On Reductions of Hintikka Sets for Higher-Order Logic

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Abstract

Steen's (2018) Hintikka set properties for Church's type theory based on primitive equality are reduced to the Hintikka set properties of Benzmüller, Brown and Kohlhase (2004) which are based on the logical connectives negation, disjunction and universal quantification.

1 Introduction and Preliminaries

Abstract consistency properties and Hintikka sets play an important role in the study (e.g., of Henkin-completeness) of proof calculi for Church's type theory [5, 2], aka. classical higher-order logic (HOL). Technically quite different definitions of these terms have been used in the literature, since they depend on the primitive logical connectives assumed in each case. The definitions of Benzmüller, Brown and Kohlhase [3, 4], for example, are based on negation, disjunction and universal quantification, while Steen [6], in the tradition of Andrews [1], works with primitive equality only. Despite their conceptual relationship, important semantical corollaries that are implied by these syntax related Hintikka properties (such as model existence theorems) can hence not be directly transferred between formalisms. Theorem 1 that we establish in this paper paves way for the convenient reuse of (e.g., model existence) results from the work of Benzmüller, Brown and Kohlhase [3, 4] in the context of Steen's setting by showing that Steen's Hintikka set properties can be reduced to those as studied by Benzmüller et al. In more general terms we illustrate how technical dependencies on particular primitive logical connectives can be overcome by providing respective reductions between technically different definitions of Hintikka sets.

Paper structure. In the remainder of this introduction we recapitulate some relevant (syntactic) notions on HOL, mainly to clarify our notation; for further details on HOL as relevant for this paper see the original publications of Steen and Benzmüller et al. [3, 4, 6]. In §2 we present the Hintikka properties as used by Benzmüller et al., and in §3 we give the related properties as used by Steen. In §3 we then show various lemmata that are implied in Steen's setting, and it are those lemmata which prepare the main reduction result of this paper (Theorem 1), which is given in §4.

Equality conventions. Different notions of equality will be used in the following: If a concept is defined (as an abbreviation), the symbol $:=$ is used. Primitive equality, written $=^\tau$, refers to a logical constant symbol from the HOL language such that $s_\tau =^\tau t_\tau$ is a term of the logic (assuming s_τ and t_τ are, where τ is a type annotation), cf. details further below. Leibniz-equality, written $\doteq$, is a defined term; usually it stands for $\lambda X_\tau.\, \lambda Y_\tau.\, \forall P_{o\tau}.\, (P\ X) \Rightarrow (P\ Y)$, where $\Rightarrow$ is a (primitive) logical connective. Meta equality $\equiv$ denotes set-theoretic identity between objects. Finally, $\equiv_\star$, for $\star \subseteq \{\beta, \eta\}$ is used for syntactic equality modulo $\beta-$, $\eta-$ and $\beta\eta$-conversion, respectively (as in the related work, α-conversion is taken as implicit).

Syntax of HOL. The set $\mathcal{T}$ of simple types is freely generated from the base types o and ι by juxtaposition. The types o and ι represent the type of Booleans and individuals, respectively. A type $\nu\tau$ represents the type of a total function from objects of type τ to objects of type ν.

Let Σ_τ be a set of constant symbols of type $\tau \in \mathcal{T}$ and let $\Sigma := \bigcup_{\tau \in \mathcal{T}} \Sigma_\tau$ be the union of all typed symbols, called a *signature*. Let further $\mathcal{V}$ denote a set of (typed) variable symbols. From these the terms of HOL are constructed by the following abstract syntax ($\tau, \nu \in \mathcal{T}$):

$$s, t ::= c_\tau \in \Sigma \mid X_\tau \in \mathcal{V} \mid \ (\lambda X_\tau.\, s_\nu)_{\nu\tau} \ \mid \ (s_{\nu\tau}\ t_\tau)_\nu$$

The terms are called *constants*, *variables*, *abstractions* and *applications*, respectively. The set of all terms of type τ over a signature Σ is denoted $\Lambda_\tau(\Sigma)$, and $\Lambda^c_\tau(\Sigma)$ is used for closed terms, respectively. The notion of free and bound variables are defined as usual, and a term t is called *closed* if t does not contain any free variables. It is assumed that the set $\mathcal{V}$ contains countably infinitely many variable symbols for each type $\tau \in \mathcal{T}$.

The type of a term is written as subscript but may be dropped by convention if clear from the context (or if not important). Also, parentheses are omitted whenever possible, and application is assumed to be left-associative. Furthermore, the scope of an λ-abstraction's body reaches as far to the right as is consistent with the remaining brackets. Nested applications $s\,t^1 \ \dots\ t^n$ may also be written in vector notation $s\,\overline{t^n}$.

In the two variants of HOL defined in §2 and §3 below, the choice of the signature Σ differs. In either case, $s \neq t$ is used as an abbreviation for $\neg(s = t)$. Also, for simplicity, the binary logical connectives may be written in infix notation; e.g., the term $p_o \vee q_o$ formally represents the application $(\vee_{ooo}\, p_o\, q_o)$. Furthermore, binder notation is used for universal and existential quantification: The term $\forall X_\tau.\, s_o$ is used as a shorthand for $\Pi^\tau\,(\lambda X_\tau.\, s_o)$, where Π^τ is a constant symbol. Finally, Leibniz-equality, denoted $\doteq$, is defined as $\doteq := \lambda X_\tau.\, \lambda Y_\tau.\, \forall P_{o\tau}.\, (P\ X) \Rightarrow (P\ Y)$. A Σ-*formula* s_o is a term from $s_o \in \Lambda_o(\Sigma)$ of type o and a Σ-*sentence* if it is a closed Σ-formula. The reference to Σ may be omitted if clear from the context.

In the following, variables are denoted by capital letters such as X_τ, Y_τ, Z_τ and, more specifically, the variable symbols P_o, Q_o and $F_{\nu\tau}, G_{\nu\tau}$ are used for predicate or Boolean variables and variables of functional type, respectively. Analogously, lower case letters s_τ, t_τ, u_τ denote general terms and $f_{\nu\tau}, g_{\nu\tau}$ are used for terms of functional type.

Semantics of HOL. The semantics of HOL, including the notions of Σ-models and Σ-Henkin models, is not discussed here; cf. [3, 6] for details.

2 Hintikka sets as defined by Benzmüller et al. [3, 4]

In the formulation of HOL as employed by Benzmüller et al. [3, 4], the set of primitive logical connectives is chosen to contain $\neg$, $\vee$ and Π^τ for every $\tau \in \mathcal{T}$. All remaining constant symbols from Σ are called parameters. A signature Σ with $\{\neg, \vee\} \cup \{\Pi^\tau \mid \tau \in \mathcal{T}\} \subseteq \Sigma$ is also referred to as $\Sigma^{\doteq}$. The remaining logical connectives can be defined as usual [3]. In their original formulation, Benzmüller et al. use $\rightarrow$ as function type constructor; we use the equivalent presentation introduced above. Moreover, we apply the convention from above and, e.g., denote general terms with lower case symbols s and t instead of upper case A and B, as used by Benzmüller et al. In order to distinguish the (primitive, defined) connectives of this variant of HOL from further variants below, the connectives and the names of the particular properties are written in red.

