

Losing ground in the income hierarchy: relative deprivation revisited*

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Abstract. The paper discusses a one-parameter generalization of individual relative deprivation measures to a two-period setting that differs from earlier approaches. The parameter is, by definition, independent of the income distributions under consideration—it is to be chosen by a social planner. Its value has an intuitive interpretation: it represents the additional weight assigned to the income shortfalls associated with agents who passed the individual in question when moving from yesterday's income distribution to today's. Therefore, the choice of this parameter represents an important value judgment on the part of a social planner regarding the relative impact of being left behind. As a special case, it is illustrated how the well-known Yitzhaki index can be extended to this environment. *Journal of Economic Literature* Classification No.: D63.

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1 Introduction

Starting with Yitzhaki’s (1979) seminal article, a rich literature on relative deprivation has emerged over the last few decades. Yitzhaki proposes to consider income as the object of relative deprivation and argues that the absolute Gini index is a suitable measure of total relative deprivation in a society. An interesting alternative interpretation appears in Hey and Lambert (1980) who express the Yitzhaki index as an average of individual relative deprivation values. The relative deprivation of an individual i in the society under consideration is defined as the sum of the (positive) gaps between i ’s income and the incomes of all members of society whose income exceeds that of person i , divided by the population size. See also Yitzhaki (1980).

Hey and Lambert’s (1980) interpretation of Yitzhaki’s (1979) index is motivated by Runciman’s (1966, p. 10) statement that “[t]he magnitude of a relative deprivation is the extent of the difference between the desired situation and that of the person desiring it.” An additional aspect of relative deprivation that has received relatively little attention in the subsequent literature also appears in Stouffer (1949, pp. 250–253) and in Runciman (1966, p. 19). Runciman proposes that “[t]he more the people a man sees promoted when he is not promoted himself, the more people he may compare himself with in a situation where the comparison will make him feel relatively deprived.” See D’Ambrosio and Frick (2012) for an empirical study that supports the hypothesis that being passed by others indeed matters to an individual. A first attempt to incorporate this phenomenon of being left behind into the derivation of an individual relative deprivation measure appears in Bossert and D’Ambrosio (2007). In that contribution, two-period income distributions are considered—one distribution observed in the previous period and one that applies to the current period. The classes of measures proposed proceed by multiplying a standard (static) index of relative deprivation for a person i by an increasing function of the number of those individuals whose incomes are less than or equal to that of i in the previous period but higher than the income of i in the current period.

While allowing the notion of being left behind to be taken into consideration by examining a two-period model, the measures obtained in Bossert and D’Ambrosio (2007) remain somewhat rudimentary. This is the case because the influence of the phenomenon in question is restricted to the number of individuals who move beyond the person under consideration. In this paper, we examine an alternative procedure of paying special attention to those whose incomes increase beyond that of person i in the transition from the past to the present. Rather than merely using the number of those who pass i , we attach a higher weight to the income gaps associated with them as compared to those whose incomes do not rise beyond that of i . Our approach employs the axiomatic method, which is well-established in the literature on social index numbers. We propose several intuitively appealing properties of a relative deprivation measure and characterize all measure that satisfy them. In particular, a one-parameter class of individual relative deprivation measures emerges, where the parameter represents the additional weight given to the income shortfalls that correspond to the individuals who pass individual i in the income distribution. As a special case, we characterize the class of measures that generalize the Yitzhaki index along the lines explored in this paper. The parameter is to be chosen by

a social planner, and it represents an important value judgment because it identifies the relative weight of being left behind when assessing relative deprivation.

The main contribution of our paper consists of the proposed new class of relative deprivation measures and its axiomatic characterization. To place our paper in the relevant literature, a few additional remarks on some earlier contributions that are inspired by Yitzhaki's work are in order to conclude this introduction. Chakravarty and Chakraborty (1984) propose a normative relative deprivation index based on a particular representation of a social welfare function. The index of Yitzhaki (1979) is included as a special case in their approach. Paul (1991) argues that an individual measure of relative deprivation should not be insensitive to transfers that take place among those whose incomes are above that of the individual under consideration. Because the Yitzhaki index as well as the generalization of Chakravarty and Chakraborty suffer from this insensitivity, Paul (1991) proposes an alternative that possesses a suitable transfer sensitivity property; see also Chakravarty and Chattopadhyay (1994), Podder (1996), Esposito (2010), Bossert and D'Ambrosio (2014), and Stark, Bielawski and Falniowski (2017). Furthermore, Kakwani (1984) introduces the relative deprivation curve and uses it to illustrate income gaps as a proportion of mean income. He proves that the area under this curve is given by the Gini coefficient. Duclos (2000) shows that the single-parameter Ginis (see Donaldson and Weymark, 1980, and Weymark, 1981) can be interpreted as indices of relative deprivation. Relative deprivation quasi-orderings are proposed in contributions by Chakravarty, Chattopadhyay and Majumder (1995), Chakravarty (1997), and Chakravarty and Moyes (2003). We note that none of these articles provides the novel systematic method of incorporating the phenomenon of being left behind that lies at the heart of our approach.

