# FROM DEFORMATION THEORY OF WHEELED PROPS TO CLASSIFICATION OF KONTSEVICH FORMALITY MAPS

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Abstract. We study the homotopy theory of the wheeled prop controlling Poisson structures on formal graded finite-dimensional manifolds and prove, in particular, that the Grothendieck-Teichmüller group acts on that wheeled prop faithfully and homotopy non-trivially. Next we apply this homotopy theory to the study of the deformation complex of an arbitrary M. Kontsevich formality map and compute the full cohomology group of that deformation complex in terms of the cohomology of a certain graph complex introduced earlier by M. Kontsevich in [K1] and studied by T. Willwacher in [W1].

## 1. Introduction

**1.1. Wheeled props, formal Poisson structures and Grothendieck-Teichmüller group.** Let $V$ be an arbitrary finite-dimensional $\mathbb{Z}$-graded vector space over a field $\mathbb{K}$ of characteristic zero (say, $V = \mathbb{R}^d$, $\mathbb{K} = \mathbb{R}$) and $V^* := \mathrm{Hom}(V, \mathbb{K})$ its dual. Then the completed symmetric algebra $\mathcal{O}_\mathcal{M} := \widehat{\odot^\bullet} V$ can be understood as the $\mathbb{K}$-algebra of formal smooth functions on the dual vector space $V^*$ understood as a formal manifold $\mathcal{M}$, and the Lie algebra of derivations of $\mathcal{O}_\mathcal{M}$,

$$\mathcal{T}\mathcal{M} := \mathrm{Der}(\mathcal{O}_V) \simeq \mathrm{Hom}(V, \widehat{\odot^\bullet} V) \simeq \prod_{m\geq 0} \mathrm{Hom}(V, \odot^m V),$$

as the Lie algebra of formal smooth vector fields on $\mathcal{M}$. A *formal graded Poisson structure* on $\mathcal{M}$ is a degree 2 element $\pi$ in the Schouten Lie algebra

$$\mathcal{T}_{poly}\mathcal{M} := \wedge^\bullet T_\mathcal{M} \simeq \prod_{m\geq 0, n\geq 0} \mathrm{Hom}(\wedge^n V, \odot^m V)[-n] = \prod_{m\geq 0, n\geq 0} \mathrm{Hom}(\odot^n (V[1]), \odot^m V) = \prod_{k\geq 0} \odot^k (V^*[-1] \oplus V)$$

of polyvector fields satisfying the Maurer-Cartan equation,

$$[\pi, \pi] = 0,$$

where the Schouten Lie bracket $[\ ,\ ]$ (of degree $-1$) originates essentially from the canonical pairing map $V^*[-1] \times V \to \mathbb{K}[-1]$. Thus a formal Poisson structure is a formal power series,

$$\pi = \sum_{n,m=0}^{\infty} \pi_n^m, \qquad \pi_n^m \in \mathrm{Hom}(\wedge^n V, \odot^m V)[2-n] \tag{1}$$

and hence can be understood as a representation in the vector space $V$,

$$\rho_\pi : \mathcal{H}olieb_{0,1}^\star \longrightarrow \mathcal{E}nd_V,$$

of a certain prop of *formal Poisson structures* which is by definition the free prop[1] generated by a collection of 1-dimensional $\mathbb{S}_m^{op} \times \mathbb{S}_n$ bimodules $\mathbf{1}_m \otimes sgn_n[n-2]$. A useful observation [Me1] is that this prop comes equipped with a natural differential $\delta^\star$ such that the Maurer-Cartan equation $[\pi, \pi] = 0$ gets encoded into the compatibility of the representation $\rho_\pi$ with the differentials. The strange notation $\mathcal{H}olieb_{0,1}^\star$ comes from the fact that this particular prop comes from the family of dg props $\mathcal{H}olieb_{c,d}^\star$ which control Maurer-Cartan elements $\pi$ in the graded commutative algebra

$$\prod_{k\geq 0} \odot^k (V^*[-d] \oplus V[-c]) \tag{2}$$

[1]See, e.g., [Ma, Me3] and the first sections of [V] for an elementary introduction into the theory of props and wheeled props.

equipped with the obvious Poisson type Lie bracket (of homological degree $-c-d$). The case $c=0$, $d=1$ corresponds to formal Poisson structures while the case $c=1$, $d=1$ corresponds to extended *homotopy Lie bialgebras* structures.

The superscript $\star$ in the notation indicates that we consider in this paper an *extended version* of the family of props $\mathcal{H}olieb_{c,d}$ studied earlier in [MW1]. The latter family controls the *truncated* version of the above formal power series (1) which allows only monomials with $m, n \geq 1$, $m+n \geq 3$; such a truncation makes perfect sense in the context of the theory of minimal resolutions of $(c,d)$ Lie bialgebras. However we are interested in this paper in the "full story" with no restrictions on the integer parameters $m$ and $n$, and that "full story" turns out to be sometimes quite different from the truncated one.

Note that under appropriate completion of the above graded commutative algebra (2) all the above structures (the convergent Lie bracket, Maurer-Cartan elements $\pi$ and associated representations $\rho_\pi$ of the props $\mathcal{H}olieb^\star_{c,d}$) make sense also for *infinite-dimensional* vector spaces $V$. An important point of this paper is that the deformation theories of these structures behave quite differently in finite and infinite dimensions. Indeed, in infinite dimensions they can be understood as representations of *ordinary* props

$$\rho_\pi : \mathcal{H}olieb^\star_{c,d} \longrightarrow \mathcal{E}nd_V$$

while in *finite* dimensions as representations of their *wheeled closures* (cf. [Me2, MMS]),

$$\rho_\pi : \mathcal{H}olieb^{\star\circlearrowright}_{c,d} \longrightarrow \mathcal{E}nd_V$$

which have quite different deformation theories or, equivalently, quite different dg Lie algebras, $\mathrm{Der}(\mathcal{H}olieb^\star_{c,d})$ and $\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d})$, of derivations (see §3 for their precise definitions). The dg prop $\mathcal{H}olieb^\star_{c,d}$ is a proper subprop of $\mathcal{H}olieb^{\star\circlearrowright}_{c,d}$, the latter containing many more universal operations (involving, roughly speaking, the trace operation $V \otimes V^* \to \mathbb{K}$ which has no sense in general when $\dim V = \infty$).

The first main purpose of this paper is the study of the deformation theory of both props $\mathcal{H}olieb^\star_{c,d}$ and $\mathcal{H}olieb^{\star\circlearrowright}_{c,d}$ (in fact of their completed versions) and the computation of the cohomologies of the associated complexes of derivations in terms of the M. Kontsevich graph complexes $\mathsf{GC}^{\geq 2}_d$ introduced[2] in [K1] and studied in [W1], and of its oriented version $\mathsf{GC}^{or}_d$ which was studied in [W2, Z]. These complexes are spanned by *connected graphs*. It is often useful [MW1, MW3] to add to these classical graph complexes an additional element $\emptyset$ placed in degree zero, "a graph with no vertices and edges", and define the *full graph complexes* of not necessarily connected graphs as the completed graded symmetric tensor algebras

$$\mathsf{fGC}^{\geq 2}_d := \widehat{\odot^\bullet}\left((\mathsf{GC}^{\geq 2}_d \oplus \mathbb{K})[-d]\right)[d], \tag{3}$$

$$\mathsf{fGC}^{or}_d := \widehat{\odot^\bullet}\left((\mathsf{GC}^{or}_d \oplus \mathbb{K})[-d]\right)[d], \tag{4}$$

the summands $\mathbb{K}$ being generated by $\emptyset$. The formal class $\emptyset$ takes care for (homotopy non-trivial) rescaling operations of the (wheeled) props under considerations, and essentially leads us in applications to the full Grothendieck-Teichmüller group $GRT = GRT_1 \rtimes \mathbb{K}^*$ (see [D]) rather than to its reduced version $GRT_1$. The Lie bracket of $\emptyset$ with elements $\Gamma$ of $\mathsf{GC}^{\geq 2}_d$ or $\mathsf{GC}^{or}_d$ is defined as the multiplication of $\Gamma$ by twice its loop number, that is, the number of vertices minus the number of edges.

**1.1.1. Proposition.** *There are morphisms of dg Lie algebras,*

$$F^\circlearrowright : \mathsf{fGC}^{\geq 2}_{c+d+1} \longrightarrow \mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d}), \qquad F : \mathsf{fGC}^{or}_{c+d+1} \longrightarrow \mathrm{Der}(\mathcal{H}olieb^\star_{c,d})$$

*which are quasi-isomorphisms.*

It was proven in [W1, W2] that $H^\bullet(\mathsf{GC}^{\geq 2}_{c+d+1}) = H^\bullet(\mathsf{GC}^{or}_{c+d+2})$ and that

$$H^0(\mathsf{fGC}^{\geq 2}_2) = H^0(\mathsf{fGC}^{or}_3) = \mathfrak{grt} \overset{\text{as a vector space}}{\simeq} \mathfrak{grt}_1 \oplus \mathbb{K},$$

[2]The symbol $\mathsf{GC}_d$ stands often in the literature for the graph complex generated by connected oriented graphs with all vertices trivalent; we denote by $\mathsf{GC}^{\geq 2}_d$ its extension which allows connected graphs with at least bivalent vertices (see §3.2 for more details and references). The latter complex has a quasi-isomorphic versions, $\mathsf{dGC}^{\geq 2}_d$, spanned by graphs with fixed directions on edges; the subcomplex of $\mathsf{dGC}^{\geq 2}_d$ spanned by *oriented* graphs, that is, directed graphs with no closed paths of directed edges, is denoted by $\mathsf{GC}^{or}_d$. These complexes have been studied in [W1, W2, Z].

where $\mathfrak{grt}$ (resp., $\mathfrak{grt}_1$) is the Lie algebra of the Grothendieck-Teichmüller group $GRT$ (resp., $GRT_1$). It is easy to see that $H^0(\mathsf{GC}_2^{or}) = 0$ and $H^0(\mathsf{GC}_3^{\geq 2}) = 0$.

**1.1.2. Corollary.** *There is an isomorphism of Lie algebras*

$$H^0(\mathrm{Der}(\mathcal{H}olieb_{0,1}^{\star\circlearrowright})) = \mathfrak{grt}$$

*that is, the Grothendieck-Teichmüller group $GRT$ acts up to homotopy faithfully (and essentially transitively) on the vertex completion of the wheeled properad $\mathcal{H}olieb_{0,1}^{\star\circlearrowright}$ governing* finite-*dimensional formal Poisson structures.*

*By contrast*

$$H^0(\mathrm{Der}(\mathcal{H}olieb_{0,1}^{\star})) = 0$$

*that is, the completion of the properad $\mathcal{H}olieb_{0,1}^{\star}$ governing* infinite-*dimensional formal Poisson structures admits no homotopy non-trivial derivations at all.*

Note that the above Proposition applied to another interesting case $c = d = 1$ gives us quite the opposite picture,

$$H^0(\mathrm{Der}(\mathcal{H}olieb_{1,1}^{\star\circlearrowright})) = 0, \qquad H^0(\mathrm{Der}(\mathcal{H}olieb_{1,1}^{\star})) = \mathfrak{grt}$$

cf. [MW1]. These results are by no means surprising — the graph complex $\mathsf{fGC}_2^{\geq 2}$ can be understood as a kind of universal incarnation of the Chevalley-Eilenberg deformation complex of the Lie algebra $\mathcal{T}_{poly}(\mathcal{M})$ for any finite-dimensional formal manifold [K1], and the fact that $H^0(\mathsf{fGC}_2^{\geq 2}) = \mathfrak{grt}$ already implies [W1] that the Grothendieck-Teichmüller group $GRT$ acts (up to homotopy) as universal $\mathcal{L}ie_\infty$ automorphisms of $\mathcal{T}_{poly}(\mathcal{M})$; this action is given in terms of certain iterations of the canonical $GL(V)$-invariant $BV$ operator on $\mathcal{T}_{poly}(\mathcal{M})$, so what the above Corollary says essentially is that even if one drops this restriction on the possible structure of linear operators acting on $\mathcal{T}_{poly}(\mathcal{M})$, the action of $GRT$ remains homotopy non-trivial. The above Proposition can be inferred from the theory of stable cohomology of the Lie algebra of polyvector fields developed in [W3] (but not immediately). In any case, our proof of Proposition 1.1.1 is very short, so we decided to show a new direct argument behind that claim in §4.1 below.

