AUTOMATION, FACTOR SHARES, AND GROWTH IN THE ERA OF POPULATION AGING

- First Version -

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Abstract: How does population aging affect factor shares and economic growth in times of declining investment good prices and increasingly automated production processes? The present paper addresses this question in a new model of automation where competitive firms perform tasks to produce output. Tasks require labor and machines as inputs. New machines embody superior technological knowledge and substitute for labor in the performance of tasks. The incentive to automate is stronger when the expected wage is higher or when the price of an automation investment is lower. Automation is shown to i) boost the aggregate demand for labor if the incentives to automate are strong enough and ii) reduce the labor share. These predictions obtain even though automation is labor-augmenting in the economy's reduced-form production function. In the short run, population aging weakens the incentives to automate and increases the labor share as individuals augment their labor supply. These implications may be neutralized if, at the same time, the price of investment goods declines. In the log-run, population aging and a lower price of investment goods are reinforcing. Both imply more automation, a lower labor share, and faster economic growth.

Keywords: Population Aging, Automation, Factor Shares, Endogenous Labor Supply, Endogenous Technical Change.

JEL-Classification: E22, J11, J22, J23, O33, O41.

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1 Introduction

Automation, i. e., the use of machines to replace human beings in the performance of tasks, has been a key driver of economic growth since the beginning of the industrial revolution (Landes (1969), Mokyr (1990), Allen (2009)). At least since the early 1960ies, this tendency has become more pronounced as technological advances in areas like robotics, information technology, digital technology, and artificial intelligence substantially widened the scale and scope of automation (Brynjolfsson and McAfee (2014), Ford (2015), Ross (2016), Goldfarb and Tucker (2017)).

In this paper I argue that the intensified implementation of automation technologies since the 1960ies reflects two main developments in the global macroeconomic environment in which firms have been operating: population aging and the decline in the relative price of investment goods. These developments are also shown to account for the widely observed decline in the labor share over this period (Piketty (2014), Karabarbounis and Neiman (2014)).

Population aging is the process by which older individuals become a proportionally larger fraction of the total population. Its two main causes, an increase in longevity and a decline in fertility (Weil (2008)), play a central role in my analysis. Figure 1.1 documents the substantial increase in the survival probability for males to age 65, a proxy for longevity, in 14 industrialized countries from 1960 - 2016.[1] Figure 1.2 shows the decline in the total fertility rate from 1969 - 2016.[2] Hence, for many industrialized countries the period from 1960 to today has been an era of population aging. This era is predicted to extend further into the 21st century (Lutz, Sanderson, and Scherbov (2008), United Nations (2015)). The effects of population aging on automation, factor shares, and economic growth identified in this paper will therefore remain of relevance.

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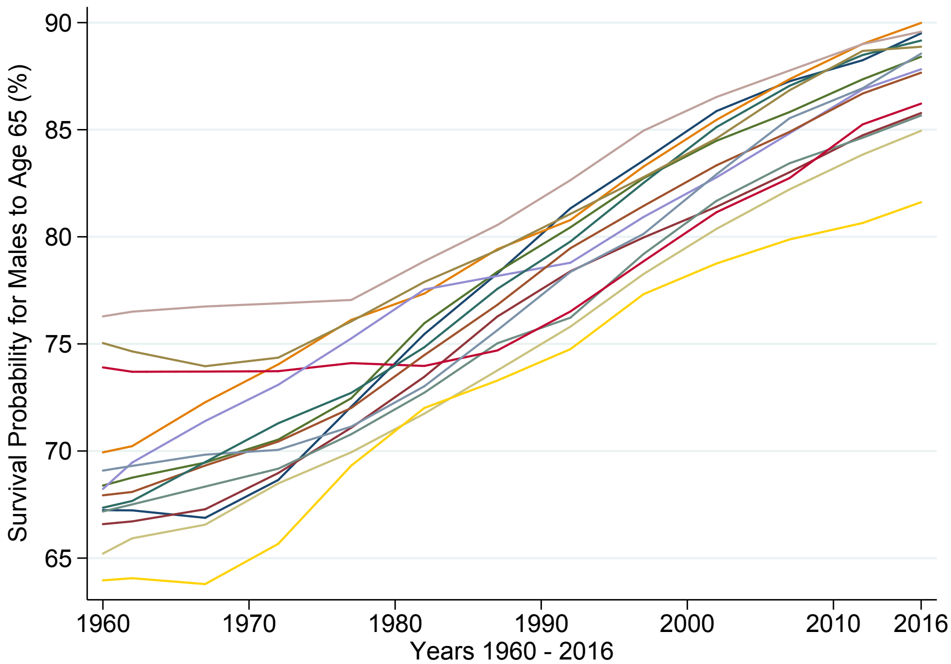
[1] The survival probability for males to age 65 is defined as the percentage of a cohort of newborn male infants that would survive to age 65, if subject to age specific mortality rates of the specified year. It has a natural counterpart in my theoretical analysis below. This motivates the choice of this variable as a proxy for longevity.

Figure 1.1 shows a sample of 14 countries: Australia, Belgium, Canada, Denmark, France, Germany, Ireland, Italy, Netherlands, Spain, Sweden, Switzerland, UK, US. The data are from United Nations (2017) accessed through $https://data.worldbank.org/indicator/SP.DYN.TO65.MA.ZS?end = 2016$. The time horizon is 1960-2016. Regressing the survival probability on a country fixed effect and years gives a slope coefficient of roughly 0.37%. Hence, over 30 years the increase in the average survival probability is 11.1%. Qualitatively similar evolutions obtain for women.

[2] The sample includes the same 14 countries as shown in Figure 1.1. The data are taken from $https://data.worldbank.org/indicator/SP.DYN.TFRT.IN?end = 2016\&locations = AU\&start = 1960\&view = chart\&year = 2015$. The total fertility rate is the number of children that would be born to a woman if she were to live to the end of her childbearing years and bear children in accordance with age-specific fertility rates of the specified year. Regressing the total fertility rate on a country fixed effect and years gives a slope coefficient of roughly -0.215. Hence, over 30 years the decline in the average total fertility rate is equal to -0.645.

Figure 1.1: **The Increase in the Survival Probability for Males to Age** 65 **in** 14 **Industrialized Countries from** 1960 **-** 2016**.**

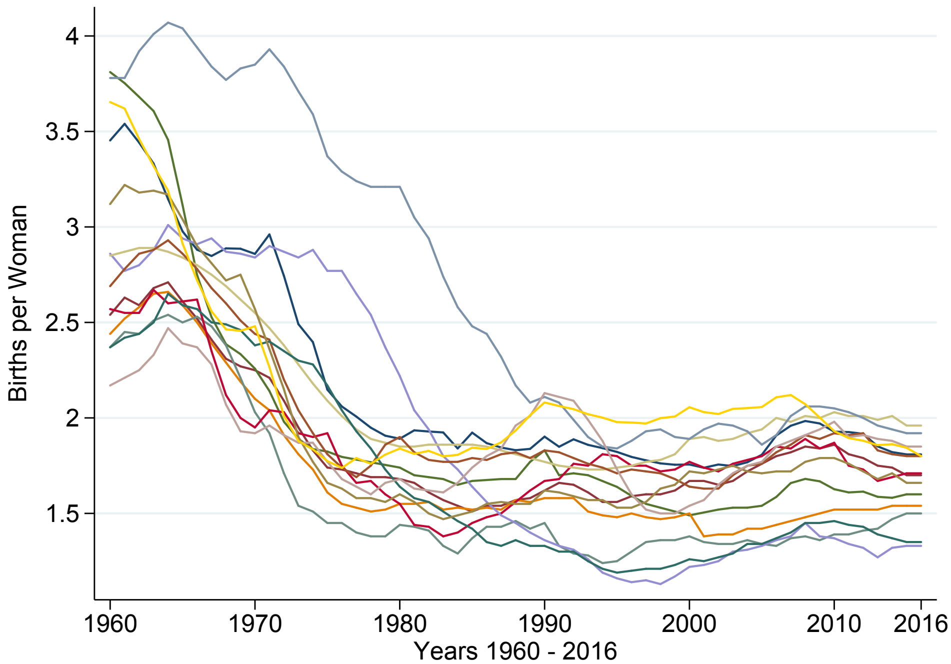


The second development that characterizes the macroeconomic environment since the 1960ies is the global decline in the *relative price of investment goods*. This decline was moderate until 1980, and then accelerated until today (Gordon (1990), Greenwood, Hercovitz, and Krusell (1997), Fisher (2006), Karabarbounis and Neiman (2014)).

I incorporate these features into a novel one-sector endogenous growth model to study their implications for automation, factor shares and economic growth. Competitive firms engage in automation investments that substitute new machines for workers in the performance of tasks. The production sector builds on and extends ideas of the so-called induced innovations literature (Hicks (1932), Drandakis and Phelps (1966), Funk (2002)). The household sector features two-period lived overlapping generations. Individuals face a survival probability when they enter the second period of their lives. Population aging corresponds to an increase in this probability and/or to a decline in fertility. These changes capture the tendencies shown in Figure 1.1 and 1.2. The per-period utility function of individuals is of the generalized log-log type recently proposed by Boppart and Krusell (2018). Hence, the labor supply is endogenous and declines at a constant rate in response to a constant wage growth (Irmen (2018)).

My key findings pertain to the long run. The economy's steady-state path is consistent with Kaldor's famous facts (Kaldor (1961)). Moreover, in line with recent empirical evidence, the supply of hours worked declines at a constant rate (Huberman and Minns (2007), Boppart and Krusell (2018)). Periodic automation investments generate sustained

Figure 1.2: **The Decline in the Total Fertility Rate in** 14 **Industrialized Countries from** 1960 **-** 2016**.**



productivity growth that supports the growth of per-capita variables in the long run. Population aging as well as a permanent decline in the price of investment goods are shown to imply more automation, a lower labor share, and faster economic growth.

The intuition behind these findings may be sketched as follows. When people expect to get older they want to work longer hours and to save a larger fraction of their earnings. Both behavioral adjustments are meant to increase consumption possibilities in old age and increase savings. This stimulates the process of capital accumulation and allows for higher wages. As labor becomes more expensive, firms respond with more automation. This reduces the labor share as automation drives a wedge between the marginal product of labor and the real wage. Accordingly, in steady state the labor share will be lower, however, the productivity of labor grows faster. A decline in the fertility rate mimics these findings though the channel is somewhat different. Savings per worker and wages will be higher in all periods following a permanent fertility decline, hence, also in steady state. As a consequence, the long run has more automation, a lower labor share, and faster economic growth. A lower real price of automation investments reduces investment costs and induces a general equilibrium effect which arises since the price of automation investments also affects the demand for labor. As the former dominates the latter, firms automate more so that labor productivity grows faster. Hence, in the long run, a lower real price of automation investments reduces the labor share and speeds up economic growth.

The derivation of the role of population aging and the decline in the relative price of automation investments for the long run requires several intermediate steps. Some of these are original and deliver novel and interesting insights in their own right. They include the following. First, a central conceptual innovation of this paper concerns the production sector which builds on and extends the one devised in Irmen (2017) and Irmen and Tabakovic (2017). Here, competitive firms undertake automation investments in new machines. These machines embody new technological knowledge that improves the productivity of labor in the performance of tasks. The incentives to automate are more pronounced the higher the wage and the lower the price of automation investments. At the level of the individual task, automation gives rise to a *rationalization effect* as fewer working hours are needed to perform a task and to a *productivity effect* as the cost per task falls. These effects imply that automating firms will expand the number of performed tasks and produce more output. It is in this sense that automation is associated with a *task expansion effect* and an *output expansion effect*.

Second, important macroeconomic implications derive directly from the behavior of the production sector. First, I establish that automation implies a higher aggregate demand for labor if and only if the real wage is sufficiently high. In other words, strong incentives to automate boost the aggregate demand for labor. Similarly, a decline in the price of automation investments boosts the aggregate demand for labor if the incentives to automate are strong. Simple back-of-the envelope calculations suggest that, irrespective of whether automation incentives are strengthened via a higher real wage or via a lower price of investments, the aggregate demand for labor will be higher if automation investments are undertaken. Second, I show that automation unequivocally reduces the labor share. As mentioned above, this obtains since automation drives a wedge between the marginal product of labor and the real wage. In a sense, workers pay for the automation investment that raises their productivity. Finally, I highlight that automation is labor-augmenting in the reduced form of the aggregate production function.

Third, I analyze the behavior of the household sector that has two-period lived overlapping generations facing a survival probability at the beginning of second period of their lives. Cohorts are endowed with a per-period utility function of the log-log type recently proposed by Boppart and Krusell (2018). I show that cohorts expand their labor supply and save more in response to an increase in the survival probability. To the best of my knowledge, the present paper is the first that incorporates preferences of the Boppart-Krusell class into a full-fledged endogenous growth model.

The fourth set of results relates to the role of automation for the equilibrium wage and the equilibrium labor share in the short run. Population aging due to a higher life expectancy increases the labor supply and, therefore, unequivocally reduces the equilibrium wage. This weakens the incentives to automate, and shifts the labor share upwards. Hence, the short-run and the long-run implications of population aging through a higher life expectancy on automation incentives are of opposite sign. As a decline in the price of automation investments increases the aggregate demand for labor, it will also increase the equilibrium wage. As a consequence, there are two reinforcing channels through which a lower price of automation investments boosts the incentives to automate in the short run: a direct effect on firms' incentives and an indirect effect through the labor market equilibrium.

The fifth set of results concerns the transitional dynamics. I show that the steady state is unique and stable over the range of parameter values that guarantee endogenous automation investments.

Finally, I present a simple calibration exercise. It highlights that the model is consistent with the orders of magnitude for long-run growth rates of per-capita variables and hours worked, the labor share, and the rental rate of capital that are found in the data.

The present paper is related to several strands of the literature. First, it contributes to the recent literature on endogenous economic growth and automation (Acemoglu and Restrepo (2018a), Acemoglu and Restrepo (2018c), Acemoglu and Restrepo (2018d), Hémous and Olsen (2018), Krenz, Prettner, and Strulik (2018)). Contrary to a commonly held opinion, I show that automation can be modelled as endogenous labor-augmenting technical change resulting from investments in new machines that substitute for human labor in a widening range of tasks.[3] This analytical strategy is consistent with well received predictions. For instance, automation increases the demand for labor if and only if the induced productivity gains are sufficiently large (see Proposition 4 below). Intuitively, this finding can be traced back to an induced (aggregate) rationalization and an induced (aggregate) task expansion effect.[4] Moreover, automation unequivocally reduces the labor share (see Proposition 6 below).

An alternative approach to modelling the distinction between "robot capital goods" and "ordinary physical capital" in a one-sector growth model was proposed by Steigum (2011). This author shows that endogenous long-run growth is possible if the elasticity of substitution between robot capital and labor is high. In contrast to this approach, the choice of using robots and their quality is endogenous in my model. Moreover, sustained growth is due to the accumulation of technological knowledge embodied in new machines rather than to a mechanism that mimics the AK-model.

[3] In view of recent studies by Sachs and Kotlikoff (2012), Nordhaus (2015), Bessen (2017), or Graetz and Michaels (2018), Acemoglu and Restrepo (2018c), p. 1, argue that common approaches of modelling technical change as factor-augmenting " ... miss a distinctive feature of automation: the use of machines to substitute for human labor in a widening range of tasks. (...) Partly as a result, factor-augmenting technologies have a limited scope to reduce the demand for labor. (...) In addition, these approaches relate the impact of technology on the labor share to the elasticity between capital and labor....". I show in Section 2.2 that technical change is labor-augmenting in my analytical framework and does not suffer from any of these potential drawbacks.

[4] Acemoglu and Restrepo (2018a) identify a displacement effect that reduces the labor demand in response to automation at the extensive margin. My results suggest that similar qualitative implications of automation obtain even in the absence of an extensive margin.

Second, the present paper complements the literature on endogenous economic growth and demographic change (Abeliansky and Prettner (2017), Acemoglu and Restrepo (2018b), Bloom, McKenna, and Prettner (2018), Irmen (2017), Prettner and Trimborn (2017)). In contrast to these contributions, the focus of my research is on the link between population aging, the individual labor supply, individual savings, and the equilibrium incentives to automate. This leads to the insight that the qualitative effects of population aging on automation incentives in the short and in the long run are of opposite sign.

Finally, my research is related to the literature that aims to explain the global decline in the labor share. Unlike the theory proposed in Piketty (2014), here, the steady-state savings rate and the economy's growth rate are endogenous (Irmen and Tabakovic (2016)). I concur with Karabarbounis and Neiman (2014) that a decline in the relative price of automation investments is a key explanatory variable. However, the mechanics these authors propose is quite different from mine. Moreover, I maintain that the decline in the labor share is a long-run consequence of population aging. Higher savings for old age foster the accumulation of capital, lead to higher wages, more automation, and a decline in the labor share. The implications of my model also differ from those derived in Zeira (1998) and Prettner (2016). In contrast to the latter, in my framework the steady-state labor share remains bounded away from zero even if if the incentives to automate are very high (see Corollary 3 below).

The remainder of this paper is organized as follows. Section 2 presents the model. Section 2.1 has a detailed discussion of the production sector. Here, the notions of tasks and of automation are introduced. Moreover, I show that the profit-maximizing production plan associates automation with a rationalization effect, a productivity effect, a task-expansion, and an output expansion effect. Section 2.2 studies important macroeconomic implications that directly derive from the behavior of firms. In particular, I characterize the relationship between automation and the aggregate demand for labor (Section 2.2.1), its role for factor shares (Section 2.2.2), and show that technical change is labor-augmenting in the reduced-form production function (Section 2.2.3). Section 2.3 introduces the household sector. Section 3 studies the inter-temporal general equilibrium. Following its definition (Section 3.1), I focus on the labor market in Section 3.2. Here, I show that the labor market equilibrium is unique and establish the short-run effects of population aging and a declining price of automation investments on the equilibrium wage and automation incentives. The dynamical system is presented in Section 3.3. Section 4 studies the steady state and the transitional dynamics. Here, I derive the long-run implications of population aging and a declining price of automation investments for automation, factor shares, and economic growth. Section 4.3 presents a simple calibration exercise. Section 5 concludes. All proof are relegated to Appendix A. Appendix B is an "online appendix." It contains additional results and generalizations referred to in the main part of the paper.

2 The Model

The economy comprises a production, a household, and an insurance sector in an infinite sequence of periods $t = 1, 2, ..., \infty$. The production sector has competitive firms that manufacture a single good. Building on Irmen (2017) and Irmen and Tabakovic (2017) the production of this good requires tasks to be performed. The manufactured good may be consumed or invested. If invested, it either serves as contemporaneous *automation investments* or as future *fixed capital*.

The household sector has overlapping generations of individuals who potentially live for two periods, youth and old age. Survival into old age is stochastic. The individual lifetime utility function features a Boppart-Krusell generalized log-log utility function (Boppart and Krusell (2018)). Hence, the labor supply is endogenous. I follow, e. g., Yaari (1965) or Blanchard (1985), and assume a perfect annuity market for insurance against survival risk.

There are four objects of exchange, the manufactured good, fixed capital, labor, and annuities. Each period has markets for these objects. Firms rent fixed capital, undertake automation investments, demand labor, and supply the manufactured good. Households demand the manufactured good for consumption and savings, supply labor, and exchange savings for annuity policies. Insurance companies sell these policies and rent the savings as fixed capital to firms producing in the next period. Without loss of generality, fixed capital fully depreciates after one period. The manufactured good serves as numéraire.

Throughout, I denote the time-invariant growth rate of some variable x_t between two adjacent periods by g_x. Moreover, I often use subscripts to write first- and second-order derivatives. For instance, the notation for the derivatives of some function $G(x, y)$ would be $G_2(x, y) \equiv \partial G(x, y)/\partial y$ or $G_{21}(x, y) \equiv \partial^2 G(x, y)/\partial y \partial x$. I shall also write G instead of $G(x, y)$ or $G(\cdot)$ whenever this does not cause confusion.

2.1 The Production Sector

The production sector has many small firms operating under perfect competition. Their behavior may be studied through the lens of a competitive representative firm. At all t, this firm has access to the production function

$$Y_t = \Gamma K_t^{\gamma} N_t^{1-\gamma}, \quad 0 < \gamma < 1. \tag{2.1}$$

Here, Y_t denotes the total output of the manufactured good, K_t the amount of fixed capital, and N_t the amount of performed tasks. The parameter $\Gamma > 0$ reflects cross-country differences in geography, technical and social infrastructure that affect the "transformation" of fixed capital and tasks into the manufactured good.

The performance of tasks requires man-hours and machines.[5] These inputs are strong substitutes with an elasticity of substitution strictly greater than unity. Machines embody technological knowledge. Automation results from investments in new machines that embody improved technological knowledge and increase the productivity of labor in the performance of tasks.

2.1.1 Tasks and Technology

Let $n \in \mathbb{R}_+$ index these tasks. At t, each task is performed once. The production function of task n is

$$1 = a_t(n)h_t(n), \tag{2.2}$$

where $h_t(n)$ is man-hours, and $a_t(n)$ is the productivity per man-hour in the performance of task n. The latter is given by

$$a_t(n) = A_{t-1}\left(1 + q_t(n)\right), \quad q_t(n) \geq 0. \tag{2.3}$$

Here, $A_{t-1} > 0$ is an aggregate indicator of the level of technological knowledge at $t-1$ to which the firm has free access at t. To fix ideas, one may think of A_{t-1} as representing the level of technological knowledge embodied in the last vintage of installed machines that may still be activated. The variable $q_t(n)$ is the growth rate of productivity per man-hour in task n at t. A growth rate $q_t(n) > 0$ requires an automation investment in a new machine at t. This machine partially replaces labor in the performance of task n. The degree to which this substitution occurs is endogenous.[6]

The invention, construction, installation, and running of a new machine for task n gives rise to investment outlays of

$$i_t(n) = \alpha q_t(n), \quad \alpha > 0, \tag{2.4}$$

units of the contemporaneous manufactured good.[7] Here, I interpret α as the real price of an automation investment generating a productivity growth rate in hours worked per

[5] I use the term "man-hour" without any gender-specific connotation.

[6] Allowing for imperfect substitution of labor with machines suggests that task n comprises a set of subtasks. Following the substitution, more of these subtasks are performed by machines.

Imperfect substitution of machines for people is in line with recent evidence, e. g., on the effect of machine learning on occupations presented in Brynjolfsson, Mitchell, and Rock (2018). These authors suggest that the focus of modelling technology should not be on full automation.

