	49.99	49.32	49.26	49.08	48.13	46.54	44.11	47.93	42.37
$E(\zeta)$		0.233	0.233	0.233	0.233	0.233	0.233	0.233	0.233	0.233	0.233	0.233	0.233

Table 1.9 - Sensitivity analysis examining the impact of changing λ_1.

We perform a sensitivity analysis focusing on changing the disaster probability parameter λ. Compared to the benchmark value of λ_1 at 1.7%, we expand the parameter space ranging from $\lambda_1 = 1.5\%$ to $\lambda_1 = 2.5\%$. Keeping all parameters inferred from data constant, namely μ, σ, and α, in each calibration exercise we gradually increase λ_1 by 0.1% percentage points, doubling λ_2 at the same time.

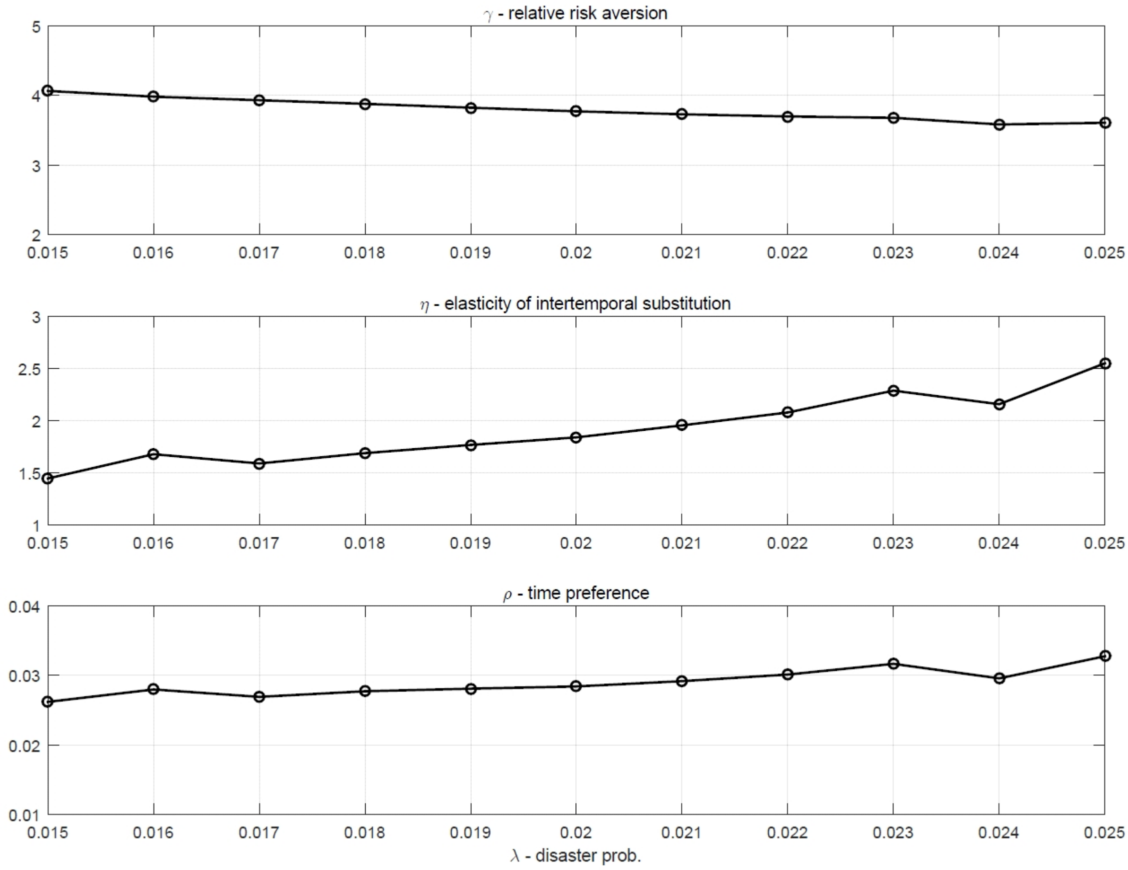

Figure 1.5 - Sensitivity analysis examining the impact of changing λ_1 (and setting $\lambda_2 = 2\lambda_1$) on preference parameters, γ, η, and ρ that provide the best fit to the four data targets, keeping μ, σ, and α constant.

Fitting the four targets of Table 1.8 through a minimum-distance approach, Figure 1.5 and Table 1.9 report how the three preference parameters, η, ρ, and γ, change as we vary the anchor value of λ_1 each time. In Table 1.9, we provide the re-calibrated parameters and the corresponding matched values. Under this sensitivity analysis, our model still performs relatively well. The model simulated values do not vary much compared to the benchmark results, and the results are quite close to the actual data targets. Regarding the preference parameters η, ρ, and γ, Figure 1.5 plots the corresponding variations, and none of the matching preference parameters change drastically. Importantly, according to Chen, Dou, and Kogan (2017, Figure 1, p. 23), we are confident that our calibration parameters are in the "acceptable calibration" area for models like ours.

1.4.3 Sensitivity analysis: sovereign-default risk

We also introduce sovereign-default risk, setting $\delta = E_{\zeta}(\zeta) = 23.28\%$, and $\pi = 40\%$, as in Barro (2006). As it is obvious from formulas (1.6) and (1.7), sovereign-default risk leaves P-D ratios unaltered, but raises r^B, as markets require a default premium. Using our calibrating values from Table 1.7, the resulting interest rates are $r_1^B = 0.75\%$ and $r_2^B = -0.50\%$. In light of our sensitivity analysis, it seems that even if sovereign-default risk is present, it is, instead market fragility that is most likely to explain recent asset-price trends. Especially if we think that higher sovereign-default risk emerged after the Lehman-Brothers collapse, this risk element would push bond rates upward instead of downward.

1.5 Conclusion: market fragility can resolve the paradox

The first part of the stock-bond-dissonance paradox refers to why bond rates have been so persistently low. Since the Lehman-Brothers collapse, the rise in stock prices creates the plausible impression that markets have increased their demand for stocks, lowering the demand for bonds. However, if fewer bonds had been demanded in the post-Lehman-Brothers era, then bond rates should have increased. Our approach to this part of the paradox has been to focus on explaining the simultaneous drop in the P-D ratio through increased market fragility, captured by the size of parameter λ in our model. Our theory says that there is no paradox: the drop in the P-D ratio implies that this substitution between stocks and bonds did not necessarily happen; *instead markets must have increased the demand for bonds, while* *decreasing the demand for stocks in a subtle manner.*

The other part of the paradox, referring to why stock prices have grown so much and so fast, can be explained by the fact that no disasters have occurred after 2008. To an extent, the dividend trend, captured by parameter μ in the model, reflects the incremental productivity growth of firms in the stock exchange. Disasters affect the perceived effective growth of dividends, so different perceptions of disaster risk before and after the financial crisis affect only the perceived but not the actual growth of dividends and prices. Therefore, *the fast rise of stock prices can be explained by the coincidence, that no disasters have occurred after 2008.*

For explaining the persistently negative bond returns, we do not rule out that the Fed policy contributed to the high demand for bonds. Yet, according to our approach, it is market fragility (perhaps bank fragility that followed the 2008 financial crisis, consistent with a rise in λ in our model), that led the Fed to its aggressive quantitative easing policy.

Our suggested market fragility explanation for resolving the paradox, points at a first message: it is crucial to avoid misinterpreting seemingly good market trends as market robustness at times of underlying market fragility. Market fragility always implies weaker investment in the real economy. This weakness alters the effects of planned fiscal and monetary policies. Our arguments in this study may serve as a starting point for new research on better identifying underlying market fragility and its sources.

1.6 Appendix

1.6.A Proof of equations (1.6) and (1.7)

Using the transformation,

$$W_t = S_{t-1}\left(D_t + P_t\right) + (1-\delta)^{\nu_t \nu_t^B} B_{t-1} \ , \tag{A.1.1}$$

the budget constraint (1.3) becomes,

$$W_{t+1} = R_{t+1}^{P} \left(W_t - C_t\right) \ , \tag{A.1.2}$$

in which R_t^P is the gross portfolio return defined as,

$$R_t^P = \phi_{t-1}^S R_t^S + \phi_{t-1}^B R_t^B \ , \tag{A.1.3}$$

with

$$R_t^S = \frac{D_t + P_t}{P_{t-1}} \quad \text{and} \quad R_t^B = \frac{(1-\delta)^{\nu_t \nu_t^B}}{Q_{t-1}} \tag{A.1.4}$$

being the gross returns of stocks and bonds, and with $\phi_t^S = P_t S_t / \left(P_t S_t + Q_t B_t\right)$ and $\phi_t^B = Q_t B_t / \left(P_t S_t + Q_t B_t\right)$ being the portfolio weights.

Using (A.1.2) the Bellman equation is,

$$V_t\left(W_t\right) = \max_{c_t \geq 0, \phi_t^S, \phi_t^B} \left\{ (1-\beta) C_t^{1-\frac{1}{\eta}} + \beta \left\{ E_t \left[V_{t+1} \left(R_{t+1}^P \left(W_t - C_t\right)\right)^{1-\gamma} \right] \right\}^{\frac{1-\frac{1}{\eta}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\eta}}} , \tag{A.1.5}$$

subject to (A.1.3) and subject to the stochastic structure given by (1.1) and (1.2). Under a general stochastic structure, the value function, $V_t(\cdot)$, is of the form,[6]

$$V\left(W_t\right) = \psi_t W_t \ , \ t = 0, 1, \dots \ . \tag{A.1.6}$$

A useful implication of (A.1.6) is,[7]

$$\frac{C_t}{W_t} = (1-\beta)^{\eta} \psi_t^{1-\eta} \ , \ t = 0, 1, \dots \ . \tag{A.1.7}$$

In addition, (A.1.6) implies the key asset pricing equation of this model, which is,[8]

$$E_t \left[\beta^{\frac{1-\gamma}{1-\frac{1}{\eta}}} \left(\frac{C_{t+1}}{C_t} \right)^{\frac{1-\gamma}{1-\eta}} \left(R_{t+1}^P\right)^{\frac{\frac{1}{\eta}-\gamma}{1-\frac{1}{\eta}}} R_{t+1}^i \right] = 1 \ , \quad i \in \{S, B\} \ . \tag{A.1.8}$$

[6] Equation (A.1.6) corresponds to Epstein and Zin (1991, p. 267, eq. 9).
[7] Equation (A.1.7) should correspond to Epstein and Zin (1991, p. 268, eq. 12), but equations (A.1.7) and Epstein and Zin (1991, p. 268, eq. 12) are different. See Appendix 1.6.B for a proof of equation (A.1.7).
[8] See Epstein and Zin (1991, p. 268, eq. 16).

In the standard textbook Lucas (1978) asset-pricing model the key simplifying assumption is that all investors are identical, all having the same amount of S_{-1} stocks in period 0, and all having $B_{t-1} = 0$ for all $t \in \{0, 1, ...\}$, i.e., bonds in zero net supply in all periods. Identical investors do not trade stocks in equilibrium. Combining these simplifying assumptions of no trade in equilibrium with the budget constraint, and also with equations (1.3) and (A.1.1), we obtain,

$$S_t = S_{-1}\ ,\ \ C_t = S_{-1} D_t\ ,\ \ W_t = S_{-1}\left(D_t + P_t\right)\ ,\ \ \phi_t^S = 1\ ,\text{ and } \phi_t^B = 0\ ,\ \ t = 0, 1, ...\ . \tag{A.1.9}$$

Combining (A.1.3) with (A.1.9) gives,

$$R_t^P = R_t^S\ , \tag{A.1.10}$$

while equation (A.1.4) implies,

$$R_{t+1}^S = \frac{1 + x_{t+1}}{x_t}\frac{D_{t+1}}{D_t}\ ,\quad \text{with}\quad x_t \equiv \frac{P_t}{D_t}\ . \tag{A.1.11}$$

In addition, equation (A.1.9) implies that $C_{t+1}/C_t = D_{t+1}/D_t$, so substituting this result into (A.1.8) for $i = S$, together with (A.1.10), (A.1.11), and (1.1), equation (A.1.8) becomes,

$$E_t\left\{\left(\beta\frac{1 + x_{t+1}}{x_t}\right)^{\frac{1-\gamma}{1-\frac{1}{\eta}}}\left[e^{\mu-\frac{\sigma^2}{2}+\sigma\varepsilon_{t+1}}\left(1-\zeta_{t+1}\right)^{\nu_{t+1}}\right]^{1-\gamma}\right\} = 1\ . \tag{A.1.12}$$

In Appendix 1.6.C we prove that, as a consequence of our assumption that variables ε_{t+1}, ν_{t+1}, and ζ_{t+1} are i.i.d. over time, the P-D ratio is also constant over time, i.e.,

$$x_t = x\ ,\ \ t = 0, 1, ...\ . \tag{A.1.13}$$

Substituting (A.1.13) into (A.1.12), proves the formula given by (1.6).

For proving equation (1.7) we substitute (A.1.4), (A.1.10), (A.1.11), and (A.1.13) into (A.1.8), for $i = B$, to obtain,

$$\beta^{\frac{1-\gamma}{1-\frac{1}{\eta}}}\left(\frac{1+x}{x}\right)^{\frac{\frac{1}{\eta}-\gamma}{1-\frac{1}{\eta}}} E_t\left\{\left[e^{\mu-\frac{\sigma^2}{2}+\sigma\varepsilon_{t+1}}\left(1-\zeta_{t+1}\right)^{\nu_{t+1}}\right]^{-\gamma}(1-\delta)^{\nu_{t+1}\nu_{t+1}^B}\frac{1}{Q_t}\right\} = 1\ . \tag{A.1.14}$$

Equation (A.1.14) implies a constant value for Q_t over time, $Q_t = Q$ for all $t \in \{0, 1, ...\}$. The bond price, Q, is set a priori to the realization of disasters and defaults. This is the reason why Q is constant over time. Yet, because of the sovereign default risk, the ex-post bond return is variable over time. Specifically, with probability $\lambda\pi$, the ex-post (post-default) maturity price of the bond is $1 - \delta$, making the ex-post bond return equal to $\underline{r}^B = (1 - \delta)/Q - 1$. So,

$$r_t^B = \begin{cases} r^B = \frac{1}{Q} - 1 & , \quad \text{with Prob. } 1 - \lambda\pi \\ \underline{r}^B = \frac{1-\delta}{Q} - 1 & , \quad \text{with Prob. } \lambda\pi \end{cases} . \tag{A.1.15}$$

In equation (1.7) we refer to the expected return implied by equation (A.1.15), which is given by $E\left(r^B\right) = (1 - \lambda\pi\delta)/Q - 1$. □

1.6.B Proof of equation (A.1.7)

Take equation (A.1.6) as an initial guess for the functional form of the value function, considering that ψ_t is an unknown stochastic process. Substituting (A.1.6) into equation (A.1.5) we obtain,

$$\psi_t W_t = \max_{c_t \geq 0, \phi_t^S, \phi_t^B} \left[(1 - \beta) C_t^{1-\frac{1}{\eta}} + \beta\omega_t \cdot (W_t - C_t)^{1-\frac{1}{\eta}} \right]^{\frac{1}{1-\frac{1}{\eta}}} , \tag{A.1.16}$$

in which, $\omega_t \equiv \left\{ E_t \left[\psi_{t+1}^{1-\gamma} \left(R_{t+1}^P\right)^{1-\gamma} \right] \right\}^{\frac{1-\frac{1}{\eta}}{1-\gamma}}$. Taking first-order conditions on equation (A.1.5) with respect to C_t gives,

$$C_t = \left(\frac{\beta}{1 - \beta} \omega_t \right)^{-\eta} (W_t - C_t) . \tag{A.1.17}$$

Equation (A.1.17) implies,

$$C_t^{1-\frac{1}{\eta}} = \left(\frac{\beta}{1 - \beta} \omega_t \right)^{1-\eta} (W_t - C_t)^{1-\frac{1}{\eta}} . \tag{A.1.18}$$

Substituting (A.1.18) into (A.1.16), imposes optimality conditions on (A.1.16). So, the max operator in (A.1.16) can be eliminated after substituting (A.1.18) into (A.1.16), which gives,

$$\psi_t^{1-\frac{1}{\eta}} W_t^{1-\frac{1}{\eta}} = \beta\omega_t \left[\left(\frac{\beta}{1-\beta}\omega_t \right)^{-\eta} + 1 \right] (W_t - C_t)^{1-\frac{1}{\eta}} \ . \tag{A.1.19}$$

Using (A.1.18) and substituting it into (A.1.19) results in,

$$\psi_t^{1-\frac{1}{\eta}} \left(\frac{C_t}{W_t} \right)^{\frac{1}{\eta}-1} = (1-\beta) \left[1 + \left(\frac{\beta}{1-\beta}\omega_t \right)^{\eta} \right] \ . \tag{A.1.20}$$

Equation (A.1.17) implies,

$$\frac{C_t}{W_t} = \left[1 + \left(\frac{\beta}{1-\beta}\omega_t \right)^{\eta} \right]^{-1} \ . \tag{A.1.21}$$

Substituting (A.1.21) into (A.1.20) gives,

$$\psi_t^{1-\frac{1}{\eta}} = (1-\beta) \left[1 + \left(\frac{\beta}{1-\beta}\omega_t \right)^{\eta} \right]^{\frac{1}{\eta}} \ . \tag{A.1.22}$$

Equation (A.1.22) reconfirms that the guess given by equation (A.1.6) is valid. Combining (A.1.21) with (A.1.22) leads to equation (A.1.7). □

1.6.C Proof that the price-dividend ratio is constant

Since variables ε_{t+1}, ν_{t+1}, and ζ_{t+1} are i.i.d. over time, through integral-variable transformation, equation (A.1.12) implies that,

$$E_t \left(\frac{1 + x_{t+1}}{x_t} \right) = E_{t+1} \left(\frac{1 + x_{t+2}}{x_{t+1}} \right) = \kappa \ , \ \ t = 0, 1, \dots \ . \tag{A.1.23}$$

To see that $x_{t+1} = x_t = x$ for $t = 0, 1, \dots$, fix some $x_t = \bar{x}_t > 0$, assuming that $\bar{x}_t$ is a solution to the asset-pricing model. Consider conditional expectations for (A.1.23), namely,

$$E_t \left(\frac{1 + x_{t+1}}{x_t} \ \mid x_t = \bar{x}_t \right) = \kappa \ . \tag{A.1.24}$$

Equation (A.1.24) implies a unique solution for $E_t(x_{t+1})$. Let that unique solution be $\bar{x}_{t+1} = E_t(x_{t+1})$. Using $\bar{x}_{t+1}$, consider equation (A.1.24) one period ahead to obtain $\bar{x}_{t+2} \equiv E_{t+1}(x_{t+2})$. Notice that, due to additive separability, and since the choice of $t \in \{0, 1, ...\}$ was arbitrary, equation (A.1.24) implies,

$$\frac{1 + \bar{x}_{t+1}}{\bar{x}_t} = \frac{1 + \bar{x}_{t+2}}{\bar{x}_{t+1}} \ , \ \ t = 0, 1, ... \ . \tag{A.1.25}$$

Using $g_{t+1} \equiv \bar{x}_{t+1}/\bar{x}_t$ equation (A.1.25) implies,

$$g_{t+2} - g_{t+1} = \frac{1}{\bar{x}_t}\left(1 - \frac{1}{g_{t+1}}\right) \ , \ \ t = 0, 1, ... \ . \tag{A.1.26}$$

If $g_{t+1} \neq 1$, since $\bar{x}_t > 0$, we can easily verify that equation (A.1.26) implies unstable dynamics for x_t. If $g_{t+1} > 1$ for some t, then (A.1.26) implies $g_{t+s} > 1$ for all $s \in \{0, 1, ...\}$, and $x_t \to \infty$. If $g_{t+1} < 1$ for some t, then eventually $g_{\hat{t}} < 0$ for some $\hat{t} > t$, leading to $x_{\hat{t}} < 0$. For $\eta \neq 1$ (which is of interest for matching the data), both of these possibilities lead to a non-well-defined value function. To see this, use equations (A.1.7), (A.1.11), (A.1.9), and (A.1.6) to obtain,

$$V_t(W_t) = \psi_t W_t = (1 - \beta)^{\frac{\eta}{\eta - 1}} (1 + x_t)^{\frac{\eta}{\eta - 1}} S_{-1} D_t \ . \tag{A.1.27}$$

For $x_t \to \infty$, either $V_t(W_t) \to \infty$ (if $\eta > 1$), or $V_t(W_t) \to 0$, (if $\eta < 1$), even if $0 < D_t < \infty$. None of these possibilities implies a well-defined value function or a maximum value, given that the EZW utility function represents a cardinal certainty-equivalent time aggregator measured in consumption units (in addition, the solution $g_t = 1$ for all $t \in \{0, 1, ...\}$, gives $V_t(W_t) > 0$ bounded away from 0 for all $t \in \{0, 1, ...\}$). Since from equation (A.1.9) $C_t/W_t = 1/(1 + x_t)$, having $x_{\hat{t}} < 0$ for some $\hat{t} > t$, implies $C_{\hat{t}} > W_{\hat{t}}$, and equation (A.1.2) then gives $W_{\hat{t}+1} < 0$ if $R^P_{\hat{t}+1} > 0$, i.e. $V_{\hat{t}}(W_{\hat{t}}) < 0$, which is unacceptable, given the consumption-unit cardinality of EZW preferences. But even if $R^P_{\hat{t}+1} < 0$, having negative stock prices in an exchange economy of identical agents is impossible in equilibrium. So, the only acceptable equilibrium solution for (A.1.26) is $g_t = 1$ for all $t \in \{0, 1, ...\}$, which leads to equation (A.1.13), proving the result. $\square$

2. CHAPTER

Time-Consistent Welfare-Maximizing Monetary Rules

2.1 Introduction

How should a Central Bank optimally respond to fiscal and productivity shocks, taking into account that households face liquidity constraints? Liquidity constraints have gained new attention recently, as more detailed consumption and wealth data have been available. Recent research emphasizes that large fractions of households, including wealthy ones, are hand-to-mouth, hence liquidity constrained (see Kaplan et al., 2014). It is reasonable to think that bond markets and overall-economy liquidity (money supply) influence the consumption choices of liquidity-constrained households. Does this interplay between bond markets and consumption choices of the liquidity constrained affect monetary policy, i.e., open-market operations by Central Banks?

To capture the role of liquidity constraints, we employ a cash-in-advance (CIA) model with two types of consumption goods: a credit-constrained and a non-credit-constrained good. More specifically in our model setup, we follow Cooley and Hansen (1989) and (1992), which build on the originally developed by Kydland and Prescott (1982) and Long and Plosser (1983) real business cycle (RBC) model. They introduce a monetary sector in the standard RBC model, while keeping the RBC model's main features: perfect competition and perfect price flexibility. Money is introduced via a cash-in-advance constraint, being used to facilitate transactions. As in Lucas and Stokey (1983, 1987), only a subset of consumption goods is CIA constraint, the so called "cash good", liquidity constraint good. We allow the Central Bank to perform open-market operations, buying government bonds in order to respond to fiscal shocks and to productivity shocks. We take the approach that Central Banks conduct monetary policy serving the ultimate goal of maximizing social welfare, as dictated by a country's constitution. In order to achieve this goal, we use concepts and algorithms from the literature on optimal time-consistent fiscal policy. More specifically, we formulate the optimal policy as a dynamic Stackelberg game between the Central Bank and private markets. In this game-theoretic formulation of the Central Bank's welfare maximization problem, we follow the literature on optimal fiscal policy and voting, which includes Krusell et al. (1997), Krusell and Rios-Rull (1999), Klein and Rios-Rull (2003), Klein et al. (2008) and Bachmann and Bai (2013).[9] In our case, our focus is on the optimal money supply growth rate. The endogenously derived optimal money supply growth rate is a function of the model's aggregate state variables. More specifically, we use a linear-quadratic approximation method, as in Cooley and Hansen (1989) for computing the equilibrium process for our model, more specifically the linear equilibrium decision rules.

We find that optimal time-consistent monetary policy has real effects along the transition to the steady state from a shock. It focuses on improving upon the ratio between the two consumption-good types that is distorted due to the presence of the liquidity constraint. Optimal time-consistent monetary policy offsets fiscal (demand) shocks, smoothing out aggregate consumption fluctuations. Finally, we find that optimal time-consistent monetary policy accommodates productivity (supply) shocks, amplifying aggregate consumption fluctuations.

The reason we focus on time-consistent monetary policy is that time-consistency establishes the credibility of monetary policy. This idea is in line with current forward-guidance monetary policy practices of Central Banks. Private markets' decision making depends on current policy and the expected course of monetary policy. Establishing credibility comes

[9] The only exception of this optimal-policy approach on monetary policy that we are aware of is the paper by Cooley and Quadrini (2004).

at the cost that time-consistent policies are not first-best policies, as first pointed out by Kydland and Prescott (1977).

Our approach differs from the conventional literature on monetary policy rules (new Keynesian models), where monetary policy rules are assumed based on empirical findings. Instead, through welfare maximization, we derive the Central Bank's optimal money supply rule endogenously. After calibrating our model to different targets than the steady-state inflation rate, the optimal steady state money supply growth rate that we find in our model economy is equal to 2.03%. This figure is consistent with the major Central Bank's price stability objective and with past empirical observations.

The paper is organized as follows. First, we describe our model economy and market clearing conditions. Then we set up the recursive formulation of the policy-setting equilibrium, before describing the game-theoretical formulation employed to derive the welfare-maximizing monetary policy rule. We calibrate the model parameters and key steady-state features of the model to US data covering the years from 1947 to 2018. We describe the cyclical behavior of our model economy. We simulate the model with the optimal endogenous money supply rule (benchmark model) and the model with a constant exogenous money supply rate (set equal to the optimal steady state money supply rate). We compare the simulations with the cyclical behavior of the US time-series.

The fiscal authority levies constant tax rates and does not adjust to exogenous shocks. We are in a monetary-fiscal policy regime of fiscal dominance. The monetary authority, the Central Bank reacts to both shocks, the productivity shock and the fiscal shock via the endogenously derived, time-consistent optimal money supply rule as depicted in the computed impulse responses. In case of a productivity shock, the endogenous monetary policy makes the economy more volatile and amplifies the cycle. This is consistent with the idea that monetary policy should accommodate supply shocks. Also, monetary policy improves the mixture between the cash good and the credit good. Therefore, monetary policy has real effects along the transition. We also observe that monetary policy has no effect on the real interest rate. In case of a fiscal shock, monetary policy tries to counteract the crowding out effect on investment and consumption and the increase in fiscal deficit. This is consistent with the idea that monetary policy should offset demand shocks. Monetary policy smooths consumption and corrects the distortion between the cash and the credit good. Also here, we can see that monetary policy has real effects. The role of the cash-in-advance constraint in monetary policy transmission is identified.

We conclude with the limitations of the model and possible directions for future work.

2.2 Model

The model is a modified stochastic version of the Cooley-Hansen (1992) economy. The key feature of our model is that a Central Bank performs open-market operations with the potential of absorbing repercussions of fiscal shocks and of productivity shocks. The central reason we use a neoclassical model without price frictions or other similar elements of post-Keynesian models is that we need to keep a clear view on the welfare-maximizing criterion of the Central Bank. A frictionless neoclassical or real-business-cycle (RBC) framework (see, e.g., Cooley and Prescott, 1995), is perhaps the ideal vehicle for numerically investigating the core question of our paper: are welfare-maximizing monetary policies stabilizing as an exogenous Taylor rule would dictate? Throughout the paper we denote variables in nominal terms by a bar. For aggregates we use uppercase letters and for micro-level variables we use lowercase letters. Therefore variable x is $\bar{X}$ if nominal and aggregate, X if real and aggregate, $\bar{x}$ if nominal and at the micro-level, and x if real and at the micro-level.

2.2.1 Government

The government finances government spending using taxes and government debt. The government issues zero-coupon bonds every period, rolling over outstanding debt from the previous period. The government budget constraint, in nominal terms, is given by,

$$\bar{G}_t = \bar{T}_t + \bar{B}_t^T - (1 + i_{t-1})\, \bar{B}_{t-1}^T + \overline{RCB}_t \ , \tag{2.1}$$

where $\bar{B}_{t-1}^T$ is the nominal value of total debt issued in period $t-1$, accounted for in prices of period $t-1$, and maturing in period t, i_{t-1} is the nominal bond rate in period t on debt issued in $t-1$, $\bar{G}_t$ and $\bar{T}_t$ are nominal government spending and government revenues in prices of period t, and $\overline{RCB}_t$ are the nominal direct receipts from the Central Bank in period t, in current prices as well. Fiscal and monetary policy are linked through the government budget constraint, consistent with Walsh (2010).

Government revenues, $\bar{T}_t$, come from three sources, taxes on consumption, on labor income and on capital income. Therefore, government revenues in real terms, T_t, are,

$$T_t = \tau_c C_t + \tau_h w_t H_t + \tau_k r_t K_t \ , \tag{2.2}$$

where C_t, H_t, and K_t are real aggregate consumption, hours worked and capital, w_t and r_t are the real wage and the real capital return, while τ_c, τ_h, and τ_k are the marginal tax rates on consumption, labor income, and capital income. We assume that all tax rates are constant over time, which is the reason they do not carry the subscript "t".[10]

Real government spending, G_t, is an exogenous random process given by,

$$G_t = \tilde{g} e^{u_t} Y_t \ , \tag{2.3}$$

[10] This assumption is based on early work by Barro (1979) and on results provided later on by Chari et al. (1994) for economies in the class within which our model falls.

in which Y_t is real output, $\tilde{g} \in (0,1)$ is a constant parameter and u_t is an AR(1) fiscal shock with,

$$u_t = \rho_u u_{t-1} + \varepsilon_{u,t} \ , \tag{2.4}$$

in which $\rho_u \in (0,1)$ and $\varepsilon_{u,t} \sim i.i.d.N\left(0, \sigma_u^2\right)$. Equations (2.3) and (2.4) fully describe the exogenous fiscal shocks that monetary policymaker wish to smooth out.

Dividing both sides of equation (2.1) by the price index, P_t, gives, the government budget constraint in real terms,

$$G_t = T_t + B_t^T - \frac{1 + i_{t-1}}{1 + \pi_t} B_{t-1}^T + RCB_t \ , \tag{2.5}$$

in which,

$$\pi_t = \frac{P_t}{P_{t-1}} - 1 \ . \tag{2.6}$$

is the inflation rate.

2.2.2 Central Bank and bonds market

The Central Bank (CB) issues fiat money (currency), $\bar{M}_t^s$, which is guaranteed through purchases of government bonds. We denote nominal holdings of fiscal debt by the CB by $\bar{B}_t^{CB}$. Taking into account interest earned by the CB, the evolution of the CB's balance sheet is given by,

$$\bar{B}_t^{CB} - \bar{B}_{t-1}^{CB} + \overline{RCB}_t = i_{t-1} \bar{B}_{t-1}^{CB} + \bar{M}_t^s - \bar{M}_{t-1}^s \ . \tag{2.7}$$

Denoting the growth rate of nominal money supply by μ_t,

$$\bar{M}_t^s = \left(1 + \mu_t\right) \bar{M}_{t-1}^s \ . \tag{2.8}$$

The main objective of this paper is to determine how the CB decides upon the optimal level of μ_t through a time-consistent welfare-maximizing monetary policy rule.

Bonds are held by the CB and the public, which is only households in our model. Therefore, the market-clearing condition for the bonds market is,

$$\bar{B}_t^T = \bar{B}_t^H + \bar{B}_t^{CB} \ , \tag{2.9}$$

where nominal bond holdings by households is denoted by $\bar{B}_t^H$. Substituting equations (2.7), (2.8) and (2.9) into (2.1) we obtain,

$$\bar{G}_t = \bar{T}_t + \bar{B}_t^H - (1 + i_{t-1})\, \bar{B}_{t-1}^H + \mu_t \bar{M}_{t-1}^s \ . \tag{2.10}$$

Equation (2.10) reveals the relationship between monetary policy, dictated by μ_t, and the bonds market. The interplay between the two is crucial in our analysis.

The balance sheet of the Central Bank is given by Table 2.1, in which $\bar{B}_t^{CB}$ is the nominal money demand of government bonds by the Central Bank, and $\bar{M}_t^S$ is the nominal money supply in period t.

Assets	Liabilities
$\bar{B}_t^{CB}$	$\bar{M}_t^S \equiv currency$

Table 2.1 - Balance sheet of the Central Bank

Table 2.1 can be summarized through,

$$\bar{B}_t^{CB} = \bar{M}_t^S \ . \tag{2.11}$$

The Central Bank exercises monetary policy through open-market operations. Purchases of government bonds lead to an expansion of the money supply. Monetary expansions and contractions can function as responses to economy-wide fiscal and productivity shocks. We describe the way that the Central Bank forms its monetary-policy objectives incorporating these considerations in a later subsection.

2.2.3 Production and representative firm

Production technology is Cobb-Douglas with,

$$Y_t = F(K_t, H_t) = e^{z_t} K_t^{\alpha} H_t^{1-\alpha} \ , \tag{2.12}$$

in which Y_t is a composite consumable good, K_t is the accumulated aggregate stock of capital in period t, H_t is aggregate labor hours, and z_t is a productivity shock that follows an AR(1) process with,

$$z_t = \rho_z z_{t-1} + \varepsilon_{z,t} \ , \tag{2.13}$$

in which $\rho_z \in (0,1)$ and $\varepsilon_{z,t} \sim i.i.d.N\left(0, \sigma_z^2\right)$. We assume that $\varepsilon_{z,t}$ and $\varepsilon_{u,t}$ have a correlation coefficient $\rho_{zu} \in (-1, 1)$.

Firms maximize profits operating in a perfectly-competitive final-goods market, hiring inputs in perfectly-competitive markets. Competitive pricing implies,[11]

$$R_t = \alpha e_t^{z_t} \left(\frac{K_t}{H_t}\right)^{\alpha-1} \ , \tag{2.14}$$

and

$$w_t = (1-\alpha) e^{z_t} \left(\frac{K_t}{H_t}\right)^{\alpha} \ . \tag{2.15}$$

2.2.4 Households

There is a large number of identical infinitely-lived households that supply labor endogenously (a source of disutility), and derive utility from two types of consumption goods, c_1 and c_2. Consumption good c_1 is a "cash good", and good c_2 is a "credit good".[12] Specifically, c_1 requires cash in advance, being restricted by the constraint,

$$(1+\tau_c)c_{1,t} \leq \frac{\bar{m}_{t-1}^H}{P_t} + \frac{(1+i_{t-1})\bar{b}_{t-1}^H}{P_t} - \frac{\bar{b}_t^H}{P_t} \ , \tag{2.16}$$

[11] Variable R_t denotes the rental cost of capital. The relationship between R_t and the capital return, r_t, is given by $r_t = R_t - \delta$, where δ is the (tax-exempt) depreciation rate (see equation (2.2) above).
[12] This distinction follows closely Cooley and Hansen (1992).

where $\bar{m}_{t-1}^H$ is the individual nominal money holdings carried over from period $t-1$ to period t, and $(1+i_{t-1})\bar{b}_{t-1}^H$ is the nominal cash amount obtained by individual household bond holdings (principal plus nominal interest), while $\bar{b}_t^H$ is the cash amount paid in period t, in order to buy newly issued bonds.

The constraint given by (2.16), shows that households allocate their resources into consumption c_1 and bond investment, $\bar{b}_t^H/P_t$. Nevertheless, as in Cooley and Hansen (1992), the bond and money trades (nominal demand of household bonds and money balances) take place in the beginning of period t, before the shocks (z_t, u_t) are revealed, and before the goods market opens. The goods market in period t, opens later within period t, after the realization of shocks (z_t, u_t). This means that while goods c_1 and $\bar{b}_t^H/P_t$ are both traded in period t, they are not traded simultaneously. Market clearing in the goods market and the asset markets later in period t determine price level P_t and inflation. The Central Bank announces its monetary policy for period t earlier than households declare their bond and money demand.

The overall budget constraint of the household is,

$$\begin{aligned}(1+\tau_c)(c_{1,t}+c_{2,t}) + k_{t+1} - (1-\delta)k_t + \frac{\bar{m}_t^H}{P_t} + \frac{\bar{b}_t^H}{P_t} \\ = (1-\tau_h)w_t h_t + (1-\tau_k)R_t k_t + \tau_k \delta k_t + \frac{\bar{m}_{t-1}^H}{P_t} + \frac{(1+i_{t-1})\bar{b}_{t-1}^H}{P_t}\ ,\end{aligned} \tag{2.17}$$

where k_t is the individual real capital claims to productive assets.

Households maximize their expected utility given by,

$$\max_{\left\{\left(c_{1,t},c_{2,t},h_t,\bar{m}_t^H,\bar{b}_t^H,k_{t+1}\right)\right\}_{t=0}^{\infty}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u\left(c_{1,t}, c_{2,t}, h_t\right) \right\}\ , \tag{2.18}$$

with $0 < \beta < 1$, and,

$$u\left(c_{1,t}, c_{2,t}, h_t\right) = \theta \log c_{1,t} + (1-\theta) \log c_{2,t} - B h_t\ . \tag{2.19}$$

with $0 < \theta < 1$, making investments in three assets: (i) productive-capital asset holdings, k_{t+1}, (ii) government bonds $\bar{b}_t^H$, and (iii) money balances $(\bar{m}_t^H)$.

The specification in (2.19) follows the concept of "indivisible labor" suggested by Hansen (1985), Rogerson (1988), and followed by Cooley and Hansen (1989). Specifically, the utility function is initially given as a function of consumption and leisure. The original version of (2.19) is,

$$\tilde{u}\left(c_{1,t}, c_{2,t}, h_t\right) = \theta \log c_{1,t} + (1-\theta) \log c_{2,t} + A \log l_t \ , \tag{2.20}$$

where l_t is leisure in period t. With households having one unit of time, $l_t + h_t = 1$. It is assumed that labor is indivisible, with households having to work either $h_o \in (0,1)$ hours, or 0 hours, with probability π_t. Therefore, the utility function (2.20) becomes,

$$\tilde{u}\left(c_{1,t}, c_{2,t}, h_t\right) = \theta \log c_{1,t} + (1-\theta) \log c_{2,t} + A\left[\pi_t \log\left(1-h_o\right) + \left(1-\pi_t\right)\log(1)\right] \ ,$$

or,

$$\tilde{u}\left(c_{1,t}, c_{2,t}, h_t\right) = \theta \log c_{1,t} + (1-\theta) \log c_{2,t} + A\pi_t \log\left(1-h_o\right) \ . \tag{2.21}$$

Setting $h_t = \pi_t h_o$, and $B = -A \log\left(1-h_o\right)/h_o$ makes (2.21) equivalent to (2.19). We emphasize that the choice of h_t is endogenous in the model. The choice of h_t represents how the effort for finding or for maintaining a job affects the probability of finding or of maintaining a job, π_t. Since this individual effort is proportional to individual productivity, labor earnings in the budget constraint given by (2.17) are $w_t h_t$.

