Factor Income Distribution
and Endogenous Economic Growth
- When Piketty meets Romer - REVISED VERSION

Abstract: What is the relationship between the economy's long-run growth rate, its capital-income ratio, and its factor income distribution? We argue that a satisfactory answer must be derived in an analytical framework that treats the growth and the savings rate as endogenous. From this perspective we scrutinize Piketty's (2014) theory put forth in his book *Capital in the Twenty-First Century* in a richly parameterized variant of Romer's (1990) seminal model with and without population growth. The economy's growth and its savings rate are exogenous in Piketty's theory and endogenous in Romer's. We find that a smaller long-run growth rate may be associated with a smaller capital-income ratio. Hence, the key implication of Piketty's *Second Fundamental Law of Capitalism* does not hold. Moreover, in contrast to Piketty's theory, a smaller long-run growth rate may go together with a greater or a smaller capital share.

Keywords: Endogenous Technological Change, Capital Accumulation, Aggregate Factor Income Distribution

JEL-Classification: E10, E21, E25, O33, O41

This Version: June 17, 2019

1 Introduction

What is the relationship between economic inequality and economic growth? Since the 19th century - if not earlier - up to the times of *Capital in the Twenty-First Century* by Thomas Piketty (2014) this question has been at the heart of many policy debates. Does economic growth cause more inequality or vice versa? Who benefits, who suffers from economic growth, capitalists or workers?

We argue that a plausible answer to these questions has to acknowledge that both, economic inequality and economic growth, are simultaneously determined. Hence, a satisfactory understanding of the determinants of economic inequality and economic growth requires an analytical framework that treats both phenomena as endogenous. This is the central methodological perspective of the present paper.

From this standpoint we scrutinize Piketty (2014)'s theory concerning the relationship between an economy's long-run growth rate, its capital-income ratio, and its factor income distribution. We accomplish this in a richly parameterized variant of Paul Romer's (1990) seminal model of endogenous technological change (with demographic growth as suggested by Jones (1995) and without it). Here, long-run growth is driven by endogenous R&D efforts of profit-maximizing research firms, and the economy's factor income distribution is endogenous, too. Hence, changes in the economic environment are bound to affect the distribution of factor incomes and economic growth simultaneously. We establish the qualitative impact of these changes on the long-run growth rate, the capital-income ratio, and the factor income distribution. More importantly, we contrast these findings with the predictions of Piketty's theory. This is why and where Piketty meets Romer.

We show that several key implications of Piketty's two fundamental laws of capitalism are violated. This discrepancy arises since Piketty treats the economy's growth rate and its savings rate as exogenous parameters whereas in our analysis both are endogenous variables. As a consequence of this change in perspective, there is no longer a direct causal effect of the steady-state growth rate of an economy on the steady-state capital-income ratio. Rather, "deep" parameters that describe the economy's technology, preferences, policy, demographics, or market structure cause both variables. This leads to the conclusion that changes in these parameters induce adjustments that are often inconsistent with Piketty's fundamental laws.

Beyond establishing contradictions to Piketty's laws, we carve out relevant policy implications that an exogenous treatment of the economic growth rate cannot elicit. From a normative point of view monopoly pricing and an inter-temporal externality associated with the creation of new technological knowledge imply too little growth in the laissez-faire equilibrium of Romer's model. Hence, the question arises how to implement the first-best allocation chosen by a benevolent planner. We show that a policy mix involving subsidies to research and a tax on capital earnings accomplishes this. However, the chosen mix has distributional side effects. In particular, the implemented planner's allocation may involve a higher labor share than under laissez-faire. Hence, if we subscribe to the common view that a larger labor share comes along with a more equal distribution of income among persons (see, e. g., Solow (2014)), then a policy mix that implements the planner's allocation enhances efficiency and may contribute to more equality.

This paper relates and contributes to the broad and growing literature on the link between the factor income distribution of an economy and its growth rate. First, it contributes to the discussion surrounding the validity of Piketty's two fundamental laws of capitalism. Acemoglu and Robinson (2014), Blume and Durlauf (2015), or Ray (2015) hint at the "endogeneity problem" of Piketty's theory. However, these authors do not provide a formal analysis which is a central issue of the present paper. Homburg (2015) and Krusell and Smith (2015) question the plausibility of Piketty's savings hypothesis. The latter authors stress that Piketty's assumption of a constant net savings rate drives his predicted explosion of the capital-output ratio as the economy's growth rate approaches zero (see Section 2 below for details). To amend this prediction these authors suggest using the savings hypothesis of Ramsey (1928), Cass (1965), and Koopmans (1965). However, with this amendment the key qualitative implication of Piketty's second fundamental law does not change: a decline in the exogenous growth rate of the economy unequivocally increases the capital-output ratio. In contrast, the present paper shows that this implication may be wrong: a decline in the endogenous growth rate of the economy induced by a change in a model parameter may reduce the capital-output ratio. This possibility arises since the economy's growth rate is endogenous.

Second, our analysis contributes to the recent literature documenting and explaining contemporaneous trends in the evolution of the factor income distribution (see, e. g., Elsby, Hobijn, and Sahin (2013), Bridgman (2014), Karabarbounis and Neiman (2014a), Karabarbounis and Neiman (2014b), Rognlie (2015), or Growiec, McAdam, and Muck (2018)). We provide a broad set of testable predictions about the long-run relationship between "deep" parameters of an economy, its speed of economic growth and its factor income distribution that are relevant for a comprehensive understanding of the real facts. Our analysis supports the view that the distinction between gross and net shares is quite relevant (Bridgman (2014)). Allowing for capital depreciation brings our analytical findings closer to reality but sometimes at a cost of some cumbersome extra algebra. More importantly, we show via simulation exercises that some qualitative results change their sign once we switch from a world with realistic levels of capital depreciation to a world void of depreciation.

Third, the present paper extends and complements previous contributions that study the income distribution-growth nexus in the AK models of Frankel (1962), Romer (1986), or Rebelo (1991). Here, parameters characterizing the economic environment that are associated with slower growth are also associated with a smaller share of capital in aggregate output (see, e. g., Bertola (1993), or Bertola, Foellmi, and Zweimüller (2006), pp. 81-87). The same qualitative result obtains in a first-generation endogenous growth model without capital accumulation sketched in Bertola, Foellmi, and Zweimüller (2006), Chapter 10. However, to the best of our knowledge the present paper is the first that conducts a comprehensive analysis of the relationship between the factor income distribution, the capital-income ratio, and economic growth in an R&D-based endogenous growth model *with* capital accumulation and provides the detailed link to Piketty's theory. In particular and unlike previous contributions, our analysis highlights that the key to the understanding of this link lies in the intricate relationship between an economy's endogenous net savings rate and its endogenous growth rate (see Proposition 6 and the ensuing discussion). Moreover, unlike the predictions derived from the AK model or the variety-expansion model without capital accumulation our analysis suggests that changes in some parameters of the economy give rise to a negative correlation between the capital share and the economy's growth rate, a finding consistent with Piketty's second law.

The remainder of this paper is organized as follows. Section 2 discusses Piketty's predictions and explains in a heuristic fashion the inherent endogeneity problem that may invalidate them. Section 3 presents the details of the model. Section 3.1 describes the economic sectors, and Section 3.2 has the definition of the dynamic general equilibrium. Our main results are established in Section 4. Here, we first characterize the equilibrium distribution of factor incomes (Section 4.1). Then, we switch to the analysis of the long-run. Section 4.2 establishes the steady-state equilibrium and derives the determinants of the steady-state growth rate. Section 4.3 studies the determinants of the capital-income ratio and relates them to Piketty's second law. Section 4.4 analyses the capital share. Section 4.5 discusses the distributional consequences associated with the implementation of the planner's allocation. Section 5 adds demographic growth to the picture. The focus of Section 5.1 is on how a change in the population growth rate affects Piketty's second law. Section 5.2 studies the effect of population growth on the capital share. Section 6 concludes. If not indicated otherwise, proofs are relegated to the Appendix, Section **??**.

2 Piketty's Predictions and the Endogeneity Problem

Piketty (2014) asserts two so-called fundamental laws of capitalism that are used to explain and predict long-run trends in the capital-income ratio, β, and in the capital share,

α. The *first fundamental law of capitalism*, henceforth "first law," is the definition of the capital share of national income, $\alpha = r \times K/Y$ (Piketty (2014), p. 52 ff.). Here, r is "the average annual rate of return on capital" (ibidem, p. 25), K is "the total market value of everything owned by the residents and the government of a given country at a given point in time" (ibidem, p. 48), and Y is national income.

Let us use a "*" to denote steady-state values. Then, the statement of Piketty's *second fundamental law of capitalism*, henceforth "second law," is $\beta^* = s/g$ where s is the exogenous savings rate defined, following Solow (1956), as $s \equiv \dot{K}/Y$, and g is the exogenous growth rate of the economy reflecting technical and demographic change. The second law holds in steady state, i. e., when the capital stock grows at the same rate as the economy, i. e., $\dot{K}/K = g$. Then, it follows from the definition of the savings rate as

$$s \equiv \frac{\dot{K}}{Y} = \frac{\dot{K}}{K} \times \frac{K}{Y} = g \times \frac{K}{Y} \quad \Leftrightarrow \quad \beta^* = \frac{s}{g}.$$

Piketty uses the second law to assert that a decline in the growth rate, g, explains a higher capital-income ratio (see, e. g., ibidem, p. 175 and p. 183). Taking the second law to extremes he argues that a society with zero population and productivity growth will see its capital-income ratio rise indefinitely (see, e. g., Piketty and Saez (2014), p. 480). We summarize these assertions as Prediction 1.