The main advantage of our study of the homotopy theory of the vertex completion $\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowright}$ of the wheeled prop $\mathcal{H}olieb_{0,1}^{\star\circlearrowright}$ is that it gives us — almost immediately! — a full insight into the homotopy theory of M. Kontsevich's formality maps which is the second main topic of this paper.

**1.2. Homotopy classification of M. Kontsevich formality maps.** M. Kontsevich formality map [K2] associates to any *finite*-dimensional formal Poisson structure $\pi$ on a formal graded manifold $\mathcal{M} = V^*$ a curved $\mathcal{A}ss_\infty$-structure on the $\mathbb{R}$-algebra $\mathcal{O}_\mathcal{M} = \widehat{\odot^\bullet} V$ of formal smooth functions on $\mathcal{M}$ which is given in terms of polydifferential operators constructed from $\pi$. In our approach $\pi$ is a representation in $V$ of the wheeled prop $\mathcal{H}olieb_{0,1}^{\star\circlearrowright}$, and the construction of polydifferentials operators from $\pi$ can be conveniently encoded into the polydifferential functor [MW3]

$$\mathcal{O}: \mathsf{Category\ of\ dg\ props} \longrightarrow \mathsf{Category\ of\ dg\ operads}$$

applied to the prop $\mathcal{H}olieb_{0,1}^{\star\circlearrowright}$: for any dg prop $\mathcal{P}$ the associated dg operad $\mathcal{O}(\mathcal{P})$ has the property that for any representation $\rho$ of $\mathcal{P}$ in a vector space $V$ the operad $\mathcal{O}(\mathcal{P})$ has a canonically associated representation $\mathcal{O}(\rho)$ in the completed graded commutative algebra $\widehat{\odot^\bullet} V$ given in terms of polydifferenial operators. Curved $\mathcal{A}ss_\infty$ algebra structures are controlled by the well-known (non-cofibrant) dg operad $c\mathcal{A}ss_\infty$ so that the Maxim Kontsevich universal formality map from [K1] (or any other universal formality map) can be understood as a morphism of dg operads

$$\mathcal{F}: c\mathcal{A}ss_\infty \longrightarrow \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowright}).$$

satisfying certain non-triviality conditions (se §5 for details). We show in this paper a very short and elementary (based essentially on the contractility of the permutahedra polytopes) proof of the following classification theorem.

**1.2.1. Theorem.** *Let* $\mathsf{Def}\left(c\mathcal{A}ss_\infty \stackrel{\mathcal{F}}{\to} \mathcal{O}(\mathcal{H}olieb_{0,1}^{\star\circlearrowright})\right)$ *be the deformation complex of any given formality map* $\mathcal{F}$ *(in particular, of the M. Kontsevich map from* [K2]*). Then there is a canonical morphism of complexes*

$$\mathsf{fGC}_2^{\geq 2} \longrightarrow \mathsf{Def}\left(c\mathcal{A}ss_\infty \stackrel{\mathcal{F}}{\to} \mathcal{O}(\mathcal{H}olieb_{0,1}^{\star\circlearrowright})\right)[1]$$

*which is a quasi-isomorphism.*

This result implies the equality of cohomology groups for any $i \in \mathbb{Z}$,

$$H^{i+1}\left(\mathsf{Def}\left(c\mathcal{A}ss_\infty \stackrel{\mathcal{F}}{\to} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowright})\right)\right) = H^i(\mathsf{fGC}_2^{\geq 2})$$

which in the special case $i = 0$ reads as

$$H^{1}\left(\mathsf{Def}\left(c\mathcal{A}ss_\infty \stackrel{\mathcal{F}}{\to} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowright})\right)\right) = H^0(\mathsf{fGC}_2^{\geq 2}) = \mathfrak{grt}$$

and hence gives us a new (very short) proof of the following remarkable Theorem by V. Dolgushev.

**1.2.2. Theorem** [Do]**.** *The Grothendieck-Teichmüller group GRT acts freely and transitively on the set of homotopy classes of universal formality morphisms.*

This Theorem implies the identification of the set of homotopy classes of formality maps with the set of V. Drinfeld associators [D].

**1.3. Some notation.** The set $\{1, 2, \ldots, n\}$ is abbreviated to $[n]$; the group of bijections $[n] \to [n]$ is denoted by $\mathbb{S}_n$; the trivial (resp., sign) one-dimensional representation of $\mathbb{S}_n$ is denoted by $\mathbf{1}_n$ (resp., $sgn_n$). The cardinality of a finite set $S$ is denoted by $\#S$. We work in this paper in the category of $\mathbb{Z}$-graded vector spaces over a field $\mathbb{K}$ of characteristic zero. If $V = \oplus_{i\in\mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ stands for the graded vector space with $V[k]^i := V^{i+k}$; for $v \in V^i$ we set $|v| := i$. If $V$ is a complex with a differential $d$, then $V[k]$ is also a complex with the differential given by $(-1)^k d$.

For the basic notions and facts of the theory of props and properads we refer to the papers [Ma, MV, V] (and references cited there) and of their wheeled versions to [MMS, Me2]. A short introduction into these theories can be found in [Me3]. We assume that every (wheeled) prop $\mathcal{P}$ we work with in this paper has the unit denote by $\uparrow \in \mathcal{P}(1,1)$.

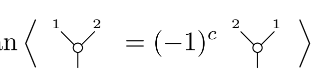
**1.4. Acknowledgements.** We are very grateful to the referees for their valuable comments and suggestions.

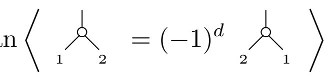
## 2. Wheeled properads of homotopy Lie bialgebras and their extensions

**2.1. Reminder on props of Lie $(c,d)$-bialgebras and their minimal resolutions.** Consider for any pair of integers $c, d \in \mathbb{Z}$ a quadratic prop [MW1]

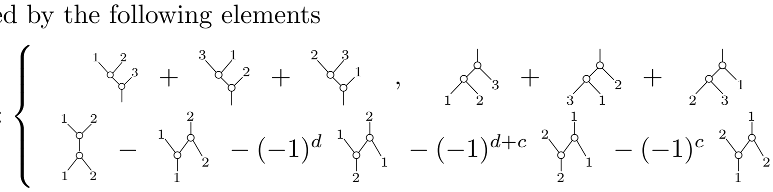
$$\mathcal{L}ieb_{c,d} := \mathcal{F}ree\langle e\rangle / \langle \mathcal{R}\rangle,$$

defined as the quotient of the free prop generated by an $\mathbb{S}$-bimodule $e = \{e(m,n)\}_{m,n\geq 0}$ with all $e(m,n) = 0$ except[3]

$e(2,1) := \mathbf{1}_1 \otimes sgn_2^c[c-1] = \mathrm{span}$ 

$e(1,2) := sgn_2^d \otimes \mathbf{1}_1[d-1] = \mathrm{span}$ 

by the ideal generated by the following elements

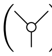
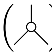
(5) $\mathcal{R}:$ 

[3]When representing elements of various props below as graphs we always assume by default that all edges and legs are *directed* with the flow running from the bottom of the graph to the top.

Thus a representation,

$$\rho : \mathcal{L}ieb_{c,d} \longrightarrow \mathcal{E}nd_V$$

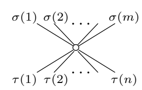
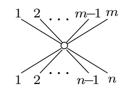
of this prop in a differential graded (dg, for short) vector space $V$ is uniquely determined by the values of $\rho$ on the generators,

$\rho\Big($  $\Big) : V[-c] \to \odot^2(V[-c])[1], \qquad \Big($  $\Big) : \odot^2(V[d]) \to V[1+d],$

which equip $V$ with (degree shifted) dg Lie algebra and Lie coalgebra structures satisfying the Drinfeld compatibility condition (which is assured by the vanishing under $\rho$ of the bottom graph in $\mathcal{R}$).

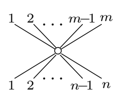
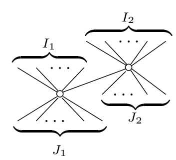
The minimal resolution of the prop $\mathcal{L}ieb_{c,d}$ is a free cofibrant prop $\mathcal{H}olieb_{c,d}$ generated by the $\mathbb{S}$-bimodule $E = \{E(m,n)\}$ with $E(m,n) \neq 0$ only for $m+n \geq 3$ and $m,n \geq 1$,

(6)

$E(m,n) := sgn_m^{\otimes |c|} \otimes sgn_n^{|d|}[c[m-1]+d[n-1]-1] =: \text{span}\Big\langle$  $= (-1)^{c|\sigma|+d|\tau|}$  $\Big\rangle_{\forall \sigma\in\mathbb{S}_m, \forall\tau\in\mathbb{S}_n}$



The differential on $\mathcal{H}olieb_{c,d}$ is given on the generators by

(7) $\delta$  $= \sum_{\substack{[1,\ldots,m]=I_1\sqcup I_2\\ |I_1|\geq 0, |I_2|\geq 1}} \sum_{\substack{[1,\ldots,n]=J_1\sqcup J_2\\ |J_1|\geq 1, |J_2|\geq 1}} \pm$ 



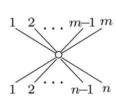
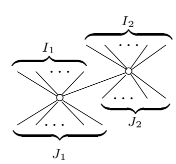
where the signs on the r.h.s are uniquely fixed for $c+d \in 2\mathbb{Z}$ by the fact that they all equal to $+1$ if $c$ and $d$ are even integers, and for $c+d \in 2\mathbb{Z}+1$ the signs are given explicitly in [Me2]. Note that the props $\mathcal{H}olieb_{c,d}$ and $\mathcal{H}olieb_{d,c}$ are canonically isomorphic to each other via the flow reversing on the generating graphs.

A representation of $\mathcal{H}olieb_{c,d}$ in a finite-dimensional vector space $V$ can be identified with a degree $c+d+1$ element $\pi$ in the completed graded commutative algebra

$$\pi = \sum_{\substack{m,n\geq 1\\ m+n\geq 3}} = \prod_{m,n\geq 1, m+n\geq 3} \mathrm{Hom}(\odot^n(V[d]), \odot^m(V[-c])) \subset \prod_{k\geq 0} \odot^k(V^*[-d]\oplus V[-c])$$

equipped with the obvious Poisson type Lie bracket of degree $-c-d$.

