Krichever–Novikov type algebras. A general review and the genus zero case

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Abstract In the first part of this survey we recall the definition and some of the constructions related to Krichever–Novikov type algebras. Krichever and Novikov introduced them for higher genus Riemann surfaces with two marked points in generalization of the classical algebras of Conformal Field Theory. Schlichenmaier extended the theory to the multi-point situation and even to a larger class of algebras. The almost-gradedness of the algebras and the classification of almost-graded central extensions play an important role in the theory and in applications. In the second part we specialize the construction to the genus zero multi-point case. This yields beside instructive examples also additional results. In particular, we construct universal central extensions for the involved algebras, which are vector field algebras, differential operator algebras, current algebras and Lie superalgebras. We point out that the recently (re-)discussed N-Virasoro algebras are nothing else as multi-point genus zero Krichever-Novikov type algebras. The survey closes with structure equations and central extensions for the three-point case.

1 Introduction

Krichever-Novikov (KN) type algebras constitutes an important class of infinite dimensional algebras. Roughly speaking, they are defined as algebras of meromorphic objects on compact Riemann surfaces, or equivalently on projective curves. The non-holomorphicity is controlled by a fixed finite set of points where algebraic poles are allowed. A splitting of this set of possible points of poles into two disjoint subsets will induce an "almost-grading" (see Definition 5.1 below). It is a weaker concept as a grading, but still powerful enough to act as a basic tool in representation theory. For example, highest weight representations still can be defined. Of course, central

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extensions of these algebras are also needed. They are forced, e.g. by representation theory and by quantization.

Examples of KN type algebras are the well-known algebras of Conformal Field Theory (CFT) [4], [22] the Witt algebra, the Virasoro algebra, the affine Lie algebras (affine Kac-Moody algebras), etc. They appear when the geometric setting consists of the Riemann Sphere, i.e. the genus zero Riemann surface, and the points of possible poles are $\{0\}$ and $\{\infty\}$. The almost-grading is now a honest grading.

Historically, starting from these well-known genus zero algebras, in 1986 Krichever and Novikov [33], [34], [35] suggested a global operator approach via KN objects. Still they only considered two possible points where poles are allowed and were dealing with the vector field and the function algebra. For work on affine algebras Sheinman [55], [56], [57], [58]. should be mentioned.

From the applications in CFT (e.g. string theory) but also from purely mathematical reasons, the need of a multi-point theory is evident. In 1990 Schlichenmaier developed a systematic theory valid for all genera (including zero) and any fixed finite set of points where poles are allowed [41], [42], [43], [44]. These extensions were not at all straight-forward. The main point was to introduce a replacement of the graded algebra structure present in the "classical" case. Krichever and Novikov found that the already mentioned almost-grading (Definition 5.1) will be enough to allow for the standard constructions in representation theory. In [43], [44] it was realized that a splitting of the set A of points where poles are allowed, into two disjoint non-empty subsets $A = I \cup O$ is crucial for introducing an almost-grading in the multi-point situation. The corresponding almost-grading was explicitly given. In contrast to the classical situation and original KN situation, where there is only one grading, we will have a finite set of non-equivalent gradings and new interesting phenomena show up. This is already true for the genus zero case (i.e. the Riemann sphere case) with more than two points where poles are allowed. These algebras will be only almost-graded, see e.g. [45], [19], [20], [54].

Also other (Lie) algebras were introduced. In fact most of them come from a *Mother Poisson Algebra* [46], the algebra of meromorphic form of all weight, see Section 4.2. This algebra carries a (weak) almost-grading which gives the almost-grading for the other algebras. For the relevant algebras almost-graded central extensions are constructed and classified. In the first part of the survey we present the basic definitions and structural results for the KN type algebras.

In the second part, starting in Section 7 we have a closer look at the genus zero multi-point situation. In this case the meromorphic objects can be given via rational functions. We obtain by them illustrative examples of KN type algebras. Furthermore, from the applications, e.g. in conformal field theory, resp. string theory, they correspond to the tree-level. Also we obtain additional results. For example we obtain universal central extensions of those algebras which are perfect [1]. We show that all cocycles are bounded cocycle classes with respect to the standard splitting, see Equation (90) for its definition. The classification result of the author for bounded cocycles gives now the universal central extension.

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[1] Most of them are perfect algebras.

In Section 8 the genus zero 3-point situation is covered in detail. Explicit structure equations (including central extensions) are given for the algebras.