Properties for Hintikka sets for $\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}$ [3, Def. 6.19]:

$\vec{\nabla}_c$: $s \notin \mathcal{H}$ or $\neg s \notin \mathcal{H}$
$\vec{\nabla}_\neg$: If $\neg\neg s \in \mathcal{H}$, then $s \in H$
$\vec{\nabla}_\beta$: If $s \in \mathcal{H}$ and $s \equiv_\beta t$, then $t \in \mathcal{H}$
$\vec{\nabla}_\eta$: If $s \in \mathcal{H}$ and $s \equiv_{\beta\eta} t$, then $t \in \mathcal{H}$
$\vec{\nabla}_\vee$: If $s \vee t \in \mathcal{H}$, then $s \in \mathcal{H}$ or $t \in \mathcal{H}$
$\vec{\nabla}_\wedge$: If $\neg(s \vee t) \in \mathcal{H}$, then $\neg s \in \mathcal{H}$ and $\neg t \in \mathcal{H}$
$\vec{\nabla}_\forall$: If $\Pi^\tau s \in \mathcal{H}$, then $(s\ t) \in \mathcal{H}$ for each closed term $t \in \Lambda^c_\tau(\Sigma)$
$\vec{\nabla}_\exists$: If $\neg\Pi^\tau s \in \mathcal{H}$, then there is a parameter $p_\tau \in \Sigma_\tau$ such that $\neg(s\ p) \in \mathcal{H}$
$\vec{\nabla}_\mathfrak{b}$: If $\neg(s \doteq^o t) \in \mathcal{H}$, then $\{s, \neg t\} \subseteq \mathcal{H}$ or $\{\neg s, t\} \subseteq \mathcal{H}$
$\vec{\nabla}_\xi$: If $\neg(\lambda X_\tau.\, s \doteq^{\nu\tau} \lambda X.\, t) \in \mathcal{H}$, then there is a parameter $w_\tau \in \Sigma_\tau$ s.t. $\neg\big([w/X]s \doteq^\nu [w/X]t\big) \in \mathcal{H}$
$\vec{\nabla}_\mathfrak{f}$: If $\neg(f \doteq^{\nu\tau} g) \in \mathcal{H}$, then there is a parameter $p_\tau \in \Sigma_\tau$ such that $\neg(f\ p \doteq^\nu g\ p) \in \mathcal{H}$

The collection of all sets satisfying all these properties is called $\mathfrak{Hint}_{\beta\mathfrak{f}\mathfrak{b}}$.

Additional properties for acceptable Hintikka sets for $\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}$ [4, Def. 6.1]:

$\vec{\nabla}_m$: If $s, t \in \Lambda^c_o(\Sigma)$ are atomic and $s, \neg t \in \mathcal{H}$, then $\neg(s \doteq^o t) \in \mathcal{H}$

$\vec{\nabla}_d$: If $\neg(h\,\overline{s^n} \doteq^\beta h\,\overline{t^n}) \in \mathcal{H}$ where $\beta \in \{o, \iota\}$ and h is a parameter, then there is an i with $1 \leq i \leq n$ such that $\neg(s^i \doteq t^i) \in \mathcal{H}$

Hintikka sets $\mathcal{H} \in \mathfrak{Hint}_{\beta\mathfrak{f}\mathfrak{b}}$ are called acceptable (in $\mathfrak{Hint}_{\beta\mathfrak{f}\mathfrak{b}}$) if they satisfy both $\vec{\nabla}_m$ and $\vec{\nabla}_d$.

3 Hintikka sets as defined by Steen [6]

In the formulation of HOL as employed by Steen [6], the equality predicate, denoted $=^\tau$, for each type τ, is assumed to be the only logical connective present in the signature Σ, i.e., $\{=^\tau \mid \tau \in \mathcal{T}\} \subseteq \Sigma$. All (potentially) remaining constant symbols from Σ are called parameters. Such signatures are also referred to as $\Sigma^=$. A formulation of HOL based on equality as sole logical connective originates from Andrew's system $\mathcal{Q}_0$, cf. [1] and the references therein. The usual logical connectives are defined as follows (technically our formulation is a modification of the one used by Andrews [1], since the order of terms in defining equations is swapped in many cases):

$$
\begin{array}{lll}
\top_o & := & =^o_{ooo} =^{ooo}_{o(ooo)(ooo)} =^o_{ooo} \\
\bot_o & := & (\lambda P_o.\, P) =^{oo} (\lambda P_o.\, \top) \\
\neg_{oo} & := & \lambda P_o.\, P =^o \bot \\
\wedge_{ooo} & := & \lambda P_o.\, \lambda Q_o.\, (\lambda F_{ooo}.\, F\ \top\ \top) =^{o(ooo)} (\lambda F_{ooo}.\, F\ P\ Q) \\
\vee_{ooo} & := & \lambda P_o.\, \lambda Q_o.\, \neg\, (\neg P \wedge \neg Q) \\
\Rightarrow_{ooo} & := & \lambda P_o.\, \lambda Q_o.\, \neg P \vee Q \\
\Leftrightarrow_{ooo} & := & \lambda P_o.\, \lambda Q_o.\, P =^o Q \\
\Pi^\tau_{o(o\tau)} & := & \lambda P_{o\tau}.\, P =^{o\tau} \lambda X_\tau.\, \top
\end{array}
$$

The connectives of this formulation of HOL are written in blue in the following.

Properties for acceptable Hintikka sets [6, Def. 3.15]

$\vec{\nabla}_c$: $s \notin \mathcal{H}$ or $\neg s \notin \mathcal{H}$.
$\vec{\nabla}_{\beta\eta}$: If $s \equiv_{\beta\eta} t$ and $s \in \mathcal{H}$, then $t \in \mathcal{H}$.
$\vec{\nabla}^r_=$: $(s \neq s) \notin \mathcal{H}$.
$\vec{\nabla}^s_=$: If $u[s]_p \in \mathcal{H}$ and $s = t \in \mathcal{H}$ then $u[t]_p \in \mathcal{H}$.
$\vec{\nabla}^+_{\mathfrak{b}}$: If $s = t \in \mathcal{H}$, then $\{s, t\} \subseteq \mathcal{H}$ or $\{\neg s, \neg t\} \subseteq \mathcal{H}$.
$\vec{\nabla}^-_{\mathfrak{b}}$: If $s \neq t \in \mathcal{H}$, then $\{s, \neg t\} \subseteq \mathcal{H}$ or $\{\neg s, t\} \subseteq \mathcal{H}$.
$\vec{\nabla}^+_{\mathfrak{f}}$: If $f_{\nu\tau} = g_{\nu\tau} \in \mathcal{H}$, then $f\ s = g\ s \in \mathcal{H}$ for any closed term $s \in \Lambda^c_\tau(\Sigma)$.
$\vec{\nabla}^-_{\mathfrak{f}}$: If $f_{\nu\tau} \neq g_{\nu\tau} \in \mathcal{H}$, then $f\ w \neq g\ w \in \mathcal{H}$ for some parameter $w \in \Sigma_\tau$.
$\vec{\nabla}_m$: If s, t are atomic and $s, \neg t \in \mathcal{H}$, then $s \neq t \in \mathcal{H}$.
$\vec{\nabla}_d$: If $h\ \overline{s^n} \neq h\ \overline{t^n} \in \mathcal{H}$, then there is an i with $1 \leq i \leq n$ such that $s^i \neq t^i \in \mathcal{H}$.

The collection of all sets satisfying these properties is called $\mathfrak{H}$. Every element $\mathcal{H} \in \mathfrak{H}$ is called acceptable.