**2.2. Non-cofibrant extensions of $\mathcal{H}olieb_{c,d}$.** Consider a dg prop $\mathcal{H}olieb^{\star}_{c,d}$ generated by the $\mathbb{S}$-bimodule $E^\star = \{E^\star(m,n)\}_{m,n\geq 0}$ with *all* $E^\star(m,n)$ non-zero and given by the same formula as in (6). The differential $\delta^\star$ on $\mathcal{H}olieb^{\star\uparrow}_{c,d}$ is given formally by the formula (7) with the summation over partitions of the sets $[m]$ and $[n]$ appropriately extended,

(8) $\delta^\star$  $= \sum_{\substack{[1,\ldots,m]=I_1\sqcup I_2\\ |I_1|\geq 0, |I_2|\geq 0}} \sum_{\substack{[1,\ldots,n]=J_1\sqcup J_2\\ |J_1|\geq 0, |J_2|\geq 0}} \pm$ 

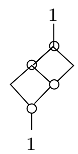


Its ideal $I_0$ generated by all $(m,n)$-corollas[4] with $m=0$ or $n=0$ is differential, and the quotient properad $\mathcal{H}olieb^{\star}_{c,d}/I^0$ is denoted by $\mathcal{H}olieb^{+}_{c,d}$ (there exists a general "plus" endofunctor , $\mathcal{P}\to\mathcal{P}^+$, in the category of props, and the non-cofibrant prop $\mathcal{H}olieb^{+}_{c,d}$ can be understood as the application of that construction to $\mathcal{H}olieb_{c,d}$).

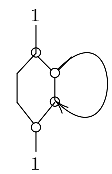
The dg prop $\mathcal{H}olieb^{+}_{c,d}$ contains in turn the differential ideal $I^+$ generated by the $(1,1)$-corolla, and the quotient prop is precisely $\mathcal{H}olieb_{c,d}$.

[4]We often call corollas of type $(0,n)$ (resp. $(m,0)$) *sources* (resp., *targets*). Note that the $(0,0)$ corolla $\bullet$ is the unique generator which is both a source and a target. The $(1,1)$-corolla is often called a *passing vertex*.

**2.3**. **Wheeled closures.** We refer to [Me2, MMS] for the full details of the wheelification functor, but as we work in this paper only with free props $\mathcal{P}$ generated by certain $(m,n)$ corollas, $m,n\in\mathbb{N}$, it is very easy to explain what is the wheeled closure $\mathcal{P}^{\circlearrowright}$ of $\mathcal{P}$: if elements of $\mathcal{P}$ are obtained in general by glueing output legs of generating corollas to input legs of other corollas in such a way that directed paths of edges in the resulting directed graph never form a cycle (a "wheel"), elements of $\mathcal{P}^{\circlearrowright}$ are constructed in the same but with latter restriction dropped. For example,




$\in \mathcal{H}olieb_{c,d}$,




$\in \mathcal{H}olieb_{c,d}^{\circlearrowright}$

where the orientation on edges is assumed to flow from bottom to the top unless explicitly shown. Clearly, $\mathcal{P}$ is a subprop of its wheeled closure $\mathcal{P}^{\circlearrowright}$. It makes sense to talk about representations of ordinary props in any vector space (finite- or infinite-dimensional), while their wheeled closures can be represented, in general, only in *finite-dimensional* vector spaces $V$ as graphs with wheels induce trace operations of the form $\mathrm{Hom}(V,V)\rightarrow\mathbb{K}$ which have no sense if $V$ is, for example, a direct limit, $V=\lim\limits_{n\longrightarrow\infty}\mathbb{K}^n$, of finite-dimensional spaces; indeed $\mathrm{Hom}(V,V)$ can contain infinite sums of the form

$$\sum_{n=1}^{\infty}\lambda_n e_n\otimes e_n^*,\quad \lambda_n\in\mathbb{K}$$

$e_n$ and $e_n^*$ being the standard (dual to each other) basis vectors of $\mathbb{K}^n$ and its dual space $(\mathbb{K}^n)^*$ whose trace $\sum_{n=1}^{\infty}\lambda_n$ diverges in general.

The wheeled closures of $\mathcal{H}olieb_{c,d}^{\star}$ and $\mathcal{H}olieb_{c,d}^{+}$ are denoted by $\mathcal{H}olieb_{c,d}^{\star\circlearrowright}$ and $\mathcal{H}olieb_{c,d}^{+\circlearrowright}$ respectively.

Denote by $\widehat{\mathcal{H}olieb}_{c,d}^{\star\circlearrowright}$ (resp., $\widehat{\mathcal{H}olieb}_{c,d}^{+\circlearrowright}$) the vertex completion of the prop $\mathcal{H}olieb_{c,d}^{\star\circlearrowright}$ (resp., $\widehat{\mathcal{H}olieb}_{c,d}^{+\circlearrowright}$). One must be careful about definitions of representations of these completed props, but for our purposes the following remark will be enough: given any representation of the prop $\mathcal{H}olieb_{0,1}^{\star\circlearrowright}$ in a finite-dimensional dg vector space $V$,

$$\rho:\mathcal{H}olieb_{c,d}^{\star\circlearrowright}\longrightarrow\mathcal{E}nd_V$$

that is, a formal Poisson structure $\pi\in\mathcal{T}_{poly}\mathcal{M}$ on $V^*$ viewed as a formal graded manifold, there is an associated *continuous* morphism of the topological props

$$\hat{\rho}:\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowright}\longrightarrow\mathcal{E}nd_V[[\hbar]]$$

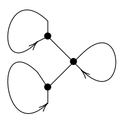
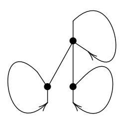
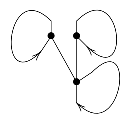
whose value on any generating corolla $e$ of $\widehat{\mathcal{H}olieb}_{0,1}$ is equal to $\hbar\rho(e)$, that is, a formal Poisson structure $\hbar\pi\in\mathcal{T}_{poly}\mathcal{M}[[\hbar]]$. Here $\hbar$ is any formal parameter of homological degree zero ("Planck constant").

**2.3.1. Proposition**. *(i) The dg subprop of $\widehat{\mathcal{H}olieb}_{c,d}^{+\circlearrowright}$ spanned by graphs with at least one ingoing or at least one outgoing legs is acyclic while its complement $\widehat{\mathcal{H}olieb}_{c,d}^{+\circlearrowright}(0,0)$ has non-trivial cohomology which is equal to $H^\bullet(\widehat{\mathcal{H}olieb}_{c,d}^{\circlearrowright}(0,0),\delta)$.*

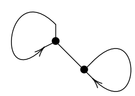
*(ii) The dg prop $\widehat{\mathcal{H}olieb}_{c,d}^{\star\circlearrowright}$ is acyclic.*

*Proof.* Consider a filtration of the complex $(\widehat{\mathcal{H}olieb}_{c,d}^{+\circlearrowright},\delta^+)$ by the number of vertices of valency $\geq 3$. The induced differential on the associated graded attaches to each leg the $(1,1)$-corolla. We can consider another filtration such that the induced differential attaches $(1,1)$-corolla only to the input (or output) leg labelled by number 1. This complex is obviously acyclic. This proves the claim for the required subprop of $\widehat{\mathcal{H}olieb}_{c,d}^{+\circlearrowright}$.

The second claim about non-triviality of $H^\bullet(\widehat{\mathcal{H}olieb}_{c,d}^{+\circlearrowright}(0,0), \delta^+)$ follows from the direct examples of non-trivial cohomology classes such as (see [Me2])

 $-$  $+$  $\in H(\widehat{\mathcal{H}olieb}_{c,d}^{+\circlearrowright}(0,0), \delta^+) \ \ \forall c,d \in \mathbb{Z}$ with $c+d \in 2\mathbb{Z}+1$,

or even a simpler one

 $\in H(\widehat{\mathcal{H}olieb}_{c,d}^{+\circlearrowright}(0,0), \delta^+) \ \ \forall c,d \in \mathbb{Z}.$

It is easy to see (cf. [W2]) that graphs containing passing vertices do not contribute to the cohomology so that

$$H^\bullet(\widehat{\mathcal{H}olieb}_{c,d}^{+\circlearrowright}(0,0), \delta^+) = H^\bullet(\widehat{\mathcal{H}olieb}_{c,d}^{\circlearrowright}(0,0), \delta).$$

Consider next a filtration of each complex $(\widehat{\mathcal{H}olieb}_{c,d}^{\star\circlearrowright}(m,n), \delta^\star)$ with $m+n \geq 1$ by the total number of vertices with no input edges or no output edges. The induced differential in the associated graded is precisely $\delta^+$ so that the argument as above proves its acyclicity.

Finally, consider the complex $(\widehat{\mathcal{H}olieb}_{c,d}^{\star\circlearrowright}(0,0), \delta^\star)$. Call univalent vertices and passing vertices (that is, vertices of type $(1,1)$, see footnote 4) of graphs from $\widehat{\mathcal{H}olieb}_{c,d}^{\star\circlearrowright}(0,0)$ *stringy* ones, and call maximal connected subgraphs (if any) of a graph $\Gamma$ from $\widehat{\mathcal{H}olieb}_{c,d}^{\star\circlearrowright}(0,0)$ consisting of stringy vertices with at least one vertex univalent *strings*. The vertices of $\Gamma$ which do not belong to strings are called *core* ones. Thus strings are subgraphs or graphs of the following three types,

(i) 

$n \geq 0$ stringy vertices (shown as black bullets)

(ii) 

$n \geq 0$ stringy vertices

(iii)  $n \geq 1$ stringy vertices

Consider a (complete, exhaustive, bounded above) filtration of $(\widehat{\mathcal{H}olieb}_{c,d}^{\star\circlearrowright}(0,0), \delta^\star)$ by the number of core vertices,

$$F_{-p} \text{ is generated by graphs with the number of core vertices } \geq p.$$

The differential in the associated graded acts non-trivially on strings of types (i) and (ii) (resp., (iii)) with *even* (resp., *odd*) number of stringy vertices only by increasing that number by one. Hence the complexes $C_{(i)}$, $C_{(ii)}$ and $C_{(iii)}$ generated by strings of type (i), (ii) and (iii) respectively are acyclic.

If the set of core vertices is empty, we are in the situation of the complex $C_{(iii)}$ so that the associated cohomology vanishes.

If the set of core vertices is non-empty, then the associated graded is isomorphic to the unordered tensor product

$$\bigotimes_v \odot^\bullet C_{(i)}^v \otimes \odot^\bullet C_{(ii)}^v$$

over the set of core vertices of the graded symmetric tensor algebras of acyclic complexes $C_{(i)}$ and $C_{(ii)}$, and hence is acyclic itself.