Prediction 1 *(Predictions for the Steady-State Capital-Income Ratio)*

1. *The smaller the economy's growth rate the greater is the capital-income ratio, i. e.,*

$$\frac{\partial \beta^*}{\partial g} < 0.$$

2. *As $g \to 0$ the capital-income ratio becomes unbounded, i. e.,*

$$\lim_{g \to 0} \beta^* = \infty.$$

The second law gives rise to a steady-state rate of return on capital $r^* = r(\beta^*)$ with $r'(\beta^*) < 0$.[1] Accordingly, the steady-state capital share is $\alpha^* = r(\beta^*) \times \beta^*$. This expres-

[1] This follows from basic neoclassical growth theory. Let national income be $Y = F(K, L) - \delta K$ where F is a neoclassical production function, L is labor, and δK is capital depreciation. In per worker terms, we may write $y = f(k) - \delta k$, where $y = Y/L$, $k = K/L$, and $f(k) \equiv F(k, 1)$ with $f'(k) > 0 > f''(k)$. Then, $\beta = k/y$ and total differentiation of the latter delivers $dk/d\beta > 0$. Due to competitive marginal cost pricing the rate of return on capital satisfies $r = f'(k(\beta)) - \delta$. Hence, $r'(\beta^*) \equiv dr/d\beta = f''(k(\beta)) \times (dk/d\beta) < 0$.

sion links the economy's growth rate to its capital share, i. e.,

$$\frac{d\alpha^*}{dg} = \frac{\partial \alpha^*}{\partial \beta} \times \frac{\partial \beta^*}{\partial g} = \left[\underbrace{r'(\beta^*) \times \beta^* + r(\beta^*)}_{(+)} \right] \times \underbrace{\frac{\partial \beta^*}{\partial g}}_{(-)} .$$

Piketty argues that a hike in β is unlikely to induce a strong decline in the rate of return on capital since the elasticity of substitution between capital and labor is greater than unity. Hence, the term in brackets is positive, and, in Piketty's words, "the volume effect will outweigh the price effect" (Piketty (2014), p. 221). Accordingly, the capital share is predicted to increase if g falls (ibidem, p. 216 or p. 220). We merge these assertions into Prediction 2.

Prediction 2 *(Predictions for the Steady-State Capital Share)*

1. *The smaller the economy's growth rate the greater is the capital share in national income, i. e.,*

$$\frac{d\alpha^*}{dg} < 0.$$

2. *The latter holds since "the volume effect will outweigh the price effect."*

Are Prediction 1 and Prediction 2 justified or justifiable? The main results of our analysis establish that both predictions are problematic since the economy's growth rate and its savings rate are endogenous. Therefore, there is no direct causal relationship between the economy's growth rate, its capital-output ratio, and its capital share. Rather, changes in the economic environment are bound to affect the capital-output ratio, the capital share, and the economy's growth rate simultaneously. We refer to this as the *endogeneity problem* of Piketty's predictions. Hereafter, we develop a heuristic example showing why the endogeneity problem matters.

In contradiction to Claim 1 of Prediction 1, we establish in the main part of the paper that a greater instantaneous discount rate, ρ, reduces the economy's steady-state growth rate *and* its capital-income ratio (see Proposition 3 in conjunction with Proposition 5).[2] The intuition comes in two steps.

[2]The literature traces differences across countries in the discount rate back to various sources. They include differences in religious believe systems (see, e. g., Weber (1930)), in the level of income per capita (see, e. g., Das (2003)), or in the geographical variation of the natural return to agricultural investments (Galor and Ömer Özak (2016)).

First, we show that in steady state the capital-income ratio can be expressed as (see Proposition 6)

$$\beta^* = \frac{s^*(\rho)}{g^*(\rho)};$$

here, $s^*(\rho)$ is the savings rate as defined by Piketty, computed for the steady state of Romer's model, and $g^*(\rho)$ is Romer's endogenous growth rate.[3] Hence, the second law holds but now both rates are endogenous and depend on ρ. This implies, in particular, that there is no direct causal effect of the steady-state growth rate of the economy on the steady-state capital-income ratio.

Second, we establish that increasing ρ lowers the steady-state capital-income ratio, i. e.,

$$\frac{d\beta^*}{d\rho} = \beta^* \left[\underbrace{\frac{(s^*)'(\rho)}{s^*(\rho)}}_{(-)} - \underbrace{\frac{(g^*)'(\rho)}{g^*(\rho)}}_{(-)} \right] < 0, \quad \text{since} \quad \left| \frac{(s^*)'(\rho)}{s^*(\rho)} \right| > \left| \frac{(g^*)'(\rho)}{g^*(\rho)} \right|.$$

This finding comes about since $(s^*)'(\rho) < 0$, $(g^*)'(\rho) < 0$, and the proportionate decline in $s^*(\rho)$ dominates the proportionate decline in $g^*(\rho)$. Hence, a greater ρ means lower growth and a smaller capital-income ratio.

We note in passing that contrary to Claim 2 of Prediction 1 the long-run capital-income ratio remains finite for an economy without growth. Intuitively, such an economy will close down its research sector and behaves very much like the textbook Ramsey model. In particular, the stationary steady state exhibits a finite capital-income ratio.

As to Prediction 2, the following findings are remarkable. Claim 1 of Prediction 2 may hold even though Claim 1 of Prediction 1 does not. For instance, a lower growth rate may lead to a higher capital share in the long run even though the capital-income ratio is predicted to decline. However, the way this result comes about must contradict Claim 2 of Prediction 2. To see this, consider again the case of an increase in ρ. As argued above, this leads to a decline in the growth rate and to a lower capital-income ratio. Moreover, it also increases the rate of return on capital, i. e., $r = r(\rho)$ with $r'(\rho) > 0$.[4] Then, the first law says that

$$\alpha^* = r^*(\rho) \times \beta^*(\rho).$$

[3] Proposition 3 and Proposition 6 show respectively that the savings rate and the growth rate will not only depend on ρ but on a whole vector of parameters that characterize the economy. For simplicity, we suppress these other parameters in the notation here.

[4] The intuition comes from the Euler condition. In steady state the rate of return on capital must adjust so that the infinitely-lived household embarks on a path along which consumption grows at rate $g(\rho)$. Hence, in the simplest case with log-utility the steady-state Euler condition reads $g^*(\rho) = r - \rho$ and $r'(\rho) \equiv dr/d\rho = 1 + g'(\rho) > 0$ as $|g'(\rho)| < 1$.

Moreover, the proportionate increase in $r^*(\rho)$ dominates the proportionate decline in $\beta^*(\rho)$ (see Proposition 7), i. e.,

$$\frac{d\alpha^*}{d\rho} = \alpha^* \left[\underbrace{\frac{(r^*)'(\rho)}{r^*(\rho)}}_{(+)} + \underbrace{\frac{(\beta^*)'(\rho)}{\beta^*(\rho)}}_{(-)} \right] > 0 \quad \text{since} \quad \frac{(r^*)'(\rho)}{r^*(\rho)} > \left| \frac{(\beta^*)'(\rho)}{\beta^*(\rho)} \right|.$$

In other words, the price effect outweighs the volume effect and not vice versa.

Our analysis establishes that, mutatis mutandis, the above heuristic is also behind the effect of other "deep" parameters on the capital-income ratio and the capital share in the long run (see, Table 1). As such and to the best of our knowledge, the present paper provides the first comprehensive analysis of the factor income distribution in Romer's seminal model.

3 The Model

We study the factor income distribution in a variant of Romer's model of endogenous technological change extended to allow for a variable degree of monopoly power of intermediate-good firms, depreciation of the physical capital stock, and for an active government. As will become clear below, these features will have an effect on the factor income distribution.

Time is continuous, i. e., $t \in [0, \infty)$. At all t, the economy has a unique final good that may be consumed or accumulated as physical capital. Besides the market for the final good, there are markets for bonds, stocks, physical capital, labor, and for intermediate goods. Moreover, there is a government that levies a tax on capital incomes and pays subsidies to innovators. In all periods its budget is balanced. This may necessitate lump-sum taxes or lump-sum transfers to the household sector. All prices are expressed in units of the contemporaneous final good.

We denote $g_x(t) = \dot{x}(t)/x(t)$ the instantaneous growth rate of some variable x at t. For notational simplicity we shall suppress the time argument whenever this is not confusing. Observe also that our notation below slightly differs from Piketty's. In particular, we follow standard textbook notation and denote by r the real interest rate on bonds, K is the stock of physical capital, and Y is the total output of the final good.