KN type algebras have a lot of interesting applications. They show up in the context of deformations of algebras, moduli spaces of marked curves, Wess-Zumino-Novikov-Witten (WZNW) models, Knizhnik-Zamolodchikov (KZ) equations, integrable systems, quantum field theories, symmetry algebras, and in many more domains of mathematics and theoretical physics. The KN type algebras carry a very rich representation theory. We have Verma modules, highest weight representations, Fermionic and Bosonic Fock representations, semi-infinite wedge forms, $b-c$ systems, Sugawara representations and vertex algebras. Unfortunately, this survey does not allow to touch on these applications and the presentation of the representation theory. Instead I refer to my book from 2014 with the title *Krichever–Novikov type algebras. Theory and applications,* [52] which collects all the results, proofs and some applications of the multi-point KN algebras. There also a quite extensive list of references can be found, including articles published by physicists on applications in the field-theoretical context. For some applications in the context of integrable systems see also Sheinman, *Current algebras on Riemann surfaces* [59].

For the proofs of the statements and more material we have to refer to the original articles and the corresponding parts of [52]. There is a certain overlap of the first part with a previous survey of mine [53].

2 The Witt and the Virasoro Algebra and their Relatives

These algebras supply important examples of non-trivial infinite dimensional Lie algebras. They are widely used in Conformal Field Theory and String Theory. We recall their conventional algebraic definitions.

The *Witt algebra* $\mathcal{W}$, sometimes also called Virasoro algebra without central term, is the Lie algebra generated as vector space over $\mathbb{C}$ by the basis elements $\{e_n \mid n \in \mathbb{Z}\}$ with Lie structure

$$[e_n, e_m] = (m-n)e_{n+m}, \qquad n, m \in \mathbb{Z}. \tag{1}$$

The algebra $\mathcal{W}$ is more than just a Lie algebra. It is a graded Lie algebra. If we set for the degree $\deg(e_n) := n$ then

$$\mathcal{W} = \bigoplus_{n\in\mathbb{Z}} \mathcal{W}_n, \qquad \mathcal{W}_n = \langle e_n \rangle_{\mathbb{C}}. \tag{2}$$

Obviously, $\deg([e_n, e_m]) = \deg(e_n) + \deg(e_m)$.

Remark 2.1 Algebraically $\mathcal{W}$ can also be given as Lie algebra of derivations of the algebra of Laurent polynomials $\mathbb{C}[z, z^{-1}]$. Moreover, in the purely algebraic context our field of definition $\mathbb{C}$ can be replaced by an arbitrary field $\mathbb{K}$ of characteristics 0.

For the Witt algebra the universal one-dimensional central extension is the *Virasoro algebra* $\mathcal{V}$. As vector space it is the direct sum $\mathcal{V} = \mathbb{C} \oplus \mathcal{W}$. If we set for $x \in \mathcal{W}$, $\hat{x} := (0, x)$, and $t := (1, 0)$ then its basis elements are $\hat{e}_n$, $n \in \mathbb{Z}$ and t with the Lie product [2]:

$$[\hat{e}_n, \hat{e}_m] = (m-n)\hat{e}_{n+m} + \frac{1}{12}(n^3 - n)\delta_n^{-m}\, t, \qquad [\hat{e}_n, t] = [t, t] = 0, \tag{3}$$

for all $n, m \in \mathbb{Z}$. By setting $\deg(\hat{e}_n) := \deg(e_n) = n$ and $\deg(t) := 0$ the Lie algebra $\mathcal{V}$ becomes a graded algebra. The algebra $\mathcal{W}$ will only be a subspace, not a subalgebra of $\mathcal{V}$. But it will be a quotient. Up to equivalence of central extensions and rescaling the central element t, this is beside the trivial (splitting) central extension, the only central extension of $\mathcal{W}$.

Given $\mathfrak{g}$ a finite-dimensional Lie algebra (e.g. a finite-dimensional simple Lie algebra) then the tensor product of $\mathfrak{g}$ with the associative algebra of Laurent polynomials $\mathbb{C}[z, z^{-1}]$ carries a Lie algebra structure via

$$[x \otimes z^n, y \otimes z^m] := [x, y] \otimes z^{n+m}. \tag{4}$$

This algebra is called *current algebra* or *loop algebra* and denoted by $\overline{\mathfrak{g}}$. Again we consider central extensions. For this let β be a symmetric, bilinear form for $\mathfrak{g}$ which is invariant (i.e. $\beta([x, y], z) = \beta(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$). Then a central extension is given by

$$[\widehat{x \otimes z^n}, \widehat{y \otimes z^m}] := \widehat{[x, y] \otimes z^{n+m}} - \beta(x, y) \cdot m\, \delta_n^{-m} \cdot t. \tag{5}$$

This algebra is denoted by $\widehat{\mathfrak{g}}$ and called *affine Lie algebra*. With respect to the classification of Kac-Moody Lie algebras, in the case of a simple $\mathfrak{g}$ they are exactly the Kac-Moody algebras of untwisted affine type, [28], [29], [37].

