Removing the saturation assumption in Bank-Weiser error estimator analysis in dimension three

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Abstract

We provide a new argument proving the reliability of the Bank-Weiser estimator for Lagrange piecewise linear finite elements in both dimension two and three. The extension to dimension three constitutes the main novelty of our study. In addition, we present a numerical comparison of the Bank-Weiser and residual estimators for a three-dimensional test case.

Keywords: finite element methods; a posteriori error estimation; saturation assumption; Bank-Weiser estimator; residual estimator
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Introduction

The Bank-Weiser error estimator was introduced in [2]. This seminal work contains a proof that the Bank-Weiser estimator is both efficient and reliable —i.e. it is both a lower and an upper bound of the error— without any restriction on the dimension or on the finite elements order. However, the argument for the upper bound was based on a fragile saturation assumption known to be tricky to assert in practice [5]. The saturation assumption was successfully removed from the upper bound proof in [8] in the case of linear finite elements in dimension two, introducing the additional term referred to as the "data oscillation". An extension of this proof to dimension three does not seem immediate although it is mentioned in the text. In particular, the proof uses the fact that Verfurth's bubble functions [10] on edges are quadratic polynomial in dimension two, which is no longer the case in dimension three. In this work, we propose a new proof that is valid both in dimensions two and three. In addition, we provide a short numerical study comparing Bank-Weiser and residual error estimators on a three dimensional test case.

1. Model problem and finite element discretization

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded domain with polygonal or polyhedral boundary $\partial\Omega$. For any subdomain $\omega \subset \Omega$ (resp. $(d-1)$-dimensional set ω), we denote by $|\omega|$ the d-dimensional (resp. $(d-1)$-dimensional) measure of ω. On the domain Ω we consider the usual functions spaces $L^2(\Omega)$ and $H^1_0(\Omega)$. The respective usual norms will be denoted $\|\cdot\|_\omega$ for $L^2(\omega)$, $\omega \subset \Omega$ and $\|\nabla\cdot\|$ for $H^1_0(\Omega)$. For the sake of simplicity, we consider the Poisson equation with homogeneous Dirichlet boundary condition

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

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with given $f \in L^2(\Omega)$. The weak form of this problem reads: find u in $H^1_0(\Omega)$ such that for any v in $H^1_0(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v. \tag{2}$$

We discretize this problem using Lagrange piecewise linear continuous finite elements. To do so, we introduce a conformal triangulation $\mathcal{T}$ on Ω composed of triangles (resp. tetrahedrons) for $d = 2$ (resp. $d = 3$) hereafter called *cells*. We assume that the triangulation $\mathcal{T}$ is regular in the sense of $h_T/\rho_T \leq \gamma$, $\forall T \in \mathcal{T}$, where h_T is the diameter of a cell T, ρ_T the diameter of its inscribed ball, and γ is positive constant fixed once and for all. For a cell $T \in \mathcal{T}$ and a non-negative integer p, we denote $\mathcal{P}_p(T)$ the set of polynomial functions of degree less than p on T and introduce the spaces of discontinuous and continuous Lagrange finite elements of order p:

$$V^{p,dG} := \{v_p \in \mathcal{P}_p(T),\ \forall T \in \mathcal{T},\ v_p = 0 \ \text{ on } \partial\Omega\}\,, \quad V^p := V^{p,dG} \bigcap H^1_0(\Omega).$$

The finite element approximation to problem (2) is: find $u_1 \in V^1$ such that

$$\int_\Omega \nabla u_1 \cdot \nabla v_1 = \int_\Omega f v_1,\ \forall v_1 \in V^1. \tag{3}$$

We now introduce some more notations needed in what follows. We call *facets* the edges of cells in $\mathcal{T}$ if $d = 2$ and the faces of cells in $\mathcal{T}$ if $d = 3$. The notion of triangulation *edges* will also be important in dimension $d = 3$. We recall that a facet $T \in \mathcal{T}$ is a triangle in this case and its boundary consists of 3 sides, called edges. The set of all interior facets of $\mathcal{T}$ is denoted by $\mathcal{F}$ and the set of all the internal vertices of $\mathcal{T}$ is denoted by $\mathcal{X}$. Finally, we use the letter C for various constants that depend only on the triangulation regularity parameter γ and are allowed to change from one occurence to another.