3.1 Economic Sectors

3.1.1 The Final-Good Sector

The final good, Y, is produced by a single competitive representative firm. Labor, $L_Y > 0$, and a continuum of $M > 0$ different intermediate goods serve as inputs in the production function

$$Y = L_Y^{1-\gamma} \left(\int_0^M x(j)^\mu \, dj \right)^{\frac{\gamma}{\mu}}, \quad 0 < \gamma \leq \mu < 1, \tag{3.1}$$

where $x(j)$, $j \in [0, M]$, denotes the quantity of intermediate good j and M is the "number" of available intermediate goods at t. The parameter μ determines the elasticity of substitution, $1/(1-\mu) > 1$, between intermediates. The output elasticity of the intermediate-good aggregate is given by γ. If $\mu \geq \gamma$, then all first-order conditions derived below are also sufficient for a profit maximum. Let w denote the real wage per (homogeneous) worker paid in the economy and $p(j)$ the price of intermediate good j. Then, the demand for labor and the demands for each intermediate good are the solution to

$$\max_{L_Y, \{x(j)\}_{j=0}^{j=M}} Y - wL_Y - \int_0^M p(j)\, x(j)\, dj \tag{3.2}$$

and given by

$$L_Y = (1-\gamma) \frac{Y}{w}, \tag{3.3}$$

$$\gamma x(j)^{\mu-1} L_Y^{1-\gamma} \left(\int_0^M x(j)^\mu \, dj \right)^{\frac{\gamma}{\mu}-1} = p(j) \quad \text{for all } j \in [0, M]. \tag{3.4}$$

3.1.2 The Intermediate-Good Sector

The intermediate-good sector comprises M monopolists with a perpetual patent for a single variety $j \in [0, M]$. The production function of all monopolists is the same and given by $x(j) = k(j)$, where $k(j)$ is the amount of capital employed by monopolist j. With R denoting the rental rate of capital, each monopolist's revenue is $\pi(j) = p(j)\, x(j) - Rk(j)$. Revenue maximization delivers $p(j) = p = R/\mu$. Hence, μ measures the monopoly power of intermediate-good firms. Then, (3.4) in conjunction with (3.1) gives

$$x(j) = x = \frac{\gamma\mu}{R} \frac{Y}{M}. \tag{3.5}$$

Moreover, maximized revenues become $\pi(j) = \pi = (1-\mu)\, px$.

3.1.3 The Research Sector

Previous to its marketing an intermediate good must be invented through research. To capture this consider a single competitive research firm with access to a technology for the creation of new intermediate-good varieties given by

$$\dot{M} = \frac{L_M}{a} M. \tag{3.6}$$

Here, $L_M \geq 0$ measures the workforce employed in the research sector, and $a > 0$ determines its productivity. The research output depends also on the current stock of technological knowledge represented by M to which access is for free.

Following its invention the blueprint of the new variety in conjunction with a perpetual patent is sold at the price v. Then, free entry into the research sector implies a zero-profit condition, so that the price received per invention cannot be greater than the cost of creating it, i. e.,

$$v \leq \frac{a}{M} (1 - \sigma) w \quad \text{with “=”, if } \dot{M} > 0, \tag{3.7}$$

where $\sigma \in (0, 1)$ is the subsidy rate of the labor cost associated with each innovation.

Since new patents are auctioned off the highest bidder pays a price per variety equal to the net present value of all future after-tax revenues. Hence, the value of a patent, v, in units of the current final good is equal to the present value of all future profits. As $\pi(j) = \pi$, these profits are the same for all varieties. Let $\tau \in (0, 1)$ denote the tax rate on capital earnings. Then, the market capitalization of each variety j at t is

$$v(t) = (1 - \tau) \int_t^{\infty} \pi(t) e^{-(s-t)(1-\tau)\bar{r}(s)} ds, \tag{3.8}$$

where $\bar{r}(s)$ is the average interest rate over the interval $[t, s]$, i. e.,

$$\bar{r}(s) = \frac{1}{s - t} \int_t^s r(\nu) d\nu, \tag{3.9}$$

and $r(t)$ is the real interest rate on bonds at t.

Since bonds, capital, and stocks are perfect substitutes as stores of value, for all t the no-arbitrage condition of the capital market must equate the after-tax returns associated with each of these assets, i. e., $(1 - \tau) r = (1 - \tau)(R - \delta) = (1 - \tau)\pi/v + g_v$. Here, δ is the instantaneous depreciation rate of physical capital, $R - \delta$ is the pre-tax rate of return on holding one unit of final-good output in capital, and $\pi/v + g_v$ is the pre-tax rate of return on holding one unit of final-good output in shares. As stated, the capital income tax is levied on the rental rate of capital net of depreciation, on paid dividends and not on the accounting profit, $\dot{v}$. Therefore, it proves useful to introduce $\delta_v \equiv -g_v/(1 - \tau)$ as the depreciation rate of share prices so that the no-arbitrage condition can be written as

$$r = R - \delta = \frac{\pi}{v} - \delta_v. \tag{3.10}$$

3.1.4 The Household Sector

There is a single representative household comprising $L(t) = L$ members. Each household member supplies one unit of homogeneous labor inelastically to the labor market. Besides its labor endowment, the household owns the capital stock, K, and all firms in the economy. The household values streams of consumption per household member, $c(t) = C(t)/L$ according to

$$U = L \int_0^\infty \frac{c(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt, \quad \rho > 0, \quad \theta > 0. \tag{3.11}$$

Here, ρ is the instantaneous discount rate, and θ is the inverse of the inter-temporal elasticity of substitution in consumption.

The flow budget constraint may be expressed as

$$Lc + \dot{K} + v\dot{M} = (1-\tau)(R-\delta) K + wL_Y + wL_M + (1-\tau)\pi M + T. \tag{3.12}$$

The right-hand side states the household's income flows net of capital depreciation. They comprise the after-tax income from physical capital, labor income earned in manufacturing and research, the after-tax dividend income from the ownership of M intermediate-good monopolists, and the lump-sum payment necessary to balance the government's budget, T. This income is spent on consumption or savings. The latter may involve net investments in the accumulation of the capital stock, $\dot{K}$, and/or the purchase of newly emitted shares, $v\dot{M}$.

Let $A \equiv K + vM$ denote the household's total assets. Then, standard arguments deliver the optimal solution to the household's optimization problem. The latter satisfies the Euler condition

$$g_c = \frac{\dot{c}}{c} = \frac{(1-\tau)r - \rho}{\theta} \tag{3.13}$$

and the transversality condition

$$\lim_{t \to \infty} A(t) \exp\left(-(1-\tau) \int_0^t r(v) dv\right) = 0 \tag{3.14}$$

as necessary and sufficient conditions.

3.2 Dynamic General Equilibrium - Definition

Given $M(0) > 0$, $K(0) > 0$, $L > 0$, and a time-invariant policy (τ, σ), a dynamic general equilibrium comprises an allocation,

$$\{Y(t), L_Y(t), x(j,t), k(j,t), M(t), K(t), L_M(t), c(t)\}_{t=0}^{\infty},$$

a price system,

$$\{w(t), r(t), p(j,t), R(t), \pi(j,t), v(t)\}_{t=0}^{\infty},$$

where $j \in [0, M(t)]$, and a path of lump-sum transfers $\{T(t)\}_{t=0}^{\infty}$ such that i) producers of the final good choose labor and the quantity of all available intermediates taking prices as given, ii) intermediate-good monopolists maximize profits taking their respective demand curves and the rental rate of capital as given, iii) research firms enter the market, take prices for new blueprints, the wage for researchers, and the subsidy rate as given, and earn zero profits, iv) firms contemplating entry into the business of producing a novel intermediate take the price of blueprints as given, finance the purchase of the blueprint through the issue of new shares, pay the promised dividends to the household sector, and earn zero inter-temporal profits, v) the household sector makes consumption and savings decisions taking prices, the tax on capital incomes and the lump-sum transfer as given, vi) all markets clear, vii) the government budget is balanced so that $T = \tau\left(\left(R-\delta\right)K + \pi M\right) - \sigma w L_M$.

4 Factor Income Distribution and Economic Growth - Piketty meets Romer

This section studies the long-run relationship between the factor income distribution and the economy's growth rate. To accomplish this it proves useful to highlight first some important and intuitive results for equilibrium factor incomes and factor shares. This is the purpose of Section 4.1. Then, we turn to the long-run and characterize the steady-state equilibrium in Section 4.2. Piketty meets Romer in Section 4.3 and 4.4. These sections contain the central findings of this paper. Section 4.3 studies the steady-state capital-income ratio and relates it to Prediction 1. The focus of Section 4.4 is on the steady-state capital share and Prediction 2. Finally, Section 4.5 derives the (first-best) planner's allocation and shows that its implementation may involve a higher labor share than under laissez-faire.

4.1 Equilibrium Factor Incomes and Factor Shares

At any t, the economy is endowed with three factors of production, labor (in two uses), $L = L_Y + L_M$, technological knowledge, M, and physical capital, K. Their respective factor incomes are as follows.

Proposition 1 *(Equilibrium Factor Incomes)*

Equilibrium factor incomes satisfy

$$wL_M = \max\left\{0, \frac{v\dot{M}}{1-\sigma}\right\}, \tag{4.1}$$

and

$$wL_Y + \pi M + RK = Y, \tag{4.2}$$

where

$$\frac{wL_Y}{Y} = 1-\gamma, \quad \frac{\pi M}{Y} = (1-\mu)\,\gamma, \quad \textit{and} \quad \frac{RK}{Y} = \mu\gamma. \tag{4.3}$$

Proposition 1 makes three important observations about the income flows in the economy. First, equation (4.1) says that the wage income of labor in research is equal to the value of the new shares that the household sector buys from new intermediate-good monopolists discounted by the fraction of the wage bill borne by research firms, $1-\sigma$. The intuition is the following. At all t the household sector buys $\dot{M}$ new shares, each at a price v. These shares are emitted by new intermediate-good monopolists that use the revenue raised from the sale of these shares to pay research firms in exchange for the blueprint. In addition, research firms receive the subsidy that covers the fraction σ of their total labor costs. In equilibrium research firms just break even. Hence, whenever $L_M > 0$ we have $v\dot{M} + \sigma wL_M = wL_M$. Trivially, if $L_M = 0$ then $wL_M = 0$ and (4.1) follows.