To complete the description let me introduce the Lie superalgebra of Neveu-Schwarz type. The centrally extended superalgebra has as basis (we drop the ˆ)

$$e_n,\ n \in \mathbb{Z}, \quad \varphi_m,\ m \in \mathbb{Z} + \frac{1}{2}, \quad t \tag{6}$$

with structure equations

$$\begin{aligned}
[e_n, e_m] &= (m-n) e_{m+n} + \frac{1}{12}(n^3 - n)\, \delta_n^{-m}\, t, \\
[e_n, \varphi_m] &= (m - \frac{n}{2})\, \varphi_{m+n}, \\
[\varphi_n, \varphi_m] &= e_{n+m} - \frac{1}{6}\,(n^2 - \frac{1}{4})\, \delta_n^{-m}\, t.
\end{aligned} \tag{7}$$

[2] Here δ_k^l is the Kronecker delta which is equal to 1 if $k = l$, otherwise zero.

By "setting $t = 0$" we obtain the non-extended superalgebra. The elements e_n (and t) are a basis of the subspace of even elements, the elements φ_m are a basis of the subspace of odd elements.

These algebras are Lie superalgebras, defined as follows.

Definition 2.2 Let $\mathcal{S}$ be a vector space which is decomposed into even and odd elements $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$, i.e. $\mathcal{S}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space. Furthermore, let $[.,.]$ be a $\mathbb{Z}/2\mathbb{Z}$-graded bilinear map $\mathcal{S}\times\mathcal{S} \to \mathcal{S}$ such that for elements x, y of pure parity

$$[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]. \tag{8}$$

Here $\bar{x}$ is the parity of x, etc. These conditions say that

$$[\mathcal{S}_{\bar{0}}, \mathcal{S}_{\bar{0}}] \subseteq \mathcal{S}_{\bar{0}}, \qquad [\mathcal{S}_{\bar{0}}, \mathcal{S}_{\bar{1}}] \subseteq \mathcal{S}_{\bar{1}}, \qquad [\mathcal{S}_{\bar{1}}, \mathcal{S}_{\bar{1}}] \subseteq \mathcal{S}_{\bar{0}}, \tag{9}$$

and that $[x, y]$ is symmetric for x and y odd, otherwise anti-symmetric. Now $\mathcal{S}$ is a *Lie superalgebra* if in addition the *super-Jacobi identity* (for x, y, z of pure parity)

$$(-1)^{\bar{x}\bar{z}}[x, [y, z]] + (-1)^{\bar{y}\bar{x}}[y, [z, x]] + (-1)^{\bar{z}\bar{y}}[z, [x, y]] = 0 \tag{10}$$

is valid. As long as the type of the arguments is different from *(even, odd, odd)* all signs can be put to $+1$ and we obtain the form of the usual Jacobi identity. In the remaining case we get

$$[x, [y, z]] + [y, [z, x]] - [z, [x, y]] = 0. \tag{11}$$

By the definitions $\mathcal{S}_0$ is a Lie algebra.

3 The Geometric Picture

A geometric description of the Witt algebra over $\mathbb{C}$ can be given as follows. Let W be the algebra of those meromorphic vector fields on the Riemann sphere $S^2 = \mathbb{P}^1(\mathbb{C})$ which are holomorphic outside $\{0\}$ and $\{\infty\}$. Its elements can be given as

$$v(z) = \tilde{v}(z)\frac{d}{dz} \tag{12}$$

where $\tilde{v}$ is a meromorphic function on $\mathbb{P}^1(\mathbb{C})$, which is holomorphic outside $\{0, \infty\}$. Those are exactly the Laurent polynomials $\mathbb{C}[z, z^{-1}]$. Consequently, this subalgebra has the set $\{e_n, n \in \mathbb{Z}\}$ with $e_n = z^{n+1}\frac{d}{dz}$ as vector space basis. The Lie bracket of vector fields calculates as

$$[v, u] = \left(\tilde{v}\frac{d}{dz}\tilde{u} - \tilde{u}\frac{d}{dz}\tilde{v}\right)\frac{d}{dz}. \tag{13}$$

Evaluated for the basis elements e_n this gives (1) and the algebra can be identified with the Witt algebra defined purely algebraically.

The subalgebra of global holomorphic vector fields is the 3-dimensional subspace $\langle e_{-1}, e_0, e_1\rangle_{\mathbb{C}}$. It is isomorphic to the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.