$\square$

## 3. Deformation complexes of wheeled props and graph complexes

**3.1. Derivations of wheeled props.** A wheeled prop $\mathcal{P}^\circlearrowright$ in the category of complexes is an $\mathbb{S}$-bimodule, that is a collection $\{\mathcal{P}^\circlearrowright(m,n)\}$ of $(\mathbb{S}_m)^{op} \times \mathbb{S}_n$ modules, equipped with two basic operations satisfying certain axioms (see §2 in [MMS] for full details, or just pictures 5 and 6 in [MMS]):

(i) the horizontal composition (“a map from disjoint union of two decorated corollas into a single corolla”)

$$\begin{array}{rccc} \circ_h: & \mathcal{P}^{\circlearrowright}(m_1,n_1)\otimes\mathcal{P}^{\circlearrowright}(m_2,n_2) & \longrightarrow & \mathcal{P}^{\circlearrowright}(m_1+m_2,n_1+n_2) \\ & a\otimes b & \longrightarrow & a\circ_h b \end{array}$$

(ii) the trace operation defined for any $m,n\geq 1$ and any $i\in[m]$, $j\in[n]$ (“gluing $i$-th output leg to the $j$-in input leg, and then contracting the resulting internal edge”),

$$\begin{array}{rccc} Tr^i_j: & \mathcal{P}^{\circlearrowright}(m,n) & \longrightarrow & \mathcal{P}^{\circlearrowright}(m-1,n-1) \\ & a & \longrightarrow & Tr^i_j(a). \end{array}$$

The Lie algebra of derivations of $\mathcal{P}^{\circlearrowright}$ is defined as the vector space $\mathrm{Der}(\mathcal{P}^{\circlearrowright})\hookrightarrow \mathrm{Hom}_{\mathbb{S}}(\mathcal{P}^{\circlearrowright},\mathcal{P}^{\circlearrowright})$ of those endomorphisms $D:\mathcal{P}^{\circlearrowright}\longrightarrow\mathcal{P}^{\circlearrowright}$ of the $\mathbb{S}$-bimodule $\mathcal{P}^{\circlearrowright}$ which satisfy the two conditions: (i) for any $a,b\in\mathcal{P}$ one has

$$D(a\circ_h b)=D(a)\circ_h f(b)+(-1)^{|D||a|}f(a)\circ_h D(b),$$

and (ii) for any $c\in\mathcal{P}(m,n)$ with $m,n\geq 1$ and any $i\in[m]$ and $j\in[n]$

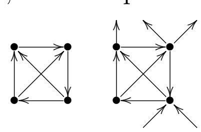
$$D(Tr^i_j(c))=Tr^i_j(D(c)).$$

If the $\delta$ is a differential in the wheeled prop $\mathcal{P}^{\circlearrowright}$, then $\delta$ is a MC element in $\mathrm{Der}(\mathcal{P}^{\circlearrowright})$ so that the latter becomes also a complex with the differential $d=[\delta,\ ]$.

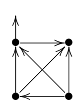
We are interested in the complex of derivations of the *completed* (by the number of vertices) prop $\widehat{\mathcal{H}olieb}^{\star\circlearrowright}_{c,d}$ but abusing notations denote it from now on by $\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d})$ (cf. [MW1]). Any derivation of $\widehat{\mathcal{H}olieb}^{\star\circlearrowright}_{c,d}$ is uniquely determined by its values on the generators of the prop $\mathcal{H}olieb^{\star\circlearrowright}_{c,d}$. Hence we have isomorphisms of graded vector spaces,

$$\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d})=\prod_{m,n\geq 0}\left(\widehat{\mathcal{H}olieb}^{\star\circlearrowright}_{c,d}(m,n)\otimes sgn_m^{\otimes|c|}\otimes sgn_n^{\otimes|d|}\right)^{\mathbb{S}_m^{op}\times\mathbb{S}_n}[1+c(1-m)+d(1-n)]. \quad (9)$$

Thus elements of this complex can be interpreted as directed (not necessarily connected) graphs that have incoming or outgoing legs and wheels, for example

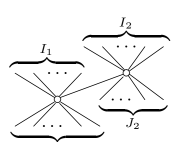
 $\in\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d})$

Its subcomplex spanned by *oriented* (i.e. with no wheels) directed graphs is precisely the derivation complex of $\mathrm{Der}(\mathcal{H}olieb^{\star}_{c,d})$, e.g.

 $\in\mathrm{Der}(\mathcal{H}olieb^{\star}_{c,d})$

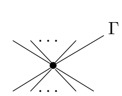
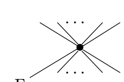
Note that the outgoing or ingoing legs (if any) of these graphs are not assigned particular numerical labels; more precisely, their numerical labels are (skew)symmetrized in accordance with the parity of the integer parameters $c$ and $d$.

The Lie algebra $\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d})$ contains a Maurer-Cartan element

$$\gamma^\star:=\sum_{m,n\geq 0}\ \sum_{\substack{[m]=I_1\sqcup I_2,[n]=J_1\sqcup J_2\\ |I_1|,|I_2|,|J_1|,|J_2|\geq 0}}\pm$$  ,



which corresponds to the differential $\delta^\star$ in $\mathcal{H}olieb^{\star\circlearrowright}_{c,d}$. Hence the differential in the complex $\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d})$ is given by

(10) $d^\star\Gamma:=[\gamma^\star,\Gamma]=\delta^\star\Gamma\pm\sum$  $\mp\sum$ 

where the differential in the first term,

$$\delta^\star\Gamma = (-1)^{|\Gamma|}\sum_v \Gamma\circ_v \bullet\!\!\rightarrow\!\!\bullet,$$

acts on the vertices of $\Gamma$ by formula[5] (8) while in the remaining two terms one attaches $(m, n+1)$-corollas and, respectively, $(m+1, n)$-corollas to each outgoing leg (if any), and, respectively each ingoing leg (if any) of $\Gamma$, and sums over all $m, n$ satisfying $m, n \geq 0$.

It is often useful (cf. [W1, MW1]) to include the graph $\uparrow$ without vertices into the complex $\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d})$ and set, in accordance with the above general formula for $d^\star$,

$$d^\star\uparrow = \sum_{m,n\geq 0}(m-n)\ \overbrace{\underbrace{\times}_{n\times}}^{m\times}, \tag{11}$$

The derivation $d^\star\uparrow$ corresponds to the universal automorphism of *any* dg wheeled prop $\mathcal{P}^\circlearrowright$ which sends every element $a\in\mathcal{P}^\circlearrowright(m,n)$ into $\lambda^{m-n}a$ for any $\lambda\in\mathbb{K}\setminus 0$.

It is important to notice that the subspace

$$\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d})_{conn}\subset\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d})$$

spanned by *connected* graphs is a subcomplex[6], and that there is an canonical isomorphism of complexes

$$\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d}) = \left(\widehat{\odot^\bullet}\left(\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d})_{conn}[-1-c-d]\right)\right)[1+c+d] \tag{12}$$

As the (completed) symmetric tensor product functor is exact, it is enough to compute the cohomology of the subcomplex $\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d})_{conn}$. We do it in the next section in terms of the cohomology of certain M. Kontsevich graph complexes [K1] which are reminded in the next subsection.

**3.2**. **Reminder on graph complexes.** A *graph* $\Gamma$ is a 1-dimensional *CW* complex whose 0-cells are called *vertices* and 1-cells are called *edges*. The set of vertices of $\Gamma$ is denoted by $V(\Gamma)$ and the set of edges by $E(\Gamma)$. A graph $\Gamma$ is called *directed* if each edge $e\in E(\Gamma)$ comes equipped with a fixed orientation. If a vertex $v$ of a directed graph has $m\geq 0$ outgoing edges and $n\geq 0$ incoming edges, then we say that $v$ is an $(m,n)$-*vertex*. A $(1,1)$-vertex is called *passing*.

Let $\mathsf{G}_{n,l}$ be the set of directed graphs $\Gamma$ with $n$ vertices and $l$ edges such that some bijections $V(\Gamma)\to[n]$ and $E(\Gamma)\to[l]$ are fixed, i.e. every edges and every vertex of $\Gamma$ has a fixed numerical label. There is a natural right action of the group $\mathbb{S}_n\times\mathbb{S}_l$ on the set $\mathsf{G}_{n,l}$ with $\mathbb{S}_n$ acting by relabeling the vertices and $\mathbb{S}_l$ by relabeling the edges. Consider a graded vector space ("directed full graph complex")

$$\mathsf{dFGC}_d = \prod_{l\geq 0}\prod_{n\geq 1}\mathbb{K}\langle\mathsf{G}_{n,l}\rangle\otimes_{\mathbb{S}_n\times\mathbb{S}_l}\left(sgn_n^{\otimes|d|}\otimes sgn_l^{|d-1|}\right)[d(1-n)+l(d-1)]$$

This space is spanned by directed graph with no numerical labels on vertices and edges but with a choice of an *orientation*: for $d$ even (resp., odd) this is a choice of ordering of edges (resp., vertices) up to an even permutation. This graded vector space has a Lie algebra structure with

$$[\Gamma_1,\Gamma_2] := \sum_{v\in V(\Gamma)}\Gamma_1\circ_v\Gamma_2 - (-1)^{|\Gamma_1||\Gamma_2|}\Gamma_2\circ_v\Gamma_1$$

where $\Gamma_1\circ_v\Gamma_2$ is defined by substituting the graph $\Gamma_2$ into the vertex $v$ of $\Gamma_1$ and taking a sum over re-attachments of dangling edges (attached earlier to $v$) to vertices of $\Gamma_2$ in all possible ways. It is easy to see that the degree 1 graph $\bullet\!\!\rightarrow\!\!\bullet$ in $\mathsf{dFGC}_d$ is a Maurer-Cartan element, so that one can make the latter into a complex with the differential

$$\delta := [\bullet\!\!\rightarrow\!\!\bullet,\ ].$$

[5]That formula might be understood as a substitution into each vertex $v$ the graph $\bullet\!\!\rightarrow\!\!\bullet$ and redistributing all edges of $v$ along the pair of new created vertices in all possible ways.

[6]The meaning of the complex $\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d})_{conn}$ is that it describes derivations of $\widehat{\mathcal{H}olieb}^{\star\circlearrowright}_{c,d}$ as a *properad* rather than as a prop.

The complex $\mathsf{dFGC}_d$ contains a subcomplex $\mathsf{FGC}_d^{or}$ spanned by *oriented graphs*, that is, graphs with no closed paths of directed edges ("wheels").