Definition 1. *A set $\mathcal{H}$ of formulas is called* saturated *iff $s \in \mathcal{H}$ or $\neg s \in \mathcal{H}$ for every closed formula s.*

Derived properties

Lemma 1 (Basic properties)**.** *Let $\mathcal{H} \in \mathfrak{H}$. Then it holds that*

(a) $\bot \notin \mathcal{H}$

(b) $\neg\top \notin \mathcal{H}$

(c) $\top = \bot \notin \mathcal{H}$

(d) If $s_o = \top \in \mathcal{H}$ or $\top = s_o \in \mathcal{H}$ ($s_o \neq \bot \in \mathcal{H}$ or $\bot \neq s_o \in \mathcal{H}$), then $\{s_o, \top\} \subseteq \mathcal{H}$ ($\{s_o, \neg\bot\} \subseteq \mathcal{H}$)

(e) If $s_o = \bot \in \mathcal{H}$ or $\bot = s_o \in \mathcal{H}$ ($s_o \neq \top \in \mathcal{H}$ or $\top \neq s_o \in \mathcal{H}$), then $\{\neg s_o, \neg\bot\} \subseteq \mathcal{H}$ ($\{\neg s_o, \top\} \subseteq \mathcal{H}$)

(f) If $\neg\bot \in \mathcal{H}$, then $\top \in \mathcal{H}$

(g) If $\top \in \mathcal{H}$, then $\neg\bot \in \mathcal{H}$

(h) If $s = t \in \mathcal{H}$ and $t = u \in \mathcal{H}$, then $s = u \in \mathcal{H}$.

Proof. Let $\mathcal{H} \in \mathfrak{H}$ be an acceptable Hintikka set.

(a) Assume $\bot \in \mathcal{H}$. By definition of $\bot$ it holds $(\lambda P.\, P) = (\lambda P.\, \top) \in \mathcal{H}$. Hence, by $\vec{\nabla}^+_{\mathfrak{f}}$ and $\vec{\nabla}_{\beta\eta}$, it follows that $w = \top \in \mathcal{H}$ for any closed term w. Taking $w \equiv \neg\top$ we obtain $\neg\top = \top \in \mathcal{H}$. But then, $\vec{\nabla}^+_{\mathfrak{b}}$ gives us a contradiction to $\vec{\nabla}_c$. Hence $\bot \notin \mathcal{H}$.

(b) Assume $\neg\top \in \mathcal{H}$. By definition of $\bot$ it holds $(= \neq =) \in \mathcal{H}$, which contradicts $\vec{\nabla}^r_=$. Hence, $\neg\top \notin \mathcal{H}$.

(c) Assume $\top{=}\bot \in \mathcal{H}$. Applying $\vec{\nabla}_{\mathfrak{b}}^{+}$ gives us that either $\{\top, \bot\} \subseteq \mathcal{H}$ or $\{\neg\top, \neg\bot\} \subseteq \mathcal{H}$. Either case is impossible by either (a) or (b) of this lemma. Hence, $\top{=}\bot \notin \mathcal{H}$.

(d) Let $s_o{=}\top \in \mathcal{H}$ or $\top{=}s_o \in \mathcal{H}$. In both cases it follows by $\vec{\nabla}_{\mathfrak{b}}^{+}$ that either $\{s, \top\} \subseteq \mathcal{H}$ or $\{\neg s, \neg\top\} \subseteq \mathcal{H}$. Since the latter case contradicts (b) from above, it follows that $\{s, \top\} \subseteq \mathcal{H}$. The negative cases are analogous using $\vec{\nabla}_{\mathfrak{b}}^{-}$.

(e) Let $s_o{=}\bot \in \mathcal{H}$ or $\bot{=}s_o \in \mathcal{H}$. In both cases it follows by $\vec{\nabla}_{\mathfrak{b}}^{+}$ that either $\{s, \bot\} \subseteq \mathcal{H}$ or $\{\neg s, \neg\bot\} \subseteq \mathcal{H}$. Since the former case contradicts (a) from above, it follows that $\{\neg s, \neg\bot\} \subseteq \mathcal{H}$. The negative case is analogous using $\vec{\nabla}_{\mathfrak{b}}^{-}$.

(f) Let $\neg\bot \in \mathcal{H}$. Then, by definition of $\bot$, $\big((\lambda P.\, P){\neq}(\lambda P.\, \top)\big) \in \mathcal{H}$. By $\vec{\nabla}_{\mathfrak{f}}^{-}$ and $\vec{\nabla}_{\beta\eta}$ it holds that $p{\neq}\top \in \mathcal{H}$ for some parameter p. By $\vec{\nabla}_{\mathfrak{b}}^{-}$ it follows that either $\{p, \neg\top\} \subseteq \mathcal{H}$ or $\{\neg p, \top\} \subseteq \mathcal{H}$. Since the former case is ruled out by (b) from above, the latter case yields the desired result.

(g) Let $\top \in \mathcal{H}$, that is, ${=}^{o}{=}^{ooo}{=}^{o} \in \mathcal{H}$. By $\vec{\nabla}_{\mathfrak{f}}^{+}$ and $\vec{\nabla}_{\beta\eta}$ it follows that $(s_o{=}t_o){=}(s_o{=}t_o) \in \mathcal{H}$ for every two closed formulas s, t. For $s \equiv t \equiv \neg\bot$ it follows that $(\neg\bot{=}\neg\bot){=}(\neg\bot{=}\neg\bot) \in \mathcal{H}$, and hence, by $\vec{\nabla}_{\mathfrak{b}}^{+}$, either $\neg\bot{=}\neg\bot \in \mathcal{H}$ or $\neg\bot{\neq}\neg\bot \in \mathcal{H}$. Since the latter case is ruled out by $\vec{\nabla}_{=}^{r}$, it follows that $\neg\bot{=}\neg\bot \in \mathcal{H}$. Again, by $\vec{\nabla}_{\mathfrak{b}}^{+}$, it follows that either $\neg\bot \in \mathcal{H}$ or $\neg(\neg\bot) \in \mathcal{H}$. The latter case is impossible by $\vec{\nabla}_{=}^{r}$ since $\neg(\neg\bot) \equiv (\bot{\neq}\bot)$ and hence $\neg\bot \in \mathcal{H}$.

(h) Let $s{=}t \in \mathcal{H}$ and $t{=}u \in \mathcal{H}$. By $\vec{\nabla}_{=}^{s}$ it follows directly that $s{=}u \in \mathcal{H}$.

□

Lemma 2 (Properties of usual connectives). *Let $\mathcal{H} \in \mathfrak{H}$. Then it holds that*

(a) If $\neg\neg s_o \in \mathcal{H}$, then $s \in \mathcal{H}$

(b) If $(s_o{\vee}t_o) \in \mathcal{H}$, then $s \in \mathcal{H}$ or $t \in \mathcal{H}$.

(c) If $(s_o{\wedge}t_o) \in \mathcal{H}$, then $s \in \mathcal{H}$ and $t \in \mathcal{H}$.

(d) If $\Pi^{\alpha}\ F \in \mathcal{H}$, then $F\ s \in \mathcal{H}$ for every closed term s.

(e) If $\neg\Pi^{\alpha}\ F \in \mathcal{H}$, then $\neg(F\ w) \in \mathcal{H}$ for some parameter $w \in \Sigma$.

Proof. Let $\mathcal{H} \in \mathfrak{H}$ be an acceptable Hintikka set.

(a) Let $\neg\neg s_o \in \mathcal{H}$. By definition of $\neg$ and $\vec{\nabla}_{\beta\eta}$ it holds $(s{\neq}\bot) \in \mathcal{H}$. Hence, by $\vec{\nabla}_{\mathfrak{b}}^{-}$, either $\{s, \neg\bot\} \subseteq \mathcal{H}$, or $\{\neg s, \bot\} \subseteq \mathcal{H}$. As the latter case is impossible by Lemma 1(a), it follows that $\{s, \neg\bot\} \subseteq \mathcal{H}$ and, in particular, that $s \in \mathcal{H}$.

