From Classical to Non-Monotonic Deontic Logic using ASPIC$^+$ ⋆

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Abstract. In this paper we use formal argumentation to design non-monotonic deontic logics, based on two monotonic deontic logics. In particular, we use the structured argumentation theory ASPIC$^+$ to define non-monotonic variants of well-understood modal logics. We illustrate the approach using argumentation about free-choice permission.

1 Using ASPIC$^+$ to design non-monotonic deontic logics

Deontic logic is the logic of obligation, prohibition and permission [7, 18]. Many axioms of deontic logic have been criticised, and non-monotonic techniques have been applied widely [11, 20, 23, 17, 21, 3]. In this paper we consider the use of so-called ASPIC$^+$ to design deontic argumentation systems and non-monotonic deontic logics and, in particular, to study strong and free-choice permission [9].

Modgil and Prakken [15] observe that "in ASPIC$^+$ and its predecessors, going back to the seminal work of John Pollock, arguments can be formed by combining strict and defeasible inference rules and conflicts between arguments can be resolved in terms of a preference relation on arguments. This results in abstract argumentation frameworks (a set of arguments with a binary relation of defeat), so that arguments can be evaluated with the theory of abstract argumentation." In this paper, we use argumentation systems to define non-monotonic logics. Our ASPIC$^+$-based methodology consists of three steps.

Arguments We take literally Modgil and Prakken's idea that "Rule-based approaches in general do not adopt a single base logic but two base logics, one for the strict and one for the defeasible rules" [15]. We use monotonic modal logics as our base logics with Hilbert-style proof theory.

Strict arguments use only strict rules defined in terms of a "lower bound" logic, in the sense that it defines the minimal inferences which must be made. We use a variant of Von Wright's standard deontic logic [24].

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Defeasible arguments use also defeasible rules defined in terms of an "upper bound" logic in the sense that it defines all possible inferences that can be made. We use a variant of Van Benthem's logic of strong permission [4].

Preferences among arguments can be generic or depend on the logical languages used to build the arguments. We focus on **Argument types** defined in ASPIC$^+$ which distinguish between defeasible and plausible arguments.

Nonmonotonic inference relations can be based on skeptical or credulous relation, and on one of the argumentation semantics.

The layout of this paper is as follows. We first introduce the running example of this paper. Then we introduce the monotonic deontic logics, and we use the logics to define ASPIC$^+$ argumentation systems. Finally we define the non-monotonic deontic logics in terms of the argumentation systems.

2 Running example: free-choice permission

There are many ways in which the relation between obligation and permission has been defined. For example, in some papers permission is used to define exceptions to general obligations and prohibitions, and in such approaches, permission overrides obligation [13]. In other approaches, we can see examples where obligations and prohibitions override permissions [4, 10]. For example, the general norm that product placement in TV programs is strongly permitted, is overridden by the particular case that product placement is forbidden in children programs. In this paper we work with an example where this latter is the case.

In particular, we take standard deontic logic without weak permission as our logic for strict rules, and for the defeasible rules we use an extension of this logic with strong permission, proposed by Van Benthem [4]. We consider three combinations of monotonic deontic logics and three ways to define the preferences, and we only consider stable semantics. So we define six non-monotonic deontic logics in this paper.

In Van Benthem's logic, what is obligated is the necessary condition of being ideal, while what is permitted is the sufficient condition for ideality. An intuitive example is the so-called "free-choice permission" [10]. If having a tea or having a coffee is permitted, then free-choice implies that both cases are permitted. Here we consider the following example in legal reasoning, and see that in what sense of non-monotonicity we say a free-choice permission holds or not.

1. It is permitted to freely use any of your property, for example, a knife.
2. It is forbidden to murder.

The question now is, is it permitted to use your knife to kill someone? The solution we adopt is that it is permitted to use the knife in normal situations, in sense of being non-defeated in ASPIC$^+$. So we can derive that (normally):

3. Knifes are not used to murder.

If we now add the information that "The knife is used to murder", or "The knife can be used to murder", and in addition we *prefer* this statement over the previous ones, then we would expect no longer to derive that (normally) knifes are not used for murder, and neither we would derive that knifed murder is permitted. However, we would still expect to predict that, for example, knifes can be used to cut the bread, because *no more preferable* argument for the contrary exists.

From a formal point of view, the problem of free-choice permission we focus on in this paper is the derivation of $P(\phi \wedge \psi)$ from $P\phi$. It has been observed by Glavaničová [8] that this is a strong rule which should not hold in case it leads to inconsistency. We adopt ASPIC$^+$ to explain it.

Example 1 (Knifed murder). Our aim is to define a logic such that the defeasible permission to use the knife, Pk, can infer the permission to cut the bread with the knife, $P(k \wedge b)$, but in some exceptional cases, for instance in case of murder, $\{Pk, O\neg m\}$ we cannot infer $P(k \wedge m)$. In the latter case, without the prohibition and in analogy with cutting the bread, the logic also derives $P(k \wedge m)$ from Pk only, and thus the logic is non-monotonic.

Each level of our approach can be analysed using the methods of that discipline, i.e. monotonic logic (e.g. possible world semantics), argumentation theory can be studied using rationality postulates [6], and non-monotonic inference can be analysed using, for example, the approach advocated by Kraus *et al.* [12].

3 Step 1: arguments based on two monotonic logics

We use two monotonic logics to define the strict and defeasible rules of ASPIC$^+$, and use the crude approach to define arguments [15]: "A crude way is to simply put all valid propositional (or first-order) inferences over your language of choice in [the strict rules] R_s. So if a propositional language has been chosen, then R_s can be defined as follows (where $\vdash_{PL}$ denotes standard propositional-logic consequence). For any finite $S \subseteq \mathcal{L}$ and any $\phi \in \mathcal{L}$: $S \rightarrow \phi \in R_s$ if and only if $S \vdash_{PL} \phi$." This method can be applied to define defeasible rules, and this application, as stated in [15], is based on some cognitional or rational criteria. By using the crude method to define strict rules in the lower-bounded logic $\mathbf{S}^-$ and to defeasible rules in the upper-bounded logic $\mathbf{S}^+$, even when Hilbert style derivations are quite long, the arguments can be short.