Similarly, the algebra $\mathbb{C}[z, z^{-1}]$ can be given as the algebra of meromorphic functions on $S^2 = \mathbb{P}^1(\mathbb{C})$ holomorphic outside of $\{0, \infty\}$.

Recall that the Riemann sphere is the (compact) Riemann surface of genus zero. In the geometric setup for the Virasoro algebra the objects are defined on the Riemann sphere and might have poles at most at two fixed points. For a global operator approach to conformal field theory and its quantization this is not sufficient. One needs Riemann surfaces of arbitrary genus. Moreover, one needs more than two points were singularities are allowed [3]. This higher genus multi-point case was systematically examined by the Schlichenmaier [41], [42], [43], [44], [45] [46], [48], [47] and is presented in a current book [52]. For some related approach and partial results, see also Sadov [40].

For the whole contribution let $\Sigma = \Sigma_g$ be a compact Riemann surface without any restriction for the genus $g = g(\Sigma)$. Furthermore, let A be a finite subset of Σ. Later we will need a splitting of A into two non-empty disjoint subsets I and O, i.e. $A = I \cup O$. Set $N := \#A$, $K := \#I$, $M := \#O$, with $N = K + M$. More precisely, let

$$I = (P_1, \ldots, P_K), \quad \text{and} \quad O = (Q_1, \ldots, Q_M) \tag{14}$$

be disjoint ordered tuples of distinct points ("marked points", "punctures") on the Riemann surface. In particular, we assume $P_i \neq Q_j$ for every pair (i, j). The points in I are called the *in-points*, the points in O the *out-points*. .

Sometimes we refer to the *classical situation*. By this we understand

$$\Sigma_0 = \mathbb{P}^1(\mathbb{C}) = S^2, \quad I = \{z = 0\}, \quad O = \{z = \infty\}. \tag{15}$$

The following figures should indicate the geometric picture. Figure 1 shows the classical situation.

Figure 2 is genus 2, but still the two-point situation.

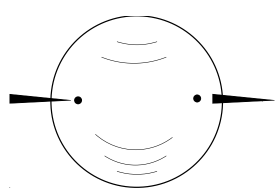

Fig. 1 Riemann surface of genus zero with one incoming and one outgoing point.

[3] The singularities correspond to points where free fields are entering the region of interaction or leaving it. In particular, from the very beginning there is a natural decomposition of the set of points into two disjoint subsets.

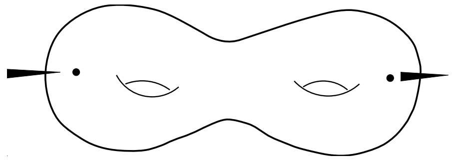

Fig. 2 Riemann surface of genus two with one incoming and one outgoing point.

Finally, in Figure 3 the case of a Riemann surface of genus 2 with two incoming points and one outgoing point is visualized.

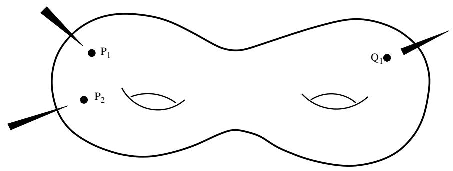

Fig. 3 Riemann surface of genus two with two incoming points and one outgoing point.

Remark 3.1 We stress the fact, that these multi-point generalizations are needed also in the case of genus zero. Even in the case of genus zero and three points interesting algebras show up. See Section 8, [54].

3.1 Meromorphic forms

To introduce the elements of the generalized algebras (later called Krichever-Novikov type algebras) we first have to discuss forms of certain (conformal) weights.

Let $\mathcal{K} = \mathcal{K}_\Sigma$ be the canonical line bundle of Σ. Its local sections are the local holomorphic differentials.

If $P \in \Sigma$ is a point and z a local holomorphic coordinate at P then a local holomorphic differential can be written as $f(z)dz$ with a local holomorphic function f defined in a neighborhood of P. A global holomorphic differential can be described locally in coordinates $(U_i, z_i)_{i\in J}$ by a system of local holomorphic functions $(f_i)_{i\in J}$, which are related by the transformation rule induced by the coordinate change map $z_j = z_j(z_i)$ and the condition $f_i dz_i = f_j dz_j$. This yields

$$f_j = f_i \cdot \left(\frac{dz_j}{dz_i}\right)^{-1}. \tag{16}$$

A meromorphic section of $\mathcal{K}$, i.e. a *meromorphic differential* is given as a collection of local meromorphic functions $(h_i)_{i\in J}$ (with respect to a coordinate covering) for which the transformation law (16) remains true.