(b) Let $(s_o{\vee}t_o) \in \mathcal{H}$. By definition of $\vee$, $\neg$ and $\vec{\nabla}_{\beta\eta}$ it holds $\big((\lambda P.\, P\ \top\ \top){\neq}\lambda P.\, P\ (\neg s)\ (\neg t)\big) \in \mathcal{H}$. Hence, by $\vec{\nabla}_{\mathfrak{f}}^{-}$ and $\vec{\nabla}_{\beta\eta}$, it follows that $(p\ \top\ \top){\neq}\big(p\ (\neg s)\ (\neg t)\big)$ for some parameter $p \in \Sigma$. By $\vec{\nabla}_{d}$ either (i) $\top{\neq}\neg s \in \mathcal{H}$ or (ii) $\top{\neq}\neg t \in \mathcal{H}$. Hence, by $\vec{\nabla}_{\mathfrak{b}}^{-}$, applied to both cases, it holds that either (i) $\neg\neg s \in \mathcal{H}$, or (ii) $\neg\neg t \in \mathcal{H}$ (because $\neg\top \notin \mathcal{H}$ by Lemma 1(b)). It follows that $s \in \mathcal{H}$ or $t \in \mathcal{H}$ by (a) of this lemma.

(c) Let $(s_o{\wedge}t_o) \in \mathcal{H}$. By definition of $\wedge$ and $\vec{\nabla}_{\beta\eta}$ it holds $(\lambda P.\, P\ \top\ \top){=}(\lambda P.\, P\ s\ t) \in \mathcal{H}$. Hence, by $\vec{\nabla}_{\mathfrak{f}}^{+}$ and $\vec{\nabla}_{\beta\eta}$, it follows that $(w\ \top\ \top){=}(w\ s\ t) \in \mathcal{H}$ for every closed term w. By $\vec{\nabla}_{\beta\eta}$, using $w \equiv \lambda x.\, \lambda y.\, x$ and $w \equiv \lambda x.\, \lambda y.\, y$, it holds $\top{=}s \in \mathcal{H}$ and $\top{=}t \in \mathcal{H}$, respectively. Application of Lemma 1(d) yields the desired result.

(d) Let $\Pi^{\alpha}\ s \in \mathcal{H}$. By definition of Π^{α} and $\vec{\nabla}_{\beta\eta}$ it holds $\big(s{=}\lambda x.\, \top\big) \in \mathcal{H}$. Hence, by $\vec{\nabla}_{\mathfrak{f}}^{+}$ and $\vec{\nabla}_{\beta\eta}$, it follows that $s\ t{=}\top \in \mathcal{H}$ for every closed term t. Application of Lemma 1(d) yields the desired result.

(e) Let $\neg\Pi^{\alpha}\ s \in \mathcal{H}$. By definition of Π^{α} and $\vec{\nabla}_{\beta\eta}$ it holds that $\big(s{\neq}(\lambda x.\, \top)\big) \in \mathcal{H}$. Hence, by $\vec{\nabla}_{\mathfrak{f}}^{-}$ and $\vec{\nabla}_{\beta\eta}$, it follows that $(s\ p){\neq}\top \in \mathcal{H}$ for some parameter p. Application of Lemma 1(e) yields the desired result.

□

Lemma 3 (Properties of Leibniz equality). *Let $\mathcal{H} \in \mathfrak{H}$. Then it holds that*

(a) If $s \doteq t \in \mathcal{H}$, then $s{=}t \in \mathcal{H}$.

(b) If $\neg(s \doteq t) \in \mathcal{H}$, then $s{\neq}t \in \mathcal{H}$.

(c) $\neg(s \doteq s) \notin \mathcal{H}$.

(d) If $u[s]_p \in \mathcal{H}$ *and* $s \doteq t \in \mathcal{H}$*, then* $u[t]_p \in \mathcal{H}$.

(e) If $s \doteq t \in \mathcal{H}$*, then* $t \doteq s \in \mathcal{H}$.

(f) If $s \doteq t \in \mathcal{H}$ *and* $t \doteq u \in \mathcal{H}$*, then* $s \doteq u \in \mathcal{H}$.

Proof. Let $\mathcal{H} \in \mathfrak{H}$ be an acceptable Hintikka set.

(a) Let $(s \doteq t) \in \mathcal{H}$. By definition of $\doteq$ and $\vec{\nabla}_{\beta\eta}$ we have $\big(\lambda P.\, (P\ s) \Rightarrow (P\ t)\big) = \big(\lambda P.\, \top\big) \in \mathcal{H}$, and hence, by $\vec{\nabla}_{\mathfrak{f}}^{+}$ and $\vec{\nabla}_{\beta\eta}$, it holds that $\big((w\ s) \Rightarrow (w\ t)\big) = \top \in \mathcal{H}$ for every closed term w. Then, $(w\ s) \Rightarrow (w\ t) \in \mathcal{H}$ by Lemma 1(d). By definition of $\Rightarrow$ and $\vec{\nabla}_{\beta\eta}$ it holds that $\neg(w\ s) \vee (w\ t) \in \mathcal{H}$ and hence, by Lemma 2(b), that $\neg(w\ s) \in \mathcal{H}$ or $(w\ t) \in \mathcal{H}$. For $w \equiv (\lambda X.\, s{=}X)$ it follows by $\vec{\nabla}_{\beta\eta}$ that $\neg(s{=}s) \in \mathcal{H}$ or $(s{=}t) \in \mathcal{H}$. Since the former case contradicts $\vec{\nabla}_{=}^{r}$, it follows that $(s{=}t) \in \mathcal{H}$.

(b) Let $\neg(s \doteq t) \in \mathcal{H}$. By definition of $\doteq$ and $\vec{\nabla}_{\beta\eta}$ we have $\big(\lambda P.\, (P\ s) \Rightarrow (P\ t)\big) \neq \big(\lambda P.\, \top\big) \in \mathcal{H}$. Hence, by $\vec{\nabla}_{\mathfrak{f}}^{-}$ and $\vec{\nabla}_{\beta\eta}$, it holds that $\big((p\ s) \Rightarrow (p\ t)\big) \neq \top \in \mathcal{H}$ for some parameter p. By Lemma 1(e) it follows that $\neg\big((p\ s) \Rightarrow (p\ t)\big) \in \mathcal{H}$. Then, by Lemma 2(a) and 2(c), it follows that $\neg\neg(p\ s) \in \mathcal{H}$ and $\neg(p\ t) \in \mathcal{H}$. Moreover, $(p\ s) \in \mathcal{H}$ by Lemma 2(a). By $\vec{\nabla}_{m}$ it then follows that $(p\ s) \neq (p\ t) \in \mathcal{H}$, and finally, by $\vec{\nabla}_{d}$, that $s \neq t \in \mathcal{H}$.

(c) Assume $\neg(s \doteq s) \in \mathcal{H}$. By (b) above it follows that $s \neq s \in \mathcal{H}$ which contradicts $\vec{\nabla}_{=}^{r}$. Hence, $\neg(s \doteq s) \notin \mathcal{H}$.

(d) Let $u[s]_p \in \mathcal{H}$ and $s \doteq t \in \mathcal{H}$. By (a) above it holds that $s{=}t \in \mathcal{H}$ and thus by $\vec{\nabla}_{=}^{s}$ it follows that $u[t]_p \in \mathcal{H}$.