To grasp the remaining results of Proposition 1 observe that in equilibrium the market for the capital input clears so that $x = K/M$. Hence, aggregate output of the final good becomes

$$Y = L_Y^{1-\gamma} M^{\gamma(1/\mu-1)} K^{\gamma}. \tag{4.4}$$

Then, equation (4.2) has the second observation: the remuneration of the three factors of production that show up in (4.4) adds up to the total output of the final good. More interestingly, the price for the service of each "unit" of technological knowledge corresponds to the dividend that the household sector receives from the respective intermediate-good firm that uses this unit.

Finally, equation (4.3) informs us about the shares of income in final-good output that accrue to industrial labor, technological knowledge, and physical capital. Intuitively, the equilibrium remuneration of industrial labor is obtained from the first-order condition (3.3). Hence, the aggregate wage bill of the final-good sector is $wL_Y = (1-\gamma)\,Y$. As expected for a Cobb-Douglas production function, the fraction of total output that goes to L_Y workers is equal to the output elasticity of these workers.

Since final-good production exhibits constant returns to scale in all rival inputs it must be that

$$\int_0^M p(j)\,x(j)\,dj = pxM = \gamma Y. \tag{4.5}$$

Consequently, the remuneration of technological knowledge and physical capital sums up to γY. But what is the split? Since $\pi = (1-\mu)px$ one finds that the remuneration of the total stock of technological knowledge amounts to $\pi M = (1-\mu)\,\gamma Y$. Ceteris paribus, a greater μ means a greater elasticity of substitution between intermediates which reduces the mark-up charged by intermediate-good monopolists. Accordingly, the share of dividends in γY falls. Finally, with (4.5) we obtain the remuneration of physical capital as $RK = \gamma Y - \pi M = \mu\gamma Y$. Hence, μ determines the breakup of γY into income for technological knowledge and physical capital.

Next, we turn to the equilibrium factor shares. Let GDP denote the economy's gross domestic product, i.e., its total value added. In equilibrium GDP satisfies

$$GDP = Y + v\dot{M}. \tag{4.6}$$

Intuitively, for any symmetric configuration of the production sector we have $GDP = (Y - Mpx) + Mpx + v\dot{M}$ where the expression in parenthesis is the value added in the final-good sector, the second and the third term show the value added in the intermediate-good sector and in the research sector. As a result, GDP is the sum of the total output of the final good and the value created by research firms.

By definition, net domestic product is $NDP \equiv GDP - \delta K$. Since the economy is closed NDP coincides with national income.

The capital share, α, and the labor share, $1-\alpha$, relate, respectively, the economy's income from asset holdings (net of capital depreciation) and its total wage bill (net of wage subsidies) to its NDP, i.e.,

$$\alpha \equiv \frac{(R-\delta)K + \pi M}{NDP} \quad \text{and} \quad 1-\alpha \equiv \frac{wL - \sigma w L_M}{NDP}. \tag{4.7}$$

These definitions are the counterparts to the (pre-tax) factor shares that Piketty considers (see, e.g., Piketty (2014), pp. 200-203). If $\sigma > 0$ then wL_M exceeds the value added of research labor by the subsidy $\sigma w L_M$ (see, equation (4.1)). The definition of $1-\alpha$ takes this into account. Hence, its numerator corresponds to the value added by labor in both sectors. To study the determinants of the factor shares we now turn to the long run.

4.2 Steady-State Equilibrium

A steady-state equilibrium is a path along which all variables of the model grow at constant, possibly different exponential rates. We denote steady-state values by a "*" and define

$$\zeta \equiv \frac{1-\sigma}{1-\tau} > 0, \quad \eta \equiv \frac{\gamma\,(1-\mu)}{\mu\,(1-\gamma)} \in (0,1], \quad \text{and} \quad \vartheta \equiv \frac{\zeta}{\eta\mu} > 0.$$

Roughly speaking, ζ states the effect of policy measures, η accounts for the fact that market power, μ, is independent of the technology parameter, γ, and ϑ captures the interaction between the two. Let g^* denote the steady-state growth rate of the economy.

Proposition 2 *(Steady-State Equilibrium)*

There exists a unique steady-state equilibrium if

$$\rho > (1-\theta)\,g^*. \tag{4.8}$$

The steady-state growth rate of technological knowledge is

$$g_M^* = \max\left\{0, \frac{\mu\left(\frac{L}{a} - \vartheta\rho\right)}{\mu + \zeta\left(\theta + \eta^{-1} - 1\right)}\right\}. \tag{4.9}$$

The steady-state growth rate of the economy is

$$g^* = g_Y^* = g_K^* = g_c^* = \eta g_M^*. \tag{4.10}$$

Moreover, it holds that

$$g_v^* = g_\pi^* = -\left(\eta^{-1} - 1\right) g^* \leq 0. \tag{4.11}$$

Proposition 2 states key properties of the steady-state equilibrium. Condition (4.8) assures that $(1-\tau)r^* > g^*$, so that the household's transversality condition holds. The steady-state growth rate of technological knowledge of (4.9) may be zero.

For a growing economy the steady state involves $g_M^* > g^*$ whenever $\mu > \gamma$ (and $\eta < 1$). To see why, recall that the equilibrium output of the final good is given by (4.4). In steady state L_Y is time-invariant, and the ratio of physical capital to final-good output must be constant, i. e., $g_K = g_Y$. Moreover, with $\mu > \gamma$ the output elasticity of M becomes strictly smaller than $1-\gamma$. This drives a wedge between g_M and g_K. Accordingly, M must grow faster than K and Y.[5]

Finally, equation (4.11) states that the steady state of a growing economy has $g_\pi^* = g_v^* < 0$ if $\mu > \gamma$. Intuitively, this reflects a declining turnover of existing intermediate-good firms

[5] For a complementary way to see this write (4.4) as $Y = \left(L_Y M^\eta\right)^{1-\gamma} K^\gamma$ so that technological knowledge appears as labor-augmenting. To have both sides of this equation grow at the same rate while $g_K = g_Y$ it is necessary that $\eta g_M = g_K$.

as $g_x^* = g_K^* - g_M^* < 0$. Hence, in steady state dividends and share prices fall at a constant rate.

From (4.10) we may write $g^* = g(\boldsymbol{\omega})$ where g is a function that maps the vector of parameters upon which g^* depends, $\boldsymbol{\omega} = (L, a, \tau, \sigma, \rho, \theta, \gamma, \mu)$, into $\mathbb{R}_+$. For further reference the following proposition summarizes the derivative properties of $g(\boldsymbol{\omega})$ for a growing economy.

Proposition 3 *(Determinants of g^*)*

Suppose $L/a > \vartheta\rho$. Then, it holds that

$$\frac{\partial g^*}{\partial L} > 0, \quad \frac{\partial g^*}{\partial a} < 0, \quad \frac{\partial g^*}{\partial \tau} < 0, \quad \frac{\partial g^*}{\partial \sigma} > 0, \quad \frac{\partial g^*}{\partial \rho} < 0, \quad \frac{\partial g^*}{\partial \theta} < 0, \quad \frac{\partial g^*}{\partial \gamma} > 0.$$

Moreover, there is ε such that, if $0 < g^* < \varepsilon$, then

$$\frac{\partial g^*}{\partial \mu} < 0.$$

According to Proposition 3 the steady-state growth rate responds in an intuitive way to the considered parameter changes: an economy with more workers grows faster (the scale effect), a lower labor productivity in the research sector slows down growth, taxing capital reduces the growth rate while subsidizing research increases it, more patience and a greater inter-temporal elasticity of substitution in consumption on the side of households spur growth, and a larger output elasticity of intermediates leads to faster growth. Moreover, for sufficiently small growth rates smaller mark-ups in the intermediate-good sector reduce the long-run growth rate.

Finally, the steady-state real interest rate is given by the Euler equation of (3.13)

$$r^* = \frac{\theta g^* + \rho}{1 - \tau}. \tag{4.12}$$

For further reference, we may also write $r^* = r(\tau, \rho, \theta, g(\boldsymbol{\omega}))$.

4.3 The Capital-Income Ratio in the Long Run

This section studies the long-run determinants of the capital-income ratio (or wealth-to-NDP ratio). Our main result contradicts Prediction 1. We establish that the long-run capital-income ratio may increase even though g^* increases. The discrepancy between our findings and Piketty's assertions is due to our treatment of the economy's savings rate and its growth rate as endogenous variables. Throughout, we follow Piketty and let β denote the capital-income ratio defined as

$$\beta \equiv \frac{A}{NDP}. \tag{4.13}$$

Consider a growing economy where $g^* > 0$, $L_M > 0$, $\dot{M} > 0$, and $GDP = Y + v\dot{M}$.[6] Since the economy has two assets in positive net supply, claims on physical capital and shares in intermediate-good firms, the capital-income ratio is

$$\beta = \frac{K + vM}{Y + v\dot{M} - \delta K}. \tag{4.14}$$

The following proposition gives β^*.

Proposition 4 *(Capital-Income Ratio in the Steady State)*

Suppose that $L/a > \vartheta\rho$. Then, the steady-state capital-income ratio is

$$\beta^* = \frac{\dfrac{\mu\gamma}{r^* + \delta} + \dfrac{(1-\mu)\gamma}{r^* + \delta_v^*}}{1 + \dfrac{\mu(1-\gamma)}{r^* + \delta_v^*} g^* - \delta \dfrac{\mu\gamma}{r^* + \delta}}. \tag{4.15}$$

Moreover,

$$\lim_{L/a - \vartheta\rho \downarrow 0} \beta^* < \infty. \tag{4.16}$$

The numerator of β^* features the present discounted value of a permanent income stream that accrues to physical capital, $R^*K^*/Y^* = \mu\gamma$, discounted at rate $r^* + \delta$ and the present discounted value of a permanent income stream that accrues to technological knowledge, $\pi^* M^*/Y^* = \mu(1-\gamma)$, discounted at rate $r^* + \delta_v^*$. The different discount rates reflect that capital depreciates at rate δ whereas the stock market value, v^*, of all M^* intermediate-good firms depreciates at rate δ_v^* where δ_v^* is endogenous. The denominator shows NDP^*/Y^*.