[7] If task n was performed in $t-1$ then $i_t(n)$ would also include the scrap costs of the old machine that is replaced. To avoid the asymmetry this would introduce for the investment outlays of tasks $n \in [0, N_{t-1}]$ and $n \in (N_{t-1}, N_t]$ if $N_t > N_{t-1}$ we neglect such expenses. This comes down to assuming that the firm can get rid of old machines without incurring a cost. In other word, the firm's production set satisfies the property of free disposal (see, Mas-Colell, Whinston, and Green (1995), p. 131). Observe further that my qualitative results extend to more general functions i as long as these are increasing and convex.

task of $q_t(n)$.[8] Investment outlays increase in the growth rate of productivity, $q_t(n)$. This reflects a higher cost associated with improved technological knowledge that the new machine embodies.

The technology described by equations (2.2) - (2.4) incorporates the notion of automation as the substitution of man-hours per task with technological knowledge. Indeed, in $(a_t(n), h_t(n))$ - space equation (2.2) has an interpretation of a unit isoquant. It states the set of necessary input combinations of man-hours and technological knowledge as

$$h_t(n) = \frac{1}{a_t(n)}, \quad a_t(n) \geq A_{t-1}. \tag{2.5}$$

This is illustrated in Figure 2.1. At $t = 1$, the relevant isoquant is the blue curve $h_1(n)$ starting at point $(A_0, 1/A_0)$. Since $A_0 > 0$ is given, task n requires at most $1/A_0$ man-hours. The use of more technological knowledge shifts $a_1(n)$ further to the right of A_0. Accordingly, the amount of man-hours shrinks along the isoquant. At $t = 2$, the relevant isoquant, $h_2(n)$, starts at $(A_1, 1/A_1)$. As depicted, $A_1 > A_0$ so that $h_2(n)$ begins to the right of A_1. Again, if more technological knowledge than A_1 is used in the performance of task n then $a_2(n)$ moves further to the right of A_1, and the amount of man-hours shrinks.

Since technological knowledge is embodied in machines the substitution of man-hours per task with technological knowledge has to occur through a substitution of man-hours with machines. Using $q_t(n) = i_t(n)/\alpha \geq 0$ from (2.4) in (2.5) delivers the unit isoquant describing the set of necessary input combinations of man-hours and investment outlays as

$$h_t(n) = \frac{1}{A_{t-1}\left(1 + \frac{i_t(n)}{\alpha}\right)}, \quad i_t(n) \geq 0. \tag{2.6}$$

Figure 2.2 shows this isoquant in the $(i_t(n), h_t(n))$ - space. Hence, automation investments substitute for man-hours. In fact, $h_t(n)$ and $i_t(n)$ are strong substitutes in the sense that the elasticity of substitution, $ES_t(n)$, between both inputs is[9]

$$ES_t(n) = 2 + \frac{\alpha}{i_t(n)} = 2 + \frac{1}{q_t(n)}. \tag{2.7}$$

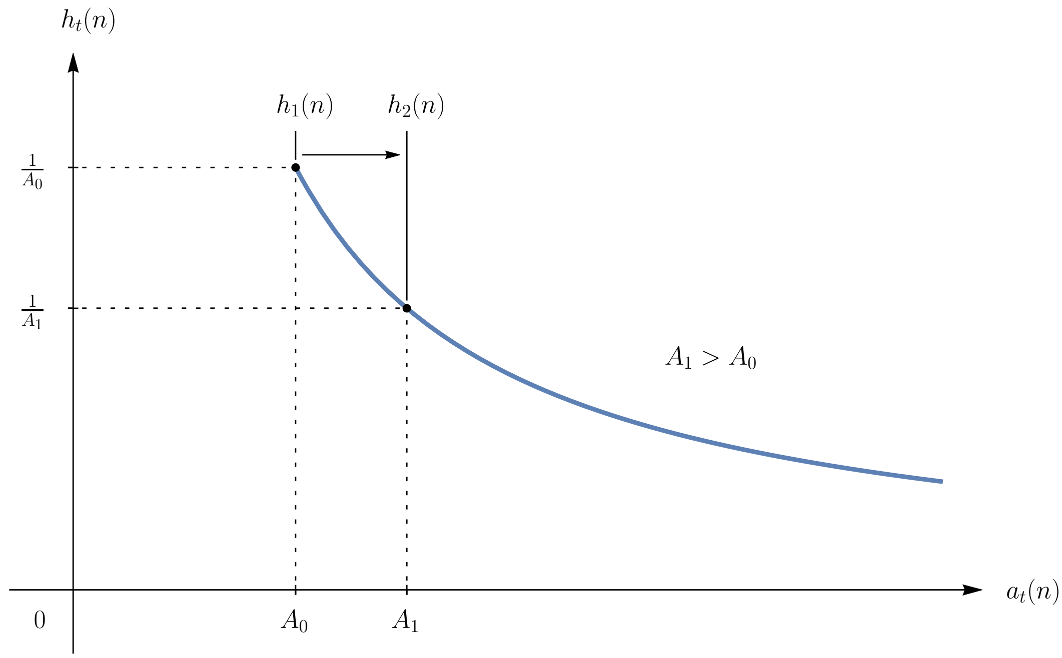
Hence, $ES_t(n) \geq 2$ and declines in the investment volume.

Finally, observe that void of an automation investment at t task n may be performed with a machine of the past vintage that embodies the technological knowledge represented by A_{t-1}, hence $h_t(n) = 1/A_{t-1}$ if $i_t(n) = 0$.

[8] In a somewhat broader sense, α parameterizes the efficiency of the activities that eventually bring the new machine into use.

[9] From (2.6) the technical rate of substitution, $TRS_t(n)$, between an automation investment and man-hours obtains as $TRS_t(n) \equiv dh_t(n)/di_t(n) = -h_t(n)/(\alpha + i_t(n))$. Then, $ES_t(n) \equiv (TRS_t(n)/\left(h_t(n)/i_t(n)\right)) \cdot \left(dTRS_t(n)/d\left(h_t(n)/i_t(n)\right)\right)^{-1}$. A detailed derivation of equation (2.7) can be found in Appendix B.1. There, I also prove that the ES exceeds unity for general functions $i_t(n) = \iota(q_t(n))$, where $\iota : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and convex (see Proposition 17). However, ES may be smaller than 2.

Figure 2.1: **Automation as the Substitution of Man-Hours per Task with Technological Knowledge.** The blue curve starting at $(A_0, 1/A_0)$ is $h_1(n)$. Hence, at $t = 1$ automation, i. e., the substitution of man-hours per task with technological knowledge, occurs along the blue curve to the right of A_0. At $t = 1$, the blue curve starting at $(A_1, 1/A_1)$ is $h_2(n)$. Automation occurs along the blue curve to the right of A_1.



2.1.2 Aggregate Technological Knowledge Growth

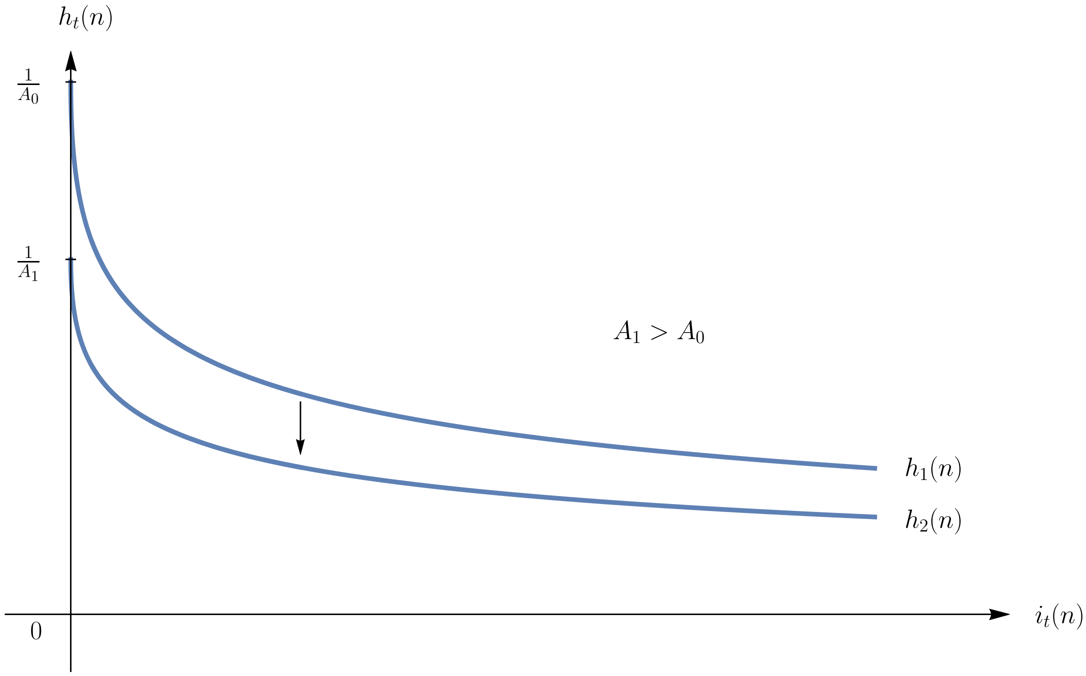
Technological knowledge embodied in a new machine is proprietary knowledge of an investing firm only in t, i. e., in the period when the investment is made. Subsequently, this knowledge becomes embodied in the indicator A_t, A_{t+1},..., with no further scope for proprietary exploitation.

The evolution of this indicator is given by

$$A_t = \max_{n \in [0, N_t]} \{a_t(n)\} = A_{t-1} \max_{n \in [0, N_t]} \{1 + q_t(n)\}. \tag{2.8}$$

Accordingly, the stock of technological knowledge to which all firms have access at the beginning of period $t + 1$ reflects the highest level of technological knowledge attained for any of the $n \in [0, N_t]$ tasks performed at t. As suggested by Figures 2.1 and 2.2, automation investments at the level of individual firms in conjunction with knowledge

Figure 2.2: **Automation as the Substitution of Man-Hours per Task with New Machines.** At $t = 1$, A_0 is given, and $h_1(n)$ is the unit isoquant of (2.6). Automation means $i_1(n) > 0$. The greater the investment outlays the fewer man-hours are needed to perform task n. At $t = 2$, A_1 is given, and $h_2(n)$ is the relevant unit isoquant. Automation means $i_2(n) > 0$. Since $A_1 > A_0$ the unit isoquant shifts downwards.



accumulation at the level of the economy as a whole will be the source of technological progress and sustained economic growth.

2.1.3 The Profit-Maximizing Production Plan

The representative firm takes the sequence $\{w_t, R_t, A_{t-1}\}_{t=1}^{\infty}$ of real wages, real rental rates of capital, and the aggregate productivity indicators as given and chooses a production plan $\left(Y_t, K_t, I_t, N_t, H_t^d, q_t(n), h_t(n), i\left(q_t(n)\right)\right)$ for all $n \in [0, N_t]$ and all t. Here, K_t is the aggregate demand for fixed capital, I_t the aggregate demand for automation investments, and H_t^d the aggregate demand for hours worked, i. e.,

$$I_t = \int_0^{N_t} i\left(q_t(n)\right) dn \quad \text{and} \quad H_t^d = \int_0^{N_t} h_t(n) dn.$$

The optimal production plan maximizes the sum of the present discounted values of profits in all periods. Since an automation investment generates proprietary technological knowledge only in the period when it is made, the inter-temporal maximization boils down to the maximization of per-period profits denoted by Π_t. Accordingly, for each period t, the optimal plan solves

$$\max_{\left(K_t, N_t, [q_t(n)]_{n=0}^{n=N_t}\right)} \quad \Pi_t = \Gamma K_t^{\gamma} N_t^{1-\gamma} - R_t K_t - \int_0^{N_t} \left[\frac{w_t}{A_{t-1}\left(1+q_t(n)\right)} + i\left(q_t(n)\right) \right] dn.$$

Here, the last term is the sum of the costs of all performed tasks. In view of (2.2) and (2.3) the time spent on the performance of task n is

$$h_t(n) = \frac{1}{A_{t-1}\left(1+q_t(n)\right)}. \tag{2.9}$$

Hence, task n gives rise to a wage cost $w_t h_t(n)$ and an investment cost $i(q_t(n))$. For further reference, let $c_t(n)$ denote these costs, i. e.,

$$c_t(n) = w_t h_t(n) + i(q_t(n)). \tag{2.10}$$

At all t, the firm's maximization problem may be split up into two parts. First, for each $n \in \mathbb{R}_+$ the firm chooses the value $q_t(n) \in \mathbb{R}_+$ that minimizes the cost of task n, i. e., it solves

$$\min_{[q_t(n)]_{n=0}^{\infty}} c_t(n). \tag{2.11}$$

Second, at minimized costs per task, the firm determines the profit-maximizing number of tasks, N_t, and the desired amount of fixed capital, K_t.

Cost-Minimization per Task

Let $\omega_t \equiv w_t / A_{t-1}$ denote the real wage per man-hour in efficiency units before any investment is being undertaken. Then, for all $n \in \mathbb{R}_+$ the respective first-order (sufficient) condition to problem (2.11) is

$$\frac{-\omega_t}{(1+q_t(n))^2} + \alpha \geq 0, \quad \text{with strict inequality only if } q_t(n) = 0. \tag{2.12}$$

This condition relates the marginal reduction of task n's wage costs to the marginal increase in its investment costs. Since this trade-off is the same for all tasks we have $q_t(n) = q_t$ where

$$q_t = q\left(\omega_t, \alpha\right) \equiv \begin{cases} -1 + \sqrt{\frac{\omega_t}{\alpha}} & \text{if } \omega_t \geq \alpha, \\ \\ 0 & \text{if } \omega_t \leq \alpha. \end{cases} \tag{2.13}$$

Hence, if $\omega_t > \alpha$ then $q_t > 0$ with $\partial q(\omega_t, \alpha)/\partial\omega_t > 0$ and $\partial q(\omega_t, \alpha)/\partial\alpha < 0$. In other words, the more expensive the man-hour under the old technology is expected to be, i. e., the higher ω_t, the higher is q_t.[10] Similarly, the lower the real price of automation investments, i. e., the lower α, the higher is q_t. However, if $\omega_t \leq \alpha$ then no automation investments are undertaken and the performance of tasks occurs with old machines that embody the technology represented by A_{t-1}. Intuitively, this corner solution arises if at $q_t(n) = 0$ the marginal reduction of the wage cost is too small compared to the marginal investment costs, $\alpha > 0$. Then, labor is so cheap that it retains its comparative advantage over new machines.

Using (2.13) in (2.4), (2.9), and (2.10) delivers the cost-minimizing choices per task of hours worked, investment outlays, and costs that I denote by h_t, i_t, and c_t.

Proposition 1 *(Cost-Minimization per Task)*

The minimization of costs per task delivers continuous, piecewise defined functions

$$q_t = q(\omega_t, \alpha), \quad h_t = \frac{h(\omega_t, \alpha)}{A_{t-1}}, \quad \textit{where} \quad h(\omega_t, \alpha) \equiv \frac{1}{1 + q(\omega_t, \alpha)},$$

$$i_t = \alpha q(\omega_t, \alpha) \equiv i(\omega_t, \alpha), \quad \textit{and} \quad c_t = \omega_t h(\omega_t, \alpha) + i(\omega_t, \alpha) \equiv c(\omega_t, \alpha).$$

For (2.2) - (2.4) the following closed-form solutions obtain in addition to (2.13):

- *if $\omega_t \geq \alpha$ then*

$$h_t = \frac{1}{A_{t-1}}\sqrt{\frac{\alpha}{\omega_t}}, \quad i_t = \sqrt{\alpha\omega_t} - \alpha, \quad \textit{and} \quad c_t = 2\sqrt{\alpha\omega_t} - \alpha,$$

- *if $\omega_t \leq \alpha$ then*

$$h_t = \frac{1}{A_{t-1}}, \quad i_t = 0, \quad \textit{and} \quad c_t = \omega_t.$$

[10] This intuition mimics a key finding of the so-called induced innovations literature of the 1960s: higher anticipated wages induce faster labor-saving technical change (see, Hicks (1932), von Weizsäcker (1962), Kennedy (1964), Samuelson (1965), Drandakis and Phelps (1966), or Funk (2002)).
One way to bring my analysis even closer to this literature is to assume that automation investments, like fixed capital investments, are undertaken in the period before they are used. Then, the investment outlays (2.4) occur at $t-1$ and lead to a cost in units of period-t output of $R_t i(q_t(n))$. Accordingly, the cost minimization per task gives rise to results that mimic Proposition 1 where ω_t is to be replaced by $\tilde{\omega}_t \equiv \omega_t/R_t$ and c_t by $\tilde{c}_t = R_t\left[2\sqrt{\alpha\tilde{\omega}_t} - \alpha\right]$. Hence, in line with the literature of the 1960s the incentive to engage in an automation investment hinges now on the (anticipated) relative price of labor with respect to machines before the investment is undertaken.

Observe that the notation $h(\omega_t, \alpha)$ is a shortcut for $h(q(\omega_t, \alpha))$ which makes the effect of induced productivity growth through automation on man-hours per task explicit. Similarly, $i(\omega_t, \alpha)$ and $c(\omega_t, \alpha)$ are abbreviations for $i(\alpha, q(\omega_t, \alpha))$ and $c(q(\omega_t, \alpha), \omega_t, \alpha)$. The latter notation highlights the presence of direct and indirect effects when ω_t and α change. Throughout this paper I shall stick to simpler notation introduced in Proposition 1.

The following corollary to Proposition 1 highlights that cost-minimizing automation investments give rise to a *rationalization effect* and to a *productivity effect*.

Corollary 1 *(Rationalization and Productivity Effect)*

If $\omega_t > \alpha$ *then*

$$h_t < \frac{1}{A_{t-1}} \quad (\textit{rationalization effect})$$

and

$$c_t < \omega_t \quad (\textit{productivity effect}).$$

Hence, if automation is profitable then it means rationalization, i. e., fewer man-hours per task. The productivity effect results since in spite of investment outlays, a cost-minimizing automation investment reduces the overall cost per task as it reduces the wage costs.

A higher wage strengthens the rationalization and the productivity effect. For the former, this holds since $\partial q / \partial \omega_t > 0$ implies $dh_t / d\omega_t = (\partial h / \partial \omega_t) / A_{t-1} < 0$. For the latter, this is true since the impact of ω_t on c_t is less than proportionate. Indeed, Proposition 1 and the envelope theorem imply

$$\frac{dc_t}{d\omega_t} = h(\omega_t, \alpha) + \left[\underbrace{\omega_t \frac{\partial h_t}{\partial q_t} + \frac{\partial i_t}{\partial q_t}}_{=0} \right] \frac{\partial q(\omega_t, \alpha)}{\partial \omega_t} = h(\omega_t, \alpha) \in (0, 1) \tag{2.14}$$

so that

$$\frac{dc_t}{d\omega_t} \frac{\omega_t}{c_t} = \frac{\omega_t h(\omega_t, \alpha)}{\omega_t h(\omega_t, \alpha) + i(\omega_t, \alpha)} < 1.$$

Accordingly, the productivity effect becomes more pronounced in response to a wage hike as both the difference $\omega_t - c_t$ and the ratio ω_t / c_t increase in ω_t.

A lower real price of automation investments, i. e., a lower α, implies a higher q_t since $\partial q / \partial \alpha < 0$. This strengthens the rationalization effect as $dh_t / d\alpha = (\partial h / \partial \alpha) / A_{t-1} > 0$, and the productivity effect as

$$\frac{dc_t}{d\alpha} = \left[\underbrace{\omega_t \frac{\partial h_t}{\partial q_t} + \frac{\partial i_t}{\partial q_t}}_{=0} \right] \frac{\partial q(\omega_t, \alpha)}{\partial \alpha} + \frac{\partial i\,(\omega_t, \alpha)}{\partial \alpha} = q_t > 0. \tag{2.15}$$

Profit-Maximization at Minimized Costs

At minimized costs per task profits become

$$\Pi_t = \Gamma K_t^{\gamma} N_t^{1-\gamma} - R_t K_t - c_t N_t,$$

and the maximization with respect to N_t and K_t delivers the first-order conditions

$$\begin{aligned} N_t &: \quad \Gamma\,(1-\gamma)\,K_t^{\gamma} N_t^{-\gamma} - c_t = 0, \\ K_t &: \quad \Gamma \gamma K_t^{\gamma-1} N_t^{1-\gamma} - R_t = 0. \end{aligned} \tag{2.16}$$

Both conditions require the respective value product to equal marginal cost. The marginal cost of task N_t is c_t. This leads to the following proposition.

Proposition 2 *(Profit-Maximizing Tasks, Output, Profits, and the Factor-Price Frontier)*

Given K_t, the profit-maximizing amounts of tasks and output at t are

$$N_t = K_t \left(\frac{\Gamma(1-\gamma)}{c_t} \right)^{\frac{1}{\gamma}} \equiv K_t N\,(c_t) \quad \textit{and} \quad Y_t = K_t \Gamma \left(\frac{\Gamma(1-\gamma)}{c_t} \right)^{\frac{1-\gamma}{\gamma}} \equiv K_t Y\,(c_t)\,.$$

Moreover,

$$R_t = \Gamma^{\frac{1}{\gamma}} \gamma \left(\frac{1-\gamma}{c_t} \right)^{\frac{1-\gamma}{\gamma}},$$

and $\Pi_t = 0$.

Proposition 2 states that, given K_t, the first-order conditions (2.16) imply that N_t, Y_t, and R_t may be expressed as functions of c_t. In particular, the functions $N\,(c_t)$ and $Y\,(c_t)$ show, respectively, how the amount of tasks per unit of fixed capital, N_t/K_t, and the productivity of fixed capital, Y_t/K_t, hinges on the minimized costs per task, c_t. At the same time, R_t and c_t are linked through the factor-price frontier.[11] One readily verifies

[11]The input demand and output supply functions of a competitive firm under constant returns to scale are not well defined. Hence, given K_t, a change in c_t requires an adjustment in the rental rate of capital, R_t, along the factor-price frontier.

that a decline in c_t increases the profit-maximizing amount of tasks since the marginal value product of tasks is equal to a lower cost per task at a greater amount of tasks. Hence, $N'(c_t) < 0$. As $Y(c_t) = N(c_t)^{1-\gamma}$ this implies $Y'(c_t) < 0$. Finally, the factor-price frontier dictates that R_t will fall in c_t, too. Finally, constant returns to scale of F imply $\Pi_t = 0$.

The following corollary to Proposition 2 establishes that automation gives rise to a *task expansion effect* and an *output expansion effect*.

Corollary 2 *(Task and Output Expansion Effect)*

If $\omega_t > \alpha$ then

$$N(c_t) > N(\omega_t) \quad \textit{(task expansion effect)}$$

and

$$Y(c_t) > Y(\omega_t) \quad \textit{(output expansion effect).}$$

The intuition behind Corollary 2 is straightforward. If $\omega_t > \alpha$ then firms undertake automation investments and the productivity effect of Corollary 1 implies $c_t < \omega_t$. Then, the *task expansion effect* and the *output expansion effect* of automation follow since $N'(c_t) < 0$ and $Y'(c_t) < 0$.

