Kurt Gödel

and the

Foundations of Set Theory

Emil Weydert

University of Luxembourg CSC/ICR

Kurt Gödel - Philosophical Views

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Roadmap

- Set theory in a nutshell
- Gödel's journey beyond ZFC
- The continuum hypothesis
- The constructible universe
- Gödel's set-theoretic realism
- Gödel's program and principles
- Visiting the new axiom zoo
- Conclusion

About set theory

Gödel and the Foundations of ... **What?**

Set theory: (Georg Cantor 1874-1883)

- abstract extensions (sets $\neq$ properties)
- iterability (forming sets of sets of sets ...)
- multiple infinities (transfinite arithmetic)
- a foundation/framework for (meta-)mathematics

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Some skeptics: non-constructivity, metaphysical dimension

- Kronecker: *Cantor as "corrupter of the youth" (19th)*
- Wittgenstein: *"utter nonsense", "laughable" (20th)*
- HoTT disciples: *inconvenient for math, computerization (21th)*

Axiomatic set theory

Foundational crisis: Russell's paradox $\{x \mid x \notin x\}$ (1901)

$\rightarrow$ responses: *Brouwer's Intuitionism, Hilbert's Formalism, ...*

Ideas: restricting set-formation, set-class distinction, axioms

Axiomatic set theory: ZFC (Zermelo-Fraenkel 1908/22)

Axioms: *extensionality, empty set, pairs, union, power set, infinity, foundation, replacement, choice* ($\sim$ enough for math)

Infinite counting: $0, 1, 2, \ldots \omega, \omega + 1, \ldots \omega + \omega, \ldots \omega_1, \ldots$

Ordinal number: set of its predecessors, $\alpha = \{\beta \mid \beta < \alpha\}$

Cumulative hierarchy: layered universe (von Neumann 1928)

Layers: $V_0 = \emptyset,\ V_{\alpha+1} = P(V_\alpha),\ V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha,\ V = \bigcup_\alpha V_\alpha$

Journey beyond ZFC

Hilbert: *"From the paradise, that Cantor created for us, no one can expel us."* (1926)

But: Is ZFC enough to capture all the fruits of the paradise?

Gödel's view: *No! But we can do something about this.*

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Gödel's contributions to transcend ZFC:

1. The incompleteness theorems

2. Introducing the inner model technique (L, HOD)

3. Relative consistency of CH: $Con(ZF) \rightarrow Con(ZFC + GCH)$

4. An optimistic ontological realism and epistemic pragmatism

5. Gödel's program (rational search for new set-theoretic axioms)

New axioms - prologue

Gödel's 2nd incompleteness theorem (1931):

No consistent formal system including elementary arithmetic [e.g. ZFC] can prove its own consistency.

A first source for (infinitely many) new justified set properties:

- $ZFC + Con(ZFC), ZFC + Con(ZFC + Con(ZFC)), \ldots$

Gödel (1946): "*... there cannot exist any formalism which would embrace all these steps; but this does not exclude that ... [they] could be described and and collected in some nonconstructive way. In set theory, e.g., ... by stronger and stronger axioms of infinity ...* "

But, are there also *natural assertions* independent from ZFC? **Yes!**

Hilbert's first problem

Continuum Hypothesis: *Every set of reals is either bijective (same size) to* $\mathbb{N} = \{0, 1, 2, ...\}$, *or to the reals* $\mathbb{R}$ (Cantor 1878).

Cantor tried hard, but failed to settle CH

CH: Nr 1 of Hilbert's 23 unsolved math. problems (1900)

Attack 1: Gödel's constructible sub-universe L (1940):

- $ZF \vdash Con(ZF) \rightarrow Con(ZFC + CH)$

Attack 2: Cohen's revolutionary forcing technique (1963):

- $ZF \vdash Con(ZF) \rightarrow Con(ZFC + \neg CH)$

Hence: *CH is independent from ZFC!*

Popular lazy conclusion: CH is not a definite problem.

Gödel: It may well be decided by new intuitive axioms!

The constructible universe

Gödel (1940): Every model of ZF includes a constructible sub-universe L verifying AC, CH, and $\mathbf{V{=}L}$: universe is constructible

$Def(X) :=$ 1st-order-definable subsets with parameters over $(X, \in)$

$$L_0 = \emptyset,\ L_{\alpha+1} = Def(L_\alpha),\ L_\lambda = \cup_{\alpha<\lambda} L_\alpha,\ L = \cup_\alpha L_\alpha$$

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V=L ... a desirable new axiom for completing ZFC?

- ZFC+V=L decides CH
- offers a detailed (but ugly) picture of definable sets of reals
- restricts the universe of sets, precludes strong axioms of infinity

Gödel's answer: **No!**

The realist's tale

Gödel - a self-described platonist and realist, looking beyond ZFC.

" ... mathematics describes a non-sensual reality ... independently of the human mind and is only perceived, and probably very incompletely by the human mind." (Gödel 1951)

" ... [the ZFC axioms] do not contain a complete description of that [well-determined mathematical] reality." (Gödel 70s)

" ... the fact that the [ZFC] axioms force themselves upon us as true. I don't see any reason why we should have less confidence in ... mathematical intuition, than in perception, ... a question not decidable now has meaning and may be decided in the future" (Gödel 1964)

" ... the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions to them suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis." (Gödel 1947)

The realist's tasks

Gödel's views were well supported by new axioms of infinity.