Proposition 4 also establishes that in the limit $L/a - \vartheta\rho \downarrow 0$, i. e., when $g_M^* \to 0$, the capital-income ratio, β^*, remains finite. Hence, in contrast to Prediction 1, as $g^* \to 0$ the capital-income ratio remains finite.

Overall, the steady-state capital-income ratio reflects the interaction between technology, preferences, policy, and market structure. The following proposition shows how these features affect β^*.

---

[6] See Irmen and Tabakovic (2016), Section 3.3.1, for the corresponding analysis of a stationary economy where $g^* = 0$.

Proposition 5 *(Determinants of β^* with Growth)*

Suppose that $L/a > \vartheta\rho$. Then, it holds that

$$\frac{d\beta^*}{d\rho} < 0, \quad \frac{d\beta^*}{d\theta} < 0, \quad \frac{d\beta^*}{d\tau} < 0, \tag{4.17}$$

$$\frac{d\beta^*}{dL} < 0, \quad \frac{d\beta^*}{da} > 0, \quad \frac{d\beta^*}{d\sigma} < 0, \tag{4.18}$$

where all derivatives are evaluated at $\mu = \gamma$ and $\delta = 0$.

Moreover, it holds that

$$\frac{d\beta^*}{d\mu} > 0, \quad \text{and} \quad \frac{d\beta^*}{d\gamma} \gtreqless 0 \quad \Leftrightarrow \quad \theta \lesseqgtr \frac{\sigma(1-\mu)(1-\tau)}{1-\sigma}, \tag{4.19}$$

where the latter derivatives are evaluated at $\mu = \gamma$, $\delta = 0$, and $g^* = 0$.

Finally, it holds for sufficiently small values of $g^* > 0$ that

$$\frac{d\beta^*}{d\delta} < 0. \tag{4.20}$$

Proposition 5 gives the long-run responses of the capital-income ratio to changes in all parameters of the model.[7] To link these findings to Piketty's Prediction 1 consider the first and the second line of Table 1 which shows the signs of the comparative statics for g^* of Proposition 3 and those for β^* of Proposition 5. Hence, Prediction 1 does not hold for ρ, θ, and τ. Changes in these parameters move g^* and β^* in the same direction. The same contradiction may hold for changes in γ. Prediction 1 does not hold for δ either. Increasing the rate of physical capital depreciation leaves g^* unaffected but reduces β^*. However, for σ, μ, γ, L, and a Prediction 1 holds true: changes in these parameters that reduce the economy's growth rate will at the same time increase the capital-income ratio.

For further reference, note that Proposition 4 allows us to express β^* as

$$\beta^* = \beta\left(\boldsymbol{\psi}, g\left(\boldsymbol{\omega}\right)\right), \tag{4.21}$$

where $\boldsymbol{\psi} = (\tau, \rho, \theta, \gamma, \mu, \delta)$ is the vector of parameters that exercise a "direct" effect on β^*, i. e., an effect that does not materialize through $g\left(\boldsymbol{\omega}\right)$.[8]

---

[7] Unfortunately, analytic results become cumbersome once we move away from $\mu = \gamma$ or allow for $\delta > 0$. However, numerical exercises reveal that the qualitative results stated in (4.17) - (4.18) hold true for a wide range of parameter values once we allow for $\mu > \gamma$ and $\delta > 0$. The same is true for the sign of $d\beta^*/d\mu$ if, in addition, we allow for sufficiently small values of $g^* > 0$. The Mathematica file where these assertions are established is available upon request.

[8] This follows since $r^* = r\left(\tau, \rho, \theta, g\left(\boldsymbol{\omega}\right)\right)$ and Proposition 2 allow us to write $\delta_v^* = \left(\eta^{-1} - 1\right) g\left(\boldsymbol{\omega}\right)/(1 - \tau)$. To obtain (4.21) replace these terms in (4.15).

Table 1: Comparative Statics of g^*, β^*, and α^* (Evaluations as in Proposition 7, 5 and 3).

Variables \ Parameters	ρ	θ	τ	σ	μ	γ	δ	L	a
g^*	$-$	$-$	$-$	$+$	$-$	$+$	/	$+$	$-$
β^*	$-$	$-$	$-$	$-$	$+$	$+/-$	$-$	$-$	$+$
α^*	$+$	$+$	$+$	$-$	$+$	$+$	$-$	$-$	$+$

How come that some parameter changes are in line with Prediction 1 while others are not? To address this question we must establish the link between Proposition 4 and Piketty's second law. This requires an appropriate definition of the following net savings rates:

$$s_K \equiv \frac{Y - Lc - \delta K}{NDP} \quad \text{and} \quad s_M \equiv \frac{v\dot{M} + \dot{v}M}{NDP}. \tag{4.22}$$

Here, $s_K \geq 0$ states net investment in the accumulation of physical capital as a fraction of NDP whereas $s_M \geq 0$ is net investment in the accumulation of shares as a fraction of NDP. Then, $s \equiv s_K + s_M$ is the net savings rate of the economy as defined by Piketty.

Proposition 6 *(Piketty's Second Law)*

At all t, the evolution of the stock of assets $A = K + vM$ may be written as

$$\dot{A} = sNDP \tag{4.23}$$

so that in steady state Piketty's second law holds, i.e.,

$$\beta^* = \frac{s^*}{g^*}. \tag{4.24}$$

Moreover, $s^* = s(\boldsymbol{\psi}, g(\boldsymbol{\omega}))$ so that β^* may be written as

$$\beta^* = \frac{s(\boldsymbol{\psi}, g(\boldsymbol{\omega}))}{g(\boldsymbol{\omega})}. \tag{4.25}$$

Proposition 6 establishes that s is appropriately defined so that the second law holds. Indeed, noting that in steady state A grows at rate g^* we just need to divide (4.23) by A and rearrange terms to get the desired relationship (4.24). Finally, (4.25) shows how the second law translates into a framework that views s^* and g^* as endogenous variables. We can use this expression to make sense of the comparative statics given in Proposition 5. For instance, consider a parameter out of $(\rho, \theta, \tau, \mu, \gamma) \in \{\boldsymbol{\psi}, \boldsymbol{\omega}\}$, say ρ. Then, the total effect of changing ρ on β^* may be decomposed as follows:

$$\frac{d\beta^*}{d\rho} = \frac{1}{g^*}\left[\underbrace{\frac{\partial s\left(\boldsymbol{\psi}, g\left(\boldsymbol{\omega}\right)\right)}{\partial\rho}}_{(-)} + \underbrace{\left(\underbrace{\frac{\partial s\left(\boldsymbol{\psi}, g\left(\boldsymbol{\omega}\right)\right)}{\partial g^*}}_{(+)} - \beta^*\right)}_{(-)}\underbrace{\frac{\partial g\left(\boldsymbol{\omega}\right)}{\partial\rho}}_{(-)}\right] < 0.$$

From Proposition 3 we know that $\partial g\left(\boldsymbol{\omega}\right)/\partial\rho < 0$. Moreover, the sign of the term in parenthesis is determined by the partial effect of g^* on β^*. It is negative since the negative direct effect of g^* on β^* outweighs the positive indirect effect that comes about through an induced increase in s^*.[9] Hence, it follows that the direct (negative) effect of ρ on s^*, i. e., $\partial s\left(\boldsymbol{\psi}, g\left(\boldsymbol{\omega}\right)\right)/\partial\rho < 0$, determines the sign of $d\beta^*/d\rho$.

So, why does Prediction 1 not hold for changes in ρ? An increase in ρ lowers g^* which increases β^*. However, there are two more channels which operate on s^*. First, a lower g^* reduces s^* and, second, a greater ρ reduces s^*. In the words used in the Introduction, the proportionate decline in s^* is stronger than the proportionate decline in g^*. Hence, β^* falls in response to a greater ρ.

For parameters like L, a, σ which are elements of $\boldsymbol{\omega}$ but not of $\boldsymbol{\psi}$ the decomposition analysis simplifies since there is no direct effect through $s\left(\boldsymbol{\psi}, g\left(\boldsymbol{\omega}\right)\right)$. For instance, consider

---

[9] Indeed, writing (4.25) as $\beta^* = s\left(\boldsymbol{\psi}, g^*\right)/g^*$ it follows that $\partial\beta^*/\partial g^* = 1/g^*\left(\partial s^*/\partial g^* - \beta^*\right)$. Evaluated at $\delta = 0$ and $\mu = \gamma$ the latter is

$$\left.\frac{\partial\beta^*}{\partial g^*}\right|_{\mu=\gamma,\,\delta=0} = -\frac{\mu\left(1-\tau\right)\left(\theta + \left(1-\mu\right)\mu\left(1-\tau\right)\right)}{\left(g_M^*\left(\theta + \left(1-\mu\right)\mu\left(1-\tau\right)\right) + \rho\right)^2} < 0,$$

where, however, $\partial s^*/\partial g^*|_{\mu=\gamma,\,\delta=0} > 0$. The negative sign of $\partial\beta^*/\partial g^*$ is consistent with an elasticity $\left(\partial s^*/\partial g^*\right)\left(g^*/s^*\right) < 1$ which is usually borne out by the data (see, e. g., Krusell and Smith (2015), pp. 738-739, for some supporting evidence).