Besides this way to define the defeasible rules, all the other definitions in this section like the arguments and the extensions are standard and taken from the handbook article of Modgil and Prakken. In particular, we consider three instantiations of ASPIC$^+$, by taking different monotonic logics ($\mathbf{D}_{-1}$ or $\mathbf{D}_{-2}$ defined later) as the basic logic and then treating either merely FCP or it together with OWP (in Table 1) as defeasible. In this section, we define the notion of argumentation theory. In the following section we use the argumentation theory to define non-monotonic logic as a combination of two selected monotonic logics $\mathbf{S}^-, \mathbf{S}^+$.

We first present a version of Van Benthem's deontic logic of obligation and permission [4]. This logic is different than Standard Deontic Logic [18], in the latter obligation and permission are a dual pair, while in the former they are the necessary and sufficient conditions of being ideal. The modal language contains the classic negation $\neg$, conjunction $\wedge$, universal modality $\Box$, as well as two additional deontic modalities, O for obligation and P for permission.

Definition 1 (Deontic Language). *Let p be any element of a given (countable) set $Prop$ of atomic propositions. The deontic language $\mathcal{L}$ of modal formulas is defined as follows:*

$$\phi := p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box\phi \mid O\phi \mid P\phi$$

The disjunction $\vee$, the material condition $\to$ and the existential modality $\Diamond$ are defined as usual: $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$, $\phi \to \psi := \neg(\phi \wedge \neg\psi)$ and $\Diamond\phi := \neg\Box\neg\phi$.

The axiomatization presented in Table 1 is a variant of Van Benthem's logic [4]. We use **D** to denote it. **D** not only takes obligation and universal modality into account, but also considers free-choice permission and the connection between obligation and permission. In the logic **D**, except the essential $\mathrm{K}_\Box$, $\mathrm{E}_\Box$, $\mathrm{T}_\Box$, $4_\Box$, $\mathrm{B}_\Box$, and $\mathrm{NEC}_\Box$, the axioms $\Box_O$ and $\Box_P$ are the core of the universal modality in normal modal logic. Moreover, $\Box_O$ claims that what is always the case is obligatory, but $\Box_P$ leaves the space for what is never the case to be permitted. The axiom D_O maintains obligation to be ideally consistent as usual. OWP states that "obligation as the weakest permission" [4, 1]. RFC is one direction of free-choice permission, and FCP is the other. For further information about the logic and its motivations, see Van Benthem's paper.

- All instances of propositional tautologies	- $\mathrm{K}_\triangle$: $\triangle(\phi \to \psi) \to (\triangle\phi \to \triangle\psi)$
- $\mathrm{E}_\Box$: $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$	- $\mathrm{T}_\Box$: $\Box\phi \to \phi$
- $4_\Box$: $\Box\phi \to \Box\Box\phi$	- $\mathrm{B}_\Box$: $\phi \to \Box\Diamond\phi$
- $\Box_O$: $\Box\phi \to O\phi$	- $\Box_P$: $\Box\neg\phi \to P\phi$
- D_O: $\neg(O\phi \wedge O\neg\phi)$	- OWP: $O\phi \wedge P\psi \to \Box(\psi \to \phi)$
- RFC: $P\phi \wedge P\psi \to P(\phi \vee \psi)$	- FCP: $P\psi \wedge \Box(\phi \to \psi) \to P\phi$
- MP: $\phi, \phi \to \psi/\psi$	- $\mathrm{NEC}_\triangle$: $\phi/\triangle\phi$
where $\triangle \in \{\Box, O\}$	

Table 1. The logic **D** of obligation and permission.

In this paper we consider sub-systems of **D** that contain a strict subset of the axioms and inference rules of **D**. In particular, we define $\mathbf{D}_{-1}$ as the axiomatization which does not contain FCP, and we define $\mathbf{D}_{-2}$ as the axiomatization which does not contain FCP and OWP.

We define the notions of derivation based on modal logic $\mathbf{S} \in \{\mathbf{D}, \mathbf{D}_{-1}, \mathbf{D}_{-2}\}$ in the usual way, see e.g. [5]. Note that modal logic provides two related kinds of derivation according to the application of necessitation, i.e. necessitation can only be applied to theorems but not to an arbitrary set of formulas. We use both notions in the formal argumentation theory.

Definition 2 (Derivations without Premises). *Let* $\mathbf{S} \in \{\mathbf{D}, \mathbf{D}_{-1}, \mathbf{D}_{-2}\}$ *be a deontic logic. A derivation for* ϕ *in* $\mathbf{S}$ *is a finite sequence* $\phi_1, \ldots, \phi_{n-1}, \phi_n$ *such that* $\phi = \phi_n$ *and for every* $\phi_i (1 \leqslant i \leqslant n)$ *in this sequence is*

1. *either an instance of one of the axioms in* $\mathbf{S}$*;*
2. *or the result of the application of one of the rules in* $\mathbf{S}$ *to those formulas appearing before* ϕ_i*.*

We write $\vdash_{\mathbf{S}} \phi$ *if there is a derivation for* ϕ *in* $\mathbf{S}$*, or,* $\vdash \phi$ *when the context of* $\mathbf{S}$ *is clear. We say* ϕ *is a theorem of* $\mathbf{S}$ *or* $\mathbf{S}$ *proves* ϕ*. We write* $Cn(\mathbf{S})$ *as the set of all theorems of* $\mathbf{S}$*.*

Definition 3 (Derivations from Premises). *Let* $\mathbf{S} \in \{\mathbf{D}, \mathbf{D}_{-1}, \mathbf{D}_{-2}\}$ *be a deontic logic. Given a set* Γ *of formulas, a derivation for* ϕ *from* Γ *in* $\mathbf{S}$ *is a finite sequence* $\phi_1, \ldots, \phi_{n-1}, \phi_n$ *such that* $\phi = \phi_n$ *and for every* $\phi_i (1 \leqslant i \leqslant n)$ *in this sequence*

1. *either* $\phi_i \in Cn(\mathbf{S}) \cup \Gamma$*;*
2. *or the result of the application of one of the rules (which is neither* $NEC_\Box$ *nor* NEC_O*) to those formulas appearing before* ϕ_i*.*

We write $\Gamma \vdash_{\mathbf{S}} \phi$ *if there is a derivation from* Γ *for* ϕ *in* $\mathbf{S}$ [3]*, or,* $\Gamma \vdash \phi$ *when the context of* $\mathbf{S}$ *is clear. We say this that* ϕ *is derivable in* $\mathbf{S}$ *from* Γ*. We write* $Cn_{\mathbf{S}}(\Gamma)$ *as the set of formulas derivable in* $\mathbf{S}$ *from* Γ*, or* $Cn(\Gamma)$ *if the context of* $\mathbf{S}$ *is clear.*

A system $\mathbf{S}$ is consistent iff $\bot \notin Cn(\mathbf{S})$; otherwise, inconsistent. A set Γ is consistent iff $\bot \notin Cn(\Gamma)$; otherwise, inconsistent. A set $\Gamma' \subseteq \Gamma$ is maximally consistent subset of Γ, denoted as $\Gamma' \in MC(\Gamma')$ iff there is no $\Gamma'' \supset \Gamma'$ such that Γ'' is consistent.