In the following λ is either an integer or a half-integer. If λ is an integer then
(1) $\mathcal{K}^\lambda := \mathcal{K}^{\otimes\lambda}$ for $\lambda > 0$,
(2) $\mathcal{K}^0 := \mathcal{O}$, the trivial line bundle, and
(3) $\mathcal{K}^\lambda := (\mathcal{K}^*)^{\otimes(-\lambda)}$ for $\lambda < 0$.
Here $\mathcal{K}^*$ denotes the dual line bundle of the canonical line bundle. This is the holomorphic tangent line bundle, whose local sections are the holomorphic tangent vector fields $f(z)(d/dz)$.

If λ is a half-integer, then we first have to fix a "square root" of the canonical line bundle, sometimes called a *theta characteristics*. . This means we fix a line bundle L for which $L^{\otimes 2} = \mathcal{K}$. After such a choice of L is done we set $\mathcal{K}^\lambda := \mathcal{K}^\lambda_L := L^{\otimes 2\lambda}$. In most cases we will drop the mentioning of L, but we have to keep the choice in mind. The fine-structure of the algebras we are about to define will depend on the choice. But the main properties will remain the same.

Remark 3.2 A Riemann surface of genus g has exactly 2^{2g} non-isomorphic square roots of $\mathcal{K}$. The choice of a theta characteristic is also called a spin structure on Σ [3]. Only for $g = 0$ we have a unique unique square root.

We set

$$\mathcal{F}^\lambda(A) := \{f \text{ is a global meromorphic section of } \mathcal{K}^\lambda \mid f \text{ is holomorphic on } \Sigma \setminus A\}. \tag{17}$$

Obviously, this is a $\mathbb{C}$-vector space. To avoid cumbersome notation, we will often drop the set A in the notation if A is fixed and clear from the context. Recall that in the case of half-integer λ everything depends on the theta characteristic L.

Definition 3.3 The elements of the space $\mathcal{F}^\lambda(A)$ are called *meromorphic forms of weight* λ (with respect to the theta characteristic L).

If f is a meromorphic λ-form it can be represented locally by meromorphic functions f_i via $f = f_i(dz_i)^{\otimes\lambda}$. If $f \not\equiv 0$ the local representing functions have only finitely many zeros and poles. Whether a point P is a zero or a pole of f does not depend on the coordinate z_i chosen. We can define for $P \in \Sigma$ the *order*

$$\mathrm{ord}_P(f) := \mathrm{ord}_P(f_i), \tag{18}$$

where $\mathrm{ord}_P(f_i)$ is the lowest nonvanishing order in the Laurent series expansion of f_i in the variable z_i around P. It will not depend on the coordinate z_i chosen.

The order $\mathrm{ord}_P(f)$ is (strictly) positive if and only if P is a zero of f. It is negative if and only if P is a pole of f. Moreover, its value gives the order of the zero and pole respectively.

By compactness of our Riemann surface Σ our $f \neq 0$ can only have finitely many zeros and poles. We define the *(sectional) degree* of f to be

$$\operatorname{sdeg}(f) := \sum_{P \in \Sigma} \operatorname{ord}_P(f). \tag{19}$$

Proposition 3.4 *Let* $f \in \mathcal{F}^\lambda$, $f \not\equiv 0$ *then*

$$\operatorname{sdeg}(f) = 2\lambda(g-1). \tag{20}$$

For this and related results see e.g. [49].

Later we will need the additional geometric data of a coordinate z_i at every point $P_i \in A$. In fact, only the first order infinitesimal neighbourhood will play a role.

4 Algebraic Structures

Next we introduce algebraic operations on the vector space of meromorphic forms of arbitrary weights. This space is obtained by summing over all weights

$$\mathcal{F} := \bigoplus_{\lambda \in \frac{1}{2}\mathbb{Z}} \mathcal{F}^\lambda. \tag{21}$$

The basic operations will allow us to introduce finally our intended algebras. We will drop the subset A in the notation.

4.1 Associative structure

The natural map of the locally free sheaves of rang one

$$\mathcal{K}^\lambda \times \mathcal{K}^\nu \to \mathcal{K}^\lambda \otimes \mathcal{K}^\nu \cong \mathcal{K}^{\lambda+\nu}, \quad (s,t) \mapsto s \otimes t, \tag{22}$$

defines a bilinear map

$$\cdot : \mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu}. \tag{23}$$

With respect to local trivialisations this corresponds to the multiplication of the local representing meromorphic functions

$$(s\, dz^\lambda, t\, dz^\nu) \mapsto s\, dz^\lambda \cdot t\, dz^\nu = s \cdot t\ dz^{\lambda+\nu}. \tag{24}$$

If there is no danger of confusion then we will mostly use the same symbol for the section and for the local representing function.