One can define an *undirected* full graph complex as

$$\mathsf{FGC}_d = \prod_{l\geq 0}\prod_{n\geq 1} \mathbb{K}\langle \mathsf{G}_{n,l}\rangle \otimes_{\mathbb{S}_n\times(\mathbb{S}_l\ltimes(\mathbb{S}_2)^l)} \left( sgn_n^{\otimes|d|}\otimes (sgn_l^{|d-1|}\otimes (sgn_2^{|d|})^{\otimes l})\right)[d(1-n)+l(d-1)]$$

where the group $(\mathbb{S}_2)^l$ acts on edges by reversing their directions. This graph complex is spanned by graphs with directions on edges forgotten for $d$ even, and fixed up to the flip and multiplication by $(-1)$ for $d$ odd. These dg Lie algebras contain dg subalgebras $\mathsf{dcGC}_d\subset \mathsf{dFGC}_d$, $\mathsf{cGC}_d^{or}\subset \mathsf{FGC}_d^{or}$ and $\mathsf{cGC}_d\subset\mathsf{FGC}_d$ spanned by *connected* graphs which in turn contain dg Lie subalgebras $\mathsf{dcGC}_d^{\geq 2}$, $\mathsf{GC}_d^{or,\geq 2}$ and, respectively, $\mathsf{GC}_d^{\geq 2}$ spanned by graphs with all vertices having valency $\geq 2$. The dg Lie algebras $\mathsf{dcGC}_d^{\geq 2}$ and $\mathsf{GC}_d^{or,\geq 2}$ (resp., $\mathsf{GC}_d^{\geq 2}$) contain in turn dg Lie subalgebras $\mathsf{dGC}_d$ and $\mathsf{GC}_d^{or}$ spanned by graphs with no passing vertices, (resp., $\mathsf{GC}_d$ spanned by graphs with all vertices at least trivalent). The canonical inclusion maps

$$\mathsf{dGC}_d \longrightarrow \mathsf{dcGC}_d^{\geq 2}\longrightarrow \mathsf{dcGC}_d,\quad \mathsf{GC}_d^{or}\longrightarrow \mathsf{GC}_d^{or,\geq 2}\longrightarrow \mathsf{cGC}_d^{or}$$

are all quasi-isomorphisms (see, e.g., [W1] and references cited there). There is also a canonical morphism of dg Lie algebras

$$\mathsf{GC}_2^{\geq 2}\longrightarrow \mathsf{dcGC}_2, \tag{13}$$

which sends a graph with no directions on edges into a sum of graphs with all possible directions on edges; it is also a quasi-isomorphism [W1]. It was proven in [W1, W2] that

$$H^\bullet(\mathsf{GC}_d^{\geq 2}) = H^\bullet(\mathsf{GC}_{d+1}^{or})$$

and that

$$H^0(\mathsf{dGC}_2) = H^0(\mathsf{GC}_2^{\geq 2}) = H^0(\mathsf{GC}_3^{or}) = \mathfrak{grt}_1,$$

where $\mathfrak{grt}_1$ is the Lie algebra of the Grothendieck-Teichmüller group $GRT_1$. It is easy to see that $H^0(\mathsf{GC}_2^{or}) = 0$ and $H^0(\mathsf{GC}_3^{\geq 2}) = 0$.

One has canonical monomorphisms of complexes

$$\odot^\bullet(\mathsf{dGC}_d[-d])[d]\to \mathsf{dFGC}_d,\quad \odot^\bullet\left(\mathsf{GC}_d^{\geq 2}[-d]\right)[d]\to\mathsf{FGC}_d,\quad \odot^\bullet(\mathsf{GC}_d^{or}[-d])[d]\to\mathsf{FGC}_d^{or}$$

which are quasi-isomorphisms. Hence it is enough to study only connected graph complexes.

It is often useful [MW1, MW3] to consider slightly extended dg Lie algebras,

$$\mathsf{dGC}_d\oplus\ \mathbb{K},\quad \mathsf{GC}_d^{\geq 2}\oplus\mathbb{K},\quad \mathsf{GC}_d^{or}\oplus\mathbb{K} \tag{14}$$

where the summand $\mathbb{K}$ is generated by an additional element $\emptyset$ concentrated in degree zero, "a graph with no vertices and edges", whose Lie bracket, $[\emptyset,\Gamma]$, with an element $\Gamma$ of $\mathsf{GC}_d^{\geq 2}$ or $\mathsf{GC}_d^{or}$ is defined as the multiplication of $\Gamma$ by twice its loop number (in particular, $\emptyset$ is a cycle with respect to the differential $\delta$). In this case the zero-th cohomology groups of the first two of these extended complexes for $d=2$ and, respectively, of the last complex for $d=3$ are all equal to the Lie algebra $\mathfrak{grt}$ of the "full" Grothendieck-Teichmüller group, rather than to its reduced version $\mathfrak{grt}_1$. This very useful fact prompts us to define the *full graph complexes* of not necessarily connected graphs as the completed graded symmetric tensor algebras (3) and (4).

## 4. Cohomology of the derivation complex of $\mathcal{H}olieb_{c,d}^{\star\circlearrowright}$

**4.1. From directed graph complex to the complex of properadic derivations.** Following [MW1] one notices that there is a natural right action of the dg Lie algebra $\mathsf{dcGC}_{c+d+1}$ on the dg wheeled *properad* $\widehat{\mathcal{H}olieb}_{c,d}^{\star\circlearrowright}$ by properadic derivations, i.e. there is a canonical morphism of dg Lie algebras,

$$\begin{array}{rccc} F^\star\colon & \mathsf{dcGC}_{c+d+1} & \to & \mathrm{Der}(\mathcal{H}olieb_{c,d}^{\star\circlearrowright})_{conn} \\ & \Gamma & \to & F(\Gamma)\end{array} \tag{15}$$

where the derivation $F^\star(\Gamma)$ has, by definition, the following values on the generators of the completed properad $\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{c,d}$

$$
\left(\begin{array}{c} {}^{1\ 2\ \cdots\ m} \\ \times \\ {}_{1\ 2\ \cdots\ n} \end{array}\right) \cdot F^\star(\Gamma) = \sum_{\substack{s:[n]\to V(\Gamma)\\ \hat{s}:[m]\to V(\Gamma)}} \begin{array}{c} {}^{1\ 2\ \cdots\ m} \\ \Gamma \\ {}_{1\ 2\ \cdots\ n} \end{array} \qquad \forall\, m,n\geq 0, \tag{16}
$$

with the sum being taken over all ways of attaching the incoming and outgoing legs to the graph $\Gamma$. The image

$$
\delta^\star := F^\star\left(\bullet\!\!\to\!\!\bullet\right)
$$

gives us the standard differential (8) in $\mathcal{H}olieb^{\star\circlearrowleft}_{c,d}$. The monomorphism $\mathsf{dGC}_{c+d+1} \hookrightarrow \mathsf{dcGC}_{c+d+1}$ is a quasi-isomorphism so that, from the cohomological viewpoint, it is enough to study the restriction of the above map to the dg Lie subalgebra $\mathsf{dGC}_{c+d+1}$ (we denote this restriction by the same symbol).

The dg prop $\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{c,d}$ (or $\widehat{\mathcal{H}olieb}^{\star}_{c,d}$) has two obvious one-parameter rescaling automorphisms given on the generators as follows,

$$
R'_\lambda : \overbrace{\underbrace{\times}_{n\times}}^{m\times} \longrightarrow \lambda^{n-1} \overbrace{\underbrace{\times}_{n\times}}^{m\times} \quad , R''_\mu : \overbrace{\underbrace{\times}_{n\times}}^{m\times} \longrightarrow \mu^{m-1} \overbrace{\underbrace{\times}_{n\times}}^{m\times}, \quad \forall \lambda,\mu\in\mathbb{K}\setminus 0, \quad \forall m,n\geq 0.
$$

The associated derivations give us the following two cycles in the derivation complex $\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowleft}_{c,d})_{conn}$

$$
r' = \frac{dR'_\lambda}{d\lambda}\,|_{\lambda=1} = \sum_{m,n\geq 0} (n-1) \overbrace{\underbrace{\times}_{n\times}}^{m\times}, \qquad r'' = \frac{dR''_\mu}{d\mu}\,|_{\mu=1} = \sum_{m,n\geq 0} (m-1) \overbrace{\underbrace{\times}_{n\times}}^{m\times},
$$

Their difference is a coboundary (see (11)) but otherwise each of them (or their sum as in [MW1]) stands for a non-trivial cohomology class[7] in $H^\bullet(\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowleft}_{c,d})_{conn})$ which does *not* belong to the image of the map $F^\star$ above. In fact, this is essentially the only class in $H^\bullet(\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowleft}_{c,d})_{conn})$ which does not come from some class in $H^\bullet(\mathsf{dGC}_{c+d+1})$.

**4.1.1. Theorem.** *For any $c,d\in\mathbb{Z}$ the morphism of dg Lie algebras*

$$
F^\star : \mathsf{dGC}_{c+d+1} \longrightarrow \mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowleft}_{c,d})_{conn} \tag{17}
$$

*is injective on cohomology. The quotient space $H^\bullet(\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowleft}_{c,d})_{conn})/F^\star(H^\bullet(\mathsf{dGC}_{c+d+1}))$ is 1-dimensional and is spanned by the rescaling class $r'$.*

*Proof.* For a graph $\Gamma$ in $\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowleft}_{c,d})$ let $V^{\lhd 2}(\Gamma)\subset V(\Gamma)$ be the subset of univalent vertices and passing vertices, and let $V^{\unrhd 2}(\Gamma)$ be its complement, i.e. the subset of non-passing vertices of valency $\geq 2$ of $\Gamma$. Consider the following filtration of the complex $\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowleft}_{c,d})$,

$$
F_{-p}(\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowleft}_{c,d})) := \text{linear span of graphs } \Gamma \text{ with } \#V^{\unrhd 2}(\Gamma)\geq p. \tag{18}
$$

For a graph $\Gamma\in\mathsf{dGC}_{c+d+1}$ one has $V(\Gamma)=V^{\unrhd 2}(\Gamma)$ so that an analogous filtration of the l.h.s. in (15) takes the form

$$
F_{-p}(\mathrm{Der}(\mathsf{dGC}_{c+d+1})) := \text{linear span of graphs } \Gamma \text{ with } \#V(\Gamma)\geq p. \tag{19}
$$

The morphism (15) respects these filtrations and hence induces the morphism of the associated graded complexes (all denoted by the same letters),

$$
\mathcal{F}^\star : (\mathsf{dcGC}_{c+d+1}, 0) \to (\mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowleft}_{c,d}), \hat{d}) \tag{20}
$$

[7] In the context of this paper in which we w rok later with an *operad* of curved $\mathcal{A}ss_\infty$ algebras, it is suitable represent this special *rescaling cohomology* class by the cycle $r'$ rather than by the sum $r'+r''$ as in [MW1].

where the induced differential in the l.h.s. is trivial while the induced differential in the r.h.s. is given by

$$\hat{d}\Gamma = \hat{\delta}\Gamma \ \ \pm \sum_{\substack{\text{in-legs}\\ \text{of }\Gamma}} \left( \begin{array}{c}\Gamma\\ \downarrow\\ \bullet\end{array} \pm \begin{array}{c}\Gamma\\ |\\ \bullet\\ |\end{array} \right) \pm \sum_{\substack{\text{out-legs}\\ \text{of }\Gamma}} \left( \begin{array}{c}\bullet\\ \uparrow\\ \Gamma\end{array} \pm \begin{array}{c}|\\ \bullet\\ |\\ \Gamma\end{array} \right)$$

where

$$\hat{\delta}\Gamma = \delta^{\star}\Gamma \text{ mod terms creating new } (m,n)\text{-corollas with } m \geq 2 \text{ or } n \geq 2.$$

Let us call ingoing or outgoing legs (if any) of graphs from (9) *hairs* and consider the following complete, exhaustive and bounded above filtration of both sides of the arrow in (20)
(21)

$$F'_{-p}(\mathrm{Der}(\mathcal{H}olieb_{c,d}^{\star\circlearrowleft})) := \text{span of graphs with \#hairs + \#univalent sources + \#univalent targets } \geq p.$$

and

$$F'_{-p}(\mathsf{dGC}_{c+d+1}) := \left\{ \begin{array}{cc} \mathsf{dGC}_{c+d+1} & \text{for } p \leq 0 \\ 0 & \text{for } p \geq 1 \end{array} \right.$$