The following example explains in what sense in monotonic logics we can say that Pk and $O\neg m$ are in conflict. Notice that the set of Pk and $O\neg m$ is consistent even in $\mathbf{D}$. This matches our intuition. We say that it is not consistent, as shown below, when it is not normal that using a knife is not a murder, i.e. $\Diamond(k \wedge m)$ holds. The conditional will play an important role in the ASPIC$^+$-based analysis of the running example.

Example 2 (Knifed murder, continued). The following derivation shows that $\{Pk, O\neg m, \Diamond(k \wedge m)\}$ is inconsistent in $\mathbf{D}_{-1}$ or $\mathbf{D}$.

1. $O\neg m \wedge \Diamond(k \wedge m)$	assumption
2. $O\neg m \wedge Pk \rightarrow \Box(k \rightarrow \neg m)$	OWP
3. $\Diamond(k \wedge m) \leftrightarrow \neg\Box(k \rightarrow \neg m)$	$E_\Box$
4. $O\neg m \wedge \Diamond(k \wedge m) \rightarrow \neg Pk$	2, 3, MP
5. $\neg Pk$	1, 4, MP

[3] Alternatively, it can be seen as a theorem $\vdash_{\mathbf{S}} \bigwedge \Gamma \rightarrow \phi$ by the so-called deduction theorem.

Since we want to represent $\{Pk, O\neg m, \Diamond(k \wedge m)\}$ in a consistent way, we use **D** only to derive conclusions which are defeasible, and we use one of the subsystems of **D** to define the monotonic conclusions.

We involve one spirit of ASPIC$^+$ by considering the inference rules which are uncertain and fallible defeasible rules, while the ones which are unfallible are strict rules. This type of uncertainty or fallibility is represented by the distinction between lower-bounded and upper-bounded logics. However, to simplify the issue addressed, namely, how we can use ASPIC$^+$ to define non-monotonic logics, we are not necessary to fully adopt all methods in ASPIC$^+$ to define arguments. We only consider a general knowledge base here. To distinguish the types of knowledge we leave it to the future work.

Definition 4 (Argumentation Theory). *Let $\mathcal{L}$ be the deontic language and $(\mathbf{S}^-; \mathbf{S}^+) \in \{(\mathbf{D}_{-2}; \mathbf{D}_{-1}), (\mathbf{D}_{-2}; \mathbf{D}), (\mathbf{D}_{-1}; \mathbf{D})\}$ be a Cartesian product of two monotonic logics. An argumentation theory AT based on $(\mathbf{S}^-; \mathbf{S}^+)$ is a tuple (AS, K) where AS is an argumentation system $(\mathcal{L}, R)$, $K \subseteq \mathcal{L}$ is a knowledge base, and $R = R_s \cup R_d$ is a set of rules, such that*

- $R_s = \{\phi_1, \ldots, \phi_n \mapsto \phi \mid \{\phi_1, \ldots, \phi_n\} \vdash_{\mathbf{S}^-} \phi\}$ *is the set of strict rules, and*
- $R_d = \{\phi_1, \ldots, \phi_n \Rightarrow \phi \mid \{\phi_1, \ldots, \phi_n\} \vdash_{\mathbf{S}^+} \phi \;\&\; \{\phi_1, \ldots, \phi_n\} \nvdash_{\mathbf{S}^-} \phi\}$ *is the set of defeasible rules.*

If the context of $(\mathbf{S}^-; \mathbf{S}^+)$ is clear, we mention AT without $(\mathbf{S}^-; \mathbf{S}^+)$.

So the requirement of $R_s \cap R_d = \varnothing$ holds.

In contrast to derivations, arguments are different structures. Although each argument corresponds to a derivation defined as a top rule, the former has to explicitly consider each step of this derivation as a finite sequence.

Definition 5 (Arguments). *Let AT be an argumentation theory with a knowledge base K and an argumentation system $(\mathcal{L}, R)$. Given each $n \in \mathbb{N}$, the set $\mathcal{A}_n$ where $n \in \mathbb{N}$ is defined by induction as follows:*

$$\mathcal{A}_0 = K$$
$$\mathcal{A}_{n+1} = \mathcal{A}_n \cup \{B_1, \ldots, B_m \rhd \psi \mid B_i \in \mathcal{A}_n \text{ for all } i \in \{1, \ldots, m\}\}$$

where for an element $B \in \mathcal{A}_i$ with $i \in \mathbb{N}$:

- *If $B \in K$, then $Prem(B) = \{\phi\}$, $Conc(B) = \phi$, $Sub(B) = \{\phi\}$, $Rules_d(B) = \varnothing$, $TopRule(B) = undefined$ where $\psi \in K$.*
- *If $B = B_1, \ldots, B_m \rhd \psi$ where $\rhd$ is $\mapsto$ then $\{Conc(B_1), \ldots, Conc(B_m)\} \mapsto \psi \in R_s$ with $Prem(B) = Prem(B_1) \cup \ldots \cup Prem(B_m)$, $Conc(B) = \psi$, $Sub(B) = Sub(B_1) \cup \ldots \cup Sub(B_m) \cup \{B\}$, $Rules_d(B) = Rules_d(B_1) \cup \ldots \cup Rules_d(B_m)$, $TopRule(B) = Conc(B_1), \ldots, Conc(B_m) \mapsto \psi$.*
- *If $B = B_1, \ldots, B_m \rhd \psi$ where $\rhd$ is $\Rightarrow$, then each condition is similar to the previous item, except that the rule is defeasible and $Rules_d(B) = Rules_d(B_1) \cup \ldots \cup Rules_d(B_m) \cup \{Conc(B_1), \ldots, Conc(B_m) \Rightarrow \psi\}$.*

We define $\mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ *as the set of arguments on the basis of* AT*, and define* $Conc(E) = \{\varphi \subseteq Conc(A) \mid A \in E\}$ *where* $E \subseteq \mathcal{A}$*.*

The following example illustrates the arguments in the running example. We consider the defeats (arrows) in Figure 1 in the following section.