2.2.5 Market clearing

We express all market-clearing conditions in terms of real variables.

Money market The only holders of money are the households, implying,

$$M_t^H = M_t^S \ . \tag{2.22}$$

Government-bond market Bonds are held by both households and the central bank, implying,

$$B_t^S = B_t^{CB} + B_t^H \ , \tag{2.23}$$

with $b_t^H = B_t^H$ due to the representative-agent assumption.

Productive-capital market Household investment in productive-asset claims equals productive capital in each period. Since households are identical,

$$k_t = K_t \ . \tag{2.24}$$

An implication of perfect competition in both the productive-capital market and the bonds market is the arbitrage condition,

$$1 + (1 - \tau_k)\, r_t = \frac{1 + i_t}{1 + \pi_t} \ . \tag{2.25}$$

Final goods market In equilibrium, aggregating the budget constraint given by (2.17) leads to

$$Y_t = C_t + X_t + G_t \ , \tag{2.26}$$

where

$$X_t = K_{t+1} - (1 - \delta)\, K_t \tag{2.27}$$

is aggregate investment, with $x_t = X_t$ ($x_t = k_{t+1} - (1 - \delta)\, k_t$), $C_t = C_{1,t} + C_{2,t}$, with $c_{1,t} = C_{1,t}$ and $c_{2,t} = C_{2,t}$.

Labor market In equilibrium,

$$h_t = H_t \ . \tag{2.28}$$

2.3 Recursive formulation of policy-setting equilibrium

As in Cooley and Hansen (1989, 1992), we detrend all nominal variables by dividing them by the nominal money supply. Therefore, a nominal variable, aggregate or individual, $\bar{X}_t$ or $\bar{x}_t$, is detrended after being transformed into $\hat{X}_t \equiv \bar{X}_t/\bar{M}_t^s$ and $\hat{x}_t \equiv \bar{x}_t/\bar{M}_t^s$. In order to define equilibrium in this optimal policy-setting environment recursively, we need to organize the variables in two groups, state variables, s, and action variables, a.

The aggregate state variables are defined as,

$$S_t = \left(z_t, u_t, \mu_t, K_t, \hat{B}_{t-1}^T \right) \ , \tag{2.29}$$

while the individual state variables are defined as,

$$s_t = \left(k_t, \hat{m}_{t-1}^H, \hat{b}_{t-1}^H \right) \ . \tag{2.30}$$

Although the two vectors given by (2.29) and (2.30) are neither of the same size nor their variables are the same kind, we will conventionally use the notation "$s_t = S_t$" in order to capture the notion of market clearing in state variables in the recursive formulations that follow. Specifically, "$s_t = S_t$" refers to the set of equations,

$$\left\{ \begin{array}{c} k_t = K_t \\ \hat{m}_{t-1}^H = 1 \\ \hat{b}_{t-1}^H = \hat{B}_{t-1}^H \\ \hat{B}_{t-1}^T = 1 + \hat{B}_{t-1}^H \end{array} \right\} \Leftrightarrow s_t = S_t \tag{2.31}$$

The four equations given by (2.31) describe equilibrium in the capital market, the money market ($\hat{m}_{t-1}^H = \bar{m}_{t-1}^H/\bar{M}_{t-1}^S = \bar{M}_{t-1}^H/\bar{M}_{t-1}^S = 1$), the bonds market (recall that $\hat{B}_{t-1}^{CB} = \bar{B}_{t-1}^{CB}/\bar{M}_{t-1}^S = 1$), and it is confirmed that individual household bond demand, $\hat{b}_{t-1}^H$, equals its aggregate counterpart. The four equations given by (2.31) refer to market clearing regarding four state variables, K_t, $\bar{M}^H_{t-1}$, $\hat{B}^H_{t-1}$ and $\hat{B}^T_{t-1}$, while the rest of the variables in (2.29), z_t, u_t, and μ_t, refer to the two exogenous aggregate shocks and to money growth, with the latter being dictated by Central Bank's monetary policy and not through market forces.

The aggregate action variables are defined as,

$$A_t = \left(H_t, X_t, \hat{P}_t, \hat{B}^T_t\right) \ , \tag{2.32}$$

and the individual action variables are,

$$a_t = \left(h_t, x_t, \hat{m}^H_t, \hat{b}^H_t\right) \ . \tag{2.33}$$

Again not all variables of two vectors given by (2.32) and (2.33) are of the same kind. Nevertheless, again, we will conventionally use the notation "$a_t = A_t$" in order to capture the notion of market clearing in action variables as,

$$\left\{\begin{array}{c} h_t = H_t \\ x_t = X_t \\ \hat{m}^H_t = 1 \\ \hat{b}^H_t = \hat{B}^H_t \\ \hat{B}^T_t = 1 + \hat{B}^H_t \end{array}\right\} \Leftrightarrow a_t = A_t \tag{2.34}$$

The five equations given by (2.34) refer to market clearing regarding five action variables, H_t, X_t, $\bar{M}^H_t$, $\hat{B}^H_t$ and $\hat{B}^T_t$, while the remaining variable in (2.32), $\hat{P}_t = P_t/\bar{M}^S_t$ is a variable that will be determined endogenously in competitive equilibrium.[13]

2.3.1 The welfare-maximizing monetary policy rule

We employ the game-theoretic formulation suggested by literature on optimal fiscal policy and voting. This literature includes Krusell et al. (1997), Krusell and Rios-Rull (1999),

[13]This formulation of vectors a_t and A_t imitates the definitions of vectors u_t and U_t in Cooley and Hansen (1989, p. 739). This formulation facilitates the recursive computation of equilibrium $\hat{P}_t$.

Klein and Rios-Rull (2003), Klein et al. (2008), and Bachmann and Bai (2013). In that optimal-fiscal policy literature the goal of the government is predominantly to determine the size of the government or its fiscal transfers. In our monetary-policy analysis the goal is to analyze optimal money supply growth rate, μ_{t+1}. Specifically, the Central Bank seeks to set an optimal time-consistent rule, Ψ, in each period, as a function of the model's aggregate state variables. Specifically,

$$\mu_{t+1} = \Psi\left(z_t, u_t, \mu_t, K_t, \hat{B}^T_{t-1}\right) = \Psi\left(S_t\right) \ . \tag{2.35}$$

One of our targets is to imitate the way Central Banks pre-announce monetary-policy strategies and then try to show commitment to the pre-announced strategies. To achieve this, we assume that monetary-policy setting takes place at the beginning of every period, i.e. *before* households and producers make their economic decisions. After policy is set, the Central Bank commits to it for one period and then revises the policy. Since, however, the revision process is history-independent, the result is a time-invariant monetary-policy rule Ψ, of the form given by (2.35). This is the setup of a dynamic-Stackelberg-game formulation with closed-loop strategies and feedback information pattern (see Basar and Olsder, 1999, Definition 5.2, p. 221), also fitting the specification of Cohen and Michel (1988) in a continuous-time setting (instantaneous precommitment at each time instant).

If Ψ has been selected by the Central Bank, then real economic activity of households depends on rule Ψ. At the same time the Central Bank needs to understand the optimal decisions of households subject to its rule Ψ, in order to calculate household utility and social welfare.

2.3.1.1 Competitive equilibrium (CE)

The economy's CE is described by the following fixed-point relations. The Bellman equation of an individual household is given by,[14]

$$V(S,s \mid \Psi) = \max_a \{u(S,s,A,a) + \beta E[V(S',s' \mid \Psi)]\} \tag{2.36}$$

subject to,

$$s' = \zeta(S,s,A,a \mid \Psi) \ ,$$

$$S' = Z(S \mid \Psi) \ ,$$

$$A = \Xi(S \mid \Psi) \ ,$$

$$\mu' = \Psi(S) \ ,$$

$$(z',u') \sim d\Phi((z',u') \mid (z,u)) \ ,$$

where primes denote variables one period ahead, $\Phi((z',u') \mid (z,u))$ is the distribution function describing the Markov chain governing the shocks (z,u). After solving (2.36), the resulting optimal individual actions,

$$a^* = \xi(S,s,A \mid \Psi) \equiv \arg\max_a \{u(S,s,A,a) + \beta E[V(S',s' \mid \Psi)]\}$$

are consistent with the aggregate actions rule $A = \Xi(S \mid \Psi)$, and the aggregate law of motion of the state variables, $S' = Z(S \mid \Psi)$, after imposing market clearing, $s = S$ and $a = A$ as defined by (2.31) and (2.34). Specifically, $A = \Xi(S \mid \Psi)$ is the implicit function solving $A = \xi(S,S,A \mid \Psi)$ and $S' = Z(S \mid \Psi)$ is consistent with $S' = \zeta(S \ , \ S \ , \ \Xi(S \mid \Psi) \ , \ \Xi(S \mid \Psi) \mid \Psi)$.

For determining the policy rule $\mu' = \Psi(S)$ we need to define an "intermediate equilibrium" that focuses on calculating the economy's (welfare) response to a one-time deviation in next-period's money growth policy μ_{t+1}, in period t, under the provision that all future policies $\{\mu_{t+j}\}_{j=2}^{\infty}$, comply with Ψ afterwards.

[14] The exact form of the utility function, $u(S,s,A,a)$, in (2.36) is given in the Appendix.

2.3.1.2 Intermediate equilibrium (IE) The economy's IE is described by the following fixed-point relations. First, we formulate the *non-Bellman equation*,

$$\tilde{V}(S, s, \mu' \mid \Psi) = \max_{a} \{u(S, s, A, a) + \beta E[V(S', s' \mid \Psi)]\} \tag{2.37}$$

subject to,

$$s' = \zeta(S, s, A, a \mid \Psi) \ ,$$

$$S' = \tilde{Z}(S, \mu' \mid \Psi) \ ,$$

$$A = \tilde{\Xi}(S, \mu' \mid \Psi) \ ,$$

$$(z', u') \sim d\Phi((z', u') \mid (z, u)) \ ,$$

where $\tilde{V}(S, s, \mu' \mid \Psi)$ is the value function that computes individual utility in equilibrium given a one-period deviation in policy μ', while $S' = \tilde{Z}(S, \mu' \mid \Psi)$ and $A = \tilde{\Xi}(S, \mu' \mid \Psi)$ describe the responsiveness of next period's state variables and aggregate actions to a one-period deviation in policy μ'. The value function on the right-hand side of equation (2.37) is the fixed point of the competitive-equilibrium Bellman equation given by (2.36). In this way, IE secures that the continuation stream of future policies complies with policy rule $\Psi(S)$ and that all decisions dictated by CE conditions are respected in the future. What IE does, is to compute the response in the evolution of the aggregate state variables one period ahead through computing $S' = \tilde{Z}(S, \mu' \mid \Psi)$. This deviation in the economy's state variables affects continuation expected utility $E[V(S', s' \mid \Psi)]$ accordingly. In turn, this impact on continuation utility is reflected in the value function $\tilde{V}(S, s, \mu' \mid \Psi)$ on the left-hand side of equation (2.37). Importantly, as in CE above, $A = \tilde{\Xi}(S, \mu' \mid \Psi)$ is the implicit function solving $A = \tilde{\xi}(S, S, A, \mu' \mid \Psi)$, with

$$\tilde{a}^* = \tilde{\xi}(S, S, A, \mu' \mid \Psi) \equiv \arg\max_{a} \{u(S, s, A, a) + \beta E[V(S', s' \mid \Psi)]\}\Big|_{\mu' \text{ is free, not equal to } \Psi(S)}$$

and $S' = \tilde{Z}(S, \mu' \mid \Psi)$ is consistent with $S' = \zeta\Big(S \ , \ S \ , \ \tilde{\Xi}(S, \mu' \mid \Psi) \ , \ \tilde{\Xi}(S, \mu' \mid \Psi) \mid \Psi\Big)$.

2.3.1.3 Setting the policy rule Once the fixed point of IE is obtained and $\tilde{V}(S, s, \mu' \mid \Psi)$, is computed, we impose market-clearing conditions on the state variables to compute,

$$\tilde{W}(S, \mu' \mid \Psi) = \tilde{V}(S, S, \mu' \mid \Psi) \ , \tag{2.38}$$

which is the welfare function of the economy as a function of a one-time deviation in next-period's money growth policy, μ', and subject to the policy rule Ψ for setting policy in future periods thereafter. Given $\tilde{W}(S, \mu' \mid \Psi)$, the policy rule Ψ is computed as,

$$\mu' = \Psi(S) = \arg\max_{\mu'} \tilde{W}(S, \mu' \mid \Psi) \ . \tag{2.39}$$

Function Ψ on the left-hand side of (2.39) must be the same as function Ψ on the right-hand side of (2.39).

2.4 Calibration and computation

The calibration process targets matching key steady state values and real business cycles features.

2.4.1 Data

We use quarterly data from the Federal Reserve Bank of St. Louis, Economic Research Division, covering years from Q1:1947-Q4:2018. We focus on four (real) time series, GDP, non-durable consumption, investment, and labor hours. The share of non-durable consumption to GDP has been 55.55% on average. We have also added durable consumption to the investment series, which brings the share of investment to GDP to 22% in our data. Nevertheless, we calibrate our model so that government spending is about 1/3 of GDP in the steady state. This strategy makes the private-investment share of GDP be plausibly smaller in our model.

2.4.2 Calibrating parameters and key steady-state features of the model

Since the data frequency is quarterly, our benchmark calibration uses $\beta = 0.99$, that reflects an annual rate of time preference equal to 4.1%. The tax rate on capital gains is $\tau_k = 50\%$, a value that, combined with the calibrated value of β gives a steady-state real interest rate of 2.02%. The value of B, set to 2.6 (following Cooley and Hansen, 1992), together with a labor-income tax rate set to $\tau_h = 23\%$, lead to a steady-state labor supply of 29%, which is well within the ballpark of the average employment data over time. The annual depreciation rate is set to 4.8%, leading to a steady-state investment share of GDP equal to 16%. The tax rate on consumption is set to $\tau_c = 8\%$, while we set the benchmark debt-to-GDP ratio to 110%. We balance the tax rates and the parameter of exogenous government spending, $\tilde{g}$, so as to achieve a steady-state government share of GDP equal to $32.86\% \simeq 1/3$.[15] The value of α is set to 43%, slightly higher than the value of 36% used by Cooley and Hansen (1992). This value of α is consistent with a notion of capital that is augmented by improved human capital in recent years. Preference parameter θ, capturing the relative preference between goods c_1 and c_2, is set to $\theta = 0.84$, following Cooley and Hansen (1992). With these parameter values, our model has a steady-state consumption share of GDP $\left(C_1^{ss} + C_2^{ss}\right)/Y = 52.82\%$, which is not far from the average 55.55% in the data (the mismatching deviation is -4.91%).

2.4.3 Business-cycle features of the model

The key business-cycle features we aim at capturing are GDP, consumption and investment volatility. The volatilities of the two shocks, are $\sigma_z = \sigma_u = 0.37\%$, for both the productivity and the fiscal shock. Given that the fiscal shock, u_t, hits the ratio G_t/Y_t, we assume that

[15] In the calibration, the value $\tilde{g}$ is derived endogenously, because it must be ensured that the debt-to-GDP ratio neither increases in the steady state nor it decreases, so as to avoid that the transversality condition of the household problem fails. We provide details on the derivation of $\tilde{g}$ with these properties in the Appendix.

the innovations to shocks u_t and z_t, $\varepsilon_{z,t}$ and $\varepsilon_{u,t}$, have a correlation coefficient $\rho_{zu} = -4\%$ which reflects the correlation observed in our database.

It is known that labor-hours volatility is difficult to be captured by a real-business cycle model, even with the labor-hoarding setup of Cooley and Hansen (1989, 1992) that we employ. Nevertheless, we report these volatilities in Table 2.2. Another set of calibration targets consists of the correlations of these macro variables (consumption, investment and labor) with output.

	Volatility				Correlation with GDP		
	Data	Model (End.)	Model (Ex.)		Data	Model (End.)	Model (Ex.)
GDP	1.60	1.61	1.56		–	–	–
Consumption	0.83	0.79	0.69		0.78	0.59	0.61
Investment	5.79	5.86	5.64		0.82	0.92	0.94
Labor Hours	1.87	1.31	1.26		0.87	0.93	0.91

Table 2.2 - Business-cycle features of the model and comparison with data. "Model (End.)" stands for the version of the benchmark model with endogenous policy while "Model (Ex.)" stands for the version of the model with exogenous, constant growth rate of money supply.

Table 2.2 shows the main calibration moments.[16] Volatility (standard deviation of log values of detrended variables, using the Hodrick-Prescott filter) of GDP is matched almost exactly by our model. Our benchmark model with endogenous monetary policy matches investment well, deviating only by 1.2% from the corresponding data moment. Nondurable consumption is slightly less successfully matched, deviating by 5.4% from the corresponding data moment.

[16]We have run 500 Monte-Carlo experiments of series of 1000 periods (quarters).

Simulated moments of correlation with output show that our model implied excess consumption smoothing and too high volatility of investment. The simulated correlation of labor with output does not deviate much from the data, deviating by 6.6% from the corresponding data moment.

The version of the model with exogenous monetary policy uses the steady-state value of the money growth rate found by the benchmark endogenous-policy model. This value is $\mu^{ss} = 2.03\%$, very close to the inflation target of the Federal Reserve (Fed) and the European Central Bank. We emphasize that μ^{ss} is endogenous in the benchmark endogenous-policy model, so μ^{ss} is the model-implied average inflation in the stationary equilibrium.

In order to compare the business-cycle features of the exogenous-monetary-policy model, we set the growth rate of money to the *constant value* $\mu^{ss} = 2.03\%$, taking it from the endogenous-monetary-policy model calculation.

The moment comparisons between the two versions of the model reveal that the model with an exogenous constant money-growth rate leads to lower first moments for all variables, GDP, consumption, investment and labor hours. Regarding the correlation of variables with GDP, the comparison between the two models gives mixed results. The model with constant exogenous money-growth rate implies weaker consumption smoothing. In addition, the model with constant exogenous money-growth rate implies more reactive investment to booms and busts. Yet, the model with constant exogenous money-growth rate implies slightly smoother reactions of employment to booms and busts.

To compare the propagation mechanisms of the two versions of the model, we plot impulse responses to the productivity and supply-side shocks.

2.4.4 Impulse responses

We compute impulse responses of the two shocks, the productivity shock, z_t, and the fiscal shock, u_t. Figures 2.1 through 2.6 below summarize the impact of these shocks on real output, Y_t, real investment, X_t, capital, K_t, hours worked, H_t, real consumption, C_t, and consumption components, $C_{1,t}$ and $C_{2,t}$, the nominal interest rate, i_t, inflation, π_t, real interest rate, r_t, the wage rate, w_t, the detrended total nominal bonds, $\hat{B}_t^T$ (after dividing by the nominal money supply), the growth rate of the nominal money supply, μ_t, the inverse of real money supply, $\hat{P}_t = P_t/\bar{M}_t^s$, real government spending, G_t, real tax revenues, $T_t = \tau_k r_t K_t + \tau_h w_t H_t + \tau_c C_t$, the detrended household-bond holdings, $\hat{B}_t^H$ (after dividing by the nominal money supply), the real total supply of bonds B_t^T, real central bank demand for bonds B_t^{CB} and real household demand for bonds B_t^H. All plotted impulse responses in Figures 1 through 6 depict percentage deviations from the model's corresponding deterministic steady-state values.

One of our goals is to detect the nature and the effects of endogenizing monetary policy, as implied by the policy rule $\mu_{t+1} = \Psi\left(z_t, u_t, \mu_t, K_t, \hat{B}_{t-1}^T\right)$. To this end, in every plot that depicts the endogenous-policy impulse responses, we also include the impulse responses of an exogenous policy rule. This exogenous policy rule uses a constant growth rate of the nominal money supply, μ, set to the implied optimal steady-state level of the endogenous model, μ^{ss}.

2.4.4.1 Productivity shock Figures 2.1, 2.2 and 2.3, summarize the effects of a 1% increase in the productivity shock, z_t, on all variables of interest of the model economy. In the panel of Figure 2.1, named "μ_t", we can see that endogenous optimal monetary policy implies a nominal growth rate of money, μ_t, that is steadily lower than its steady-state value after some periods (blue line). This means that optimal time-consistent policy targets a path of decelerating prices. Indeed, the panel of Figure 2.2 named "π_t" reveals that endogenous monetary policy achieves this price deceleration, because inflation drops below its steady-state value throughout transition to the steady state (blue line). The effect of this endogenous policy on price deceleration is particularly strong, as revealed by the panel of Figure 2.2 named "$\hat{P}_t$". Because $\hat{P}_t = P_t/\bar{M}_t^s$, the drop in the impulse response of $\hat{P}_t$ reveals that prices decelerate more than nominal money supply, $\bar{M}_t^s$, i.e., $\pi_t < \mu_t$ along the transition.

The fine tuning of endogenous monetary policy implied by $\pi_t < \mu_t$ is reflected in the panel of Figure 2.3 named "B_t^{CB}".[17] Specifically, the Central Bank intends to achieve an intervention in the bonds market that discourages bondholding by households, an effect revealed by the panel of Figure 2.3 named "B_t^H". Equation (2.16) shows that when households buy fewer bonds and when this is combined with decelerating prices, the consumption of the liquidity constrained good, $C_{1,t}$, will increase. This increase is profound in the panel of Figure 2.1 named "$C_{1,t}$". The impulse response of $C_{1,t}$ reveals the strongest effect that endogenous monetary policy has on real variables, after comparing the blue and the red impulse-response lines of all real-variable panels of Figure 2.1.

To understand why endogenous monetary policy aims at having a sizable impact on the consumption of the liquidity constrained good, $C_{1,t}$, we must examine the key mechanics of the model. The key feature of this model is that money influences real variables because of the binding constraint of the liquidity-constrained good, $C_{1,t}$ (see equation (2.16)). If one removes the liquidity constraint given by equation (2.16), that leads to a corner solution, then money would not have any effects on real variables.

[17] As $B_t^{CB} = \bar{B}_t^{CB}/P_t$ and equation (2.11) implies $\bar{B}_t^{CB} = \bar{M}_t^s$, $B_t^{CB} = 1/\hat{P}_t$. Therefore, the impulse response of B_t^{CB} is the flipped version of the impulse response of $\hat{P}_t$.

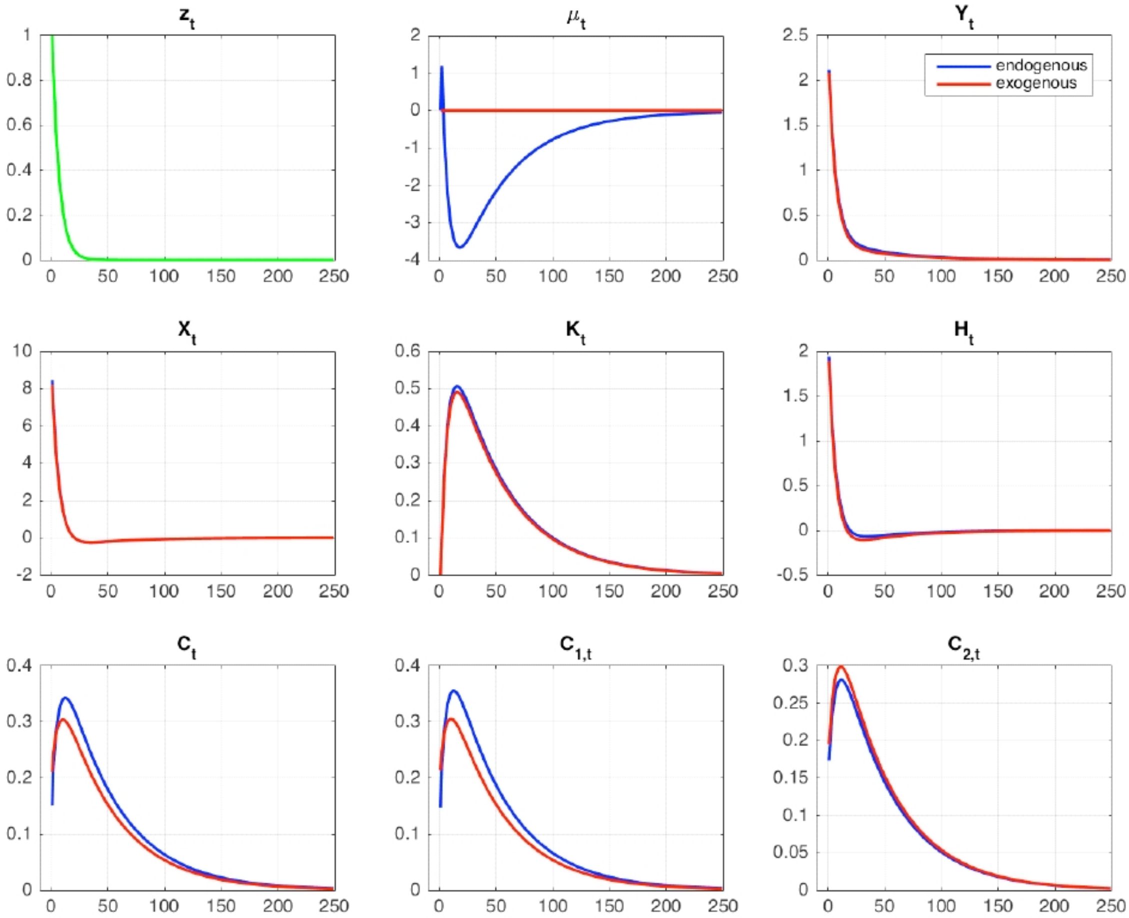

Figure 2.1 - Impulse responses of growth rate of the nominal money supply, μ_t, real output, Y_t, real investment, X_t, capital, K_t, hours worked, H_t, real consumption, C_t, and consumption components, $C_{1,t}$ and $C_{2,t}$ to a 1% increase in productivity shock z_t (with endogenous and exogenous monetary policy).

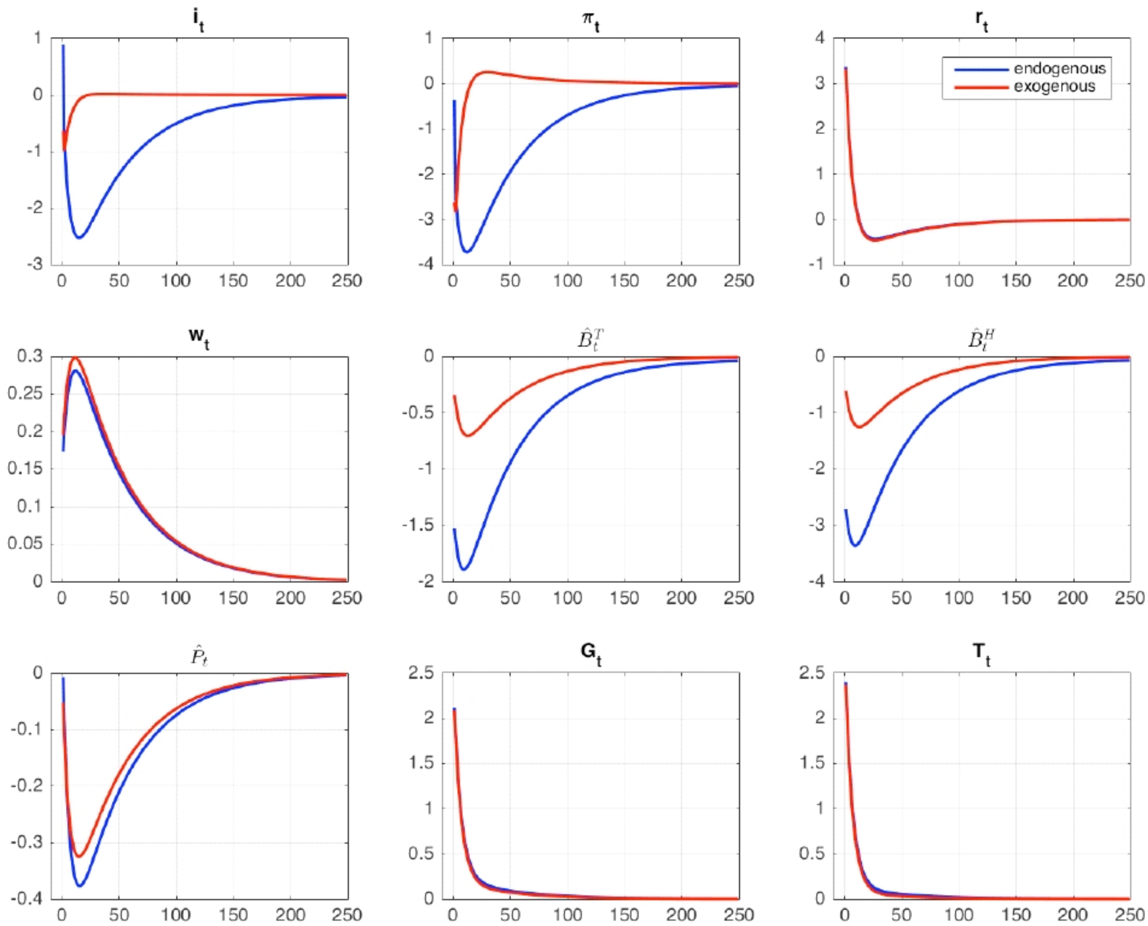

Figure 2.2 - Impulse responses of nominal interest rate, i_t, inflation, π_t, real interest rate, r_t, the wage rate, w_t, the detrended total nominal bonds, $\hat{B}_t^T$, the detrended household-bond holdings, $\hat{B}_t^H$, the inverse of real money supply, $\hat{P}_t$, real government spending, G_t and real tax revenues, T_t to a 1% increase in productivity shock z_t (with endogenous and exogenous monetary policy).

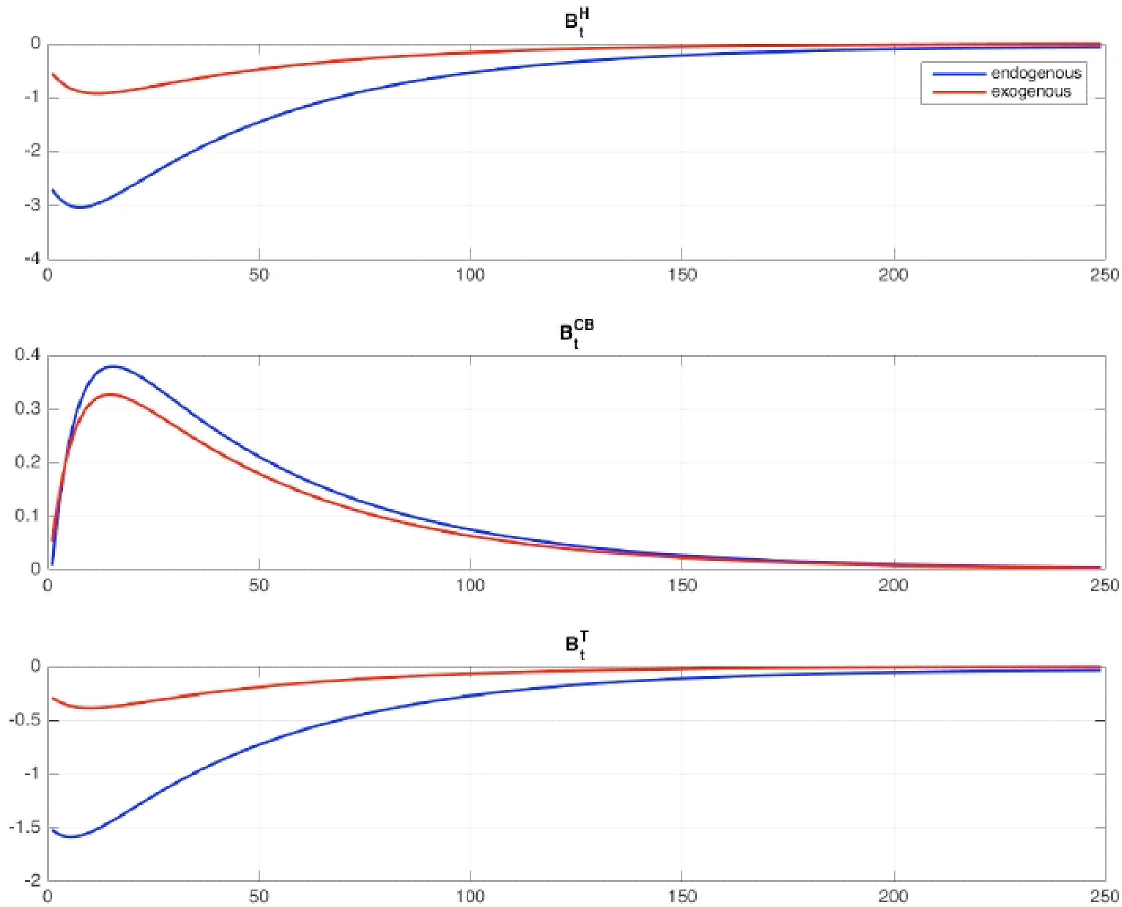

Figure 2.3 - Impulse responses of household real bond holdings B_t^H, central bank real bond holdings B_t^{CB} and total real bonds B_t^T to a 1% increase in productivity shock z_t (with endogenous and exogenous monetary policy).

Endogenous time-consistent monetary policy tries to relax the effects of the binding liquidity constraint given by equation (2.16), in order to make the corner solution more tolerable for households during transitions from shocks. Endogenous policy achieves this goal by reducing the growth rate of nominal money supply (see Figure 2.1). This policy will, eventually, reduce the inflation rate (see Figure 2.2). Given that the real interest rate under exogenous monetary policy is similar to the real interest rate under endogenous monetary policy, through the Fisher equation given by (2.25), a drop in the inflation rate implies a drop in the nominal interest rate. This drop in the nominal interest rate makes bonds less attractive for households. Through equation (2.16), buying fewer bonds and having real incomes increased by decelerating prices increases $C_{1,t}$ relative to $C_{2,t}$. In addition, aggregate consumption, $C_t = C_{1,t} + C_{2,t}$, is higher under endogenous monetary policy, revealing a source of welfare gains.

2.4.4.2 Fiscal shock Figures 2.4, 2.5 and 2.6, summarize the effects of a 1% increase in the fiscal shock, u_t, on all variables. Focusing on the exogenous-monetary policy version of the model is informative about the transmission mechanism of fiscal shocks to the private economy in the absence of Central-Bank reactions (regime of constant money-supply growth rate, all magenta lines in the figures). The crucial feature of a fiscal shock is that it increases deficits and creates a need for collecting more taxes in the future. This immediate impact of a fiscal shock creates a foreseen welfare loss. This welfare loss is internalized by households. Observing that leisure is the linear component of a quasilinear utility function, these welfare losses are reflected in a decrease in leisure.[18] This decrease in leisure is reflected in the panel of Figure 2.4 named "H_t" which depicts a substantial increase in labor supply, H_t. This increase in labor supply increases output. In turn, this increase in output pushes the price index, P_t, drastically down. This decrease in the price index is reflected by the reaction of inflation, π_t, appearing in the panel of Figure 2.5 named "π_t".

The drop in inflation upon the impact of the fiscal shock is substantial. For the economy with exogenous fiscal policy inflation drops by 25% of its steady-state value, i.e., it falls

[18]Notice that $u(c_{1,t}, c_{2,t}, h_t) = \theta \log c_{1,t} + (1-\theta) \log c_{2,t} - Bh_t = \theta \log c_{1,t} + (1-\theta) \log c_{2,t} + B(1-h_t) - B$, i.e., leisure is the numeraire, the linear component of this quasilinear utility function.

to $2.03\% \cdot (1 - 25\%) = 1.52\%$. This price deceleration immediately mitigates the negative impact of the fiscal shock (the foreseen welfare loss), providing both fiscal space to the government to avoid running deficits and the ability to financial markets to adjust. Specifically, in the panel of Figure 2.5 named "i_t" we can see that the initial drop in π_t causes a sharp drop in the nominal interest rate, i_t. This drop is implied by Fisher's equation, despite the increase in the real interest rate, seen in the panel of Figure 2.5 named "r_t" (this increase in r_t stems from the drop in the capital-labor ratio, K_t/H_t). This sharp drop in the nominal interest rate allows the government to avoid increasing the debt sharply. Instead, the panel of Figure 2.6 named "B_t^T" reveals that the sharp drop in the nominal interest rate allows for total real debt to decrease.

The immediate market reactions to a fiscal shock, the internalization of the welfare loss by households that leads to higher labor supply, to higher output and to lower nominal interest rates, give the opportunity for smoother transition paths. Specifically, the transition paths are characterized by higher nominal rates and higher inflation than their corresponding steady-state levels. Yet, this adjustment speeds up the recovery of the crowded-out investment, X_t, and the reduction in consumption.