2. A posteriori error estimators

Let $\mathcal{I} : V^{2,dG} \to V^{1,dG}$ be the cell by cell Lagrange interpolation operator. The first step in the definition of the Bank-Weiser estimator is to introduce the finite element space $V^{\mathrm{bw}} := \ker(\mathcal{I}) = \left\{v_2 \in V^{2,dG},\ \mathcal{I}(v_2) = 0\right\}$. By definition, the functions of V^{bw} are piecewise quadratic polynomials on the triangulation that vanish at the vertices. Let e_{bw} in V^{bw} be the solution to

$$\sum_{T\in\mathcal{T}} \int_T \nabla e_{\mathrm{bw}} \cdot \nabla v_{\mathrm{bw}} = \sum_{T\in\mathcal{T}} \int_T f v_{\mathrm{bw}} + \sum_{F\in\mathcal{F}} \int_F J_F \left\{v_{\mathrm{bw}}\right\} \quad \forall v_{\mathrm{bw}} \in V^{\mathrm{bw}}, \tag{4}$$

where $J_F := \left[\!\left[\dfrac{\partial u_1}{\partial n}\right]\!\right]$ is the jump of the normal derivative of u_1 on F and $\{\cdot\}$ denotes the average across edges. More precisely, the jump is defined as $J_F = (\nabla u_1|_{T_{F,2}} - \nabla u_1|_{T_{F,1}}) \cdot n$ where $T_{F,1}, T_{F,2}$ are the triangulation cells sharing the facet F and n is the unit normal directed from $T_{F,1}$ to $T_{F,2}$. The Bank-Weiser estimator is then defined as

$$\eta_{\mathrm{bw}}{}^2 := \sum_{T\in\mathcal{T}} \|\nabla e_{\mathrm{bw}}\|^2_T. \tag{5}$$

We also recall the explicit residual error estimator

$$\eta_{\mathrm{res}}{}^2 := \sum_{T\in\mathcal{T}} h_T^2 \|f\|^2_T + \sum_{F\in\mathcal{F}} h_F \|J_F\|^2_F. \tag{6}$$

and the data oscillation indicator

$$\mathrm{osc}^2(f) := \sum_{T\in\mathcal{T}} h_T^2 \|f - f_T\|^2_T, \tag{7}$$

with $h_T = \operatorname{diam} T$ and $f_T = \frac{1}{|T|}\int_T f$.

The following theorem establishes the equivalence of the two error estimators (5) and (6) modulo a data oscillation term. This proves the equivalence of the Bank-Weiser estimator to the true error $\|\nabla u - \nabla u_1\|_\Omega$ since such an equivalence is well known to hold for the residual estimator [10].

Theorem 1. *Let u_1 be the solution of (3) for a regular triangulation $\mathcal{T}$. Let $\eta_{\rm bw}$ and $\eta_{\rm res}$ be the Bank-Weiser and explicit residual estimators defined respectively in (5) and (6). Let $\operatorname{osc}(f)$ be the oscillation of f defined in (7). Then, there exists two constants c and C only depending on the triangulation regularity such that*

$$\eta_{\rm bw} \leqslant c\,\eta_{\rm res}, \qquad \text{(8a)} \qquad\qquad \eta_{\rm res} \leqslant C\big(\eta_{\rm bw} + \operatorname{osc}(f)\big). \qquad \text{(8b)}$$

Proof. The arguments to prove (8a) can be found in [10], applied to a slightly different version of the Bank-Weiser estimator. These arguments consist in putting $v_{\rm bw} = e_{\rm bw}$ in (5) and noting (by scaling and equivalence of norms) that $\|v_{\rm bw}\|_T \leqslant C h_T \|\nabla v_{\rm bw}\|_T$ and $\|v_{\rm bw}\|_F \leqslant C\sqrt{h_T}\|\nabla v_{\rm bw}\|_T$ for any $v_{\rm bw} \in V^{\rm bw}$, $T \in \mathcal{T}$ and F a facet of T.
The proof of (8b) essentially proceeds in five steps.
Step 1) Substracting (3) from (2) and integrating by parts we get

$$\sum_{T\in\mathcal{T}} \int_T f v_1 + \sum_{F\in\mathcal{F}} \int_F J_F v_1 = \int_\Omega \nabla(u-u_1)\cdot\nabla v_1 = 0, \quad \forall v_1 \in V^1. \tag{9}$$