Example 3 (Knifed murder, continued). We illustrate the argumentation theory in our running example shown in Figure 1. Let $(\mathbf{S}^-;\mathbf{S}^+) \in \{(\mathbf{D}_{-2};\mathbf{D}_{-1}), (\mathbf{D}_{-2};\mathbf{D}), (\mathbf{D}_{-1};\mathbf{D})\}$ be a pair of two monotonic logics, and AT be an argumentation theory based on $(\mathbf{S}^-;\mathbf{S}^+)$ that takes $K = \{\Diamond(k \wedge m), O\neg m, Pk\}$ where k (using knife) and m (murder) are atomic propositions. We know that $K \vdash_{\mathbf{S}} \bot$ for any consistent system $\mathbf{S} \in \{\mathbf{D}_{-1}, \mathbf{D}\}$. Three arguments refer to the knowledge that it is forbidden to kill (A), it is permitted to use the knife (B), and knifed murder is possible (C). In the logic $\mathbf{D}_{-2}$, we can derive, for example, that knifed murder is forbidden from the premise that murder is forbidden. This derivation is the strict rule to construct the argument A''' (in closed circle) from the knowledge A. Also, $TopRule(A''')$ is this derivation. In the stronger logic $\mathbf{D}_{-1}$, we can derive, for example, that knifed murders are not permitted from the premises that murder is forbidden and knifed murder is possible. This derivation as a defeasible rule together with the knowledges A and C construct the argument A'' (in densely dashed circle). In the strongest logic $\mathbf{D}$, we can derive, for example, that knifed murders are permitted (loosely dashed circles for B'', B'''').

$A = O\neg m$ $\qquad$ $B = Pk$

$A' = A, C \Rightarrow \neg Pk$ $\qquad$ $B' = B, C \Rightarrow \neg O\neg m$

$A'' = A, C \Rightarrow \neg P(k \wedge m)$ $\qquad$ $B'' = B \Rightarrow P(k \wedge m)$

$A''' = A \mapsto O\neg(k \wedge m)$ $\qquad$ $B''' = B, C \Rightarrow \neg O\neg(k \wedge m)$

$C' = A, B \Rightarrow \neg\Diamond(k \wedge m)$ $\qquad$ $B'''' = B \Rightarrow P(k \wedge b)$

$C = \Diamond(k \wedge m)$

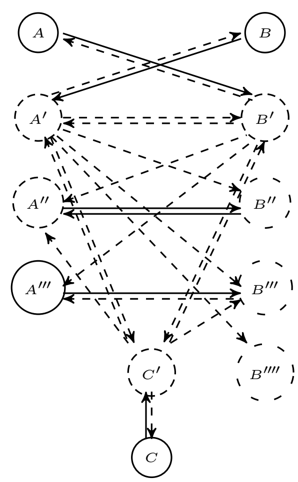

Fig. 1. It shows some of the arguments and defeats. Closed circles are arguments of $\mathbf{D}_{-2}$, densely dashed circles are arguments of $\mathbf{D}_{-1}$, and loosely dashed circles are arguments of $\mathbf{D}$. Straight arrows are defeats among these arguments in the rule-based ordering for $S^+ = \mathbf{D}_{-2}$, and the dashed arrows are defeats in all orderings.

4 Step 2: preferences among arguments

In this paper we consider three orders: universal, rule-based, and premise-based. They emphasize different perspectives of selecting proper arguments for constructing non-monotonic inferences.

Definition 6 (Argument Properties). *Let A be an argument and E a set of arguments. Then A is strict if $Rules_d(A) = \varnothing$; defeasible if $Rules_d(A) \neq \varnothing$; firm if $Prem(A) \subseteq K$; plausible if $Prem(A) \cap K \neq \varnothing$. We define $Concs(E) = \{Conc(A) \mid A \in E\}$. The partial order $\leqslant$ rule-based iff we have $A \leqslant B$ iff A is defeasible; and premise-based iff $A \leqslant B$ iff A is plausible.*

We use $\leqslant^{\tau}$ to denote the τ-ordering with $\tau \in \{r, p\}$, where r for rule-based and p for premise-base. Next we introduce the notion of defeat. The first is a rebuttal while the second is a undermining [15]. In the next section this distinction will give different consequences in non-monotonic reasoning.

Definition 7 (Argumentation Frameworks). *An abstract argumentation framework AF corresponding to $\langle AT, \leqslant\rangle$ is a pair $(\mathcal{A}, \mathcal{D})$, where $\mathcal{D}$ is a set of pairs of arguments in which argument A defeats argument B is defined as:*

- *either $Conc(A) = \neg\phi$ for some $B' \in Sub(B)$ and $TopRule(B') \in R_d$, $Conc(B') = \phi$ and $A \nleqslant B'$.*
- *or $Conc(A) = \neg\phi$ for knowledge $\phi \in Prem(B)$ of B and $A \nleqslant \phi$.*

As shown in Figure 1, A'' rebuts B'' but B' does not undermine A''' when it is in the rule-based ordering. This shows a case of obligation overriding permission but not vice versa. In the universal and premise-based ordering, we then have permission overrides obligation, that is B' undermines A'''.