2 Two-period relative deprivation measures

We adopt the notational convention $\sum_{j \in \emptyset} a_j = 0$. Consider a society $N = \{1, \dots, n\}$ of $n \in \mathbb{N} \setminus \{1\}$ individuals. The vector consisting of n ones is denoted by $\mathbf{1}_n$ and the origin of \mathbb{R}^n is $\mathbf{0}_n$. A two-period income distribution is a vector

$$(y^0, y^1) = ((y_1^0, \dots, y_n^0), (y_1^1, \dots, y_n^1)) \in \mathbb{R}_+^{2n}.$$

For $x, z \in \mathbb{R}_+^n$ and a subset M of N , the vector $w = (x|_M, z|_{N \setminus M})$ is defined as follows. For all $j \in N$,

$$w_j = \begin{cases} x_j & \text{if } j \in M, \\ z_j & \text{if } j \in N \setminus M. \end{cases}$$

For all permutations $\pi: N \rightarrow N$ and for all $(y^0, y^1) \in \mathbb{R}_+^{2n}$, the pair (y_π^0, y_π^1) is obtained by applying π to all components of y^0 and y^1 .

An (intertemporal) individual measure of relative deprivation for individual $i \in N$ is a function

$$D_i: \mathbb{R}_+^{2n} \rightarrow \mathbb{R}_+.$$

For $x \in \mathbb{R}_+^n$, $B_i(x) = \{j \in N \mid x_j > x_i\}$ is the set of individuals with a higher income than i in the income distribution x .

A traditional (static) measure of individual relative deprivation can be thought of as the restriction of D_i to pairs of distributions such that $y^0 = y^1$. For example, Yitzhaki's (1979) well-known static index can be expressed by setting, for all $y^1 \in \mathbb{R}_+^n$,

$$D_i(y^1, y^1) = \frac{1}{n} \sum_{j \in B_i(y^1)} (y_j^1 - y_i^1).$$

The purpose of this paper is to develop a systematic method of extending a static index to the full domain of D_i with the underlying idea that the income shortfalls with respect to those who passed individual i in the move from the previous period's distribution y^0 to the current distribution y^1 should receive a higher weight than the shortfalls associated with the remaining members of $B_i(y^1)$. We propose to proceed as follows. Let $\alpha \in \mathbb{R}_{++}$ be a parameter and suppose that a static measure of individual relative deprivation is given, that is, the values $D_i(y^1, y^1)$ are known for all $y^1 \in \mathbb{R}_+^n$. We propose to extend this static measure by defining, for all $(y^0, y^1) \in \mathbb{R}_+^{2n}$,

$$D_i^\alpha(y^0, y^1) = D_i(y^1, y^1) + \alpha \sum_{j \in B_i(y^1) \setminus B_i(y^0)} (y_j^1 - y_i^1). \quad (1)$$

These measures provide a simple and intuitive way of taking into consideration Runciman's (1966, p. 10) postulate. Clearly, if $B_i(y^1) \setminus B_i(y^0) = \emptyset$, no one passed individual i and, thus, the measure defined in (1) reduces to a standard static index. It is immediate that the higher the value of α , the more importance is attached to the income shortfalls with respect to those who passed individual i when moving from the previous distribution y^0 to the current distribution y^1 .

The parameter α is to be chosen by the social planner. It represents an important feature of the measure in that it identifies the relative importance that is to be attached to the phenomenon of being left behind.

In order to derive the class of measures identified in (1), we employ the axiomatic method. That is, we identify a list of plausible properties that a relative deprivation index is assumed to possess and show that the members of our class are the only ones that satisfy the requisite conditions.

Positivity is a property that requires the index to assume a positive value if and only if the set of those whose current incomes exceed that of individual i is non-empty. This is a standard requirement in relative deprivation measurement.