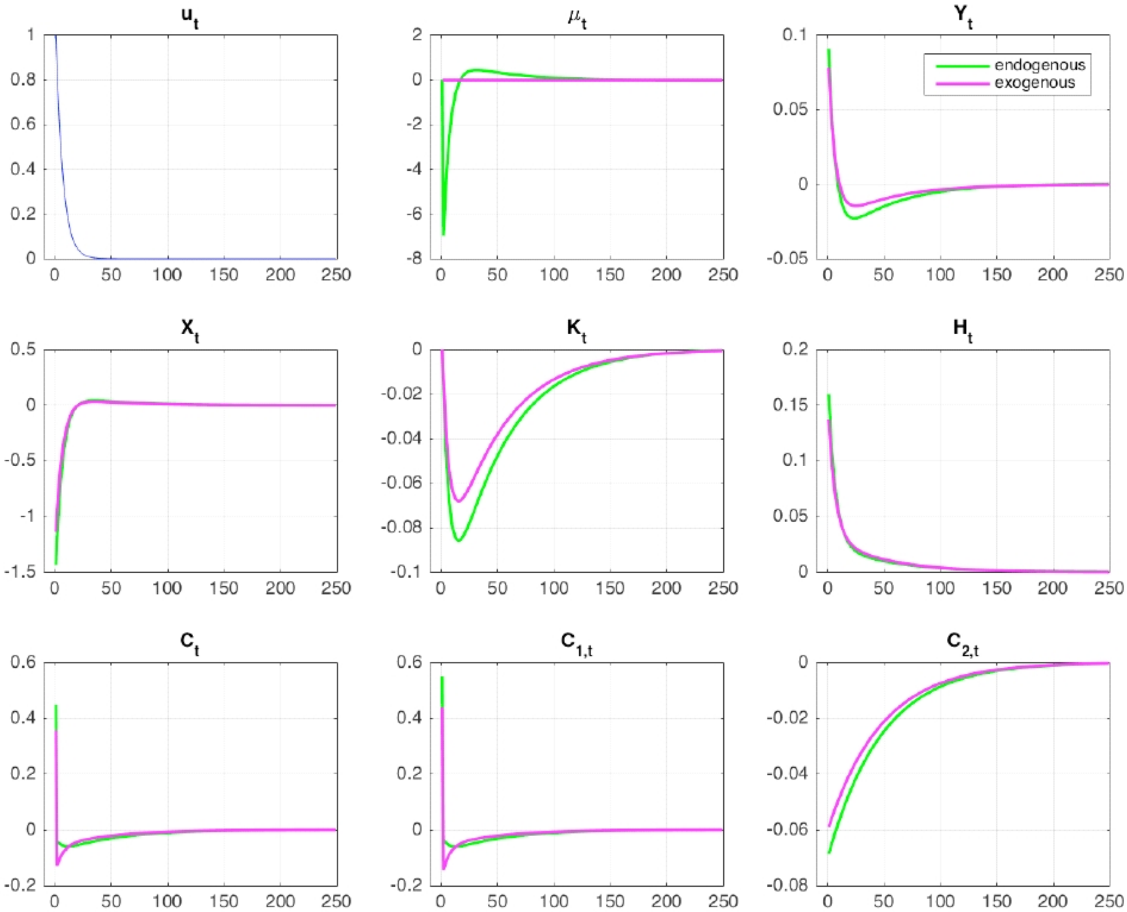

Figure 2.4 - Impulse responses of growth rate of the nominal money supply, μ_t, real output, Y_t, real investment, X_t, capital, K_t, hours worked, H_t, real consumption, C_t, and consumption components, $C_{1,t}$ and $C_{2,t}$ to a 1% increase in fiscal shock u_t (with endogenous and exogenous monetary policy).

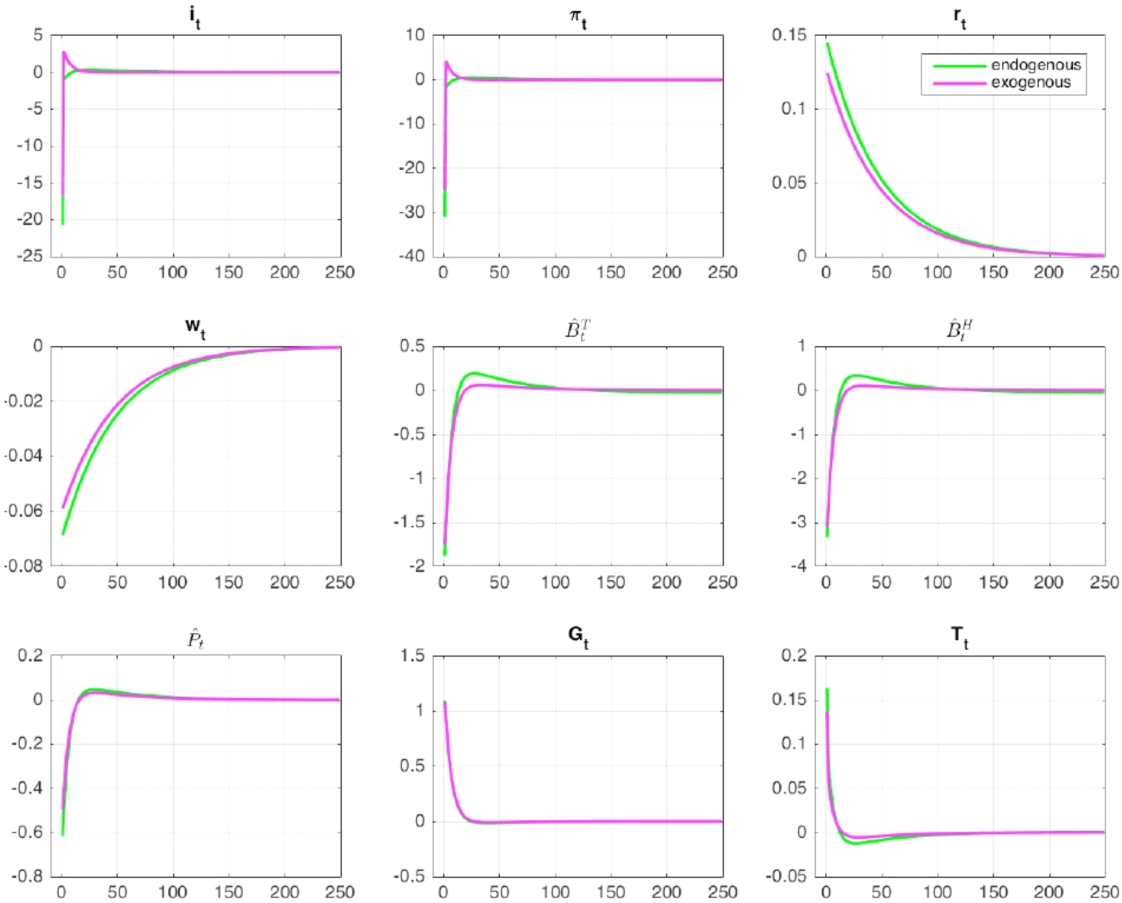

Figure 2.5 - Impulse responses of nominal interest rate, i_t, inflation, π_t, real interest rate, r_t, the wage rate, w_t, the detrended total nominal bonds, $\hat{B}_t^T$, the detrended household-bond holdings, $\hat{B}_t^H$, the inverse of real money supply, $\hat{P}_t$, real government spending, G_t and real tax revenues, T_t to a 1% increase in fiscal shock u_t (with endogenous and exogenous monetary policy).

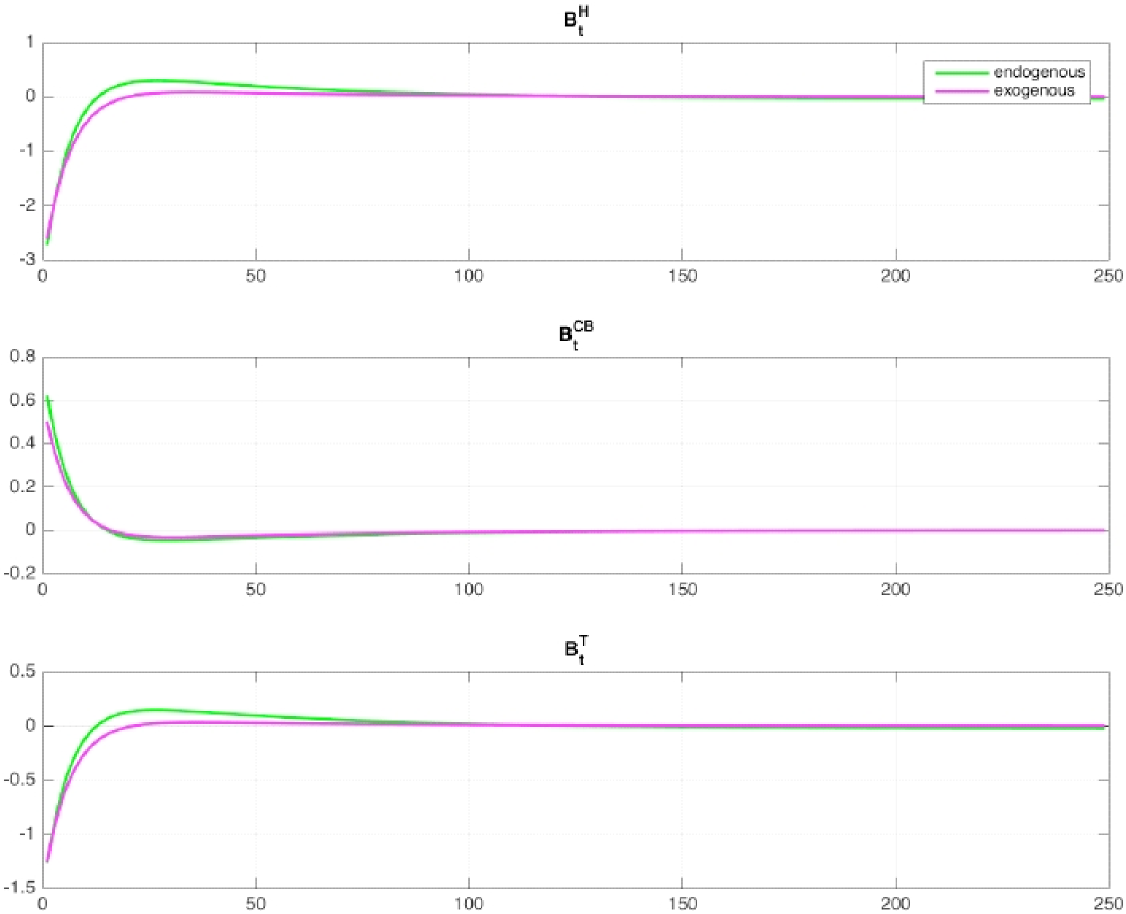

Figure 2.6 - Impulse responses of household real bond holdings B_t^H, central bank real bond holdings B_t^{CB} and total real bonds B_t^T to a 1% increase in fiscal shock u_t (with endogenous and exogenous monetary policy).

The model's mechanics of adjustment to fiscal shocks reflected by the exogenous-fiscal-policy impulse responses (magenta lines) are utilized by the Central Bank in the case of endogenous monetary policy (green lines). Specifically, the Central Bank decreases money-supply growth (see the panel of Figure 2.4 named "μ_t") and causes even sharper drops in inflation and in nominal interest rate in the period of the fiscal shock's impact (see the panels of Figure

2.5 named "π_t" and "i_t"). In this way, the Central Bank achieves a speeding up of the adjustment, characterized by two features. First, by fine-tuning money-supply growth, μ_t, it achieves inflation rates and nominal interest rates below their steady-state levels throughout the transition. This achieves a consumption-smoothing effect that increases welfare (see the panel of Figure 2.4 named "C_t"). Second, it achieves a more balanced mixture between the liquidity-constrained good, $C_{1,t}$, and the other good, $C_{2,t}$. Since households prefer mixtures of these two goods, monetary policy provides welfare gains through this channel as well.

2.5 Conclusion

We have studied a cash-in-advance model with two consumption goods, in which one type of consumption is liquidity-constrained. In equilibrium, the liquidity-constrained good always hits the liquidity constraint. As preferences over these two goods are strictly quasi-concave, if the liquidity constraint was absent, households would prefer a specific optimal mixture of these two goods. This optimal mixture is distorted by the presence of the liquidity constraint and the corner solution, causing a welfare loss. Due to this welfare loss, a Central Bank has the opportunity to improve welfare by improving this mixture through monetary policy. This cash-in-advance idea, without price rigidities, is one of the classic attempts to micro-found money demand and potential benefits of monetary policy. Nevertheless, little has been done to identify the optimal monetary policy. This paper has focused on this research question.

Because optimal policies face a credibility challenge by the presence of time inconsistency, we have focused on credible, time-consistent monetary policy rules. Specifically, among monetary-policy rules that respond to changes in the state of the economy, we have focused on forward-looking monetary-policy rules that are consistent with the forward-guidance practices of Central Banks. This forward-guidance practice encouraged us to formulate the optimal policy as a dynamic Stackelberg Markovian game between the Central Bank and private markets. Therefore, the recursive formulation for computing these optimal monetary policies tracked the optimal time-consistent fiscal-policy literature.

We focused on how optimal time-consistent rules of increasing the money supply through open-market operations respond to two types of shocks: (a) a total-factor productivity shock, and (b) a fiscal shock to government spending. An unsurprising part of our findings is that, for both types of shocks, the Central Bank tries to improve the mixture of the two consumption goods that is distorted by the presence of the liquidity constraint. Among all real variables, monetary policy mostly affects the dynamics of the two consumption goods. Yet, some differences in Central-Bank policy in dealing with the two types of shocks are informative.

A productivity shock offers production opportunities that eventually lead to a demand expansion, after some periods in which households have worked more and have saved more. Under a constant money-supply growth regime, this demand expansion leads to some above-steady-state inflation, that tightens the constraint on the liquidity-constrained good. In order to relax this constraint tightness, optimal policy achieves a low-inflation path that helps aggregate-consumption demand to expand even further. Therefore, optimal monetary policy improves the mixture of the two goods, but it increases consumption volatility. Apparently, the welfare gains of improving the mixture of the two goods exceed the welfare losses of higher consumption volatility.

A fiscal shock to government spending implies higher taxes for the future. Under a regime of constant money-supply growth, government spending eventually crowds out investment and leads to increased nominal interest rates and higher inflation. These fiscal-shock effects make the liquidity constraint relatively tighter and also cause substantial consumption fluctuations. The Central Bank tries to mitigate these effects by decreasing money growth supply initially and fine-tuning interest rates in order to both improve the mixture between the two types of goods and to smooth consumption.

A key feature in our model is the absence of price rigidities. This absence facilitates a clearer understanding of the role that cash-in-advance constraints play in the transmission of monetary policy. In this paper we have seen the role that cash-in-advance constraints play in the making of optimal time-consistent monetary policy. Nevertheless, the absence of price rigidities, combined with rational expectations, allows for some vivid market reactions in the first few periods that a shock occurs. These vivid market reactions are the outcome of internalized calculations of future impulse responses to shocks that are foreseen, in an environment that price adjustments are free of cost. The impulse responses to both shocks imply an over-reaction in the first periods that the shock hits the economy: some variables move toward the opposite direction in the beginning and soon rebound to the intuitive side of the shock response. These over-reactions, that the Central Bank internalizes in the model, are a feature of the model that may be inconsistent with data observations. Therefore, one direction for future research would be to introduce micro-founded price rigidities, e.g., by having menu costs as in Golosov and Lucas (2007).

Finally, a key direction for future work would be to introduce financial intermediaries, such as a banking sector, in order to partially disconnect prices of financial assets from the real economy. Such an extension should shed light on the way monetary-policy transmission reaches the real economy. In addition, introducing banks to a model with cash-in-advance constraints would help us understand whether Central Banks can internalize the interplay between the effects of their monetary-policy instruments and the real shocks to the economy.

2.6 Appendix

Before describing the algorithm, we formulate the model's constraints in a way that can accommodate the Bellman-equation formulation presented in the main body of the paper. From (2.7) and (2.11) we obtain $\overline{RCB}_t = i_{t-1}\bar{B}^{CB}_{t-1}$, which can be introduced into (2.1) in order to obtain,

$$\bar{G}_t = \bar{T}_t + \bar{B}^T_t - (1+i_{t-1})\,\bar{B}^T_{t-1} + i_{t-1}\bar{B}^{CB}_{t-1}\ . \tag{A.2.1}$$

Equation (2.23) implies,

$$\bar{B}^T_{t-1} = \bar{B}^H_{t-1} + \bar{B}^{CB}_{t-1}\ . \tag{A.2.2}$$

Combining (A.2.1) with (A.2.2) we obtain,

$$\bar{G}_t = \bar{T}_t + \bar{B}^T_t - (1+i_{t-1})\,\bar{B}^H_{t-1} - \bar{B}^{CB}_{t-1}\ . \tag{A.2.3}$$

After rewriting equation (A.2.3) in real terms, by dividing both sides of (A.2.3) by P_t, we use the transformation $\hat{x}_t = \bar{x}_t/\bar{M}^S_t$, in order to obtain,

$$G_t = T_t + \frac{\hat{B}^T_t}{\hat{P}_t} - \frac{(1+i_{t-1})\,\hat{B}^H_{t-1}}{(1+\mu_t)\hat{P}_t} - \frac{\hat{B}^{CB}_{t-1}}{(1+\mu_t)\hat{P}_t}\ . \tag{A.2.4}$$

Using condition (2.11), equation (A.2.2) becomes $\hat{B}^T_{t-1} = \hat{B}^H_{t-1} + 1$ and therefore equation (A.2.4) becomes,

$$G_t = T_t + \frac{\hat{B}^T_t}{\hat{P}_t} - \frac{(1+i_{t-1})\left(\hat{B}^T_{t-1} - 1\right)}{(1+\mu_t)\hat{P}_t} - \frac{1}{(1+\mu_t)\hat{P}_t}\ . \tag{A.2.5}$$

Using (A.2.5) together with (2.2), (2.3), (2.12), (2.14), (2.15) and (2.26), equation (A.2.5) becomes,

$$\begin{aligned}\frac{1+i_{t-1}}{1+\mu_t} \quad = \quad & \Big\{[\tau_c(1-\alpha) + \tau_k\alpha + \tau_c - \tilde{g}e^{u_t}(1+\tau_c)]\, e^{z_t}K^\alpha_t H^{1-\alpha}_t \\ & -\tau_k\delta K_t - \tau_c X_t + \frac{\hat{B}^T_t}{\hat{P}_t} - \frac{1}{(1+\mu_t)\hat{P}_t}\Big\}\,\frac{\hat{P}_t}{\left(\hat{B}^T_{t-1} - 1\right)}\ .\end{aligned} \tag{A.2.6}$$

Transforming the cash in advance constraint (2.16) we obtain,

$$(1+\tau_c)c_{1,t} = \frac{\hat{m}^H_{t-1}}{(1+\mu_t)\hat{P}_t} + \frac{1+i_{t-1}}{1+\mu_t}\frac{\hat{b}^H_{t-1}}{\hat{P}_t} - \frac{\hat{b}^H_t}{\hat{P}_t}\ . \tag{A.2.7}$$

Combining (A.2.7) with (A.2.6) gives us an expression for $c_{1,t}$,

$$c_{1,t} = \frac{1}{(1+\tau_c)}\left\{\frac{\hat{m}^H_{t-1}}{\hat{P}_t(1+\mu_t)} + \frac{\hat{b}^H_{t-1}}{\left(\hat{B}^T_{t-1}-1\right)}\left\{\left[\tau_c(1-\alpha)+\tau_k\alpha+\tau_c-\tilde{g}\ e^{u_t}(1+\tau_c)\right]\right.\right.$$

$$\left.\left. e^{z_t}K_t^{\alpha}H_t^{1-\alpha} - \tau_k\delta K_t - \tau_c X_t + \frac{\hat{B}^T_t}{\hat{P}_t} - \frac{1}{\hat{P}_t(1+\mu_t)}\right\} - \frac{\hat{b}^H_t}{\hat{P}_t}\right\}\ , \tag{A.2.8}$$

which is convenient for formulating momentary utility as a function of all state and control variables in the Bellman equation.

The reduced form of the household's budget constraint is obtained by combining (2.16) and (2.17). Expressed in real terms and detrended using the $\hat{x}_t = \bar{x}_t/\bar{M}^S_t$ transformation, it is given by,

$$(1+\tau_c)c_{2,t} + x_t + \frac{\hat{m}^H_t}{\hat{P}_t} = (1-\tau_h)w_t h_t + (1-\tau_k)R_t k_t + \tau_k\delta k_t. \tag{A.2.9}$$

Using (A.2.9) together with (2.14), (2.15), gives us a convenient expression for $c_{2,t}$,

$$c_{2,t} = \frac{1}{(1+\tau_c)}\left\{(1-\tau_h)(1-\alpha)e^{z_t}\left(\frac{K_t}{H_t}\right)^{\alpha} h_t + (1-\tau_k)\alpha e^{z_t}\left(\frac{K_t}{H_t}\right)^{\alpha-1} k_t + \right.$$

$$\left. +\tau_k\delta k_t - x_t - \frac{\hat{m}^H_t}{\hat{P}_t}\right\}\ , \tag{A.2.10}$$

facilitating the implementation of our numerical technique through the use of the Bellman equation given by (2.36).

Households' utility maximization and deterministic steady-state

For the quadratic approximation (QA) in the algorithm we need the steady state values of the state and action variables, that is why we solve the households' maximization problem using equations in real terms and apply the transformation $\hat{x}_t = \bar{x}_t / \bar{M}_t^S$. More specifically, (2.18) corresponds to,

$$\max_{\left\{\left(c_{1,t},c_{2,t},h_t,\hat{m}_t^H,\hat{b}_t^H,k_{t+1}\right)\right\}_{t=0}^{\infty}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u\left(c_{1,t}, c_{2,t}, h_t\right) \right\}$$

with $0 < \beta < 1$. Preferences are given by the following utility function,

$$u\left(c_{1,t}, c_{2,t}, h_t\right) = \theta \log c_{1,t} + (1-\theta) \log c_{2,t} - Bh_t$$

with $0 < \theta < 1$.

Households maximize their expected utility subject to (2.16) and (2.17), expressed in real terms and detrended using the $\hat{x}_t = \bar{x}_t / \bar{M}_t^S$ transformation.

We formulate the households' Lagrangian to solve the maximization problem:

$$\begin{aligned} L &= E_0 \left[\left\{ \sum_{t=0}^{\infty} \beta^t \left(\theta \log c_{1,t} + (1-\theta) \log c_{2,t} - Bh_t\right) \right\} + \right. \\ & + \sum_{t=0}^{\infty} \lambda_t \left((1-\tau_h) w_t h_t + (1-\tau_k) R_t k_t + \tau_k \delta k_t + \frac{\hat{m}_{t-1}^H}{\hat{P}_t (1+\mu_t)} \right. \\ & \left. + \frac{(1+i_{t-1}) \hat{b}_{t-1}^H}{\hat{P}_t (1+\mu_t)} - (1+\tau_c)(c_{1,t} + c_{2,t}) - k_{t+1} + (1-\delta) k_t - \frac{\hat{m}_t^H}{\hat{P}_t} - \frac{\hat{b}_t^H}{\hat{P}_t} \right) \\ & \left. + \sum_{t=0}^{\infty} \gamma_t \left(\frac{\hat{m}_{t-1}^H}{\hat{P}_t (1+\mu_t)} + \frac{(1+i_{t-1}) \hat{b}_{t-1}^H}{\hat{P}_t (1+\mu_t)} - \frac{\hat{b}_t^H}{\hat{P}_t} - (1+\tau_c) c_{1,t} \right) \right] \end{aligned}$$

First order conditions

FOC. w.r.t. $c_{1,t}$:

$$\begin{aligned} \frac{\beta^t \theta}{c_{1,t}} &= (\lambda_t + \gamma_t)(1+\tau_c) \\ \gamma_t &= \frac{\beta^t \theta}{c_{1,t}(1+\tau_c)} - \lambda_t \qquad \text{(A.2.11)} \end{aligned}$$

FOC. w.r.t. $c_{2,t}$:

$$\begin{aligned} \frac{\beta^t (1-\theta)}{c_{2,t}} &= \lambda_t (1+\tau_c) \\ \lambda_t &= \frac{\beta^t (1-\theta)}{c_{2,t}(1+\tau_c)} \qquad \text{(A.2.12)} \end{aligned}$$

FOC. w.r.t. h_t:

$$\begin{aligned} \beta^t B &= \lambda_t(1-\tau_h)w_t \\ \lambda_t &= \frac{\beta^t B}{(1-\tau_h)w_t} \end{aligned} \tag{A.2.13}$$

Using (A.2.12) and (A.2.13), we get,

$$c_{2,t} = \frac{(1-\theta)(1-\tau_h)w_t}{B(1+\tau_c)} \tag{A.2.14}$$

FOC. w.r.t. $\hat{m}_t^H$:

$$\frac{\lambda_t}{\hat{P}_t} = \frac{\lambda_{t+1}+\gamma_{t+1}}{\hat{P}_{t+1}(1+\mu_{t+1})} \ ,$$

using (A.2.11) and (A.2.13),

$$\begin{aligned} \frac{\beta^t B}{\hat{P}_t(1-\tau_h)w_t} &= \frac{\beta^{t+1}\theta}{c_{1,t+1}(1+\tau_c)\hat{P}_{t+1}(1+\mu_{t+1})} \\ c_{1,t+1} &= \frac{\beta\theta\hat{P}_t(1-\tau_h)w_t}{B(1+\tau_c)\hat{P}_{t+1}(1+\mu_{t+1})} \end{aligned} \tag{A.2.15}$$

Note that,

$$1+\pi_t = \frac{P_t}{P_{t-1}} \qquad \Longleftrightarrow \qquad 1+\pi_t = \frac{\hat{P}_t(1+\mu_t)}{\hat{P}_{t-1}} \tag{A.2.16}$$

$c_{1,t+1}$ can be rewritten using (A.2.16),

$$c_{1,t+1} = \frac{\beta\theta(1-\tau_h)w_t}{B(1+\tau_c)(1+\pi_{t+1})} \tag{A.2.17}$$

or $c_{1,t+1}$ can be rewritten using (A.2.14),

$$c_{1,t+1} = \frac{\beta\theta c_{2,t}}{(1-\theta)(1+\pi_{t+1})} \ . \tag{A.2.18}$$

FOC. w.r.t. $\hat{b}_t^H$:

$$\frac{\lambda_t+\gamma_t}{\hat{P}_t} = \frac{(\lambda_{t+1}+\gamma_{t+1})(1+i_t)}{\hat{P}_{t+1}(1+\mu_{t+1})} \ ,$$

using (A.2.11),

$$c_{1,t+1} = \frac{\beta(1+i_t)c_{1,t}\hat{P}_t}{\hat{P}_{t+1}(1+\mu_{t+1})} \ . \tag{A.2.19}$$

$c_{1,t+1}$ can be rewritten using (A.2.16),

$$c_{1,t+1} = \frac{\beta(1+i_t)c_{1,t}}{(1+\pi_{t+1})} \tag{A.2.20}$$

Using (A.2.18) and (A.2.20), we get,

$$c_{1,t} = \frac{\theta c_{2,t}}{(1-\theta)(1+i_t)} \tag{A.2.21}$$

Using (A.2.17) and (A.2.20), we get,

$$c_{1,t} = \frac{\theta w_t(1-\tau_h)}{B(1+\tau_c)(1+i_t)} \tag{A.2.22}$$

(A.2.22) period ahead,

$$c_{1,t+1} = \frac{\theta w_{t+1}(1-\tau_h)}{B(1+\tau_c)(1+i_{t+1})}$$

together with (A.2.17), is equal to,

$$\frac{w_{t+1}}{\beta w_t} = \frac{1+i_{t+1}}{1+\pi_{t+1}} \tag{A.2.23}$$

Total household consumption is equal to,

$$c_t = c_{1,t} + c_{2,t} \tag{A.2.24}$$

FOC. w.r.t. k_{t+1}:

$$\lambda_t = \lambda_{t+1}((1-\tau_k)R_{t+1} + \tau_k\delta + (1-\delta)) \ ,$$

using (A.2.13),

$$\frac{w_{t+1}}{\beta w_t} = 1 + (1-\tau_k)(R_{t+1} - \delta) \tag{A.2.25}$$

Combining (A.2.23) and (A.2.25), gives us the Fisher equation (approx. $i_t \approx (1-\tau_k)r_t + \pi_t$),

$$\frac{1+i_t}{1+\pi_t} = (1 + (1-\tau_k)r_t) \tag{A.2.26}$$

(A.2.26) is reasonable. The nominal government bond return in $t+1$, i_t is determined in t, the moment the government bond is bought. The real return of this government bond is known in $t+1$.

The nominal bond rate, i_t is equal to,

$$i_t = (1 + (1 - \tau_k)r_t)(1 + \pi_t) - 1$$

and the real capital return, r_t is equal to,

$$r_t = \frac{1}{1 - \tau_k} \left(\frac{1 + i_t}{1 + \pi_t} - 1 \right) \ .$$

Note that the net real capital return is equal to $(1 - \tau_k)r_t$.

Deterministic steady state

In the deterministic steady state all real variables are constant and markets clear. For real money balances to be constant in the deterministic steady state, the growth rate of money supply, μ_t needs to be equal to the inflation rate, π_t. (2.8) in real terms, is given by,

$$M_t^S = \frac{(1 + \mu_t)}{(1 + \pi_t)} M_{t-1}^S \ ,$$

and shows that for

$$M_t^S = M_{t-1}^S \Longrightarrow \mu_t = \pi_t \ .$$

So in the deterministic steady-state,

$$\mu^{ss} = \pi^{ss} \tag{A.2.27}$$

Also if (A.2.27) holds, then

$$\hat{P}_t = \hat{P}_{t-1} = \hat{P}^{ss}$$

This can be shown by,

$$\begin{aligned}
\hat{P}_t &= \hat{P}_{t-1} \\
\frac{P_t}{\bar{M}_t^S} &= \frac{P_{t-1}}{\bar{M}_{t-1}^S} \\
\frac{(1+\pi_t)\,P_{t-1}}{(1+\mu_t)\bar{M}_{t-1}^S} &= \frac{P_{t-1}}{\bar{M}_{t-1}^S} \\
1+\pi_t &= 1+\mu_t
\end{aligned}$$

Also, under market clearing of the money market,(2.22),

$$\hat{m}_t^H = \hat{m}_{t-1}^H = \left(\hat{m}^H\right)^{ss} = 1 \ .$$

Using (A.2.25),

$$\frac{1}{\beta} = 1 + (1-\tau_k)(R^{ss} - \delta)$$

Steady-state marginal product of capital, R^{ss},

$$R^{ss} = \frac{1-\beta}{\beta(1-\tau_k)} + \delta \ . \tag{A.2.28}$$

Steady-state real capital return, $r^{ss} = R^{ss} - \delta$,

$$^{r}ss = \frac{1-\beta}{\beta(1-\tau_k)} \ . \tag{A.2.29}$$

Using (2.14) and (A.2.28),

$$\frac{K^{ss}}{H^{ss}} = \left\{ \frac{1}{\alpha e^{z^{ss}}} \left(\frac{1-\beta}{\beta(1-\tau_k)} + \delta \right) \right\}^{\frac{1}{\alpha-1}} \ . \tag{A.2.30}$$

Steady-state wage, w^{ss}, using (2.15), is equal to,

$$w^{ss} = (1-\alpha)e^{z^{ss}} \left(\frac{K^{ss}}{H^{ss}} \right)^{\alpha} \tag{A.2.31}$$

Using (A.2.14), steady-state consumption of "credit good", C_2^{ss}, is equal to,

$$C_2^{ss} = \frac{(1-\theta)(1-\tau_h)w^{ss}}{B(1+\tau_c)} \tag{A.2.32}$$

Using (A.2.17) and (A.2.27), steady-state consumption of "cash good", C_1^{ss}, is equal to,

$$C_1^{ss} = \frac{\beta\theta(1-\tau_h)w^{ss}}{B(1+\tau_c)(1+\mu^{ss})} \tag{A.2.33}$$

Using (A.2.32) and (A.2.33), steady-state total consumption, C^{ss}, is equal to,

$$C^{ss} = C_1^{ss} + C_2^{ss} \tag{A.2.34}$$

Using (A.2.26), (A.2.27) and (A.2.29), steady-state nominal bond rate, i^{ss} is equal to,

$$i^{ss} = \frac{1+\mu^{ss}}{\beta} - 1 \tag{A.2.35}$$

Using (2.27), steady-state investment is equal to,

$$X^{ss} = \delta K^{ss} \tag{A.2.36}$$

To derive steady-state productive capital stock, use (2.3), (2.12) and (A.2.36) in (2.26),

$$Y^{ss} = C^{ss} + X^{ss} + G^{ss}$$

$$\begin{aligned} e^{z^{ss}}\left(K^{ss}\right)^{\alpha}\left(H^{ss}\right)^{1-\alpha} - \delta K^{ss} - \tilde{g}e^{u^{ss}}e^{z^{ss}}\left(K^{ss}\right)^{\alpha}\left(H^{ss}\right)^{1-\alpha} &= C^{ss} \\ K^{ss}\left\{e^{z}\left(\frac{K^{ss}}{H^{ss}}\right)^{\alpha-1} - \delta - \tilde{g}e^{u^{ss}}e^{z^{ss}}\left(\frac{K^{ss}}{H^{ss}}\right)^{\alpha-1}\right\} &= C^{ss} \end{aligned}$$

$$\begin{aligned} K^{ss} &= C^{ss}\left\{e^{z^{ss}}\left(\frac{K^{ss}}{H^{ss}}\right)^{\alpha-1} - \delta - \tilde{g}e^{u^{ss}}e^{z^{ss}}\left(\frac{K^{ss}}{H^{ss}}\right)^{\alpha-1}\right\}^{-1} \\ K^{ss} &= C^{ss}\left\{e^{z^{ss}}\left(\frac{K^{ss}}{H^{ss}}\right)^{\alpha-1}\left(1-\tilde{g}e^{u^{ss}}\right) - \delta\right\}^{-1} \end{aligned} \tag{A.2.37}$$

Using (A.2.30) and (A.2.37), steady-state hours worked are equal to,

$$H^{ss} = \left(\frac{K^{ss}}{H^{ss}}\right)^{-1} K^{ss} \tag{A.2.38}$$

Using (A.2.9), we derive steady-state $\hat{P}^{ss}$,

$$\begin{aligned}(1+\tau_c)C_2^{ss} &= (1-\tau_h)w^{ss}H^{ss}+(1-\tau_k)R^{ss}K^{ss}+\tau_k\delta K^{ss}-X^{ss}-\frac{1}{\hat{P}^{ss}} \\ \hat{P}^{ss} &= \{(1-\tau_h)w^{ss}H^{ss}+(1-\tau_k)R^{ss}K^{ss}+\tau_k\delta K^{ss}-X^{ss}-(1+\tau_c)C_2^{ss}\}^{-1}\end{aligned} \tag{A.2.39}$$

Steady-state real money balances is equal to,

$$\left(M^S\right)^{ss}=\left(\hat{P}^{ss}\right)^{-1}$$

Using the cash-in-advance constraint, (A.2.7) in the steady state, we derive the steady-state $\left(\hat{B}^H\right)^{ss}$,

$$\begin{aligned}(1+\tau_c)\,C_1^{ss} &= \frac{1}{\hat{P}^{ss}\,(1+\mu^{ss})}\left(1+(1+i^{ss})\left(\hat{B}^H\right)^{ss}-(1+\mu^{ss})\left(\hat{B}^H\right)^{ss}\right) \\ \left(\hat{B}^H\right)^{ss} &= \frac{(1+\tau_c)\,C_1^{ss}\hat{P}^{ss}\,(1+\mu^{ss})-1}{i-\mu^{ss}}\ .\end{aligned} \tag{A.2.40}$$

Using (2.11),we derive the steady-state $\left(\hat{B}^{CB}\right)^{ss}$,

$$\left(\hat{B}^{CB}\right)^{ss}=1\ . \tag{A.2.41}$$

The sum of (A.2.40) and (A.2.41), gives us the steady-state $\left(\hat{B}^T\right)^{ss}$,

$$\left(\hat{B}^T\right)^{ss}=\left(\hat{B}^H\right)^{ss}+\left(\hat{B}^{CB}\right)^{ss} \tag{A.2.42}$$

Endogenizing $\tilde{g}$

Use (A.2.36) and (A.2.39), such that,

$$\frac{1}{\hat{P}^{ss}}=\underbrace{-(1+\tau_c)C_2^{ss}}_{\psi_1}+\underbrace{\left[(1-\tau_h)w^{ss}\left(\frac{K^{ss}}{H^{ss}}\right)^{-1}+(1-\tau_k)R^{ss}-(1-\tau_k)\delta\right]K^{ss}}_{\psi_2}\ . \tag{A.2.43}$$

From $Y^{ss} = C^{ss} + X^{ss} + G^{ss}$, we derive,

$$K^{ss} = \frac{C^{ss}}{\underbrace{e^{z^{ss}}\left(\frac{K^{ss}}{H^{ss}}\right)^{\alpha-1} - \delta}_{\psi_3} - \underbrace{e^{u^{ss}+z^{ss}}\left(\frac{K^{ss}}{H^{ss}}\right)^{\alpha-1}}_{\psi_4}\tilde{g}} \ . \tag{A.2.44}$$

Using (A.2.43) and (A.2.44), gives us,

$$\frac{1}{\hat{P}^{ss}} = \psi_1 + \frac{\psi_2 C^{ss}}{\psi_3 - \psi_4 \tilde{g}} \ . \tag{A.2.45}$$

We know that nominal direct receipts from the Central Bank in period t are equal to,

$$\overline{RCB}_t = i_{t-1}\overline{M}_{t-1^S} \ . \tag{A.2.46}$$

In real terms, (A.2.46), is equal to,

$$RCB_t = \frac{i_{t-1}M^S_{t-1}}{1+\pi_t} = \frac{i_{t-1}}{(1+\pi_t)\,\hat{P}_{t-1}} \ ,$$

and in the steady state,

$$RCB^{ss} = \frac{i}{(1+\mu^{ss})\,\hat{P}^{ss}} \tag{A.2.47}$$

(A.2.45) together with (A.2.47), is equal to,

$$RCB^{ss} = \frac{i^{ss}}{1+\mu^{ss}}\left(\psi_1 + \frac{\psi_2 C^{ss}}{\psi_3 - \psi_4 \tilde{g}}\right) \ . \tag{A.2.48}$$

We use (2.5) in the steady state,

$$\tilde{g}e^{u^{ss}}Y^{ss} = T^{ss} + \left(B^T\right)^{ss}\left(1 - \frac{1+i^{ss}}{1+\pi^{ss}}\right) + RCB^{ss} \ ,$$

together with (2.2) in the steady state and (A.2.35), gives

$$\tilde{g}e^{u^{ss}}Y^{ss} = \tau_c C^{ss} + \tau_h w^{ss} H^{ss} + \tau_k \left(R^{ss} - \delta\right) K^{ss} + \left(B^T\right)^{ss}\left(1 - \frac{1}{\beta}\right) + RCB^{ss} \ . \tag{A.2.49}$$

We rewrite (A.2.49) using (2.12), (2.14), (2.15), (2.26) and (2.27) in the steady state,

$$\tilde{g} = \underbrace{\frac{1}{(1+\tau_c)\, e^{u^{ss}}}}_{\xi_1} \frac{RCB^{ss}}{Y^{ss}} +$$
$$+ \underbrace{\frac{1}{(1+\tau_c)\, e^{u^{ss}}} \left(\tau_c - \frac{\tau_c \delta}{e^{z^{ss}}} \left(\frac{K^{ss}}{H^{ss}} \right)^{1-\alpha} + \tau_h (1-\alpha) + \tau_k \alpha - \frac{\tau_k \delta}{e^{z^{ss}}} \left(\frac{K^{ss}}{H^{ss}} \right)^{1-\alpha} + \frac{\left(B^T\right)^{ss}}{Y^{ss}} \left(1 - \frac{1}{\beta} \right) \right)}_{\xi_2}$$

$$\tilde{g} = \xi_1 \frac{RCB^{ss}}{Y^{ss}} + \xi_2 \qquad \text{(A.2.50)}$$

Using (2.14) in the steady state,

$$Y^{ss} = \frac{1}{\alpha} R^{ss} K^{ss} \ ,$$

together with (A.2.44), is equal to,

$$Y^{ss} = \frac{1}{\alpha} R^{ss} \frac{C^{ss}}{\psi_3 - \psi_4 \tilde{g}} \ . \qquad \text{(A.2.51)}$$

Use (A.2.48) and (A.2.51) in (A.2.50),

$$\tilde{g} = \xi_1 \frac{\left(\frac{i^{ss}}{1+\mu^{ss}} \left(\psi_1 + \frac{\psi_2 C^{ss}}{\psi_3 - \psi_4 \tilde{g}} \right) \right)}{\left(\frac{1}{\alpha} R^{ss} \frac{C^{ss}}{\psi_3 - \psi_4 \tilde{g}} \right)} + \xi_2$$

$$\tilde{g} = \xi_1 \frac{\alpha i^{ss}}{(1+\mu^{ss})\, R^{ss}} \left[\psi_2 + \frac{\psi_1}{C^{ss}} (\psi_3 - \psi_4 \tilde{g}) \right] + \xi_2$$
$$\tilde{g} = \underbrace{\xi_1 \frac{\alpha i^{ss}}{(1+\mu^{ss})\, R^{ss}} \left(\psi_2 + \frac{\psi_1 \psi_3}{C^{ss}} \right) + \xi_2}_{\eta_1} - \underbrace{\xi_1 \frac{\alpha i^{ss}}{(1+\mu)\, R^{ss}} \frac{\psi_1 \psi_4}{C^{ss}}}_{\eta_2} \tilde{g}$$

$$\tilde{g} = \frac{\eta_1}{1+\eta_2}$$

Algorithm

Step 1 The outer loop, called the "μ^{ss} loop" is a loop that guarantees accuracy of quadratic approximations. Specifically, in the process of calculating the optimal money supply rule, Ψ, given by,

$$\mu_{t+1} = \Psi \left(z_t, u_t, \mu_t, K_t, \hat{B}^T_{t-1} \right) = \Psi (S_t) \ ,$$

we approximate all functions around the steady-state values of state variables and action variables. In order to ensure that the implied steady-state variable for the optimal money-supply growth rate, μ^{ss}, is accurately computed, we check if

$$\mu^{ss} = \Psi\left(z^{ss}, u^{ss}, \mu^{ss}, K^{ss}, \left(\hat{B}^{T}\right)^{ss}\right) = \Psi\left(S^{ss}\right) \ ,$$

holds, where the superscript "ss" denotes steady-state values. Before entering the μ^{ss} loop, we take a guess on μ^{ss}.