The following is obvious

Proposition 4.1 *The space* $\mathcal{F}$ *is an associative and commutative graded (over* $\frac{1}{2}\mathbb{Z}$*) algebra. Moreover,* $\mathcal{A} = \mathcal{F}^0$ *is a subalgebra and the* $\mathcal{F}^\lambda$ *are modules over* $\mathcal{A}$.

Of course, $\mathcal{A}$ is the algebra of those meromorphic functions on Σ which are holomorphic outside of A.

4.2 Lie and Poisson algebra structure

Next we define a Lie algebra structure on the space $\mathcal{F}$. The structure is induced by the map

$$\mathcal{F}^{\lambda}\times\mathcal{F}^{\nu}\to\mathcal{F}^{\lambda+\nu+1},\qquad (e,f)\mapsto[e,f], \tag{25}$$

which is defined in local representatives of the sections by

$$(e\,dz^{\lambda},f\,dz^{\nu})\mapsto[e\,dz^{\lambda},f\,dz^{\nu}]:=\left((-\lambda)e\frac{df}{dz}+\nu f\frac{de}{dz}\right)dz^{\lambda+\nu+1}, \tag{26}$$

and bilinearly extended to $\mathcal{F}$.

Proposition 4.2 *[52, Prop. 2.6 and 2.7] The prescription* $[.,.]$ *given by (26) is well-defined and defines a Lie algebra structure on the vector space* $\mathcal{F}$.

Proposition 4.3 *[52, Prop. 2.8] The subspace* $\mathcal{L}=\mathcal{F}^{-1}$ *is a Lie subalgebra, and the* $\mathcal{F}^{\lambda}$*'s are Lie modules over* $\mathcal{L}$.

The $\mathcal{L}$ consists of meromorphic vector fields on Σ and the Lie module structure is the Lie derivative. of forms.

We have the Leibniz rule

$$\forall e,f,g\in\mathcal{F}:[e,f\cdot g]=[e,f]\cdot g+f\cdot[e,g]. \tag{27}$$

relating the associative and the Lie structure. Hence, by definition

Theorem 4.4 *[52, Thm. 2.10] The space* $\mathcal{F}$ *with respect to* $\cdot$ *and* $[.,.]$ *is a Poisson algebra.*

Next we consider important substructures. We already encountered the subalgebras $\mathcal{A}$ and $\mathcal{L}$. But there are more structures around.

4.3 The algebra of differential operators

If we look at $\mathcal{F}$, considered as Lie algebra, more closely, we see that $\mathcal{F}^0$ is an abelian Lie subalgebra and the vector space sum $\mathcal{F}^0\oplus\mathcal{F}^{-1}=\mathcal{A}\oplus\mathcal{L}$ is also a Lie subalgebra. In an equivalent way it can also be constructed as semidirect sum of $\mathcal{A}$ considered as abelian Lie algebra and $\mathcal{L}$ operating on $\mathcal{A}$ by taking the derivative.

Definition 4.5 The *Lie algebra of differential operators of degree* ≤ 1 is defined as the semidirect sum of $\mathcal{A}$ with $\mathcal{L}$ and is denoted by $\mathcal{D}^1$.

In terms of elements the Lie product is

$$[(g,e),(h,f)] = (e\,.\,h - f\,.\,g\,,\ [e,f]). \tag{28}$$

The projection on the second factor $(g,e)\mapsto e$ is a Lie homomorphism and we obtain a short exact sequences of Lie algebras

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{D}^1 \longrightarrow \mathcal{L} \longrightarrow 0\,. \tag{29}$$

Hence, $\mathcal{A}$ is an (abelian) Lie ideal of $\mathcal{D}^1$ and $\mathcal{L}$ a quotient Lie algebra. Obviously, $\mathcal{L}$ is also a subalgebra of $\mathcal{D}^1$.

The vector spaces $\mathcal{F}^\lambda$ become Lie modules over $\mathcal{D}^1$ by the operation

$$(g,e).f := g\cdot f + e.f, \quad (g,e)\in\mathcal{D}^1(A), f\in\mathcal{F}^\lambda(A). \tag{30}$$

Differential operators of arbitrary degree can be constructed via universal constructions, see e.g. [52].

4.4 Lie superalgebras of half forms

Recall from Definition 2.2 the notion of a Lie superalgebra.