Positivity. For all $(y^0, y^1) \in \mathbb{R}_+^{2n}$,

$$D_i(y^0, y^1) > 0 \Leftrightarrow B_i(y^1) \neq \emptyset.$$

The focus axiom defined below differs from that usually encountered in the context of static measures of relative deprivation. In addition to the sets of those with higher incomes than i in the current period, the sets of those who pass i in the move from period 0 to period 1 have to coincide for the conclusion of the axiom to apply. This is a

natural generalization that is in line with the intertemporal interpretation chosen in our framework.

Focus. For all $(y^0, y^1), (z^0, z^1) \in \mathbb{R}_+^{2n}$, if $B_i(y^1) \setminus B_i(y^0) = B_i(z^1) \setminus B_i(z^0)$, $B_i(y^1) = B_i(z^1)$ and $y_j^1 = z_j^1$ for all $j \in B_i(y^1) \cup \{i\}$, then

$$D_i(y^0, y^1) = D_i(z^0, z^1).$$

The axioms of anonymity and linear homogeneity are the natural generalizations of the requisite conditions commonly employed in the static framework.

Anonymity. For all $(y^0, y^1) \in \mathbb{R}_+^{2n}$ and for all permutations $\pi: N \rightarrow N$ such that $\pi(i) = i$,

$$D_i(y_\pi^0, y_\pi^1) = D_i(y^0, y^1).$$

Linear homogeneity. For all $(y^0, y^1) \in \mathbb{R}_+^{2n}$ and for all $\lambda \in \mathbb{R}_{++}$,

$$D_i(\lambda y^0, \lambda y^1) = \lambda D_i(y^0, y^1).$$

Another commonly employed property is translation invariance. In our setting, however, it is not sufficient to use a standard translation-invariance property with a translative constant that applies in both periods. The reason is that such an axiom is silent if a constant has to be subtracted from the incomes in a given period and there are incomes in the other period that are below the value to be subtracted; such a move would take us out of our (non-negative) domain. Thus, we employ the following two axioms instead—one translation-invariance property for each period in which the incomes in the other period are assumed to remain unchanged.

Previous-period translation invariance. For all $(y^0, y^1) \in \mathbb{R}_+^{2n}$ and for all $\delta \in \mathbb{R}$ such that $(y^0 + \delta \mathbf{1}_n) \in \mathbb{R}_+^n$,

$$D_i(y^0 + \delta \mathbf{1}_n, y^1) = D_i(y^0, y^1).$$

Current-period translation invariance. For all $(y^0, y^1) \in \mathbb{R}_+^{2n}$ and for all $\delta \in \mathbb{R}$ such that $(y^1 + \delta \mathbf{1}_n) \in \mathbb{R}_+^n$,

$$D_i(y^0, y^1 + \delta \mathbf{1}_n) = D_i(y^0, y^1).$$

As shown below, the previous-period translation-invariance property is redundant; it is implied by the focus axiom.

Our final property postulates that the effect of individuals who pass agent i in the move from the previous distribution to the current distribution is of an additive nature and, moreover, can be performed iteratively—one individual at a time. Note that the incomes of the agents other than i and j in the second term of the following sum are (hypothetically) fixed at y_i^1 ; this is a natural choice, especially in the presence of the focus

axiom. Analogous constructions are used in other contributions to the measurement of relative deprivation, such as those of Ebert and Moyes (2000) and Bossert and D'Ambrosio (2006).

Recursivity. For all $(y^0, y^1) \in \mathbb{R}_+^{2n}$,

(i) if $B_i(y^1) \setminus B_i(y^0) = \emptyset$, then

$$D_i(y^0, y^1) = D_i(y^1, y^1);$$

(ii) if $B_i(y^1) \setminus B_i(y^0) \neq \emptyset$, then, for all $j \in B_i(y^1) \setminus B_i(y^0)$,

$$\begin{aligned} D_i(y^0, y^1) &= D_i((y^0|_{(B_i(y^1) \setminus B_i(y^0)) \setminus \{j\}}, y^1|_{N \setminus ((B_i(y^1) \setminus B_i(y^0)) \setminus \{j\})}), y^1) \\ &+ D_i((y^0|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}}), (y^1|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}})). \end{aligned}$$

As a preliminary observation, the following lemma shows that previous-period translation invariance is implied by the focus axiom.