Step 2 The next loop, called the "Ψ loop", iterates on the optimal money supply rule, Ψ. We start with an initial guess for the policy rule Ψ. We denote the initial guess as $\Psi^{(0)}$, while the iteration index is n_{ψ}. Therefore, the symbol $\Psi^{\left(n_{\psi}\right)}$ stands for the policy-rule update. The form of the policy rule $\Psi^{\left(n_{\psi}\right)}$ ($\mu' = \Psi^{\left(n_{\psi}\right)}\left(z, u, \mu, K, \hat{B}^{T}\right)$) is given by,

$$\mu' = \bar{\boldsymbol{\Psi}}^{\left(n_{\psi}\right)} \cdot \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ \hat{B}^{T} \end{bmatrix} = \underbrace{\begin{bmatrix} \psi_{cst.}^{\left(n_{\psi}\right)} & \psi_{z}^{\left(n_{\psi}\right)} & \psi_{u}^{\left(n_{\psi}\right)} & \psi_{\mu}^{\left(n_{\psi}\right)} & \psi_{K}^{\left(n_{\psi}\right)} & \psi_{\hat{B}^{T}}^{\left(n_{\psi}\right)} \end{bmatrix}}_{\substack{\shortparallel \\ \bar{\boldsymbol{\Psi}}^{\left(n_{\psi}\right)}}} \cdot \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ \hat{B}^{T} \end{bmatrix} , \tag{A.2.52}$$

Step 3

In this step we enter a new loop, the competitive-equilibrium loop, or the "CE loop", iterating on the value function of the representative household, imposing market-clearing conditions in order to ensure that aggregate variables and prices are consistent with individual decision rules. The Bellman equation of the representative household in competitive equilibrium is given by,

$$V\left(z, u, \mu, K, k, \hat{m}^H, \hat{B}^T, \hat{b}^H \mid \Psi^{\left(n_\psi\right)}\right) =$$

$$= \max_{h, x, \hat{m}^{H\,\prime}, \hat{b}^{H\,\prime}} \left\{ u\left(z, u, \mu, K, k, \hat{m}^H, \hat{B}^T, \hat{b}^H, h, x, \hat{m}^{H\,\prime}, \hat{b}^{H\,\prime}, H, X, \hat{P}, \hat{B}^{T\,\prime}\right) + \right.$$

$$\left. + \beta E_t \left[V\left(z', u', \mu', K', k', \hat{m}^{H\,\prime}, \hat{B}^{T\,\prime}, \hat{b}^{H\,\prime} \mid \Psi^{\left(n_\psi\right)}\right)\right] \right\} \ , \quad \text{(A.2.53)}$$

in which,

$$u\left(z, u, \mu, K, k, \hat{m}^H, \hat{B}^T, \hat{b}^H, h, x, \hat{m}^{H\,\prime}, \hat{b}^{H\,\prime}, H, X, \hat{P}, \hat{B}^{T\,\prime}\right) = \theta \log c_1 + (1-\theta) \log c_2 - Bh \ , \quad \text{(A.2.54)}$$

with c_1 given by equation (A.2.8) and c_2 given by equation (A.2.10). Bellman equation (A.2.53) represents equation (2.36) in the main body of the paper. Equation (A.2.54) combined with (A.2.8) and (A.2.10), give the functional form of the momentary utility function in equation (2.36) in the main body of the paper.

The fixed point of (A.2.53) is calculated using quadratic approximation around the steady-state values of action and state variables calculated above. Some of these steady-state values depend on the (guessed) value μ^{ss} of the outer loop introduced by Step 1 above.

We take a guess on the value function's quadratic form, denoted by $Q^{(0)}_{V \mid \Psi^{\left(n_\psi\right)}}$, and use the right-hand-side (RHS) of the above Bellman equation as a mapping. A good guess on the quadratic form is a negative-definite matrix, namely $Q^{(0)}_{V \mid \Psi^{\left(n_\psi\right)}} = -I_9$, in which I_9 is a 9×9 identity matrix. This guess implies that the RHS of the Bellman equation corresponds to a strictly concave objective with respect to maximizers, enabling the existence of a global maximum. In brief, the Bellman equation acts as a contraction mapping, which, for the n_{CE}-th step of the updating process, can be written as,

$$
\begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \end{bmatrix}^T Q^{(n_{CE}+1)}_{V|\Psi^{\left(n_\psi\right)}} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \end{bmatrix} = \max_{h,x,\hat{m}^{H\,\prime},\hat{b}^{H\,\prime}} \left\{ \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \\ h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \\ H \\ X \\ \hat{P} \\ \hat{B}^{T\,\prime} \\ 1 \\ z' \\ u' \\ \mu' \\ K' \\ k' \\ \hat{m}^{H\,\prime} \\ \hat{B}^{T\,\prime} \\ \hat{b}^{H\,\prime} \end{bmatrix}^T \begin{bmatrix} Q_u & \mathbf{0} \\ \mathbf{0} & \beta Q^{(n_{CE})}_{V|\Psi^{\left(n_\psi\right)}} \end{bmatrix} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \\ h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \\ H \\ X \\ \hat{P} \\ \hat{B}^{T\,\prime} \\ 1 \\ z' \\ u' \\ \mu' \\ K' \\ k' \\ \hat{m}^{H\,\prime} \\ \hat{B}^{T\,\prime} \\ \hat{b}^{H\,\prime} \end{bmatrix} \right\}, \tag{A.2.55}
$$

subject to the laws of motion (2.4), (2.13), (A.2.52) and (2.27).

Next, we incorporate the constraints (2.4), (2.13), (A.2.52) and (2.27), into the right-hand-side of (A.2.55). This is achieved through matrix Ω_1, namely,

$\Omega_1 =$

$$
\begin{array}{cc}
\begin{array}{ccccccccccccccccc}
1 & z & u & \mu & K & k & \hat{m}^H & \hat{B}^T & \hat{b}^H & h & x & \hat{m}^{H\,\prime} & \hat{b}^{H\,\prime} & H & X & \hat{P} & \hat{B}^{T\,\prime} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} & \\
\left[\begin{array}{ccccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\psi^{(n_\psi)}_{cst.} & \psi^{(n_\psi)}_{z} & \psi^{(n_\psi)}_{u} & \psi^{(n_\psi)}_{\mu} & \psi^{(n_\psi)}_{K} & 0 & 0 & \psi^{(n_\psi)}_{\hat{B}^T} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1-\delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1-\delta & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right] &
\begin{array}{c}
\leftarrow 1 \\ \leftarrow z \\ \leftarrow u \\ \leftarrow \mu \\ \leftarrow K \\ \leftarrow k \\ \leftarrow \hat{m}^H \\ \leftarrow \hat{B}^T \\ \leftarrow \hat{b}^H \\ \leftarrow h \\ \leftarrow x \\ \leftarrow \hat{m}^{H\,\prime} \\ \leftarrow \hat{b}^{H\,\prime} \\ \leftarrow H \\ \leftarrow X \\ \leftarrow \hat{P} \\ \leftarrow \hat{B}^{T\,\prime} \\ \leftarrow 1 \\ \leftarrow z' \\ \leftarrow u' \\ \leftarrow \mu' \\ \leftarrow K' \\ \leftarrow k' \\ \leftarrow \hat{m}^{H\,\prime} \\ \leftarrow \hat{B}^{T\,\prime} \\ \leftarrow \hat{b}^{H\,\prime}
\end{array}
\end{array}
$$

We define a 17×17 matrix,

$$
\Phi_1^{(n_{CE})} = \Omega_1^T \begin{bmatrix} Q_u & \mathbf{0} \\ \mathbf{0} & \beta Q^{(n_{CE})}_{V|\Psi^{(n_\psi)}} \end{bmatrix} \Omega_1 \qquad \text{(A.2.56)}
$$

The RHS of (A.2.55) can we rewritten as an unconstrained maximization problem, incorporating the constraints by using (A.2.56), namely,

$$\max_{h,\, x,\, \hat{m}^{H\,\prime},\, \hat{b}^{H\,\prime}} \left\{ \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \\ h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \\ H \\ X \\ \hat{P} \\ \hat{B}^{T\,\prime} \end{bmatrix}^T \Phi_1^{(n_{CE})} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \\ h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \\ H \\ X \\ \hat{P} \\ \hat{B}^{T\,\prime} \end{bmatrix} \right\} \tag{A.2.57}$$

The first-order conditions of (A.2.57) give,

$$\Phi_{1\ [10:13,1:17]}^{(n_{CE})} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \\ h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \\ H \\ X \\ \hat{P} \\ \hat{B}^{T\,\prime} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{A.2.58}$$

where the subscript "[10:13, 1:17]" denotes the submatrix of $\Phi_1^{(n_{CE})}$ comprised by the rows from 10 to 13, and by the columns from 1 to 17. In order to impose the market-clearing conditions we construct a 17×10 matrix MC_{CE}, given by,

$$MC_{CE} \quad = \quad \begin{array}{c} \begin{array}{cccccccccc} 1 & z & u & \mu & K & \hat{B}^T & H & X & \hat{P} & \hat{B}^{T\,\prime} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \left[\begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \end{array} \begin{array}{l} \\ \\ \leftarrow 1 \\ \leftarrow z \\ \leftarrow u \\ \leftarrow \mu \\ \leftarrow K \\ \leftarrow k \\ \leftarrow \hat{m}^H \\ \leftarrow \hat{B}^T \\ \leftarrow \hat{b}^H \\ \leftarrow h \\ \leftarrow x \\ \leftarrow \hat{m}^{H\,\prime} \\ \leftarrow \hat{b}^{H\,\prime} \\ \leftarrow H \\ \leftarrow X \\ \leftarrow \hat{P} \\ \leftarrow \hat{B}^{T\,\prime} \end{array} \tag{A.2.59}$$

To impose market clearing combine (A.2.59) with (A.2.58) in the following way,

$$\Phi_{1\;[10:13,1:17]}^{(n_{CE})}\; MC_{CE} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ \hat{B}^T \\ H \\ X \\ \hat{P} \\ \hat{B}^{T\,\prime} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{A.2.60}$$

in order to ensure that individual decision rules comply with aggregate laws of motion. Specifically, we impose the set of conditions concisely denoted by "$s_t = S_t$" and "$a_t = A_t$" in equations (2.31) and (2.34) in the main body of the paper.

Define a 4×10 matrix,

$$\Phi_2^{(n_{CE})} = \Phi_{1\;[10:13,1:17]}^{(n_{CE})}\; MC_{CE},$$

such that we can derive from (A.2.60), the optimal law of motion for H, X, $\hat{P}$ and $\hat{B}^{T\,\prime}$,

$$\begin{bmatrix} H \\ X \\ \hat{P} \\ \hat{B}^{T\,\prime} \end{bmatrix} = \Phi_3^{(n_{CE})} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ \hat{B}^{T} \end{bmatrix}. \tag{A.2.61}$$

where the 4×6 matrix $\Phi_3^{(n_{CE})}$ is given by,

$$\Phi_3^{(n_{CE})} = -\left[\Phi_{2\,[1:4,7:10]}^{(n_{CE})}\right]^{-1} \Phi_{2\,[1:4,1:6]}^{(n_{CE})} .$$

Matrix $\Phi_3^{(n_{CE})}$ summarizes the optimal decision rules of the aggregate action variables $A = \left(H, X, \hat{P}, \hat{B}^{T\,\prime}\right)$. Given these decision rules, we reformulate (A.2.55), by initiating a matrix which includes all household constraints (2.4), (2.13), (A.2.52) and (2.27), and all decision rules given by $\Phi_3^{(n_{CE})}$ through equation (A.2.61), namely,

$\Omega_2 =$

$$
\begin{array}{c}
\begin{array}{ccccccccccccc}
1 & z & u & \mu & K & k & \hat{m}^H & \hat{B}^T & \hat{b}^H & h & x & \hat{m}^{H\,\prime} & \hat{b}^{H\,\prime} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array}
\\
\left[\begin{array}{ccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\Phi^{(n_{CE})}_{3\,[1,1]} & \Phi^{(n_{CE})}_{3\,[1,2]} & \Phi^{(n_{CE})}_{3\,[1,3]} & \Phi^{(n_{CE})}_{3\,[1,4]} & \Phi^{(n_{CE})}_{3\,[1,5]} & 0 & 0 & \Phi^{(n_{CE})}_{3\,[1,6]} & 0 & 0 & 0 & 0 & 0 \\
\Phi^{(n_{CE})}_{3\,[2,1]} & \Phi^{(n_{CE})}_{3\,[2,2]} & \Phi^{(n_{CE})}_{3\,[2,3]} & \Phi^{(n_{CE})}_{3\,[2,4]} & \Phi^{(n_{CE})}_{3\,[2,5]} & 0 & 0 & \Phi^{(n_{CE})}_{3\,[2,6]} & 0 & 0 & 0 & 0 & 0 \\
\Phi^{(n_{CE})}_{3\,[3,1]} & \Phi^{(n_{CE})}_{3\,[3,2]} & \Phi^{(n_{CE})}_{3\,[3,3]} & \Phi^{(n_{CE})}_{3\,[3,4]} & \Phi^{(n_{CE})}_{3\,[3,5]} & 0 & 0 & \Phi^{(n_{CE})}_{3\,[3,6]} & 0 & 0 & 0 & 0 & 0 \\
\Phi^{(n_{CE})}_{3\,[4,1]} & \Phi^{(n_{CE})}_{3\,[4,2]} & \Phi^{(n_{CE})}_{3\,[4,3]} & \Phi^{(n_{CE})}_{3\,[4,4]} & \Phi^{(n_{CE})}_{3\,[4,5]} & 0 & 0 & \Phi^{(n_{CE})}_{3\,[4,6]} & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\psi^{(n_\psi)}_{cst.} & \psi^{(n_\psi)}_{z} & \psi^{(n_\psi)}_{u} & \psi^{(n_\psi)}_{\mu} & \psi^{(n_\psi)}_{K} & 0 & 0 & \psi^{(n_\psi)}_{\hat{B}^T} & 0 & 0 & 0 & 0 & 0 \\
\Phi^{(n_{CE})}_{3\,[2,1]} & \Phi^{(n_{CE})}_{3\,[2,2]} & \Phi^{(n_{CE})}_{3\,[2,3]} & \Phi^{(n_{CE})}_{3\,[2,4]} & \Phi^{(n_{CE})}_{3\,[2,5]} + 1 - \delta & 0 & 0 & \Phi^{(n_{CE})}_{3\,[2,6]} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\Phi^{(n_{CE})}_{3\,[4,1]} & \Phi^{(n_{CE})}_{3\,[4,2]} & \Phi^{(n_{CE})}_{3\,[4,3]} & \Phi^{(n_{CE})}_{3\,[4,4]} & \Phi^{(n_{CE})}_{3\,[4,5]} & 0 & 0 & \Phi^{(n_{CE})}_{3\,[4,6]} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
\begin{array}{l}
\\ \\
\leftarrow 1 \\ \leftarrow z \\ \leftarrow u \\ \leftarrow \mu \\ \leftarrow K \\ \leftarrow k \\ \leftarrow \hat{m}^H \\ \leftarrow \hat{B}^T \\ \leftarrow \hat{b}^H \\ \leftarrow h \\ \leftarrow x \\ \leftarrow \hat{m}^{H\,\prime} \\ \leftarrow \hat{b}^{H\,\prime} \\
\leftarrow H \\ \leftarrow X \\ \leftarrow \hat{P} \\ \leftarrow \hat{B}^{T\,\prime} \\
\leftarrow 1 \\ \leftarrow z' \\ \leftarrow u' \\ \leftarrow \mu' \\ \leftarrow K' \\ \leftarrow k' \\ \leftarrow \hat{m}^{H\,\prime} \\ \leftarrow \hat{B}^{T\,\prime} \\ \leftarrow \hat{b}^{H\,\prime}
\end{array}
$$

Therefore, the 13×13 matrix,

$$\Phi_4^{(n_{CE})} = \Omega_2^T \begin{bmatrix} Q_u & \mathbf{0} \\ \mathbf{0} & \beta Q^{(n_{CE})}_{V|\Psi^{(n_\psi)}} \end{bmatrix} \Omega_2 \ , \tag{A.2.62}$$

imposes all household constraints and all aggregate decision rules on the RHS of (A.2.55) transforming (A.2.55) into an unconstrained optimization problem that respects market-clearing conditions. Therefore, maximizing,

$$\max_{h,\, x,\, \hat{m}^{H\,\prime},\, \hat{b}^{H\,\prime}} \left\{ \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \\ h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \end{bmatrix}^T \Phi_4^{(n_{CE})} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \\ h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \end{bmatrix} \right\}, \tag{A.2.63}$$

and after deriving the first-order conditions, we obtain the individual optimality conditions,

$$\begin{bmatrix} h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \end{bmatrix} = \Phi_5^{(n_{CE})} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \end{bmatrix} \tag{A.2.64}$$

where the 4×9 matrix $\Phi_5^{(n_{CE})}$, summarizing the optimality conditions, is given by,

$$\Phi_5^{(n_{CE})} = -\left[\Phi_{4\ [10:13,10:13]}^{(n_{CE})}\right]^{-1} \Phi_{4\ [10:13,1:9]}^{(n_{CE})} .$$

We impose these conditions given by (A.2.64) on the RHS of the Bellman equation in order to obtain an update of the quadratically approximated value function, $Q_{V|\Psi^{(n_\psi)}}^{(n_{CE})}$. Specifically, we formulate the following 13×9 matrix,

$\Omega_3 =$

$$\begin{array}{c@{}c}
\begin{array}{ccccccccc}
1 & z & u & \mu & K & k & \hat{m}^H & \hat{B}^T & \hat{b}^H \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} & \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\Phi_{5\,[1,1]}^{(n_{CE})} & \Phi_{5\,[1,2]}^{(n_{CE})} & \Phi_{5\,[1,3]}^{(n_{CE})} & \Phi_{5\,[1,4]}^{(n_{CE})} & \Phi_{5\,[1,5]}^{(n_{CE})} & \Phi_{5\,[1,6]}^{(n_{CE})} & \Phi_{5\,[1,7]}^{(n_{CE})} & \Phi_{5\,[1,8]}^{(n_{CE})} & \Phi_{5\,[1,9]}^{(n_{CE})} \\
\Phi_{5\,[2,1]}^{(n_{CE})} & \Phi_{5\,[2,2]}^{(n_{CE})} & \Phi_{5\,[2,3]}^{(n_{CE})} & \Phi_{5\,[2,4]}^{(n_{CE})} & \Phi_{5\,[2,5]}^{(n_{CE})} & \Phi_{5\,[2,6]}^{(n_{CE})} & \Phi_{5\,[2,7]}^{(n_{CE})} & \Phi_{5\,[2,8]}^{(n_{CE})} & \Phi_{5\,[2,9]}^{(n_{CE})} \\
\Phi_{5\,[3,1]}^{(n_{CE})} & \Phi_{5\,[3,2]}^{(n_{CE})} & \Phi_{5\,[3,3]}^{(n_{CE})} & \Phi_{5\,[3,4]}^{(n_{CE})} & \Phi_{5\,[3,5]}^{(n_{CE})} & \Phi_{5\,[3,6]}^{(n_{CE})} & \Phi_{5\,[3,7]}^{(n_{CE})} & \Phi_{5\,[3,8]}^{(n_{CE})} & \Phi_{5\,[3,9]}^{(n_{CE})} \\
\Phi_{5\,[4,1]}^{(n_{CE})} & \Phi_{5\,[4,2]}^{(n_{CE})} & \Phi_{5\,[4,3]}^{(n_{CE})} & \Phi_{5\,[4,4]}^{(n_{CE})} & \Phi_{5\,[4,5]}^{(n_{CE})} & \Phi_{5\,[4,6]}^{(n_{CE})} & \Phi_{5\,[4,7]}^{(n_{CE})} & \Phi_{5\,[4,8]}^{(n_{CE})} & \Phi_{5\,[4,9]}^{(n_{CE})}
\end{bmatrix}
&
\begin{array}{l}
\leftarrow 1 \\ \leftarrow z \\ \leftarrow u \\ \leftarrow \mu \\ \leftarrow K \\ \leftarrow k \\ \leftarrow \hat{m}^H \\ \leftarrow \hat{B}^T \\ \leftarrow \hat{b}^H \\ \leftarrow h \\ \leftarrow x \\ \leftarrow \hat{m}^{H\,\prime} \\ \leftarrow \hat{b}^{H\,\prime}
\end{array}
\end{array} ,$$

and we update the value function's quadratic form through,

$$Q_{V|\Psi\left(n_\psi\right)}^{(n_{CE}+1)} = \mathbf{\Omega}_3^T \cdot \Phi_5^{(n_{CE})} \cdot \mathbf{\Omega}_3 \ . \qquad \text{(A.2.65)}$$

If the distance between $Q_{V|\Psi\left(n_\psi\right)}^{(n_{CE}+1)}$ and $Q_{V|\Psi\left(n_\psi\right)}^{(n_{CE})}$ is not arbitrarily small, we return to Step 3 and we continue until we have found the fixed point, $Q_{V|\Psi\left(n_\psi\right)}^{*}$, and the individual decision rules, which are expressed by Φ_5^*.

Step 4

In this step we compute the intermediate-equilibrium ("IE"). We formulate the non-Bellman equation of the household given by equation (2.37) in the main body of the paper, leaving μ' free for one period ahead only, which is given by,

$$\tilde{V}\left(z,u,\mu,\mu',K,k,\hat{m}^H,\hat{B}^T,\hat{b}^H \mid \Psi^{(n_\psi)}\right) =$$

$$\max_{h,x,\hat{m}^{H\,\prime},\hat{b}^{H\,\prime}} \Big\{ u(z,u,\mu,K,k,\hat{m}^H,\hat{B}^T,\hat{b}^H,h,x,\hat{m}^{H\,\prime},\hat{b}^{H\,\prime},H,X,\hat{P},\hat{B}^{T\,\prime})+$$

$$+\beta E_t\left[V^*\left(z',u',\mu',K',k',\hat{m}^{H\,\prime},\hat{B}^{T\,\prime},\hat{b}^{H\,\prime} \mid \Psi^{(n_\psi)}\right)\right]\Big\}\ , \quad \text{(A.2.66)}$$

The value function V^* used on the RHS of (A.2.66), is the fixed point of the competitive-equilibrium Bellman equation, given by (A.2.65) after the "CE loop" has converged. The expanded quadratic form of (A.2.66) is given by,

$$\begin{bmatrix} 1 \\ z \\ u \\ \mu \\ \mu' \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \end{bmatrix}^T \tilde{Q}_{V|\Psi^{(n_\psi)}} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ \mu' \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \end{bmatrix} = \max_{h,x,\hat{m}^{H\,\prime},\hat{b}^{H\,\prime}} \left\{ \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \\ h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \\ H \\ X \\ \hat{P} \\ \hat{B}^{T\,\prime} \\ 1 \\ z' \\ u' \\ \mu' \\ K' \\ k' \\ \hat{m}^{H\,\prime} \\ \hat{B}^{T\,\prime} \\ \hat{b}^{H\,\prime} \end{bmatrix}^T \begin{bmatrix} Q_u & \mathbf{0} \\ \mathbf{0} & \beta Q^*_{V|\Psi^{(n_\psi)}} \end{bmatrix} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \\ h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \\ H \\ X \\ \hat{P} \\ \hat{B}^{T\,\prime} \\ 1 \\ z' \\ u' \\ \mu' \\ K' \\ k' \\ \hat{m}^{H\,\prime} \\ \hat{B}^{T\,\prime} \\ \hat{b}^{H\,\prime} \end{bmatrix} \right\} . \quad \text{(A.2.67)}$$

Since μ' is free for one period ahead only, we impose all constraints of the household's problem on the RHS of (A.2.67) except (A.2.52). Specifically, we impose constraints (2.4), (2.13), and (2.27). This is achieved through the 26×18 matrix ω_1 given by,

$\omega_1 =$

$$
\begin{array}{cc}
\begin{array}{cccccccccccccccccc}
\underset{\downarrow}{1} & \underset{\downarrow}{z} & \underset{\downarrow}{u} & \underset{\downarrow}{\mu} & \underset{\downarrow}{\mu'} & \underset{\downarrow}{K} & \underset{\downarrow}{k} & \underset{\downarrow}{\hat{m}^H} & \underset{\downarrow}{\hat{B}^T} & \underset{\downarrow}{\hat{b}^H} & \underset{\downarrow}{h} & \underset{\downarrow}{x} & \underset{\downarrow}{\hat{m}^{H\,\prime}} & \underset{\downarrow}{\hat{b}^{H\,\prime}} & \underset{\downarrow}{H} & \underset{\downarrow}{X} & \underset{\downarrow}{\hat{P}} & \underset{\downarrow}{\hat{B}^{T\,\prime}}
\end{array} & \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1-\delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1-\delta & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix} &
\begin{array}{l}
\leftarrow 1 \\ \leftarrow z \\ \leftarrow u \\ \leftarrow \mu \\ \leftarrow K \\ \leftarrow k \\ \leftarrow \hat{m}^H \\ \leftarrow \hat{B}^T \\ \leftarrow \hat{b}^H \\ \leftarrow h \\ \leftarrow x \\ \leftarrow \hat{m}^{H\,\prime} \\ \leftarrow \hat{b}^{H\,\prime} \\ \leftarrow H \\ \leftarrow X \\ \leftarrow \hat{P} \\ \leftarrow \hat{B}^{T\,\prime} \\ \leftarrow 1 \\ \leftarrow z' \\ \leftarrow u' \\ \leftarrow \mu' \\ \leftarrow K' \\ \leftarrow k' \\ \leftarrow \hat{m}^{H\,\prime} \\ \leftarrow \hat{B}^{T\,\prime} \\ \leftarrow \hat{b}^{H\,\prime}
\end{array}
\end{array}
$$

Transforming the RHS of (A.2.67) as follows,

$$\phi_1^{(n_{IE})} = \omega_1^T \begin{bmatrix} Q_u & \mathbf{0} \\ \mathbf{0} & \beta Q^*_{V|\Psi\left(n_\psi\right)} \end{bmatrix} \omega_1 \tag{A.2.68}$$

leads to the following unconstrained-optimization version of the optimization problem of (A.2.67):

$$\max_{h,x,\hat{m}^{H\,\prime},\hat{b}^{H\,\prime}} \left\{ \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ \mu' \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \\ h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \\ H \\ X \\ \hat{P} \\ \hat{B}^{T\,\prime} \end{bmatrix}^T \phi_1^{(n_{IE})} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ \mu' \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \\ h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \\ H \\ X \\ \hat{P} \\ \hat{B}^{T\,\prime} \end{bmatrix} \right\}. \tag{A.2.69}$$

The first-order conditions are summarized by,

$$\phi^{(n_{IE})}_{1\;[11:14,1:18]} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \\ h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \\ H \\ X \\ \hat{P} \\ \hat{B}^{T\,\prime} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{A.2.70}$$

Imposing the market-clearing conditions given by equations (2.31) and (2.34) in the main body of the paper through matrix MC_{IE},

$$
MC_{IE} \quad = \quad \begin{array}{cc}
\begin{array}{ccccccccccc}
1 & z & u & \mu & \mu' & K & \hat{B}^T & H & X & \hat{P} & \hat{B}^{T\,\prime} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} & \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} &
\begin{array}{l}
\leftarrow 1 \\
\leftarrow z \\
\leftarrow u \\
\leftarrow \mu \\
\leftarrow \mu' \\
\leftarrow K \\
\leftarrow k \\
\leftarrow \hat{m}^H \\
\leftarrow \hat{B}^T \\
\leftarrow \hat{b}^H \\
\leftarrow h \\
\leftarrow x \\
\leftarrow \hat{m}^{H\,\prime} \\
\leftarrow \hat{b}^{H\,\prime} \\
\leftarrow H \\
\leftarrow X \\
\leftarrow \hat{P} \\
\leftarrow \hat{B}^{T\,\prime}
\end{array}
\end{array} \tag{A.2.71}
$$

applied on $\phi_{1\,[11:14,1:18]}^{(n_{IE})}$ as follows,

$$
\phi_{1\,[11:14,1:18]}^{(n_{IE})} \; MC_{IE} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ \mu' \\ K \\ \hat{B}^T \\ H \\ X \\ \hat{P} \\ \hat{B}^{T\,\prime} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{A.2.72}
$$

we obtain a new matrix, $\phi_2^{(n_{IE})}$, that contains this market-clearing information:

$$
\phi_2^{(n_{IE})} = \phi_{1\,[11:14,1:18]}^{(n_{IE})} \; MC_{IE} \;.
$$

Therefore, the aggregate decision rules for IE are given by,

$$\begin{bmatrix} H \\ X \\ \hat{P} \\ \hat{B}^{T\,\prime} \end{bmatrix} = \phi_3^{(n_{IE})} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ \mu' \\ K \\ \hat{B}^T \end{bmatrix} . \tag{A.2.73}$$

where,

$$\phi_3^{(n_{IE})} = -\left[\phi^{(n_{IE})}_{2\ [1:4,8:11]}\right]^{-1} \ \phi^{(n_{IE})}_{2\ [1:4,1:7]} \ .$$

Notice that the difference between $\phi_3^{(n_{IE})}$ and $\Phi_3^{(n_{CE})}$ in the CE loop is that $\phi_3^{(n_{IE})}$ also depends on μ', which is a free policy variable, capturing the impact of any one-period deviation of monetary policy on the state variables. In order to impose that influence of the aggregate decision rules for IE on the RHS of (A.2.67) we use matrix ω_2, given by,

$\omega_2 =$

$$
\begin{array}{c}
\begin{array}{cccccccccccccc}
1 & z & u & \mu & \mu' & K & k & \hat{m}^H & \hat{B}^T & \hat{b}^H & h & x & \hat{m}^{H\,\prime} & \hat{b}^{H\,\prime} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} \\
\left[\begin{array}{cccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\phi^{(n_{IE})}_{3\,[1,1]} & \phi^{(n_{IE})}_{3\,[1,2]} & \phi^{(n_{IE})}_{3\,[1,3]} & \phi^{(n_{IE})}_{3\,[1,4]} & \phi^{(n_{IE})}_{3\,[1,5]} & \phi^{(n_{IE})}_{3\,[1,6]} & 0 & 0 & \phi^{(n_{IE})}_{3\,[1,7]} & 0 & 0 & 0 & 0 & 0 \\
\phi^{(n_{IE})}_{3\,[2,1]} & \phi^{(n_{IE})}_{3\,[2,2]} & \phi^{(n_{IE})}_{3\,[2,3]} & \phi^{(n_{IE})}_{3\,[2,4]} & \phi^{(n_{IE})}_{3\,[2,5]} & \phi^{(n_{IE})}_{3\,[2,6]} & 0 & 0 & \phi^{(n_{IE})}_{3\,[2,7]} & 0 & 0 & 0 & 0 & 0 \\
\phi^{(n_{IE})}_{3\,[3,1]} & \phi^{(n_{IE})}_{3\,[3,2]} & \phi^{(n_{IE})}_{3\,[3,3]} & \phi^{(n_{IE})}_{3\,[3,4]} & \phi^{(n_{IE})}_{3\,[3,5]} & \phi^{(n_{IE})}_{3\,[3,6]} & 0 & 0 & \phi^{(n_{IE})}_{3\,[3,7]} & 0 & 0 & 0 & 0 & 0 \\
\phi^{(n_{IE})}_{3\,[4,1]} & \phi^{(n_{IE})}_{3\,[4,2]} & \phi^{(n_{IE})}_{3\,[4,3]} & \phi^{(n_{IE})}_{3\,[4,4]} & \phi^{(n_{IE})}_{3\,[4,5]} & \phi^{(n_{IE})}_{3\,[4,6]} & 0 & 0 & \phi^{(n_{IE})}_{3\,[4,7]} & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi^{(n_{IE})}_{3\,[2,1]} & \phi^{(n_{IE})}_{3\,[2,2]} & \phi^{(n_{IE})}_{3\,[2,3]} & \phi^{(n_{IE})}_{3\,[2,4]} & \phi^{(n_{IE})}_{3\,[2,5]} & \phi^{(n_{IE})}_{3\,[2,6]} + 1 - \delta & 0 & 0 & \phi^{(n_{IE})}_{3\,[2,7]} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1-\delta & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\phi^{(n_{IE})}_{3\,[4,1]} & \phi^{(n_{IE})}_{3\,[4,2]} & \phi^{(n_{IE})}_{3\,[4,3]} & \phi^{(n_{IE})}_{3\,[4,4]} & \phi^{(n_{IE})}_{3\,[4,5]} & \phi^{(n_{IE})}_{3\,[4,6]} & 0 & 0 & \phi^{(n_{IE})}_{3\,[4,7]} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
\begin{array}{l}
\\ \\
\leftarrow 1 \\ \leftarrow z \\ \leftarrow u \\ \leftarrow \mu \\ \leftarrow K \\ \leftarrow k \\ \leftarrow \hat{m}^H \\ \leftarrow \hat{B}^T \\ \leftarrow \hat{b}^H \\ \leftarrow h \\ \leftarrow x \\ \leftarrow \hat{m}^{H\,\prime} \\ \leftarrow \hat{b}^{H\,\prime} \\
\leftarrow H \\ \leftarrow X \\ \leftarrow \hat{P} \\ \leftarrow \hat{B}^{T\,\prime} \\
\leftarrow 1 \\ \leftarrow z' \\ \leftarrow u' \\ \leftarrow \mu' \\ \leftarrow K' \\ \leftarrow k' \\ \leftarrow \hat{m}^{H\,\prime} \\ \leftarrow \hat{B}^{T\,\prime} \\ \leftarrow \hat{b}^{H\,\prime}
\end{array}
$$

Using ω_2, define a 14×14 matrix $\phi_4^{(n_{IE})}$ which summarizes the RHS of (A.2.66) as an unconstrained problem that includes the information of market clearing conditions as follows

$$\phi_4^{(n_{IE})} = \omega_2^T \begin{bmatrix} Q_u & \mathbf{0} \\ \mathbf{0} & \beta Q^*_{V|\Psi^{\left(n_\psi\right)}} \end{bmatrix} \omega_2 \ , \qquad \text{(A.2.74)}$$

perform optimization,

$$\max_{h,\, x,\, \hat{m}^{H\,\prime},\, \hat{b}^{H\,\prime}} \left\{ \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ \mu' \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \\ h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \end{bmatrix}^T \phi_4^{(n_{IE})} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ \mu' \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \\ h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \end{bmatrix} \right\}, \tag{A.2.75}$$

and after taking first-order conditions obtain the individual optimality conditions given by,

$$\begin{bmatrix} h \\ x \\ \hat{m}^{H\,\prime} \\ \hat{b}^{H\,\prime} \end{bmatrix} = \phi_5^{(n_{IE})} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ \mu' \\ K \\ k \\ \hat{m}^H \\ \hat{B}^T \\ \hat{b}^H \end{bmatrix}, \tag{A.2.76}$$

where $\phi_5^{(n_{IE})}$ is given by,

$$\phi_5^{(n_{IE})} = -\left[\phi^{(n_{IE})}_{4\,[11:14,11:14]}\right]^{-1} \phi^{(n_{IE})}_{4\,[11:14,1:10]}\,.$$

In order to derive the quadratically-approximated version of value function $\tilde{V}$, impose the individual household optimal decisions on the RHS of (A.2.66), through transforming matrix $\phi_4^{(n_{IE})}$. Specifically, create the 14×10 matrix ω_3,