Taking v_2 in V^2 (continuous) and denoting $v_1 = \mathcal{I}(v_2)$, v_1 belongs to V^1 and can be used in (9). In addition, $v_2 - \mathcal{I}(v_2)$ belongs to $V^{\rm bw}$ and can be used in (4) to get

$$\begin{aligned}\int_\Omega \nabla e_{\rm bw}\cdot\nabla\big(v_2 - \mathcal{I}(v_2)\big) &= \sum_{T\in\mathcal{T}}\int_T f\,(v_2 - \mathcal{I}(v_2)) + \sum_{F\in\mathcal{F}}\int_F J_F\,(v_2 - \mathcal{I}(v_2)) \\ &= \sum_{T\in\mathcal{T}}\int_T f v_2 + \sum_{F\in\mathcal{F}}\int_F J_F v_2, \quad \forall v_2 \in V^2\end{aligned} \tag{10}$$

by linearity of the right hand side and (9).
Step 2) For any vertex $x \in \mathcal{X}$, one can construct $\psi_x \in V^2$ such that $\psi_x(x) = 1$, $\psi_x = 0$ outside of the patch ω_x of triangulation cells sharing x and

$$\int_F \psi_x = 0 \quad \forall F \in \mathcal{F}. \tag{11}$$

In dimension $d = 3$, we can simply take $\psi_x = \phi_x$ the shape function of V^2 associated to x and thus vanishing on the edges midpoints. Indeed, denoting by $\mathcal{M}_F$ the set of edge midpoints on the facet F (a triangle in this case) we recall that the quadrature rule $\int_F v = \frac{1}{3}|F|\sum_{m\in\mathcal{M}_F} v(m)$ is exact on polynomials of degree lower than two. In dimension $d = 2$, we take $\psi_x = \phi_x - \frac{1}{4}\sum_{m\in\mathcal{M}_x}\phi_m$, where ϕ_x is again the shape function of V^2 associated to x, $\mathcal{M}_x$ is the set of midpoints of the edges sharing x, and ϕ_m are the shape functions of V^2 associated to these midpoints. The equality (11) is checked for this ψ_x by applying Simpson's quadrature rule on the edges.

Using (11) and the fact that J_F is constant over any facet F, (10) with $v_2 = \psi_x$ is reduced to

$$\int_{\omega_x} \nabla e_{\rm bw}\cdot\nabla\big(\psi_x - \mathcal{I}(\psi_x)\big) = \int_{\omega_x} f\psi_x = \sum_{T\in\omega_x} f_T\int_T\psi_x + \sum_{T\in\omega_x}\int_T (f - f_T)\psi_x.$$

Reordering the terms and using both Cauchy-Schwarz and triangle inequalities give

$$\left|\sum_{T\in\omega_x} f_T \int_T \psi_x\right| \leqslant \|\nabla e_{\rm bw}\|_{\omega_x}\|\nabla(\psi_x - \mathcal{I}(\psi_x))\|_{\omega_x} + \sum_{T\in\omega_x}\left|\int_T (f - f_T)\psi_x\right|. \tag{12}$$

Using e.g. quadrature rules for any T in ω_x we can compute $\int_T \psi_x = -\frac{1}{6}|T|$ in dimension $d = 2$ and $\int_T \psi_x = -\frac{1}{20}|T|$ in dimension $d = 3$. Applying the Cauchy-Schwarz inequality twice we also get

$$\sum_{T\in\omega_x}\left|\int_T (f - f_T)\psi_x\right| \leqslant \left(\sum_{T\in\omega_x}\|f - f_T\|_T^2\right)^{1/2}\|\psi_x\|_{\omega_x}.$$

Moreover, continuity of $\mathrm{id} - \mathcal{I}$, Poincaré's inequality as well as a scaling argument give

$$\left\|\nabla\big(\psi_x - \mathcal{I}(\psi_x)\big)\right\|_{\omega_x} \leqslant \frac{C}{h_x}\sqrt{|\omega_x|} \quad \text{and} \quad \|\psi_x\|_{\omega_x} \leqslant C\sqrt{|\omega_x|},$$

where h_x is the size of the longest edge in ω_x. Then, we finally get

$$\left|\sum_{T\in\omega_x} |T| f_T\right| \leqslant C\left(\frac{1}{h_x}\|\nabla e_{\mathrm{bw}}\|_{\omega_x} + \left(\sum_{T\in\omega_x}\|f - f_T\|_T^2\right)^{1/2}\right)\sqrt{|\omega_x|}. \tag{13}$$