Lemma 1 *If an intertemporal individual relative deprivation index D_i satisfies focus, then D_i satisfies previous-period translation invariance.*

Proof. Suppose that D_i satisfies focus and let $(y^0, y^1) \in \mathbb{R}_+^{2n}$ be such that $(y^0 + \delta \mathbf{1}_n) \in \mathbb{R}_+^n$. It follows that $B_i(y^0 + \delta \mathbf{1}_n) = B_i(y^0)$ and hence

$$B_i(y^1) \setminus B_i(y^0 + \delta \mathbf{1}_n) = B_i(y^1) \setminus B_i(y^0).$$

Furthermore, we (trivially) have $y_j^1 = y_j^1$ for all $j \in B_i(y^1)$ and $y_i^1 = y_i^1$ so that the focus axiom implies $D_i(y^0 + \delta \mathbf{1}_n, y^1) = D_i(y^0, y^1)$. ■

The following theorem is our main result.

Theorem 1 *Let D_i be an intertemporal individual relative deprivation index such that the values $D_i(y^1, y^1)$ are determined for all $y^1 \in \mathbb{R}_+^n$. If D_i satisfies positivity, focus, anonymity, linear homogeneity, current-period translation invariance and recursivity, then there exists $\alpha \in \mathbb{R}_{++}$ such that $D_i = D_i^\alpha$.*

Proof. Suppose D_i satisfies the requisite axioms and $D_i(y^1, y^1)$ is determined for all $y^1 \in \mathbb{R}_+^n$. If $B_i(y^1) \setminus B_i(y^0) = \emptyset$, part (i) of recursivity immediately implies (1). To establish the claim for the case $B_i(y^1) \setminus B_i(y^0) \neq \emptyset$, we proceed by induction on the cardinality of $B_i(y^1) \setminus B_i(y^0)$. Suppose that $|B_i(y^1) \setminus B_i(y^0)| = 1$ and let $B_i(y^1) \setminus B_i(y^0) = \{j\}$. Thus, $y_j^0 \leq y_j^0$ and $y_j^1 > y_j^1$. Clearly, $(B_i(y^1) \setminus B_i(y^0)) \setminus \{j\} = \emptyset$ and, therefore,

$$(y^0|_{(B_i(y^1) \setminus B_i(y^0)) \setminus \{j\}}, y^1|_{N \setminus ((B_i(y^1) \setminus B_i(y^0)) \setminus \{j\})}) = y^1.$$

Thus, part (ii) of recursivity requires that

$$D_i(y^0, y^1) = D_i(y^1, y^1) + D_i((y^0|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}}), (y^1|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}})). \quad (2)$$

Focus implies

$$D_i(y^0, y^1) = D_i(y^0, (y^1|_{B_i(y^1)}, y_i^1 \mathbf{1}_n|_{N \setminus B_i(y^1)})), \quad (3)$$

$$D_i(y^1, y^1) = D_i((y^1|_{B_i(y^1)}, y_i^1 \mathbf{1}_n|_{N \setminus B_i(y^1)}), (y^1|_{B_i(y^1)}, y_i^1 \mathbf{1}_n|_{N \setminus B_i(y^1)})) \quad (4)$$

and

$$D_i((y^0|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}}), (y^1|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}})) = D_i(\mathbf{0}_n, (y^1|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}})). \quad (5)$$

Note that (5) follows because $B_i(\mathbf{0}_n) = \emptyset$ and therefore $j \notin B_i(\mathbf{0}_n)$, which implies

$$\begin{aligned} B_i(y^1|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}}) \setminus B_i(y^0|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}}) &= \{j\} \\ &= B_i(y^1|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}}) \setminus B_i(\mathbf{0}_n). \end{aligned}$$

Substituting (3), (4) and (5) into (2), it follows that we must have

$$\begin{aligned} D_i(y^0, (y^1|_{B_i(y^1)}, y_i^1 \mathbf{1}_n|_{N \setminus B_i(y^1)})) &= D_i((y^1|_{B_i(y^1)}, y_i^1 \mathbf{1}_n|_{N \setminus B_i(y^1)}), (y^1|_{B_i(y^1)}, y_i^1 \mathbf{1}_n|_{N \setminus B_i(y^1)})) \\ &\quad + D_i(\mathbf{0}_n, (y^1|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}})). \end{aligned} \quad (6)$$