$\omega_3 =$

$$
\begin{array}{c}
\begin{array}{cccccccccc}
\overset{1}{\downarrow} & \overset{z}{\downarrow} & \overset{u}{\downarrow} & \overset{\mu}{\downarrow} & \overset{\mu'}{\downarrow} & \overset{K}{\downarrow} & \overset{k}{\downarrow} & \overset{\hat{m}^H}{\downarrow} & \overset{\hat{B}^T}{\downarrow} & \overset{\hat{b}^H}{\downarrow}
\end{array} \\
\left[\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\phi^{(n_{IE})}_{5\ [1,1]} & \phi^{(n_{IE})}_{5\ [1,2]} & \phi^{(n_{IE})}_{5\ [1,3]} & \phi^{(n_{IE})}_{5\ [1,4]} & \phi^{(n_{IE})}_{5\ [1,5]} & \phi^{(n_{IE})}_{5\ [1,6]} & \phi^{(n_{IE})}_{5\ [1,7]} & \phi^{(n_{IE})}_{5\ [1,8]} & \phi^{(n_{IE})}_{5\ [1,9]} & \phi^{(n_{IE})}_{5\ [1,10]} \\
\phi^{(n_{IE})}_{5\ [2,1]} & \phi^{(n_{IE})}_{5\ [2,2]} & \phi^{(n_{IE})}_{5\ [2,3]} & \phi^{(n_{IE})}_{5\ [2,4]} & \phi^{(n_{IE})}_{5\ [2,5]} & \phi^{(n_{IE})}_{5\ [2,6]} & \phi^{(n_{IE})}_{5\ [2,7]} & \phi^{(n_{IE})}_{5\ [2,8]} & \phi^{(n_{IE})}_{5\ [2,9]} & \phi^{(n_{IE})}_{5\ [2,10]} \\
\phi^{(n_{IE})}_{5\ [3,1]} & \phi^{(n_{IE})}_{5\ [3,2]} & \phi^{(n_{IE})}_{5\ [3,3]} & \phi^{(n_{IE})}_{5\ [3,4]} & \phi^{(n_{IE})}_{5\ [3,5]} & \phi^{(n_{IE})}_{5\ [3,6]} & \phi^{(n_{IE})}_{5\ [3,7]} & \phi^{(n_{IE})}_{5\ [3,8]} & \phi^{(n_{IE})}_{5\ [3,9]} & \phi^{(n_{IE})}_{5\ [3,10]} \\
\phi^{(n_{IE})}_{5\ [4,1]} & \phi^{(n_{IE})}_{5\ [4,2]} & \phi^{(n_{IE})}_{5\ [4,3]} & \phi^{(n_{IE})}_{5\ [4,4]} & \phi^{(n_{IE})}_{5\ [4,5]} & \phi^{(n_{IE})}_{5\ [4,6]} & \phi^{(n_{IE})}_{5\ [4,7]} & \phi^{(n_{IE})}_{5\ [4,8]} & \phi^{(n_{IE})}_{5\ [4,9]} & \phi^{(n_{IE})}_{5\ [4,10]}
\end{array}\right]
\begin{array}{l}
\leftarrow 1 \\ \leftarrow z \\ \leftarrow u \\ \leftarrow \mu \\ \leftarrow \mu' \\ \leftarrow K \\ \leftarrow k \\ \leftarrow \hat{m}^H \\ \leftarrow \hat{B}^T \\ \leftarrow \hat{b}^H \\ \leftarrow h \\ \leftarrow x \\ \leftarrow \hat{m}^{H\ \prime} \\ \leftarrow \hat{b}^{H\ \prime}
\end{array}
\end{array} .
$$

Obtain the quadratically approximated individual-utility function, $\tilde{V}$, through,

$$\tilde{Q}_{V|\Psi\left(n_\psi\right)} = \omega_3^T \phi_4^{(n_{IE})} \omega_3 \ .$$

In order to derive the quadratically approximated version of the welfare function $\tilde{W}\left(S, \mu' \mid \Psi\right)$ as defined by equation (2.38) in the main body of the paper, we construct the 10×7 matrix $MC_{\mu'}$, given by,

$$
MC_{\mu'} \quad = \quad
\begin{array}{c}
\begin{array}{ccccccc}
\overset{1}{\downarrow} & \overset{z}{\downarrow} & \overset{u}{\downarrow} & \overset{\mu}{\downarrow} & \overset{K}{\downarrow} & \overset{\hat{B}^T}{\downarrow} & \overset{\mu'}{\downarrow}
\end{array} \\
\left[\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\begin{array}{l}
\leftarrow 1 \\ \leftarrow z \\ \leftarrow u \\ \leftarrow \mu \\ \leftarrow \mu' \\ \leftarrow K \\ \leftarrow k \\ \leftarrow \hat{m}^H \\ \leftarrow \hat{B}^T \\ \leftarrow \hat{b}^H
\end{array}
\end{array}
\qquad (A.2.77)
$$

that imposes the market-clearing conditions described by (2.31) in the main body of the paper. Therefore, the quadratically approximated version of the welfare function $\tilde{W}\left(S, \mu' \mid \Psi\right)$

is given by the 7×7 matrix,

$$\tilde{Q}_{W|\Psi^{\left(n_\psi\right)}} = MC_{\mu'}^T \tilde{Q}_{V|\Psi^{\left(n_\psi\right)}} MC_{\mu'} \ .$$

Therefore, to obtain optimal money supply μ', we solve the maximization problem,

$$\max_{\mu'} \left\{ \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ \hat{B}^T \\ \mu' \end{bmatrix}^T \tilde{Q}_{W|\Psi^{\left(n_\psi\right)}} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ \hat{B}^T \\ \mu' \end{bmatrix} \right\} , \tag{A.2.78}$$

with first order conditions given by,

$$\tilde{Q}_{W|\Psi^{\left(n_\psi\right)}\,[7,1:7]} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ \hat{B}^T \\ \mu' \end{bmatrix} = 0 \tag{A.2.79}$$

and optimal solution,

$$\mu' = -\left[\tilde{Q}_{W|\Psi^{\left(n_\psi\right)}\,[7,7]}\right]^{-1} \tilde{Q}_{W|\Psi^{\left(n_\psi\right)}\,[7,1:6]} \begin{bmatrix} 1 \\ z \\ u \\ \mu \\ K \\ \hat{B}^T \end{bmatrix} ,$$

which provides the update of the optimal monetary-policy rule Ψ,

$$\bar{\mathbf{\Psi}}^{\left(n_\psi+1\right)} = -\left[\tilde{Q}_{W|\Psi^{\left(n_\psi\right)}\,[7,7]}\right]^{-1} \tilde{Q}_{W|\Psi^{\left(n_\psi\right)}\,[7,1:6]} \ .$$

If the distance between $\bar{\mathbf{\Psi}}^{\left(n_\psi+1\right)}$and $\bar{\mathbf{\Psi}}^{\left(n_\psi\right)}$ is not arbitrarily small, we update Ψ using an overshooting convex-combination adjustment between $\bar{\mathbf{\Psi}}^{\left(n_\psi+1\right)}$and $\bar{\mathbf{\Psi}}^{\left(n_\psi\right)}$ that places more weight on $\bar{\mathbf{\Psi}}^{\left(n_\psi\right)}$, and we return to Step 2, continuing until we reach the fixed point $\bar{\mathbf{\Psi}}^{*}$.

Step 5

Once the fixed point $\bar{\mathbf{\Psi}}^{*}$ is reached, we introduce the steady-state value of μ, the value μ^{ss} used in the outer loop, to the equation,

$$\mu^{ss} = \bar{\mathbf{\Psi}}^{*} \begin{bmatrix} 1 \\ z^{ss} \\ u^{ss} \\ \mu^{ss} \\ K^{ss} \\ \left(\hat{B}^{T}\right)^{ss} \end{bmatrix}, \tag{A.2.80}$$

in order to see if the value μ^{ss} is the same on both sides. If equation (A.2.80) holds, then we stop, as quadratic approximations in all loops are taken around a consistently accurate steady state. Otherwise, we update our previous μ^{ss} value with that implied by the RHS of equation (A.2.80), using again a convex-combination overshooting adjustment that places more weight on the previous value, and we return to Step 1, continuing until convergence.

3. CHAPTER

Jump Shocks, Endogenous Investment Leveraging and Asset Prices: Analytical Results

3.1 Introduction

In today's globalized world, the financial system may be too much leveraged. Financial institutions invest using not only their own equity, but mostly borrowed money from other sources, the so called "inside money". In a typical investment bank, the leverage ratio, which is the ratio between inside money and own equity, varies between 20 to 30, meaning that own equity can be as low as 3% of total assets. Such high leverage ratios mean that, in case a rare disaster hits some investments of the bank causing damage beyond 3% of total assets, the bank will become bankrupt.

Such bankruptcy concerns are especially alarming for countries with enormous banking sectors, like Luxembourg. While EU agreements such as Basel II and III intend to regulate leverage ratios, the crucial question is: are they enough? This paper intends to contribute methods for answering such questions. The reason one needs to develop an analytical approach to financial stability is that asset prices and leveraging are both endogenous and they co-determine each other. Simultaneously determining leverage ratios and the value of productive assets held by financial institutions in a micro-founded framework consisting of only preference-and-technology fundamentals is a challenging task. In this paper we undertake this task in the presence of jump risks hitting (production) fundamentals. We focus on these risks because we want to understand how market imperfections may lead to an inadequacy of markets to respond to the presence of these jump risks. We therefore build a financial-stability model with jump risks and incomplete capital markets. As the task of simultaneously endogenizing asset pricing and leveraging is cumbersome, the main goal and scope of this paper is to deliver analytical results and methods that will facilitate future work based on simulations.

The micro-founded model approach that we take goes beyond standard business-cycle model analysis. We do so as the 2008 global financial crisis has shown that we need to better understand the sources of financial instability. Existing dynamic stochastic general-equilibrium models have failed in replicating the substantial and long-lasting contraction in economic activity that followed the crisis. One reason for this failure is that most dynamic stochastic general-equilibrium models focus only on the approximate-economy dynamics around a prior measurement of a projected economic trend, viewed as "the steady state". This strategy of measuring the economic trend is adequate for predicting usual business cycle shock-propagation phenomena, but rather inadequate for studying the long-lasting impact of credit-cycle shocks causing an economic crisis that leads to misallocation. A way to deal with this inadequacy, is to endogenize the economic trend. A new generation of models attempting to both endogenize economic growth and to deal with financial-crisis mechanics is Brunnermeier and Sannikov (2014, 2017).

The approach led by Brunnermeier and Sannikov (2014, 2017) uses continuous-time general-equilibrium stochastic models combining the macroeconomy with the banking sector. The techniques of Brunnermeier and Sannikov (2014, 2017) follow several of the ideas in He and Krishnamurthy (2012, 2013), with the crucial difference that they adapt the analysis to an endogenous-growth framework. This class of models uses, as in the literature developed around the Black and Scholes (1973) option-pricing model, stochastic calculus for the non-linear specifications of economic dynamics. Since we know that credit-cycle shocks have non-linear effects (see, for example, Bernanke et al., 1999, and Kiyotaki and Moore, 1997), these continuous-time techniques are appropriate.

The key contribution of this paper is that it focuses on how markets deal with the fact that some financial-crisis (credit-cycle) shocks can be sizeable. The size of these shocks is a feature that can contribute to their long-lasting effects. As credit cycles exhibit a shock-propagation mechanism that involves vicious and virtuous circles of bank-leveraging and asset-price movements, the size of a credit-cycle shock can give rise to a financial crisis. Therefore, we focus on introducing jump shocks. The analysis of Brunnermeier and Sannikov (2014, 2017) focuses only on diffusion shocks (normally-distributed shocks) to growth rates of economic fundamentals. Jumps increase the kurtosis of growth rates to economic fundamentals. Facing extreme shocks more frequently than those implied by the normal distribution can lead to more frequent incidents where the equity of banks can be entirely destroyed. The dramatically destroyed equity cannot be entirely rebuilt in the next period. Only in the long run, through retained earnings, banks can restore their equity. In addition, the bank shock can be amplified through price movements.

The crucial feature of the class of models we study is the joint determination of asset prices and leverage ratios of banks. For example, high stock prices can reinforce further demand for stocks by banks, which further increases stock prices. Such reinforcing mechanisms can amplify stock-market booms and can also make these booms persistent. The problem is that the same mechanism can work on the downside: falling stock prices can lead to a rapid reduction of demand by banks; banks deleverage too fast and become hit by the further stock-price decline. Such dramatic downward paths can become much worse in the presence of rare disasters. The reason is that rare disasters introduce more downside risk. More interestingly, high leveraging during stock-market booms can make banks even more exposed and vulnerable to this downside risk. Such scenarios necessitate the extension of the Brunnermeier and Sannikov (2014) framework to rare disasters in order to understand financial stability and crisis prevention. More specifically, we need to understand how banks with rational expectations leverage up in an environment with jump shocks.[19]

Our prototype model is also an endogenous-growth model. We study how financial (in)stability influences the amount of resources channeled from the financial sector to the real economy. During a financial crisis, because of leveraging, a financial system may block real-economy investments. Such a blockage lowers economic growth in the long run. In order to eliminate the possibility of a disaster, leverage ratios may be rather conservative. Nevertheless, taking medium-level risk (e.g. with regulation allowing for higher leverage ratios), could stimulate higher economic growth in the long run. This trade-off defines an exciting agenda for future research: could it be desirable to allow an economy to be moderately exposed to some rare-disaster risk? Is financial regulation welfare increasing? Can we achieve a Pareto improvement? Although these questions are left unexplored in this paper, establishing the technical features of this framework can help in building the platform for answering these questions.

In our analysis, we first describe the structure of our model economy. As in Brunnermeier and Sannikov (2014), our model is a variation of the Basak and Cuoco (1998) model. This model allows to study endogenous stock prices and endogenous leverage ratios in an endowment economy. We explain the roles of the two types of agents in our model and we formulate their expected utility optimization problems and derive their optimal choices. Using logarithmic preferences, we derive a closed form solution. In a next step, we analytically derive equilibrium conditions for agents with recursive Epstein-Zin preferences. We

[19]Documented by Reinhart and Rogoff (2009), similar persistent declines and long-lasting changes in asset prices, output or investment were observed following previous financial crises. The fact that "we have been there before", motivates the importance of understanding financial instability in order to make the financial system more resilient. This is the best protection to limit the economic consequences of future disaster shocks.

explain the homotopy analysis method, which is appropriate for a numerical implementation in Matlab. Then we extent our baseline model to a general–equilibrium model by introducing production based on capital investments, with adjustment costs for investment changes. We derive analytical solutions for agents with recursive Epstein-Zin preferences and a closed form solution for agents with logarithmic preferences. We conclude with reflections on how to proceed.

3.2 The baseline model

As in Basak and Cuoco (1998), we consider a continuous-time endowment economy on an infinite time span. We assume two different types of agents. Within each agent type we have a large number of homogenous agents. Therefore, all agents are price-takers. Throughout, we speak about a representative household and a so-called representative "expert". The household is subject to a non-convexity: it is restricted to investing only in bonds and not in the stock market; on the contrary, only the expert has access to managing portfolios. This non-convexity is the reason the resulting competitive allocation is not Pareto optimal. In addition to that, a financial friction is assumed on the expert's behavior. The expert can only finance risky asset holdings through debt by issuing bonds, not by issuing new equity.

All goods are accounted for in real terms, i.e. in units of consumable goods. The composition of the balance sheets of the expert and the household are summarized as follows:

Expert		**Household**	
Assets	**Liabilities**	**Assets**	**Liabilities**
$P_t = q_t D_t$	B_t	B_t	N_t^H
	N_t		

Table 3.1 - Balance sheet of the expert and the household in the baseline model.

The household has net wealth of N_t^H, with which it buys bonds, B_t, issued by the expert.

The household lends money (deposits) to the expert at the bond rate, which is determined endogenously. The expert invests the amount of the issued bonds, B_t, together with its net wealth (equity), N_t, in the aggregate stock market, which is a simplification of the production sector. Therefore, only experts hold risky assets, which they finance mostly through debt (short-selling of the bonds). The stock market can be seen as a stock index. The risky asset pays a stochastic dividend, D_t, which is described by the following stochastic differential equation,

$$dD_t = gD_t dt + [(1-\zeta) D_t - D_t] \, dn_t \ , \tag{3.1}$$

which can be simplified to,

$$\frac{dD_t}{D_t} = gdt - \zeta dn_t \tag{3.2}$$

Equation (3.2) reveals that D_t is governed by a geometric Poisson process. In the endowment-economy version of our model, dividends are the core exogenous random variable and jump shocks are the only shocks to fundamentals (the dividends). The deterministic part of (3.1) shows that D_t grows at a constant, positive and unbounded growth rate, g, over time (drift). Parameter $\zeta > 0$, represents the jump size of the negative shock, while n_t is the geometric Poisson process with arrival rate λ, of the shock.

$$dn_t = \begin{cases} 1, \text{ with Pr. } \lambda dt \\ 0, \text{ with Pr. } 1 - \lambda dt \end{cases} \tag{3.3}$$

In a Poisson process the sequence of inter-arrival times (time between two shocks) is a sequence of independent random variables. Each random variable has an exponential distribution with parameter λ. The past doesn't count in determining the probability of a new shock innovation (no memory). This can be seen as a renewal assumption, which means that the process probabilistically restarts at each arrival time and at each fixed time. Therefore, the shocks in this model occur continuously and independently (of the past), a key feature of a Lévy process. The expected total number of arrivals, n_t, within a given time-interval is distributed proportionately to the time-interval, being equal to λt.

The dividend process, and thereby the risk for a negative shock in our case, is exogenous. The expert, who is the only agent holding risky assets, has limited ability to absorb risk through net wealth, N_t. In case a downturn in the stock market exceeds the expert's net wealth, N_t, then the expert becomes unable to pay back its debt.

The macroeconomic environment is represented by two variables, q_t and η_t, where q_t is the price-dividend ratio,

$$q_t \equiv \frac{P_t}{D_t} \tag{3.4}$$

with P_t being the stock market value (i.e. market capitalization),

$$P_t = q_t D_t \tag{3.5}$$

and η_t is the fraction of the expert's net wealth divided by the value of total assets. Specifically, fraction η_t is,

$$\eta_t \equiv \frac{N_t}{q_t D_t} \in [0, 1] \ . \tag{3.6}$$

In our model, η_t is a state variable. Later on, we will use η_t in order to characterize how decisions in our endowment economy evolve. State variable η_t is convenient, as it describes the distribution of net wealth. Notably, η_t is also equal to the reciprocal of the leverage ratio. Note that if η_t drops, leverage increases and the expert becomes more balance-sheet constrained.

Both the price-dividend ratio and the leverage ratio $(1/\eta_t)$ are determined endogenously. The corresponding debt-equity ratio is equal to,

$$\frac{q_t D_t - N_t}{N_t} = \frac{B_t}{N_t} = \frac{1}{\eta_t} - 1 \tag{3.7}$$

Both types of agents have the same preferences.

3.2.1 Representative household's decision with logarithmic utility

The representative household maximizes its life-time utility under uncertainty. In a household problem the household decides on how much to consume and how much to invest in bonds. The household's preferences are represented by a time-additive von Neumann-Morgenstern utility function. Indeed, the household maximizes its expected utility facing a stochastic bond rate r_t, which depends on the macroeconomy.

$$\max_{(C_t^H, B_t)_{t\geq 0}} \quad E_0\left[\int_0^\infty e^{-\rho t} U\left(C_t^H\right) dt\right] \tag{3.8}$$

$$s.t. \qquad dB_t = \left(r_t B_t - C_t^H\right) dt \tag{3.9}$$

and subject to an initial bond endowment and a suitable transversality condition. In order to obtain closed-form solutions, we assume a logarithmic utility function, i.e.,

$$U(C_t^H) \equiv \ln(C_t^H) \ .$$

Parameter $\rho > 0$ is the discount factor (rate of time preference). As we will see below, with logarithmic utility, consumption will be a fixed fraction, ρ, of net wealth.

The household takes a decision under uncertainty, because the bond rate, r_t, is a function of the random variable η_t, namely,

$$r_t = R(\eta_t) \ . \tag{3.10}$$

The dynamics of η_t are governed by the recursion,

$$d\eta_t = F(\eta_t)dt + G(\eta_t)dn_t \ . \tag{3.11}$$

At this stage, the functional forms of $R(\cdot)$, $F(\cdot)$ and $G(\cdot)$ are unknown.

The most suitable tool for optimization under uncertainty is the Hamilton-Jacobi-Bellman (HJB) equation. For the above stochastic dynamic optimization problem, we formulate the corresponding HJB equation denoted by HJB^H, in which the state variables are two, (B_t, η_t):

$$\begin{aligned}\rho J^H(B_t,\eta_t) &= \max_{C_t^H \geqslant 0} \left\{ \ln(C_t) + J_B^H(B_t,\eta_t)\cdot\left[R(\eta_t)B_t - C_t^H\right] + J_\eta^H(B_t,\eta_t)F(\eta_t) + \right. \\ &\quad \left. + \lambda\left[J^H(B_t,\eta_t+G(\eta_t)) - J^H(B_t,\eta_t)\right]\right\} \qquad (3.12)\end{aligned}$$

Consistent with the property of logarithmic utility, the household's optimal consumption rule is equal to:[20]

$$C_t^H = \rho B_t \qquad (3.13)$$

Equation (3.13) implies that the household consumes a fixed fraction of his net wealth regardless of the bond rate.

3.2.2 Representative expert's decision with logarithmic utility

Since the expert is allowed to invest in stocks, we are solving a Merton-type portfolio choice model with risky assets (stocks). Before specifying the expert's optimization problem, we define ϕ_t as the ratio of the expert's total assets over its net wealth,

$$\phi_t \equiv \frac{q_t D_t}{N_t} = \frac{1}{\eta_t}\ . \qquad (3.14)$$

Since all experts are the same, in equilibrium, ϕ_t is the reciprocal of η_t and thereby the economy-wide leverage ratio. Yet, in the formulation of an individual expert's problem, we will disconnect ϕ_t from η_t, i.e., we will not associate these two variables through equation (3.14). As for the representative expert's optimization problem, the objective is to maximize lifetime utility derived by shareholder consumption. The representative expert's shareholder preferences are represented by a time-additive von Neumann-Morgenstern utility function. The expert's optimization differs from the household's by its budget constraint.

[20] For details on the representative household's optimization problem, and more specifically on equation (3.13), see the technical appendix, 3.6.A.1., equation (A.3.6)

As mentioned above, the expert invests only in the stock market, which gives him a higher return $\bar{r}_t^D$ than the bond rate r_t. The expert cannot keep the whole return on his investment $q_t D_t \bar{r}_t^D$, but needs to reimburse the borrowed amount from the household with its return $r_t B_t$. In his optimization problem, the expert takes into account the possibility of an exogenous shock, which would have a negative effect on his net wealth. The expert maximizes his shareholders' expected utility. This is a standard Merton (1969) portfolio choice problem with simultaneous decisions on how much to consume, by picking C_t^E, and on how much to invest in the stock market, by picking ϕ_t, under uncertainty. We postulate that q_t, the price-dividend ratio and $\bar{r}_t^D$, the expected return of the investment in the stock market, are both functions of η_t,

$$q_t = Q(\eta_t)$$

$$\bar{r}_t^D \equiv E(r_t^D) = R^D(\eta_t)$$

The functional forms $Q(\cdot)$ and $R^D(\cdot)$ are unknown at this stage.

The expert's optimization problem is as follows,

$$\max_{(C_t^E, N_t, \phi_t)_{t\geq 0}} \quad E_0\left[\int_0^\infty e^{-\rho t} U(C_t^E) dt\right]$$

$$s.t. \quad dN_t = \left\{\left[\phi_t(R^D(\eta_t) - R(\eta_t)) + R(\eta_t)\right] N_t - C_t^E\right\} dt + \underbrace{\left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right] \phi_t N_t dn_t}_{dN_t^{Shock} = d(q_t^{Shock} D_t^{Shock})}$$

subject to given initial conditions on η and N and subject to an appropriate transversality condition. Again,

$$U(C_t^E) \equiv \ln(C_t^E) \ .$$

In the expert's budget constraint, dN_t^{Shock} is the shock term of dN_t.[21] Indeed, dN_t depends on its trend and on the stochastic exogenous shock. Not only there is a direct effect on

[21] For details on how the shock term is derived, see the technical appendix, 3.6.A.2, equation (A.3.7).

D_t when a shock appears ($dn_t = 1$), but also an indirect effect to the price-dividend ratio, $Q(\eta_t)$, which amplifies the initial shock and further reduces the expert's net wealth N_t. The direct effect of an exogenous shock to D_t on N_t, can be seen as "fundamental risk". The indirect effect, triggered by the initial shock to D_t on N_t, can be seen as an "endogenous risk" component. The presence of endogenous risk introduces an amplification mechanism to shocks through price movements. Deleveraging of banks puts downward pressure on the price-dividend ratio. N_t decreases beyond the size of the initial shock, $-\zeta D_t$.

The expert's wealth share, η_t, decreases when a negative shock occurs, because the expert needs to absorb the exogenous shock and additional stock market movements with his net wealth N_t.

For the stochastic dynamic optimization problem, we formulate the Hamilton-Jacobi-Bellman equation of the expert, HJB^E (two state variables (N_t, η_t)):

$$\begin{aligned} \rho J^E(N_t,\eta_t) \;=\; & \max_{C_t^E,\phi_t} \Bigg\{ \ln(C_t^E) + J_N^E(N_t,\eta_t)\left\{\left[\phi_t(R^D(\eta_t) - R(\eta_t)) + R(\eta_t)\right] N_t - C_t^E\right\} \\ & + J_\eta^E(N_t,\eta_t)F(\eta_t) + \\ & + \lambda \left[J^E\left(N_t + \left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right]\phi_t N_t, \eta_t + G(\eta_t)\right) - J^E(N_t,\eta_t)\right]\Bigg\} \end{aligned}$$

The expert's optimal consumption rule is obtained by the first-order condition (F.O.C.) with respect to C_t^E. Consistently with the usual property of logarithmic utility, optimal consumption is a fixed fraction of the expert's net wealth:[22]

$$C_t^E = \rho N_t \tag{3.15}$$

The expert's optimal portfolio allocation rule (leveraging) is obtained by the F.O.C. with

[22] For details on the representative expert's optimization problem, and more specifically on the optimal consumption rule, (3.15), see the technical appendix, 3.6.A.2., equation (A.3.15).

respect to ϕ_t:[23]

$$\phi_t = -\frac{1}{\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right]} - \frac{\lambda}{R^D(\eta_t) - R(\eta_t)} \tag{3.16}$$

Equation (3.16) shows that the decision to leverage ($\phi_t > 1$) depends on the size of the disaster shock and also on the price effect. In addition, equation (3.16) shows that an increase in the risk premium, $R^D(\eta_t) - R(\eta_t)$, has a first-order effect on leverage, pushing it upward.

3.2.3 Equilibrium conditions

An equilibrium is defined as a set of consumption choices of all agents, $\{C_t^H, C_t^E\}$, and an asset-holding decision of the expert, ϕ_t, together with price processes, $\{q_t, r_t, \bar{r}_t^D\}$, such that (1.) all markets clear and (2.) all agents maximize their expected utility.

Given the balance sheet of the representative household, the following equation must hold,

$$B_t = N_t^H \ . \tag{3.17}$$

Similarly, the balance sheet of the representative expert implies,

$$q_t D_t = B_t + N_t \ . \tag{3.18}$$

Using the household's and expert's optimal consumption rule together with (3.18), we derive aggregate consumption, $\mathbf{C}_t$:

$$\mathbf{C}_t = C_t^H + C_t^E = \rho(B_t + N_t) = \rho(q_t D_t) \ . \tag{3.19}$$

In an endowment economy, market clearing implies that aggregate consumption must be equal to aggregate output Y_t, i.e.,

$$\mathbf{C}_t = Y_t \ . \tag{3.20}$$

[23]For more details on the optimal portfolio allocation rule, (3.16), see the technical appendix, 3.6.A.2., equation (A.3.13).

Yet, in a pure endowment economy aggregate output is equal to D_t, therefore,

$$Y_t = D_t \ . \tag{3.21}$$

Combining (3.19), (3.20) and (3.21) allows us to derive the equilibrium price-dividend ratio (equilibrium price of the risky asset per unit flow of dividend),

$$q_t = \frac{1}{\rho} = q \ , \tag{3.22}$$

which is constant. This tells us, up front, that in an endowment economy where agents have logarithmic preferences, there will be no endogenous risk, i.e., there is no amplification of the disaster shock. The equilibrium price-dividend ratio is independent of η_t, it is constant and equal to the inverse of ρ.

In a next step, we characterize asset prices. In order to derive the functional form of $R(\cdot)$ for the equilibrium bond rate r_t, one first needs to derive the equilibrium expected value of r_t^D, $R^D(\eta_t)$. The dynamics of the risky asset return r_t^D are equal to:

$$dr_t^D = \frac{D_t}{P_t}dt + \frac{dP_t}{P_t} \ .$$

One can decompose the risky asset return into two parts: (1) the return from the dividend process (the dividend yield $1/q_t = D_t/P_t$), and (2) the return from stock market price movement (the capital gains rate, dP_t/P_t). With $q_t = \frac{1}{\rho}$ one can show that the expected value of r_t^D, is equal to:[24]

$$R^D(\eta_t) = \int_0^1 E(dr_t^D)dt = \rho + g - \lambda\zeta = \bar{r}^D. \tag{3.23}$$

With logarithmic preferences, we have a constant risky asset return in equilibrium. A change in the risk premium can come only from a variation in the bond rate, r_t. Using (3.16), the

[24]For more details on the derivation of the equilibrium expected value of risky asset return, (3.23), see the technical appendix, 3.6.A.3., equation (A.3.18).

optimal asset-allocation decision, ϕ_t, of the expert with constant q and constant $\bar{r}^D$, yields,

$$\phi_t = \frac{1}{\zeta} - \frac{\lambda}{\bar{r}^D - R(\eta_t)} \ . \tag{3.24}$$

From equation (3.24) one can deduce that leveraging, ϕ_t, and the risk premium, $\left(\bar{r}^D - R(\eta_t)\right)$, are positively related. Intuitively, this positive relationship tells us that whenever there is a disaster shock, i.e., whenever ϕ_t jumps up, $R(\eta_t)$ needs to decrease, given that $\bar{r}^D$ is constant.

We now re-arrange (3.24) using $\phi_t = 1/\eta_t$ in order to derive the equilibrium condition for the bond rate, $R(\eta_t)$, in order to specify how $R(\eta_t)$ depends on η_t,[25]

$$R(\eta_t) = \rho + g - \lambda\zeta - \frac{\lambda}{\frac{1}{\zeta} - \frac{1}{\eta_t}} \ . \tag{3.25}$$

The risk premium $\left(\bar{r}^D - R(\eta_t)\right)$ increases with smaller η_t. In order to complete the characterization of all equilibrium conditions, we need to describe how jump shocks affect η_t, as all equations are functions of the state variable η_t. The stochastic differential equation of $d\eta_t$, can be derived from,

$$d\eta_t = d\left(\frac{N_t}{q_t D_t}\right) = \rho d\left(\frac{N_t}{D_t}\right) \ . \tag{3.26}$$

More specifically, in order to derive the law of motion of η_t, we must first derive the law of motions of N_t and D_t. The stochastic differential equation governing the motion of N_t, is given by,[26]

$$dN_t = \underbrace{\left\{\left(\frac{1}{\eta_t} - 1\right)\frac{\lambda}{\frac{1}{\zeta} - \frac{1}{\eta_t}} + \bar{r}^D - \rho\right\} N_t}_{\mu^N(N_t,D_t)} dt \underbrace{-\frac{\zeta}{\rho} D_t}_{b^N(N_t,D_t)} dn_t \ , \tag{3.27}$$

while the stochastic differential equation driving the motion of D_t is,

$$dD_t = \underbrace{gD_t}_{\mu^D(N_t,D_t)} \ dt + \underbrace{(1-\zeta)\, D_t - D_t}_{b^D(N_t,D_t)} dn_t \ . \tag{3.28}$$

[25] For more details on the derivation of the equilibrium condition for the bond rate, (3.25), see the technical appendix, 3.6.A.3., equation (A.3.19).

[26] For more details on the derivation of the law of motion of N_t, (3.27), see the technical appendix, 3.6.A.3., equation (A.3.20).

Using (3.27) and (3.28) and applying *Itô*'s Lemma for Poisson processes, we can derive, $d\left(N_t/D_t\right)$, namely,[27]

$$d\left(\frac{N_t}{D_t}\right)=\frac{N_t}{D_t}\left[\left(\frac{1}{\eta_t}-1\right)\frac{\lambda\zeta}{1-\frac{\zeta}{\eta_t}}-\lambda\zeta\right]dt+\left[\frac{N_t-\frac{\zeta}{\rho}D_t}{D_t-\zeta D_t}-\frac{N_t}{D_t}\right]dn_t\ . \tag{3.29}$$

By using (3.29) in (3.26), we can derive the law of motion of η_t,[28]

$$\frac{d\eta_t}{\eta_t}=-\lambda\zeta\left(\frac{1-\eta_t}{\zeta-\eta_t}+1\right)dt-\left(1-\frac{1-\frac{\zeta}{\eta_t}}{1-\zeta}\right)dn_t\ , \tag{3.30}$$

with,

$$F(\eta_t)=-\lambda\zeta\left(\frac{1-\eta_t}{\zeta-\eta_t}+1\right)\eta_t\ ,$$

and,

$$G(\eta_t)=\left(\frac{1-\frac{\zeta}{\eta_t}}{1-\zeta}-1\right)\eta_t\ .$$

In a nutshell, in the case of logarithmic preferences, we were able to derive a closed-form solution for $Q(\cdot)$, $R(\cdot)$, $R^D(\cdot)$, $F(\cdot)$ and $G(\cdot)$.

3.3 Recursive Epstein-Zin preferences

We extend our model by generalizing the preferences of both agents, the representative household and the representative expert, to recursive Epstein-Zin (Duffie-Epstein, 1992a,b) preferences. The goal is to separate risk and time preferences, as there is no fundamental economic reason behind assuming that they are the same. We would like to remark that, in the case of logarithmic utility, if economic agents are averse to variation of consumption over time, then it is implied that they are also averse to consumption variation across different states at a particular point of time. By assuming that the agents have recursive Epstein-Zin preferences over consumption, we allow for independent specification of the elasticity of

[27]For more details on the derivation of $d\left(\frac{N_t}{D_t}\right)$, (3.29), see the technical appendix, 3.6.A.3., equation (A.3.22).
[28]For more details on how to derive the law of motion of η_t, (3.30), see the technical appendix, 3.6.A.3., equation (A.3.24).

intertemporal substitution and the coefficient of risk aversion. From the finance literature we know, that recursive Epstein-Zin preferences help to better examine the role of risk in determining price-dividend ratios.

The continuous-time version of recursive Epstein-Zin preferences is provided by Duffie and Epstein (1992a,b). Specifically, expected utility is defined as,[29]

$$J_t = E_t \left[\int_t^{\infty} f(C(\tau), J(\tau)) d\tau \right] ,$$

in which $f(C, J)$ is a normalized aggregator function of continuation utility, J, and current consumption, C, that takes the form,

$$f(C, J) \equiv \rho(1-\gamma) J \frac{\left\{ \frac{C}{[(1-\gamma)J]^{\frac{1}{1-\gamma}}} \right\}^{1-\frac{1}{\alpha}} - 1}{1 - \frac{1}{\alpha}} ,$$

where ρ is the rate of time preference, γ is the coefficient of relative risk aversion and α is the elasticity of intertemporal substitution. It is interesting to mention that if $\alpha = 1/\gamma$, the normalized aggregator $f(C, J)$ reduces to the additive power, constant relative-risk-aversion (CRRA) utility function, from which the logarithmic utility function can be obtained by taking the limit (as $\gamma \to 1$). All utility parameters ρ, γ and α are strictly positive, i.e., $\rho, \gamma, \alpha > 0$.

3.3.1 Homotopy approach for a numerical solution with Epstein-Zin preferences

In the technical appendix, section 3.6.A.4 - 3.6.A.6 , we derive the representative household's and the representative expert's optimal decisions with recursive Epstein-Zin preferences. We determine market-clearing conditions and derive analytically all equilibrium conditions needed to proceed to the formulation of an algorithm for the numerical solution. We use

[29] The utility is recursive, because the current utility J_t depends on expected values of future utility $J_{s,s>t}$.

a homotopy approach to numerically solve for the functional forms of $Q(\eta_t)$, $F(\eta_t)$, $G(\eta_t)$, $R(\eta_t)$, $R^D(\eta_t)$, $\psi^H(\eta_t)$ and $\psi^E(\eta_t)$, as explained in the following steps:

1. Take the analytically derived equilibrium solutions from the time-separable-logarithmic-utility setup of the model as a first guess for $Q(\eta_t)$, $F(\eta_t)$, $G(\eta_t)$ and $R(\eta_t)$, and match them by exponential projection method.

$$Q(\eta_t) = \frac{1}{\rho} \;\&\; Q(\eta_t) \simeq e^{\sum_{i=0}^{\nu} \xi_i^Q [\ln(\eta_t)]^i}$$

$$F(\eta_t) = -\lambda\zeta\eta_t \left(\frac{1-\eta_t}{\zeta-\eta_t} + 1\right) \;\&\; F(\eta_t) \simeq e^{\sum_{i=0}^{\nu} \xi_i^F [\ln(\eta_t)]^i}$$

$$G(\eta_t) = \eta_t \left(\frac{1-\frac{\zeta}{\eta_t}}{1-\zeta} - 1\right) \;\&\; G(\eta_t) \simeq e^{\sum_{i=0}^{\nu} \xi_i^G [\ln(\eta_t)]^i}$$

$$R(\eta_t) = \rho + g - \lambda\zeta - \frac{\lambda}{\frac{1}{\zeta} - \frac{1}{\eta_t}} \;\&\; R(\eta_t) \simeq e^{\sum_{i=0}^{\nu} \xi_i^R [\ln(\eta_t)]^i}$$

2. Then use these in

$$R^D(\eta_t) \equiv E(r_t^D) = \frac{1}{Q(\eta_t)} + g + \frac{Q_{\eta_t}(\eta_t)}{Q(\eta_t)} F(\eta_t) + \lambda \left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right]$$

to obtain $R^D(\eta_t)$.