Note that the unique graph $\bullet$ consisting of the zero valent vertex is counted twice — once as a source and once as a target — so that $\bullet$ belongs to $F'_{-2}(\mathrm{Der}(\mathcal{H}olieb_{c,d}^{\star\circlearrowleft}))$; similarly, the derivation $\uparrow$ (see §3.1) is assumed by definition to have two hairs and hence also belongs to $F'_{-2}(\mathrm{Der}(\mathcal{H}olieb_{c,d}^{\star\circlearrowleft}))$. The map $\mathcal{F}^{\star}$ respects both filtrations and hence induces a morphism (denoted by the same letter again) of the associated graded complexes

$$\mathcal{F}^{\star} : (\mathsf{dcGC}_{c+d+1}, 0) \to (\mathrm{Der}(\mathcal{H}olieb_{c,d}^{\star\circlearrowleft}), d_0) =: (C, d_0) \tag{22}$$

where the induced differential $d_0$ is given on two exceptional graphs by

$$d_0 \bullet = \begin{array}{c}\bullet\\ \uparrow\\ \bullet\end{array} \ , \quad d_0 \uparrow = \begin{array}{c}\uparrow\\ \bullet\end{array} - \begin{array}{c}\bullet\\ \uparrow\end{array}$$

and on all other graphs by the formula

$$d_0\Gamma = \delta^{+}\Gamma \ \ \pm \sum_{\substack{\text{in-legs}\\ \text{of }\Gamma}} \left( \begin{array}{c}\Gamma\\ \downarrow\\ \bullet\end{array} \pm \begin{array}{c}\Gamma\\ |\\ \bullet\\ |\end{array} \right) \pm \sum_{\substack{\text{out-legs}\\ \text{of }\Gamma}} \left( \begin{array}{c}\bullet\\ \uparrow\\ \Gamma\end{array} \pm \begin{array}{c}|\\ \bullet\\ |\\ \Gamma\end{array} \right) \tag{23}$$

where

$$\delta^{+}\Gamma \quad := \quad \hat{\delta}\Gamma \text{ modulo term creating new univalent vertices.}$$

By analogy to the proof of Proposition **2.3.1**, let us call the univalent vertices and passing bivalent vertices of graphs $\Gamma$ from $\mathrm{Der}(\mathcal{H}olieb_{c,d}^{\star\circlearrowleft})$ *stringy*; the maximal connected subgraphs (if any) of a graph $\Gamma$ consisting of stringy vertices with at least one univalent vertex or with at least one hair are called *strings*. Let us call the non-passing vertices of valency $\geq 2$ which do not belong to the strings (if any) the *core vertices*, and let $\Gamma^{core}$ be the full subgraph of $\Gamma$ spanned by the core vertices; in principle any graph from the set of generators of $\mathsf{dcGC}_{c+d+1}$ can occur as a core graph $\Gamma^{core}$ of some graph $\Gamma \in \mathrm{Der}(\mathcal{H}olieb_{c,d}^{\star\circlearrowleft})$. A string is a subgraph (if any) of $\Gamma$ of one of the following eight types (we classify the unique graph in $\mathrm{Der}(\mathcal{H}olieb_{c,d}^{\star\circlearrowleft})$ consisting consisting of the zero valency vertex $\bullet$ as well as the graph with no vertices $\uparrow$ as *strings* as well — they correspond to the element $\alpha_1^\bullet$ and $\alpha_0^{\uparrow\downarrow}$ listed below),

$$\alpha_n^\bullet \simeq \quad \bullet\!\longrightarrow\!\bullet\!\longrightarrow\!\bullet\!\longrightarrow \cdots \longrightarrow\!\bullet\!\longrightarrow\!\bullet\!\longrightarrow\!\bullet \quad n \geq 1 \text{ stringy vertices}$$

$$\alpha_n^\uparrow \simeq \quad \bullet\!\longrightarrow\!\bullet\!\longrightarrow\!\bullet\!\longrightarrow \cdots \longrightarrow\!\bullet\!\longrightarrow\!\bullet\!\longrightarrow \quad n \geq 1 \text{ stringy vertices}$$

$$\alpha_n^\downarrow \simeq \quad \longrightarrow\!\bullet\!\longrightarrow\!\bullet\!\longrightarrow \cdots \longrightarrow\!\bullet\!\longrightarrow\!\bullet\!\longrightarrow\!\bullet \quad n \geq 1 \text{ stringy vertices}$$

$$\alpha_n^{\uparrow\downarrow} \simeq \quad \longrightarrow\!\bullet\!\longrightarrow\!\bullet\!\longrightarrow \cdots \longrightarrow\!\bullet\!\longrightarrow\!\bullet\!\longrightarrow \quad n \geq 0 \text{ stringy vertices}$$

$$\beta_n^{\bullet,\uparrow} \simeq \quad ⓥ\!\longrightarrow\!\bullet\!\longrightarrow \cdots \longrightarrow\!\bullet\!\longrightarrow\!\bullet \quad n \geq 1 \text{ stringy vertices (shown as black bullets)}$$

$$\beta_n^{\uparrow} \simeq \quad ⓥ\!\longrightarrow\!\bullet\!\longrightarrow \cdots \longrightarrow\!\bullet\!\longrightarrow \quad n \geq 0 \text{ stringy vertices}$$

$$\beta_n^{\bullet,\downarrow} \simeq \quad ⓥ\!\longleftarrow\!\bullet\!\longleftarrow \cdots \longleftarrow\!\bullet\!\longleftarrow\!\bullet \quad n \geq 1 \text{ stringy vertices}$$

$$\beta_n^{\downarrow} \simeq \quad ⓥ\!\longleftarrow\!\bullet\!\longleftarrow \cdots \longleftarrow\!\bullet\!\longleftarrow \quad n \geq 0 \text{ stringy vertices}$$

$$\gamma_n \simeq \quad ⓥ\!\longrightarrow\!\bullet\!\longrightarrow \cdots \longrightarrow\!\bullet\!\longrightarrow ⓦ \quad n \geq 1 \text{ passing vertices (shown as black bullets)}$$

where $v$ and $w$ stand for any pair of (not necessary distinct) arbitrary *core* vertices. Note that $\beta_0^{\bullet,\uparrow} \equiv \beta_0^{\bullet,\downarrow}$ stand for one and the same element — a core vertex $v$ with no strings attached.

The associated graded complex $C$ in the r.h.s. of (22) splits into a direct sum

$$C = C_{\text{empty core}} \quad \oplus \quad C_{\text{non-empty core}} \tag{24}$$

where the first (resp., second) summand is spanned by graphs $\Gamma$ with the set $V(\Gamma^{core})$ empty (resp., non-empty). Thus

$$C_{\text{empty core}} := \text{span}\left\langle \alpha_n^\bullet\ , \alpha_n^\uparrow\ , \alpha_n^\downarrow\ , \alpha_n^{\uparrow\downarrow}\ , \uparrow \right\rangle_{n\geq 1}$$

with the induced differential $d_0$ given on the generators by

$$d_0\alpha_n^\bullet = \begin{cases} 0 & \text{if } n \text{ is even} \\ \pm\alpha_{n+1}^\bullet & \text{if } n \text{ is odd} \end{cases}, \quad d_0\alpha_n^{\uparrow\downarrow} = \begin{cases} \pm\alpha_{n+1}^\uparrow \pm \alpha_{n+1}^\downarrow \pm \alpha_{n+1}^{\uparrow\downarrow} & \text{if } n \text{ is odd} \\ \pm\alpha_{n+1}^\uparrow \pm \alpha_{n+1}^\downarrow & \text{if } n \text{ is even} \end{cases}$$

$$d_0\alpha_n^\uparrow = \begin{cases} \pm\alpha_{n+1}^\bullet & \text{if } n \text{ is odd} \\ \pm\alpha_{n+1}^\bullet \pm \alpha_{n+1}^\uparrow & \text{if } n \text{ is even} \end{cases}, \quad d_0\alpha_n^\downarrow = \begin{cases} \pm\alpha_{n+1}^\bullet & \text{if } n \text{ is odd} \\ \pm\alpha_{n+1}^\bullet \pm \alpha_{n+1}^\downarrow & \text{if } n \text{ is even} \end{cases}$$

It is easy to see that the cohomology of this complex is one-dimensional and can be understood as spanned by the cycle $\bullet + \underset{\bullet}{\uparrow}$ (representing the derivation $r'$) or by the cycle $\bullet + \overset{\bullet}{\uparrow}$ (representing the derivation $r''$) or by their sum

$$\left(\bullet + \underset{\bullet}{\uparrow}\right) + \left(\bullet + \overset{\bullet}{\uparrow}\right) = 2\bullet + \underset{\bullet}{\uparrow} + \overset{\bullet}{\uparrow} \tag{25}$$

The difference of the first two cycles is a coboundary as $d_0 \uparrow = \underset{\bullet}{\uparrow} - \overset{\bullet}{\uparrow}$ ).

Consider next the second complex $(C_{\text{non-empty core}}, d_0)$. It decomposes into the completed direct sum (parameterized by arbitrary graphs $\Gamma^{core}$ from $\mathsf{dcGC}_{c+d+1}$) of the tensor products of complexes

$$C_{\text{non-empty core}} \simeq \prod_{\Gamma^{core}} C_{\Gamma^{core}}, \qquad C_{\Gamma^{core}} := \bigotimes_{v\in V(\Gamma^{core})} X_v \bigotimes_{e\in E(\Gamma^{core})} X_e$$

where

- for each edge $X_e := \mathbb{K}[0] \oplus \text{span}\langle\gamma_n\rangle_{n\geq 1}$ with differential given on generators by $d(1 \in \mathbb{K}[0]) = 0$ and $d\gamma_n = \pm\gamma_{n+1}$ so that $H^\bullet(X_e) = \mathbb{K}[0]$; hence the factors $X_e$ can be ignored in the above formula for the complex $C_{\Gamma^{core}}$;
- the complexes $X_v$ can be different for different core vertices $v$ but their classification is rather simple and is discussed next.

For each core vertex $v$ consider two complexes,

$$C_v^\uparrow := \text{span}\left\langle \beta_n^{\bullet,\uparrow}\ , \beta_m^\uparrow\ , \right\rangle_{n\geq 1, m\geq 0}, \qquad C_v^\downarrow := \text{span}\left\langle \beta_n^{\bullet,\downarrow}\ ,\ \beta_m^\downarrow\ \right\rangle_{n\geq 1, m\geq 0}$$

equipped with the differentials given by

$$d_v\beta_n^{\bullet,\uparrow} = \begin{cases} \pm\beta_{n+1}^{\bullet,\uparrow} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}, \quad d_v\beta_n^{\uparrow} = \begin{cases} \pm\beta_{n+1}^{\bullet,\uparrow} & \text{if } n \text{ is even} \\ \pm\beta_{n+1}^{\bullet,\uparrow} \pm \beta_{n+1}^{\uparrow} & \text{if } n \text{ is odd} \end{cases}$$

$$d_v\beta_n^{\bullet,\downarrow} = \begin{cases} \pm\beta_{n+1}^{\bullet,\downarrow} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}, \quad d_v\beta_n^{\downarrow} = \begin{cases} \pm\beta_{n+1}^{\bullet,\downarrow} & \text{if } n \text{ is even} \\ \pm\beta_{n+1}^{\bullet,} \pm \beta_{n+1}^{\downarrow} & \text{if } n \text{ is odd} \end{cases}$$

It is easy that that both complexes $C_v^{\uparrow}$ and $C_v^{\downarrow}$ are acyclic. Indeed, consider a filtration of, say, the complex $C_v^{\uparrow}$ by the number of vertices of the form $\beta_n^{\bullet,\uparrow}$; the cohomology of the associated graded complex is 2-dimensional and is spanned by $\beta_1^{\bullet,\uparrow}$ and $\beta_0^{\uparrow}$ with the induced differential given on the generators by the isomorphism $\beta_0^{\uparrow} \to \beta_1^{\bullet,\uparrow}$.