Step 3) Now, for any facet $F \in \mathcal{F}$ and any cell $T \in \mathcal{T}$ such that $F \subset \partial T$, one can construct $\psi_{F,T} \in V^{\mathrm{bw}}$ such that $\psi_{F,T} = 1$ at all the edge midpoints on F (one midpoint if $d = 2$ and 3 midpoints if $d = 3$), $\psi_{F,T} = 0$ outside of T and

$$\int_{F'} \psi_{F,T} = 0 \quad \forall F' \in \mathcal{F},\ F' \neq F. \tag{14}$$

In dimension $d = 2$, we take $\psi_{F,T}$ as the usual bubble function associated to the facet F setting $\psi_{F,T} = 0$ on all the facets of T other than F. In dimension $d = 3$, we put $\psi_{F,T}(m) = -\frac{1}{2}$ if m is the midpoint of any edge of T that does not belong to F. Using once again the quadrature rule on triangles $\int_{F'} \psi_{F,T} = \frac{1}{3}|F'|\sum_{m\in\mathcal{M}_{F'}} \psi_{F,T}(m)$, we can check that this construction does the job.

Now we consider any facet $F \in \mathcal{F}$, denote $T_{F,1}$, $T_{F,2}$ the two adjacent triangulation cells and take $v_{\mathrm{bw}} = \psi_{F,T_{F,1}} - \psi_{F,T_{F,2}}$ in (4). The integral of the average $\{v_{\mathrm{bw}}\}$ then vanishes on all the facets and we get

$$\int_{T_{F,1}} \nabla e_{\mathrm{bw}}\cdot\nabla\psi_{F,T_{F,1}} - \int_{T_{F,2}} \nabla e_{\mathrm{bw}}\cdot\nabla\psi_{F,T_{F,2}} = \int_{T_{F,1}} f\psi_{F,T_{F,1}} - \int_{T_{F,2}} f\psi_{F,T_{F,2}}. \tag{15}$$

Introducing the average of f on cells and reordering the terms give

$$\begin{aligned} f_{T_{F,1}}\int_{T_{F,1}}\psi_{F,T_{F,1}} - f_{T_{F,2}}\int_{T_{F,2}}\psi_{F,T_{F,2}} \quad &= \quad \int_{T_{F,1}} \nabla e_{\mathrm{bw}}\cdot\nabla\psi_{F,T_{F,1}} - \int_{T_{F,2}} \nabla e_{\mathrm{bw}}\cdot\nabla\psi_{F,T_{F,2}} \\ &\quad + \int_{T_{F,1}} \left(f_{T_{F,1}} - f\right)\psi_{F,T_{F,1}} - \int_{T_{F,2}} \left(f_{T_{F,2}} - f\right)\psi_{F,T_{F,2}}. \end{aligned}$$

Using quadrature rules in dimensions $d = 2$ and $d = 3$ give respectively $\int_T \psi_{F,T} = \frac{2}{3}|T|$ and $\int_T \psi_{F,T} = \frac{3}{10}|T|$. In addition, by Poincaré's inequality and scaling arguments we have

$$\|\nabla\psi_{F,T}\|_T \leqslant \frac{C}{h_T}\sqrt{|T|}, \quad \text{and} \quad \|\psi_{F,T}\|_T \leqslant C\sqrt{|T|}. \tag{16}$$

By the precedent quadrature computations, (16) and Cauchy-Schwarz inequality we get

$$\big||T_{F,1}|f_{T_{F,1}} - |T_{F,2}|f_{T_{F,2}}\big| \leqslant C\left(\frac{1}{h_F}\|\nabla e_{\mathrm{bw}}\|_{\omega_F} + \left(\sum_{T\in\omega_F}\|f - f_T\|_T^2\right)^{1/2}\right)\sqrt{|\omega_F|} \tag{17}$$