σ. Then,

$$\frac{d\beta^*}{d\sigma} = \frac{1}{g^*}\left(\underbrace{\underbrace{\frac{\partial s\left(\boldsymbol{\psi}, g\left(\boldsymbol{\omega}\right)\right)}{\partial g^*}}_{(+)} - \beta^*}_{(-)}\right)\underbrace{\frac{\partial g\left(\boldsymbol{\omega}\right)}{\partial \sigma}}_{(+)} < 0.$$

Hence, a higher research subsidy increases the growth rate of the economy but reduces β^* since the partial effect of g^* on β^* is negative. Accordingly, Prediction 1 holds.

Finally, for the depreciation rate of physical capital the comparative statics simplify even further since δ is an element of $\boldsymbol{\psi}$ but not of $\boldsymbol{\omega}$. Here, we have

$$\frac{d\beta^*}{d\delta} = \frac{1}{g^*}\underbrace{\frac{\partial s\left(\boldsymbol{\psi}, g\left(\boldsymbol{\omega}\right)\right)}{\partial \delta}}_{(-)} < 0$$

since a higher depreciation rate reduces the economy's net savings rate. Prediction 1 does not hold since changes in δ leave g^* unchanged.

4.4 The Capital Share in the Long Run

This section studies the determinants of the steady-state capital share in a growing economy and relates them to Prediction 2 of Piketty's theory. Our analysis tends to confirm Piketty's assertion that slower long-run growth is associated with a greater capital share. Importantly, the underlying intuition is quite different from Piketty's.

Recall that the capital share is defined as the fraction of total income from asset holdings net of capital depreciation in NDP. Letting $\tilde{r} \equiv ((R-\delta)K + \pi M)\,/\,(K + vM)$ denote the average rate of return on assets as a percentage of their total value, the capital share of equation (4.7) may be expressed as

$$\begin{aligned} \alpha &= \left(\frac{(R-\delta)K + \pi M}{K + vM}\right) \times \left(\frac{K + vM}{NDP}\right) \\ &= \tilde{r} \times \beta, \end{aligned} \tag{4.26}$$

which is a restatement of Piketty's first law.[10]

[10] Note that the steady-state average rate of return on assets, $\tilde{r}^*$, exceeds r^* whenever $\mu > \gamma$ (and $\delta_v^* > 0$). This is because the no-arbitrage condition requires $\pi^*/v^* > r^*$ to make investors willing to hold shares that loose value over time.

The following proposition establishes the comparative statics of the steady-state capital share in a growing economy.[11]

Proposition 7 *(Determinants of the Steady-State Capital Share with Growth)*

It holds that

$$\frac{d\alpha^*}{d\rho} > 0, \quad \frac{d\alpha^*}{d\theta} > 0, \quad \frac{d\alpha^*}{d\tau} > 0, \quad \frac{d\alpha^*}{d\delta} < 0, \tag{4.27}$$

$$\frac{d\alpha^*}{dL} < 0, \quad \frac{d\alpha^*}{da} > 0, \quad \frac{d\alpha^*}{d\sigma} < 0, \tag{4.28}$$

where all derivatives are evaluated at $\mu = \gamma$ and $\delta = 0$. Moreover, it holds that

$$\frac{d\alpha^*}{d\mu} > 0 \quad \text{and} \quad \frac{d\alpha^*}{d\gamma} > 0, \tag{4.29}$$

where the latter are evaluated at $\mu = \gamma$, $\delta = 0$, and $g^* = 0$.

Proposition 7 provides the comparative statics of α^* with respect to all parameters of the model.[12] To interpret these findings in light of Piketty's Prediction 2 compare the first and the third line of Table 1. It follows that Prediction 2 holds true for almost all parameters, i. e., parameter changes that lead to a decline in g^* also imply an increase in α^*. The exceptions are γ and δ. A smaller γ reduces g^* and α^* whereas a smaller δ increases α^* while leaving g^* unaffected.

The comparative statics of Proposition 7 may be decomposed into a price and a volume effect. Do the results reflect the dominance of the volume effect over the price effect? This depends on the parameter. For instance, consider the effect of ρ and σ on α^*.[13] From

[11] See Irmen and Tabakovic (2016), Section 3.4.1, for the corresponding analysis of a stationary economy where $g^* = 0$.

[12] Again, analytic results become cumbersome once we move away from the indicated evaluation. Nevertheless, numerical exercises reveal that the qualitative results in (4.27) remain valid when $\mu > \gamma$ and $\delta > 0$ is allowed for. The same holds true for changes in γ if, in addition, we allow for sufficiently small values of $g^* > 0$. However, all signs in (4.28) may change when we allow for $\mu > \gamma$ and $\delta > 0$. The same is true for the sign of $d\alpha^*/d\mu$. A Mathematica file where these assertions are established is available upon request.

[13] To simplify the notation we suppress the information about where a particular derivative is evaluated. It is understood that all decompositions hold if the evaluation is as in Proposition 7, 5, and 3.

(4.26) the total effect of ρ on α^* may be decomposed in a price and a volume effect

$$\frac{d\alpha^*}{d\rho} = \underbrace{\frac{d\tilde{r}^*}{d\rho}}_{(+)} \beta^* + \underbrace{\frac{d\beta^*}{d\rho}}_{(-)} \tilde{r}^* > 0.$$

From Proposition 5 we know that $d\beta^*/d\rho < 0$. Therefore, $d\alpha^*/d\rho > 0$ must be due to $d\tilde{r}^*/d\rho > 0$. Hence, contrary to Prediction 2, it is the price effect that outweighs the volume effect (as already hinted at in Section 2).

Similarly, the effect of a change in σ on α^* may be decomposed as follows

$$\frac{d\alpha^*}{d\sigma} = \underbrace{\frac{d\tilde{r}^*}{d\sigma}}_{(+)} \beta^* + \underbrace{\frac{d\beta^*}{d\sigma}}_{(-)} \tilde{r}^* < 0.$$

From Proposition 5 we know that $d\beta^*/d\sigma < 0$ while $d\tilde{r}^*/d\sigma > 0$. Hence, the negative sign of $d\alpha^*/d\sigma$ is brought about by a volume effect that dominates the price effect which is consistent with Prediction 2.

Finally, consider the effect of an increase in τ on the steady-state after-tax capital share, $(1-\tau)\alpha^*$. One readily verifies that this effect is negative since a tax hike reduces both the after-tax average rate of return, $(1-\tau)\tilde{r}^*$, and the capital-income ratio, β^*. Hence, we arrive at the conclusion that a higher tax slows down growth and reduces the after-tax capital share.

4.5 The Functional Income Distribution of the Planner's Allocation

This section derives the steady-state functional income distribution that is consistent with the (first-best) allocation chosen by a benevolent planner. It results from the implementation of the planner's steady-state allocation in the economy with government activity of Section 3, i. e., through the use of subsidies to research, taxes on capital earnings, and lump-sum taxation. We show that, depending on the policy mix, the labor share may be higher than the one of the laissez-faire equilibrium that has $\sigma = \tau = 0$.

Given $K(0) > 0$ and $M(0) > 0$, the planner maximizes the lifetime utility of the representative household (3.11) subject to the economy's technology for the creation of new intermediate-good varieties (3.6) and its resource constraints for capital and labor,[14]

$$\dot{K}(t) = L_Y(t)^{1-\gamma} M(t)^{\gamma(1/\mu-1)} K(t)^{\gamma} - c(t)L \quad \text{and} \quad L_Y(t) + L_M(t) = L. \tag{4.30}$$

[14]Details on the derivation of the planner's solution are available from the authors upon request.

Here, use is made of the fact that due to a falling marginal product of $x(j)$ in the production function of the final good (3.1) the planner maximizes output Y by choosing $x(j) = x$. Moreover, a symmetric partitioning of the capital stock implies $x = K/M$. We use the superscripts P and LF to denote variables that belong to the planner's allocation or obtain under laissez-faire.

One readily verifies that the planner's allocation involves

$$g_M^P = \max\left\{0, \frac{1}{\theta}\left(\frac{L}{a} - \frac{\rho}{\eta}\right)\right\} \quad \text{and} \quad L_Y^P = L\left(1 - \frac{1}{\theta}\right) + \frac{a\rho}{\eta\theta}. \tag{4.31}$$

Moreover, $g_M^{LF} \leq g_M^P$, i. e., the planner chooses a faster growth rate than under laissez-faire. This reflects an inter-temporal externality in the research process. An additional variety increases the future productivity of labor in research. However, this benefit is not reflected in the market prices of the laissez-faire equilibrium. Moreover, research is an input purchased by a sector endowed with monopoly power. This drives a wedge between the marginal social product of an input and its market price. Solving $g_M^P = g_M^*$ for ζ using (4.31) reveals that a policy mix satisfying

$$\sigma^P = 1 - \left(1 - \tau^P\right) \frac{\eta\mu L_Y^P}{L - (1-\eta) L_Y^P} \tag{4.32}$$

implements the planner's allocation in the decentralized economy with government activity. The fraction on the right-hand side is strictly smaller than unity. Therefore, $\sigma^P > 0$ is necessary to implement the planner's solution. As τ^P becomes positive and increases the equilibrium of the decentralized economy with government activity has a smaller growth rate of intermediate-good varieties. Then, σ^P also increases to support a constant g_M^P (see Proposition 3). The following proposition establishes that the implementation of the planner's allocation may imply a higher labor share than under laissez-faire.