To complete the discussion of the profit-maximizing production plan let me note that, given K_t, Proposition 1 and 2 give rise to aggregate demands for automation investments and for hours worked that may be expressed as

$$I_t = i_t N_t = K_t i(\omega_t, \alpha) N(c(\omega_t, \alpha)) \text{ and } H_t^d = h_t N_t = \left(\frac{K_t}{A_{t-1}}\right) h(\omega_t, \alpha) N(c(\omega_t, \alpha)). \quad (2.17)$$

The following proposition takes stock of the findings of this section.

Proposition 3 *(Unique Profit-Maximizing Production Plan)*

For a given sequence $\{w_t, R_t, A_{t-1}\}_{t=1}^{\infty}$ there is a unique profit-maximizing production plan of the representative firm. It satisfies Proposition 1, 2 and equation (2.17).

2.2 Macroeconomic Implications of Automation - A Static View

In this section I establish and discuss three key macroeconomic consequences of automation that result directly from the aggregate behavior of firms in a given period t where K_t and A_{t-1} are given. In particular, I show that[12]

i) automation reduces the aggregate demand for hours worked if and only if the induced labor productivity growth is small (Section 2.2.1),

ii) automation unequivocally reduces the labor share (Section 2.2.2), and

iii) automation is labor-augmenting for the economy as a whole (Section 2.2.3).

2.2.1 Automation and the Aggregate Demand for Hours Worked

Does automation increase or decrease the aggregate demand for hours worked? To address this question I consider some period t and assume $\omega_t > \alpha$. Then, I compare the aggregate demand for hours worked resulting under profit-maximizing automation investments to the demand obtained without automation. I denote the former demand by $H_t^{d1}(\omega_t, \alpha)$ and the latter by $H_t^{d2}(\omega_t)$.

Using Proposition 1, 2, and (2.17), the aggregate demand for hours worked under profit-maximizing automation investments is

$$H_t^{d1}(\omega_t, \alpha) = \left(\frac{K_t}{A_{t-1}}\right) h\left(\omega_t, \alpha\right) N\left(c\left(\omega_t, \alpha\right)\right) = \left(\frac{K_t}{A_{t-1}}\right) \sqrt{\frac{\alpha}{\omega_t}} \left(\frac{\Gamma\left(1-\gamma\right)}{2\sqrt{\alpha\omega_t} - \alpha}\right)^{\frac{1}{\gamma}}. \quad (2.18)$$

Without automation investments $h\left(\omega_t, \alpha\right) = 1$ and $c\left(\omega_t, \alpha\right) = \omega_t$ such that $N\left(c\left(\omega_t, \alpha\right)\right) = N\left(\omega_t\right)$. Hence, the aggregate demand for hours worked becomes

$$H_t^{d2}(\omega_t) = \left(\frac{K_t}{A_{t-1}}\right) N\left(\omega_t\right) = \left(\frac{K_t}{A_{t-1}}\right) \left(\frac{\Gamma\left(1-\gamma\right)}{\omega_t}\right)^{\frac{1}{\gamma}}. \quad (2.19)$$

[12] These three propositions are difficult to reconcile with a representation of automation as exogenous, factor-augmenting technical change in a neoclassical aggregate production function $F\left(BK, AL\right)$. On the one hand, if automation is labor-augmenting, i. e., for a higher A, the demand for labor falls only if the capital share exceeds the elasticity of substitution between capital and labor. Moreover, the labor share falls only if the elasticity of substitution exceeds unity. With a range for the capital share of about $0.3 - 0.4$ and for the elasticity of substitution of about $0.5 - 1$ (see Oberfield and Raval (2014)) both predictions seem unreasonable. On the other hand, if automation is capital-augmenting, i. e., for a higher B, then the demand for labor increases as both arguments in F are complements, whereas, for an elasticity of substitution smaller than unity, the labor share falls (Irmen (2014), Acemoglu and Restrepo (2018c)).
See, e. g., Sachs and Kotlikoff (2012) or Nordhaus (2015) for recent attempts to model automation as exogenous, capital-augmenting technical change.

Both demands decline in the real wage. However, their slopes differ. The slope of $H_t^{d1}(\omega_t, \alpha)$ reflects an *induced aggregate rationalization effect* as well as an *induced aggregate task expansion effect*. Both effects are negative. As w_t, respectively ω_t, increases, the former effect means that fewer man-hours per task will be demanded for all performed tasks. The latter effect reflects the increase in the cost per task and the concomitant decline in the total number of performed tasks.

Analytically, these effects appear as

$$\begin{aligned}\frac{dH^{d1}(\omega_t,\alpha)}{d\omega_t} &= \underbrace{\left(\frac{K_t}{A_{t-1}}\right) N\left(c_t\right) \underbrace{\frac{\partial h\left(\omega_t,\alpha\right)}{\partial \omega_t}}_{(-)}}_{\text{Induced Aggregate Rationalization Effect}} + \underbrace{\left(\frac{K_t}{A_{t-1}}\right) h\left(\omega_t,\alpha\right) \underbrace{\frac{\partial N\left(c_t\right)}{\partial c_t}}_{(-)} \underbrace{\frac{dc\left(\omega_t,\alpha\right)}{d\omega_t}}_{(+)}}_{\text{Induced Aggregate Task Expansion Effect}} \\ &= \left(\frac{K_t}{A_{t-1}}\right)\left[N\left(c_t\right)\frac{\partial h\left(\omega_t,\alpha\right)}{\partial \omega_t} + \left[h\left(\omega_t,\alpha\right)\right]^2 \frac{\partial N\left(c_t\right)}{\partial c_t}\right] < 0,\end{aligned}$$

where the second line follows from (2.14).

Void of automation, a higher real wage affects $H_t^{d2}(\omega_t)$ only through the task expansion effect, i. e.,

$$\frac{dH^{d2}(\omega_t)}{d\omega_t} = \left(\frac{K_t}{A_{t-1}}\right)\frac{\partial N\left(\omega_t\right)}{\partial \omega_t} < 0.$$

The following proposition collects the relevant economic implications of the comparison between $H_t^{d1}(\omega_t, \alpha)$ and $H_t^{d2}(\omega_t)$.[13] Figure 2.3 provides an illustration.

Proposition 4 *(Automation and the Aggregate Demand for Hours Worked)*

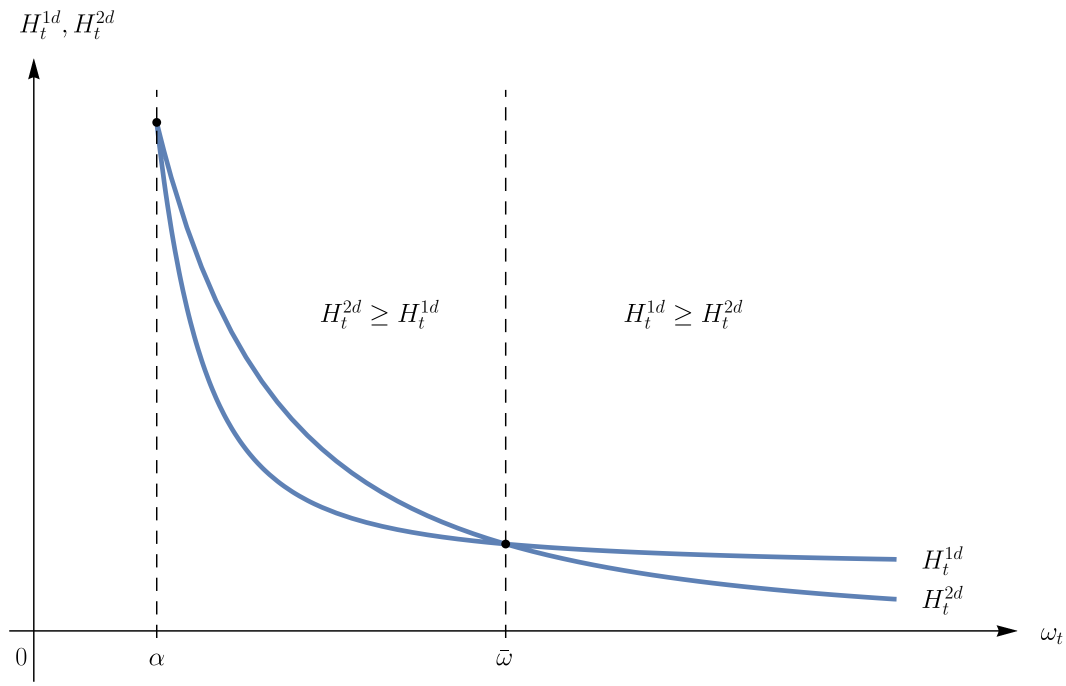
Consider $H_t^{d1}(\omega_t, \alpha)$ and $H_t^{d2}(\omega_t)$ for $\omega_t \geq \alpha$. Then, the following holds:

1. $H_t^{d1}(\alpha, \alpha) = H_t^{d2}(\alpha)$,
2. *the slopes of $H^{d1}(\omega_t, \alpha)$ and $H^{d2}(\omega_t)$ satisfy*

$$\lim_{\omega_t \to \alpha} \frac{\partial H_t^{d1}(\omega_t,\alpha)}{\partial \omega_t} < \lim_{\omega_t \to \alpha} \frac{\partial H_t^{d2}(\omega_t)}{\partial \omega_t},$$

[13] This comparison is meaningful if one admits that even without automation investments at t the firm owns at least $N(\alpha)$ machines that embody a level of technological knowledge represented by A_{t-1}. Moreover, observe that, irrespective of whether there is automation or not, any change in ω_t requires an adjustment of R_t along the factor-price frontier.

Figure 2.3: **Automation and the Aggregate Demand for Hours Worked.** If $\omega_t \geq \alpha$ and automation investments are undertaken then the aggregate demand for hours worked is H_t^{1d}, H_t^{2d} is the aggregate demand for $\omega_t \geq \alpha$ without automation investments. It holds that $H_t^{2d} \geq H_t^{1d}$ for $\omega_t \in [\alpha, \bar{\omega}]$ and $H_t^{1d} > H_t^{2d}$ for $\omega_t > \bar{\omega}_t$.



3. *there is* $\bar{\omega} \in (\alpha, \infty)$*, such that*

$$H_t^{d2}(\omega_t) \geq H_t^{d1}(\omega_t, \alpha) \quad \textit{if } \alpha \leq \omega_t \leq \bar{\omega},$$

$$H_t^{d1}(\omega_t, \alpha) \geq H_t^{d2}(\omega_t) \quad \textit{if } \omega_t \geq \bar{\omega}.$$

Claim 1 recalls that there are no automation investments if $\omega_t = \alpha$. Then, H_t^{d1} and H_t^{d2} coincide. However, according to Claim 2, H_t^{d1} declines faster than H_t^{d2} if ω_t exceeds α by a triffle. Interestingly, in the limit $\omega_t \to \alpha$ one finds $c_t = \alpha$ and

$$\frac{dH_t^{d1}(\alpha, \alpha)}{d\omega_t} = \left(\frac{K_t}{A_{t-1}}\right)\left[-\frac{N(\alpha)}{2\alpha} - \frac{N(\alpha)}{\gamma\alpha}\right] < \frac{dH_t^{d2}(\alpha)}{d\omega_t} = \left(\frac{K_t}{A_{t-1}}\right)\left[\frac{-N(\alpha)}{\gamma\alpha}\right].$$

Here, $-N(\alpha)/(\gamma\alpha)$ represents the aggregate task expansion effect identified above. It appears in both demands. However, in the presence of automation investments the induced aggregate rationalization effect does not vanish and is responsible for the stronger

Table 1: **Changing ω_t: Aggregate Demand for Hours Worked and the Critical Induced Productivity Growth Rate.** The table shows the critical induced productivity growth rate per man-hour, $q(\bar{\omega})$, and its annualized counterpart, $\tilde{q}\,(\bar{\omega})$, computed for a period length of 30 years, for varying values of γ. If $q_t > q(\bar{\omega})$ then automation will increase $H_t^{d1}(\omega)$.

γ	1/3	1/4	1/5	1/6	1/7	1/8
$q(\bar{\omega})$	0.64	0.39	0.28	0.22	0.18	0.15
$\tilde{q}\,(\bar{\omega})$	0.017	0.0111	0.008	0.0067	0.0056	0.0048

decline of H^{d1}.[14] This view gives support to the often encountered public opinion according to which automation will reduce the demand for labor due to rationalization. Indeed, this is why $H_t^{d1}(\omega_t, \alpha) < H_t^{d2}(\omega_t)$ for values of ω_t greater than but close to α. Here, wages are not expected to be too high, q_t is small but positive, and automation investments are fairly small-seized.

The above logic no longer holds if w_t, respectively, ω_t is sufficiently high so that induced automation investments per task become large. This is the essence of Claim 3. If $\omega_t > \bar{\omega}$ then automation investments boost the aggregate demand for hours worked to an extent that $H_t^{d1}(\omega_t, \alpha) > H_t^{d2}(\omega_t)$. Hence, if profit-maximizing automation investments generate a high growth rate of labor productivity then the aggregate demand for hours worked will be higher with automation than without.

What is the predicted order of magnitude of the critical induced productivity growth rate per man-hour, $q\,(\bar{\omega})$, and its annualized counterpart, $\tilde{q}\,(\bar{\omega})$, above which the aggregate demand for hours worked with automation is greater than without? Table 1 shows that this critical growth rate, computed for a period length of 30 years, is quite small and varies with γ. However, for all considered values of this parameter actual automation investments that bring about an annual productivity growth of roughly 2% would increase the aggregate demand for hours worked.[15]

Finally, let me turn to the effect of a change in the price of automation investments, α, on the aggregate demand for hours worked, $H_t^{d1}(\omega_t, \alpha)$. From the first expression of (2.18)

[14] This result obtains since $\partial h(\omega_t, \alpha)/\partial\omega_t = -\left[h(\omega_t, \alpha)\right]^2 \cdot \partial q(\omega_t, \alpha)/\partial\omega_t$ with $\lim_{\omega_t \to \alpha} \partial h(\omega_t, \alpha)/\partial\omega_t = (-1) \cdot \partial q(\alpha, \alpha)/\partial\omega_t$ and $\partial q(\alpha, \alpha)/\partial\omega_t > 0$. It does not hinge on the linear specification of investment outlays but extends to any increasing and convex function $i_t = \alpha\iota(q_t)$ where $\lim_{q_t \to 0} \iota'(q_t) = 0$. See Appendix B.2 for a proof of this claim.

[15] The computations underlying Table 1 are explained in the Proof of Proposition 4.

one readily verifies that

$$\begin{aligned}\frac{dH_t^{d1}(\omega_t,\alpha)}{d\alpha} &= \underbrace{\left(\frac{K_t}{A_{t-1}}\right)N\left(c_t\right)\underbrace{\frac{\partial h\left(\omega_t,\alpha\right)}{\partial\alpha}}_{(+)}}_{\text{Induced Aggregate Rationalization Effect}} + \underbrace{\left(\frac{K_t}{A_{t-1}}\right)h\left(\omega_t,\alpha\right)\underbrace{\frac{\partial N\left(c_t\right)}{\partial c_t}}_{(-)}\underbrace{\frac{\partial c\left(\omega_t,\alpha\right)}{\partial\alpha}}_{(+)}}_{\text{Induced Aggregate Task Expansion Effect}} \\ &= \left(\frac{K_t}{A_{t-1}}\right)\left[N\left(c_t\right)\frac{\partial h\left(\omega_t,\alpha\right)}{\partial\alpha}+h\left(\omega_t,\alpha\right)\frac{\partial N\left(c_t\right)}{\partial c_t}q\left(\omega_t,\alpha\right)\right]\gtreqless 0, \qquad (2.20)\end{aligned}$$

where the second line uses (2.15).

Hence, a decline in α has two opposing effects on H_t^{d1}. On the one hand, the incentive to automate become more pronounced. Accordingly, there will be more rationalization, and H_t^{d1} falls. This is captured by the induced aggregate rationalization effect. On the other hand, the cost per task falls so that more tasks will be performed and H_t^{d1} increases. This is captured by the aggregate task expansion effect. The following proposition makes this more precise.

Proposition 5 *(Real Price of Automation Investments and the Aggregate Demand for Hours Worked)*

Consider $H_t^{d1}(\omega_t,\alpha)$ for $\omega_t \geq \alpha$. Then, it holds that

$$\frac{dH_t^{d1}(\omega_t,\alpha)}{d\alpha} \lesseqgtr 0 \quad \Leftrightarrow \quad \frac{\omega_t}{\alpha} \gtreqless \left(\frac{2-\gamma}{2(1-\gamma)}\right)^2.$$

Hence, a lower α increases the aggregate demand for hours worked if ω_t/α is sufficiently large. The reason is that the induced aggregate rationalization effect in (2.20) dominates the induced aggregate task expansion effect only for small values of ω_t.[16]

To gauge the sign of $dH_t^{d1}/d\alpha$ observe that Proposition 1 allows to write

$$\frac{dH_t^{d1}(\omega_t,\alpha)}{d\alpha} \lesseqgtr 0 \quad \Leftrightarrow \quad \gamma \lesseqgtr \frac{2\left(\sqrt{\frac{\omega_t}{\alpha}}-1\right)}{2\sqrt{\frac{\omega_t}{\alpha}}-1} \quad \Leftrightarrow \quad q_t \gtreqless \frac{\gamma}{2(1-\gamma)} \equiv q(\gamma).$$

Hence, a lower α increases the aggregate demand for hours worked if $q_t > q(\gamma)$. Table 2 shows that the latter condition is already satisfied for very small annual productivity growth rates, $\tilde{q}(\gamma)$, computed for a period of 30 years. For instance, if $\gamma = 1/4$, then a decline in α boosts $H_t^{d1}(\omega_t,\alpha)$ for annual productivity growth rates exceeding 0.5%.

[16] In the limit $\omega_t \to \alpha$, it holds that

$$\lim_{\omega_t\to\alpha}\frac{dH^{d1}(\omega_t,\alpha)}{d\alpha} = \left(\frac{K_t}{A_{t-1}}\right)N\left(\alpha\right)\frac{\partial h\left(\alpha,\alpha\right)}{\partial\alpha} > 0,$$

Table 2: **Changing α: Aggregate Demand for Hours Worked and the Critical Induced Productivity Growth Rate.** The table shows the critical growth rate of productivity per man-hour, $q(\gamma)$, and its annualized counterpart, $\tilde{q}(\gamma)$, computed for a period length of 30 years. If $q_t > q(\gamma)$ then a small decline in α increases $H_t^{d1}(\omega_t, \alpha)$.

γ	1/3	1/4	1/5	1/6	1/7	1/8
$q(\gamma)$	0.25	0.167	0.125	0.1	0.083	0.071
$\tilde{q}(\gamma)$	0.007	0.005	0.004	0.003	0.0026	0.002

2.2.2 Automation and the Labor Share

Does automation reduce the labor share? Since $\Pi_t = 0$, the economy satisfies $Y_t - R_t K_t - c_t N_t = Y_t - R_t K_t - w_t h_t N_t - i_t N_t = 0$. Accordingly, total income is equal to net output, i. e., $w_t H_t + R_t K_t = Y_t - I_t$, and the labor share is defined as

$$LS_t \equiv \frac{w_t H_t}{Y_t - I_t}. \tag{2.21}$$

In what follows, I compare the labor share under profit-maximizing automation investments, LS_t^1, to the one obtained without automation, LS_t^2.

Proposition 6 *(Automation and the Labor Share)*

If $\omega_t > \alpha$ then

$$LS_t^2 = 1 - \gamma > LS_t^1 = (1-\gamma)\left(\frac{w_t h_t}{w_t h_t + \gamma i_t}\right).$$

Hence, automation unequivocally reduces the labor share because it involves investment outlays, $i_t > 0$. Since the labor and the capital share add up to one, automation will increase the capital share.[17]

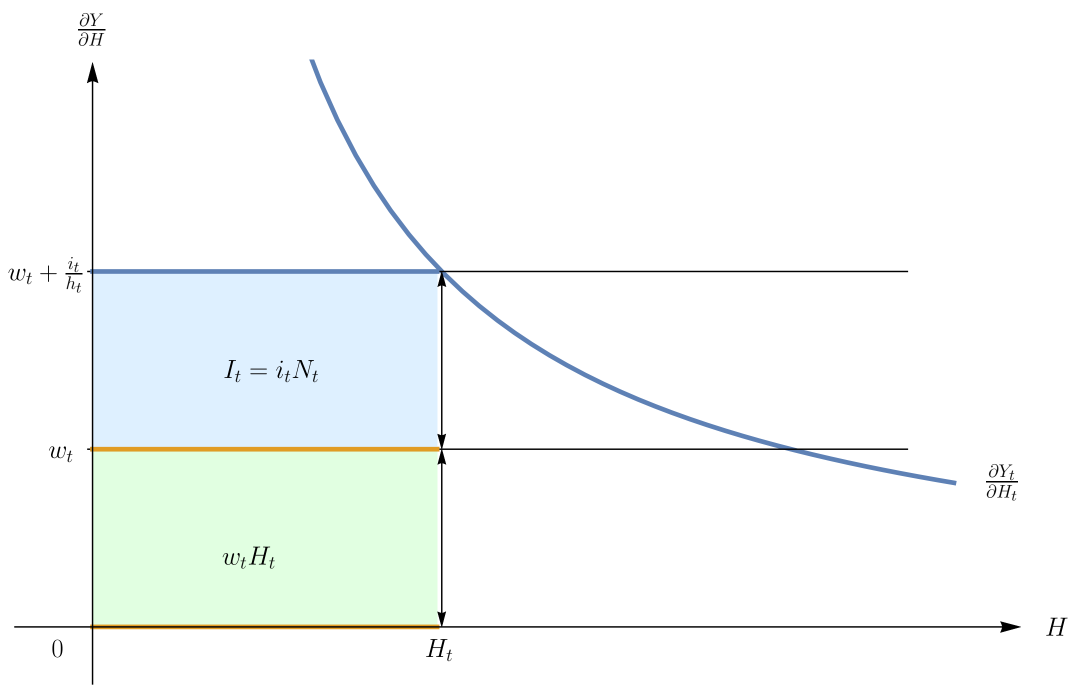
The intuition for Proposition 6 is a follows. First, observe that irrespective of whether there is automation or not the first-order condition for N_t in (2.16) implies

$$\frac{\partial Y_t}{\partial N_t} = c_t \quad \text{and} \quad \frac{\partial Y_t}{\partial N_t}\frac{N_t}{Y_t} = \frac{c_t N_t}{Y_t} = 1 - \gamma. \tag{2.22}$$

i. e., the induced aggregate task expansion effect vanishes whereas the induced aggregate rationalization effect does not.