Next we have to consider several types of non-empty core graphs.

Case 1: the case $\Gamma^{core} = \bullet$, the single vertex without any edges. In this case

$$C_{\Gamma^{core}} = \prod_{p+q\geq 3} \odot^p C_v^{\uparrow} \otimes \odot^q C_v^{\downarrow} \quad \oplus \quad \odot^2 C_v^{\uparrow} \quad \oplus \quad \odot^2 C_v^{\downarrow}.$$

Due to the acyclicity of the complexes $C_v^{\downarrow}$ and $C_v^{\uparrow}$ and exactness of the (symmetric) tensor product functor, we conclude that $H^\bullet(C_{\Gamma^{core}}) = 0$ in this case.

Case 2: the core graph $\Gamma^{core}$ contains at least one vertex $v$ which either has valency one or is a passing[8] vertex. Then $C_{\Gamma^{core}}$ has the following tensor factor

$$X_v = \begin{cases} \prod_{p+q\geq 2} \odot^p C_v^{\uparrow} \otimes \odot^q C_v^{\downarrow} \ \oplus \ C_v^{\uparrow} & \text{if } |v| = |v|_{out} = 1 \\ \prod_{p+q\geq 2} \odot^p C_v^{\uparrow} \otimes \odot^q C_v^{\downarrow} \ \oplus \ C_v^{\downarrow} & \text{if } |v| = |v|_{in} = 1 \\ \prod_{p+q\geq 1} \odot^p C_v^{\uparrow} \otimes \odot^q C_v^{\downarrow} & \text{if } |v| = 2, |v|_{in} = |v|_{out} = 1 \end{cases}$$

which is in all cases acyclic, $H^\bullet(X_v) = 0$, so that $H^\bullet(C_{\Gamma^{core}}) = 0$.

Thus we conclude that only generators $\Gamma^{core}$ of the subspace $\mathsf{dGC}_{c+d+1}$ can contribute to $H^\bullet(C_{\text{non-empty core}})$.

Case 3: Consider finally the case when $\Gamma^{core}$ is a generator of $\mathsf{dGC}_{c+d+1}$. Then

$$C_{\Gamma^{core}} := \bigotimes_{v\in V(\Gamma^{core})} X_v \bigotimes_{e\in E(\Gamma^{core})} X_e \quad \text{with} \quad X_v = \prod_{p,q\geq 0} \odot^p C_v^{\uparrow} \otimes \odot^q C_v^{\downarrow} \quad \text{and} \quad X_e := \mathbb{K}[0] \oplus \mathrm{span}\,\langle \gamma_n \rangle_{n\geq 1}$$

We conclude that for each $v \in V(\Gamma^{core})$ (resp., each edge $e \in E(\Gamma^{core})$) the associated cohomology group $H^\bullet(X_v)$ (resp., $H^\bullet(X_e)$) is concentrated in degree zero and is equal to $\mathbb{K}$ so that $H^\bullet(C_{\Gamma^{core}}) = \mathrm{span}\,\langle\Gamma^{core}\rangle$ and hence

$$H^\bullet(C_{\text{non-empty core}}) \simeq \mathsf{dGC}_{c+d+1}.$$

Moreover, this isomorphism holds true at the level complexes when turning the page of our spectral sequence. By the spectral sequences comparison theorem we conclude that the map of the original complexes (17) is a quasi-isomorphism up to one rescaling class $r^\star$. □

**4.1.2. Remarks.** (i) In terms of representations of the prop $\mathcal{H}olieb_{0,1}^{\star\circlearrowleft}$, that is, in terms of formal Poisson structures, the rescaling class $r' + r''$ corresponds to the following universal automorphism

$$\pi = \sum_{m,n\geq 0} \pi_n^m \longrightarrow \pi^{new} = \sum_{m,n\geq 0} \lambda^{m+n-2}\pi_n^m, \quad \forall\, \lambda \in \mathbb{K}^\star,$$

of the set of formal (finite- or infinite-dimensional) Poisson structures.

(ii) In terms of the extended graph complexes (14) the above result can be re-written as a quasi-isomorphism of dg Lie algebras

$$\mathsf{dGC}_{c+d+1} \oplus \mathbb{K} \longrightarrow \mathrm{Der}(\mathcal{H}olieb_{c,d}^{\star\circlearrowleft})_{conn}$$

where the generator $\emptyset$ of $\mathbb{K}$ is mapped into the rescaling class. Note that the l.h.s. is *not* a direct sum of Lie algebras, only of graded vector spaces.

[8] Note that a passing vertex in $\Gamma^{core}$ can *not* be passing in $\Gamma$.

(iii) Composing the quasi-isomorphism (13) with the quasi-isomorphism (15) and using equalities (3) and (12) we obtain a canonical quasi-isomorphism of dg Lie algebras

$$F^{\circlearrowright} : \mathsf{fGC}^{\geq 2}_{c+d+1} \longrightarrow \mathrm{Der}(\mathcal{H}olieb^{\star\circlearrowright}_{c,d})$$

and hence prove the first part of Proposition 1.1.1.

Similarly one can study the deformation theory of the ordinary (non-wheeled) properad $\mathcal{H}olieb^{\star}_{c,d}$ and obtain the following result.

**4.1.3. Proposition.** *There is a quasi-isomorphism of dg Lie algebras*

$$F : \mathsf{fGC}^{or}_{c+d+1} \longrightarrow \mathrm{Der}(\mathcal{H}olieb^{\star}_{c,d})$$

We skip the details which are identical to the arguments used in the proof of Theorem 4.1.1.

## 5. Classification of universal quantizations of Poisson structures

**5.1. Polydifferential functor on wheeled props.** There is a polydifferential functor[9] [MW2]

$$\mathcal{O} : \text{Category of dg props} \longrightarrow \text{Category of dg operads}$$

which verbatim extends (on the l.h.s.) to the category of dg wheeled props and has the property that for any dg (wheeled) prop $\mathcal{P}$ and its any representation, $\rho : \mathcal{P} \to \mathcal{E}nd_V$, in a dg vector space $V$ the associated dg operad $\mathcal{O}(\mathcal{P})$ has an associated representation, $\mathcal{O}(\rho) : \mathcal{O}(\mathcal{P}) \to \mathcal{E}nd_{\widehat{\odot^\bullet} V}$, in the (completed) graded commutative algebra $\widehat{\odot^\bullet} V$ given in terms of polydifferential (with respect to the standard multiplication in $\widehat{\odot^\bullet} V$) operators. We refer to §5.2 of [MW2] for full details and explain here only the explicit structure of the dg operad

$$\mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowright}_{0,1}) = \left\{ \mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowright}_{0,1})(k) \right\}_{k\geq 0}$$

Each element $a$ in the $\mathbb{S}_k$-module $\mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowright}_{0,1})(k)$ is a linear combination,

$$\lambda_1 \hat{e}_1 + \ldots + \lambda_N \hat{e}_N, \quad \lambda_1, \ldots, \lambda_N \in \mathbb{K},$$

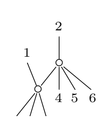
where each generator, say, $\hat{e}_s \in \mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowright}_{0,1})(k)$, $s \in [N]$, is constructed from some graph $e_s \in \widehat{\mathcal{H}olieb}^{\star\circlearrowright}_{0,1}(m_s, n_s)$ as follows:

(i) draw new $k$ big white vertices labelled from 1 to k (the “inputs” of $\hat{e}_s$) and one extra output big white vertex,
(ii) symmetrize all $m_s$ outputs legs of $e_s$ (if $m_s \geq 1$) and attach them to the unique output white vertex; if $m_s = 0$, the output big white vertex receives no incoming edges;
(iii) partition the set $[n_s]$ if input legs of $e_s$ into $k$ ordered disjoint subsets

$$[n_s] = I_1 \sqcup \ldots \sqcup I_k, \qquad \#I_i \geq 0, i \in [k],$$

and then symmetrize the legs in each subset $I_i$ and attach them (if any) to the $i$-labelled input white vertex.

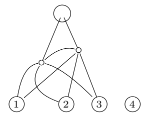
For example, the element

$e =$ 

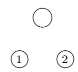
$\in \widehat{\mathcal{H}olieb}^{\star\circlearrowright}_{0,1}(2,6)$

[9]In fact in the subsequent paper [MW3] the functor $\mathcal{O}$ was further extended to a *polydifferential endofunctor* $\mathcal{D}$ in the category of (wheeled) props such that $\mathcal{O}$ is an operadic part of $\mathcal{D}$.

can generate the following element


$\in \mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1})(4)$

in the associated polydifferential operad. If we erase the top big white vertex and its all attached edges, then we get from elements of $\mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1})$ precisely M. Kontsevich graphs from [K2]. The operad $\mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1})$ admits a filtration by the number of small white vertices (that is, by the number of vertices coming from the underlying generators of $\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1}$) which we call from now on *internal vertices*. The big white vertices of graphs from $\mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1})$ are called the *external* ones. Note that incoming external vertices are *not* ordered from left to right as one might infer from the pictures above — they are only labelled by distinct integers. Note also that elements of $\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1}$ may contain elements with no internal vertices at all, for example,


$\in \widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1}(2).$

The latter graph admits an automorphism which swaps numerical labels of vertices (cf. [MW2, MW3]) and controls the canonical graded commutative multiplication in $\widehat{\odot^\bullet} V$. For any $i \in [n]$ the operadic composition

$$\begin{array}{rccc} \circ_i: & \mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1})(n)\otimes \mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1})(m) & \longrightarrow & \mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1})(m+n-1) \\ & \Gamma_1\otimes\Gamma_2 & \longrightarrow & \Gamma_1\circ_i\Gamma_2 \end{array}$$

is defined by substituting the graph $\Gamma_2$ (with the output external vertex erased so that all edges, if any, connected to that external vertex become "dangling in the air")) inside the big circle of the $i$-labelled external vertex of $\Gamma_1$ and erasing that big circle (so that all edges of $\Gamma_1$ connected to the $i$-th external vertex, if any, also become "dangling in the air"), and then taking the sum over all possible ways to do the following operations

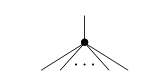
(i) glueing some (or all or none) hanging edges of $\Gamma_2$ to some hanging edges of $\Gamma_1$,
(ii) attaching some (or all or none) hanging edges of $\Gamma_2$ to the output external vertex of $\Gamma_1$,
(iii) attaching some (or all or none) hanging edges of $\Gamma_1$ to the external input vertices of $\Gamma_2$,

in such a way that no dangling edges are left. We refer to [MW2, MW3] for concrete examples.