Definition 8 (Dung Extensions). *Let $AF = (\mathcal{A}, \mathcal{D})$ and $E \subseteq \mathcal{A}$ is a set of arguments. Then*

- *E is conflict-free iff $\forall A, B \in E$ we have $(A, B) \notin \mathcal{D}$.*
- *$A \in \mathcal{A}$ is acceptable w.r.t. E iff when $B \in \mathcal{A}$ such that $(B, A) \in \mathcal{D}$ then $\exists C \in E$ such that $(C, B) \in \mathcal{D}$.*
- *E is an admissible set iff E is conflict-free and if $A \in E$ then A is acceptable w.r.t. E.*
- *E is a complete extension iff E is admissible and if $A \in \mathcal{A}$ is acceptable w.r.t. E then $A \in E$.*
- *E is a stable extension iff E is conflict-free and $\forall B \notin E$ $\exists A \in E$ such that $(A, B) \in \mathcal{D}$.*

The following example illustrates a different sense of consistency in ASPIC$^+$ by using stable extensions, in order to explain, given the inconsistent knowledge base K, why $B \Rightarrow P(k \wedge m)$ is sometimes defeated and why $B \Rightarrow P(k \wedge b)$ is always non-defeated.

Example 4 (Knifed murder, continued). Consider the arrows in Figure 1. The straight arrows represent defeat relations under the rule-based ordering, and the dashed arrows represent additional defeat relations under the premise-based or universal ordering. Under the rule-based ordering the arguments A, B and C will not be defeated and thus in every extension, whereas in the premise-based or universal ordering, they will not. For this reason, we prefer the rule-based ordering in this example. Furthermore, under the rule-based ordering, we at least have two stable extensions, one contains $B \Rightarrow P(k \wedge m)$ and another $A, C \Rightarrow \neg P(k \wedge m)$. As $B'''' = B \Rightarrow P(k \wedge b)$ will be non-defeated, we have B'''' in every stable extension. Similarly, arguments in form of $A_1, \ldots, A_n \mapsto Pk \vee O\neg m \vee \Diamond(k \wedge m)$ are contained in every stable extension.

Apart from comparing plausible and defeasible arguments in the preference ordering, factual statements can be preferred over deontic statements, prohibitions over permissions, or vice versa. We leave such further investigations for the journal extension of this paper.

5 Step 3: Designing Non-Monotonic Logics

Our non-monotonic logics are designed by using the stable extensions regarding to different monotonic logics and to different orderings. In order to do so, the following proposition provides a guideline to search for these stable extensions. In the case of universal/premise-based ordering, strict rules are as equally preferable as defeasible rules. So a stable extension can be considered as a maximally consistent subset of the knowledge base K. We call this the *undermining* mechanism, see e.g. [2, 22]. But this is not enough to capture the case of rule-based ordering, in which the defeasible argument is less preferable than the others. So the second item of this proposition provides a general method to construct the desired extensions, stable extensions. We construct each stable extension in the style of Lindenbaum's Lemma [5]. That is, we first consider the maximally consistent subset K' of the knowledge base w.r.t. the lower-bounded logic $\mathbf{S}^-$ for strict rules, and then a consistent subset of K' w.r.t. the upper-bounded logic $\mathbf{S}^+$ for defeasible rules, such that no argument w.r.t. $\mathbf{S}^+$ defeat that w.r.t. $\mathbf{S}^-$ and it is a maximal set satisfying these two conditions. This is called the *rebuttal* mechanism. See the following for details.

Proposition 1. [4] *Consider the deontic language $\mathcal{L}$ and a combination of two monotonic logics $(\mathbf{S}^-; \mathbf{S}^+) \in \{(\mathbf{D}_{-2}; \mathbf{D}_{-1}), (\mathbf{D}_{-2}; \mathbf{D}), (\mathbf{D}_{-1}; \mathbf{D})\}$. Let AF corresponding to $\langle AT, \leqslant^\tau\rangle$ be an abstract argumentation framework $(\mathcal{A}, \mathcal{D})$, such that AT is based on $(\mathbf{S}^-; \mathbf{S}^+)$, K is a knowledge base, and $\tau \in \{p, r\}$. We define $F(D) = Prem(D) \cup \{Conc(D)\}$ where $D \in \mathcal{A}$. Let $F(E) = \bigcup\{F(D) \mid D \in E \subseteq \mathcal{A}\}$.*

[4] For the proof please check: `https://pan.zju.edu.cn/share/793a363c53083fbf2c00433b1b`.

1. *When* $\tau = p$ *then* $E = \{D \in \mathcal{A} \mid Conc(D) \in Cn_{\mathbf{S}^+}(\Gamma)\}$ *is a stable extension if and only if* Γ *is a maximally consistent subset of the knowledge base* K *in* AT *w.r.t.* $\mathbf{S}^+$*.*
2. *We define* $E_1 = \{D \in \mathcal{A} \mid F(D) \subseteq Cn_{\mathbf{S}^-}(\Gamma_1)\}$ *where* Γ_1 *is a maximally consistent subset of* K *w.r.t.* $\mathbf{S}^-$*. Let* $E_2 = \{D \in \mathcal{A} \mid F(D) \subseteq Cn_{\mathbf{S}^+}(\Gamma_1)\}$ *such that (i)* $F(D)$ *is consistent w.r.t.* $\mathbf{S}^+$*; (ii)* $F(D) \cup \{\varphi\}$ *is consistent w.r.t.* $\mathbf{S}^-$ *where* $\varphi \in Cn_{\mathbf{S}^-}(\Gamma_1)$*; (iii) there is no* $\Gamma \supset F(D)$ *such that* Γ *is consistent w.r.t.* $\mathbf{S}^+$*, and for any* $\varphi \in Cn_{\mathbf{S}^-}(\Gamma_1)$ *we have* $\Gamma \cup \{\varphi\}$ *be* $\mathbf{S}^-$*-consistent. If* $\tau \in \{p, r\}$ *then* $E = E_1 \cup E_2$ *is a stable extension.*

Given the knowledge base $K = \{Pk, O\neg m, \Diamond(k \wedge m)\}$ of the running example, ASPIC$^+$ provides a mechanism to decide whether the two arguments $A, C \Rightarrow \neg P(k \wedge m)$ and $A, C \Rightarrow \neg Pk$ can be accepted. In the case of premise-based, the undermining together with stability is a mechanism to ensure that in conflict like K the maximally consistent subsets form the stable extensions. In the case of rule-based ordering, we cannot use the undermining mechanism to ensure that we derive the first but not the second. Instead, we need to use the rebuttal mechanism. Rebuttal corresponds to closed world assumption. So the above two arguments hold, unless there is a proof to the contrary. That is how the two are then distinguished in the logics.