[17]Since investment outlays are treated here as an ordinary flow input, no income accrues to machines. However, the qualitative result of Proposition 6 and Corollary 3 below remain unchanged if new machines are treated as a stock and deliver rental income to their capitalist owners. See Footnote 18 for more details.

Figure 2.4: **Automation and the Functional Income Distribution.** Profit-Maximization with respect to N_t implies a wedge between the real wage, w_t, and the marginal product of man-hours, $\partial Y_t/\partial H_t$. The wedge is equal to i_t/h_t and is just enough to cover aggregate investment outlays as $i_t H_t/h_t = i_t h_t N_t/h_t = i_t N_t = I_t$.



Hence, tasks earn their marginal product, and, since the aggregate production function is Cobb-Douglas, the share of total output that accrues to tasks is equal to $1-\gamma$. Moreover, since the choice of h_t does not depend on N_t, the expression for the total amount of hours worked, $H_t = h_t N_t$, implies $dH_t = h_t dN_t$. Using this property in (2.22) gives

$$\frac{\partial Y_t}{\partial H_t} = \frac{c_t}{h_t} \quad \text{and} \quad \frac{\partial Y_t}{\partial H_t}\frac{H_t}{Y_t} = \frac{c_t}{h_t}\frac{H_t}{Y_t} = 1-\gamma. \tag{2.23}$$

The latter reveals that the cost per hour worked, c_t/h_t, is equal to the marginal product of total hours, and, since $(\partial Y_t/\partial N_t)\, N_t = (\partial Y_t/\partial H_t)\, H_t$ the share of hours worked in total output is $1-\gamma$.

The implications of (2.22) and (2.23) for the labor share with and without automation are as follows. Without automation, costs per task are equal to $c_t = w_t h_t$ and (2.23) delivers

$$\frac{\partial Y_t}{\partial H_t} = w_t \quad \text{and} \quad \frac{w_t H_t}{Y_t} = 1-\gamma. \tag{2.24}$$

Hence, hours worked earn their marginal product and $LS_t^2 = 1-\gamma$.

In the presence of automation investments it holds that $c_t = w_t h_t + i_t$. Using this in (2.23) reveals that

$$\frac{\partial Y_t}{\partial H_t} = w_t + \frac{i_t}{h_t} \quad \Leftrightarrow \quad w_t = \frac{\partial Y_t}{\partial H_t} - \frac{i_t}{h_t}. \tag{2.25}$$

The latter highlights that automation investments drive a wedge between the wage per hour worked and the marginal product of total hours worked. This wedge is equal to the investment outlays per hour worked, i_t/h_t. Upon multiplication of (2.25) by H_t one finds

$$w_t H_t = \frac{\partial Y_t}{\partial H_t} H_t - i_t N_t. \tag{2.26}$$

Hence, instead of earning $(\partial Y_t/\partial H_t)\, H_t$ workers pay the price for all automation investments (see Figure 2.4 for an illustration). As a consequence, the labor share falls in the presence of automation. To derive this formally use $(\partial Y_t/\partial H_t)\, H_t = (1-\gamma)\, Y_t$ from (2.23) in (2.26) to express net output as $Y_t - I_t = (w_t H_t + \gamma I_t)\,/(1-\gamma)$. Then, LS_t^1 as stated in the proposition is obtained with the definition of the labor share (2.21). Since $i_t > 0$, it holds that $LS_t^2 > LS_t^1$.

Finally, let me note that the labor share, LS_t^1, declines the stronger the incentives to automate are, i. e., the higher the anticipated wage or the cheaper automation investments become. However, LS_t^1 is bounded from below.

Corollary 3 *(Automation Incentives and the Lower Bound of the Labor Share)*

If $\omega_t > \alpha$ then it holds that

$$\frac{\partial LS_t^1}{\partial \omega_t} < 0 \quad \textit{and} \quad \frac{\partial LS_t^1}{\partial \alpha} > 0.$$

Moreover,

$$\lim_{\omega_t \to \infty} LS_t = \lim_{\alpha \to 0} LS_t = \frac{1-\gamma}{1+\gamma}.$$

Hence, a higher anticipated wage induces more automation and reduces the labor share. Moreover, in line with the findings of Karabarbounis and Neiman (2014) a decline in the price of automation investments, i. e., a lower α, reduces the labor share. However, the labor share remains strictly positive even if the incentives to automate become very strong. This follows since in both limits the fraction $w_t h_t/\,(w_t h_t + \gamma i_t)$ converges to $1/(1+\gamma)$ as $i_t/(w_t h_t)$ converges to 1. Hence, in the limit investment outlays per task are equal to the wage costs per task. If $\gamma = 1/4$ then the lower bound of the labor share is .6, for $\gamma = 1/3$ it falls to $1/2$.[18]

[18]As a follow-up to Footnote 10 and 17 one may wonder whether the qualitative results of Section 2.2.2

2.2.3 Labor-Augmenting Automation

What type of technical change does automation imply? According to the rationalization effect identified above automation is labor-saving in the sense that fewer hours of labor are needed in the performance of a task. However, in the aggregate production function (2.1) automation is labor-augmenting, i. e., it increases the productivity of all hours worked. This follows since cost minimization implies $q_t = q(n_t)$ so that the production function of each task is $1 = A_{t-1}(1+q_t)h_t$ (see equation (2.2)). Accordingly, if N_t tasks are performed then

$$N_t = N_t A_{t-1}(1+q_t)h_t = A_{t-1}(1+q_t)H_t^d$$

since $H_t^d = h_t N_t$. Using this in (2.1) gives

$$Y_t = \Gamma K_t^{\gamma}\left(A_{t-1}(1+q_t)H_t^d\right)^{1-\gamma}. \tag{2.27}$$

Hence, technical change augments the total amount of employed hours worked.

Interestingly, this finding does not hinge on the Cobb-Douglas form of the aggregate production function. Since it only reflects the minimization of costs per task and the definition of the firm's demand for hours worked it generalizes to any production function $F(K_t, N_t)$. For these functions, the counterpart to (2.27) is

$$Y_t = F\left(K_t, A_{t-1}(1+q_t)H_t^d\right).$$

Accordingly, the term labor-augmenting technical change is meaningful here.

2.3 The Household Sector

Individuals live for possibly two periods, young and old age. When young, they supply labor, earn wage income, enjoy leisure and consumption, and save. At the onset of old

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survive if automation investments, like fixed capital investments, are undertaken in the period before they are used. The answer is in the affirmative. With this modification zero profits imply $Y_t = R_t(K_t + I_t) + w_t H_t$ and the labor share becomes $\tilde{LS}_t \equiv w_t H_t / Y_t$. Using the first-order condition for profit-maximization $\tilde{c}_t N_t = (1-\gamma) Y_t$ one readily verifies that

$$\tilde{LS}_t = (1-\gamma)\left(\frac{w_t \tilde{h}_t}{w_t \tilde{h}_t + R_t \tilde{i}_t}\right) = (1-\gamma)\left(\frac{\sqrt{\tilde{\omega}_t}}{2\sqrt{\tilde{\omega}_t} - \sqrt{\alpha}}\right),$$

where $\tilde{h}_t = \sqrt{\alpha/\tilde{\omega}_t}/A_{t-1}$ and $\tilde{i}_t = \sqrt{\alpha\tilde{\omega}_t} - \alpha$. Hence, automation reduces $\tilde{LS}_t$ since $\tilde{i}_t > 0$. Moreover, $\tilde{LS}_t$ falls in $\tilde{\omega}_t$, increases in α, and is bounded from below as $\lim_{\omega_t \to \infty} \tilde{LS}_t = \lim_{\alpha \to 0} \tilde{LS}_t = (1-\gamma)/2 \in (0, \infty)$. With $\gamma = 1/4$ ($\gamma = 1/3$) this bound is equal to 0.375 (1/3).

age, they face a survival probability $\mu \in (0,1)$. Surviving old individuals retire and consume their wealth.[19]

The population at t consists of L_t young (cohort t) and μL_{t-1} old individuals (survivors of cohort $t-1$). Due to birth and other demographic factors the number of young individuals between two adjacent periods grows at rate $g_L > (-1)$. For short, I shall refer to g_L as the fertility rate.

My measure of population aging is the old-age dependency ratio at t defined as

$$OADR_t \equiv \frac{\mu L_{t-1}}{L_t} = \frac{\mu}{1+g_L}. \tag{2.28}$$

Hence, $OADR_t$ is determined by the survival probability and the fertility rate of cohort $t-1$. There is population aging between period $t-1$ and t if $OADR_t > OADR_{t-1}$. Accordingly, an increase in the survival probability of cohort $t-1$ and/or a decline in the fertility rate of this cohort implies population aging.

For cohort t, denote consumption when young and old by c_t^y and c_{t+1}^o, and leisure time enjoyed when young by l_t. The per-period time endowment is normalized to unity. Then, $l_t = 1 - h_t^s$, where $h_t^s \in [0,1]$ is man-hours supplied by cohort t when young.

Individuals of all cohorts assess bundles $\left(c_t^y, l_t, c_{t+1}^o\right)$ according to an expected lifetime utility function, U, featuring a periodic utility function of the generalized log-log type proposed by Boppart and Krusell (2018). The utility after death is set equal to zero. Accounting for retirement when old, i. e., $l_{t+1} = 1$, cohort t's expected utility is

$$U\left(c_t^y, l_t, c_{t+1}^o\right) = \ln c_t^y + \ln\left(1 - \phi\left(1 - l_t\right)\left(c_t^y\right)^{\frac{\nu}{1-\nu}}\right) + \mu\beta \ln c_{t+1}^o, \tag{2.29}$$

where $0 < \beta < 1$ is the discount factor, $\phi > 0$ and $\nu \in (0,1)$. For ease of notation, I use henceforth $x_t \equiv (1 - l_t)\left(c_t^y\right)^{\frac{\nu}{1-\nu}}$.

The term $\ln(1 - \phi x_t)$ reflects the disutility of labor when young. The parameter ϕ captures characteristics of the labor market that affect the disutility of labor in the population irrespective of the amount of hours worked and the level of consumption. These include, e. g., the level of occupational safety regulations and the climatic conditions under which

[19]Hence, by assumption aging may affect the endogenous labor supply when young but not the timing of retirement. To a first approximation, this does not seem too far from reality. For instance, Bloom, Canning, and Fink (2010), p. 5-6, report for a sample of 43 mostly developed countries that the average male life expectancy increased between 1965 and 2005 by 8.8 years whereas the average legal male retirement age increased by less than half a year. More strikingly, the correlation between the change in male life expectancy and the change in the retirement age over this time-span is small and negative. While recent years have seen political initiatives to increase the statutory retirement age, e. g., in the EU-27, there is often substantial political resistance (see, e. g., New York Times (2011) on France). Whether and how such changes impact on the effective retirement age that people choose is likely to depend on the future evolution of life expectancy and on institutional details of the retirement scheme (Gruber and Wise (2004)). I shall get back to this issue in Section 5.

labor is done (Landes (1998)). As shown in Irmen (2018), $\nu \in (0,1)$ assures that consumption and leisure are complements in the sense that $\partial^2 U / \partial c_t^y \partial l_t > 0$.

Expected utility, U, is strictly monotone and strictly concave if

$$1 - 2\nu - (1-\nu)\phi x_t > 0. \tag{2.30}$$

This condition requires $\nu < 1/2$. Henceforth, I refer to the set of bundles $\left(c_t^y, l_t, c_{t+1}^o\right) \in \mathbb{R}_{++} \times [0,1] \times \mathbb{R}_{++}$ that satisfy (2.30) as the set of permissible bundles and denote it by $\mathcal{P}$.

At the end of their young age, individuals of cohort t deposit their entire savings with life insurers in exchange for annuity policies. These insurers rent the savings out as fixed capital to the firms producing in $t+1$. In return, the latter pay a (perfect foresight) real rental rate R_{t+1} per unit of savings. Perfect competition among risk-neutral life insurers guarantees a gross return to a surviving old at $t+1$ of R_{t+1}/μ. This rate exceeds the rental rate of capital and compensates individuals for the risk of dying before having their savings withdrawn. Hence, cohort t faces the per-period budget constraints

$$c_t^y + s_t \leq w_t(1-l_t) \quad \text{and} \quad c_{t+1}^o \leq \frac{R_{t+1}}{\mu} s_t. \tag{2.31}$$

I refer to $\left(c_t^y, l_t, c_{t+1}^o, s_t, h_t^s\right)$ as a plan of cohort t. The optimal plan solves

$$\max_{\left(c_t^y, l_t, c_{t+1}^o, s_t\right) \in \mathcal{P} \times \mathbb{R}} U\left(c_t^y, l_t, c_{t+1}^o\right) \quad \text{subject to (2.31)} \tag{2.32}$$

and includes the utility maximizing supply of man-hours as $h_t^s = 1 - l_t$. Before I fully characterize the solution to this problem the following assumption must be introduced.

Assumption 1 *For all t it holds that*

$$w_t > w_c \equiv \left(\frac{(1+\mu\beta)(1-\nu)}{\left(\phi\left(1+(1+\mu\beta)(1-\nu)\right)\right)^{1-\nu}\left(1-\nu(1+\mu\beta)\right)^{\nu}} \right)^{\frac{1}{\nu}}$$

and

$$0 < \nu < \bar{\nu}(\mu\beta) \equiv \frac{3+\mu\beta - \sqrt{5+\mu\beta(2+\mu\beta)}}{2(1+\mu\beta)}.$$

As will become clear in the Proof of Proposition 7, Assumption 1 assures two things. First, if the real wage exceeds the critical level w_c then cohort t's demand for leisure is strictly positive. Second, the unique bundle identified by the Lagrangian associated with problem (2.32) satisfies condition (2.30).[20] Hence, it is a global maximum on the choice set $\mathcal{P} \times \mathbb{R}$.

[20]The function $\bar{\nu}(\mu\beta)$ is strictly positive and declining in $\mu\beta$ with $\bar{\nu}(0) \approx 0.382$ and $\bar{\nu}(1) \approx 0.293$. Hence, Assumption 1 imposes a tighter constraint on ν than just $\nu < 1/2$ which is necessary for (2.30) to hold.

Proposition 7 *(Optimal Plan of Cohort t)*

Suppose Assumption 1 holds. Then, the optimal plan of cohort $t = 1, 2, ..., \infty$ is

$$\begin{aligned}
h_t^s &= w_c^\nu w_t^{-\nu}, \\
c_t^y &= \frac{1-\nu(1+\mu\beta)}{(1+\mu\beta)(1-\nu)} w_c^\nu w_t^{1-\nu}, \\
c_{t+1}^o &= \frac{\beta R_{t+1}}{(1+\mu\beta)(1-\nu)} w_c^\nu w_t^{1-\nu}, \\
s_t &= \frac{\mu\beta}{(1+\mu\beta)(1-\nu)} w_c^\nu w_t^{1-\nu}.
\end{aligned}$$

For surviving members of cohort 0*, consumption when old is $c_1^o = R_1 s_0/\mu > 0$ where $s_0 > 0$ is given.*

According to Proposition 7 cohort t's supply of hours worked declines in the wage with an elasticity equal to ν. As a consequence, the positive response of c_t^y, s_t, and c_{t+1}^o to a wage hike is less than proportionate. Observe that c_t^y and s_t may be expressed, respectively, as the product of a marginal (and average) propensity to consume or to save and the wage income, i. e.,

$$c_t^y = \frac{1-\nu(1+\mu\beta)}{(1+\mu\beta)(1-\nu)} w_t h_t^s \quad \text{and} \quad s_t = \frac{\mu\beta}{(1+\mu\beta)(1-\nu)} w_t h_t^s. \tag{2.33}$$

This helps to understand how a change in the life expectancy affects the optimal plan.

Proposition 8 *(Life-Expectancy and the Optimal Plan of Cohort t)*

If $w_t > w_c$ then it holds that

$$\frac{\partial h_t^s}{\partial \mu} > 0, \quad \frac{\partial c_t^y}{\partial \mu} < 0, \quad \frac{\partial s_t}{\partial \mu} > 0, \quad \frac{\partial c_{t+1}^o}{\partial \mu} < 0.$$

Hence, a higher life expectancy increases the supply of hours worked. This reflects the appreciation of the utility when old relative to the utility when young. Through this channel the demand for leisure declines and h_t^s increases.[21]

[21] A higher μ also reduces the gross rate of return to a surviving old, R_{t+1}/μ. However, for U of (2.29) the substitution and the income effect associated with such a reduction on h_t^s, c_t^y, and s_t cancel out. See Irmen (2018) for details.

The effect of a higher life expectancy on consumption when young is the result of two opposing channels. On the one hand, for a given wage income, the propensity to consume in (2.33) falls. This reflects the desire to shift resources into the second period of life which now has more weight. On the other hand, there will be more income since the supply of hours worked increases. Then, consumption smoothing requires more consumption when young. Overall, the former effect dominates so that c_t^y falls in μ.

The same two channels determine the effect of a higher life expectancy on savings. However, now they are reinforcing. Indeed, for a given wage income, the propensity to save in (2.33) increases. Moreover, a higher wage income and consumption smoothing imply more savings, too. Hence, s_t increases in μ.

Finally, consumption when old declines with a higher life expectancy. Again, two channels of opposite sign are at work. On the one hand, savings increase pushing c_{t+1}^o upwards. On the other hand, the rate of return on savings for a surviving old, R_{t+1}/μ, falls. As the latter dominates, c_{t+1}^o declines in μ.

3 Inter-temporal General Equilibrium

3.1 Definition

A *price system* corresponds to a sequence $\{w_t, R_t\}_{t=1}^{\infty}$. An *allocation* is a sequence

$$\{c_t^y, l_t, c_t^o, s_t, h_t^s, Y_t, K_t, N_t, H_t^d, I_t, q_t(n), a_t(n), h_t(n), i\left(q_t(n)\right)\}_{t=1}^{\infty}$$

for all tasks $n \in [0, N_t]$. It comprises a plan $\{c_t^y, l_t, c_t^o, s_t, h_t^s\}_{t=1}^{\infty}$ for all cohorts, consumption of the old at $t = 1$, c_1^o, and a plan $\{Y_t, K_t, N_t, H_t^d, I_t, q_t(n), a_t(n), h_t(n), i\left(q_t(n)\right)\}_{t=1}^{\infty}$ for the production sector.

For an exogenous evolution of the labor force, $L_t = L_1 \left(1 + g_L\right)^{t-1}$ with $L_1 > 0$ and $g_L > (-1)$, and initial levels of fixed capital, $K_1 > 0$, and technological knowledge, $A_0 > 0$, an *inter-temporal general equilibrium with perfect foresight* corresponds to a price system, an allocation, and a sequence $\{A_t\}_{t=1}^{\infty}$ of the aggregate technological knowledge indicator that comply with the following conditions for all $t = 1, 2, ..., \infty$:

(E1) The production sector satisfies Proposition 3.

(E2) The indicator A_t evolves according to (2.8).

(E3) The plan of each cohort satisfies Proposition 7.

(E4) The market for the manufactured good clears, i. e.,

$$\mu L_{t-1} c_t^o + L_t c_t^y + I_t + I_t^K = Y_t, \tag{3.1}$$

where I_t^K is aggregate investment in fixed capital.

(E5) There is full employment of labor, i. e.,

$$h_t N_t = h_t^s L_t. \tag{3.2}$$

(E1) assures the optimal behavior of the production sector and zero profits. In conjunction with (E2) the evolution of technological knowledge for the economy as a whole boils down to

$$A_t = a_t = A_{t-1}\left(1 + q_t\right), \quad \text{for all } t \text{ given } A_0 > 0. \tag{3.3}$$

(E3) guarantees the optimal behavior of the household sector under perfect foresight. Since the old own the capital stock, their consumption at $t = 1$ is $\mu L_0 c_1^o = R_1 K_1$ and $s_0 = K_1/L_0$. (E4) states that the aggregate demand for the manufactured good produced at t is equal to its supply. Aggregate demand at t comprises aggregate consumption, $\mu L_{t-1} c_t^o + L_t c_t^y$, aggregate automation investments, I_t, and aggregate investment in fixed capital, I_t^K. On the supply side, (3.1) reflects the (innocuous) assumption that fixed capital fully depreciates after one period. According to (E5) the aggregate demand for hours worked must be equal to its supply. Here, use is made of (E1) in that $h_t(n) = h_t$ for all n.

3.2 The Labor Market

The labor market requires a special treatment for two reasons. First, I need to make sure that the labor market equilibrium is unique. This is not obvious since both the aggregate demand for and the aggregate supply of labor fall in the real wage. Accordingly, there may be none, one, or multiple wage levels at which demand is equal to supply. Second, since the supply of hours worked is endogenous, the equilibrium wage, hence, the incentive to engage in automation investments, will depend on demographic, technological, *and* preference parameters.

To focus the discussion on the role of automation and leisure for growth I henceforth restrict attention to constellations that satisfy

Assumption 2 *For all t it holds that*

$$w_t > \alpha A_{t-1} > w_c.$$

The inequality $w_t > \alpha A_{t-1}$ is equivalent to $\omega_t > \alpha$. Hence, it assures that automation investments are profit-maximizing and Proposition 1 and 2 apply. The requirement $w_t > w_c$ means that the optimal plan of all cohorts involves a strictly positive demand for leisure (see Proposition 7). Finally, assuming $\alpha A_{t-1} > w_c$ simplifies the analysis of the transitional dynamics since, in a typical scenario, A_{t-1} grows over time. Hence, if $\alpha A_{t-1} > w_c$ for all t then the distinction between the three regimes $\alpha A_{t-1} < w_c$, $\alpha A_{t-1} = w_c$, and $\alpha A_{t-1} > w_c$ can be neglected.

Before I characterize the labor market equilibrium it proves useful to introduce the following notation:

$$k_t \equiv \frac{K_t}{A_{t-1}^{1-\nu} L_t}, \quad \underline{k}_c \equiv \frac{\alpha^{\frac{1-2\nu}{2}}}{\Lambda}, \quad \text{and} \quad \Lambda \equiv \left(\frac{\Gamma(1-\gamma)}{\alpha^{\frac{2-\gamma}{2}} w_c^{\nu\gamma}} \right)^{\frac{1}{\gamma}}.$$

From now on, I shall refer to k_t as the *efficient capital intensity*. Below, it will serve as the state variable of the dynamical system. The parameter $\underline{k}_c$ denotes a critical level of the efficient capital intensity ensuring that $w_t > \alpha A_{t-1}$ holds in equilibrium whenever $k_t > \underline{k}_c$. Finally, the parameter Λ summarizes technological and preference parameters that affect the relationship between k_t and ω_t defined by the labor-market equilibrium (see equation (3.4) below).