Proposition 8 *(Functional Income Distribution - Planner's Allocation versus Laissez-Faire)*

Suppose $\gamma = \mu$ and $\delta = 0$. If $g_M^{LF} > 0$ then for any policy mix $\left(\sigma^P, \tau^P\right) \in (0,1)^2$ it holds that

$$\sigma^P > \frac{\gamma L}{a\rho}\tau^P \quad \Leftrightarrow \quad \alpha^P < \alpha^{LF}. \tag{4.33}$$

According to Proposition 8, a policy intervention that implements the planner's allocation with a sufficiently high subsidy rate reduces the capital share. Hence, it increases the labor share. The intuition comes in two steps. First, Proposition 7 informs us that a higher σ reduces the capital share whereas a higher τ increases it. Second, from (4.32) we know that the implementation of the planner's allocation requires $\sigma^P > 0$. Therefore, an implementation involving $\sigma^P = 1 - \gamma L_Y^P/L$ and $\tau^P = 0$ implies $\alpha^P < \alpha^{LF}$. As σ^P and τ^P increase further in accordance with (4.32), α^P also increases so that $\alpha^P \geq \alpha^{LF}$ holds for $\sigma^P \leq \gamma L/(a\rho)\tau^P$. Hence, policy mixes involving a high τ^P lift the capital share above its laissez-faire level.[15]

Proposition 8 also highlights that a violation of Piketty's Prediction 2 may be due to policy. Indeed, shifting an economy from the laissez-faire equilibrium to a first-best allocation using a policy mix that satisfies $\sigma^P < \gamma L/(a\rho)\tau^P$ implies faster growth and a higher capital share.

5 Demographic Growth

Since Piketty's g comprises demographic and technical change it is obvious that changes in demographic growth will affect the steady-state capital-income ratio and the capital share. The purpose of this section is to study this relationship in light of Prediction 1 and 2.

To incorporate population growth we extend the model of Section 3 along the lines suggested by Jones (1995). Accordingly, the representative household comprises $L(t) = L(0)\exp(tg_L) > 0$ members where $g_L \in \mathbb{R}_+$ is the instantaneous population growth rate. Moreover, the technology for the creation of new intermediate-good varieties is now given by

$$\dot{M} = \frac{L_M}{a} M^{\phi} l_M^{\lambda-1}, \quad \phi < 1, \quad 0 < \lambda \leq 1. \tag{5.1}$$

Here, l_M captures an externality due to duplication in the R&D process, and $l_M = L_M$ holds in equilibrium.[16]

Let g_J^* denote the steady-state growth rate of per-capita variables. The following proposition characterizes the steady state of this economy.

Proposition 9 *(Steady-State Equilibrium with Population Growth)*

[15] However, one readily verifies that no permissible policy mix can lift the after-tax capital share above its laissez-faire level, i. e., it always holds that $(1-\tau^P)\alpha^P < \alpha^{LF}$.

[16] The economy of this section is very close to the one for which Scrimgeour (2015) studies the effect of changes in tax rates on government revenue. While this author is not concerned with issues related to the factor income distribution his and our analytical settings differ only in the way how accounting profits and losses associated with changing share prices are treated for tax purposes. In Scrimgeour (2015) the tax on capital earnings applies also to accounting profits and losses whereas in our analysis it does not (compare the no-arbitrage condition of Scrimgeour (2015), p. 705, to our equation (3.10)).

Let $\phi < 1$, $0 < \lambda \leq 1$, and $g_L \geq 0$. Then, there exists a unique steady-state equilibrium if

$$\rho > (1-\theta)\left(g_J^* + g_L\right). \tag{5.2}$$

The steady-state growth rate of technological knowledge is

$$g_M^* = \frac{\lambda}{1-\phi} g_L. \tag{5.3}$$

The steady-state growth rate of per-capita variables is

$$g_J^* = \eta g_M^* \tag{5.4}$$

whereas economic aggregates grow at rate $g_J^* + g_L$. Moreover, it holds that

$$g_{vJ}^* = g_{\pi J}^* = -\left(\eta^{-1} - 1\right) g_J^* + g_L \gtreqless 0. \tag{5.5}$$

Condition (5.2) assures the transversality condition. The steady-state growth rate of technological knowledge in (5.3) follows immediately from the research technology (5.1). For the reason discussed in the context of Proposition 2 the steady state has $g_M^* > g_J^*$ whenever $\mu > \gamma$. Finally, equation (5.5) reveals that in steady state the share price and the dividend of intermediate-good firms need not decline even if $\mu > \gamma$. Intuitively, in the presence of positive population growth, $g_L > 0$, the turnover of intermediate-good producers increases as the market size for intermediates grows. This may even offset the tendency of a declining turnover arising from $g_M^* > g_J^*$.

5.1 The Capital-Income Ratio in the Long Run

Let β_J^* denote the steady-state capital-income ratio. By construction, it is still given by the right-hand side of (4.15) with g^* and δ_v^* being respectively replaced by g_J^* and $\delta_{vJ}^* = -g_{vJ}^*/(1-\tau)$. Since $g_L \to 0$ implies $g_J^* \to 0$ and $g_v^* \to 0$ we obtain with Proposition 4 that

$$\lim_{g_L \to 0} \beta_J^* = \lim_{L/a - \vartheta\rho \downarrow 0} \beta^* < \infty, \tag{5.6}$$

i. e., void of population growth the steady-state capital-income ratio is the one of the stationary economy, and, contrary to Prediction 1, finite.

The following proposition shows that the effect of population growth on the steady-state capital-labor ratio is ambiguous.[17]

[17] In addition to the population growth rate the steady-state capital-labor ratio as well as the capital share below depend on a broad set of model parameters. Here, we focus on the role of the population growth rate. See Irmen and Tabakovic (2016) for a comprehensive analysis of the effect of the remaining parameters on β_J^* and α_J^*.

Proposition 10 *(The Effect of Population Growth on β_J^*)*

There is $\varepsilon > 0$ such that $0 < g_L < \varepsilon$ and

$$\frac{d\beta_J^*}{dg_L} \gtreqless 0. \tag{5.7}$$

Hence, depending on the circumstances faster population growth may increase or decrease the long-run capital-income ratio. Since faster population growth unequivocally increases the growth rate of the economy this violates Prediction 1.

To provide an intuition for this result consider Piketty's second law from Proposition 6. Since now A grows at rate $g_A^* = g_J^* + g_L$ we obtain from $\dot{A} = sNDP$ that

$$\beta_J^* = \frac{s_J^*}{g_J^* + g_L}, \tag{5.8}$$

where s_J^* denotes the endogenous steady-state saving rate. Analogous to Proposition 6, we may express $s_J^* = s_J\left(g_L, g_J(g_L)\right)$. Moreover, with (5.3) and(5.4) we have $g_J^* = g_J(g_L)$. Hence, β_J^* results as

$$\beta_J^* = \frac{s_J\left(g_L, g_J(g_L)\right)}{g_J(g_L) + g_L}. \tag{5.9}$$

Then, the the total effect of changing g_L on β_J^* may be decomposed into a direct effect and two indirect effects via the saving rate and the per-capita growth rate. Formally, the decomposition is as follows:

$$\frac{d\beta^*}{dg_L} = \left(\frac{1}{g_J(g_L) + g_L}\right) \times \left[\left(\underbrace{\frac{\partial s_J\left(g_L, g_J(g_L)\right)}{\partial g_L} - \beta_J^*}_{(+)}\right) + \left(\underbrace{\frac{\partial s_J\left(g_L, g_J(g_L)\right)}{\partial g_J^*} - \beta_J^*}_{(-)}\right)\underbrace{\frac{\partial g_J(g_L)}{\partial g_L}}_{(+)}\right]$$

The first parenthesis in the bracket turns out to be positive. It shows the difference between the partial effect of g_L on s_J^* and its direct effect on β_J^*. The last product in brackets captures the effect of g_L on the capital-output ratio via the growth rate of per-capita variables and can be shown to be negative. As a consequence, the sign of $d\beta^*/dg_L$ is in general indeterminate which is inconsistent with Prediction 1.

5.2 The Capital Share in the Long Run

What is the effect of demographic growth on the steady-state capital share? Let α_J^* denote the steady-state capital share in the model with population growth. By construction α_J^* is given by the right-hand side of (4.26) where $\tilde{r}^*$ is replaced by $\tilde{r}_J^*$ and β^* by β_J^*. Hence, we may write

$$\alpha_J^* = \tilde{r}_J^* \times \beta_J^*.$$

Since $g_L \to 0$ implies $\delta_v^* = -g_{vJ}/(1-\tau) \to 0$ it becomes obvious with (5.6) that

$$\lim_{g_L \to 0} \alpha_J^* = \lim_{L/a - \vartheta\rho \downarrow 0} < \infty, \tag{5.10}$$

i. e., void of population growth the factor income distribution is the one of the stationary economy. Hence, contrary to Prediction 2 without growth the steady-state capital share remains finite.

Again, the total effect of a change in the population growth rate may be decomposed into a price effect and a volume effect as follows:

$$\frac{d\alpha_J^*}{dg_L} = \frac{d\tilde{r}_J^*}{dg_L}\beta_J^* + \frac{d\beta_J^*}{dg_L}\tilde{r}_J^* \gtreqless 0. \tag{5.11}$$

As seen above, g_L has an ambiguous effect on β_J^*, i. e., sign of the the volume effect may be positive or negative. Similarly, it is straightforward to derive that the effect of g_L on $\tilde{r}_J^*$, i. e., the price effect, is also ambiguous. Thus, we find that the total effect of demographic growth on the steady-state capital share is not unequivocal either.