With the help of our associative product (22) we will obtain examples of Lie superalgebras. First we consider

$$\cdot\ \mathcal{F}^{-1/2}\times\mathcal{F}^{-1/2}\to\mathcal{F}^{-1}=\mathcal{L}\,, \tag{31}$$

and introduce the vector space $\mathcal{S}$ with the product

$$\mathcal{S} := \mathcal{L}\oplus\mathcal{F}^{-1/2}, \quad [(e,\varphi),(f,\psi)] := ([e,f]+\varphi\cdot\psi, e\,.\,\varphi - f\,.\,\psi). \tag{32}$$

The elements of $\mathcal{L}$ are denoted by $e, f, \ldots$, and the elements of $\mathcal{F}^{-1/2}$ by $\varphi, \psi, \ldots$.

The definition (32) can be reformulated as an extension of $[.,.]$ on $\mathcal{L}$ to a superbracket (denoted by the same symbol) on $\mathcal{S}$ by setting

$$[e,\varphi] := -[\varphi,e] := e\,.\,\varphi = (e\frac{d\varphi}{dz} - \frac{1}{2}\varphi\frac{de}{dz})(dz)^{-1/2} \tag{33}$$

and

$$[\varphi,\psi] := \varphi\cdot\psi\ . \tag{34}$$

The elements of $\mathcal{L}$ are elements of even parity, and the elements of $\mathcal{F}^{-1/2}$ are elements of odd parity. For such elements x we denote by $\bar{x}\in\{\bar{0},\bar{1}\}$ their parity.

The sum (32) can also be described as $\mathcal{S}=\mathcal{S}_{\bar{0}}\oplus\mathcal{S}_{\bar{1}}$, where $\mathcal{S}_{\bar{i}}$ is the subspace of elements of parity $\bar{i}$.

Proposition 4.6 *[52, Prop. 2.15], [51] The space $\mathcal{S}$ with the above introduced parity and product is a Lie superalgebra.*

Leidwanger and Morier-Genoux introduced in [36] a *Jordan superalgebra* in our geometric setting. They put

$$\mathcal{J} := \mathcal{F}^0 \oplus \mathcal{F}^{-1/2} = \mathcal{J}_{\bar{0}} \oplus \mathcal{J}_{\bar{1}}. \tag{35}$$

Recall that $\mathcal{A} = \mathcal{F}^0$ is the associative algebra of meromorphic functions. They define the (Jordan) product $\circ$ via the algebra structures for the spaces $\mathcal{F}^\lambda$ by

$$\begin{aligned} f \circ g &:= f \cdot g \quad && \in \mathcal{F}^0, \\ f \circ \varphi &:= f \cdot \varphi \quad && \in \mathcal{F}^{-1/2}, \\ \varphi \circ \psi &:= [\varphi, \psi] \quad && \in \mathcal{F}^0. \end{aligned} \tag{36}$$

By rescaling the second definition with the factor 1/2 one obtains a *Lie anti-algebra* as introduced by Ovsienko [38]. See [36] for more details and additional results on representations.

4.5 Higher genus current algebras

We fix an arbitrary finite-dimensional complex Lie algebra $\mathfrak{g}$. Our goal is to generalize the classical current algebra to higher genus. For this let (Σ, A) be the geometric data consisting of the Riemann surface Σ and the subset of points A used to define $\mathcal{A}$, the algebra of meromorphic functions which are holomorphic outside of the set $A \subseteq \Sigma$.

Definition 4.7 The *higher genus current algebra* associated to the Lie algebra $\mathfrak{g}$ and the geometric data (Σ, A) is the Lie algebra $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}(A) = \overline{\mathfrak{g}}(\Sigma, A)$ given as vector space by $\overline{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}$ with the Lie product

$$[x \otimes f, y \otimes g] = [x, y] \otimes f \cdot g, \qquad x, y \in \mathfrak{g}, \quad f, g \in \mathcal{A}. \tag{37}$$

Sometimes this algebra is also called *loop algebra*.

Proposition 4.8 $\overline{\mathfrak{g}}$ *is a Lie algebra.*

As usual we will suppress the mentioning of (Σ, A) if not needed. The elements of $\overline{\mathfrak{g}}$ can be interpreted as meromorphic functions $\Sigma \to \mathfrak{g}$ from the Riemann surface Σ to the Lie algebra $\mathfrak{g}$, which are holomorphic outside of A.

Later we will introduce central extensions for these current algebras. They will generalize affine Lie algebras, respectively affine Kac-Moody algebras of untwisted type.