Proposition 9 *(Labor Market Equilibrium)*

Suppose $\alpha A_{t-1} > w_c$ holds. Then, a unique labor market equilibrium, $(\hat{w}_t, \hat{H}_t)$, with $\hat{w}_t > \alpha A_{t-1}$ exists for all $t = 1, 2, ..., \infty$ if and only if

$$k_t > \underline{k}_c.$$

Moreover, the labor market equilibrium defines a function $\hat{\omega} : (\underline{k}_c, \infty) \to (\alpha, \infty)$ such that

$$\hat{\omega}_t = \omega(k_t) \quad \textit{with} \quad \omega'(k_t) > 0.$$

Proposition 9 makes two important points. First, it establishes the conditions for the existence and the uniqueness of the labor market equilibrium consistent with Assumption 2. As shown in Figure 3.1, existence and uniqueness follow since the aggregate supply of hours worked, H_t^s, is sufficiently flatter than the aggregate demand for hours worked, H_t^d. Moreover, $k_t > \underline{k}_c$ assures that H_t^d is sufficiently large relative to H_t^s so that at $w_t = \alpha A_{t-1}$ it holds that $H_t^d > H_t^s$. Accordingly, $\hat{\omega}_t > \alpha$ and the equilibrium wage satisfies $\hat{w}_t > \alpha A_{t-1}$.

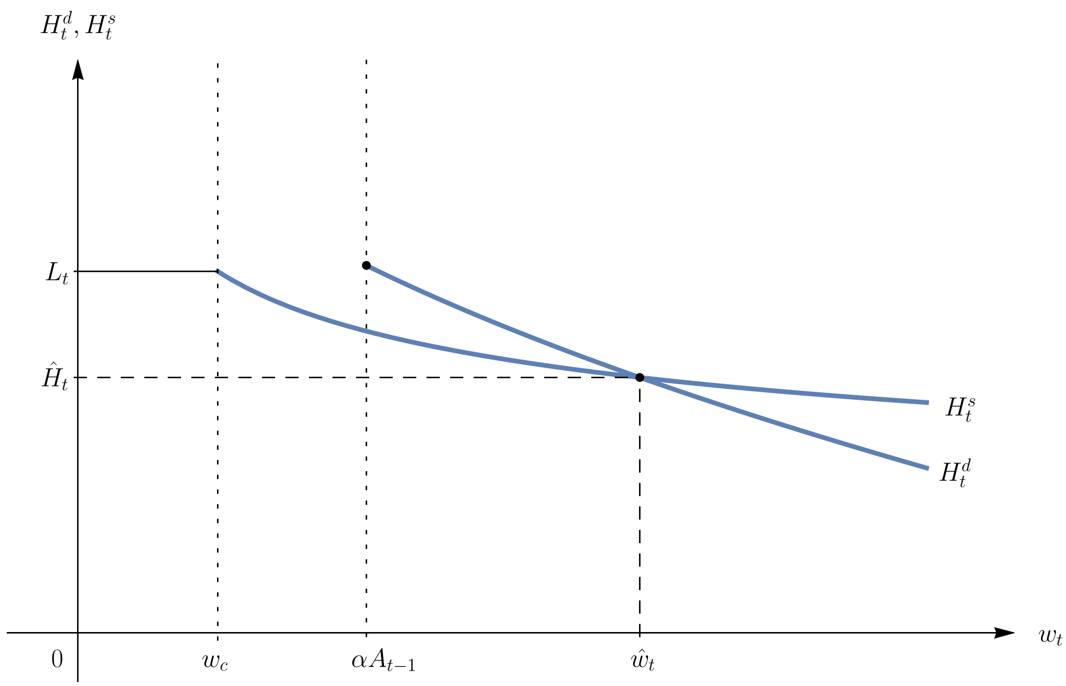
Second, it lays open that the labor market equilibrium can be expressed as a function of k_t. This obtains since $H_t^d = H_t^s$ may be stated as

$$k_t = \frac{\hat{\omega}_t^{\frac{1-2\nu}{2}}}{\Lambda} \left(2\sqrt{\frac{\hat{\omega}_t}{\alpha}} - 1 \right)^{\frac{1}{\gamma}}. \tag{3.4}$$

This condition delivers a unique $\hat{\omega}_t > \alpha$ if and only if $k_t > \underline{k}_c$ and implicitly defines the function $\omega(k_t)$. Hence, $\hat{w}_t = A_{t-1}\omega(k_t)$. In Figure 3.1, a greater K_t shifts H_t^d upwards, a lower L_t shifts H_t^s downwards. Both changes imply a higher level of $\hat{w}_t$, $\hat{\omega}_t$, and of k_t. This is captured by the derivative $\omega'(k_t) > 0$.

The following proposition shows the short-run responses of the labor market equilibrium to changes in the survival probability, μ, and in the real price of automation investments, α.

Figure 3.1: **The Labor-Market Equilibrium** $(\hat{w}_t, \hat{H}_t)$**.** By Assumption 2 it holds that $\alpha A_{t-1} > w_c$. The unique labor market equilibrium satisfies $\hat{w}_t > \alpha A_{t-1}$. Intuitively, existence follows since i) at $w_t = \alpha A_{t-1}$, $H_t^d > H_t^s$, and ii), for $w_t > \alpha A_{t-1}$, H_t^s is flatter than H_t^d so that for large values of w_t, $H_t^s > H_t^d$.



Proposition 10 *(Short-Run Determinants of the Labor Market Equilibrium)*

Consider the labor market equilibrium of Proposition 9. Given k_t, it holds that

$$\frac{\partial \hat{\omega}_t}{\partial \mu} < 0, \quad \textit{and} \quad \frac{\partial \hat{\omega}_t}{\partial \alpha} \lesseqgtr 0 \quad \Leftrightarrow \quad \gamma \lesseqgtr \frac{2\left(\sqrt{\frac{\hat{\omega}_t}{\alpha}} - 1\right)}{2\sqrt{\frac{\hat{\omega}_t}{\alpha}} - 1}.$$

Hence, if cohort t expects to live longer, then $\hat{\omega}_t$ and the equilibrium wage fall while the equilibrium amount of hours worked, $\hat{H}_t$, increases (see Figure 3.2 for an illustration). This follows since, in accordance with Proposition 8, an expected increase in longevity increases the individual supply of hours worked so that H_t^s increases, too. In conjunction with Proposition 1, this has the important implication that a higher μ reduces the incentive to automate in the short-run.[22] Recall that a higher μ at t means that more members of cohort t survive into period $t+1$, and that the old-age dependency ratio in $t+1$ increases. Hence, the conclusion is that the short-run effect of (expected) population aging between period t and $t+1$ reduces the incentives to automate in period t.

A lower α reduces the cost of automation, affects the aggregate demand for hours worked, and the equilibrium wage. From Proposition 5 it is evident that the condition for the sign of $\partial\hat{\omega}_t/\partial\alpha$ is the one that determines the sign of $\partial H_t^d(\omega_t,\alpha)/\partial\alpha$. Hence, a lower α shifts the equilibrium wage up if and only if it increases $H_t^d(\omega_t,\alpha)$. According to Table 2 this is already the case for fairly small annual productivity growth rates, $\tilde{q}(\gamma)$.

To assess the short-run effect of a decline in α on equilibrium automation incentives let me denote the cost-minimizing productivity growth rate of (2.13) evaluated at the labor-market equilibrium as $\hat{q}_t = q(\hat{\omega}_t,\alpha)$. Then,

$$\frac{d\hat{q}_t}{d\alpha} = \underbrace{\frac{\partial q(\hat{\omega}_t,\alpha)}{\partial\omega_t}}_{(+)}\underbrace{\frac{\partial\hat{\omega}_t}{\partial\alpha}}_{(+/-)} + \underbrace{\frac{\partial q(\hat{\omega}_t,\alpha)}{\partial\alpha}}_{(-)}. \tag{3.5}$$

Here, the first term reflects the short-run general equilibrium effect of α on the automation incentives induced through the labor market. The last term reflects the negative direct effect of α on the cost-minimizing level of q_t. Proposition 10 implies that these effects are reinforcing if $\hat{\omega}_t$ is sufficiently larger than α and of opposite sign otherwise. However, some tedious but straightforward algebra reveals that $d\hat{q}_t/d\alpha < 0$ holds unequivocally for all permissible parameter constellations. Hence, in the short-run, a decline in α increases the incentives to automate and implies faster growth technological knowledge.[23]

Finally, observe that Proposition 9 and 10 allow for the analysis of the short-run determinants of the labor share. Denote by $\hat{LS}_t$ the labor share evaluated at the labor-market equilibrium. Then, the following holds.

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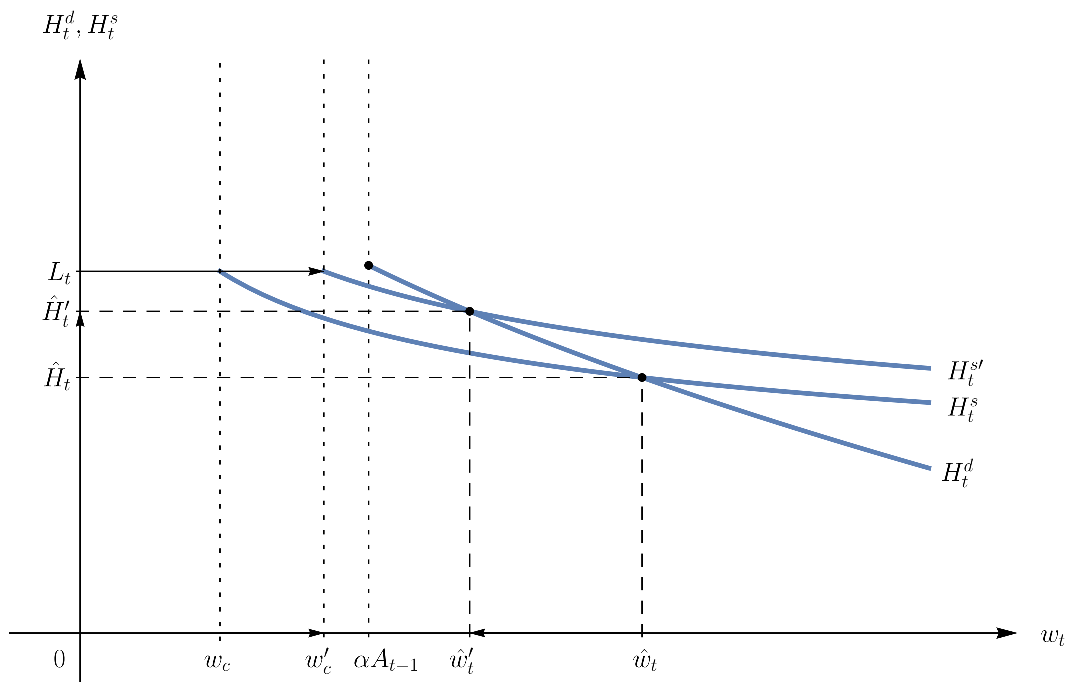
[22]A similar logic applies to the comparative statics $\partial\hat{\omega}_t/\partial\phi > 0$ and $\partial\hat{\omega}_t/\partial\Gamma > 0$, which are not mentioned in Proposition 10. Intuitively, for a higher ϕ the disutility of labor increases and, accordingly, the individual and the aggregate supply of hours worked falls. In Figure 3.2 this would correspond to a leftward shift of H_t^s as w_c falls. It implies a higher $\hat{\omega}_t$, a higher equilibrium wage, $\hat{w}_t$, and a lower equilibrium amount of hours worked, $\hat{H}_t$. A higher Γ increases the marginal product of tasks which boosts the aggregate demand for hours worked, i. e., H_t^d would shift upwards in Figure 3.2. Accordingly, in equilibrium fewer hours of work will be demanded at a higher wage. In line with Proposition 1 this means that an increase in ϕ and Γ strengthens the incentive to automate.

[23]Implicit differentiation of (3.4) delivers

$$\frac{\partial\hat{\omega}_t}{\partial\alpha} = \frac{\hat{\omega}_t}{\alpha\gamma}\cdot\left(\frac{\frac{\sqrt{\hat{\omega}}}{2\sqrt{\hat{\omega}}-\sqrt{\alpha}} - 1 + \frac{\gamma}{2}}{\frac{1}{\gamma}\frac{\sqrt{\hat{\omega}}}{2\sqrt{\hat{\omega}}-\sqrt{\alpha}} + \frac{1}{2} - \nu}\right). \tag{3.6}$$

Computing the direct effects in (3.5) from (2.13) delivers $d\hat{q}_t/d\alpha \lesseqgtr 0 \Leftrightarrow \gamma \lesseqgtr 1/\nu$. Then, $d\hat{q}_t/d\alpha < 0$ follows since $0 < \nu < 1/2$ and $0 < \gamma < 1$.

Figure 3.2: **Population Aging and the Labor-Market Equilibrium.** A higher life-expectancy, $\mu' > \mu$, shifts the aggregate supply of hours worked to the right as $w_c' > w_c$. Accordingly, the equilibrium wage falls and the equilibrium amount of hours worked increases, i. e., $\hat{w}_t' < \hat{w}_t$ and $\hat{H}_t' > \hat{H}_t$.



Corollary 4 *(Short-Run Determinants of the Labor Share)*

Consider the short-run labor share $\hat{LS}_t$. It holds that

$$\frac{d\hat{LS}_t}{d\mu} > 0 \quad \textit{and} \quad \frac{d\hat{LS}_t}{d\alpha} > 0.$$

Hence, in the short run, population aging associated with a higher life expectancy of cohort t increases the labor share. Intuitively, a higher μ boosts the supply of hours worked, reduces the equilibrium wage and, hence, the incentives to automate.

The effect of a decline in α is somewhat more involved as

$$\frac{d\hat{LS}_t}{d\alpha} = \underbrace{\frac{\partial \hat{LS}_t}{\partial \omega_t}}_{(-)} \underbrace{\frac{\partial \hat{\omega}_t}{\partial \alpha}}_{(+/-)} + \underbrace{\frac{\partial \hat{LS}_t}{\partial \alpha}}_{(+)} > 0, \tag{3.7}$$

where the signs of the effects are from Corollary 3 and Proposition 10. Hence, even though the indirect effect through the labor market may be negative the overall effect remains positive. As a consequence, a decline in the real price of automation investments reduces the labor share in the short run.

3.3 The Dynamical System

The transitional dynamics of the inter-temporal general equilibrium can be analyzed through the evolution of a single state variable, k_t. To derive the equilibrium sequence $\{k_t\}_{t=1}^{\infty}$ observe that conditions (E3) and (E4) require investments in fixed capital to equal savings, i. e., $I_t^K = s_t L_t = K_{t+1}$, or

$$\frac{\mu\beta}{(1+\mu\beta)(1-\nu)} w_t h_t^s L_t = K_{t+1}, \quad \text{for all } t = 1, 2, ..., \infty. \tag{3.8}$$

Using Proposition 1 and 7 the latter equation may be expressed as

$$\Omega \omega_t^{\frac{1-\nu}{2}} = k_{t+1}, \quad \text{for } t = 1, 2, ..., \infty, \tag{3.9}$$

where,

$$\Omega \equiv \frac{\alpha^{\frac{1-\nu}{2}} \mu\beta w_c^{\nu}}{(1+\mu\beta)(1-\nu)(1+g_L)}$$

summarizes technological, preference, and demographic parameters that affect the relationship between ω_t and k_{t+1}. Henceforth, I shall refer to equation (3.9) as the capital market equilibrium condition. The equilibrium difference equation results from replacing ω_t of (3.9) with the labor market clearing condition $\hat{\omega}_t = \omega(k_t)$ of Proposition 9. This gives

$$k_{t+1} = \Omega \left[\omega(k_t)\right]^{\frac{1-\nu}{2}}. \tag{3.10}$$

A difficulty arises since the labor market equilibrium has to be such that $\hat{\omega}_t > \alpha$. From Proposition 9 this requires $k_t > \underline{k}_c$ for all t. Hence, (3.9) is to deliver a value $k_{t+1} > \underline{k}_c$. A necessary and sufficient condition for this is $\Omega\alpha^{\frac{1-\nu}{2}} > \underline{k}_c$. Hence, if the latter inequality is satisfied then for any $k_t > \underline{k}_c$ it also holds that $k_{t+1} > \underline{k}_c$ and the labor market equilibrium at $t+1$ satisfies $\hat{\omega}_{t+1} > \alpha$. For notational simplicity define

$$\overline{k}_c \equiv \Omega\alpha^{\frac{1-\nu}{2}}.$$

Then, the following proposition holds.

Proposition 11 *(Dynamical System)*

Consider initial values $(K_1, L_1, A_0) > 0$ such that $k_1 > \underline{k}_c$. Then, the transitional dynamics of the inter-temporal general equilibrium is governed by the autonomous first-order, non-linear difference equation (3.10). If $\overline{k}_c > \underline{k}_c$ it gives rise to a unique equilibrium sequence $\{k_t\}_{t=1}^{\infty}$ with $k_t > \underline{k}_c$ for all t.

Figure 3.3: **The Dynamical System, Steady State, and Transitional Dynamics.** For any $k_1 > \underline{k}_c$ the labor market at $t = 1$ delivers $\omega_1 = \hat{\omega}_1 > \alpha$. Since $\overline{k}_c > \underline{k}_c$, using $\hat{\omega}_1$ in the capital market equilibrium condition for $t = 1$ delivers $k_2 > \underline{k}_c$ and so forth.

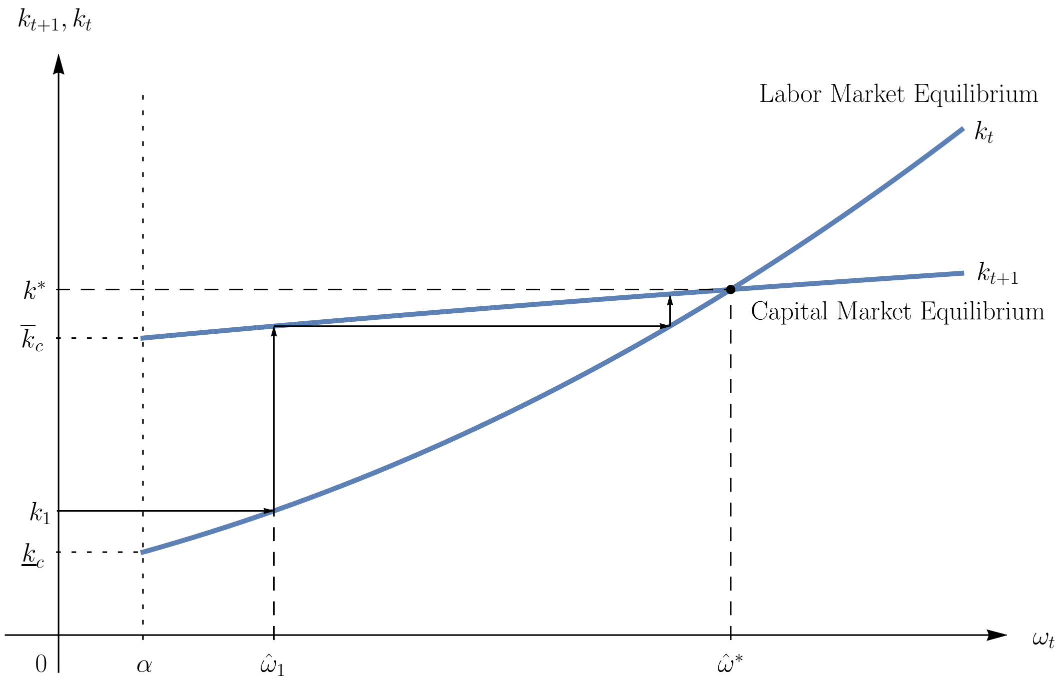


Figure 3.3 illustrates the intuition behind Proposition 11. It depicts the labor market equilibrium at t of equation (3.4) and the capital market equilibrium at t of equation (3.9) for $\overline{k}_c > \underline{k}_c$. Then, for any $k_1 > \underline{k}_c$ the labor market at $t = 1$ delivers $\omega_1 = \hat{\omega}_1 > \alpha$. Using $\hat{\omega}_1$ in the capital market equilibrium condition at $t = 1$ delivers $k_2 > \underline{k}_c$. Clearly, these steps apply to any pair $(k_t, k_{t+1}) > \underline{k}_c$.

4 Steady-State and Transitional Dynamics

4.1 Existence, Uniqueness, and Stability of the Steady State

Figure 3.3 suggests that the steady state is unique and stable. The following proposition confirms this impression.

Proposition 12 *(Steady State - Existence, Uniqueness, Stability)*

If $\overline{k}_c > \underline{k}_c$ then the dynamical system of Proposition 11 has a unique steady state

$$k^* > \overline{k}_c.$$

Moreover, for any $k_1 > \underline{k}_c$ the equilibrium sequence $\{k_t\}_{t=1}^{\infty}$ is monotonous with $\lim_{t\to\infty} k_t = k^*$.

Hence, for $k_1 > \underline{k}_c$ the steady state is unique and asymptotically stable with $k^* > \overline{k}_c$. Intuitively, these properties follow since $\overline{k}_c > \underline{k}_c$ and the right-hand side of (3.10) is increasing and sufficiently concave in k_t.

4.2 Structural Properties and Comparative Statics of the Steady State

The following proposition states the steady-state evolution of all endogenous variables.

Proposition 13 *(Dynamic Properties of the Steady State)*

Consider the steady state of Proposition 12. Then, it holds that aggregate technological knowledge grows at rate $q^* > 0$. Moreover,

a) $$\frac{a_{t+1}}{a_t} = 1 + q^*, \quad \frac{h_{t+1}}{h_t} = \frac{1}{1+q^*}, \quad i_t = i^* > 0, \quad c_t = c^*,$$

b) $$\frac{w_{t+1}}{w_t} = \frac{\hat{w}_{t+1}}{\hat{w}_t} = 1 + q^*, \quad R_t = R^* > 0,$$

c) $$\frac{h^s_{t+1}}{h^s_t} = \frac{1}{(1+q^*)^{\nu}}, \quad \frac{c^y_{t+1}}{c^y_t} = \frac{c^o_{t+1}}{c^o_t} = \frac{s_{t+1}}{s_t} = (1+q^*)^{1-\nu},$$

d) $$\frac{\hat{H}_{t+1}}{\hat{H}_t} = (1+q^*)^{-\nu}(1+g_L), \quad \frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = \frac{I_{t+1}}{I_t} = \frac{N_{t+1}}{N_t} = (1+q^*)^{1-\nu}(1+g_L).$$

The intuition is as follows. Since $\omega^* > \alpha$ firms undertake automation investments that support a strictly positive growth rate of aggregate technological knowledge, $q^* > 0$. On the production side, this means that the productivity of labor in the performance of tasks increases at that rate. Accordingly, there will be rationalization, i. e., $h_{t+1}/h_t < 1$. Automation investments per task remain constant over time. The real wage inherits the growth rate of aggregate technological knowledge since by definition $\hat{w}_t = A_{t-1}\hat{\omega}^*$. As wages and productivity per man-hour grow at the same rate and $i_t = i^*$ the costs per task are time-invariant, i. e., $c_t = c^*$.