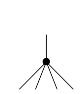
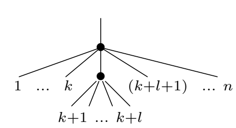
**5.2**. **Kontsevich formality map as a morphism of dg operads.** M. Kontsevich formality map from [K2] provides us with a universal quantization of arbitrary (formal) graded Poisson structures. It can understood as a morphism of dg operads[10],

$$\mathcal{F}: c\mathcal{A}ss_\infty \longrightarrow \mathcal{O}(\mathcal{H}olieb^{\star\circlearrowleft}_{0,1}) \tag{26}$$

satisfying a certain non-triviality condition (which is given explicitly below). Here $c\mathcal{A}ss_\infty$ is a dg operad of *curved* $A_\infty$*-algebras* defined as the free operad generated by the $\mathbb{S}$-module

$E(n) := \mathbb{K}[\mathbb{S}_n][n-2] = \text{span}\Big($  $\Big)_{\sigma\in\mathbb{S}_n}, \quad \forall\, n\geq 0$

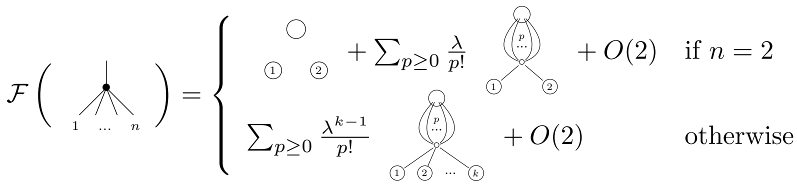
and equipped with the differential given on the generators by the formula

$\delta$  $= \sum_{k=0}^{n}\sum_{l=0}^{n-k}(-1)^{k+l(n-k-l)+1}$  .

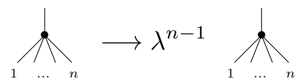
It is non-cofibrant and acyclic (as the dg operad $\mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1})$).

[10]Similarly, a universal formality map behind Drinfeld's deformation quantizations of Lie bialgebras can be understood as a morphism of dg props, see [MW3].

The non-triviality condition on the map (26) reads as the following approximation on the values of $\mathcal{F}$ on the generating $n$ corollas of $c\mathcal{A}ss_\infty$ for any $n \geq 0$ (modulo graphs with the number of internal vertices $\geq 2$ whose linear span is denoted below by $O(2)$)

(27) 

where $\lambda$ is any non-zero number and the summations $\sum_{p\geq 0}$ run over the number of edges connecting the internal vertex to the external out-vertex. A morphism of dg operads (26) satisfying the above non-triviality condition is called *a Kontsevich formality map* (after [K2]). Using the standard rescaling automorphism of the operad $c\mathcal{A}ss_\infty$,



one can always adjust a formality map in such a way that the boundary condition holds for $\lambda = 1$.

**5.3. Deformation complexes of morphisms of props.** Let $\mathcal{P}$ be an arbitrary dg free prop, and $\mathcal{Q}$ an arbitrary dg prop, and $f : \mathcal{P} \to \mathcal{Q}$ a morphism between them. Then there is a standard construction of the *deformation complex* $\mathsf{Def}(\mathcal{P} \stackrel{f}{\to} \mathcal{Q})$ of the morphism $f$ described in several ways in [MV]; in general, $\mathsf{Def}(\mathcal{P} \stackrel{f}{\to} \mathcal{Q})$ is a filtered $\mathcal{L}ie_\infty$ algebra. This construction builds on earlier works which describe deformation complexes of morphisms of dg *operads* [KS, VdL]. The constructions in [MV] generalize straightforwardly to the case when $\mathcal{P}$ and $\mathcal{Q}$ are dg *wheeled* props. For example, when $\mathcal{P} = \mathcal{Q} = \widehat{\mathcal{H}olieb}_{c,d}^{\star\circlearrowleft}$ and $f$ is the identity map, then the associated deformation complex (constructed following the standard procedure developed in [MV])

$$\mathsf{Def}(\widehat{\mathcal{H}olieb}_{c,d}^{\star\circlearrowleft} \stackrel{\mathrm{Id}}{\longrightarrow} \widehat{\mathcal{H}olieb}_{c,d}^{\star\circlearrowleft})[1] \simeq \mathrm{Der}(\mathcal{H}olieb_{c,d}^{\star\circlearrowleft})$$

is, up to the degree shift, is precisely the derivation complex of $\widehat{\mathcal{H}olieb}_{c,d}^{\star\circlearrowleft}$ (but the Lie algebra structure is different!). The machinery of [KS, MV, VdL] gives us a well-defined dg Lie algebra

$$\mathsf{Def}\left(c\mathcal{A}ss_\infty \stackrel{\mathcal{F}}{\longrightarrow} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})\right)$$

which controls the deformation theory of any formality map $\mathcal{F}$. Our second main result in this paper is the computation of its cohomology in terms of the M. Kontsevich graph complex $\mathsf{fGC}_2^{\geq 2}$.

**5.4. Theorem (Classification of formality maps).** *For any Kontsevich formality morphism* $\mathcal{F}$

$$\mathcal{F} : c\mathcal{A}ss_\infty \longrightarrow \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})$$

*there is a canonically associated morphism of complexes*

$$f_{\mathcal{F}} : \mathsf{fGC}_2^{\geq 2} \longrightarrow \mathsf{Def}\left(c\mathcal{A}ss_\infty \stackrel{\mathcal{F}}{\longrightarrow} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})\right)[1]$$

*which is a quasi-isomorphism.*

*Proof.* The proof of this Theorem is very similar to the proof of Proposition 5.4.1 in [MW3] and is based essentially on the contractibility of the permutahedra polytopes. Let us first explain the naturality of the morphism $f_{\mathcal{F}}$. Any derivation of the dg wheeled prop $\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft}$, that is, any deformation $D$ of the identity automorphism of $\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft}$,

$$D \in \mathsf{Def}\left(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft} \stackrel{\mathrm{Id}}{\longrightarrow} \widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft}\right)$$

induces an associated deformation of the identity automorphism of $\mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})$,

$$D \in \mathsf{Def}\left(\mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft}) \stackrel{\mathrm{Id}}{\longrightarrow} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})\right)$$

and hence, via the composition of $D$ with the given map $\mathcal{F}$, gives us a canonical morphism of complexes

$$g_{\mathcal{F}}: \mathsf{Def}\left(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft} \xrightarrow{\mathrm{Id}} \widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft}\right) \longrightarrow \mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})\right) \tag{28}$$

or, equivalently,

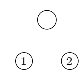
$$g_{\mathcal{F}}: \mathrm{Der}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft}) \longrightarrow \mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})\right)[1] \tag{29}$$

Composing this map $g_{\mathcal{F}}$ with the canonical quasi-isomorphism $F$ from Proposition 1.1.1, we obtain the required map $f_{\mathcal{F}}$. Thus to prove the theorem it is enough to prove that the map $g_{\mathcal{F}}$ is a quasi-isomorphism. Which is easy.

Both complexes in (29) admits filtrations by the number of edges in the graphs, and the map $g_{\mathcal{F}}$ preserves these filtrations, and hence induces a morphism of the associated spectral sequences,

$$g^r_{\mathcal{F}}: (\mathcal{E}_r\mathrm{Der}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft}), d_r) \longrightarrow \left(\mathcal{E}_r\mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})\right)[1], \delta_r\right).$$

The induced differential $d_0$ on the initial page of the spectral sequence of the l.h.s. is trivial, $d_0 = 0$. The induced differential on the initial page of the spectral sequence of the r.h.s. is not trivial and is determined by the following summand in $\mathcal{F}$ (see (27)),



Hence the differential $\delta_0$ acts only on big input white vertices of graphs from $\mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})\right)[1]$ by splitting each such big white vertex ⓥ into two big white vertices $(v')$ $(v'')$ and redistributing all edges (if any) attached to $v$ in all possible ways among the new vertices $v'$ and $v''$. The cohomology

$$\mathcal{E}_1\mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})\right)[1] = H\left(\mathcal{E}_0\mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})\right)[1], \delta_0\right)$$

is spanned by graphs all of whose white vertices are precisely univalent and skew symmetrized (see, e.g., Theorem 3.2.4 in [Me4] where this result is obtained from the cell complexes of permutahedra, or Appendix A in [W1] for another purely algebraic argument) and hence is isomorphic (after erasing these no more needed big white vertices) to $\mathrm{Der}(\mathcal{H}olieb_{0,1}^{\star\circlearrowleft})$ as a graded vector space. The boundary condition (27) says that the induced differential $\delta_1$ in the complex $\mathcal{E}_1\mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})\right)[1]$ agrees precisely with the induced differential $d_1$ in $\mathcal{E}_1\mathrm{Der}(\mathcal{H}olieb_{0,1}^{\star\circlearrowleft})$ so that the induced morphism of the next pages of the spectral sequences,

$$g^1_{\mathcal{F}}: (\mathcal{E}_1\mathrm{Der}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft}), d_1) \longrightarrow \left(\mathcal{E}_1\mathsf{Def}(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft}), \delta_1\right)$$

is an isomorphism. By the spectral sequence comparison theorem, the morphism $g_{\mathcal{F}}$ is a quasi-isomorphism. □

We conclude that for any $i \in \mathbb{Z}$,

$$H^{i+1}\left(\mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})\right)\right) = H^i(\mathsf{fGC}_2^{\geq 2})$$

The special case $i = 0$ reads as

$$H^1\left(\mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})\right)\right) = H^0(\mathsf{fGC}_2^{\geq 2}) = \mathfrak{grt}$$

To complete a new proof of the main result of the remarkable paper [Do] by V. Dolgushev we have to show that every infinitesimal deformation $\zeta \in H^1\left(\mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})\right)\right)$ of any given Kontsevich formality map $\mathcal{F}$ exponentiates to a genuine Kontsevich formality map $\mathcal{F}^\zeta$, and then apply Lemma 3 from Thomas Willwacher paper [W3]. In principle, there could be obstructions belonging to the space

$$H^2\left(\mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowleft})\right)\right) \simeq H^2(\mathsf{fGC}_2^{\geq 2})$$

which is not yet proven to be zero (this is a folklore conjecture), but *not* in our case thanks to two different Lie algebra structures on the underlying cohomology space — one is in degree zero and induced from the Lie brackets on $\mathsf{Def}\left(c\mathcal{A}ss_\infty \stackrel{\mathcal{F}}{\to} \mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1})\right)$ and another in degree $-1$ and induced from the quasi-isomorphic (as a vector space only!) Lie brackets on $\mathrm{Der}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1})$. Indeed, the dg Lie algebra $\mathrm{Der}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1})$ is positively graded so that any degree zero cycle $\zeta$ in $\mathrm{Der}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1})$ can be exponentiated to a genuine automorphism $e^\zeta$ of the dg prop $\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1}$ which in turns gives us a genuine Maurer-Cartan element of another dg Lie algebra $\mathsf{Def}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{c,d} \xrightarrow{\mathrm{Id}} \widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{c,d})$. The map (28), being a morphism of dg Lie algebras, gives us finally the required formality morphism $\mathcal{F}^\zeta := g_{\mathcal{F}}(e^\zeta)$.

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