(e) Let $(s \doteq t) \in \mathcal{H}$. By definition of $\doteq$ and $\vec{\nabla}_{\beta\eta}$ we have $\big(\lambda P.\, (P\ s) \Rightarrow (P\ t)\big) = \big(\lambda P.\, \top\big) \in \mathcal{H}$. Hence, by $\vec{\nabla}_{\mathfrak{f}}^{+}$ and $\vec{\nabla}_{\beta\eta}$, it holds that $\big((w\ s) \Rightarrow (w\ t)\big) = \top \in \mathcal{H}$ for every closed term w. Then, $(w\ s) \Rightarrow (w\ t) \in \mathcal{H}$ by Lemma 1(d). By definition of $\Rightarrow$ and $\vec{\nabla}_{\beta\eta}$ it holds that $\neg(w\ s) \vee (w\ t) \in \mathcal{H}$ and hence by Lemma 2(b) that $\neg(w\ s) \in \mathcal{H}$ or $(w\ t) \in \mathcal{H}$. For $w \equiv (\lambda X.\, t \doteq s)$, it follows by $\vec{\nabla}_{\beta\eta}$ that $\neg(t \doteq s) \in \mathcal{H}$ or $(t \doteq s) \in \mathcal{H}$. Assume $\neg(t \doteq s) \in \mathcal{H}$. Since by Lemma 3(a) it holds that $s{=}t \in \mathcal{H}$, it follows by $\vec{\nabla}_{=}^{s}$ that $\neg(t \doteq t) \in \mathcal{H}$, which contradicts (c) above. Hence, $(t \doteq s) \in \mathcal{H}$.

(f) Let $(s \doteq t) \in \mathcal{H}$ and $(t \doteq u) \in \mathcal{H}$. By (d) above it follows that $(s \doteq u) \in \mathcal{H}$.

□

Lemma 4 (Sufficient conditions for saturatedness)**.** *Let* $\mathcal{H} \in \mathfrak{H}$*. Then it holds that*

(a) If $\top \in \mathcal{H}$*, then* $\mathcal{H}$ *is saturated.*

(b) If $\neg s \in \mathcal{H}$ *for some closed term* s*, then* $\mathcal{H}$ *is saturated.*

(c) If $s \vee t \in \mathcal{H}$ *for some closed terms* s, t*, then* $\mathcal{H}$ *is saturated.*

(d) If $s \wedge t \in \mathcal{H}$ *for some closed terms* s, t*, then* $\mathcal{H}$ *is saturated.*

(e) If $\Pi^{\tau}\ P \in \mathcal{H}$ *for some closed term* $P_{\tau \to o}$*, then* $\mathcal{H}$ *is saturated.*

(f) If $s \doteq t \in \mathcal{H}$ *for some closed terms* s, t*, then* $\mathcal{H}$ *is saturated.*

Proof. Let $\mathcal{H} \in \mathfrak{H}$ be an acceptable Hintikka set.

(a) Let $\top \in \mathcal{H}$, that is, $=^{o} =^{ooo} =^{o} \in \mathcal{H}$. By $\vec{\nabla}_{\mathfrak{f}}^{+}$ and $\vec{\nabla}_{\beta\eta}$ it follows that $(s_o{=}t_o) = (s_o{=}t_o) \in \mathcal{H}$ for every two closed formulas s, t. For $s \equiv t \equiv c$ for some closed term c, it follows that $(c{=}c) = (c{=}c) \in \mathcal{H}$ and thus, by $\vec{\nabla}_{\mathfrak{b}}^{+}$ and $\vec{\nabla}_{=}^{r}$, it holds that $c{=}c \in H$. By $\vec{\nabla}_{\mathfrak{b}}^{+}$ it follows that $c \in \mathcal{H}$ or $\neg c \in \mathcal{H}$. Hence, $\mathcal{H}$ is saturated.

(b) If $\neg s \in \mathcal{H}$ for some closed term s, then $s{=}\bot \in \mathcal{H}$. By $\vec{\nabla}_{\mathfrak{b}}^{+}$, it follows that either $\{s, \bot\} \subseteq \mathcal{H}$ or $\{\neg s, \neg\bot\} \subseteq \mathcal{H}$. Since the former case is ruled out by Lemma 1(a), it follows that $\neg\bot \in \mathcal{H}$. By Lemma 1(f) it follows that $\top \in \mathcal{H}$ and by (a) above it follows that $\mathcal{H}$ is saturated.

(c) If $s \vee t \in \mathcal{H}$ for some closed terms s, t, then by definition of $\vee$ and $\vec{\nabla}_{\beta\eta}$ we have $\neg(\neg s \wedge \neg t) \in \mathcal{H}$. An application of (b) yields the desired result.

(d) If $s \wedge t \in \mathcal{H}$ for some closed terms s, t, then by definition of $\wedge$ and $\vec{\nabla}_{\beta\eta}$ it holds $(\lambda g.\, g\ s\ t) = (\lambda g.\, g \top \top) \in \mathcal{H}$. By $\vec{\nabla}^+_{\mathfrak{f}}$ and $\vec{\nabla}_{\beta\eta}$ it follows that $s = \top \in \mathcal{H}$ (take $\lambda x.\, \lambda y.\, x$). By $\vec{\nabla}^+_{\mathfrak{b}}$, it follows that either $\{s, \top\} \subseteq \mathcal{H}$ or $\{\neg s, \neg\top\} \subseteq \mathcal{H}$. Since the latter case is ruled out by Lemma 1(b), it follows that $\top \in \mathcal{H}$. An application of (a) above yields the desired result.

(e) If $\Pi^\tau\ s \in \mathcal{H}$ for some closed terms s, then by definition of Π^τ and $\vec{\nabla}_{\beta\eta}$ it holds that $s = (\lambda x.\, \top) \in \mathcal{H}$. By $\vec{\nabla}^+_{\mathfrak{f}}$ and $\vec{\nabla}_{\beta\eta}$ it follows that $(s\ w) = \top \in \mathcal{H}$ for every closed term w. By $\vec{\nabla}^+_{\mathfrak{b}}$, it follows that either $\{(s\ w), \top\} \subseteq \mathcal{H}$ or $\{\neg(s\ w), \neg\top\} \subseteq \mathcal{H}$. Since the latter case is ruled out by Lemma 1(b), it follows that $\top \in \mathcal{H}$. An application of (a) above yields the desired result.

(f) Let $s \doteq t \in \mathcal{H}$. By definition of Π^τ and $\vec{\nabla}_{\beta\eta}$ it holds that $\Pi\big(\lambda P.\, (P\ s) \Rightarrow (P\ t)\big) \in \mathcal{H}$. An application of (e) yields the desired result.

□

Corollary 1. *Let $\mathcal{H} \in \mathfrak{H}$ and let $s \neq t \in \mathcal{H}$ or $\neg(s \doteq t) \in \mathcal{H}$ for some closed terms s, t. Then, $\mathcal{H}$ is saturated.*

Proof. As $(s \neq t) \equiv \neg(s = t)$, both cases are a special instance of Lemma 4(b). □

Lemma 5 (Saturated sets properties)**.** *Let $\mathcal{H} \in \mathfrak{H}$ and let $\mathcal{H}$ be saturated. Then it holds that*

(a) If $s = t \in \mathcal{H}$, then $s \doteq t \in \mathcal{H}$.

(b) If $s \neq t \in \mathcal{H}$, then $\neg(s \doteq t) \in \mathcal{H}$.

(c) If $s = t \in \mathcal{H}$ then $t = s \in \mathcal{H}$.

(d) $s = s \in \mathcal{H}$ for every closed term s.

(e) $s \doteq s \in \mathcal{H}$ for every closed term s.