Current-period translation invariance with $\delta = -y_i^1$ requires

$$D_i(y^0, ((y^1 - y_i^1 \mathbf{1}_n)|_{B_i(y^1)}, \mathbf{0}_n|_{N \setminus B_i(y^1)})) = D_i(y^0, (y^1|_{B_i(y^1)}, y_i^1 \mathbf{1}_n|_{N \setminus B_i(y^1)}))$$

and, using (6), we obtain

$$\begin{aligned} &D_i(((y^1 - y_i^1 \mathbf{1}_n)|_{B_i(y^1)}, \mathbf{0}_n|_{N \setminus B_i(y^1)}), ((y^1 - y_i^1 \mathbf{1}_n)|_{B_i(y^1)}, \mathbf{0}_n|_{N \setminus B_i(y^1)})) \\ &\quad + D_i(\mathbf{0}_n, ((y^1 - y_i^1 \mathbf{1}_n)|_{\{i,j\}}, \mathbf{0}_n|_{N \setminus \{i,j\}})) \\ &= D_i((y^1|_{B_i(y^1)}, y_i^1 \mathbf{1}_n|_{N \setminus B_i(y^1)}), (y^1|_{B_i(y^1)}, y_i^1 \mathbf{1}_n|_{N \setminus B_i(y^1)})) \\ &\quad + D_i(\mathbf{0}_n, (y^1|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}})). \end{aligned} \quad (7)$$

Previous-period translation invariance (which follows from the focus axiom—see Lemma 1) and current-period translation invariance with $\delta = -y_i^1$ together imply

$$\begin{aligned} &D_i(((y^1 - y_i^1 \mathbf{1}_n)|_{B_i(y^1)}, \mathbf{0}_n|_{N \setminus B_i(y^1)}), ((y^1 - y_i^1 \mathbf{1}_n)|_{B_i(y^1)}, \mathbf{0}_n|_{N \setminus B_i(y^1)})) \\ &= D_i((y^1|_{B_i(y^1)}, y_i^1 \mathbf{1}_n|_{N \setminus B_i(y^1)}), (y^1|_{B_i(y^1)}, y_i^1 \mathbf{1}_n|_{N \setminus B_i(y^1)})) \end{aligned}$$

and, thus, the first terms of the two sides in (7) cancel each other out. Therefore, (7) simplifies to

$$D_i(\mathbf{0}_n, ((y^1 - y_i^1 \mathbf{1}_n)|_{\{i,j\}}, \mathbf{0}_n|_{N \setminus \{i,j\}})) = D_i(\mathbf{0}_n, (y^1|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}})). \quad (8)$$

By anonymity, the position of j in the requisite distributions is irrelevant and, therefore, (8) can be expressed as

$$D_i(\mathbf{0}_n, ((y_j^1 - y_i^1), \mathbf{0}_{n-1})) = D_i(\mathbf{0}_n, (y^1|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}})).$$

Using linear homogeneity with $\lambda = 1/(y_j^1 - y_i^1) > 0$, we obtain

$$D_i(\mathbf{0}_n, (1, \mathbf{0}_{n-1})) = \frac{1}{y_j^1 - y_i^1} D_i(\mathbf{0}_n, (y^1|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}}))$$

or, equivalently,

$$D_i(\mathbf{0}_n, (y^1|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}})) = \alpha(y_j^1 - y_i^1)$$

where $\alpha := D_i(\mathbf{0}_n, (1, \mathbf{0}_{n-1}))$. By positivity, the parameter α is greater than zero and by anonymity, α cannot depend on j . Substituting back into (5) and then into (2), it follows that

$$\begin{aligned} D_i(y^0, y^1) &= D_i(y^1, y^1) + \alpha(y_j^1 - y_i^1) \\ &= D_i(y^1, y^1) + \alpha \sum_{j \in B_i(y^1) \setminus B_i(y^0)} (y_j^1 - y_i^1) \\ &= D_i^\alpha(y^0, y^1) \end{aligned}$$

which establishes the result for the case $|B_i(y^1) \setminus B_i(y^0)| = 1$.