" ... these axioms [inaccessible and Mahlo cardinals] show clearly, not only that the axiomatic system of set theory as known today is incomplete, ... can be supplemented without arbitrariness by new axioms which are only the natural continuation of those set up so far." (Gödel 1947)

But less so for CH. Gödel's direct arguments against CH based on geometric intuition (Gödel 1947) were met with skepticism - just like the Banach-Tarski "paradox" ultimately did not discredit AC.

On the other hand, while Gödel believed in rational mathematical intuition, he was realistic:

" ... mathematical intuition need not be conceived as a faculty giving immediate knowledge of the objects concerned." (Gödel 70s)

The realist's progress

Gödel expected and hoped for new *intrinsically justified* principles and resulting axioms. But he was aware that conceptual insight is evolving and needs differentiation.

" ... probably ... there exist other [axioms] based on hitherto unknown principles ... which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts." (Gödel 1947)

" ... strong axioms of infinity of an entirely new kind [measurable cardinals] ... implies the existence of non-constructible sets. ... implied by the general concept of set in the same sense as Mahlo's has not been made clear yet. ... However they are supported by strong arguments from analogy ... ." (Gödel 1966)

Gödels program

Gödel was also ready to consider *extrinsic justifications*, at least as a guide to mathematical reality.

" ... There might exist axioms so abundant in their verifiable [including mathematical intuition] consequences ... that quite irrespective of their intrinsic necessity they would have to be assumed in the same sense as any well-established physical theory." (Gödel 1947)

A highly prescient remark, extensively confirmed, and well in line with the current turn in the philosophy of mathematics towards mathematical practice.

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Gödel's program (in a nutshell): *Discover new intrinsically or extrinsically justified set-theoretic axioms to elucidate the concept of set, and in particular to decide CH.*

Gödel's principles

In the 70s, Gödel identified **5 principles for axiom choice**:

1. Intuitive range. Overviewable multitudes can be represented by sets, which essentially justifies ZFC.

2. Closure principle. Closure properties of the universe are also realized within specific sets (e.g. inaccessible, Mahlo cardinals).

3. Reflection principle. The universe of sets is structurally indefinable, indiscernible from its initial segments "*in any logic of finite or transfinite type, including infinitary logics*". See Reinhardt's strong reflection axioms (1974).

4. Extensionalization. Generalize principles for sets based on defining properties to arbitrary extensions.

5. Uniformity. Properties of smaller infinite sets or cardinals, e.g. of the first infinite ordinal ω, reappear - suitably adapted - at arbitrary high levels (link with reflection).

Axioms for the 21st century

Three major types of new axioms:

- **Large cardinal axioms**
  - implement the idea of maximality ($\to$ reflection, uniformity)

Promising evidence: large cardinals are (essentially) linearly ordered according to their relative consistency strength:

$$A1 \preceq_{con} A2 \text{ iff } Con(ZFC + A2) \vdash_{ZFC} Con(ZFC + A1)$$

A quasi-universal yardstick for measuring consistency strength!

- **Axioms of determinacy**
  - existence of winning strategies for some infinite integer games
- **Forcing axioms**
  - make forcing possible in broader contexts

Large cardinals

When is κ a large cardinal? *Reflection and Uniformity*

- *Weak reflection*: V_κ is like V, e.g. inaccessible cardinals
- *Finite-infinite analogy*: e.g. strong compactness, i.e. T with $|T| \geq \kappa$ is satisfiable in the infinitary logic $\mathcal{L}_{\kappa,\kappa}$ iff all the subsets $S \subseteq T$ with $|S| < \kappa$ are satisfiable
- *Strong reflection/uniformity principles*, e.g. 1-extendibility: elementary embedding $j : V_{\kappa+1} \mapsto V_{\lambda+1}$ $(\kappa < \lambda)$ ("κ like λ")

Sad fact: *Standard large cardinals cannot decide CH*

... because they are invariant under forcing CH or ¬CH

Axioms of determinacy

Axiom of determinacy (AD): Any perfect information number game of length ω is determined, i.e. one player has a winning strategy (always $(n_0, m_1, n_2, \ldots) \in Win_I$, or ... $\in Win_{II} = \mathbb{R} - Win_I$)

- ZFC: *limited determinacy*, only for simple (e.g. closed) Win_I
- ZFC: *AD fails*, inconsistent with the axiom of choice

ZFC + suitable large cardinals:

- $L(\mathbb{R}) \models ZF + AD$ and $V \models PD$ (det. for projective sets)
- $Th(L(\mathbb{R}))$ is invariant under forcing
- Nice properties for definable sets of reals (e.g. Lebesgue measurable).

Forcing axioms

Forcing: (sloppily) adding a new set G to $M \models ZFC$

Tool for relative consistency/independence proofs (Cohen 63)

Forcing axioms: more forcing opportunities (indep. of ZFC)

Martin's maximum: decides CH ($2^{\aleph_0} = \aleph_2$) + implies PD
+ solves Whitehead's problem (no) $\rightarrow$ *extrinsic justification!*

Beyond: ZFC + very large cardinals + very strong forcing:
can decide much of 3rd-order arithmetic (where CH lives)

$\rightarrow$ *These are amazing results!*

Conclusion

Gödel's program today:

An active, far-reaching, and multi-faceted research endeavour

- helping to elucidate Cantor's set concept,
- pushing back the frontiers of decidability/knowledge,
- closing up on CH, and
- implementing a large-scale experiment probing the viability and limits of mathematical realism!

A century after Gödel chased us from Hilbert's promised land,
in some way, his ideas help us to find a way back