3. Use $R^D(\eta_t)$ and the derived simplification of the HJB^E,

$$\begin{aligned} 0 \;=\; & \frac{\rho}{\theta}\left[\rho^{\frac{\theta(1-\gamma)}{1-\theta(1-\gamma)}} \psi^E(\eta_t)^{\frac{\theta}{\theta(1-\gamma)-1}} - 1\right]\psi^E(\eta_t) \\ & +(1-\gamma)\psi^E(\eta_t)\left[\frac{1}{\eta_t}\left[R^D(\eta_t) - R(\eta_t)\right] + R(\eta_t) - \rho^{\frac{1}{1-\theta(1-\gamma)}} \psi^E(\eta_t)^{\frac{\theta}{\theta(1-\gamma)-1}}\right] \\ & + \left(\psi^E\right)'(\eta_t) F(\eta_t) \\ & +\lambda\left\{\psi^E(\eta_t + G(\eta_t))\left\{1 + \left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right]\frac{1}{\eta_t}\right\}^{1-\gamma} - \psi^E(\eta_t)\right\} \quad (3.31) \end{aligned}$$

and the derived FOC w.r.t. ϕ_t, which, in general equilibrium complies with $\phi_t = 1/\eta_t$,

$$\phi_t = \frac{\left\{ \frac{-\lambda\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]\psi^E(\eta_t+G(\eta_t))}{\psi^E(\eta_t)[R^D(\eta_t)-R(\eta_t)]} \right\}^{\frac{1}{\gamma}} - 1}{\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right]} \tag{3.32}$$

jointly with $\psi^E(\eta_t)$ and $\left(\psi^E\right)'(\eta_t)$, by using the exponential projection method to approximate $\psi^E(\eta_t)$,

$$\psi^E(\eta_t) \simeq e^{\sum_{i=0}^{\nu} \xi_i^E [\ln(\eta_t)]^i}$$

and $\left(\psi^E\right)'(\eta_t)$,

$$\left(\psi^E\right)'(\eta_t) \simeq \frac{\psi^E(\eta_t)}{\eta_t} e^{\sum_{i=0}^{\nu} i\xi_i^E [\ln(\eta_t)]^{i-1}},$$

in order to obtain an update for $R^{(n)}(\eta_t)$, $\psi^{E,(n)}(\eta_t)$ and $C^{E,(n)}(N_t, \eta_t)$.

4. Use the updated $R^{(n)}(\eta_t)$ and the derived simplification of the HJB^H,

$$\begin{aligned} 0 \;\; = \;\; & \frac{\rho}{\theta}\left[\rho^{\frac{\theta(1-\gamma)}{1-\theta(1-\gamma)}} \psi^H(\eta_t)^{\frac{\theta}{\theta(1-\gamma)-1}} - 1\right] \psi^H(\eta_t) \\ & +(1-\gamma)\psi^H(\eta_t)\left[R(\eta_t) - \rho^{\frac{1}{1-\theta(1-\gamma)}} \psi^H(\eta_t)^{\frac{\theta}{\theta(1-\gamma)-1}}\right] \\ & +(\psi^H)'(\eta_t)\cdot F(\eta_t) + \lambda[\psi^H(\eta_t + G(\eta_t)) - \psi^H(\eta_t)] \end{aligned} \tag{3.33}$$

jointly with $\psi^H(\eta_t)$ and $\left(\psi^H\right)'(\eta_t)$, by using the exponential projection method to approximate $\psi^H(\eta_t)$,

$$\psi^H(\eta_t) \simeq e^{\sum_{i=0}^{\nu} \xi_i^H [\ln(\eta_t)]^i}$$

and $\left(\psi^H\right)'(\eta_t)$,

$$\left(\psi^H\right)'(\eta_t) \simeq \frac{\psi^H(\eta_t)}{\eta_t} e^{\sum_{i=0}^{\nu} i\xi_i^H [\ln(\eta_t)]^{i-1}}$$

in order to obtain an update for $\psi^{H,(n)}(\eta_t)$ and $C^{H,(n)}(N_t, \eta_t)$.

5. Use $\psi^{E,(n)}(\eta_t)$ and $\psi^{H,(n)}(\eta_t)$ in order to update $\Psi_E^{(n)}(\eta_t)$ and $\Psi_H^{(n)}(\eta_t)$,

$$\Psi_E^{(n)}(\eta_t) = \rho^{\frac{1}{1-\theta(1-\gamma)}} \psi_E^{\frac{\theta}{\theta(1-\gamma)-1}}$$

$$\Psi_H^{(n)}(\eta_t) = \rho^{\frac{1}{1-\theta(1-\gamma)}} \psi_H^{\frac{\theta}{\theta(1-\gamma)-1}}$$

which are used to update $Q^{(n+1)}(\eta_t)$,

$$Q^{(n+1)}(\eta_t) = \left[\Psi_H^{(n)}(\eta_t) \cdot (1-\eta_t) + \Psi_E^{(n)}(\eta_t) \cdot \eta_t\right]^{-1} \tag{3.34}$$

6. Use from $d\eta_t$, $F(\eta_t)$ and $G(\eta_t)$ in order to obtain an update for $F^{(n+1)}(\eta_t)$ and $G^{(n+1)}(\eta_t)$.

$$F^{(n+1)}(\eta_t) = \eta_t \left[\frac{1}{\eta_t}(R^{D(n)}(\eta_t) - R^{(n)}(\eta_t)) + R^{(n)}(\eta_t) - \underbrace{\frac{C^E(N_t, \eta_t)}{N_t}}_{\Psi_E^{(n)}(\eta_t)} - g - \frac{Q'^{(n)}(\eta_t)}{Q^{(n)}(\eta_t)} F^{(n)}(\eta_t) - g\right] \tag{3.35}$$

$$G^{(n+1)}(\eta_t) = \eta_t \left[\frac{\frac{\eta_t - 1}{\eta_t} + \frac{1}{\eta_t}(1-\zeta)\frac{Q^{(n)}(\eta_t + G(\eta_t))}{Q^{(n)}(\eta_t)}}{(1-\zeta)\frac{Q^{(n)}(\eta_t + G(\eta_t))}{Q^{(n)}(\eta_t)}} - 1\right] \tag{3.36}$$

7. If

$$\parallel Q^{(n+1)} - Q^{(n)} \parallel + \parallel F^{(n+1)} - F^{(n)} \parallel + \parallel G^{(n+1)} - G^{(n)} \parallel < \varepsilon$$

stop;

otherwise, go to step 2.

The above seven-step procedure is the way to extend the analysis to Epstein-Zin recursive preferences using numerical methods. Crucial in this procedure is the first guess suggested by the logarithmic-utility case. Using outer loops to the seven-step procedure described above, both preference parameters, γ and α are gradually changed. Initially, both γ and α are set to values close to 1 and the formulas derived for the logarithmic case are used as the first guess. These outer loops, gradually, using small steps, change the values of γ and α away from 1, while using the fixed-point outcomes for functions Q, F, and G from the previous step of the outer-loop iterations referring to values of γ and α. For this homotopy procedure to work, the first guess is crucial. The contribution of this paper is to recommend this first guess and to deliver the above seven-step algorithm that implements the homotopy approach. Numerical simulations for data applications using the endowment-economy exercise are beyond the scope of this paper. Yet, we recommend an extension to a production economy in the next section.

3.4 General-equilibrium model with production

We extend the baseline model by introducing an "AK" production function, $Y_t = AK_t$, where $A > 0$ is the level of technology. We have a production economy with endogenous growth. The representative expert is, as before, the shareholder of the banks. As the banks own (invest in) the equity of the firms, the expert indirectly also becomes the shareholder of the firms. This implies that the expert not only decides on optimal consumption and asset allocation, but also on optimal investment. As can be seen in the balance sheet below, we now have the aggregate amount of capital, K_t, which is entirely hold by the expert at price q_t on its assets' side (q_t is similar to Tobin's Q).

Expert		**Household**	
Assets	**Liabilities**	**Assets**	**Liabilities**
$P_t = q_t K_t$	B_t	B_t	N_t^H
	N_t		

Table 3.2 - Balance sheet of the expert and the household in the general equilibrium model.

The valuation of firms' productive capital takes place in the stock market. The capital stock, K_t, are the units of capital entering production, while q_t is the price of one unit of capital. The law of motion of the aggregate amount of capital is described by the following stochastic differential equation,

$$dK_t = (\Phi(\iota_t) - \delta)K_t dt + [(1-\zeta)K_t - K_t]\, dn_t$$

where ι_t is the reinvestment rate per unit of capital, $\iota_t = I_t/K_t$,

$$\Longrightarrow \frac{dK_t}{K_t} = (\Phi(\iota_t) - \delta)dt - \zeta dn_t$$

which is driven by a geometric Poisson process n_t,

$$dn_t = \begin{cases} 1, \text{ with Pr. } \lambda dt \\ 0, \text{ with Pr. } 1-\lambda dt \end{cases}.$$

The aggregate amount of capital, K_t, is subject to negative exogenous shocks of size ζK_t with arrival rate λ. The deterministic growth rate of aggregate capital is determined by investment in capital and its depreciation rate δ, which is constant over time. The growth-rate effect of investment on capital depends on the investment function $\Phi(\iota_t)$. As in Bernanke et al. (1999) we introduce nonlinear costs in adjustment of capital, in order to have an amplification effect of the exogenous shock. $\Phi(\iota_t)$ is an investment function with convex costs in adjustments to the amount of capital and the investment function is such that $\Phi(0) = 0$, $\Phi' > 0$ (increasing) and $\Phi'' \leq 0$ (concave). The concavity of $\Phi(\iota_t)$ creates technological illiquidity.

The investment function, $\Phi(\iota_t)$, can be seen as a financial friction. Technological liquidity is the same as reversibility of an investment, with limited price impact. Technological illiquidity refers to the difficulty to undo investment or to increase investment. Technological illiquidity leads to variations in Tobin's Q, which is equal to q_t, as will be shown later. In the setup of our model, as in Brunnermeier and Sannikov (2014), the price impact is driven by shifts in the state variable η_t, which describes the distribution of wealth and is defined as,

$$\eta_t = \frac{N_t}{q_t K_t}$$

with $\eta_t \in [0, 1]$, and with,

$$q_t = Q(\eta_t) \ .$$

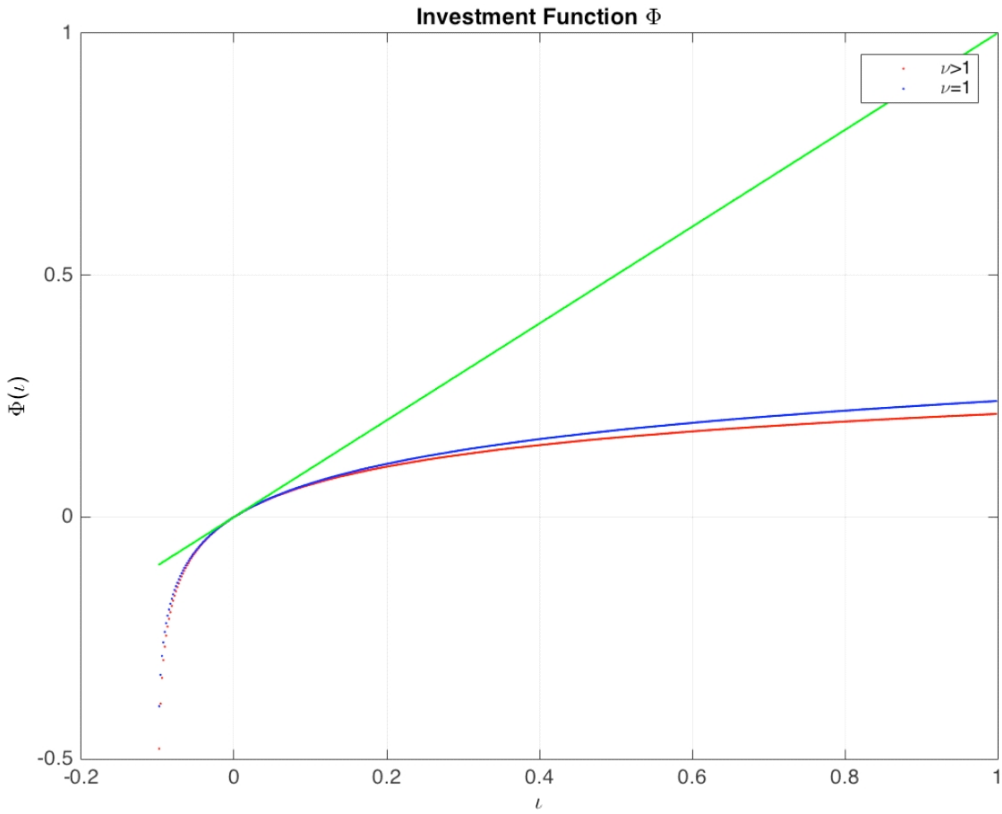

Figure 3.1 - Investment function $\Phi(\iota_t)$ with $\nu > 1$ and $\nu = 1$.

We assume the following functional form for the investment function:

$$\Phi(\iota_t) = \frac{1}{\kappa}\frac{(1+\kappa\iota_t)^{1-\nu} - 1}{1-\nu} \ , \nu \geq 0 \tag{3.37}$$

with κ being an adjustment cost parameter and with ν, a parameter to increase the curvature and thereby technological illiquidity. For the special case, where $\nu = 1$, we have,

$$\Phi(\iota_t) = \frac{\ln(1+\kappa\iota_t)}{\kappa} \tag{3.38}$$

$\Phi(\iota_t)$ is depicted in Figure 3.1 with $\nu > 1$ and $\nu = 1$. One can see that by increasing ν, the curvature of the investment function increases, which means that the adjustment costs for investment increase. In the numerical example depicted by Figure 3.1, $\kappa = 10$ and the employed value for $\nu > 1$ is $\nu = 1.10$.

3.4.1 Representative household's utility maximization

The household's expected utility maximization in the production economy does not differ from the one in the endowment economy. The change in the household's net wealth depends on the return the household receives from lending to the expert and on how much it consumes, i.e.,

$$dB_t = (r_t B_t - C_t^H)dt \ .$$

The optimal consumption rule of the household with recursive Epstein-Zin preferences, is given by a consumption rate depending on the state variable η_t times its net wealth B_t,[30]

$$C_t^H = \Psi^H(\eta_t)\, B_t \ , \tag{3.39}$$

while in the special case of logarithmic preferences, the optimal consumption rule of the household has a consumption rate equal to the discount factor, ρ, i.e.,[31]

$$C_t^H = \rho B_t \ . \tag{3.40}$$

3.4.2 Representative expert's utility maximization

The representative expert has to solve two maximization problems in the production economy, one as the shareholder of the firms and one as the shareholder of the banks. From the optimization problem of the expert as the shareholder of the firms, we derive the optimal investment rate rule,

$$\Phi'(\iota_t) = \frac{1}{Q(\eta_t)} \ , \tag{3.41}$$

[30] For the intermediate steps, we refer to the representative household's decision with recursive Epstein-Zin preferences in the endowment economy in the technical appendix, section 3.6.A.4., equation (A.3.26).
[31] For the intermediate steps, we refer to the representative household's decision with logarithmic utility in the endowment economy in the technical appendix, section 3.6.A.1., equation (A.3.6).

with (3.41) being the same in all cases, the general recursive Epstein-Zin preferences case and the special case with logarithmic preferences.[32]

The function $Q(\eta_t)$ is can be interpreted as Tobin's Q. From the functional form of $\Phi(\iota_t)$, we can derive $\Phi'(\iota_t)$, and solve for the optimal investment rate ι_t,

$$\begin{aligned} Q(\eta_t) &= (1+\kappa\iota_t)^{\nu} \\ \iota_t &= \frac{[Q(\eta_t)]^{\frac{1}{\nu}}-1}{\kappa} \end{aligned} \tag{3.42}$$

For the special case where $\nu = 1$,

$$\begin{aligned} Q(\eta_t) &= (1+\kappa\iota_t) \\ \iota_t &= \frac{Q(\eta_t)-1}{\kappa} \end{aligned} \tag{3.43}$$

As in the endowment economy, the expert maximizes the banks' shareholders' expected utility from consumption, by selecting the leveraging rate, ϕ_t, and consumption, C_t^E. The law of motion of the representative expert's net wealth N_t is given by,

$$dN_t = \left\{\left[\phi_t(\bar{r}_t^K - R(\eta_t)) + R(\eta_t)\right] N_t - C_t^E\right\} dt + \left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right]\phi_t N_t dn_t$$

where $\bar{r}_t^K$ is the expected return of the aggregate amount of capital and it is postulated to be a function of η_t,

$$\bar{r}_t^K \equiv E(r_t^K) \ ,$$

expressed as a function of η_t,

$$\bar{r}_t^K = R^K(\eta_t) \ . \tag{3.44}$$

[32]For details on the optimization problem of the expert as the shareholder of the firms with logarithmic preferences and with recursive Epstein-Zin preferences, see the technical appendix, section 3.6.B.1., equations (A.3.37) and (A.3.38).

The optimal consumption rule for the expert with recursive Epstein-Zin preferences, is given by a consumption rate depending on the state variable η_t times its net wealth N_t,[33]

$$C_t^E = \Psi^E(\eta_t) N_t \ , \tag{3.45}$$

while with logarithmic utility, the consumption rate is equal to the discount factor ρ,[34]

$$C_t^E = \rho N_t \ . \tag{3.46}$$

The optimal portfolio allocation rule for the expert with recursive Epstein-Zin preferences, is equal to,[35]

$$\phi_t = \frac{\left\{ \frac{-\lambda\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]\psi^E(\eta_t+G(\eta_t))}{\psi^E(\eta_t)[R^K(\eta_t)-R(\eta_t)]} \right\}^{\frac{1}{\gamma}} - 1}{\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right]} \tag{3.47}$$

while with logarithmic utility,[36]

$$\phi_t = -\frac{1}{\left((1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right)} - \frac{\lambda}{R^K(\eta_t) - R(\eta_t)} \ . \tag{3.48}$$

3.4.3 Equilibrium conditions in the production economy

The way we defined an equilibrium in the endowment economy, also holds for the production economy. The balance sheets of both agents always need to be balanced at all times. This balance-sheet balancing condition means that the representative household has,

$$B_t = N_t^H \ , \tag{3.49}$$

[33] For the intermediate steps, we refer to the representative expert's decision with recursive Epstein-Zin preferences in the endowment economy in the technical appendix, section 3.6.A.5., equation (A.3.28).
[34] For the intermediate steps, we refer to the representative expert's decision with logarithmic utility in the endowment economy in the technical appendix, section 3.6.A.2., equation (??).
[35] For the intermediate steps, we refer to the representative expert's decision with recursive Epstein-Zin preferences in the endowment economy in the technical appendix, section 3.6.A.5., equation (A.3.29).
[36] For the intermediate steps, we refer to the representative expert's decision with logarithmic utility in the endowment economy in the technical appendix, section 3.6.A.2., equation (A.3.13).

while the representative expert has,

$$q_t K_t = B_t + N_t \ . \tag{3.50}$$

For the market-clearing condition to hold in a closed economy with no government purchases, the sum of aggregate consumption, $C_t = C_t^H + C_t^E$, and investment, I_t must be equal to aggregate output, Y_t, which is obtained by the AK production function, such that,

$$\begin{aligned} C_t + I_t &= AK_t \qquad (3.51) \\ C_t^H + C_t^E &= AK_t - I_t = (A - \iota_t)K_t \end{aligned}$$

with $I_t = \iota_t K_t$. With recursive Epstein-Zin preferences, aggregate consumption, C_t, is equal to,

$$C_t = \Psi^H(\eta_t) B_t + \Psi^E(\eta_t) N_t \tag{3.52}$$

while with logarithmic preferences, C_t, is equal to,

$$C_t = \rho (B_t + N_t) \ ,$$

or, equivalently,

$$C_t = \rho Q(\eta_t) K_t \ . \tag{3.53}$$

Under market-clearing conditions, given optimal consumption choices from both agents and given the optimal reinvestment rate, the equilibrium condition for the price of capital, $Q(\eta_t)$ for recursive Epstein-Zin preferences is equal to,[37]

$$Q(\eta_t) = \frac{A - \left(\frac{Q(\eta_t)^{\frac{1}{\nu}} - 1}{\kappa}\right)}{\eta_t \Psi^E(\eta_t) + (1 - \eta_t)\Psi^H(\eta_t)} \ , \tag{3.54}$$

[37]For the intermediate steps, on the derivation of the equilibrium condition for the price of capital, see the technical appendix, section 3.6.B.2., equation (A.3.39).

where the optimal reinvestment rate per unit of capital is equal to,

$$\iota_t = \frac{1}{\kappa}\left[Q(\eta_t)^{\frac{1}{\nu}} - 1\right] \ . \tag{3.55}$$

For the special case where $\nu = 1$, the equilibrium condition for the price of capital is equal to,[38]

$$Q(\eta_t) = \frac{A + \frac{1}{\kappa}}{\Psi^H(\eta_t)(1-\eta_t) + \Psi^E(\eta_t)\eta_t + \frac{1}{\kappa}} \tag{3.56}$$

and the equilibrium condition for the reinvestment rate per unit of capital is equal to,[39]

$$\iota_t = \frac{A - \Psi^H(\eta_t)(1-\eta_t) - \Psi^E(\eta_t)\eta_t}{\kappa\left[\Psi^H(\eta_t)(1-\eta_t) + \Psi^E(\eta_t)\eta_t\right] + 1} \tag{3.57}$$

With logarithmic utility, the equilibrium price of capital is a constant, computed as an implicit function of,[40]

$$q = \frac{A - \left(\frac{q^{\frac{1}{\nu}} - 1}{\kappa}\right)}{\rho} \tag{3.58}$$

The equilibrium price in the production economy for agents with logarithmic utility is a constant. Moreover, in the production economy with logarithmic preferences, the equilibrium reinvestment rate per unit of capital ι is also constant. If $\nu \neq 1$, ι can be calculated by combining equations (3.58) and (3.55). For the special case where $\nu = 1$, the equilibrium price of capital is given by a closed-form,[41]

$$q = \frac{A\kappa + 1}{\rho\kappa + 1} \tag{3.59}$$

and the equilibrium reinvestment rate per unit of capital is equal to,[42]

$$\iota = \frac{A - \rho}{\rho\kappa + 1} \tag{3.60}$$

[38]For the intermediate steps, on the derivation of the equilibrium condition for the price of capital in the special case where $\nu = 1$, see the technical appendix, section 3.6.B.2., equation (A.3.41).
[39]For the intermediate steps, on the derivation of the equilibrium reinvestment rate in the special case where $\nu = 1$, see the technical appendix, section 3.6.B.2., equation (A.3.42).
[40]For the intermediate steps, on the derivation of the equilibrium condition for the price of capital, see the technical appendix, section 3.6.B.2., equation (A.3.44).
[41]For the intermediate steps, on the derivation of the equilibrium condition for the price of capital in the special case where $\nu = 1$, see the technical appendix, section 3.6.B.2., equation (A.3.45).
[42]For the intermediate steps, on the derivation of the equilibrium reinvestment rate in the special case where $\nu = 1$, see the technical appendix, section 3.6.B.2., equation (A.3.46).

In Figure 3.2 we depict the equilibrium price of capital and equilibrium reinvestment rate per unit of capital with logarithmic utility for two curvature levels of the investment function, representing higher and lower investment adjustment costs $\nu = 1.1 > 1$ and $\nu = 1$. Both capital prices, $Q(\eta)$, and investment, ι, do not depend on λ and ζ. This holds because, in the case of logarithmic utility $\Psi^H(\eta_t) = \Psi^E(\eta_t) = \rho$. With higher adjustment cost of investment, driven by parameter ν, the reinvestment rate per unit of capital goes down (investment is discouraged) and the price for capital goes up.

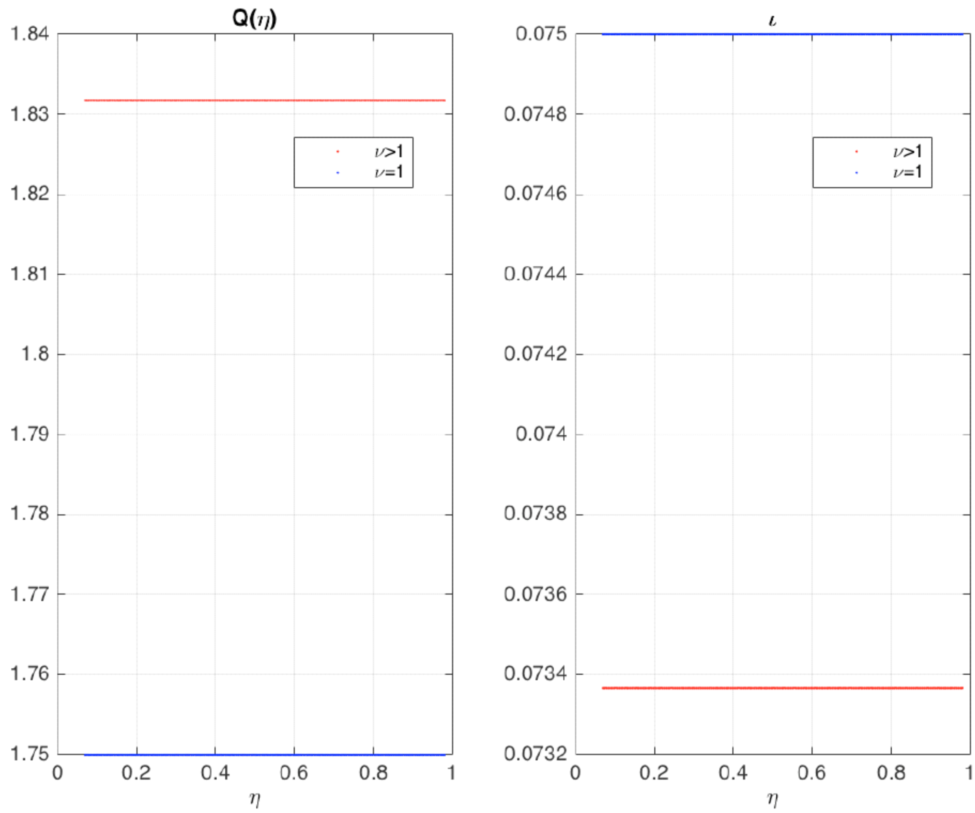

Figure 3.2 - Equilibrium price of capital and reinvestment rate per unit of capital

The rate of return that the expert earns from holding capital has the following law of motion,

$$dr_t^K = \underbrace{\frac{(A - \iota_t) K_t}{Q(\eta_t) K_t} dt}_{\substack{\shortparallel \\ \text{dividend yield}}} + \underbrace{\frac{d(Q(\eta_t) K_t)}{Q(\eta_t) K_t}}_{\substack{\shortparallel \\ \text{Capital gains rate}}} \tag{3.61}$$

The equilibrium condition for $R^K(\eta_t)$ for agents with recursive Epstein-Zin preferences, is given by,[43]

$$\begin{aligned} R^K(\eta_t) &= \frac{1}{\kappa}\left[\frac{A\kappa + 1}{Q(\eta_t)} + \frac{\nu Q(\eta_t)^{\frac{1-\nu}{\nu}} - 1}{1-\nu}\right] - \delta + \\ &\quad + \frac{Q_{\eta_t}(\eta_t) F(\eta_t)}{Q(\eta_t)} + \lambda\left[\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)}(1-\zeta) - 1\right] \end{aligned} \tag{3.62}$$

In the special case, where $\nu = 1$, the equilibrium condition for the expected rate of return on capital, $R^K(\eta_t)$, for agents with recursive Epstein-Zin preferences, is given by,[44]

$$\begin{aligned} R^K(\eta_t) &= \frac{1}{\kappa}\left[\frac{A\kappa + 1}{Q(\eta_t)} + \ln(Q(\eta_t))\right] - \delta + \\ &\quad + \frac{Q_{\eta_t}(\eta_t) F(\eta_t)}{Q(\eta_t)} + \lambda\left[\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)}(1-\zeta) - 1\right] \end{aligned} \tag{3.63}$$

The expected equilibrium rate of return on capital, for agents with logarithmic preferences, is given by,[45]

$$E\left(r_t^K\right) = \frac{1}{\kappa}\left(\frac{A\kappa + 1}{q} + \frac{\nu q^{\frac{1-\nu}{\nu}} - 1}{1-\nu}\right) - \delta - \lambda\zeta = \bar{r}_t^K \tag{3.64}$$

$\bar{r}_t^K$ is, as in the endowment economy, constant for agents with logarithmic preferences. In the special case, where $\nu = 1$, the expected equilibrium rate of return on capital, for agents

[43]For the intermediate steps, on the derivation of the equilibrium condition for $R^K(\eta_t)$ for agents with recursive Epstein-Zin preferences, see the technical appendix, section 3.6.B.2., equation (A.3.57).

[44]For the intermediate steps, on the derivation of the equilibrium condition for $R^K(\eta_t)$ for agents with recursive Epstein-Zin preferences for the special case where $\nu = 1$, see the technical appendix, section 3.6.B.2., equation (A.3.58).

[45]For the intermediate steps, on the derivation of the expected equilibrium rate of return on capital for agents with logarithmic preferences, see the technical appendix, section 3.6.B.2., equation (A.3.61).

with logarithmic preferences, is given by,[46]

$$E\left(r_t^K\right) = \frac{1}{\kappa}\left(\rho\kappa + 1 + \ln\left(A\kappa + 1\right) - \ln\left(\rho\kappa + 1\right)\right) - \delta - \lambda\zeta = \bar{r}_t^K \tag{3.65}$$

In a next step, we use the optimal portfolio allocation ϕ_t, derived from the expert's expected utility maximization problem, in order to derive the equilibrium condition of the bond rate $R(\eta_t)$ for agents with recursive Epstein-Zin preferences,[47]

$$R(\eta_t) = R^K(\eta_t) + \frac{\lambda\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right]\psi^E(\eta_t + G(\eta_t))}{\psi^E(\eta_t)\left[\frac{1}{\eta_t}\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right] + 1\right]^\gamma} \tag{3.66}$$

where $R^K(\eta_t)$ is given by (3.62) and in the special case, where $\nu_t = 1$, $R^K(\eta_t)$ is given by (3.63).

For agents with logarithmic preferences, the equilibrium condition of the bond rate $R(\eta_t)$ is given by,[48]

$$R\left(\eta_t\right) = \bar{r}_t^K + \frac{\lambda}{\frac{1}{\eta_t} - \frac{1}{\zeta}} \tag{3.67}$$

where $\bar{r}_t^K$ is given by (3.64) and in the special case, where $\nu = 1$, $\bar{r}_t^K$ is given by (3.65). In Figure 3.3, $R^K(\eta_t)$, $R(\eta_t)$ and the risk premium (denoted by "R-P(η)") are depicted for the special case of logarithmic preferences. The three panels comprising the first row of Figure 3.3 depict $R^K(\eta_t)$, $R(\eta_t)$ and the risk premium for the benchmark parameter values, $\kappa = 10$, $\rho = 2\%$, $A = 11\%$, $\delta = 3\%$, $\lambda = 3\%$, and $\zeta = 5\%$. The blue line depicts the case of logarithmic investment adjustment costs ($\nu = 1$), while the red line depicts higher adjustment costs setting $\nu = 1.1$. The pure message is that higher adjustment costs make

[46]For the intermediate steps, on the derivation of the expected equilibrium rate of return on capital for agents with logarithmic preferences for the special case where $\nu = 1$, see the technical appendix, section 3.6.B.2., equation (A.3.64).

[47]For the intermediate steps, on the derivation of the equilibrium condition for $R\left(\eta_t\right)$ for agents with recursive Epstein-Zin preferences, see the technical appendix, section 3.6.B.2., equation (A.3.65).

[48]For the intermediate steps, on the derivation of the equilibrium condition for $R\left(\eta_t\right)$ for agents with logarithmic preferences, see the technical appendix, section 3.6.B.2., equation (A.3.67).

both expected stock returns, $R^K(\eta_t)$, and the bond rate, $R(\eta_t)$ to drop. Yet, as implied by (3.67), the risk premium, $R^K(\eta_t) - R(\eta_t)$ is not affected by changes in the value of ν.

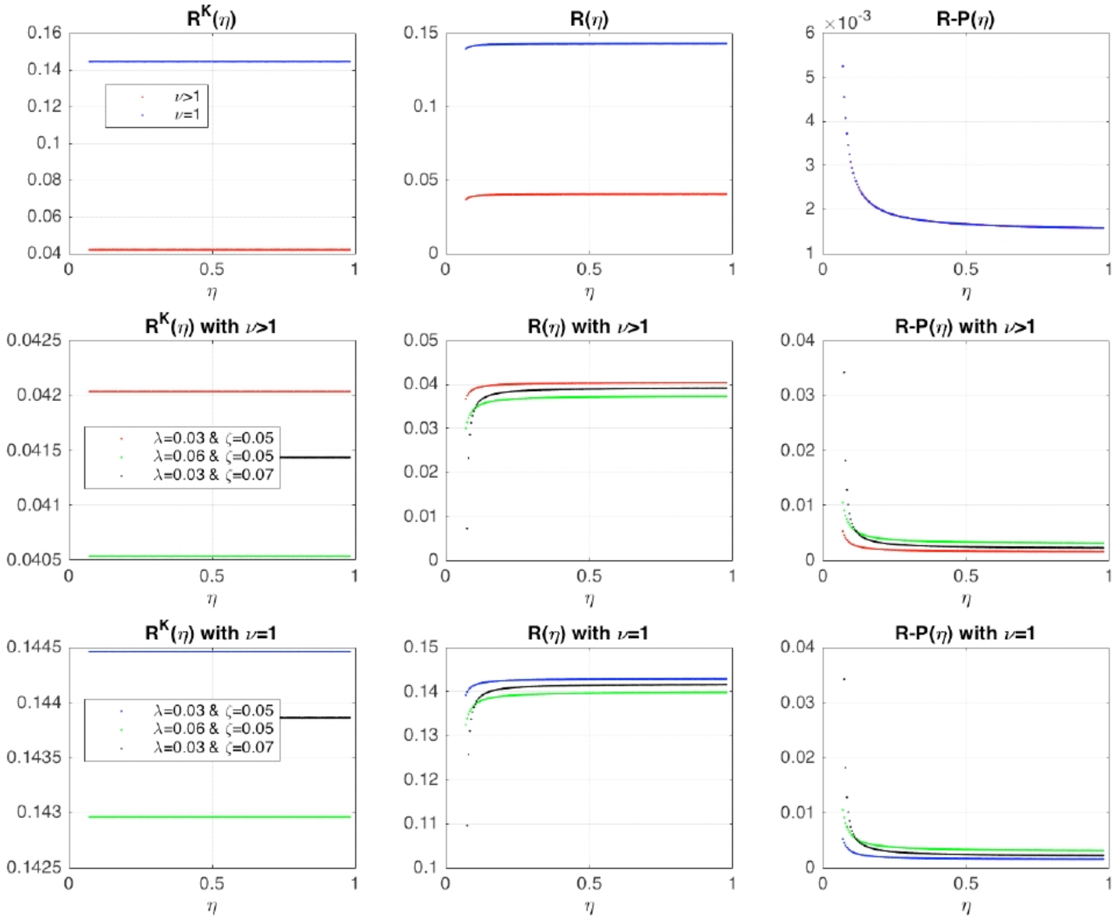

Figure 3.3 - Expected equilibrium rate of return on capital, equilibrium bond rate and the risk premium (denoted by "R-P(η)") for agents with logarithmic preferences.

The three panels comprising the second row of Figure 3.3 depict $R^K(\eta_t)$, $R(\eta_t)$ and the risk premium for all the benchmark parameter values except for setting $\nu = 1.1$ and except for varying the values of λ and ζ. Introducing higher disaster risk, λ, ceteris paribus (the green line) decreases both $R^K(\eta_t)$ and $R(\eta_t)$ but increases the risk premium (as this is also implied by (3.67)). These are reasonable effects, as stock returns become less attractive and investors ask for a higher risk premium for the higher partial-default risk (risk spread). Unsurprisingly, the direction of changes is qualitatively the same for introducing higher disaster impact, ζ, ceteris paribus (the black line). The three panels comprising the second row of Figure 3.3 depict $R^K(\eta_t)$, $R(\eta_t)$ and the risk premium for $\nu = 1$, with the same qualitative conclusions. Quantitatively, in the case of $\nu = 1$, $R^K(\eta_t)$ and $R(\eta_t)$ are at higher levels compared to the case of $\nu = 1.1$, while the risk premium comparisons are the same for $\nu = 1$ and $\nu = 1.1$, as implied by (3.67).

As a final step, we derive the equilibrium condition for the law of motion of the state variable η_t, for agents with recursive Epstein-Zin preferences,[49]

$$\begin{aligned}\frac{d\eta_t}{\eta_t} \quad = \quad & \underbrace{\left[\left[\frac{1}{\eta_t}(R^K(\eta_t) - R(\eta_t)) + R(\eta_t)\right] - \Psi^E(\eta_t) - \Phi^*(\iota_t) + \delta - \frac{Q_{\eta_t}(\eta_t) F(\eta_t)}{Q(\eta_t)}\right]}_{\overset{\shortparallel}{\frac{1}{\eta_t}F(\eta_t)}} dt + \\ & + \underbrace{\frac{\left(\frac{1}{\eta_t} - 1\right)\left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right]}{\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)}(1-\zeta)}}_{\overset{\shortparallel}{\frac{1}{\eta_t}G(\eta_t)}} dn_t\end{aligned} \tag{3.68}$$

where $\Phi^*(\iota_t)$ corresponds to $\Phi(\iota_t)$ with its optimal reinvestment rate. For agents with logarithmic preferences, the equilibrium law of motion is equal to,[50]

$$\frac{d\eta_t}{\eta_t} = \underbrace{\left[\left[\frac{1}{\eta_t}(\bar{r}_t^K - R(\eta_t)) + R(\eta_t)\right] - \rho - \Phi^*(\iota_t) + \delta\right]}_{\overset{\shortparallel}{\frac{1}{\eta_t}F(\eta_t)}} dt + \underbrace{\frac{\left(\frac{1}{\eta_t} - 1\right)(-\zeta)}{1-\zeta}}_{\overset{\shortparallel}{\frac{1}{\eta_t}G(\eta_t)}} dn_t \tag{3.69}$$

[49]For the intermediate steps, on the derivation of the equilibrium condition for the law of motion of η_t for agents with recursive Epstein-Zin preferences, see the technical appendix, section 3.6.B.2., equation (A.3.70).
[50]For the intermediate steps, on the derivation of the equilibrium condition for the law of motion of η_t for agents with logarithmic preferences, see the technical appendix, section 3.6.B.2., equation (A.3.72).