with $\omega_F = T_{F,1} \cup T_{F,2}$. Now, if we denote by $\mathcal{F}_x$ the set of all facets having the vertex x in common and if we consider the finite dimensional vectorial space $E_x := \{(a_T)_{T\in\omega_x}\} = \mathbb{R}^{\#\omega_x}$, the following applications $n_1(a) := \sum_{T\in\omega_x}|a_T|$ and $n_2(a) := \left|\sum_{T\in\omega_x} a_T\right| + \sum_{F\in\mathcal{F}_x}|a_{T_{F,1}} - a_{T_{F,2}}|$, define norms on E_x. Then, using norm equivalence in finite dimension as well as the regularity of the triangulation, we prove the existence of a constant C only depending on triangulation regularity such that

$$\sum_{T\in\omega_x}|T||f_T| \leqslant C\left(\left|\sum_{T\in\omega_x}|T|f_T\right| + \sum_{F\in\mathcal{F}_x}\big||T_{F,1}|f_{T_{F,1}} - |T_{F,2}|f_{T_{F,2}}\big|\right). \tag{18}$$

Step 4) We can now bound the right-hand side of (18) using (13) and (17) to get, for any node $x \in \mathcal{X}$,

$$\sum_{T\in\omega_x} |T||f_T| \leqslant C\left(\frac{1}{h_x}\|\nabla e_{\mathrm{bw}}\|_{\omega_x} + \left(\sum_{T\in\omega_x}\|f - f_T\|_T^2\right)^{1/2}\right)\sqrt{|\omega_x|}\,. \tag{19}$$

Taking the square of (19) and using triangulation regularity, the fact that f_T is constant over the cells and convexity of the square yields

$$\sum_{T\in\omega_x} h_T^2\|f_T\|_T^2 \leqslant C\left(\|\nabla e_{\mathrm{bw}}\|_{\omega_x}^2 + \sum_{T\in\omega_x} h_T^2\|f - f_T\|_T^2\right).$$

Summing this over all the vertices and applying once more the triangle inequality as well as triangulation regularity leads to

$$\sum_{T\in\mathcal{T}} h_T^2\|f\|_T^2 \leqslant C\left({\eta_{\mathrm{bw}}}^2 + \mathrm{osc}^2(f)\right). \tag{20}$$

Step 5) It remains to bound the edge term of the residual estimator (6). To this end, we use the functions $\psi_{F,T} \in V^{\mathrm{bw}}$ again. For any facet $F \in \mathcal{F}$, take $v_{\mathrm{bw}} = \psi_{F,T_{F,1}} + \psi_{F,T_{F,2}}$ in (4) to get

$$\left|J_F\int_F v_{\mathrm{bw}}\right| \leqslant \sum_{T\in\omega_F}\left|\int_T \nabla e_{\mathrm{bw}}\cdot\nabla v_{\mathrm{bw}}\right| + \sum_{T\in\omega_F}\left|\int_T f v_{\mathrm{bw}}\right|.$$

Then, by the same quadrature rules as before we have for $T = T_{F,1},\ T_{F,2}$, $\int_F \psi_{F,T} = \frac{2}{3}|F|$ in dimension $d = 2$ and $\int_F \psi_{F,T} = |F|$ in dimension $d = 3$. In addition, Cauchy-Schwarz inequality as well as (16) give

$$|F||J_F| \leqslant C\left(\frac{1}{h_F}\|\nabla e_{\mathrm{bw}}\|_{\omega_F} + \|f\|_{\omega_F}\right)\sqrt{|\omega_F|}$$

so that

$$h_F\|J_F\|_F^2 \leqslant C(\|\nabla e_{\mathrm{bw}}\|_{\omega_F} + h_F^2\|f\|_{\omega_F}^2). \tag{21}$$