Proof. Let $\mathcal{H} \in \mathfrak{H}$ and let $\mathcal{H}$ be saturated.

(a) Let $s = t \in \mathcal{H}$ and assume $s \doteq t \notin \mathcal{H}$. Since $\mathcal{H}$ is saturated we have $\neg(s \doteq t) \in \mathcal{H}$. Then, by Lemma 3(b), it follows that $s \neq t \in \mathcal{H}$, and thus $\{s = t, s \neq t\} \subseteq \mathcal{H}$, which contradicts $\vec{\nabla}_c$. Hence, $s \doteq t \in \mathcal{H}$.

(b) Let $s \neq t \in \mathcal{H}$ and assume $\neg(s \doteq t) \notin \mathcal{H}$. Since $\mathcal{H}$ is saturated we have that $(s \doteq t) \in \mathcal{H}$. Then, by Lemma 3(a), it follows that $s = t \in \mathcal{H}$, and thus $\{s = t, s \neq t\} \subseteq \mathcal{H}$, which contradicts $\vec{\nabla}_c$. Hence, $\neg(s \doteq t) \in \mathcal{H}$.

(c) Let $s = t \in \mathcal{H}$ and assume $t = s \notin \mathcal{H}$. Since $\mathcal{H}$ is saturated we have $t \neq s \in \mathcal{H}$. Then, by $\vec{\nabla}^s_=$, it follows that $t \neq t \in \mathcal{H}$ which contradicts $\vec{\nabla}^r_=$. Hence, $t = s \in \mathcal{H}$.

(d) Let s be a closed term of some type and assume that $s = s \notin \mathcal{H}$. Since $\mathcal{H}$ is saturated we have that $s \neq s \in \mathcal{H}$. Since this contradicts $\vec{\nabla}^r_=$ it follows that $s = s \in \mathcal{H}$.

(e) Let s be a closed term of some type and assume that $s \doteq s \notin \mathcal{H}$. Since $\mathcal{H}$ is saturated we have that $\neg(s \neq s) \in \mathcal{H}$. Since this is impossible by Lemma 3(c) it follows that $s \doteq s \in \mathcal{H}$.

□

Definition 2 (Leibniz-free)**.** *Let S be a set of formulas. S is called* Leibniz-free *iff $s \doteq t \notin S$ for any terms s, t.*

Corollary 2 (Impredicativity Gap)**.** *Let $\mathcal{H} \in \mathfrak{H}$. $\mathcal{H}$ is saturated or Leibniz-free.*

Proof. Assume that $\mathcal{H}$ is not Leibniz-free. Then, there exists some formula $s \doteq t \in \mathcal{H}$. An application of Lemma 4(f) yields the desired result. □

Summary of properties of equality and Leibniz-equality. The following table contains an overview of the implied properties of $=$ and $\doteq$, respectively. A property that holds unconditionally is marked with ✓, a property that holds for saturated Hintikka sets is marked with sat..

Property	$\star \equiv =$	$\star \equiv \doteq$
$s \star s \in \mathcal{H}$	sat.	sat.
$\neg(s \star s) \notin \mathcal{H}$	✓	✓
If $s \star t \in \mathcal{H}$ and $t \star u \in \mathcal{H}$, then $s \star u \in \mathcal{H}$	✓	✓
If $s \star t \in \mathcal{H}$, then $t \star s \in \mathcal{H}$	sat.	✓
If $u[s]_p \in \mathcal{H}$ and $s \star t \in \mathcal{H}$, then $u[t]_p \in \mathcal{H}$	✓	✓

4 Reduction of $\mathfrak{H}$ (Steen) to $\mathfrak{Hint}_{\beta\mathfrak{fb}}$ (Benzmüller et al.)

We reduce the notion of Hintikka sets of Steen to the notion of Hintikka sets by Benzmüller et al. To that end, an translation scheme is employed as follows: Let $\mathcal{H}^{\rightarrow}$ be the set that is constructed from a set $\mathcal{H}$ of closed formulas by replacing all occurrences of blue connectives by their corresponding red connective. Note that this might replace defined blue connectives by primitive red connectives, e.g. $\vee$ by $\vee$, and defined blue connectives by defined red connectives, e.g. $\doteq$ by $\doteq$. In case of equality, the primitive blue equality connective $=$ is replaced by a red symbol $=$ which might be a primitive logical connective (if the target language is with equality) or simply a parameter (if the target language is without equality). The (recursively) translated formula associated with s is written s (in red).

Since the notion of Hintikka sets by Benzmüller et al. in [4] does not assume equality $=$ to be a logical connective available in the signature $\Sigma^{\doteq}$, we first transform the set $\mathcal{H}$ of closed formulas using "Leibnizification" into a set $\mathcal{H}[\doteq]$ that does not contain primitive equality (except in defined logical connectives): Let $\mathcal{H}$ be a set of closed formulas and let $\mathcal{H}[\doteq]$ denote the *Leibnizification of* $\mathcal{H}$, given by

$$\mathcal{H}[\doteq] := \{s[\doteq] \mid s \in \mathcal{H}\}$$

where $s[\doteq]$ is the formula that is created by replacing all occurrences of $=$ in s that are not part of the definition of a defined logical connective by $\doteq$.

For $\mathcal{H} \in \mathfrak{H}$ we now construct a translated Hintikka set based on $\mathcal{H}[\doteq]$. For that purpose, let $\mathcal{H}[\doteq]^{\rightarrow}$ be a set that is constructed by translating all blue connectives to red connectives in $\mathcal{H}[\doteq]$ as described further above.

As an example, let $\mathcal{H} := \{\ldots, ((p\ \wedge) = (p =^o)) = ((p\ \vee) \doteq (p =^o)), \ldots\}$, where $p \in \Sigma^{=}$ is some parameter of appropriate type. $\mathcal{H}[\doteq]$ is then constructed by systematically replacing all occurrences of primitive equality by Leibniz equality if not occurring as part of the definition of a defined logical connective, i.e., we have that $\mathcal{H}[\doteq] \equiv \{\ldots, ((p\ \wedge) \doteq (p \doteq^o)) \doteq ((p\ \vee) \doteq (p \doteq^o)), \ldots\}$. Subsequently, $\mathcal{H}[\doteq]$ is translated to the languages based on $\Sigma^{\doteq}$, i.e., using the red connectives: $\mathcal{H}[\doteq]^{\rightarrow} \equiv \{\ldots, ((p\ \wedge) \doteq (p \doteq^o)) \doteq ((p\ \vee) \doteq (p \doteq^o)), \ldots\}$.

In the following, we write $s[\doteq]$ for the term s' such that $s' \equiv s[\doteq]$, i.e., the term that is created from $s \in \Lambda(\Sigma^{=})$ by first replacing the blue primitive equalities as indicated above and then translating the connectives to their red counterpart.

Lemma 6. *Let $\mathcal{H} \in \mathfrak{H}$ be a Hintikka set and let $\mathcal{H}[\doteq]^{\rightarrow}$ be its translation as indicated above. Then, it holds that*

(a) If $(s[\doteq] \doteq t[\doteq]) \in \mathcal{H}[\doteq]^{\rightarrow}$, then $(s{=}t) \in \mathcal{H}$.

(b) If $\neg(s[\doteq] \doteq t[\doteq]) \in \mathcal{H}[\doteq]^{\rightarrow}$, then $(s{\neq}t) \in \mathcal{H}$.

Proof. Let $\mathcal{H} \in \mathfrak{H}$ be a Hintikka set and let $\mathcal{H}[\doteq]^{\rightarrow}$ its translation as indicated above.