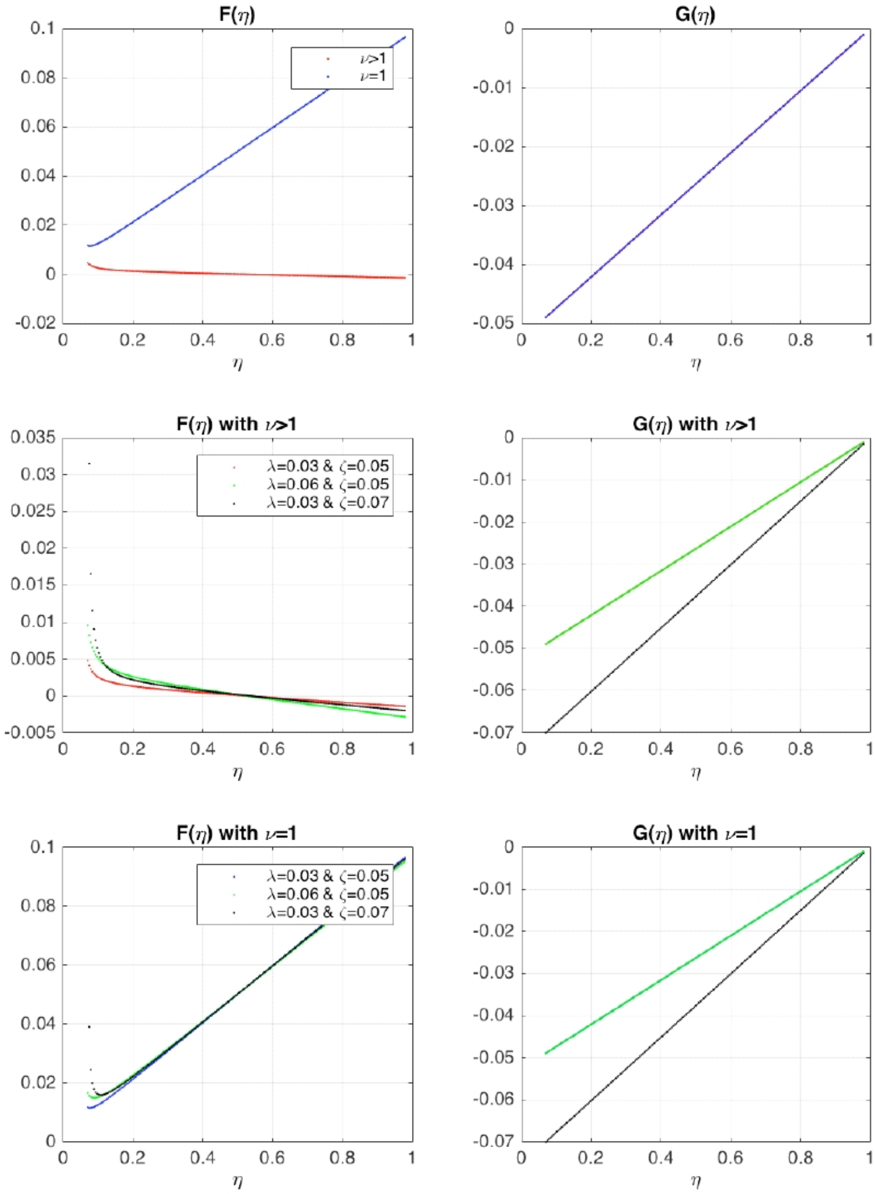

Figure 3.4 - Law of motion of η for different values of λ and ζ, logarithmic investment costs ($\nu = 1$), vs. higher adjustment costs ($\nu > 1$).

In Figure 3.4, $F(\eta_t)$ and $G(\eta_t)$ for logarithmic preferences are depicted. For financial stability, it is crucial to have a region of negative values for function $F(\eta)$. The main messages from Figure 3.4 are that increasing ν produces such a negative region, while increasing disaster risk, λ, and the negative disaster impact, ζ, increase the negative slope of $F(\eta)$. Further research focusing on numerical simulations can investigate the role of such parameters, especially in the presence of regulating constraints on leveraging policies of financial intermediaries. This extension requires numerical techniques that handle corner solutions for such sharp constraints, such as viscosity-solution approaches.

3.5 Conclusion

We provided analytical results on introducing jump/disaster risk to a new class of models of financial stability. This class of models is micro-founded, relying only on preference-and-technology fundamentals in order to co-determine the leveraging of financial institutions and asset prices. As excess leveraging may boost asset prices and leveraging may expose the system to a sudden drop in asset-prices, the core feature to investigate is the extent to which markets internalize these features. As the framework we study has incomplete markets, facing a certain degree of market failure is inevitable. In our view, introducing disaster shocks to such markets is at the heart of the broader research question of financial stability.[51] The place at which markets fail to internalize the risk of such disaster shock is the place where the need for financial-stability regulation starts.

We have not attempted a policy analysis in this paper. The task of introducing disaster risk to this class of financial stability models is cumbersome and it requires the development of robust numerical techniques. The contribution of this paper is to deliver analytical results that can serve as guides for developing homotopy approaches in order to study such economies. We made this contribution for two versions of the model. The first is a

[51]This model complements other "workhorse" models such as the Bernanke et al. (1999) model, in which the possibility of disasters is not studied.

Basak and Cuoco (1998) version of the model, an endowment economy. The second, is a production-economy version with endogenous growth. In the endogenous-growth version, the economy-wide growth rate depends on the level of endogenous risk in the economy. Endogenous risk is a model feature that stems from the level of exposure to external shocks due to leveraging.

In our model, the inverse of the leverage ratio, η_t, will not be stationary for most parameter values. This feature allows us to study the role and urgency of regulation in the form of Basel II and Basel III requirements in future research. Further research is needed to see which parameters and policies make η_t stationary. A provisional idea is to introduce regulation in the form of imposing a minimum leverage ratio (lower bound) that implies a penalty if violated. Specifically, one needs to investigate the case in which experts will have to pay a penalty to the households. By taking this punishment mechanism into account while solving their utility-maximization problem, experts may decrease their leverage ratio even more than the imposed minimum-leverage ratio. Whether this policy may lead to more stock-price stability is an additional extension for future research.

Our model opens the potential of studying the role that monetary policy can play for preventing bank runs. Borrowed money by banks is inside money in the balance sheets of banks. Money injections to the balance sheets of banks is outside money. Understanding the mechanism driving the dynamics of inside money and also the ways banks perceive monetary-policy rules, or surprising monetary-policy announcements, gives the chance to answer specific questions such as: "how does quantitative easing work?"

A key challenge of these models is the matching of real-world asset-price features. Specifically, one goal is to match price-dividend ratios of the financial sector, bond-prices and aggregate-stock-index returns simultaneously. Once this goal is achieved, then, by varying model parameters, one can study whether the financial system tends to be unstable. In future work, we plan to introduce outside money to our model (central bank money), allowing for the central bank to rescue banks. Furthermore, we plan to introduce model uncertainty. Model uncertainty implies that there are "blind spots" in disaster anticipations that may allow for even higher leveraging. Such high leverage may exacerbate the potential for financial crises.

3.6 Technical appendix

3.6.A Endowment economy

3.6.A.1 Representative household's decision with logarithmic utility

$$\max_{(C_t^H, B_t)_{t\geq 0}} \quad E_0\left[\int_0^\infty e^{-\rho t} U(C_t^H) dt\right]$$

$$s.t. \qquad dB_t = (r_t B_t - C_t^H)dt$$

where $U(C_t^H) \equiv \ln(C_t^H)$. The bond rate r_t is a function of η_t,

$$r_t = R(\eta_t)$$

and the dynamics of η_t are in the form of,

$$d\eta_t = F(\eta_t)dt + G(\eta_t)dn_t$$

At this stage, the functional forms of $R(\cdot)$, $F(\cdot)$ and $G(\cdot)$ are unknown.

We solve the stochastic dynamic optimization problem through the Hamilton-Jacobi-Bellman (HJB^H) equation,

$$\begin{aligned} \rho J^H(B_t, \eta_t) \quad = \quad & \max_{C_t^H \geqslant 0} \left\{\ln(C_t^H) + J_B^H(B_t, \eta_t)\cdot\left[R(\eta_t)B_t - C_t^H\right] + J_\eta^H(B_t, \eta_t)F(\eta_t) + \right. \\ & \left. + \lambda\left[J^H(B_t, \eta_t + G(\eta_t)) - J^H(B_t, \eta_t)\right]\right\} \end{aligned} \tag{A.3.1}$$

in which $J_B^H(B_t, \eta_t)$ denotes the partial derivative with respect to variable B_t and $J_\eta^H(B_t, \eta_t)$ denotes the partial derivative with respect to variable η_t.

We take a guess on the functional form for the household's value function $J^H(B_t, \eta_t)$,

$$J^H(B_t, \eta_t) = H(\eta_t) + b\ln(B_t)$$

FOC w.r.t. C_t^H:

$$\begin{aligned}\left(C_t^H\right)^{-1} &= J_B^H(B_t, \eta_t) \\ C_t^H &= b^{-1}B_t \qquad \text{(A.3.2)}\end{aligned}$$

Now plugging this equation into (A.3.1) and using the function form of the value function gives,

$$\begin{aligned}\rho H(\eta_t) + \rho b\ln(B_t) &= \ln(b^{-1}) + \ln(B_t) + bR(\eta_t) - 1 \\ &+ H'(\eta_t)F(\eta_t) + \lambda\left[H(\eta_t + G(\eta_t)) - H(\eta_t)\right] \qquad \text{(A.3.3)}\end{aligned}$$

In order to make the equation hold, the following conditions have to be secured,

$$\rho b\ln(B_t) = \ln(B_t)$$

$$b = \frac{1}{\rho} \qquad \text{(A.3.4)}$$

and

$$\rho H(\eta_t) = \ln(b^{-1}) + bR(\eta_t) - 1 + H'(\eta_t)F(\eta_t) + \lambda\left[H(\eta_t + G(\eta_t)) - H(\eta_t)\right] \qquad \text{(A.3.5)}$$

We can further simplify the (A.3.1), using both conditions, (A.3.4) and (A.3.5) in one another,

$$\begin{aligned}\rho H(\eta_t) &= \ln(\rho) + \frac{1}{\rho}R(\eta_t) - 1 + H'(\eta_t)F(\eta_t) + \lambda\left[H(\eta_t + G(\eta_t)) - H(\eta_t)\right] \\ 0 &= \rho H(\eta_t) - \ln(\rho) - \frac{1}{\rho}R(\eta_t) + 1 - H'(\eta_t)F(\eta_t) - \lambda\left[H(\eta_t + G(\eta_t)) - H(\eta_t)\right]\ .\end{aligned}$$

$H(\eta_t)$ can be solved once we know the functional forms of $R(\cdot)$, $F(\cdot)$ and $G(\cdot)$.

(A.3.4) in (A.3.2) gives us the household's optimal consumption rule as a fixed fraction, ρ, of his net wealth B_t,

$$C_t^H = \rho B_t \tag{A.3.6}$$

3.6.A.2 Representative expert's decision with logarithmic utility

$$\max_{(C_t^E, N_t, \phi_t)_{t\geq 0}} \quad E_0 \left[\int_0^\infty e^{-\rho t} U(C_t^E) dt \right]$$

$$s.t. \quad dN_t = \left\{ \left[\phi_t(\bar{r}_t^D - r_t) + r_t \right] N_t - C_t^E \right\} dt + \left[(1-\zeta) \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1 \right] \phi_t N_t dn_t$$

where $U(C_t^E) \equiv \ln(C_t^E)$ and $\phi_t = \frac{1}{\eta_t}$.

The shock term of dN_t,

$$dN_t^{Shock} = d(q_t^{Shock} D_t^{Shock}) = \left[(1-\zeta) \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1 \right] \phi_t N_t dn_t \tag{A.3.7}$$

can be derived applying *Itô*'s Lemma for Poisson processes. We recall the stochastic differential equation for D_t, dD_t,

$$dD_t = gD_t dt + \left[(1-\zeta) D_t - D_t \right] dn_t$$

and postulate that q_t depends on the state variable η_t,

$$q_t = Q(\eta_t)$$

with the law of motion equal to,

$$d\eta_t = F(\eta_t)dt + G(\eta_t)d\eta_t$$

Applying *Itô*'s Lemma, we derive the stochastic differential equation of $Q(\eta_t)$,

$$dQ(\eta_t) = Q_{\eta_t}(\eta_t)F(\eta_t)dt + \left[Q(\eta_t + G(\eta_t)) - Q(\eta_t) \right] dn_t \tag{A.3.8}$$

in which $Q_{\eta_t}(\eta_t)$ is the derivative with respect to η_t.

Next we derive the stochastic differential equation of $f(Q(\eta_t), D_t) = Q(\eta_t)D_t$,

$$\begin{aligned}
df(Q(\eta_t), D_t) &= \left[f_{Q(\eta_t)}Q_{\eta_t}F(\eta_t) + f_{D_t}gD_t\right]dt \\
&\quad + \left[f(Q(\eta_t) + Q(\eta_t + G(\eta_t)) - Q(\eta_t), D_t + (1-\zeta)D_t - D_t) - f(Q(\eta_t), D_t)\right]dn_t \\
df(Q(\eta_t), D_t) &= \left[D_tQ_{\eta_t}F(\eta_t) + Q(\eta_t)gD_t\right]dt + \left[f(Q(\eta_t + G(\eta_t)), (1-\zeta)D_t) - f(Q(\eta_t), D_t)\right]dn_t \\
df(Q(\eta_t), D_t) &= \left[D_tQ_{\eta_t}F(\eta_t) + Q(\eta_t)gD_t\right]dt + \left[Q(\eta_t + G(\eta_t))(1-\zeta)D_t - Q(\eta_t)D_t\right]dn_t
\end{aligned} \tag{A.3.9}$$

$$\begin{aligned}
d(Q(\eta_t)^{Shock}D_t^{Shock}) &= \left[Q(\eta_t + G(\eta_t))(1-\zeta)D_t - Q(\eta_t)D_t\right]dn_t \\
&= \left[\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)}(1-\zeta) - 1\right]Q(\eta_t)D_tdn_t \\
&= \left[\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)}(1-\zeta) - 1\right]\phi_tN_tdn_t
\end{aligned} \tag{A.3.10}$$

$\bar{r}_t^D$ is the expected return from the stock market and it is postulated to be a function of η_t:

$$\bar{r}_t^D \equiv E(r_t^D) = R^D(\eta_t)$$

For the expert's stochastic dynamic optimization problem, we formulate the corresponding HJB^E:

$$\begin{aligned}
\rho J^E(N_t, \eta_t) = \max_{C_t^E, \phi_t} \Big\{ \ln(C_t^E) + J_N^E(N_t, \eta_t)\left\{\left[\phi_t(R^D(\eta_t) - R(\eta_t)) + R(\eta_t)\right]N_t - C_t^E\right\} + \\
J_\eta^E(N_t, \eta_t)F(\eta_t) + +\lambda\left[J^E\left(N_t + \left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right]\phi_tN_t, \eta_t + G(\eta_t)\right) - J^E(N_t, \eta_t)\right]\Big\}
\end{aligned} \tag{A.3.11}$$

We take a guess on the functional form for the expert's value function $J^E(N_t, \eta_t)$,

$$J^E(N_t, \eta_t) = L(\eta_t) + \kappa\ln(N_t)$$

FOC w.r.t. C_t:

$$
\begin{aligned}
\left(C_t^E\right)^{-1} &= J_N^E\left(N_t, \eta_t\right) \\
C_t^E &= \frac{1}{\kappa} N_t
\end{aligned} \tag{A.3.12}
$$

FOC w.r.t. ϕ_t:

$$
\begin{aligned}
J_N^E\left(N_t, \eta_t\right)\left(R^D(\eta_t) - R(\eta_t)\right) N_t &= -\lambda\left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right] N_t \\
&\quad J_N^E\left(N_t + \left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right]\phi_t N_t, \eta_t + G(\eta_t)\right) \\
\kappa(R^D(\eta_t) - R(\eta_t)) &= \frac{-\lambda\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right]\kappa}{1 + \left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right]\phi_t} \\
\left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right]\phi_t &= -1 - \frac{\lambda\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right]}{(R^D(\eta_t) - R(\eta_t))} \\
\phi_t &= -\frac{1}{\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right]} - \frac{\lambda}{(R^D(\eta_t) - R(\eta_t))}
\end{aligned} \tag{A.3.13}
$$

Simplify (A.3.11) by using the functional form of the value function $J^E(N_t, \eta_t)$, its partial derivatives, (A.3.12) and (A.3.13),

$$
\begin{aligned}
\rho L(\eta_t) + \rho\kappa\ln(N_t) &= \ln(\frac{1}{\kappa}) + \ln(N_t) + \\
&\quad + \kappa\frac{1}{N_t}\left\{\left[\phi_t(\bar{r}_t^D - r_t) + r_t\right] N_t - \frac{1}{\kappa}N_t\right\} + L'(\eta_t)F(\eta_t) + \lambda\left[L(\eta_t + G(\eta_t))\right. \\
&\quad \left. + \kappa\ln\left(N_t + \left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right]\phi_t N_t\right) - L(\eta_t) - \kappa\ln(N_t)\right]
\end{aligned}
$$

$$
\begin{gathered}
\rho L(\eta_t) + \rho\kappa\ln(N_t) = \ln(\frac{1}{\kappa}) + \ln(N_t) + L'(\eta_t)F(\eta_t) + \\
+\left[\left(-\frac{1}{\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right]} - \frac{\lambda}{(R^D(\eta_t) - R(\eta_t))}\right)(R^D(\eta_t) - R(\eta_t)) + R(\eta_t)\right]\kappa - 1 + \\
+\lambda\left[L(\eta_t + G(\eta_t)) + \kappa\ln(N_t) + \kappa\ln\left(-\frac{\lambda\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right]}{(R^D(\eta_t) - R(\eta_t))}\right) - L(\eta_t) - \kappa\ln(N_t)\right]
\end{gathered}
$$

$$\begin{aligned}\rho L(\eta_t)+\rho\kappa\ln(N_t)&=\ln(\frac{1}{\kappa})+\ln(N_t)+L'(\eta_t)F(\eta_t)+\\&+\left[\left(-\frac{(R^D(\eta_t)-R(\eta_t))}{\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]}-\lambda\right)+R(\eta_t)\right]\kappa-1+\\&+\lambda\left[L(\eta_t+G(\eta_t))+\kappa\ln\left(-\frac{\lambda\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]}{(R^D(\eta_t)-R(\eta_t))}\right)-L(\eta_t)\right]\end{aligned}$$

To make the equation hold, we need to secure that,

$$\begin{aligned}\rho\kappa\ln(N_t)&=\ln(N_t)\\\kappa&=\frac{1}{\rho}\end{aligned}\tag{A.3.14}$$

and

$$\begin{aligned}\rho L(\eta_t)&=\ln(\frac{1}{\kappa})+L'(\eta_t)F(\eta_t)+\left[\left(-\frac{(R^D(\eta_t)-R(\eta_t))}{\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]}-\lambda\right)+R(\eta_t)\right]\kappa-1+\\&+\lambda\left[L(\eta_t+G(\eta_t))+\kappa\ln\left(-\frac{\lambda\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]}{(R^D(\eta_t)-R(\eta_t))}\right)-L(\eta_t)\right]\end{aligned}$$

(A.3.14) in (A.3.12) gives us the expert's optimal consumption rule as a fixed fraction, ρ, of his net wealth N_t:

$$C_t^E=\rho N_t\tag{A.3.15}$$

3.6.A.3 Equilibrium conditions

In order to derive the equilibrium expected value of the risky asset return, $R^D(\eta_t)$, we need to determine the dynamics of the risky asset return r_t^D,

$$\begin{aligned}
dr_t^D &= \frac{D_t}{P_t}dt + \frac{d(P_t)}{P_t} \\
dr_t^D &= \frac{D_t}{P_t}dt + \frac{d(qD_t)}{qD_t} \\
dr_t^D &= \rho dt + \frac{d(\frac{1}{\rho}D_t)}{\frac{1}{\rho}D_t} \\
dr_t^D &= (\rho + g)dt - \zeta dn_t
\end{aligned} \tag{A.3.16}$$

The expected value of dr_t^D, $E\left(dr_t^D\right)$ in equilibrium is equal to,

$$E(dr_t^D) = (\rho + g - \lambda\zeta)dt \tag{A.3.17}$$

and the expected value of r_t^D, $R^D(\eta_t)$ in equilibrium is equal to,

$$R^D(\eta_t) = \int_0^1 E(dr_t^D)dt = \rho + g - \lambda\zeta = \bar{r}_t^D \tag{A.3.18}$$

To derive the equilibrium bond return $R(\eta_t)$, we use (A.3.13) and constant q, in

$$\begin{aligned}
\phi_t &= \frac{1}{\eta_t} \\
-\frac{1}{\left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right]} - \frac{\lambda}{(\bar{r}_t^D - R(\eta_t))} &= \frac{1}{\eta_t} \\
\frac{1}{\zeta} - \frac{\lambda}{(\rho + g - \lambda\zeta - R(\eta_t))} &= \frac{1}{\eta_t} \\
\frac{1}{\zeta} - \frac{1}{\eta_t} &= \frac{\lambda}{(\rho + g - \lambda\zeta - R(\eta_t))} \\
(\rho + g - \lambda\zeta - R(\eta_t)) &= \frac{\lambda}{\frac{1}{\zeta} - \frac{1}{\eta_t}}
\end{aligned}$$

$$R(\eta_t) = \rho + g - \lambda\zeta - \frac{\lambda}{\frac{1}{\zeta} - \frac{1}{\eta_t}} \tag{A.3.19}$$

In order to derive the law of motion of η_t, rewrite η_t as,

$$\eta_t \equiv \frac{N_t}{q_t D_t} = \rho\frac{N_t}{D_t}$$

so that the stochastic differential equation of η_t, $d\eta_t$, reduces to,

$$d\eta_t = \rho d\left(\frac{N_t}{D_t}\right)$$

First we derive the stochastic differential equation of N_t, dN_t, with $\phi_t = \frac{1}{\eta_t}$ and (A.3.15), (3.22), (A.3.18) and (A.1.4):

$$\begin{aligned}
dN_t &= \left\{\left[\phi_t(\bar{r}_t^D - r_t) + r_t\right] N_t - C_t^E\right\} dt + \left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right]\phi_t N_t dn_t \\
dN_t &= \left\{\left[\frac{1}{\eta_t}(\bar{r}_t^D - R(\eta_t)) + R(\eta_t) - \rho\right] N_t\right\} dt - \zeta\frac{N_t}{\eta_t} dn_t \\
dN_t &= \left\{\left[\left(1 - \frac{1}{\eta_t}\right)\left(\rho + g - \lambda\zeta - \frac{\lambda}{\frac{1}{\zeta} - \frac{1}{\eta_t}}\right) + \frac{1}{\eta_t}(\rho + g - \lambda\zeta) - \rho\right] N_t\right\} dt - \frac{\zeta}{\rho} D_t dn_t \\
dN_t &= \left[\left(\frac{1}{\eta_t} - 1\right)\frac{\lambda}{\frac{1}{\zeta} - \frac{1}{\eta_t}} + \rho + g - \lambda\zeta - \rho\right] N_t dt - \frac{\zeta}{\rho} D_t dn_t \\
dN_t &= \underbrace{\left[\left(\frac{1}{\eta_t} - 1\right)\frac{\lambda}{\frac{1}{\zeta} - \frac{1}{\eta_t}} + \bar{r}_t^D - \rho\right] N_t}_{\overset{\shortparallel}{\mu^N(N_t, D_t)}} dt \underbrace{-\frac{\zeta}{\rho} D_t}_{\overset{\shortparallel}{b^N(N_t, D_t)}} dn_t \qquad \text{(A.3.20)} \\
dN_t &= \mu^N(N_t, D_t)dt + b^N(N_t, D_t)dn_t
\end{aligned}$$

Then we recall the stochastic differential equation of D_t, dD_t:

$$\begin{aligned}
dD_t &= \underbrace{gD_t}_{\overset{\shortparallel}{\mu^D(N_t, D_t)}} dt + \underbrace{\left[(1-\zeta) D_t - D_t\right]}_{\overset{\shortparallel}{b^D(N_t, D_t)}} dn_t \\
dD_t &= \mu^D(N_t, D_t)dt + b^D(N_t, D_t)dn_t \qquad \text{(A.3.21)}
\end{aligned}$$

We define the function $f(N_t, D_t)$,

$$f(N_t, D_t) \equiv \frac{N_t}{D_t}$$

and apply *Itô*'s Lemma for Poisson processes, in order to derive $d\left(\frac{N_t}{D_t}\right)$,

$$\begin{aligned}
df(N_t, D_t) &= \left[f_D(N_t, D_t)\cdot\mu^D(N_t, D_t) + f_N(N_t, D_t)\cdot\mu^N(N_t, D_t)\right]dt + \\
&\quad \left[f(N_t + b^N(N_t, D_t), D_t + b^D(N_t, D_t)) - f(N_t, D_t)\right]dn_t \\
df(N_t, D_t) &= \left\{-\frac{N_t}{D_t^2}gD_t + \frac{1}{D_t}\left[\left(\frac{1}{\eta_t}-1\right)\frac{\lambda}{\frac{1}{\zeta}-\frac{1}{\eta_t}} + \bar{r}_t^D - \rho\right]N_t\right\}dt + \\
&\quad + \left\{\frac{N_t - \frac{\zeta}{\rho}D_t}{D_t + \left[(1-\zeta)D_t - D_t\right]} - \frac{N_t}{D_t}\right\}dn_t \\
df(N_t, D_t) &= \frac{N_t}{D_t}\left\{\left(\frac{1}{\eta_t}-1\right)\frac{\lambda}{\frac{1}{\zeta}-\frac{1}{\eta_t}} + \bar{r}_t^D - (\rho+g)\right\}dt + \left[\frac{N_t - \frac{\zeta}{\rho}D_t}{D_t - \zeta D_t} - \frac{N_t}{D_t}\right]dn_t \\
df(N_t, D_t) &= \frac{N_t}{D_t}\left[\left(\frac{1}{\eta_t}-1\right)\frac{\lambda\zeta}{1-\frac{\zeta}{\eta_t}} - \lambda\zeta\right]dt + \left[\frac{N_t - \frac{\zeta}{\rho}D_t}{D_t - \zeta D_t} - \frac{N_t}{D_t}\right]dn_t
\end{aligned} \tag{A.3.22}$$

Now we can derive the law of motion of η_t:

$$\begin{aligned}
d\eta_t &= \rho dH(N_t, D_t) \\
d\eta_t &= \frac{\rho N_t}{D_t}\cdot\left\{\left(\frac{1}{\eta_t}-1\right)\frac{\lambda\zeta}{1-\frac{\zeta}{\eta_t}} - \lambda\zeta\right\}dt + \left[\frac{\frac{\rho N_t}{D_t}-\zeta}{1-\zeta} - \frac{\rho N_t}{D_t}\right]dn_t \\
d\eta_t &= \eta_t\left[\left(\frac{1}{\eta_t}-1\right)\frac{\lambda\zeta}{1-\frac{\zeta}{\eta_t}} - \lambda\zeta\right]dt + \left(\frac{\eta_t-\zeta}{1-\zeta} - \eta_t\right)dn_t \\
d\eta_t &= \underbrace{-\lambda\zeta\eta_t\left(\frac{1-\eta_t}{\zeta-\eta_t}+1\right)}_{\substack{\shortparallel \\ F(\eta_t)}}dt + \underbrace{\eta_t\left(\frac{1-\frac{\zeta}{\eta_t}}{1-\zeta}-1\right)}_{\substack{\shortparallel \\ G(\eta_t)}}dn_t
\end{aligned} \tag{A.3.23}$$

The dynamics of η_t:

$$\frac{d\eta_t}{\eta_t} = -\lambda\zeta\left(\frac{1-\eta_t}{\zeta-\eta_t}+1\right)dt - \left(1 - \frac{1-\frac{\zeta}{\eta_t}}{1-\zeta}\right)dn_t \tag{A.3.24}$$

3.6.A.4 Representative household's decision with recursive Epstein-Zin preferences

Expected utility is defined as,

$$J_t^H = E_t\left[\int_t^\infty f\left(C^H(\tau), J^H(\tau)\right) d\tau\right]$$

in which $f\left(C^H, J^H\right)$ is a normalized aggregator of continuation utility, J^H, and current consumption C^H,

$$f(C_t^H, J_t^H) \equiv \rho(1-\gamma)J_t^H \frac{\left\{\frac{C_t^H}{\left[(1-\gamma)J_t^H\right]^{\frac{1}{1-\gamma}}}\right\}^{1-\frac{1}{\alpha}} - 1}{1-\frac{1}{\alpha}}$$

Derivative of $f(C_t^H, J_t^H)$ w.r.t. C_t^H:

$$f_c(C_t^H, J_t^H) = \rho\left[(1-\gamma)J_t^H\right]^{1-\frac{1-\frac{1}{\alpha}}{1-\gamma}}\left(C_t^H\right)^{-\frac{1}{\alpha}}$$

We define,

$$\begin{aligned}\theta &\equiv \frac{1-\frac{1}{\alpha}}{1-\gamma}\\ 1-\frac{1}{\alpha} &= \theta(1-\gamma)\end{aligned}$$

We postulate that the bond rate r_t is a function of η_t,

$$r_t = R(\eta_t)$$

and the dynamics of η_t are of the form,

$$d\eta_t = F(\eta_t)dt + G(\eta_t)dn_t\ .$$

The expert maximizes his expected utility subject to,

$$dB_t = \left(R(\eta_t)B_t - C_t^H\right)dt$$

At this stage, the functional forms of $R(\cdot)$, $F(\cdot)$ and $G(\cdot)$ are unknown.

For the household's stochastic dynamic optimization problem with recursive Epstein-Zin preferences, we formulate the corresponding HJB^H:

$$0 = \max_{C_t^H} \left\{ f(C_t^H, J_t^H) + J_B^H(B_t, \eta_t) \cdot \left[R(\eta_t) B_t - C_t^H \right] + J_\eta^H(B_t, \eta_t) F(\eta_t) + \right.$$
$$\left. + \lambda \left[J^H(B_t, \eta_t + G(\eta_t)) - J^H(B_t, \eta_t) \right] \right\} \tag{A.3.25}$$

We take a guess on the functional form of $J^H(B_t, \eta_t)$,

$$J^H(B_t, \eta_t) = \psi^H(\eta_t) \frac{B_t^{1-\gamma}}{1-\gamma}$$

FOC w.r.t. C_t^H:

$$\begin{aligned} f_c(C_t^H, J_t^H) &= J_B^H(B_t, \eta_t) \\ \rho[(1-\gamma)J_t^H]^{1-\theta} \left(C_t^H\right)^{\theta(1-\gamma)-1} &= \psi^H(\eta_t) B_t^{-\gamma} \\ \rho(1-\gamma)^{1-\theta} \psi^H(\eta)^{1-\theta} \frac{B_t^{(1-\gamma)(1-\theta)}}{(1-\gamma)^{1-\theta}} \left(C_t^H\right)^{\theta(1-\gamma)-1} &= \psi^H(\eta) B_t^{-\gamma} \\ \rho \psi^H(\eta)^{-\theta} C_t^{\theta(1-\gamma)-1} &= B_t^{-(1-\gamma)+\theta(1-\gamma)-\gamma} \end{aligned}$$

We obtain the optimal consumption rule for the representative household:

$$\begin{aligned} C_t^H &= \underbrace{\rho^{\frac{1}{1-\theta(1-\gamma)}} \psi^H(\eta_t)^{\frac{\theta}{\theta(1-\gamma)-1}}}_{\overset{\shortparallel}{\Psi_H(\eta_t)}} B_t \\ C_t^H &= \Psi_H(\eta_t) B_t \end{aligned} \tag{A.3.26}$$

Rewrite $f(C_t^H, J_t^H)$ for the simplification of (A.3.25),

$$
\begin{aligned}
f(C_t^H, J_t^H) &\equiv \rho(1-\gamma)J_t^H \frac{\left\{\frac{C_t^H}{\left[(1-\gamma)J_t^H\right]^{\frac{1}{1-\gamma}}}\right\}^{1-\frac{1}{\alpha}} - 1}{1-\frac{1}{\alpha}} \\
f(C_t^H, J_t^H) &= -\frac{\rho}{\theta}J_t^H + \frac{\rho}{\theta}(1-\gamma)^{-\theta}\left(J_t^H\right)^{1-\theta}\left(C_t^H\right)^{\theta(1-\gamma)} \\
f(C_t^H, J_t^H) &= \frac{\rho}{\theta}\left[-\psi^H(\eta)\frac{B_t^{1-\gamma}}{1-\gamma} + (1-\gamma)^{-\theta}\psi^H(\eta)^{1-\theta}\frac{B^{(1-\gamma)(1-\theta)}}{(1-\gamma)^{1-\theta}}\rho^{\frac{\theta(1-\gamma)}{1-\theta(1-\gamma)}}\psi^H(\eta)^{\frac{\theta^2(1-\gamma)}{\theta(1-\gamma)-1}}B_t^{\theta(1-\gamma)}\right] \\
f(C_t^H, J_t^H) &= \frac{\rho}{\theta}\psi^H(\eta)\frac{B_t^{1-\gamma}}{1-\gamma}\left[-1 + \rho^{\frac{\theta(1-\gamma)}{1-\theta(1-\gamma)}}\psi^H(\eta)^{\frac{\theta^2(1-\gamma)-\theta^2(1-\gamma)+\theta}{\theta(1-\gamma)-1}}\right] \\
f(C_t^H, J_t^H) &= \frac{\rho}{\theta}[\rho^{\frac{\theta(1-\gamma)}{1-\theta(1-\gamma)}}\psi^H(\eta_t)^{\frac{\theta}{\theta(1-\gamma)-1}} - 1]\psi^H(\eta_t)\frac{B_t^{1-\gamma}}{1-\gamma}
\end{aligned}
$$

and replace it in (A.3.25),

$$
\begin{aligned}
0 &= \frac{\rho}{\theta}\left[\rho^{\frac{\theta(1-\gamma)}{1-\theta(1-\gamma)}}\psi^H(\eta_t)^{\frac{\theta}{\theta(1-\gamma)-1}} - 1\right]\psi^H(\eta_t) \\
&\quad + (1-\gamma)\psi^H(\eta_t)\left[R(\eta_t) - \rho^{\frac{1}{1-\theta(1-\gamma)}}\psi^H(\eta_t)^{\frac{\theta}{\theta(1-\gamma)-1}}\right] \\
&\quad + (\psi^H)'(\eta_t)\cdot F(\eta_t) + \lambda[\psi^H(\eta_t + G(\eta_t)) - \psi^H(\eta_t)] \qquad \text{(A.3.27)}
\end{aligned}
$$

(A.3.27) will be used in the numerical solution to derive $\psi^H(\eta_t)$ and $C_t^H(N_t, \eta_t)$.

3.6.A.5 Representative expert's decision with recursive Epstein-Zin preferences

The expert maximizes his expected utility subject to,

$$dN_t = \left\{\left[\phi_t(\bar{r}_t^D - r_t) + r_t\right]N_t - C_t^E\right\}dt + \left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right]\phi_t N_t dn_t$$

dN_t^{Shock} is derived as in (A.3.7) and we postulate that the price-dividend ratio is a function of η_t,

$$q_t = Q(\eta_t)$$

$\bar{r}_t^D$ is the expected return of the investment in the stock market and it is a function of η_t,

$$\bar{r}_t^D \equiv E(r_t^D) = R^D(\eta_t)$$

For the expert's stochastic dynamic optimization problem with recursive Epstein-Zin preferences, we formulate the corresponding HJB^E:

$$0 = \max_{C_t^E, \phi_t} \left\{ f(C_t^E, J_t^E) + J_N^E(N_t, \eta_t) \left\{ \left[\phi_t (R^D(\eta_t) - R(\eta_t)) + R(\eta_t) \right] N_t - C_t^E \right\} + J_\eta^E (N_t, \eta_t) F(\eta_t) \right.$$
$$\left. + \lambda \left[J^E \left(\left\{ 1 + \left[(1-\zeta) \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1 \right] \phi_t \right\} N_t, \eta_t + G(\eta_t) \right) - J^E(N_t, \eta_t) \right] \right\}$$

We take a guess for the functional form of $J^E(N_t, \eta_t)$:

$$J^E(N_t, \eta_t) = \psi^E(\eta_t) \frac{N_t^{1-\gamma}}{1-\gamma}$$

FOC w.r.t. C_t:

$$\begin{aligned} f_c(C_t^E, J_t^E) &= J_N^E(N_t, \eta_t) \\ C_t^E &= \underbrace{\rho^{\frac{1}{1-\theta(1-\gamma)}} \psi^E(\eta_t)^{\frac{\theta}{\theta(1-\gamma)-1}}}_{\substack{\shortparallel \\ \Psi_E(\eta_t)}} N_t \\ C_t^E &= \Psi_E(\eta_t) N_t \end{aligned} \tag{A.3.28}$$

FOC w.r.t. ϕ_t:

$$\begin{aligned} J_N^E(N_t, \eta_t) N_t (R^D(\eta_t) - R(\eta_t)) &= -\lambda N_t \left[(1-\zeta) \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1 \right] \\ &\quad J_N^E \left(\left\{ 1 + \left[(1-\zeta) \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1 \right] \phi_t \right\} N_t, \eta_t + G(\eta_t) \right) \\ \psi^E(\eta) N_t^{-\gamma} \left[R^D(\eta_t) - R(\eta_t) \right] &= -\lambda \left[(1-\zeta) \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1 \right] \psi^E(\eta_t + G(\eta_t)) \\ &\quad \left\{ 1 + \left[(1-\zeta) \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1 \right] \phi_t \right\}^{-\gamma} N_t^{-\gamma} \\ 1 + \left[(1-\zeta) \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1 \right] \phi_t &= \left\{ \frac{-\lambda \left[(1-\zeta) \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1 \right] \psi^E(\eta_t + G(\eta_t))}{\psi^E(\eta_t) \left[R^D(\eta_t) - R(\eta_t) \right]} \right\}^{\frac{1}{\gamma}} \\ \phi_t &= \frac{\left\{ \frac{-\lambda \left[(1-\zeta) \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1 \right] \psi^E(\eta_t + G(\eta_t))}{\psi^E(\eta_t) [R^D(\eta_t) - R(\eta_t)]} \right\}^{\frac{1}{\gamma}} - 1}{\left[(1-\zeta) \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1 \right]} \end{aligned} \tag{A.3.29}$$

Simplification of the HJB^E,

$$
\begin{aligned}
0 \;=\; & \frac{\rho}{\theta}\left[\rho^{\frac{\theta(1-\gamma)}{1-\theta(1-\gamma)}}\psi^E(\eta_t)^{\frac{\theta}{\theta(1-\gamma)-1}}-1\right]\psi^E(\eta_t) \\
& +(1-\gamma)\psi^E(\eta_t)\left[\frac{1}{\eta}\left[R^D(\eta_t)-R(\eta_t)\right]+R(\eta_t)-\rho^{\frac{1}{1-\theta(1-\gamma)}}\psi^E(\eta_t)^{\frac{\theta}{\theta(1-\gamma)-1}}\right] \\
& +\left(\psi^E\right)'(\eta_t)F(\eta_t) \\
& +\lambda\left\{\psi^E(\eta_t+G(\eta_t))\left\{1+\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]\frac{1}{\eta_t}\right\}^{1-\gamma}-\psi^E(\eta_t)\right\}
\end{aligned} \tag{A.3.30}
$$

Both (A.3.29) and (A.3.30) are used in the numerical solution to derive $R(\eta_t)$ and $\psi^E(\eta_t)$.