On the household side, wage growth implies a declining individual supply of hours worked. The key implication is that wage income, $w_t h^s_t$, grows at a factor $(1+q^*)^{1-\nu}$, which is also the growth factor of consumption in both periods of life and of individual savings.

At the level of economic aggregates, the evolution of the equilibrium amount of hours worked reflects a decline at the intensive margin, $(1+q^*)^{-\nu}$, and an expansion at the extensive margin, $1+g_L$. From the accumulation equation (3.8) it is obvious that fixed capital grows with a factor $(1+q^*)^{1-\nu}(1+g_L)$. Total output, Y_t, the aggregate demand for automation investments, I_t, and the number of tasks, N_t, inherit this trend.[24]

Finally, observe that the steady state is a balanced growth path as the labor share, the ratios K_t/Y_t, I_t/Y_t, and $\left(\mu L_{t-1}c_t^o + L_t c_t^y\right)/Y_t$ as well as the real rental rate of capital remain constant over time.

How does population aging and the price of automation investments affect the growth rate of technological knowledge and the functional income distribution in the long run? To address this question let me denote the steady state growth rate of aggregate technological knowledge by $q^* = q(\hat{\omega}^*, \alpha) = q(\omega(k^*), \alpha)$. Then, the following holds.

Proposition 14 *(Determinants of the Steady-State Growth Rate of Technological Knowledge)*

Consider the steady state of Proposition 12. It holds that

$$\frac{dq^*}{d\mu} > 0, \qquad \frac{dq^*}{dg_L} < 0, \quad \textit{and} \quad \frac{dq^*}{d\alpha} < 0.$$

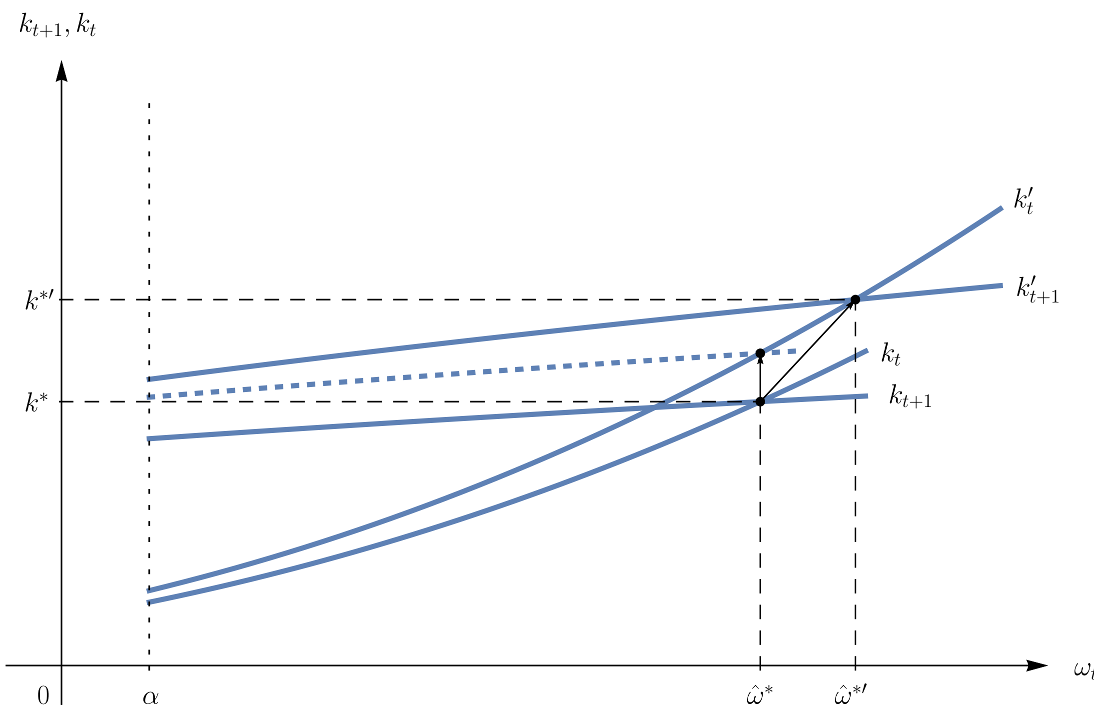
Proposition 14 makes two important points. The first concerns the relationship between population aging and long-run growth. Of two otherwise identical economies the one with a higher life expectancy and/or a lower fertility rate enjoys faster steady-state growth of technological knowledge and more rationalization through automation. As μ and g_L determine the OADR the prediction is that in the long run, the older economy grows faster. The second point concerns the price of automation investments: a decline in this price shifts q^* up.

To develop the intuition behind these findings recall that the steady state, hence q^*, is determined by the interaction between the labor and the capital market as shown in Figure 3.3.

The effect of a permanent increase in μ on q^* reflects three channels. They are illustrated in Figure 4.1. Initially the economy has a life expectancy equal to μ and starts in the steady

[24]Growth factors may be classified according to the following rule. The growth factor of per-capita variables is the one of efficient individual hours worked, $A_t h_t^s$. It is equal to $(1+q^*)^{1-\nu}$ and reflects the attenuating effect of a declining labor supply. The growth factor of aggregate variables is the one of aggregate efficient hours worked, $A_t h_t^s L_t$ and equal to $(1+q^*)^{1-\nu}(1+g_L)$.

Figure 4.1: **The Effect of a Higher Life Expectancy on the Steady State.** In response to a permanent increase in life expectancy from μ to μ' the steady state of the economy switches from (ω^*, k^*) to $(\omega^{*\prime}, k^{*\prime})$. The labor market equilibrium condition (3.4) denoted, respectively, by k_t and k_t' shifts upwards as a higher life expectancy increases the supply of hours worked (see Proposition 8 and 10). The capital market equilibrium locus of (3.9) denoted, respectively, by k_{t+1} and k_{t+1}' shifts upwards for two reasons (see Proposition 8). First, individual wage income increases as individuals work more hours, and, second, the propensity to save increases. The dashed blue line shows the upward shift of the capital market equilibrium locus that reflects only the increase in the supply of hours worked. This shift leaves ω^* unchanged.



state (ω^*, k^*). The new steady state corresponding to $\mu' > \mu$ is $(\omega^{*\prime}, k^{*\prime})$. The first channel concerns the labor market. According to Proposition 8 a higher μ increases the individual supply of hours worked, hence, H_t^s increases. This puts pressure on the equilibrium wage, and, given k_t the labor market equilibrium locus shifts leftwards. The second and third channel concern the capital market. Here, a greater μ increases savings for two reasons. On the one hand, the wage income increases with the individual supply of hours worked (see Proposition 3.9). Given ω_t, this shifts the capital market equilibrium locus in Figure 4.1 upwards (second channel). On the other hand, the individual propensity to save increases (again, see Proposition 8). In Figure 4.1, this effect shifts the capital market equilibrium locus even further upwards (third channel). As a result, the new steady state has $k^{*\prime} > k^*$, $\hat{\omega}^{*\prime} > \hat{\omega}^*$, and, $q^{*\prime} > q^*$. Hence, population aging through increased longevity induces faster steady-state growth of technological knowledge and more automation.[25]

A decline in the fertility rate, g_L, means higher savings per unit of next period's workers, i. e., Ω increases in (3.9). This shifts the capital market equilibrium locus in Figure 3.3 upwards while leaving the labor market equilibrium locus unaffected. Accordingly the new steady has $k^{*\prime} > k^*$, $\hat{\omega}^{*\prime} > \hat{\omega}^*$, and, $q^{*\prime} > q^*$. Hence, in the long run population aging through a decline in fertility leads to faster growth of technological knowledge and more automation.

[25] The underlying computations reveal that, as shown in Figure 4.1, the first and the second channel shift k^* upwards while leaving $\hat{\omega}^*$ unaffected. To see this analytically, consider a small increase in life expectancy from μ to μ' where $\mu' > \mu$ at the steady state (ω^*, k^*). First, I show that, given ω^*, this change shifts the labor market equilibrium locus upwards. Second, I show that, given ω^*, the capital market equilibrium locus shifts upwards through an expansion of the labor supply and the associated increase in the wage income. Finally, I show that these two effects coincide.

At (ω^*, k^*), the labor market equilibrium of (3.4) satisfies

$$k^* = \frac{(\hat{\omega}^*)^{\frac{1-2\nu}{2}}}{\Lambda} \left(2\sqrt{\frac{\hat{\omega}^*}{\alpha}} - 1\right)^{\frac{1}{\gamma}}.$$

Then, given ω^* the effect of $d\mu = \mu' - \mu > 0$ on k^* is

$$dk^* = \frac{(\hat{\omega}^*)^{\frac{1-2\nu}{2}}}{\Lambda} \left(2\sqrt{\frac{\hat{\omega}^*}{\alpha}} - 1\right)^{\frac{1}{\gamma}} \left(\frac{\partial w_c}{\partial \mu} \frac{\nu}{w_c}\right) d\mu = k^* \left(\frac{\partial w_c}{\partial \mu} \frac{\nu}{w_c}\right) d\mu > 0 \tag{4.1}$$

as $\partial w_c / \partial \mu > 0$. At (ω^*, k^*), the capital market equilibrium of (3.9) satisfies

$$k^* = (\omega^*)^{\frac{1-\nu}{2}} \Omega.$$

Then, given ω^*, and keeping the savings rate, $\mu\beta / ((1+\mu\beta)(1-\nu))$, constant I have

$$dk^* = (\omega^*)^{\frac{1-\nu}{2}} \frac{\partial \Omega}{\partial w_c^\nu} \frac{\partial w_c^\nu}{\partial \mu} = (\omega^*)^{\frac{1-\nu}{2}} \Omega \left(\frac{\partial w_c}{\partial \mu} \frac{\nu}{w_c}\right) d\mu = k^* \left(\frac{\partial w_c}{\partial \mu} \frac{\nu}{w_c}\right) d\mu. \tag{4.2}$$

From (4.1) and (4.2) it is obvious that at ω^* a small change, $d\mu > 0$, shifts the loci of the labor and the capital market upwards by the same amount.

The impact of a change in α on q^* is given by

$$\frac{dq^*}{d\alpha} = \underbrace{\frac{\partial q\left(\hat{\omega}^*, \alpha\right)}{\partial \omega}}_{+} \underbrace{\frac{d\hat{\omega}^*}{d\alpha}}_{(+/-)} + \underbrace{\frac{\partial q\left(\hat{\omega}^*, \alpha\right)}{\partial \alpha}}_{-}. \tag{4.3}$$

Hence, in addition to the general equilibrium effect, $\left(\partial q\left(\hat{\omega}^*, \alpha\right) / \partial \omega\right)\left(d\hat{\omega}^* / d\alpha\right)$, there is a negative direct effect as $\partial q\left(\hat{\omega}^*, \alpha\right) / \partial \alpha < 0$. The general equilibrium effect is not unequivocal.[26] However, whatever its sign, the total effect is negative. Hence, a lower price of automation investments means faster steady-state growth of technological knowledge.

Finally, I turn to the long-run determinants of the functional income distribution. Let LS^* denote the steady-state labor share.

Proposition 15 *(Determinants of the Steady-State Functional Income Distribution)*

Consider the steady state of Proposition 12. It holds that

$$\frac{dLS^*}{d\mu} < 0, \qquad \frac{dLS^*}{dg_L} > 0, \quad \textit{and} \quad \frac{dLS^*}{d\alpha} > 0.$$

Hence, population aging reduces the steady-state labor share irrespective of whether it is due to a higher life expectancy or a decline in the fertility rate. Moreover, a permanent decline in the price of automation investments diminishes the steady-state labor share. In light of Proposition 14, one may correctly argue that a parameter change that speeds up q^* also lowers LS^*. A deeper intuition can be gained from writing the steady-state labor share as (see Proposition 6)

$$LS^* = (1-\gamma)\left(\frac{1}{1+\gamma\frac{i_t}{w_t h_t}}\right).$$

Hence, what matters is how the respective parameter change affects the ratio of investment outlays and wage costs per task in steady state. A higher μ, a lower g_L as well as a lower α increase this ratio. Accordingly, LS^* declines.

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[26]To see this consider the capital market equilibrium locus. Here, α appears on the left-hand side of the capital market equilibrium condition (3.9) as savings at t are expressed in units of $A_t^{1-\nu}L_{t+1}$ and $A_t^{1-\nu} = A_{t-1}^{1-\nu}\left(1+q_t\right)^{1-\nu} = A_{t-1}^{1-\nu}\left(\omega_t/\alpha\right)^{(1-\nu)/2}$ where the last step uses Proposition 1. Hence, as a lower α induces a higher q_t, savings per unit of $A_t^{1-\nu}L_{t+1}$ and Ω fall. Accordingly, a lower α shifts the capital market equilibrium locus in Figure 3.3 downwards. At the same time, a change in α also affects the labor market through its effect on the aggregate demand for hours worked. Proposition 5 implies for reasonable parameter values that a decline in α will increase the aggregate demand for hours worked. In Figure 3.3, this shifts the labor market equilibrium locus rightwards. As a consequence, $d\hat{\omega}^*/d\alpha$ may be positive or negative.

4.3 A Simple Calibration

The purpose of this section is to show that the model and its steady state lend themselves to a reasonable calibration. Throughout, a period corresponds to 30 years. The calibration delivers an annual steady-state growth rate of per-capita output, per-capita consumption and savings of 2%, an annual rate of decline in the individual supply of hours worked of 0.658%, an annual real rental rate of capital of 4.21%, and a labor share of slightly less than 2/3.

On the *production side* I set

$$\Gamma = 6.15, \quad \gamma = \frac{1}{4}, \quad \text{and} \quad \alpha = 1.$$

For the *household sector* preferences and demographics are characterized by the following parameter values:

$$\mu = 70\%, \quad \beta = \frac{10}{21}, \quad \nu = \frac{1}{4}, \quad \phi = \frac{1}{2}\left(\frac{3}{2}\right)^{\frac{1}{3}}, \quad \text{and} \quad g_L = 35\%.$$

I proxy μ with the probability at birth for males of reaching the age of 65 as shown in Figure 1.1. In line with the literature, the chosen value for β corresponds to an annual discount factor of roughly 0.976 (see, e. g., Prescott (1986), Blanchard and Fischer (1989), p. 147, or Barro and Sala-í-Martin (2004), p. 197). The wage elasticity of hours worked, ν, is in line with the value suggested by Boppart and Krusell (2018). The preference parameter ϕ is chosen such that $w_c = 1$. Then, Assumption 1 holds for $w_t > 1$ as $\nu = 1/4 < \bar{\nu}(\mu\beta) = 0.348612$. Finally, the fertility rate, g_L, implies that cohorts grow at an annual rate of 1%.

Proposition 16 *(Steady State of the Calibrated Economy)*

Let $A_0 > 1$ and suppose that the calibrated economy embarks on a steady state in $t = 1$. Then, the steady state satisfies $w_t > \alpha A_{t-1} > w_c$ for all $t = 1, 2, ..., \infty$. Moreover, it holds that

$$k^* = 0.447278, \quad \omega^* = 4.8762623457,$$

and

$$q^* = 1.20823, \quad g^*_{h^s} = -0.170669, \quad LS^* = 0.659754, \quad R^* = 3.78354.$$

Proposition 16 shows that the calibration of the model of Section 2 delivers reasonable results. To confirm this impression observe that $q^* = 1.20823$ means that technological knowledge and of the real wage grow at an annual rate of 2.67581%. Moreover, from

Proposition 13, the growth factor of per-capita output, per-capita consumption and savings satisfies $(1+q^*)^{1-\nu} = 2.20823^{3/4}$ which implies an annual growth rate of 2%. The order of magnitude of the labor share is also in line with the empirical evidence.

From Proposition 13 the growth rate of the individual supply of hours worked is $g^*_{h^s} = 1/(1+q^*)^{\nu} - 1 = -0.170669$. This corresponds to an annual growth rate of -0.657984%.[27]

Finally, the real rental rate R^* corresponds to an annual rate of 5.35577%. Considering an annual depreciation rate of fixed capital equal to 5% gives an annual net rental rate of 4.85577%.

5 Concluding Remarks

People who recognize that they are likely to get older adapt their behavior. As aging affects the population as a whole the resulting behavioral changes have macroeconomic implications. Hence, population aging alters the environment, in which firms operate, and will have an effect on investment, hiring, and output supply decisions. The present paper disentangles the repercussions between these behavioral changes and derives the consequences for automation, factor shares, and economic growth in a novel dynamic competitive endogenous growth model where technical change is labor-augmenting.

In the short run, the expectation of getting older induces people to expand their labor supply. This puts pressure on the equilibrium wage and reduces the incentive to engage in automation investments. As a consequence, the labor share increases. However, these effects may be offset if, at the same time, the real price of automation investments falls. For reasonable parameter values, I find that such a decline boosts the aggregate demand for hours worked, increases the equilibrium wage, and strengthens the incentives to undertake automation investments. Hence, in the short run, the effect of population aging on automation, factor shares, and economic growth may be neutralized by a decline in the real price of automation investments. These finding contrast with the long run.

In the long run, population aging, i. e., a higher life-expectancy and/or a decline in fertility, and the decline in the real price of automation investment are reinforcing: each

[27]For the US the PWT 9.0 estimates average annual hours per person engaged in 1960 to equal 1863. In 2010 the corresponding number is 1695 (Feenstra, Inklaar, and Timmer (2015)). This corresponds to an average annual growth rate of -0.19%. The estimates for the annual hours worked per worker of Huberman and Minns (2007) for the US are much higher than the numbers in the PWT 9.0. According to Huberman and Minns (2007) annual hours worked per worker in 1960 were 2033 whereas this number plunges to 1878 for the year 2000. However, the average annual growth rate of roughly -0.1845% is in line with the one found for the PWT data. The chosen calibration implies an average annual growth rate of hours worked of -0.657984% which is closer to the estimate that Boppart and Krusell (2018) derive for the sample of countries included in Figure 1.1 over the time span 1870-2000. This confirms the view that the US evolution of hours worked per worker is an outlier.

change induces more automation, a lower labor share, and faster growth of per-capita variables. A higher life-expectancy encourages savings and capital accumulation as i) the expansion of the individual labor supply increases earnings and ii) the propensity to save increases. In the long run, this leads to a higher efficient capital intensity, higher real wages, more automation, a smaller labor share, and faster economic growth. A decline in the fertility rate mimics these findings as savings per worker and the efficient capital intensity will be higher is all periods following the fertility decline. A lower real price of automation investments induces more automation since automating firms face lower investment costs. Hence, in the long run, a lower real price of automation investments reduces the labor share and speeds up economic growth.

The present paper gives rise to several interesting questions that a comprehensive understanding of the determinants of automation, factor shares, and economic growth in the era of population aging needs to address. For instance, one may take into account that labor markets are not competitive. The historical experience suggests that workers do not freely choose their amount of working hours. Rather, the average work week is fixed by law or by negotiations between employers and employee representatives such as trade-unions (Huberman and Minns (2007)). This raises the question of whether a growing franchise or the behavior of unions can have a macroeconomic effect on the incentive to automate. Related is the question about the possible effect of a binding minimum wage on automation investments (Hellwig and Irmen (2001)).

Another characteristic of the era of population aging is the increase in individual educational attainments (Goldin and Katz (2008)). Intuition suggests that the expectation of a longer work-life increases the rate of return of an educational investment. Hence, educational attainments may increase in response to population aging. However, the potential effect of this tendency on the incentive to automate remains elusive.

Finally, one may want to allow for alternative ways to expand the supply of hours worked in response to aging. They include an endogenous retirement age and/or an extensive margin. The results of the present paper suggest that individuals will want to retire later and expand the extensive margin in response to a higher life-expectancy. For the short and the long run, one may conjecture that these tendencies are likely to have similar qualitative effects as those derived in the analysis above. I leave the detailed analysis of these issues for future research.

A Proofs

A.1 Proofs of Propositions

A.1.1 Proof of Proposition 1

Given $q(\omega_t, \alpha)$ of (2.13), equation (2.9) delivers $h_t = 1/\left(A_{t-1}\left(1+q\left(\omega_t, \alpha\right)\right)\right) \equiv h\left(\omega_t, \alpha\right)/A_{t-1}$, where $h\left(\omega_t, \alpha\right) \equiv 1/\left(1+q\left(\omega_t, \alpha\right)\right)$. From (2.4) $i_t = i\left(q\left(\omega_t\right), \alpha\right) \equiv i\left(\omega_t, \alpha\right)$. Since the wage cost per task is $w_t h_t = \omega_t h\left(\omega_t, \alpha\right)$, we have $c_t = \omega_t h\left(\omega_t, \alpha\right) + i\left(\omega_t, \alpha\right) \equiv c\left(\omega_t, \alpha\right)$. Continuity of these functions follows since $\lim_{\omega_t \downarrow \alpha} q\left(\omega_t, \alpha\right) = 0$. The remaining arguments that complete the proof are straightforward or given in the main text. ■

A.1.2 Proof of Proposition 2

Given in the main text.

A.1.3 Proof of Proposition 3

Given in the main text.

A.1.4 Proof of Proposition 4

I consider each claim in turn.