We now present the central definition of the paper, namely the definition of the non-monotonic logic in terms of the formal argumentation theory. This is well in line with current practice in ASPIC$^+$. We first take the desired conclusions in each stable extension (as shown in Proposition 1) and then the intersection of all the stable extensions.

Definition 9 (Non-Monotonic Inferences). *Let* $\Gamma \subseteq \mathcal{L}$, $\phi \in \mathcal{L}$, $(\mathbf{S}^-; \mathbf{S}^+) \in \{(\mathbf{D}_{-2}; \mathbf{D}_{-1}), (\mathbf{D}_{-2}; \mathbf{D}), (\mathbf{D}_{-1}; \mathbf{D})\}$ *be a Cartesian product of two monotonic logics, and* $\leqslant^\tau$ *be a* τ*-ordering such that* $\tau \in \{r, p\}$*. Let* AT *be the* Γ*-argumentation theory based on* $(\mathbf{S}^-; \mathbf{S}^+)$ *iff the argumentation theory* AT *obtains with* $K = \Gamma$*, and* $AF^\tau = \langle AT, \leqslant^\tau \rangle$*. The non-monotonic inference* $|\!\!\sim^\tau_{\mathbf{S}^-;\mathbf{S}^+}$ *is defined as follows:*

- $\Gamma |\!\!\sim^\tau_{\mathbf{S}^-;\mathbf{S}^+} \phi$ *iff every stable extension of the* Γ*-*AT *based on* $(\mathbf{S}^-; \mathbf{S}^+)$ *corresponded by* AF^τ *contains an argument* A *with* $Conc(A) = \phi$*.*

We define the closure operator corresponding to this inference relation as usual: $\mathcal{C}^\tau_{\mathbf{S}^-;\mathbf{S}^+}(\Gamma) = \{\phi \mid \Gamma |\!\!\sim^\tau_{\mathbf{S}^-;\mathbf{S}^+} \phi\}$*. Moreover, we write* $|\!\!\sim^\tau_{\mathbf{S}^-;\mathbf{S}^+} \phi$ *when* $\varnothing |\!\!\sim^\tau_{\mathbf{S}^-;\mathbf{S}^+} \phi$*.*

The resulting non-monotonic inference relations are standard relations among sets of formulas of the logical language, i.e. they no longer refer to ASPIC$^+$. An alternative way to define non-monotonic logics is to first consider the intersection of all stable extensions and then the conclusions. For instance, $Pk \vee O\neg m \vee \Diamond(k \wedge m)$ is an element in $\mathcal{C}^\tau_{\mathbf{D}_{-2};\mathbf{D}}(\{Pk, O\neg m, \Diamond(k \wedge m)\}$ where $\tau \in \{p, r\}$. If the proposed order is reversed, this cannot be inferred. Because it is possible to have many different arguments which contain the same conclusion but from different premises.

The following proposition offers a detailed explanation of the mechanisms we proposed. First, the undermining mechanism states that the non-monotonic consequences are the intersection of all maximally consistent subsets of the knowledge base under the universal or premise-based ordering. Second, and more generally, the rebuttal mechanism states that the non-monotonic consequences are encased by all unions of a maximally consistent subset of the knowledge base w.r.t. the lower-bounded logic and a consistent subset of it w.r.t. the upper-bounded logic in certain maximal behavior.

Proposition 2. *Let $\Gamma \subseteq \mathcal{L}$, $(\mathbf{S}^-; \mathbf{S}^+) \in \{(\mathbf{D}_{-2}; \mathbf{D}_{-1}), (\mathbf{D}_{-2}; \mathbf{D}), (\mathbf{D}_{-1}; \mathbf{D})\}$ be a Cartesian product of two monotonic logics, $\leqslant^\tau$ be a τ-ordering such that $\tau \in \{r, p\}$, and K be a knowledge base of AT. Then*

1. *$\mathcal{C}^\tau_{\mathbf{S}^-;\mathbf{S}^+}(K) = \bigcap_{\Gamma \in MC(K)} Cn_{\mathbf{S}^+}(\Gamma)$, where $\tau = p$.*
2. *We define $\Gamma' \in M(\Gamma)$ as follows: (i) Γ' is a consistent subset of Γ w.r.t. $\mathbf{S}^+$, (ii) $Cn_{\mathbf{S}^+}(\Gamma')$ is consistent with φ w.r.t. $\mathbf{S}^-$ where $\varphi \in Cn_{\mathbf{S}^-}(\Gamma)$, and (iii) there is no $\Gamma'' \subseteq \Gamma'$ such that Γ'' is $\mathbf{S}^+$-consistent and for all $\varphi \in Cn_{\mathbf{S}^-}(\Gamma)$ we have $\Gamma'' \cup \{\varphi\}$ be $\mathbf{S}^-$-consistent. Then*

$$\mathcal{C}^r_{\mathbf{S}^-;\mathbf{S}^+}(K) = \bigcap_{\Gamma \in MC(K)} \bigcap_{\Gamma' \in M(\Gamma)} (Cn_{\mathbf{S}^-}(\Gamma) \cup Cn_{\mathbf{S}^+}(\Gamma')),$$

   where $\tau \in \{r, p\}$.

To prove Proposition 2, as inspired by Proposition 1, we first consider the maximally consistent subset of the knowledge base w.r.t the lower-bounded logic $\mathbf{S}^-$, and then consider the consistent subset of the knowledge base w.r.t. the upper-bounded logic $\mathbf{S}^+$, such that this set is maximal in the sense that it is consistent with each element of the previous set w.r.t. the lower-bounded logic. Moreover, Proposition 2.2 illustrates a new understanding of maximality of consistency, which not only has to consider the consistency of the upper-bounded logic but also the consistency with each element in the lower-bounded logic.

A formal analysis of the non-monotonic inference relation is left to further research, as well as the development of alternative non-monotonic relations in terms of the formal argumentation theory.

Example 5 (Knifed murder, continued). Given the set $K = \{\Diamond(k \wedge m), O\neg m, Pk\}$ as the premises, we have different non-monotonic consequences shown in Table 2, depending on the combinations of monotonic logics and the orderings. They are non-monotonic, in the sense that, even given Pk as one premise, $P(k \wedge m)$ is excluded in every non-monotonic consequences, while $P(k \wedge b)$ is a non-monotonic consequence w.r.t. $(\mathbf{D}_{-2}; \mathbf{D})$ under the rule-based ordering.