3.6.A.6 Numerical solution with recursive Epstein-Zin preferences

A numerical solution has to be applied in order to solve the model with recursive Epstein-Zin preferences. As we do not know the exact solution of the dynamic system, the first idea is to start from the equilibrium conditions. In equilibrium, aggregated consumption must be equal to output, namely, the dividend realized at a specific time.

$$\mathbf{C}_t^E(N_t,\eta_t)+\mathbf{C}_t^H(B_t,\eta_t)=D_t$$

By definition:

$$\eta_t=\frac{N_t}{Q(\eta_t)D_t}$$

and

$$
\begin{aligned}
1-\eta_t &= \frac{Q(\eta_t)D_t}{Q(\eta_t)D_t}-\frac{N_t}{Q(\eta_t)D_t} \\
&= \frac{B_t}{Q(\eta_t)D_t}
\end{aligned}
$$

Solving for N_t and B_t,

$$N_t=\eta_tQ(\eta_t)D_t$$

$$B_t = (1 - \eta_t)\, Q(\eta_t) D_t$$

Thereby we transform the two state variables N_t and B_t into functions of η_t. The equilibrium condition equation can be rewritten as,

$$C_t^E(\eta_t Q(\eta_t) D_t, \eta_t) + C_t^H((1 - \eta_t)\, Q(\eta_t) D_t, \eta_t) = D_t$$

From (A.3.26) and (A.3.28), we know that,

$$C_t^H = \Psi_H(\eta_t) B_t \text{ and } C_t^E = \Psi_E(\eta_t) N_t$$

We further simplify the equilibrium condition equation, in order to obtain equilibrium $Q(\eta_t)$,

$$\begin{aligned}
\Psi_H(\eta_t)\,(1 - \eta_t)\, Q(\eta_t) D_t + \Psi_E(\eta_t)\eta_t Q(\eta_t) D_t &= D_t \\
\Psi_H(\eta_t)\,(1 - \eta_t)\, Q(\eta_t) + \Psi_E(\eta_t)\eta_t Q(\eta_t) &= 1 \\
Q(\eta_t) &= \frac{1}{\Psi_H(\eta_t)\,(1 - \eta_t) + \Psi_E(\eta_t)\eta_t} \qquad \text{(A.3.31)}
\end{aligned}$$

We recall the dynamics of η_t,

$$d\eta_t = F(\eta_t)dt + G(\eta_t)dn_t$$

Applying *Itô*'s lemma for Poisson processes to derive the stochastic differential equation of $Q(\eta_t)$, $dQ(\eta_t)$,

$$\begin{aligned}
dQ(\eta_t) &= Q_{\eta_t}(\eta_t)F(\eta_t)dt + [Q(\eta_t + G(\eta_t)) - Q(\eta_t)]\, dn_t \\
dQ(\eta_t) &= Q_{\eta_t}(\eta_t)F(\eta_t)dt + \\
&\quad +([\psi_H(\eta_t + G(\eta_t))\,(1 - (\eta_t + G(\eta_t))) + \psi_E(\eta_t + G(\eta_t))\,(\eta_t + G(\eta_t))]^{-1} \\
&\quad - [\Psi_H(\eta_t)\,(1 - \eta_t) + \Psi_E(\eta_t)\eta_t]^{-1})dn_t
\end{aligned}$$

The dynamics of returns from investing in the stock market, are given by,

$$\begin{aligned}
dr_t^D &= \frac{D_t}{P_t}dt + \frac{d(Q(\eta_t)D_t)}{Q(\eta_t)D_t} \qquad \text{(A.3.32)} \\
dr_t^D &= \frac{1}{Q(\eta_t)}dt + \frac{d(Q(\eta_t)D_t)}{Q(\eta_t)D_t}
\end{aligned}$$

with (A.3.9),

$$
\begin{aligned}
dr_t^D &= \frac{1}{Q(\eta_t)}dt + \frac{\left[Q_{\eta_t}(\eta_t)F(\eta_t) + Q(\eta_t)g\right]}{Q(\eta_t)}dt + \frac{\left[Q(\eta_t + G(\eta_t))(1-\zeta) - Q(\eta_t)\right]}{Q(\eta_t)}dn_t \\
dr_t^D &= \left[\frac{1}{Q(\eta_t)} + g + \frac{Q_{\eta_t}(\eta_t)}{Q(\eta_t)}F(\eta_t)\right]dt + \left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right]dn_t \qquad \text{(A.3.33)}
\end{aligned}
$$

The expected value of dr_t^D is equal to:

$$
E(dr_t^D) = \left\{\frac{1}{Q(\eta_t)} + g + \frac{Q'(\eta_t)}{Q(\eta_t)}F(\eta_t) + \lambda\left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right]\right\}dt \qquad \text{(A.3.34)}
$$

and the expected value of r_t^D is equal to:

$$
\bar{r}_t^D \equiv E(r_t^D) = \frac{1}{Q(\eta_t)} + g + \frac{Q'(\eta_t)}{Q(\eta_t)}F(\eta_t) + \lambda\left[(1-\zeta)\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right] \qquad \text{(A.3.35)}
$$

To derive the law of motion of η_t, $d\eta_t$, recall the differentials of $Q(\eta_t)D_t$, $d\left(Q(\eta_t)D_t\right)$,

$$
d(Q(\eta_t)D_t) = \underbrace{\left[Q_{\eta_t}(\eta_t)F(\eta_t) + Q(\eta_t)g\right]D_t}_{\mu^y}dt + \underbrace{\left[Q(\eta_t + G(\eta_t))(1-\zeta) - Q(\eta_t)\right]D_t}_{b^y}dn_t
$$

and N_t, dN_t,

$$
dN_t = \underbrace{\left\{\left[\frac{1}{\eta_t}(R_t^D(\eta_t) - R_t(\eta_t)) + R_t(\eta_t)\right]N_t - C_t^E(N_t, \eta_t)\right\}}_{\mu^X}dt + \underbrace{\left[(1-\zeta)\,\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1\right]\frac{N_t}{\eta_t}}_{b^x}dn_t
$$

By *Itô*'s Lemma for Poisson processes x_t and y_t, we have:

$$
d\left(\frac{x_t}{y_t}\right) = \left(\frac{1}{y_t}\mu^x - \frac{x_t}{y_t^2}\mu^y\right)dt + \left(\frac{x_t + b^x}{y_t + b^y} - \frac{x_t}{y_t}\right)dn_t
$$

Applied to $d\eta_t = d\left(\frac{N_t}{Q(\eta_t)D_t}\right)$, gives us the law of motion of η_t,

$$\begin{aligned}
d\eta_t &= \left[\frac{1}{Q(\eta_t)D_t}\left\{\left[\frac{1}{\eta_t}(R_t^D(\eta_t)-R_t(\eta_t))+R_t(\eta_t)\right]N_t - C_t^E(N_t,\eta_t)\right\}-\right.\\
&\left. -\frac{N_t}{(Q(\eta_t)D_t)^2}\left[Q_{\eta_t}(\eta_t)F(\eta_t)+Q(\eta_t)g\right]D_t\right]dt+\\
&+\left[\frac{N_t+\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]\frac{N_t}{\eta_t}}{Q(\eta_t)D_t+\left[Q(\eta_t+G(\eta_t))(1-\zeta)-Q(\eta_t)\right]D_t}-\frac{N_t}{Q(\eta_t)D_t}\right]dn_t\\
d\eta_t &= \eta_t\left[\left\{\left[\frac{1}{\eta_t}(R_t^D(\eta_t)-R_t(\eta_t))+R_t(\eta_t)\right]-\frac{C_t^E(N_t,\eta_t)}{N_t}\right\}-\left(\frac{Q_{\eta_t}(\eta_t)F(\eta_t)}{Q(\eta_t)}+g\right)\right]dt+\\
&+\eta_t\left[\frac{\left(1+\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]\frac{1}{\eta_t}\right)}{\left[1+\frac{[Q(\eta_t+G(\eta_t))(1-\zeta)-Q(\eta_t)]}{Q(\eta_t)}\right]}-1\right]dn_t\\
d\eta_t &= \underbrace{\eta_t\left[\frac{1}{\eta_t}(R_t^D(\eta_t)-R_t(\eta_t))+R_t(\eta_t)-\frac{C_t^E(N_t,\eta_t)}{N_t}-\frac{Q_{\eta_t}(\eta_t)F(\eta_t)}{Q(\eta_t)}-g\right]}_{= F(\eta_t)}dt+\\
&+\underbrace{\eta_t\left[\frac{\left(\frac{\eta_t-1}{\eta_t}+\frac{1}{\eta_t}(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}\right)}{(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}}-1\right]}_{= G(\eta_t)}dn_t
\end{aligned} \tag{A.3.36}$$

3.6.B Production economy (general-equilibrium model)

3.6.B.1 Representative expert's utility maximization

The expert has to solve two maximization problems in the production economy, one as the shareholder of the firms and one as the shareholder of the banks.

Optimization problem of the expert as the shareholder of the firms:

The expert maximizes his expected utility, subject to,

$$C_t^E + C_t^H + I_t = AK_t$$

given the law of motions of the state variables K_t, η_t and B_t:

$$dK_t = (\Phi(\iota_t) K_t - \delta K_t) dt + ((1-\zeta) K_t - K_t) dn_t$$

$$d\eta_t = F(\eta_t) dt + G(\eta_t) dn_t$$

$$dB_t = \left(R(\eta_t) B_t - C_t^H\right) dt$$

For the expert's stochastic dynamic optimization problem with logarithmic utility, $U\left(C_t^E\right) = \ln\left(C_t^E\right)$, we formulate the corresponding HJB^E:

$$\begin{aligned}
\rho J^E(K_t, \eta_t, B_t) &= \max_{\iota_t} \left\{U\left(AK_t - C_t^H - \iota_t K_t\right)\right. \\
&\quad + J_{K_t}^E(K_t, \eta_t, B_t)(\Phi(\iota_t) K_t - \delta K_t) + J_{\eta_t}^E(K_t, \eta_t, B_t) F(\eta_t) \\
&\quad + J_{B_t}^E(K_t, \eta_t, B_t)\left(R(\eta_t) B_t - C_t^H\right) \\
&\quad \left. + \lambda\left[J^E(K_t + (1-\zeta) K_t - K_t, \eta_t + G(\eta_t), B_t) - J^E(K_t, \eta_t, B_t)\right]\right\}
\end{aligned}$$

For our purpose to formally derive the optimal investment rate, we only need the FOC w.r.t. ι_t:

$$\begin{aligned}
U'\left(C_t^E\right) K_t &= J_{K_t}^E(K_t, \eta_t, B_t) \Phi'(\iota_t) K_t \\
\Phi'(\iota_t) &= \frac{U'\left(C_t^E\right)}{J_{K_t}^E(K_t, \eta_t, B_t)} = \frac{1}{Q(\eta_t)} \\
\Phi'(\iota_t) &= \frac{1}{Q(\eta_t)} \qquad \text{(A.3.37)}
\end{aligned}$$

For consistency, we also formulate the HJB^E for the expert's stochastic dynamic optimization problem with recursive Epstein-Zin preferences:

$$\begin{aligned}
0 &= \max_{\iota_t} \left\{f\left(AK_t - C_t^H - \iota_t K_t, J^E(K_t, \eta_t, B_t)\right)\right. \\
&\quad + J_{K_t}^E(K_t, \eta_t, B_t)(\Phi(\iota_t) K_t - \delta K_t) + J_{\eta_t}^E(K_t, \eta_t, B_t) F(\eta_t) \\
&\quad + J_{B_t}^E(K_t, \eta_t, B_t)\left(R(\eta_t) B_t - C_t^H\right) \\
&\quad \left. + \lambda\left[J^E(K_t + (1-\zeta) K_t - K_t, \eta_t + G(\eta_t), B_t) - J^E(K_t, \eta_t, B_t)\right]\right\}
\end{aligned}$$

FOC w.r.t. ι_t:

$$\begin{aligned}
f_{C_t^E}\left(C_t^E, J^E\right) K_t &= J_{K_t}^E\left(K_t, \eta_t, B_t\right) \Phi'\left(\iota_t\right) K_t \\
\Phi'\left(\iota_t\right) &= \frac{f_{C_t^E}\left(C_t^E, J^E\right)}{J_{K_t}^E\left(K_t, \eta_t, B_t\right)} = \frac{1}{Q\left(\eta_t\right)} \\
\Phi'\left(\iota_t\right) &= \frac{1}{Q\left(\eta_t\right)} \qquad \text{(A.3.38)}
\end{aligned}$$

3.6.B.2 Equilibrium conditions in the production economy

Derivation of the equilibrium condition for the price of capital $Q\left(\eta_t\right)$ and the optimal investment rate ι_t **with recursive Epstein-Zin preferences:**

From the market clearing condition (3.51), using (3.52), we derive the equilibrium condition for the price of capital $Q\left(\eta_t\right)$,

$$\begin{aligned}
C_t + I_t &= AK_t \\
C_t^H + C_t^E &= \left(A - \iota_t\right) K_t \\
\Psi^H\left(\eta_t\right) B_t + \Psi^E\left(\eta_t\right) N_t &= \left(A - \iota_t\right) K_t \\
\Psi^H\left(\eta_t\right)\left(1 - \eta_t\right) + \Psi^E\left(\eta_t\right) \eta_t &= \frac{A - \iota_t}{Q\left(\eta_t\right)} \\
Q\left(\eta_t\right) &= \frac{A - \iota_t}{\Psi^H\left(\eta_t\right)\left(1 - \eta_t\right) + \Psi^E\left(\eta_t\right) \eta_t} \,. \qquad \text{(A.3.39)}
\end{aligned}$$

Using the optimal reinvestment rate per unit of capital,

$$\iota_t = \frac{\left[Q\left(\eta_t\right)\right]^{\frac{1}{\nu}} - 1}{\kappa}$$

in (A.3.39), we have,

$$Q\left(\eta_t\right) = \frac{A - \left(\frac{Q\left(\eta_t\right)^{\frac{1}{\nu}} - 1}{\kappa}\right)}{\Psi^H\left(\eta_t\right)\left(1 - \eta_t\right) + \Psi^E\left(\eta_t\right) \eta_t} \qquad \text{(A.3.40)}$$

In the special case where $\nu = 1$, (A.3.40) is equal to,

$$\begin{aligned}
Q(\eta_t) &= \frac{A - \left(\frac{Q(\eta_t)-1}{\kappa}\right)}{\Psi^H(\eta_t)(1-\eta_t) + \Psi^E(\eta_t)\eta_t} \\
A &= Q(\eta_t)\left(\Psi^H(\eta_t)(1-\eta_t) + \Psi^E(\eta_t)\eta_t\right) + \left(\frac{Q(\eta_t)-1}{\kappa}\right) \\
Q(\eta_t) &= \frac{A + \frac{1}{\kappa}}{\Psi^H(\eta_t)(1-\eta_t) + \Psi^E(\eta_t)\eta_t + \frac{1}{\kappa}}\,.
\end{aligned} \tag{A.3.41}$$

and the optimal reinvestment rate per unit of capital, (3.43) is equal to,

$$\begin{aligned}
\iota_t &= \frac{Q(\eta_t)-1}{\kappa} \\
\iota_t &= \frac{\frac{A+\frac{1}{\kappa}}{\Psi^H(\eta_t)(1-\eta_t)+\Psi^E(\eta_t)\eta_t+\frac{1}{\kappa}}}{\kappa} - \frac{1}{\kappa} \\
\iota_t &= \frac{A + \frac{1}{\kappa} - \Psi^H(\eta_t)(1-\eta_t) - \Psi^E(\eta_t)\eta_t - \frac{1}{\kappa}}{\kappa\left[\Psi^H(\eta_t)(1-\eta_t) + \Psi^E(\eta_t)\eta_t\right] + 1} \\
\iota_t &= \frac{A - \Psi^H(\eta_t)(1-\eta_t) - \Psi^E(\eta_t)\eta_t}{\kappa\left[\Psi^H(\eta_t)(1-\eta_t) + \Psi^E(\eta_t)\eta_t\right] + 1}
\end{aligned} \tag{A.3.42}$$

Derivation of the equilibrium condition for the price of capital $Q(\eta_t)$ and the optimal reinvestment rate per unit of capital ι_t **with logarithmic preferences:**

From the market clearing condition (3.51), using (3.53), we derive the equilibrium condition for the price of capital $Q(\eta_t)$,

$$\begin{aligned}
C_t + I_t &= AK_t \\
\rho(B_t + N_t) &= AK_t - I_t \\
\rho Q(\eta_t) K_t &= (A - \iota_t) K_t \\
Q(\eta_t) &= \frac{(A - \iota_t)}{\rho}
\end{aligned} \tag{A.3.43}$$

and with the optimal reinvestment rate per unit of capital (3.42) in (A.3.43),

$$q = \frac{A - \left(\frac{q^{\frac{1}{\nu}}-1}{\kappa}\right)}{\rho} \tag{A.3.44}$$

implies that the equilibrium price of capital q is constant.

In the special case where $\nu = 1$, (A.3.44) is equal to,

$$\begin{aligned} q &= \frac{A - \left(\frac{q-1}{\kappa}\right)}{\rho} \\ q &= \frac{A}{\rho} - \frac{q}{\rho\kappa} + \frac{1}{\rho\kappa} \\ q &= \frac{A\kappa + 1}{\rho\kappa + 1} \end{aligned} \tag{A.3.45}$$

and the optimal reinvestment rate per unit of capital, (3.43) is equal to,

$$\begin{aligned} \iota &= \frac{q-1}{\kappa} \\ \iota &= \frac{\frac{A\kappa+1}{\rho\kappa+1} - 1}{\kappa} \\ \iota &= \frac{A - \rho}{\rho\kappa + 1} \end{aligned} \tag{A.3.46}$$

Derivation of the equilibrium condition for the expected return of the aggregate amount of capital:

The dynamics of capital return, are given by the following law of motion,

$$dr_t^K = \frac{D_t}{Q(\eta_t) K_t} dt + \frac{d(Q(\eta_t) K_t)}{Q(\eta_t) K_t} \tag{A.3.47}$$

with $D_t \equiv (A - \iota_t) K_t$ being the dividend. In order to derive the equilibrium equation of $R^K(\eta_t)$, we first derive the stochastic differential equation of $Q(\eta_t)$,

$$dQ(\eta_t) = Q_{\eta_t}(\eta_t) F(\eta_t) dt + [Q(\eta_t + G(\eta_t)) - Q(\eta_t)] dn_t \tag{A.3.48}$$

and we recall the stochastic differential equation of K_t,

$$dK_t = (\Phi(\iota_t) - \delta) K_t dt + ((1-\zeta) K_t - K_t) dn_t \tag{A.3.49}$$

in order to use both, $dQ(\eta_t)$ and dK_t, to derive $d(Q(\eta_t) K_t)$, by applying *Itô*'s Lemma for

Poisson processes,

$$
\begin{aligned}
d\left(Q\left(\eta_t\right)K_t\right) &= \left[K_t Q_{\eta_t}\left(\eta_t\right)F\left(\eta_t\right) + Q_t\left(\Phi(\iota_t)-\delta\right)K_t\right]dt + \\
&+ \left[\left(Q\left(\eta_t\right) + \left[Q\left(\eta_t + G\left(\eta_t\right)\right) - Q\left(\eta_t\right)\right]\right)\left(K + \left(\left(1-\zeta\right)K_t - K_t\right)\right)\right. \\
&\left. - Q\left(\eta_t\right)K_t\right]dn_t \\
d\left(Q\left(\eta_t\right)K_t\right) &= \left[\frac{Q_{\eta_t}\left(\eta_t\right)F\left(\eta_t\right)}{Q\left(\eta_t\right)} + \Phi(\iota_t) - \delta\right]Q\left(\eta_t\right)K_t dt + \\
&+ \left[Q\left(\eta_t + G\left(\eta_t\right)\right)\left(1-\zeta\right)K_t - Q\left(\eta_t\right)K_t\right]dn_t \\
\frac{d\left(Q\left(\eta_t\right)K_t\right)}{Q\left(\eta_t\right)K_t} &= \left[\frac{Q_{\eta_t}\left(\eta_t\right)F\left(\eta_t\right)}{Q\left(\eta_t\right)} + \Phi(\iota_t) - \delta\right]dt + \left[\frac{Q\left(\eta_t + G\left(\eta_t\right)\right)}{Q\left(\eta_t\right)}\left(1-\zeta\right) - 1\right]dn_t
\end{aligned}
\tag{A.3.50}
$$

Used in (A.3.47), gives us,

$$
\begin{aligned}
dr_t^K &= \left[\frac{\left(A-\iota_t\right)}{Q\left(\eta_t\right)} + \Phi(\iota_t) - \delta + \frac{Q_{\eta_t}\left(\eta_t\right)F\left(\eta_t\right)}{Q\left(\eta_t\right)}\right]dt + \\
&+ \left[\frac{Q\left(\eta_t + G\left(\eta_t\right)\right)}{Q\left(\eta_t\right)}\left(1-\zeta\right) - 1\right]dn_t
\end{aligned}
\tag{A.3.51}
$$

We rewrite the optimal reinvestment rate per unit of capital ι_t,

$$
\begin{aligned}
\iota_t &= \frac{Q\left(\eta_t\right)^{\frac{1}{\nu}} - 1}{\kappa} \\
1 + \kappa\iota_t &= Q\left(\eta_t\right)^{\frac{1}{\nu}}
\end{aligned}
$$

and use it in the investment function (3.37),

$$
\Phi(\iota_t) = \frac{1}{\kappa}\frac{Q\left(\eta_t\right)^{\frac{1-\nu}{\nu}} - 1}{1-\nu}\ . \tag{A.3.52}
$$

Using this in (A.3.51),

$$
\begin{aligned}
dr_t^K &= \left\{\underbrace{\frac{A}{Q\left(\eta_t\right)} - \frac{Q\left(\eta_t\right)^{\frac{1}{\nu}} - 1}{\kappa Q\left(\eta_t\right)} + \frac{1}{\kappa}\frac{Q\left(\eta_t\right)^{\frac{1-\nu}{\nu}} - 1}{1-\nu}}_{\overset{\shortparallel}{\Omega(\eta_t)}} - \delta + \frac{Q_{\eta_t}\left(\eta_t\right)F\left(\eta_t\right)}{Q\left(\eta_t\right)}\right\}dt + \\
&+ \left\{\frac{Q\left(\eta_t + G\left(\eta_t\right)\right)}{Q\left(\eta_t\right)}\left(1-\zeta\right) - 1\right\}dn_t
\end{aligned}
\tag{A.3.53}
$$

Reformulate $\Omega(\eta_t)$ in order to simplify (A.3.53),

$$\begin{aligned}
\Omega(\eta_t) &= \frac{A}{Q(\eta_t)} - \frac{Q(\eta_t)^{\frac{1}{\nu}} - 1}{\kappa Q(\eta_t)} + \frac{1}{\kappa}\frac{Q(\eta_t)^{\frac{1-\nu}{\nu}} - 1}{1-\nu} \\
&= \frac{A\kappa + 1}{\kappa Q(\eta_t)} + \frac{1}{\kappa} Q(\eta_t)^{\frac{1-\nu}{\nu}} \left(\frac{1}{1-\nu} - 1\right) - \frac{1}{\kappa(1-\nu)} \\
&= \frac{A\kappa + 1}{\kappa Q(\eta_t)} + \frac{1}{\kappa}\left[Q(\eta_t)^{\frac{1-\nu}{\nu}} \left(\frac{1}{1-\nu} - 1\right) - \frac{1}{(1-\nu)}\right] \\
&= \frac{1}{\kappa}\frac{A\kappa + 1}{Q(\eta_t)} + \frac{1}{\kappa}\left[\frac{Q(\eta_t)^{\frac{1-\nu}{\nu}} - \left(Q(\eta_t)^{\frac{1-\nu}{\nu}} - \nu Q(\eta_t)^{\frac{1-\nu}{\nu}}\right) - 1}{1-\nu}\right] \\
&= \frac{1}{\kappa}\left[\frac{A\kappa + 1}{Q(\eta_t)} + \frac{\nu Q(\eta_t)^{\frac{1-\nu}{\nu}} - 1}{1-\nu}\right] \qquad \text{(A.3.54)}
\end{aligned}$$

Substituting (A.3.54) in (A.3.53), gives us,

$$\begin{aligned}
dr_t^K &= \left\{\frac{1}{\kappa}\left[\frac{A\kappa + 1}{Q(\eta_t)} + \frac{\nu Q(\eta_t)^{\frac{1-\nu}{\nu}} - 1}{1-\nu}\right] - \delta + \frac{Q_{\eta_t}(\eta_t) F(\eta_t)}{Q(\eta_t)}\right\} dt + \\
&\quad + \left\{\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)}(1-\zeta) - 1\right\} dn_t \qquad \text{(A.3.55)}
\end{aligned}$$

$$R^K(\eta_t) = \int_0^1 E\left(dr_t^K\right) dt = E\left(r_t^K\right) \qquad \text{(A.3.56)}$$

The equilibrium condition for the expected return of the aggregate amount of capital, $R^K(\eta_t)$ for agents with recursive Epstein-Zin preferences, is derived using (A.3.55) and (A.3.56),

$$\begin{aligned}
R^K(\eta_t) &= \frac{1}{\kappa}\left[\frac{A\kappa + 1}{Q(\eta_t)} + \frac{\nu Q(\eta_t)^{\frac{1-\nu}{\nu}} - 1}{1-\nu}\right] - \delta + \\
&\quad + \frac{Q_{\eta_t}(\eta_t) F(\eta_t)}{Q(\eta_t)} + \lambda\left[\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)}(1-\zeta) - 1\right] \qquad \text{(A.3.57)}
\end{aligned}$$

In (A.3.57) we consider the special case of $\nu = 1$, so that

$$\lim_{\nu \to 1} \frac{\nu Q(\eta_t)^{\frac{1-\nu}{\nu}} - 1}{1-\nu} = \ln(Q(\eta_t)) \ ,$$

which gives us the equilibrium equation of $R^K(\eta_t)$,

$$\begin{aligned} R^K(\eta_t) &= \frac{1}{\kappa}\left[\frac{A\kappa+1}{Q(\eta_t)} + \ln\left(Q(\eta_t)\right)\right] - \delta + \\ &+ \frac{Q_{\eta_t}(\eta_t) F(\eta_t)}{Q(\eta_t)} + \lambda\left[\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)}(1-\zeta) - 1\right] \end{aligned} \tag{A.3.58}$$

The dynamics of the capital return, with logarithmic utility are derived using (3.42) in (A.3.47) and taking into account that q is constant,

$$\begin{aligned} dr_t^K &= \frac{D_t}{qK_t}dt + \frac{d(qK_t)}{qK_t} \\ dr_t^K &= \frac{(A-\iota_t)}{q}dt + \frac{dK_t}{K_t} \\ dr_t^K &= \left[\frac{(A-\iota_t)}{q} + \Phi(\iota_t) - \delta\right]dt - \zeta dn_t \\ dr_t^K &= \left[\frac{\left[A - \left(\frac{q^{\frac{1}{\nu}}-1}{\kappa}\right)\right]}{q} + \frac{1}{\kappa}\frac{q^{\frac{1-\nu}{\nu}}-1}{1-\nu} - \delta\right]dt - \zeta dn_t \\ dr_t^K &= \left[\frac{A}{q} - \frac{q^{\frac{1-\nu}{\nu}}}{\kappa} + \frac{1}{q\kappa} + \frac{1}{\kappa}\frac{q^{\frac{1-\nu}{\nu}}-1}{1-\nu} - \delta\right]dt - \zeta dn_t \\ dr_t^K &= \left[\frac{A\kappa+1}{q\kappa} + \frac{1}{\kappa}\left(\frac{q^{\frac{1-\nu}{\nu}} - 1 - q^{\frac{1-\nu}{\nu}} + \nu q^{\frac{1-\nu}{\nu}}}{1-\nu}\right) - \delta\right]dt - \zeta dn_t \\ dr_t^K &= \left[\frac{1}{\kappa}\left(\frac{A\kappa+1}{q} + \frac{\nu q^{\frac{1-\nu}{\nu}}-1}{1-\nu}\right) - \delta\right]dt - \zeta dn_t \end{aligned} \tag{A.3.59}$$

The expected value of dr_t^K is equal to,

$$E\left(dr_t^K\right) = \left[\frac{1}{\kappa}\left(\frac{A\kappa+1}{q} + \frac{\nu q^{\frac{1-\nu}{\nu}}-1}{1-\nu}\right) - \delta - \lambda\zeta\right]dt \tag{A.3.60}$$

and the expected value of r_t^K is equal to,

$$E\left(r_t^K\right) = \frac{1}{\kappa}\left(\frac{A\kappa+1}{q} + \frac{\nu q^{\frac{1-\nu}{\nu}}-1}{1-\nu}\right) - \delta - \lambda\zeta \tag{A.3.61}$$

with

$$q = \frac{\left(A - \left(\frac{q^{\frac{1}{\nu}}-1}{\kappa}\right)\right)}{\rho}. \tag{A.3.62}$$

In (A.3.61), we consider the special case, where $\nu = 1$,

$$\lim_{\nu\to 1}\frac{\nu q^{\frac{1-\nu}{\nu}}-1}{1-\nu} = \ln(q) \ ,$$

which gives us,

$$E\left(r_t^K\right) = \frac{1}{\kappa}\left[\frac{A\kappa+1}{q} + \ln(q)\right] - \delta - \lambda\zeta \tag{A.3.63}$$

In (A.3.63) use the equilibrium price q, derived for logarithmic preferences, (A.3.45), in order to obtain the equilibrium expected rate of return on capital,

$$\begin{aligned}
E\left(r_t^K\right) &= \frac{1}{\kappa}\left(\rho\kappa + 1 + \ln\left(\frac{A\kappa+1}{\rho\kappa+1}\right)\right) - \delta - \lambda\zeta \\
E\left(r_t^K\right) &= \frac{1}{\kappa}\left(\rho\kappa + 1 + \ln(A\kappa+1) - \ln(\rho\kappa+1)\right) - \delta - \lambda\zeta
\end{aligned} \tag{A.3.64}$$

Derivation of the equilibrium condition for the bond return:

We rewrite the optimal portfolio allocation rule, (3.47) for an expert with recursive Epstein-Zin preferences,

$$\begin{aligned}
\phi_t &= \frac{\left\{\frac{-\lambda\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]\psi^E(\eta_t+G(\eta_t))}{\psi^E(\eta_t)[R^K(\eta_t)-R(\eta_t)]}\right\}^{\frac{1}{\gamma}} - 1}{\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right]} \\
\phi_t\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)} - 1\right] + 1 &= \left\{\frac{-\lambda\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]\psi^E(\eta_t+G(\eta_t))}{\psi^E(\eta_t)\left[R^K(\eta_t)-R(\eta_t)\right]}\right\}^{\frac{1}{\gamma}} \\
\left\{\psi^E(\eta_t)\left[R^K(\eta_t)-R(\eta_t)\right]\right\}^{\frac{1}{\gamma}} &= \frac{\left\{-\lambda\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]\psi^E(\eta_t+G(\eta_t))\right\}^{\frac{1}{\gamma}}}{\phi_t\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]+1} \\
\psi^E(\eta_t)\left[R^K(\eta_t)-R(\eta_t)\right] &= \frac{-\lambda\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]\psi^E(\eta_t+G(\eta_t))}{\left[\phi_t\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]+1\right]^{\gamma}}
\end{aligned}$$

such that,

$$R(\eta_t) = R^K(\eta_t) + \frac{\lambda\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]\psi^E(\eta_t+G(\eta_t))}{\psi^E(\eta_t)\left[\frac{1}{\eta_t}\left[(1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right]+1\right]^{\gamma}} \tag{A.3.65}$$

with optimal $R^K(\eta_t)$ given by (A.3.57) and for the special case where $\nu = 1$, with optimal $R^K(\eta_t)$ given by (A.3.58).

For an expert with logarithmic preferences, we rewrite the optimal portfolio allocation rule, (3.48) in order to derive $R(\eta_t)$,

$$\begin{aligned}
\phi_t &= -\frac{1}{\left((1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right)} - \frac{\lambda}{R^K(\eta_t) - R(\eta_t)} \\
\frac{\lambda}{R^K(\eta_t) - R(\eta_t)} &= -\frac{1}{\left((1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right)} - \phi_t \\
R^K(\eta_t) - R(\eta_t) &= -\frac{\lambda}{\frac{1}{\left((1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right)} + \phi_t} \\
R(\eta_t) &= R^K(\eta_t) + \frac{\lambda}{\frac{1}{\left((1-\zeta)\frac{Q(\eta_t+G(\eta_t))}{Q(\eta_t)}-1\right)} + \phi_t} \qquad \text{(A.3.66)}
\end{aligned}$$

We know that with logarithmic utility, q is constant, so that the equation of (A.3.66) is equal to,

$$R(\eta_t) = R^K(\eta_t) + \frac{\lambda}{\frac{1}{\eta_t} - \frac{1}{\zeta}} \qquad \text{(A.3.67)}$$

with optimal $R^K(\eta_t)$ given by (A.3.61) and (A.3.62) and for the special case where $\nu = 1$, with optimal $R^K(\eta_t)$ given by (A.3.64).

Derivation of the law of motion of η_t:

To derive the law of motion of η_t, recall that

$$\frac{d\eta_t}{\eta_t} = \frac{d\left(\frac{N_t}{Q(\eta_t)K_t}\right)}{\frac{N_t}{Q(\eta_t)K_t}}$$

and recall from (A.3.50) the stochastic differential equations of $Q(\eta_t)K_t$,

$$\begin{aligned}
d(Q(\eta_t)K_t) &= \left\{\Phi(\iota_t) - \delta + \frac{Q_{\eta_t}(\eta_t)F(\eta_t)}{Q(\eta_t)}\right\} Q(\eta_t)K_t dt + \\
&+ \left\{\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)}(1-\zeta) - 1\right\} Q(\eta_t)K_t dn_t \qquad \text{(A.3.68)}
\end{aligned}$$

and N_t,

$$dN_t = \left\{ \left[\phi_t(R^K(\eta_t) - R(\eta_t)) + R(\eta_t) \right] N_t - C_t^E \right\} dt + \left[(1-\zeta) \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1 \right] \phi_t N_t dn_t \tag{A.3.69}$$

Using both, (A.3.68) and (A.3.69), we can derive the law of motion of η_t for recursive Epstein-Zin preferences, with (A.3.52),

$$\begin{aligned}
\frac{d\eta_t}{\eta_t} &= \left[\left\{ \left[\frac{1}{\eta_t} (R^K(\eta_t) - R(\eta_t)) + R(\eta_t) \right] - \Psi^E(\eta_t) \right\} - \left\{ \Phi(\iota_t) - \delta + \frac{Q_{\eta_t}(\eta_t) F(\eta_t)}{Q(\eta_t)} \right\} \right] dt + \\
&+ \frac{\left[(1-\zeta) \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1 \right] \frac{1}{\eta_t} - \left\{ \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} (1-\zeta) - 1 \right\}}{1 + \left\{ \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} (1-\zeta) - 1 \right\}} dn_t \\
\frac{d\eta_t}{\eta_t} &= \underbrace{\left[\left[\frac{1}{\eta_t} (R^K(\eta_t) - R(\eta_t)) + R(\eta_t) \right] - \Psi^E(\eta_t) - \frac{1}{\kappa} \frac{Q(\eta_t)^{\frac{1-\nu}{\nu}} - 1}{1-\nu} + \delta - \frac{Q_{\eta_t}(\eta_t) F(\eta_t)}{Q(\eta_t)} \right]}_{\frac{1}{\eta_t} F(\eta_t)} dt + \\
&+ \underbrace{\frac{\left(\frac{1}{\eta_t} - 1 \right) \left[(1-\zeta) \frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} - 1 \right]}{\frac{Q(\eta_t + G(\eta_t))}{Q(\eta_t)} (1-\zeta)}}_{\frac{1}{\eta_t} G(\eta_t)} dn_t
\end{aligned} \tag{A.3.70}$$

using (A.3.57) for $R^K(\eta_t)$ and (A.3.65) for $R(\eta_t)$.

For the special case where $\nu = 1$, we use the optimal ι_t from (3.43), in order to rewrite the investment function as,

$$\Phi(\iota_t) = \frac{\ln(Q(\eta_t))}{\kappa} \tag{A.3.71}$$

which is then used in the law of motion of η_t together with (A.3.58) for $R^K(\eta_t)$ and (A.3.65) for $R(\eta_t)$.

The law of motion of η_t for logarithmic utility, with (A.3.52), is given by,

$$
\begin{aligned}
\frac{d\eta_t}{\eta_t} &= \left[\left[\frac{1}{\eta_t}(\bar{r}_t^K - R(\eta_t)) + R(\eta_t)\right] - \rho - (\Phi(\iota_t) - \delta)\right] dt + \frac{\left(\frac{1}{\eta_t} - 1\right)(-\zeta)}{1-\zeta} dn_t \\
\frac{d\eta_t}{\eta_t} &= \underbrace{\left[\left[\frac{1}{\eta_t}(\bar{r}_t^K - R(\eta_t)) + R(\eta_t)\right] - \rho - \frac{1}{\kappa}\frac{q^{\frac{1-\nu}{\nu}} - 1}{1-\nu} + \delta\right]}_{\overset{\shortparallel}{\frac{1}{\eta_t}F(\eta_t)}} dt + \underbrace{\frac{\left(\frac{1}{\eta_t} - 1\right)(-\zeta)}{1-\zeta}}_{\overset{\shortparallel}{\frac{1}{\eta_t}G(\eta_t)}} dn_t
\end{aligned}
\tag{A.3.72}
$$

together with (A.3.61) for $\bar{r}_t^K$ and (A.3.67) for $R(\eta_t)$. For the special case where $\nu = 1$, we use the law of motion of η_t, (A.3.72), together with (A.3.71) for $\Phi(\iota_t)$, (A.3.64) for $\bar{r}_t^K$ and (A.3.67) for $R(\eta_t)$.

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