1. This follows immediately from (2.18) and (2.19) evaluated at $\omega_t = \alpha$.
2. Given in the main text.
3. From (2.18) and (2.19) $H_t^{d2}(w_t) \geq H_t^{d1}(w_t, \alpha)$ holds if and only if
$$\left(\frac{1}{\omega_t}\right)^{\frac{1}{\gamma}} \geq \sqrt{\frac{\alpha}{\omega_t}}\left(\frac{1}{2\sqrt{\alpha\omega_t} - \alpha}\right)^{\frac{1}{\gamma}}. \tag{A.1}$$
Rearranging using $z \equiv \omega_t/\alpha \geq 1$ reveals that the latter condition boils down to
$$0 \geq z^{\frac{2-\gamma}{2}} - 2z^{\frac{1}{2}} + 1 \equiv RHS(z). \tag{A.2}$$
One readily verifies that $RHS(1) = 0$ and $RHS'(1) < 0$. Hence, there are values $z > 1$ so that $(A.1)$ holds with strict inequality. Observe further that $RHS(z)$ attains a minimum at
$$z_{min} = \left(\frac{2}{2-\gamma}\right)^{\frac{2}{1-\gamma}} > 1$$
since $\gamma > 0$. Moreover, for $z > z_{min}$, $RHS(z)$ monotonically increases with $\lim_{z\to\infty} RHS(z) = \infty$. The latter follows as $RHS(z)$ may be written as
$$RHS(z) = z^{\frac{1}{2}}\left(z^{\frac{1-\gamma}{2}} - 2 + z^{\frac{-1}{2}}\right)$$
and $\gamma < 1$. Accordingly, there is a unique $\bar{\omega} \in (\alpha, \infty)$ such that (A.2), hence, (A.1), is violated for $\omega_t > \bar{\omega}$.
The values of $q(\bar{\omega})$ and $\tilde{q}(\bar{\omega})$ in Table 1 are derived as follows. From (A.2) I compute the critical $z_c > 1$ that satisfies $RHS(z_c) = 0$. The latter is then used to compute $q(\bar{\omega})$ using Proposition 1. Finally, $\tilde{q}(\bar{\omega}) = (1 + q(\bar{\omega}))^{1/30} - 1$. ■

A.1.5 Proof of Proposition 5

Follows directly from the second expression of (2.18) and (2.20). ■

A.1.6 Proof of Proposition 6

Given in the main text. ■

A.1.7 Proof of Proposition 7

For ease of notation I shall most often suppress the time argument. Consider problem (2.32). Since preferences are increasing in c^o both per-period budget constraints will hold as equalities and can be merged. Accordingly, the Lagrangian of this problem is

$$\mathcal{L} = \ln c^y + \ln \left(1 - \phi (1-l) (c^y)^{\frac{\nu}{1-\nu}}\right) + \mu\beta \ln c^o + \lambda \left[w (1-l) - c^y - \frac{\mu c^o}{R}\right]. \tag{A.3}$$

Corner solutions involving $c^y = c^o = 0$ and $l = 1$ can be excluded since U satisfies the Inada conditions and $l = 1$ implies no income. Hence, with $x \equiv (1-l)(c^y)^{\frac{\nu}{1-\nu}}$ the respective first-order Kuhn-Tucker conditions read as follows:

$$\frac{\partial \mathcal{L}}{\partial c^y} = \frac{1-\nu-\phi x}{c^y(1-\nu)(1-\phi x)} - \lambda = 0, \tag{A.4}$$

$$\frac{\partial \mathcal{L}}{\partial l} = \frac{\phi (c^y)^{\frac{\nu}{1-\nu}}}{1-\phi x} - \lambda w \leq 0, \quad \text{with strict inequality if } l_t = 0, \tag{A.5}$$

$$\frac{\partial \mathcal{L}}{\partial c^o} = \frac{\beta}{c^o} - \frac{\lambda}{R} = 0, \tag{A.6}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = w(1-l) - c^y - \frac{\mu c^o}{R} = 0. \tag{A.7}$$

Suppose $l > 0$. Then, upon multiplication by $(1-l)$, condition (A.5) may be written as

$$\frac{\phi x}{(1-\phi x)\, w\, (1-l)} = \lambda. \tag{A.8}$$

Using the latter to replace λ in (A.4) and (A.6) delivers, respectively,

$$c^y = \left(\frac{1}{\phi x} - \frac{1}{1-\nu}\right) w(1-l) \tag{A.9}$$

and

$$c^o = \beta R \left(\frac{1}{\phi x} - 1\right) w(1-l). \tag{A.10}$$

With (A.9) and (A.10) in the budget constraint (A.7) I obtain

$$\phi x_t = \phi x = \frac{(1+\mu\beta)(1-\nu)}{1+(1+\mu\beta)(1-\nu)} \in (0,1). \tag{A.11}$$

Using (A.11) in (A.9), (A.10), and (2.31) delivers (2.33). Since the optimal plan satisfies Assumption 1 I have $1 > \nu(1+\mu\beta)$, hence, $c^y > 0$.

From the definition of x with $h^s = 1 - l$ it holds that $c^y = \left(x\left(h^s\right)^{-1}\right)^{\frac{1-\nu}{\nu}}$. Replacing c^y with this expression in (2.33) and solving for h^s delivers h_t^s. Using the latter in (2.33) delivers c_t and s_t. Then, c_{t+1}^o is obtained from the budget when old. Clearly, $h_t^s \leq 1$ as long as $w_t \geq w_c$. In accordance with this, $w < w_c$ implies a strict inequality in (A.5).

To see that the solution identified by the Lagrangian (A.3) is indeed a global maximum if $\nu < \bar{\nu}\left(\mu\beta\right)$ consider first the leading principal minors of the Hessian matrix of $U\left(c^y, l, c^o\right)$, i. e.,

$$\begin{aligned}
D_1\left(c^y, l, c^o\right) &= -\frac{\left(1-\nu-\phi x\right)^2 + \nu\phi x\left(1-\phi x\right)}{\left(c^y(1-\nu)\left(1-\phi x\right)\right)^2}, \\
D_2\left(c^y, l, c^o\right) &= \frac{\phi^2\left(1-2\nu-(1-\nu)\phi x\right)}{\left(c^y\right)^{\frac{2(1-2\nu)}{1-\nu}}\left(1-\nu\right)^2\left(1-\phi x\right)^3}, \\
D_3\left(c_t^y, l_t, c_{t+1}^o\right) &= -\frac{\mu\beta}{\left(c^o\right)^2} D_2\left(c_t^y, l_t, c_{t+1}^o\right).
\end{aligned}$$

First, we have $-D_1\left(c^y, l, c^o\right) > 0$. Second, observe that $D_2\left(c^y, l, c^o\right) > 0$ and $-D_3\left(c^y, l, c^o\right) > 0$ hold if and only if condition (2.30) holds. Hence, U is strictly concave for all $\left(c^y, l, c^o\right) \in \mathcal{P}$.

What remains to be shown is that the solution identified by the Lagrangian satisfies condition (2.30). With ϕx of (A.11) this is the case if and only if

$$\frac{1-2\nu}{1-\nu} > \frac{(1+\mu\beta)(1-\nu)}{1+(1+\mu\beta)(1-\nu)}$$

or

$$\nu^2(1+\mu\beta) - \nu(3+\mu\beta) + 1 > 0.$$

It is not difficult to show that the latter condition is satisfied if and only if $\nu < \bar{\nu}\left(\mu\beta\right)$ as stated in Assumption 1.

Finally, observe that surviving members of cohort 0 satisfy their budget constraint when old as equality, i. e., we have $c_1^o = R_1 s_0 / \mu > 0$. ■

A.1.8 Proof of Proposition 8

Some straightforward algebra reveals that

$$\frac{\partial w_c}{\partial \mu} = \frac{\beta w_c}{\nu(1+\mu\beta)(1+(1-\nu)(1+\mu\beta))(1-\nu(1+\mu\beta)} > 0.$$

It follows that $\partial h_t^s / \partial \mu > 0$. From the definition of w_c and Proposition 7 c_t^y may be written as

$$c_t^y = \left(\frac{1-\nu\left(1+\mu\beta\right)}{\phi\left(1+\left(1+\mu\beta\right)\left(1-\nu\right)\right)}\right)^{1-\nu} w_t^{1-\nu}.$$

Hence,

$$\frac{\partial c_t^y}{\partial \mu} = \frac{-\left(1-\nu\right)\beta w_t^{1-\nu}}{\phi^{1-\nu}(1+(1-\nu)(1+\mu\beta))^{2-\nu}(1-\nu(1+\mu\beta))^{\nu}} < 0.$$

The sign of $\partial s_t / \partial \mu > 0$ follows since the marginal propensity to save in (2.33) increases in μ and $\partial h_t^s / \partial \mu > 0$. Finally, using s_t in the budget constraint of a surviving old delivers

$$c_{t+1}^o = \frac{\beta R_{t+1} w_t^{1-\nu}}{\phi^{1-\nu}(1+(1-\nu)(1+\mu))^{1-\nu}(1-\nu(1+\mu))^{\nu}}.$$

Hence, by Assumption 1

$$\frac{\partial c_{t+1}^o}{\partial \mu} = -\frac{\left(\nu^2(1+\mu\beta) - \nu(3+\mu\beta) + 1\right)\beta^2 R_{t+1} w_t^{1-\nu}}{\phi^{1-\nu}(1+(1+\nu)(1+\mu\beta))^{2-\nu}(1-\nu(1+\mu\beta))^{1+\nu}} < 0.$$

■

A.1.9 Proof of Proposition 9

Under Assumption 2 the aggregate demand for hours worked is $H_t^d = H_t^{1d}(\omega_t, \alpha)$ of (2.18). The aggregate supply of hours worked, $H_t^s = h_t^s L_t$, follows with Proposition 7. Hence, the labor market equilibrium requires $H_t^d = H_t^s$ or

$$\frac{K_t}{A_{t-1}}\sqrt{\frac{\alpha}{\omega_t}}\left(\frac{\Gamma(1-\gamma)}{2\sqrt{\alpha\omega_t}-\alpha}\right)^{\frac{1}{\gamma}} = L_t w_c^{\nu} w_t^{-\nu}.$$

Rearranging and using $k_t \equiv K_t/\left(A_{t-1}^{1-\nu}L_t\right)$ delivers equation (3.4) which I restate here for convenience,

$$k_t = \frac{\hat{\omega}_t^{\frac{1-2\nu}{2}}}{\Lambda}\left(2\sqrt{\frac{\hat{\omega}_t}{\alpha}}-1\right)^{\frac{1}{\gamma}}.$$

Denote the right-hand side of this equation by $RHS(\omega_t)$ where $RHS : [\alpha, \infty) \to [\underline{k}_c, \infty)$. Then, $RHS(\alpha) = \underline{k}_c > 0$. Moreover, since $\nu < 1/2$ we have $RHS'(\omega_t) > 0$ for $\omega_t > \alpha$ and $\lim_{\omega_t \to \infty} RHS(\omega_t) = \infty$. The left-hand side of (3.4) is strictly positive for any permissible parameter constellation. Hence, for equation (3.4) to be satisfied for any value $\omega_t > \alpha$ it is necessary and sufficient to have $RHS(\alpha) < k_t$ or $k_t > \underline{k}_c$. Then, the properties of $RHS(\omega_t)$ assure that there is indeed a unique $\hat{\omega}_t > \alpha$ that satisfies (3.4). The unique equilibrium wage is then $\hat{w}_t = \hat{\omega}_t A_{t-1} > \alpha A_{t-1}$, the equilibrium amount of hours worked is $\hat{H}_t = H_t^{1d}(\hat{\omega}_t, \alpha) < L_t$.

From (3.4) it is also obvious that there is a function $\omega(k_t)$ with the indicated properties. ■

A.1.10 Proof of Proposition 10

Consider equation (3.4). Since $\partial w_c/\partial\mu > 0$ I have $\partial\Lambda/\partial\mu < 0$. Hence, $\partial\hat{\omega}_t/\partial\mu < 0$. Since H_t^d does not depend on μ, $\hat{H}_t$ increases in μ.

Total differentiation of (3.4) with respect to α and $\hat{\omega}_t$ delivers

$$\frac{\partial\hat{\omega}_t}{\partial\alpha} = \frac{\frac{\partial\Lambda}{\partial\alpha}k_t + \Lambda k_t \frac{\sqrt{\hat{\omega}_t}}{\alpha\gamma\left(2\sqrt{\hat{\omega}_t}-\sqrt{\alpha}\right)}}{Z},$$

where $Z > 0$ stems from the derivative of the right-hand side of (3.4) with respect to $\hat{\omega}_t$. Taking the derivative $\partial\Lambda/\partial\alpha$ reveals the condition for $\hat{\omega}_t$ (and $\hat{H}_t$) as stated in the Proposition. ■

A.1.11 Proof of Proposition 11

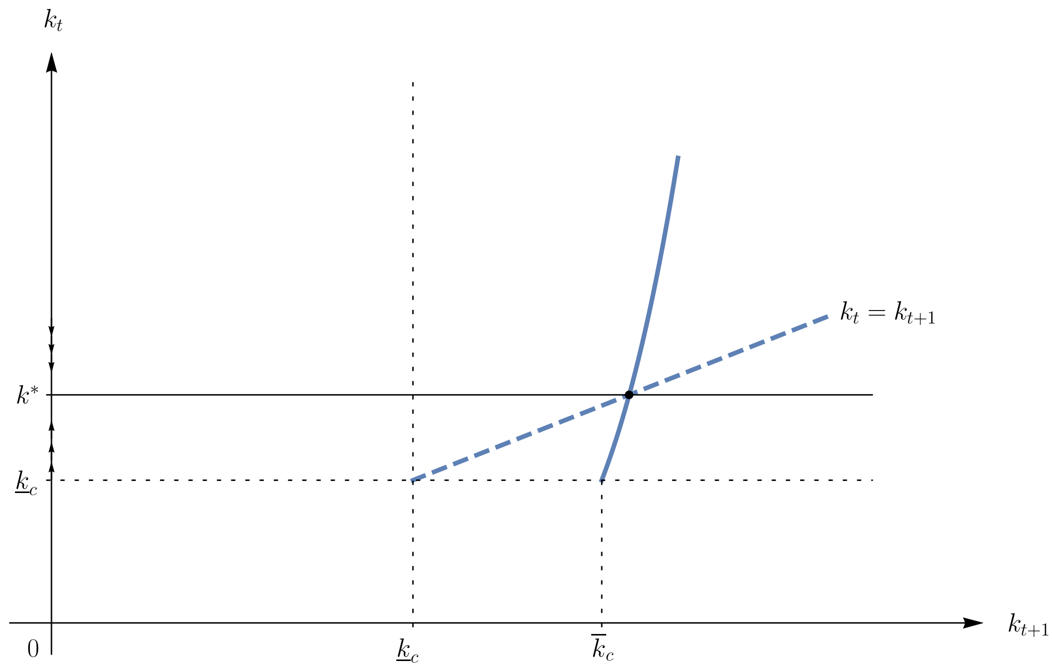
First, observe that k_t is a state variable of the inter-temporal general equilibrium. Indeed, given k_t, the labor market determines $\hat{\omega}_t = A_{t-1}w_t > \alpha$. Hence, Proposition 1 delivers q_t, a_t, i_t, and c_t. Proposition 2 and (2.17) determine N_t, Y_t, I_t, and H_t^d. Hence, (2.16) delivers R_t. On the household side, Proposition 7 gives h_t^s, l_t, c_t^y, c_t^o, s_t. Finally, K_{t+1} follows from (3.8).

Second, consider (3.4) and replace ω_t by $\omega_t = (k_{t+1}/\Omega)^{2/(1-\nu)}$ from (3.9). This gives the equilibrium difference equation (3.10) as (see Figure A.1 for an illustration)

$$k_t = \frac{1}{\Lambda}\left(\frac{k_{t+1}}{\Omega}\right)^{\frac{1-2\nu}{1-\nu}}\left(\frac{2}{\sqrt{\alpha}}\left(\frac{k_{t+1}}{\Omega}\right)^{\frac{1}{1-\nu}}-1\right)^{\frac{1}{\gamma}}. \tag{A.12}$$

Denote the right-hand side of the latter equation by $RHS(k)$. Since, $k > \bar{k}_c$ and $\lim_{k\downarrow\bar{k}_c} RHS(k) = \underline{k}_c$ I have $RHS : \left(\bar{k}_c, \infty\right) \to (\underline{k}_c, \infty)$.

Figure A.1: **The Equilibrium Difference Equation (A.12).** Since $\overline{k}_c > \underline{k}_c$, there is for any $k_t > \underline{k}_c$ a unique $k_{t+1} > \underline{k}_c$ such that $\hat{\omega}_t > \alpha$ for all t. Moreover, $\overline{k}_c > \underline{k}_c$ implies a unique steady state $k^* \in (\overline{k}_c, \infty)$ and, for any $k_1 > \underline{k}_c$, there is monotonic convergence to the steady state, k^*.



One readily verifies that $RHS'(k) > 0$. To see that $RHS''(k) > 0$ define $\zeta \equiv k/\Omega$. Then, $RHS(k) = RHS(\zeta)$ and $RHS'(k) = RHS'(\zeta) \cdot (d\zeta/dk)$. Moreover,

$$RHS''(k) = RHS'(\zeta)\frac{d^2\zeta}{dk^2} + RHS''(\zeta)\left(\frac{d\zeta}{dk}\right)^2 = RHS''(\zeta)\left(\frac{d\zeta}{dk}\right)^2$$

since $d^2\zeta/dk^2 = 0$. Hence, $RHS''(k) > 0$ follows if $RHS''(\zeta) > 0$. To see that the latter holds define

$$z \equiv \frac{2}{\sqrt{\alpha}}\zeta^{\frac{1}{1-\nu}} - 1.$$

Then, $RHS'(\zeta)$ becomes

$$RHS'(\zeta) = \frac{RHS(\zeta)}{(1-\nu)\zeta}\left[1 - 2\nu + \frac{\frac{2}{\gamma\sqrt{\alpha}}\zeta^{\frac{1}{1-\nu}}}{z}\right].$$

Hence, the sign of $RHS''(\zeta)$ is given by the sign of

$$\frac{\partial\left(\frac{RHS(\zeta)}{\zeta}\right)}{\partial\zeta}\left[1 - 2\nu + \frac{\frac{2}{\gamma\sqrt{\alpha}}\zeta^{\frac{1}{1-\nu}}}{z}\right] + \frac{RHS(\zeta)}{\zeta}\left(\frac{2}{\gamma\sqrt{\alpha}}\right)\frac{\partial\left(\frac{\zeta^{\frac{1}{1-\nu}}}{z}\right)}{\partial\zeta}. \tag{A.13}$$

Since

$$\frac{RHS(\zeta)}{\zeta} = \frac{z^{\frac{1}{\gamma}}}{\Lambda\zeta^{\frac{\nu}{1-\nu}}}$$

I have

$$\frac{\partial\frac{RHS(\zeta)}{\zeta}}{\partial\zeta} = \frac{z^{\frac{1}{\gamma}}}{\Lambda\gamma\zeta^{\frac{1}{1-\nu}}}\left(\frac{\partial z}{\partial\zeta}\frac{\zeta}{z} - \frac{\gamma\nu}{1-\nu}\right).$$

The term in parenthesis is strictly positive since

$$\frac{2}{\sqrt{\alpha}}\zeta^{\frac{1}{1-\nu}}\left(1-\nu\gamma\right) > -\nu\gamma.$$

Hence, $\partial\left(RHS(\zeta)/\zeta\right)/\partial\zeta > 0$. Since $\nu < 1/2$, equation (A.13) is strictly positive if

$$\frac{\partial\left(\frac{RHS(\zeta)}{\zeta}\right)}{\partial\zeta}\left[\frac{\zeta^{\frac{1}{1-\nu}}}{z}\right] + \frac{RHS(\zeta)}{\zeta}\frac{\partial\left(\frac{\zeta^{\frac{1}{1-\nu}}}{z}\right)}{\partial\zeta} > 0. \tag{A.14}$$

To see that this is indeed the case observe that

$$\begin{aligned}\frac{\partial\left(\frac{RHS(\zeta)}{\zeta}\right)}{\partial\zeta}\left[\frac{\zeta^{\frac{1}{1-\nu}}}{z}\right] &= \frac{z^{\frac{1-\gamma}{\gamma}}}{\Lambda\gamma}\left(\frac{\partial z}{\partial\zeta}\frac{\zeta}{z} - \frac{\nu\gamma}{1-\nu}\right),\\ \frac{RHS(\zeta)}{\zeta}\frac{\partial\left(\frac{\zeta^{\frac{1}{1-\nu}}}{z}\right)}{\partial\zeta} &= \frac{z^{\frac{1-\gamma}{\gamma}}}{\Lambda}\left(\frac{1}{1-\nu} - \frac{\partial z}{\partial\zeta}\frac{\zeta}{z}\right).\end{aligned}$$

The sum of these terms delivers inequality (A.14) as

$$\frac{z^{\frac{1-\gamma}{\gamma}}}{\Lambda}\left(\frac{\partial z}{\partial\zeta}\frac{\zeta}{z}\left(\frac{1}{\gamma}-1\right)+1\right) > 0.$$

The latter holds since $\partial z/\partial\zeta > 0$ and $0 < \gamma < 1$.

Strict convexity of $RHS(k)$ also implies that $\lim_{k\to\infty} RHS(k) = \infty$. Hence, for any $k_t \in (\underline{k}_c, \infty)$ (A.12) delivers a unique value $k_{t+1} \in (\underline{k}_c, \infty)$. ■

A.1.12 Proof of Proposition 12

Consider the function $RHS(k)$ defined in the proof of Proposition 11. Since this function is monotonically increasing with $\lim_{k\to\infty} RHS(k) = \infty$, a sufficient condition for the existence of a unique steady state $k^* > \underline{k}_c$ is $\overline{k}_c > \underline{k}_c$. A simple graphical argument using Figure A.1 reveals that the strict convexity of $RHS(k)$ delivers $k^* > \overline{k}_c$ as well as the stability of k^* in the indicated sense. Then, $k^*$is a fixed point of the difference equation (A.12) and given by

$$k^* = \left(\frac{\left(\frac{2}{\sqrt{\alpha}}\left(\frac{k^*}{\Omega}\right)^{\frac{1}{1-\nu}} - 1\right)^{\frac{1-\nu}{\gamma}}}{\Lambda^{1-\nu}\Omega^{1-2\nu}}\right)^{\frac{1}{\nu}}.$$

■

A.1.13 Proof of Proposition 13

From Proposition 12 the steady state has $k_t = k^* > \bar{k}_c > \underline{k}_c$ so that Proposition 9 implies $\omega_t = \hat{\omega}^* = \omega(k^*) > \alpha$. Then, from (2.13) I have $q_t = q^* = q(\hat{\omega}^*, \alpha) > 0$, and the results listed under a) - d) follow from Proposition 1, Proposition 2, Proposition 7, Proposition 9, and equations (2.16), (2.17) and (3.8). ■

A.1.14 Proof of Proposition 14

First, consider a change in μ and g_L. I show how such a change affects $\hat{\omega}^*$. Then, the statements in the proposition follow since $\partial q(\hat{\omega}^*, \alpha)/\partial alpha > 0$ (see Proposition 1).

Consider the labor market equilibrium condition (3.4) and the capital market condition (3.9) in steady state. Solving both equations for k^* and substitution delivers

$$\Lambda\Omega = (\hat{\omega}^*)^{\frac{-\nu}{2}} \left(2\sqrt{\frac{\hat{\omega}^*}{\alpha}} - 1\right)^{\frac{1}{\gamma}}, \tag{A.15}$$

where

$$\Lambda\Omega = \left(\frac{\Gamma(1-\gamma)}{\alpha^{1-\gamma+\frac{\gamma\nu}{2}}}\right)^{\frac{1}{\gamma}} \frac{\mu\beta}{(1+\mu\beta)(1-\nu)(1+g_L)}.$$

The right-hand side of equation (A.15) defines a continuous function $RHS(\omega)$ with the following properties:

$$\begin{aligned} RHS(\alpha) &= \alpha^{\frac{-\nu}{2}}, \\ RHS'(\omega) &= \Lambda\Omega \left(\frac{2\omega(1-\gamma\nu) + \gamma\nu\sqrt{\alpha\omega}}{2\gamma\omega^{\frac{3}{2}}\left(2\sqrt{\omega} - \sqrt{\alpha}\right)}\right) > 0, \\ \lim_{\omega\to\infty} RHS(\omega) &= \lim_{\omega\to\infty} \left(\frac{2\omega^{\frac{1-\gamma\nu}{2}}}{\sqrt{\alpha}} - \omega^{\frac{-\gamma\nu}{2}}\right)^{\frac{1}{\gamma}} = \infty. \end{aligned}$$

Hence, for any $\Lambda\Omega > \alpha^{-\nu/2}$, which is equivalent to $\bar{k}_c > \underline{k}_c$, there is a unique $\hat{\omega}^* > \alpha$, hence, a unique $q^* = q(\hat{\omega}^*, \alpha) > 0$.