6 Related work

Concerning the formalization of non-monotonic reasoning about norms, obligations and permissions, there is a large amount of work. For instance, Horty [11]

	Order	K	Example of Consequences $\{\Diamond(k \wedge m), O\neg m, Pk\}$
$(\mathbf{D}_{-2};\mathbf{D}_{-1})$	p	T_u	$\bigvee K$
$(\mathbf{D}_{-2};\mathbf{D}_{-1})$	r	T_r^1	$\Diamond(k \wedge m), O\neg m, Pk, O\neg(k \wedge m), \bigvee K$
$(\mathbf{D}_{-2};\mathbf{D})$	p	T_u	$\bigvee K$
$(\mathbf{D}_{-2};\mathbf{D})$	r	T_r^2	$\Diamond(k \wedge m), O\neg m, Pk, O\neg(k \wedge m), P(k \wedge b), \bigvee K$
$(\mathbf{D}_{-1};\mathbf{D})$	p, r	T_u	$\bigvee K$

Table 2. Examples of the non-monotonic inferences in the case of knifed murder. We have $T_u = \bigcap_{\Gamma \in MC(K)} Cn_{\mathbf{D}_{-1}}(\Gamma)$ and $T_r^1 = \bigcap_{\Gamma \in M(K)}(Cn_{\mathbf{D}_{-2}}(K) \cup Cn_{\mathbf{D}_{-1}}(\Gamma))$ and $T_r^2 = \bigcap_{\Gamma \in M(K)}(Cn_{\mathbf{D}_{-2}}(K) \cup Cn_{\mathbf{D}}(\Gamma))$.

formalized the reasoning in the presence of conflicting obligations and reasoning with conditional obligations based on default logic and a model preference logic, Prakken [20] proposed a combination of standard deontic logic with an early-generation formal argumentation system to formalize defeasible deontic reasoning, and Prakken and Sartor [21] formulated arguments about norms as the application of argument schemes to knowledge bases of facts and norms, among others. Our work is in line with the existing methodology by using non-monotomic formalisms to deal with the conflicts between norms, obligations and permissions. Besides this point, our work focuses more on how to capture the intuition of reasoning about free-choice permission, by using different monotonic logics (lower bound and upper bound) to define strict rules and defeasible rules, and different types of arguments (rule-based, premise-based and universal) to define the preference relation between arguments.

Connecting formal argumentation and deontic logic is an increasingly active research topic in recent years [19]. In the direction of using argumentation to represent various non-monotonic logics, Young *et al.* [25] proposed an approach to represent prioritized default logic by using ASPIC$^+$. Liao et al [14] represented three logics of prioritized norms by using argumentation. While existing works use argumentation to represent existing non-monotonic logics or non-monotonic reasoning, this paper uses argumentation to define new logics. A recent work that is close to our work is by Straßer and Arieli [22], which presented an argumentative approach to normative reasoning by using standard deontic logic as base logic. Similar to this paper, our logic $\mathbf{D}_{-2}$ is also a variant of standard deontic logic. The difference is that we use the extension $\mathbf{D}_{-1}$ and $\mathbf{D}$ to capture the permission by using FCP and OWP.

7 Summary and concluding remarks

In this paper, ASPIC$^+$ relates formal argumentation to non-monotonic logic. We believe this approach benefits both areas. For formal argumentation, the resulting non-monotonic logics can be studied to provide new insights in the adopted argumentation systems, for example in the effect of the adopted argumentation semantics. For the non-monotonic logics, the underlying argumentation theory can be used for explaining deontic conclusions. Our case-study with the logic of obligations and permissions provides first evidence for this.

Within this general ambitious setting, the contributions of this paper are as follows. First, concerning the definitions, in Definition 4 we show how to use two logics in ASPIC$^+$, and in Definition 9 we show how to build a non-monotonic logic on top of ASPIC$^+$. For the formal results, Proposition 1 and 2 characterise the consequences of the non-monotonic logic. Finally, the example illustrates how to apply this approach to formalise the analysis of Glavaničová [8] of strong and free-choice permission.

The relation between the argumentation system and the non-monotonic logic can be studied in more detail. Consider the possibility of post-rationalization in law. The models describing decision making as people deliberate and argue and then, at the end, a group decision is proposed might be considered as naive: we cannot identify the cases of post-rationalization, where a decision is made first, then arguments in favour of that decision are sought. The interaction between the argumentation system and the non-monotonic logic is not a trivial relation where one is the master and the other is the slave, but both the argumentation system and the non-monotonic logic should be seen as different conceptualizations with different concerns, which are related, but one cannot be reduced to the other.

The main tool for studying formal argumentation in the setting of ASPIC$^+$ is based on the use of rationality postulates [6]. It immediately follows from the two propositions of this paper, that all rationality postulates are satisfied. This can also be proven as a corollary of the more general theorems of Caminada, and of Modgil and Prakken.

Our study opens up many lines of further research. For example, as done by Beirlaen *et al.* [3], we can consider the alternatives of monotonic combinations, like $\mathbf{D}$ minus K_O and NEC_O as the logic for strict rules and SDL minus weak permission as the logic for defeasible rules. In this case, we can go for the approach of permission overriding obligation. Second, we can study the sophisticated method rather than the crude method by using the natural deduction proof theories [16], in which we take axioms as the knowledge base and the others as the rules for arguments. Also, we then can explore the challenge of obtaining a normalizing system of natural deduction for deontic logic with the sub-formula property. Further, we have discussed the distinction of strict/defeasible rules in this paper, and have checked the relation of the non-monotonic inferences with the monotonic one. We can distinguish the premises of arguments from strict to defeasible, and then study the relation with supra-classical logics. We believe that this future work will bring us an interesting insight of non-monotonicity.

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Appendix

(This appendix is not included in the LORI 2019 version published by Springer.)