(a) Let $(s[\doteq] \doteq t[\doteq]) \in \mathcal{H}[\doteq]^{\rightarrow}$, then by definition $(s[\doteq] \doteq t[\doteq]) \in \mathcal{H}[\doteq]$ and thus it follows that (i) $(s{=}t) \in \mathcal{H}$, or (ii) $(s \doteq t) \in \mathcal{H}$. In the former case, we are done. In the latter case, it follows by Lemma 3(a) that $(s{=}t) \in \mathcal{H}$.

(b) Let $\neg(s[\doteq] \doteq t[\doteq]) \in \mathcal{H}[\doteq]^{\rightarrow}$, then by definition $\neg(s[\doteq] \doteq t[\doteq]) \in \mathcal{H}[\doteq]$ and thus it follows that (i) $(s{\neq}t) \in \mathcal{H}$, or (ii) $\neg(s \doteq t) \in \mathcal{H}$. In the former case, we are done. In the latter case, it follows by Lemma 3(b) that $(s{\neq}t) \in \mathcal{H}$.

□

We now show that if $\mathcal{H} \in \mathfrak{H}$ then $\mathcal{H}[\doteq]^{\rightarrow} \in \mathfrak{Hint}_{\beta\mathfrak{fb}}$ and $\mathcal{H}[\doteq]^{\rightarrow}$ is acceptable in $\mathfrak{Hint}_{\beta\mathfrak{fb}}$ (i.e.,$\mathcal{H}[\doteq]^{\rightarrow}$ fulfils all $\vec{\nabla}$ from §2).

Theorem 1 (Reduction of $\mathfrak{H}$ to $\mathfrak{Hint}_{\beta\mathfrak{fb}}$). *If $\mathcal{H} \in \mathfrak{H}$, then there exists an acceptable $\mathcal{H}' \in \mathfrak{Hint}_{\beta\mathfrak{fb}}$ such that $\mathcal{H}[\doteq]^{\rightarrow} \equiv \mathcal{H}'$.*

Proof. Let $\mathcal{H} \in \mathfrak{H}$ be a Hintikka set according §3, i.e., fulfilling all $\vec{\nabla}$ properties.
We verify every $\vec{\nabla}$-property for $\mathcal{H}[\doteq]^{\rightarrow}$:

$\vec{\nabla}_c$: This follows immediately by definition of $\mathcal{H}[\doteq]^{\rightarrow}$ and $\vec{\nabla}_c$.

$\vec{\nabla}_\neg$: Let $\neg\neg s[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$, then by definition $\neg\neg s \in \mathcal{H}$. By Lemma 2(a) it follows that $s \in \mathcal{H}$ and hence $s[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$.

$\vec{\nabla}_\beta$: Let $s[\doteq] \in \mathcal{H}^{\rightarrow}$ and $s[\doteq] \equiv_\beta t[\doteq]$. Then, $s \in \mathcal{H}$ and $s \equiv_\beta t$, and hence $t \in \mathcal{H}$ by $\vec{\nabla}_{\beta\eta}$. It follows that $t[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$.

$\vec{\nabla}_\eta$: *Analogous to the previous case.*

$\vec{\nabla}_\vee$: Let $s \vee t[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$. Then, it holds that $s \vee t \in \mathcal{H}$. By Lemma 2(b) we have that $s \in \mathcal{H}$ or $t \in \mathcal{H}$ and hence $s[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$ or $t[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$.

$\vec{\nabla}_\wedge$: Let $\neg(s \vee t)[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$. Then, $\neg(s \vee t) \in \mathcal{H}$ and by definition $\neg\neg((\neg s) \wedge (\neg t)) \in \mathcal{H}$. By Lemma 2(a), it follows that $(\neg s) \wedge (\neg t) \in \mathcal{H}$. By Lemma 2(c), it follows that $\neg s \in \mathcal{H}$ and $\neg t \in \mathcal{H}$. Hence, $\neg s[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$ and $\neg t[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$.

$\vec{\nabla}_\forall$: Let $\Pi^\alpha s[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$. Then, $\Pi^\alpha\ s \in \mathcal{H}$. By Lemma 2(d) it follows that $s\ t \in \mathcal{H}$ for every closed term t. Hence, $(s\ t)[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$ for every closed term t.

$\vec{\nabla}_\exists$: Let $\neg\Pi^\alpha s[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$. Then, $\neg\Pi^\alpha\ s \in \mathcal{H}$. By Lemma 2(e) it follows that $\neg(s\ w) \in \mathcal{H}$ for some parameter w. Hence, $\neg(s\ w)[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$ for some parameter w.

$\vec{\nabla}_\mathfrak{b}$: Let $\neg(s \doteq^o t)[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$. Then by Lemma 6 it follows that $s \neq t \in \mathcal{H}$, from which we get by $\vec{\nabla}_\mathfrak{b}^-$ that $\{s, \neg t\} \subseteq \mathcal{H}$ or $\{\neg s, t\} \subseteq \mathcal{H}$. Hence, $\{s[\doteq], \neg t[\doteq]\} \subseteq \mathcal{H}[\doteq]^{\rightarrow}$ or $\{\neg s[\doteq], t[\doteq]\} \subseteq \mathcal{H}[\doteq]^{\rightarrow}$.

$\vec{\nabla}_\xi$: Let $\neg(\lambda X_\alpha.\, M \doteq^{\alpha\rightarrow\beta} \lambda X.\, N)[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$. Then by Lemma 6 it follows that $(\lambda X_\alpha.\, M) \neq^{\alpha\rightarrow\beta} (\lambda X.\, N) \in \mathcal{H}$. By $\vec{\nabla}_\mathfrak{f}^-$ and $\vec{\nabla}_{\beta\eta}$ it then follows that $[w/X]M \neq [w/X]N \in \mathcal{H}$ for some parameter w. By Corollary 1, it follows that $\mathcal{H}$ is saturated, and by Lemma 5 it hence follows that $\neg([w/X]M \doteq [w/X]N) \in \mathcal{H}$. This implies $\neg([w/X]M \doteq [w/X]N)[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$

$\vec{\nabla}_\mathfrak{f}$: *Analogous to the previous case.*

$\vec{\nabla}_m$: Let $s[\doteq], \neg t[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$, where $s[\doteq], t[\doteq]$ atomic. Then, $s, \neg t \in \mathcal{H}$. By $\vec{\nabla}_m$ it follows that $s \neq t \in \mathcal{H}$. Moreover, since $\neg t \in \mathcal{H}$, it follows from by Lemma 4(b) that $\mathcal{H}$ is saturated. Hence, by Lemma 5(b), it holds that $\neg(s \doteq t) \in \mathcal{H}$ and consequently $\neg(s \doteq t)[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$.

$\vec{\nabla}_d$: Let $\neg(h\,\overline{s^n} \doteq^\beta h\,\overline{t^n})[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$, where $\beta \in \{o, \iota\}$ and h is a parameter. Then by Lemma 6 it follows that $(h\,\overline{s^n} \neq^\beta h\,\overline{t^n}) \in \mathcal{H}$. By $\vec{\nabla}_d$ there exists some $1 \leq i \leq n$, s.t. $s^i \neq t^i \in \mathcal{H}$. It follows by Corollary 1 that $\mathcal{H}$ is saturated, and hence by Lemma 5(b), it holds that $\neg(s^i \doteq t^i) \in \mathcal{H}$. Consequently, $\neg(s^i \doteq t^i)[\doteq] \in \mathcal{H}[\doteq]^{\rightarrow}$.

□

We claim that this result analogously holds if the original definitions of Andrews for the logical connectives are assumed instead of the slightly modified ones introduced in 3 and used in [6]; a technical proof remains future work.