To complete the induction proof, suppose that $|B_i(y^1) \setminus B_i(y^0)| = K \geq 2$ and, for all $(z^0, z^1) \in \mathbb{R}_+^{2n}$ such that $|B_i(z^1) \setminus B_i(z^0)| \in \{1, \dots, K-1\}$, we have

$$D_i(z^0, z^1) = D_i(z^1, z^1) + \alpha \sum_{k \in B_i(z^1) \setminus B_i(z^0)} (z_k^1 - z_i^1). \quad (9)$$

Let $j \in B_i(y^1) \setminus B_i(y^0)$. By part (ii) of recursivity, we have

$$\begin{aligned} D_i(y^0, y^1) &= D_i((y^0|_{(B_i(y^1) \setminus B_i(y^0)) \setminus \{j\}}, y^1|_{N \setminus ((B_i(y^1) \setminus B_i(y^0)) \setminus \{j\})}), y^1) \\ &\quad + D_i((y^0|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}}), (y^1|_{\{i,j\}}, y_i^1 \mathbf{1}_n|_{N \setminus \{i,j\}})). \end{aligned} \quad (10)$$

Because $y_j^1 > y_i^1$, it follows that

$$j \in B_i((y^0|_{(B_i(y^1) \setminus B_i(y^0)) \setminus \{j\}}, y^1|_{N \setminus ((B_i(y^1) \setminus B_i(y^0)) \setminus \{j\})}))$$

and hence

$$j \notin B_i(y^1) \setminus B_i((y^0|_{(B_i(y^1) \setminus B_i(y^0)) \setminus \{j\}}, y^1|_{N \setminus ((B_i(y^1) \setminus B_i(y^0)) \setminus \{j\})})).$$

Therefore,

$$|B_i(y^1) \setminus B_i((y^0|_{(B_i(y^1) \setminus B_i(y^0)) \setminus \{j\}}, y^1|_{N \setminus ((B_i(y^1) \setminus B_i(y^0)) \setminus \{j\})}))| = K - 1$$

and, by (9),

$$\begin{aligned} D_i((y^0|_{(B_i(y^1) \setminus B_i(y^0)) \setminus \{j\}}, y^1|_{N \setminus ((B_i(y^1) \setminus B_i(y^0)) \setminus \{j\})}), y^1) \\ = D_i(y^1, y^1) + \alpha \sum_{k \in (B_i(y^1) \setminus B_i(y^0)) \setminus \{j\}} (y_k^1 - y_i^1). \end{aligned}$$

Substituting into (10), an argument analogous to that employed in the initial induction step can be applied to conclude that

$$\begin{aligned} D_i(y^0, y^1) &= D_i(y^1, y^1) + \alpha \sum_{k \in (B_i(y^1) \setminus B_i(y^0)) \setminus \{j\}} (y_k^1 - y_i^1) + \alpha(y_j^1 - y_i^1) \\ &= D_i(y^1, y^1) + \alpha \sum_{k \in B_i(y^1) \setminus B_i(y^0)} (y_k^1 - y_i^1) \\ &= D_i^\alpha(y^0, y^1) \end{aligned}$$

and the proof is complete. ■

The above theorem can alternatively be formulated as an if-and-only-if statement. In that case, it needs to be explicitly stated that the restriction of D_i to the set of distribution pairs $\{(y^0, y^1) \in \mathbb{R}_+^{2n} \mid y^0 = y^1\}$ satisfies the restrictions of the requisite axioms to this set of pairs.

3 The Yitzhaki index revisited

A natural special case emerges if the static relative deprivation measure $D_i(y^1, y^1)$ is given by the Yitzhaki index. In this case, we obtain the indexes $D_i^{Y,\alpha}$ defined by setting, for all $(y^0, y^1) \in \mathbb{R}_+^{2n}$,

$$D_i^{Y,\alpha}(y^0, y^1) = \frac{1}{n} \sum_{j \in B_i(y^1)} (y_j^1 - y_i^1) + \alpha \sum_{j \in B_i(y^1) \setminus B_i(y^0)} (y_j^1 - y_i^1).$$

The significance of the parameter value α now becomes even more apparent. If $\alpha = 1/n$, the income shortfalls from those whose incomes pass beyond that of individual i in the move from the previous period to the current period receive twice as much weight as the shortfalls associated with the other members of $B_i(y^1)$. A possible choice might be to select a value of α below $1/n$ but, clearly, this is not the only possibility; if the negative sentiment of being left behind is rather strong, it may very well be plausible to set α equal to $1/n$, or even at a value that exceeds $1/n$. Clearly, our class allows for sufficient flexibility to accommodate any of these options. The class of measures $\{D_i^{Y,\alpha} \mid \alpha \in \mathbb{R}_{++}\}$ can be characterized by adding a suitable set of axioms imposed on the restriction of D_i to the set of pairs $\{(y^0, y^1) \in \mathbb{R}_+^{2n} \mid y^0 = y^1\}$ while, of course, avoiding any redundancies. Natural candidates for such an extension are the axiomatization of the Yitzhaki index by Ebert and Moyes (2000) and that provided by Bossert and D'Ambrosio (2006). We conclude this section with a characterization of the new class that is based on the latter.