Proposition 11 *(The Effect of Population Growth on the Capital Share)*

There is $\varepsilon > 0$ such that, if $0 < g_L < \varepsilon$, then

$$\frac{d\alpha_J^*}{dg_L} \quad \gtreqless \quad 0. \tag{5.12}$$

6 Concluding Remarks

According to David Ricardo the principal problem in Political Economy is to discover the laws which regulate the distribution of income (see Ricardo (1821), preface). Thomas Piketty's *Capital in the Twenty-First Century* (2014) is a forceful reminder of this assessment. More so, the author comprehensively documents the relevant empirical phenomena and presents two "fundamental laws of capitalism" that are meant to explain a large part of these stylized facts. Yet, are these laws what Ricardo had hoped for? Should we use these laws to formulate predictions about the future?

The present paper argues that a central weakness of Piketty's laws and the conclusions he draws from them is an endogeneity problem. The variables that explain the factor income distribution in the long run, namely, the real rate of return on assets, the economy's savings rate and its growth rate, are *all* endogenous variables. Therefore, in contrast to Piketty's predictions, what matters for the steady-state capital share is not that the economy's growth rate falls, what matters is why it falls. The cause of the decline in the economy's growth rate will affect the equilibrium of the economy as a whole, including its factor income distribution.

Our analysis identifies cases where the implications of Piketty's second law are violated. Due to an exogenous shock the steady-state capital-income ratio may well increase if the economy's growth rate increases. In spite of this violation, our analysis of Romer's model tends to confirm Piketty's assertion that slower long-run growth goes together with a greater capital share. However, as the comparison between the equilibrium under laissez-faire and the one that implements the planner's allocation demonstrates, a simultaneous use of two policy instruments may increase the economy's growth rate and its capital share. Moreover, it is worth mentioning that in the model with population growth slower long-run growth may be associated with a greater or a smaller capital share. On the whole, we conclude that neither Prediction 1 about the implications of the second law, nor Prediction 2 on the role of the growth rate for the capital share, should be uncritically used to forecast the capital-income ratio and the factor income distribution.

Clearly, there are important channels that our research does not touch upon even though they are likely to be relevant for the determination of the factor income distribution in the long run. For instance, our analysis is mute on the role of housing as an important determinant of the share of capital in net income (see, e. g., Bonnet, Bono, Chapelle, and Wasmer (2014), Rognlie (2015), Grossmann and Steger (2016)). It also neglects the role of wage bargaining as opposed to marginal product pricing or open economy issues. At a more technical level, our findings rely on a Cobb-Douglas production function of the final-good sector. Therefore, the factor shares for capital, technological knowledge, and industrial labor in final-good production are constant (see Proposition 1). This raises the question of how our qualitative findings would change under a more general production function allowing for an elasticity of substitution between the composite of all intermediates and industrial labor different from unity.[18] An acceptable solution to this problem

[18]A problem that a generalization along these lines faces is that a steady-state path with a positive rate of technical change can no longer exist since technical change is "capital-augmenting." In an economy where capital accumulates and the aggregate production function of the final-good has constant returns to scale in

may also open the gate to a reasonable quantitative analysis of the model's implications.

Overall, our results suggest that technology, preferences, policy, demographics, and market structure shape the factor income distribution. Yet, we concur with Blume and Durlauf (2015) and others that these dimensions are themselves endogenous and determined, e. g., by advances in scientific and medical knowledge (Fogel (2004)) or by institutional changes (Acemoglu and Robinson (2014)). The development of a comprehensive understanding of the laws that govern the distribution of income must also take these features into account.

References

ACEMOGLU, D., AND J. A. ROBINSON (2014): "The Rise and Decline of General Laws of Capitalism," *Journal of Economic Perspectives*, 29(1), 3–28.

BERTOLA, G. (1993): "Factor Shares and Savings in Endogenous Growth," *American Economic Review*, 83, 1184–1198.

BERTOLA, G., R. FOELLMI, AND J. ZWEIMÜLLER (2006): *Income Distribution in Macroeconomic Models*. Princeton University Press, Princeton.

BLUME, L. E., AND S. N. DURLAUF (2015): "Capital in the Twenty-First Century: A Review Essay," *Journal of Political Economy*, 123(4), 749 – 777.

BONNET, O., P.-H. BONO, G. CHAPELLE, AND E. WASMER (2014): "Does Housing Capital Contribute to Inequality? A Comment on Thomas Piketty's Capital in the 21st Century," Sciences Po Economics Discussion Papers 2014-07, Sciences Po Departement of Economics.

BRIDGMAN, B. (2014): "Is Labor's Loss Capital's Gain? Gross versus Net Labor Shares," BEA Working Papers 0114, Bureau of Economic Analysis.

CASS, D. (1965): "Optimum Growth in an Aggregative Model of Capital Accumulation," *The Review of Economic Studies*, 32, 233–240.

DAS, M. (2003): "Optimal Growth with Decreasing Marginal Impatience," *Journal of Economic Dynamics and Control*, 27(10), 1881–1898.

ELSBY, M., B. HOBIJN, AND A. SAHIN (2013): "The Decline of the U.S. Labor Share," *Brookings Papers on Economic Activity*, 47, 1–63.

---

capital and labor, Uzawa's theorem applies so that there can be no capital-augmenting technical change in steady state (see, e. g., Uzawa (1961) or Irmen (2018)).

FOGEL, R. W. (2004): *The Escape from Hunger and Premature Death, 1700 - 2100*. Cambridge University Press, Cambridge, UK.

FRANKEL, M. (1962): "The Production Function in Allocation and Growth: A Synthesis," *American Economic Review*, 52(5), 995–1022.

GALOR, O., AND ÖMER ÖZAK (2016): "The Agricultural Origins of Time Preference," *American Economic Review*, 106(10), 3064–3103.

GROSSMANN, V., AND T. STEGER (2016): "Das House-Kapital: A Theory of Wealth-to-Income Ratios," CESifo Working Paper 5844, CESifo Munich.

GROWIEC, J., P. MCADAM, AND J. MUCK (2018): "Endogenous Labor Share Cycles: Theory and Evidence," *Journal of Economic Dynamics and Control*, 87(C), 74–93.

HOMBURG, S. (2015): "Critical Remarks on Piketty's Capital in the Twenty-First Century," *Applied Economics*, 47(14), 1401–1406.

IRMEN, A. (2018): "A Generalized Steady-State Growth Theorem," *Macroeconomic Dynamics*, 22, 779–804.

IRMEN, A., AND A. TABAKOVIC (2016): "Factor Income Distribution and Endogenous Economic Growth - When Piketty Meets Romer -," CESifo Working Paper Series 6217, CESifo Group Munich.

JONES, C. I. (1995): "R&D-Based Models of Economic Growth," *Journal of Political Economy*, 103(4), 759–784.

KARABARBOUNIS, L., AND B. NEIMAN (2014a): "Capital Depreciation and Labor Shares Around the World: Measurement and Implications," NBER Working Papers 20606, National Bureau of Economic Research, Inc.

——— (2014b): "The Global Decline of the Labor Share," *The Quarterly Journal of Economics*, 129(1), 61–103.

KOOPMANS, T. (1965): "On the Concept of Optimal Economic Growth," in *(Study Week on) The Economic Approach to Development Planning*, ed. by P. A. S. S. Varia, pp. 225–287. North Holland, Amsterdam.

KRUSELL, P., AND A. A. SMITH (2015): "Is Piketty's "Second Law of Capitalism" Fundamental?," *Journal of Political Economy*, 123(4), 725–748.

PIKETTY, T. (2014): *Capital in the Twenty-First Century*. Harvard University Press, Cambridge, MA.

PIKETTY, T., AND E. SAEZ (2014): "Inequality in the Long Run," *Science*, 344(6186), 838–843.

RAMSEY, F. P. (1928): "A Mathematical Theory of Savings," *The Economic Journal*, 38, 543–559.

RAY, D. (2015): "*Nit-Piketty*: A Comment on Thomas Piketty's *Capital in the Twenty First Century*," *CESifo Forum*, 16(1), 19–25.

REBELO, S. (1991): "Long-Run Policy Analysis and Long-Run Growth," *Journal of Political Economy*, 99, 500–521.

RICARDO, D. (1821): *On The Principles of Political Economy and Taxation*. John Murray, London, 3rd edn.

ROGNLIE, M. (2015): "Deciphering the Fall and Rise in the Net Capital Share: Accumulation or Scarcity?," *Brookings Papers on Economic Activity*, pp. 1–69.

ROMER, P. M. (1986): "Increasing Returns and Long-Run Growth," *Journal of Political Economy*, 94, 1002–1037.

SCRIMGEOUR, D. (2015): "Dynamic Scoring in a Romer-Style Economy," *Southern Economic Journal*, 81(3), 697–723.

SOLOW, R. (2014): "Thomas Piketty Is Right," *The New Republic, April 22, 2014*.

SOLOW, R. M. (1956): "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, 70(1), 65–94.

UZAWA, H. (1961): "Neutral Inventions and the Stability of Growth Equilibrium," *The Review of Economic Studies*, 28(2), 117–124.

WEBER, M. (1930): *The Protestant Ethic and the Spirit of Capitalism*. reprinted by Routledge Classic 2001, London.