5 Use Case: Bridging Model Existence

In this section, we apply the above reduction to get a model existence theorem for $\Sigma^=$-Hintikka sets. First, we recapitulate relevant results by Benzmüller et al. [3, 4].

The properties $\mathfrak{b}$, $\mathfrak{f}$, and $\mathfrak{q}$ used in the subscript of model classes $\mathfrak{M}$ and elsewhere refer to the following properties of a HOL model M [3, Def. 3.46]:

- $\mathfrak{b}$: There are only two truth values (M satisfies Boolean extensionality).
- $\mathfrak{f}$: M is functional (M satisfies functional extensionality).
- $\mathfrak{q}$: For all $\tau \in \mathcal{T}$, there exists an element $\mathfrak{q}^\tau$ in the respective domain such that $\mathfrak{q}^\tau(x, y) \equiv \top$ iff $x \equiv y$ for all x, y in the domain of τ.

Property $\mathfrak{q}$ is always assumed, cf. [3, Remark 3.52].

Theorem 2 (Model existence for $\mathfrak{Hint}_{\beta\mathfrak{fb}}$ [4, Theorem 8.12])**.** *Let $\mathcal{H} \in \mathfrak{Hint}_{\beta\mathfrak{fb}}$ be an $\Sigma^{\doteq}$-Hintikka set that is acceptable in $\mathfrak{Hint}_{\beta\mathfrak{fb}}$. Then there exists a $\Sigma^{\doteq}$-model $\mathcal{M} \in \mathfrak{M}_{\beta\mathfrak{fb}}$ such that $\mathcal{M} \models \mathcal{H}$.*

Theorem 3 (Henkin models for $\mathfrak{M}_{\beta\mathfrak{fb}}$)**.** *Let $\mathcal{M} \in \mathfrak{M}_{\beta\mathfrak{fb}}$ be a Σ-model. Then, there exists an Σ-Henkin model $\mathcal{M}^H$ that is isomorphic to $\mathcal{M}$.*

Proof. Let $\mathcal{M} \in \mathfrak{M}_{\beta\mathfrak{fb}}$ be a Σ-model. Then, there exists an isomorphic Σ-model $\mathcal{M}^{fr}$ over a frame [3, Theorem 3.68]. $\mathcal{M}^{fr}$ satisfies properties $\mathfrak{q}$, $\mathfrak{f}$ and $\mathfrak{b}$ (since $\mathcal{M}$ does) [3, Lemma 3.67] and hence is a Σ-Henkin model. □

The two above Theorems can be combined in a straight-forward manner, yielding

Corollary 3. *Let $\mathcal{H} \in \mathfrak{Hint}_{\beta\mathfrak{fb}}$ be an $\Sigma^{\doteq}$-Hintikka set that is acceptable in $\mathfrak{Hint}_{\beta\mathfrak{fb}}$. Then, there exists a $\Sigma^{\doteq}$-Henkin model $\mathcal{M}$ such that $\mathcal{M} \models \mathcal{H}$.*

Model existence for Hintikka sets $\mathfrak{H}$. All we have to show is a rather technical lemma, stating that if there exists a $\Sigma^{\doteq}$-Henkin model satisfying $\mathcal{H}[\doteq]^{\rightarrow}$ then, there exists a $\Sigma^{=}$-Henkin model satisfying $\mathcal{H}$. Note that this can be shown quite easily in contrast to model existence theorems in general.

Lemma 7. *Let $\mathcal{H} \subseteq \Lambda^c(\Sigma^=)$ be a set of closed formulas over $\Sigma^=$. Let $\mathcal{M}$ be a $\Sigma^{\doteq}$-Henkin model, such that $\mathcal{M} \models \mathcal{H}[\doteq]^{\rightarrow}$. Then there exists a $\Sigma^=$-Henkin model $\mathcal{M}^{\leftarrow}$ such that $\mathcal{M}^{\leftarrow} \models \mathcal{H}$.*

Proof. Let $\mathcal{M} \equiv (\mathcal{D}, \mathcal{I})$ be a $\Sigma^{\doteq}$-Henkin model, such that $\mathcal{M} \models \mathcal{H}[\doteq]^{\rightarrow}$.
Let $\Sigma^= \equiv \{p \mid p \in \Sigma^{\doteq} \text{ is a parameter}\} \cup \{=^\tau \mid \tau \in \mathcal{T}\}$. We construct $\mathcal{M}^{\leftarrow} = (\mathcal{D}^{\leftarrow}, \mathcal{I}^{\leftarrow})$ over $\Sigma^=$ as follows:

$$\mathcal{D}^{\leftarrow} := \mathcal{D}$$

$$\mathcal{I}^{\leftarrow} := c_\tau \mapsto \begin{cases} \mathcal{I}(c) \in \mathcal{D}_\tau & \text{if } c \text{ is a parameter from } \Sigma^{\doteq} \\ \mathsf{q}^\tau \in \mathcal{D}_\tau & \text{if } \tau \equiv o\nu\nu \text{ for some type } \nu \in \mathcal{T} \text{ and } c \equiv =^\nu \end{cases}$$

It is immediate that $\mathcal{M}^{\leftarrow}$ is a $\Sigma^=$-model. In particular, q^τ with $\mathsf{q}^\tau(a,b) \equiv \top$ if and only if $a \equiv b$ for every $a, b \in \mathcal{D}_\tau$ is guaranteed to exist by the assumed property q.

Still, we need to verify that $\mathcal{M}^{\leftarrow} \models \mathcal{H}$ indeed holds. Let $s \in \mathcal{H}$. By definition $\mathcal{M}^{\leftarrow} \models s$ if and only if $\|s\|^{\mathcal{M}^{\leftarrow},g} \equiv \top$ for every g. By assumption we know that $\mathcal{M} \models s'$ and hence $\|s'\|^{\mathcal{M},g} \equiv \top$, for $s' \equiv s[\doteq]$. A simple induction over the structure of $s \in \Lambda(\Sigma^=)$ gives us that $\|s\|^{\mathcal{M}^{\leftarrow},g} \equiv \|s'\|^{\mathcal{M},g}$ for every variable assignment g, where $s' \equiv s[\doteq]$. Hence it follows that $\mathcal{M}^{\leftarrow} \models s$, and thus $\mathcal{M}^{\leftarrow} \models \mathcal{H}$. □

Finally, we can apply the above lemma to achieve a model existence theorem for $\Sigma^=$-Hintikka sets.

Theorem 4 (Bridged Model Existence)**.** *Let $\mathcal{H} \in \mathfrak{H}$ be a $\Sigma^=$-Hintikka set. Then there exists a $\Sigma^=$-Henkin model $\mathcal{M}$ such that $\mathcal{M} \models \mathcal{H}$.*

Proof. Let $\mathcal{H} \in \mathfrak{H}$ be a $\Sigma^=$-Hintikka set. By Theorem 1 there exists a $\Sigma^{\doteq}$-Hintikka set $\mathcal{H}' \in \mathfrak{Hint}_{\beta\mathfrak{f}\mathfrak{b}}$ such that $\mathcal{H}[\doteq]^{\rightarrow} \equiv \mathcal{H}'$. By Corollary 3 there exists a $\Sigma^{\doteq}$-Henkin model $\mathcal{M}$ such that $\mathcal{M} \models \mathcal{H}[\doteq]^{\rightarrow}$. By Lemma 7 it follows that there exists a $\Sigma^=$-Henkin model $\mathcal{M}^{\leftarrow}$ such that $\mathcal{M}^{\leftarrow} \models \mathcal{H}$. □

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