Bossert and D'Ambrosio (2006) characterize the static Yitzhaki index by means of five independent axioms. They are a static version of the focus axiom, translation invariance, linear homogeneity, a normalization property and an additive-decomposability condition. This system of axioms can be simplified in our dynamic setting because the properties already introduced imply some of the arguments required in establishing the static part of the index.

The focus axiom employed by Bossert and D'Ambrosio (2006) is implied by our version. This is immediate because it is obtained whenever $y^0 = y^1$, that is, whenever the restriction of the index to a static setting is considered. The static version of linear homogeneity clearly is implied by our variant because its scope covers the case in which $y^0 = y^1$.

Moreover, there is no need to impose the static version of translation invariance because it is implied by virtue of current-period translation invariance and the focus axiom. To see that this is the case, suppose that $y^1 \in \mathbb{R}_+^n$ and $\delta \in \mathbb{R}$ are such that $(y^1 + \delta \mathbf{1}_n) \in \mathbb{R}_+^n$.

By current-period translation invariance, it follows that

$$D_i(y^1, y^1 + \delta \mathbf{1}_n) = D_i(y^1, y^1)$$

and, applying previous-period translation invariance (which is implied by the focus axiom; again, see Lemma 1), we obtain

$$D_i(y^1 + \delta \mathbf{1}_n, y^1 + \delta \mathbf{1}_n) = D_i(y^1, y^1 + \delta \mathbf{1}_n).$$

Combining these two observations, it follows immediately that

$$D_i(y^1 + \delta \mathbf{1}_n, y^1 + \delta \mathbf{1}_n) = D_i(y^1, y^1),$$

as was to be established.

This leaves us with the axioms of normalization and additive decomposability, defined as follows (see Bossert and D'Ambrosio, 2006, pp. 424–425).

Normalization. For all $y^1 \in \mathbb{R}_+^n$, if there exists $j \in N \setminus \{i\}$ such that $y_j = 1$ and $y_k = 0$ for all $k \in N \setminus \{j\}$, then

$$D_i(y^1, y^1) = \frac{1}{n}.$$

Additive decomposability. For all $y^1 \in \mathbb{R}_+^n$ and for all $C^1, C^2 \subseteq B_i(y^1)$, if $C^1 \cap C^2 = \emptyset$ and $C^1 \cup C^2 = B_i(y^1)$, then

$$D_i(y^1, y^1) = D_i((y_i \mathbf{1}_n|_{C^1}, y|_{N \setminus C^1}), (y_i \mathbf{1}_n|_{C^1}, y|_{N \setminus C^1})) + D_i((y_i \mathbf{1}_n|_{C^2}, y|_{N \setminus C^2}), (y_i \mathbf{1}_n|_{C^2}, y|_{N \setminus C^2})).$$

Note that anonymity is not required in the requisite (static) result. This is the case because the normalization property applies to any choice of j and, therefore, an additional impartiality condition would be redundant. In addition, our positivity requirement can be weakened so as to apply only to non-static situations—that is, to cases in which y^0 and y^1 differ. This follows from the observation that the conjunction of normalization and additive decomposability takes care of the requisite requirement in the static case. Moreover, the anonymity property employed in Theorem 1 can be restricted to situations in which the distributions in the previous period and the current period are different. The resulting weaker axioms are defined as follows.

Restricted positivity. For all $(y^0, y^1) \in \mathbb{R}_+^{2n}$ such that $y^0 \neq y^1$,

$$D_i(y^0, y^1) > 0 \Leftrightarrow B_i(y^1) \neq \emptyset.$$

Restricted anonymity. For all $(y^0, y^1) \in \mathbb{R}_+^{2n}$ such that $y^0 \neq y^1$ and for all permutations $\pi: N \rightarrow N$ such that $\pi(i) = i$,

$$D_i(y^0_\pi, y^1_\pi) = D_i(y^0, y^1).$$

The following theorem characterizes the class of extended Yitzhaki indexes $D_i^{Y,\alpha}$. Its proof follows from Theorem 1, the characterization result of Bossert and D'Ambrosio (2006) and the above remarks.