Proposition 1. *Given the deontic language $\mathcal{L}$ and a combination of two monotonic logics $(\mathbf{S}^-; \mathbf{S}^+) \in \{(\mathbf{D}_{-2}; \mathbf{D}_{-1}), (\mathbf{D}_{-2}; \mathbf{D}), (\mathbf{D}_{-1}; \mathbf{D})\}$. Let AF corresponding to $\langle AT, \leqslant^\tau \rangle$ be an abstract argumentation framework $(\mathcal{A}, \mathcal{D})$, such that AT is based on $(\mathbf{S}^-; \mathbf{S}^+)$, K is a knowledge base, and $\tau \in \{u, p, r\}$. We define $F(D) = Prem(D) \cup \{Conc(D)\}$ where $D \in \mathcal{A}$.*

1. *When $\tau \in \{u, p\}$ then $E = \{D \in \mathcal{A} \mid Conc(D) \in Cn_{\mathbf{S}^+}(\Gamma)\}$ is a stable extension if and only if Γ is a maximally consistent subset of the knowledge base K in AT w.r.t. $\mathbf{S}^+$.*
2. *We define $E_1 = \{D \in \mathcal{A} \mid F(D) \subseteq Cn_{\mathbf{S}^-}(\Gamma_1)\}$ where Γ_1 be a maximally consistent subset of K w.r.t. $\mathbf{S}^-$. Let $E_2 = \{D \in \mathcal{A} \mid F(D) \subseteq Cn_{\mathbf{S}^+}(\Gamma_1)\}$ such that (i) $F(D)$ is consistent w.r.t. $\mathbf{S}^+$; (ii) $F(D) \cup \{\varphi\}$ is consistent w.r.t. $\mathbf{S}^-$ where $\varphi \in Cn_{\mathbf{S}^-}(\Gamma_1)$; (iii) there is no $\Gamma \supset F(D)$ such that Γ is consistent w.r.t. $\mathbf{S}^+$, and for any $\varphi \in Cn_{\mathbf{S}^-}(\Gamma_1)$ we have $\Gamma \cup \{\varphi\}$ be $\mathbf{S}^-$-consistent. If $\tau \in \{u, p, r\}$ then $E = E_1 \cup E_2$ is a stable extension.*

Proof. 1. We first prove the right-to-left direction. E is conflict-free. Otherwise there are $A, B \in E$ such that $(A, B) \in \mathcal{D}$. As $Prem(C) \subseteq K$ for each $C \in \mathcal{A}$, we know that for all $C, C' \in \mathcal{A}$ that $C \leqslant C'$. Because of this, no matter in which case of defeats, we know that $Conc(Sub(A)) \cup Conc(Sub(B)) \vdash \bot$. As $Conc(Sub(A)), Conc(Sub(B)) \subseteq Cn_{\mathbf{S}^+}(\Gamma)$, it implies that $\Gamma \vdash \bot$, which contradicts the consistency of Γ. Similarly, another case of defeats also implies contradiction. So E is conflict-free. Now we show that E is stable. Otherwise, there is a $B \notin E$ such that for all $A \in E$ we have $(A, B) \notin \mathcal{D}$. It then means that $Conc(Sub(B)) \cup \Gamma \nvdash \bot$. As Γ is maximally consistent, we have $Conc(Sub(B)) \subseteq Cn_{\mathbf{S}^+}(\Gamma)$. But it contradicts the construction of E. So E is a stable extension.
We then prove the left-to-right direction. Suppose E is a stable extension. First we prove that Γ is consistent. If not, then $A \mapsto \varphi$ and $A \mapsto \neg\varphi$ are both contained in E where $A \in \mathcal{A}$. But then E is not conflict-free, which contradicts the assumption. We then prove that Γ is a maximally consistent

subset. If not, we assume that $\Gamma' \supset \Gamma$ is a consistent. Let $\varphi \in \Gamma'$ and φ be consistent with Γ. Then $F(B) \cup F(A) \nvdash \bot$. So there is a $B \notin E$ but for all $A \in E$ that $(A, B) \notin E$. This implies that E is not a stable extension, which contradicts the assumption. Thus Γ is a maximally consistent subset.

2. Let $E = E_1 \cup E_2$ be defined as given in the assumption. We define $F(E) = \bigcup\{F(A) \mid A \in E \subseteq \mathcal{A}\}$.
First E is conflict-free. Otherwise, there are $A, B \in E$ such that $(A, B) \in \mathcal{D}$. This implies that $F(A) \cup F(B) \vdash \bot$ where $\vdash$ is either $\vdash_{\mathbf{S}^-}$ or $\vdash_{\mathbf{S}^+}$. As both E_1 and E_2 are constructed from consistent sets, it implies that A and B are elements from different E_i where $i \in \{1, 2\}$. We assume that $A \in E_1$ and $B \in E_2$. But then it contradicts the condition (ii).
Now we prove that E is a stable extension. For each $B \notin E$ we need to find out a $A \in E$ such that $(A, B) \in \mathcal{D}$. Given $B \notin E$, we either have $F(E_1) \cup F(B) \vdash_{\mathbf{S}^-} \bot$ or $F(E_2) \cup F(B) \vdash_{\mathbf{S}^+} \bot$, because of the condition (iii) of E_2. If it is the former case, then there is a $A \in E_1$ such that $F(A) \cup F(B) \vdash_{\mathbf{S}^-} \bot$ and $A \geqslant B$, because $TopRule(A) \in R_s$. So $(A, B) \in \mathcal{D}$. If it is the latter case, then there is a $A \in E_2$ such that $F(A) \cup F(B) \vdash_{\mathbf{S}^+} \bot$. Notice that $TopRule(A) \in R_d$. If $TopRule(B) \in R_d$, then $A \geqslant B$, and then we have $(A, B) \in \mathcal{D}$. If $TopRule(B) \in R_s$, then $A < B$. Suppose $Cn_{\mathbf{S}^-}(\Gamma_1) \cup F(B) \nvdash_{\mathbf{S}^-} \bot$. Then $B \in E_1$ and thus contradicts $B \notin E$. So $Cn_{\mathbf{S}^-}(\Gamma_1) \cup F(B) \vdash_{\mathbf{S}^-} \bot$. This gives that there is a $D \in E_1$ such that $Cn_{\mathbf{S}^-}(\Gamma_1) \cup F(B) \vdash_{\mathbf{S}^-} \bot$ and $D \geqslant B$. We then have $(D, B) \in \mathcal{D}$. Now we can conclude that E is a stable extension.