Then, straightforward implicit differentiation of (A.15) delivers $d\hat{\omega}^*/d\mu > 0$ and $d\hat{\omega}^*/dg_L < 0$.

Second, consider a change in α as stated in (4.3). Here, Proposition 1 delivers the partial effects. Evaluated at the steady state, these are

$$\frac{\partial q(\hat{\omega}^*, \alpha)}{\partial \omega} = \frac{1}{2\sqrt{\hat{\omega}^*\alpha}} \quad \text{and} \quad \frac{\partial q(\hat{\omega}^*, \alpha)}{\partial \alpha} = \frac{-\sqrt{\hat{\omega}^*}}{2\alpha^{3/2}}.$$

It follows that

$$\frac{dq^*}{d\alpha} \gtreqless 0 \quad \Leftrightarrow \quad \frac{d\hat{\omega}^*}{d\alpha}\frac{\alpha}{\hat{\omega}^*} \gtreqless 1.$$

To derive $d\hat{\omega}^*/d\alpha$, consider conditions (3.4) and (3.9) in steady state. Solving for k^* and substitution delivers

$$\Gamma\Omega\,(\hat{\omega}^*)^{\frac{\nu}{2}} - \left(\sqrt{\frac{\hat{\omega}^*}{\alpha}} - 1\right)^{\frac{1}{\gamma}} = 0.$$

Total differentiation of the latter gives

$$\frac{d\hat{\omega}^*}{d\alpha} = \frac{\hat{\omega}^*\left((2\gamma - 1 - \nu\gamma)\left(\sqrt{\hat{\omega}^*} - \sqrt{\alpha}\right) + \sqrt{\alpha}\right)}{\alpha\left(\sqrt{\hat{\omega}^*} - \nu\gamma\left(\sqrt{\hat{\omega}^*} - \sqrt{\alpha}\right)\right)}.$$

Here, the denominator is strictly positive whereas the numerator may be positive or negative. It follows that

$$\frac{d\hat{\omega}^*}{d\alpha}\frac{\alpha}{\hat{\omega}^*} \gtreqless 1 \quad \Leftrightarrow \quad \frac{(2\gamma - 1 - \nu\gamma)\left(\sqrt{\hat{\omega}^*} - \sqrt{\alpha}\right) + \sqrt{\alpha}}{\sqrt{\hat{\omega}^*} - \nu\gamma\left(\sqrt{\hat{\omega}^*} - \sqrt{\alpha}\right)} \gtreqless 1.$$

One readily derives that $(d\hat{\omega}^*/d\alpha)(\alpha/\hat{\omega}^*) < 1$ must hold since $\gamma < 1$. Hence, $dq^*/d\alpha < 0$. ■

A.1.15 Proof of Proposition 15

From equation (A.19) the steady state labor share is

$$LS^* = (1-\gamma)\left(\frac{\sqrt{\hat{\omega}^*}}{\sqrt{\hat{\omega}^*} + \gamma\left(\sqrt{\hat{\omega}^*} - \sqrt{\alpha}\right)}\right).$$

Then, one readily verifies that

$$\frac{\partial LS^*}{\partial \omega} = -\frac{(1-\gamma)\gamma\sqrt{\alpha}}{2\sqrt{\hat{\omega}^*}\left((1+\gamma)\sqrt{\hat{\omega}^*} - \gamma\sqrt{\alpha}\right)^2} < 0, \quad \text{and} \quad \frac{\partial LS^*}{\partial \alpha} = \frac{(1-\gamma)\gamma\sqrt{\hat{\omega}^*}}{2\sqrt{\alpha}\left((1+\gamma)\sqrt{\hat{\omega}^*} - \gamma\sqrt{\alpha}\right)^2} > 0.$$

From the proof of Proposition 14 I know that $\partial\hat{\omega}^*/\partial\mu > 0$ and $\partial\hat{\omega}^*/\partial g_L < 0$. Hence,

$$\frac{dLS^*}{d\mu} = \frac{\partial LS^*}{\partial \omega}\frac{\partial\hat{\omega}^*}{\partial\mu} < 0 \quad \text{and} \quad \frac{dLS^*}{dg_L} = \frac{\partial LS^*}{\partial \omega}\frac{\partial\hat{\omega}^*}{\partial g_L} > 0$$

which are the stated effects for μ and g_L.

The case of α is more intricate since changing α gives rise to a direct effect on LS^*. Hence, the total effect may be written as

$$\frac{dLS^*}{d\alpha} = \frac{\partial LS^*}{\partial \omega}\frac{d\hat{\omega}^*}{d\alpha} + \frac{\partial LS^*}{\partial \alpha}.$$

Using the partial effects above delivers the total effect as

$$\frac{dLS^*}{d\alpha} = \frac{(1-\gamma)\gamma}{2\left((1+\gamma)\sqrt{\hat{\omega}^*} - \gamma\sqrt{\alpha}\right)^2}\left[-\frac{\sqrt{\alpha}}{\sqrt{\hat{\omega}^*}}\frac{d\hat{\omega}^*}{d\alpha} + \frac{\sqrt{\hat{\omega}^*}}{\sqrt{\alpha}}\right].$$

It follows that

$$\frac{dLS^*}{d\alpha} \gtreqless 0 \quad \Leftrightarrow \quad 1 \gtreqless \frac{d\hat{\omega}^*}{d\alpha}\frac{\alpha}{\hat{\omega}^*}.$$

From the proof of Proposition 14 I know that $(d\hat{\omega}^*/d\alpha)(\alpha/\hat{\omega}^*) < 1$ must hold. Hence, $dLS^*/d\alpha > 0$. ■

A.1.16 Proof of Proposition 16

Consider the labor and the capital market.[28] For the chosen parameter constellation $\Lambda = 452.632$ and $\underline{k}_c = 0.0022093$. Moreover, the labor-market equilibrium condition (3.4) reads

$$k_t = 0.0022093\,(2\sqrt{\omega_t} - 1)^4\,\omega_t^{\frac{1}{4}}. \tag{A.16}$$

[28] All computations were executed in *Mathematica*. The relevant notebooks are available upon request.

At the same time, $\Omega = 0.246914 = \overline{k}_c$. Hence, it holds that $\overline{k}_c = 0.246914 > \underline{k}_c = 0.0022093$, and the capital market equilibrium condition (3.9) becomes

$$k_{t+1} = 0.246914 \cdot \omega_t^{\frac{3}{8}}. \tag{A.17}$$

Equations (A.16) and (A.17) determine the equilibrium difference equation (3.10) as

$$k_t = 0.00561338 \cdot k_{t+1}^{2/3} \left(12.9113 \cdot k_{t+1}^{4/3} - 1\right)^4. \tag{A.18}$$

The evaluation of (A.16) and (A.17) at $k_t = k_{t+1} = k^*$ and $\omega_t = \omega^*$ delivers k^* and ω^* as stated in the proposition. Since $\omega^* > 1$ and $\alpha = 1$, the steady state satisfies $w_t > A_{t-1}$ for all $t = 1, 2, ..., \infty$. Since $w_c = 1$ it also satisfies $\alpha A_{t-1} > w_c$ if $A_{t-1} > 1$.

Using ω^* in Proposition 1 delivers the indicated value of q^*, using ω^* in (A.19) gives the stated labor share, LS^*. With ω^* in Proposition 1 one also finds

$$i^* = 1.20823 \quad \text{and} \quad c^* = 3.41645.$$

With the latter in Proposition 2 one obtains

$$\frac{N_t}{K_t} = \left(\frac{\Gamma(1-\gamma)}{c^*}\right)^{\frac{1}{\gamma}} = 3.32234.$$

For $N_t/K_t = 3.32234$ the first-order condition for K_t in (2.16) gives R^*.

Finally, from Proposition 1 the growth factor of the supply of hours worked is $1/(1+q^*)^\nu = 0.820331$. ■

A.2 Proofs of Corollaries

A.2.1 Proof of Corollary 1

If $\omega_t > \alpha$ then $q_t > 0$ and the rationalization effect follows since

$$\frac{1}{A_{t-1}(1+q_t)} < \frac{1}{A_{t-1}}.$$

The productivity effect follows since c_t is the solution to (2.11) and $c(\omega_t, \alpha)\,|_{\omega_t = \alpha} = \omega_t$. ■

A.2.2 Proof of Corollary 2

Follows from Proposition 2 and Corollary 1. ■

A.2.3 Proof of Corollary 3

With Proposition 1 one readily verifies that the labor share may be expressed as

$$LS_t^1 = (1-\gamma)\left(\frac{\sqrt{\omega_t}}{\sqrt{\omega_t} + \gamma\left(\sqrt{\omega_t} - \sqrt{\alpha}\right)}\right). \tag{A.19}$$

Then, the corollary follows from straightforward derivations and from an application of l'Hôpital's rule to

$$\lim_{\omega_t \to \infty}\left(\frac{\sqrt{\omega_t}}{\sqrt{\omega_t} + \gamma\left(\sqrt{\omega_t} - \sqrt{\alpha}\right)}\right) = \lim_{\alpha \to 0}\left(\frac{\sqrt{\omega_t}}{\sqrt{\omega_t} + \gamma\left(\sqrt{\omega_t} - \sqrt{\alpha}\right)}\right) = \frac{1}{1+\gamma}.$$

■

A.2.4 Proof of Corollary 4

From (A.19) in the proof of Corollary 3 I have

$$\hat{LS}_t = (1-\gamma)\left(\frac{\sqrt{\hat{\omega}_t}}{\sqrt{\hat{\omega}_t} + \gamma\left(\sqrt{\hat{\omega}_t} - \sqrt{\alpha}\right)}\right).$$

Then, $d\hat{LS}_t/d\mu = (\partial LS_t/\partial\omega_t)(\partial\hat{\omega}_t/\partial\mu)$ where $\partial LS_t/\partial\omega_t < 0$. Then $d\hat{LS}_t/d\mu > 0$ follows with Proposition 10.

From Corollary 3 I also have

$$\begin{aligned}
\frac{\partial LS_t}{\partial\omega_t} &= -\frac{LS_t}{2\omega_t}\cdot\frac{1}{\sqrt{\frac{\omega}{\alpha}} + \gamma\left(\sqrt{\frac{\omega}{\alpha}} - 1\right)} < 0, \\
\frac{\partial LS_t}{\partial\alpha_t} &= \frac{LS_t}{2\alpha}\cdot\frac{\gamma}{\sqrt{\frac{\omega}{\alpha}} + \gamma\left(\sqrt{\frac{\omega}{\alpha}} - 1\right)} > 0.
\end{aligned}$$

Then, using the above and (3.6) in (3.7) delivers

$$\frac{d\hat{LS}_t}{d\alpha} \gtreqless 0 \quad \Leftrightarrow \quad \frac{\gamma(1-\gamma) + 2(1-\nu\gamma^2)}{2(1-\gamma)} \gtreqless \frac{\sqrt{\frac{\hat{\omega}_t}{\alpha}}}{2\sqrt{\frac{\hat{\omega}_t}{\alpha}} - 1}. \tag{A.20}$$

Denote the left-hand side of this inequality by $LHS(\gamma,\nu)$. One finds $LHS(0,\nu) = 1$, $\lim_{\gamma\to 1} LHS(\gamma,\nu) = \infty$, and

$$\frac{\partial LHS(\gamma,\nu)}{\partial\gamma} = \frac{\gamma^2(2\nu+1) - 2\gamma(2\nu+1) + 3}{2(1-\gamma)^2} > 0.$$

Hence, $LHS(\gamma,\nu) > 1$ for all $0 < \gamma < 1$. At the same time the right-hand side of (A.20) is strictly smaller than unity. Hence, $d\hat{LS}_t/d\alpha > 0$. ∎

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B Additional Results

B.1 The Elasticity of Substitution between Automation Investments and Hours Worked per Task

This section derives the elasticity of substitution between automation investments and hours worked by task. Section B.1.1 has the case $i_t(n) = \alpha q_t(n)$ that is used in the main text (see equation (2.4)). Section B.1.2 studies the general case $i_t(n) = \iota(q_t(n))$ where $\iota : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and convex. Henceforth, I shall drop the argument n and the time index t. Then, the two cases boil down to $i = \alpha q$ and $i = \iota(q)$, respectively.

B.1.1 The Linear Case, $i = \alpha q$

Consider the isoquant defined by equation (2.6). Dropping the argument n and the time index t this can be written as

$$A\left(1 + \frac{i}{\alpha}\right)h = 1.$$

Then, the technical rate of substitution, TRS, between h and i is

$$TRS \equiv \frac{dh}{di} = -\frac{h}{\alpha + i}. \tag{B.1}$$

The elasticity of substitution, ES, between h and i is defined as

$$ES \equiv \left(\frac{TRS}{h/i}\right)\left(\frac{dTRS}{d(h/i)}\right)^{-1}. \tag{B.2}$$

With (B.1) the first factor is $-i/(i + \alpha)$. To obtain the second factor observe that the differential $d(h/i)$ is

$$d\left(\frac{h}{i}\right) = \frac{h}{i}\left(\frac{dh}{h} - \frac{di}{i}\right). \tag{B.3}$$

From (B.1), I have

$$dTRS = \frac{-dh}{\alpha + i} \quad \text{or} \quad \frac{dh}{h} = -\frac{\alpha + i}{h} dTRS,$$

$$dTRS = \frac{h}{(\alpha + i)^2} di \quad \text{or} \quad \frac{di}{i} = -\frac{(\alpha + i)^2}{hi} dTRS.$$

Using the latter in (B.3) delivers

$$\left(\frac{dTRS}{d(h/i)}\right)^{-1} = -\frac{(2i + \alpha)(i + \alpha)}{i^2}.$$

It follows that,

$$\begin{aligned} ES &= \left(-\frac{i}{\alpha + i}\right)\left(-\frac{(2i + \alpha)(i + \alpha)}{i^2}\right) \\ &= 2 + \frac{\alpha}{i}, \\ &= 2 + \frac{1}{q}, \end{aligned}$$

where the last step uses (2.4).

B.1.2 The General Case $i = \iota(q)$

Consider the general case $i = \iota(q)$ where $\iota : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$\lim_{q \to 0} \iota(q) = 0, \quad \iota'(q) > 0 \text{ for all } q > 0, \quad \lim_{q \to 0} \iota'(q) \geq 0, \quad \iota''(q) \geq 0. \tag{B.4}$$

Hence, ι is one-to-one with range $\mathbb{R}_+$. Then, the inverse of ι, $q = \iota^{-1}(i)$, where $\iota^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$, exists and satisfies

$$\lim_{i \to 0} \iota(i) = 0, \quad \left(\iota^{-1}\right)'(i) = \frac{1}{\iota'(q)} > 0 \text{ for all } q > 0, \quad \lim_{i \to 0} \left(\iota^{-1}\right)'(i) \leq \infty, \quad \left(\iota^{-1}\right)''(i) = -\frac{\iota''\left(\iota^{-1}(i)\right)}{\left[\iota'\left(\iota^{-1}(i)\right)\right]^3} \leq 0. \tag{B.5}$$

Proposition 17 *(ES for General Functional Forms of $\iota(q)$)*

Suppose automation investments are given by $i = \iota(q)$ that satisfies condition (B.4). Then, for $q > 0$ it holds that

$$ES > 1.$$

Proof of Proposition 17

The isoquant corresponding to equation (2.6) becomes

$$A\left(1 + \iota^{-1}(i)\right) h = 1.$$

Accordingly, the technical rate of substitution, TRS, between h and i is

$$TRS \equiv \frac{dh}{di} = -\frac{h\left(\iota^{-1}\right)'(i)}{1 + \iota^{-1}(i)}. \tag{B.6}$$

The elasticity of substitution, ES, between h and i is defined as in (B.2). The first factor of ES is equal to

$$\frac{TRS}{h/i} = -\frac{i\left(\iota^{-1}\right)'(i)}{1 + \iota^{-1}(i)}. \tag{B.7}$$

To obtain the second factor, $(dTRS/d(h/i))^{-1}$, use (B.6) to derive the differentials

$$\begin{aligned} dTRS &= -\frac{\left(\iota^{-1}\right)'(i)}{1 + \iota^{-1}(i)} dh, \\ dTRS &= -\frac{h}{1 + \iota^{-1}(i)} \left(\left(\iota^{-1}\right)''(i) - \frac{\left(\left(\iota^{-1}\right)'(i)\right)^2}{1 + \iota^{-1}(i)} \right) di, \end{aligned}$$

or

$$\begin{aligned} \frac{dh}{h} &= -\frac{1 + \iota^{-1}(i)}{h\left(\iota^{-1}\right)'(i)} dTRS, \\ \frac{di}{i} &= -\frac{1 + \iota^{-1}(i)}{hi} \left(\left(\iota^{-1}\right)''(i) - \frac{\left(\left(\iota^{-1}\right)'(i)\right)^2}{1 + \iota^{-1}(i)} \right)^{-1} dTRS. \end{aligned}$$

Using these expressions in (B.3) delivers

$$d\left(\frac{h}{i}\right) = \frac{h}{i}\left[-\frac{1+\iota^{-1}(i)}{h\left(\iota^{-1}\right)'(i)}dTRS + \frac{1+\iota^{-1}(i)}{hi}\left(\left(\iota^{-1}\right)''(i) - \frac{\left(\left(\iota^{-1}\right)'(i)\right)^2}{1+\iota^{-1}(i)}\right)^{-1} dTRS\right]$$

$$= \frac{1+\iota^{-1}(i)}{i}\left[-\frac{1}{\left(\iota^{-1}\right)'(i)} + \frac{1}{i}\left(\left(\iota^{-1}\right)''(i) - \frac{\left(\left(\iota^{-1}\right)'(i)\right)^2}{1+\iota^{-1}(i)}\right)^{-1}\right] dTRS.$$

Hence,

$$\left(\frac{dTRS}{d(h/i)}\right)^{-1} = \frac{1+\iota^{-1}(i)}{i}\left[-\frac{1}{\left(\iota^{-1}\right)'(i)} + \frac{1}{i}\left(\left(\iota^{-1}\right)''(i) - \frac{\left(\left(\iota^{-1}\right)'(i)\right)^2}{1+\iota^{-1}(i)}\right)^{-1}\right]. \tag{B.8}$$

It follows from (B.7) and (B.8) that

$$ES = \left(\frac{TRS}{h/i}\right)\left(\frac{dTRS}{d(h/i)}\right)^{-1}$$

$$= -\frac{i\left(\iota^{-1}\right)'(i)}{1+\iota^{-1}(i)}\left[\frac{1+\iota^{-1}(i)}{i}\left(-\frac{1}{\left(\iota^{-1}\right)'(i)} + \frac{1}{i}\left(\left(\iota^{-1}\right)''(i) - \frac{\left(\left(\iota^{-1}\right)'(i)\right)^2}{1+\iota^{-1}(i)}\right)^{-1}\right)\right]$$

$$= 1 + \frac{\left(\iota^{-1}\right)'(i)}{i\left(\frac{\left(\left(\iota^{-1}\right)'(i)\right)^2}{1+\iota^{-1}(i)} - \left(\iota^{-1}\right)''(i)\right)}.$$

Then, the proposition follows from (B.5). ■

Examples

In addition to the linear case of the previous section, examples of functions $\iota(q)$ that satisfy (B.4) include:

- $\iota(q) = \alpha q^2$, $\alpha > 0$. Then, $ES = 2 + 1/(1+q) \geq 2$.
- $\iota(q) = \alpha\left(e^{q^z} - 1\right)$, $\alpha > 0$, $z > 1$. Then,

$$ES = 1 + \frac{z(\alpha+i)\ln\left(\frac{\alpha+i}{\alpha}\right)\left(\sqrt[z]{\ln\left(\frac{\alpha+i}{\alpha}\right)} + 1\right)}{(1+z)\ln\left(\frac{\alpha+i}{\alpha}\right) + (i+z-1)\sqrt[z]{\ln\left(\frac{\alpha+i}{\alpha}\right)} + z\ln\left(\frac{\alpha+i}{\alpha}\right) + z - 1}$$

  with

$$\lim_{i\to 0} ES(i) = 1.$$

B.2 The Induced Aggregate Rationalization Effect for $i = \iota(q)$

Consider $i = \iota(q)$ characterized in (B.4). The following result can be stated and proved.

Proposition 18 *(Induced Aggregate Rationalization Effect for $i = \iota(q)$)*

If $\lim_{q\to 0}\iota'(q) = 0$, then the aggregate rationalization effect is strictly positive.

Proof of Proposition 18

Recall that cost minimization requires

$$q_t = \text{argmin}_{q \geq 0} \quad \frac{\omega}{1+q} + \iota(q).$$

If $\lim_{q \to 0} \iota'(q) = 0$ then the first-order condition

$$\frac{-\omega}{(1+q)^2} + \iota'(q) \geq 0$$

delivers an interior solution $q_t = q(\omega)$ for all $\omega > 0$. Moreover, $\lim_{\omega \to 0} q(\omega) = 0$ and

$$q'(\omega) \equiv \frac{dq_t}{d\omega} = \frac{1 + q(\omega)}{2\omega + (1 + q(\omega))^3 \iota''(q(\omega))} > 0. \tag{B.9}$$

From the discussion of Proposition 4 in Footnote 14 I know that the induced aggregate rationalization effect is strictly positive if $dh(\omega)/d\omega = -[h(\omega)]^2 \cdot q'(\omega) > 0$. For $\omega > 0$ it holds that $h(\omega) < 1$ and $q'(\omega) > 0$. Since $\lim_{\omega \to 0} h(\omega) = 1$ it is sufficient for the proposition to hold that $\lim_{\omega \to 0} q'(\omega) > 0$. From (B.9) it is immediate that

$$\lim_{\omega \to 0} q'(\omega) = \frac{1}{\lim_{\omega \to 0} \iota''(q(\omega))}.$$

Hence, the proposition holds if $\lim_{\omega \to 0} \iota''(q(\omega)) \geq 0$. The latter is satisfied as $\iota(q)$ is convex for $q \geq 0$. ■