Theorem 2 *An intertemporal individual relative deprivation index D_i satisfies restricted positivity, focus, restricted anonymity, linear homogeneity, current-period translation invariance, recursivity, normalization and additive decomposability if and only if there exists $\alpha \in \mathbb{R}_{++}$ such that $D_i = D_i^{Y,\alpha}$.*

4 Discussion

In the main body of the paper, we work within a fixed-population environment. Thus, it is not necessary to explicitly note that the parameter α may depend on the population size n under consideration. Once variable-population variants of our measures are to be discussed, an entire sequence of parameters is required—one parameter α^n for each possible population size $n \geq 2$.

A common variable-population requirement is to demand that an index be replication invariant—that is, an m -fold replica of a distribution pair (y^0, y^1) should be associated with the same relative deprivation as the original distribution. To use a concrete example, consider the extension of the Yitzhaki index discussed in Section 3. Suppose that we require the index to be replication invariant so that, for all possible population sizes n , for all pairs (y^0, y^1) and for all replica sizes m , it must be the case that

$$\begin{aligned} & \frac{1}{nm} m \sum_{j \in B_i(y^1)} (y_j^1 - y_i^1) + \alpha^{nm} m \sum_{j \in B_i(y^1) \setminus B_i(y^0)} (y_j^1 - y_i^1) \\ &= \frac{1}{n} \sum_{j \in B_i(y^1)} (y_j^1 - y_i^1) + \alpha^n \sum_{j \in B_i(y^1) \setminus B_i(y^0)} (y_j^1 - y_i^1); \end{aligned}$$

this condition follows because replicating (y^0, y^1) leads to corresponding replications of the sets $B_i(y^0)$ and $B_i(y^1)$ as well as the associated income shortfalls. Simplifying, this implies that we must have $\alpha^{nm} = \alpha^n/m$ and also $\alpha^{nm} = \alpha^m/n$ because the roles of n and m can be interchanged. Thus, it must be true that $\alpha^n/m = \alpha^m/n$ and, setting $n = 2$ and solving for α^m , it follows that

$$\alpha^m = \frac{2\alpha^2}{m}$$

for all $m \in \mathbb{N} \setminus \{1\}$. For instance, if $\alpha^2 = 1/4 = 1/(2 \cdot 2)$, the requisite remaining parameter values that ensure replication invariance are given by $\alpha^m = 1/(2m)$.

The only role played by the income distribution of the previous period is that it identifies those individuals in $B_i(y^1)$ who should be given higher weights when calculating the requisite value of the measure. Thus, there is no additional complexity involved in arriving at the dynamic variant of a standard static measure of relative deprivation; all that needs to be done is to add a weighted sum to the static index where this sum is given by the aggregate shortfall associated with a subgroup of those in $B_i(y^1)$ —namely, those who moved ahead of i in the transition from yesterday to today. A change in the positions occupied by the individuals in the income hierarchy only has an impact on the measure if it moves a previously lower-ranked person above the individual i under consideration.

A phenomenon related to but different from the subject matter analyzed here is what is often referred to as the tunnel effect; see Hirschman and Rothschild (1973). According to this effect, advances of others may, at least in the short term, be beneficial from the

viewpoint of an individual. If one wants to capture this response, it is possible to amend our measures accordingly by subtracting instead of adding the second term in the requisite expression.

As a final remark, note that the two-period model analyzed in this paper has some parallels with the measurement of income mobility; see, for instance, Maasoumi (1998), Fields and Ok (1999) and Jäntti and Jenkins (2014) for comprehensive surveys. While income differences play a crucial role in that literature as well, there are some features that set the two apart. When assessing mobility, these differences are the (absolute values of the) within-person differences $y_i^1 - y_i^0$ from one period to the next. In contrast, the two-period approach to relative deprivation utilizes across-person differences such as $y_j^1 - y_i^1$ within the same time period under consideration. In addition, the response to certain changes in the distribution differs under numerous circumstances, depending on the chosen interpretation. For instance, a *ceteris-paribus* change (increase or decrease) of person j 's income in period one that reduces the absolute value of the difference between y_j^1 and y_j^0 will reduce mobility as measured by an index based on these differences, but the impact of such a change on the relative deprivation of another person i depends on whether person j is among those whose incomes exceeded that of i prior to and after the change. Again, what matters for relative deprivation measurement is the relationship between y_j^1 and y_i^1 rather than that between y_j^1 and y_j^0 .

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