Summing this over all the facets and combining with (20) we get (8b). □

3. Numerical results

We consider a three-dimensional domain Ω with a L-shaped polyhedral boundary, $\Omega := (-0.5, 0.5)^3 \setminus ([0, -0.5] \times [0, -0.5] \times [-0.5, 0.5])$. We solve (1) with f chosen in order to get the following analytical solution defined on Ω and given, in cylindrical coordinates, by $u(r, \theta, z) = \phi(r, \theta, z)r^{2/3}\sin\left(\frac{2\theta}{3}\right)$, where ϕ is a polynomial cut-off function defined (in cartesian coordinates) by $\phi(x, y, z) = 0.25^{-6}(0.25 - x^2)^2(0.25 - y^2)^2(0.25 - z^2)^2$. This solution belongs to $H^{5/3-\varepsilon}(\Omega)$ for any $\varepsilon > 0$ and its gradient admits a singularity along the re-entrant edge [7]. We consider here two adaptive refinement algorithms respectively driven by the Bank-Weiser estimator and the residual estimator. Each of these algorithms works as follow: 1) The primal problem (3) is discretised using piecewise linear Lagrange finite elements. 2) The error is measured using implementations of the estimator (either Bank-Weiser or residual) in the FEniCS Project [1]. 3) The triangulation is marked according to the local contributions of the estimator and using the Dörfler marking strategy. The marking strategy consists in finding the smallest subset $\mathcal{M}$ in $\mathcal{T}$ such that, $\sum_{T\in\mathcal{M}}\eta_T^2 \geqslant \theta^2\eta^2$ where $\eta = \eta_{\mathrm{bw}}$ or η_{res} and $(\eta_T)_{T\in\mathcal{T}}$ are the local contributions of the estimator, for a given parameter $\theta \in (0, 1)$. Here, we chose the value $\theta = 0.5$ [4] in both cases. 4) Finally, we refine the triangulation using the Plaza-Carey algorithm present in FEniCS [9]. Details on the implementations can be found in [3]. On the left hand side of Fig. 1 we can see the final triangulation obtained with the Bank-Weiser driven algorithm after four refinement steps. We notice that, as expected, strong refinement occurs near the re-entrant corner edge around the origin. The choice of the a posteriori error estimator does not have a strong influence on

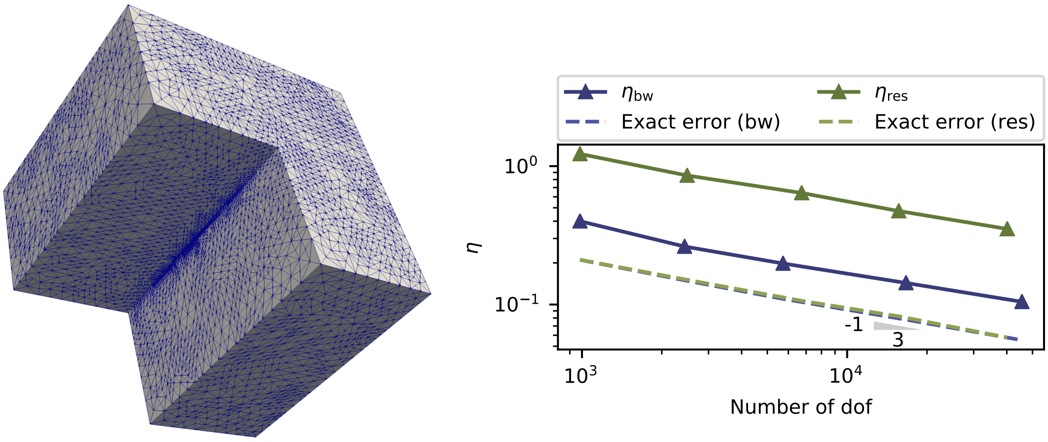

Figure 1: On the left: The triangulation after four refinement steps of the adaptive algorithm driven by the Bank-Weiser estimator. On the right: The convergence curves for the estimator and exact error for each adaptive refinement algorithm.

the mesh hierarchy. On the right of Fig. 1, the convergence curves of the estimator and the corresponding exact error are plotted for both refinement algorithms. The approximated rates of convergence (estimated with least squares) are, in the case of the Bank-Weiser driven adaptive algorithm, -0.33 for the exact error and -0.31 for the Bank-Weiser estimator and, in the case of the residual driven adaptive algorithm, -0.34 for the exact error and -0.33 for the residual estimator. In the case of the Bank-Weiser driven adaptive algorithm, we have also computed the residual estimator on the same mesh hierarchy in order to compare estimators efficiencies $\eta/\|\nabla(u-u_h)\|$ for $\eta = \eta_{\mathrm{bw}}$ or η_{res}. Estimators efficiencies on the last refinement step are respectively 1.9 for the Bank-Weiser estimator and 5.9 for the residual estimator. We can notice that the Bank-Weiser estimator is much sharper than the residual estimator but it is not as sharp as for two-dimensional problems (see e.g. [2]). The fact that the Bank-Weiser estimator is not asymptotically exact for non-structured meshes is known and was proved in [6].

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