For some applications it is useful to extend the definition by considering differential operators (of degree ≤ 1) associated to $\overline{\mathfrak{g}}$. We define $\mathcal{D}^1_{\mathfrak{g}} := \overline{\mathfrak{g}} \oplus \mathcal{L}$ and take in the summands the Lie product defined there and put additionally

$$[e, x \otimes g] := -[x \otimes g, e] := x \otimes (e.g). \tag{38}$$

This operation can be described as semidirect sum of $\overline{\mathfrak{g}}$ with $\mathcal{L}$ and we get

Proposition 4.9 *[52, Prop. 2.15] $\mathcal{D}^1_{\mathfrak{g}}$ is a Lie algebra.*

4.6 Krichever–Novikov type algebras

Above we could even allow that the set A of points where poles are allowed is arbitrary, even non-finite. In case that A is finite and moreover $\#A \geq 2$ the constructed algebras we a called Krichever–Novikov (KN) type algebras. In this way we get the KN vector field algebra, the function algebra, the current algebra, the differential operator algebra, the Lie superalgebra, etc. The reader might ask what is so special about this situation so that these algebras deserve special names. In fact in this case we can endow the algebra with a (strong) almost-graded structure. This will be discussed in the next section. The almost-grading is a crucial tool for representation theory and for extending the classical result to higher genus. Recall that in the classical case we have genus zero and $\#A = 2$.

Strictly speaking, a KN type algebra should be considered to be one of the above algebras for $2 \leq \#A < \infty$ together with a fixed chosen almost-grading, induced by the splitting $A = I \cup O$ into two disjoint non-empty subset, see Definition 5.1.

5 Almost-Graded Structure

5.1 Definition of almost-gradedness

In the classical situation discussed in Section 2 the algebras introduced in the last section are graded algebras. In the higher genus case and even in the genus zero case with more than two points where poles are allowed there is no non-trivial grading anymore. As realized by Krichever and Novikov [33] there is a weaker concept, an almost-grading, which to a large extend is a valuable replacement of a honest grading. Such an almost-grading is induced by a splitting of the set A into two non-empty and disjoint sets I and O. The (almost-)grading is fixed by exhibiting certain basis elements in the spaces $\mathcal{F}^\lambda$ as homogeneous.

Definition 5.1 Let $\mathcal{L}$ be a Lie or an associative algebra such that $\mathcal{L} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n$ is a vector space direct sum, then $\mathcal{L}$ is called an *almost-graded* (Lie-) algebra if

(i) $\dim \mathcal{L}_n < \infty$,

(ii) There exists constants $L_1, L_2 \in \mathbb{Z}$ such that

$$\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathcal{L}_h, \qquad \forall n, m \in \mathbb{Z}.$$

The elements in $\mathcal{L}_n$ are called *homogeneous* elements of degree n, and $\mathcal{L}_n$ is called *homogeneous subspace* of degree n.

If $\dim \mathcal{L}_n$ is bounded with a bound independent of n we call $\mathcal{L}$ *strongly almost-graded*. If we drop the condition that $\dim \mathcal{L}_n$ is finite dimensional we call $\mathcal{L}$ *weakly almost-graded*.

In a similar manner almost-graded modules over almost-graded algebras are defined. We can extend in an obvious way the definition to superalgebras, respectively even to more general algebraic structures. Note that this definition makes complete sense also for more general index sets $\mathbb{J}$. In fact we will consider the index set $\mathbb{J} = (1/2)\mathbb{Z}$ in the case of superalgebras. The even elements (with respect to the super-grading) will have integer degree, the odd elements half-integer degree.

5.2 Separating cycle and Krichever-Novikov pairing

Before we give the almost-grading we introduce an important structure.

First we recall the splitting of A (14) into two non-empty disjoint subsets I and O. Let C_i be positively oriented (deformed) circles on Σ around the points P_i in I, $i = 1, \ldots, K$ and C_j^* positively oriented circles on Σ around the points Q_j in O, $j = 1, \ldots, M$.

A cycle C_S on Σ is called a *separating cycle* if it is smooth, positively oriented of multiplicity one and if it separates the in-points from the out-points. It might have more than one component. In the following we will integrate meromorphic differentials on Σ without poles in $\Sigma \setminus A$ over closed curves C. Hence, we might consider C and C' as equivalent if $[C] = [C']$ in $\mathrm{H}_1(\Sigma \setminus A, \mathbb{Z})$. In this sense we write for every separating cycle

$$[C_S] = \sum_{i=1}^{K} [C_i] = -\sum_{j=1}^{M} [C_j^*]. \tag{39}$$

The minus sign appears due to the opposite orientation. Another way for giving such a C_S is via level lines of a “proper time evolution”, for which I refer to [52, Section 3.9].

Given such a separating cycle C_S (respectively cycle class) we define a linear map

$$\mathcal{F}^1 \to \mathbb{C}, \qquad \omega \mapsto \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega. \tag{40}$$

The map will not depend on the separating line C_S chosen, as two of such will be homologous and the poles of ω are only located in I and O.

Consequently, the integration of ω over C_S can also be described over the special cycles C_i or equivalently over C_j^*. This integration corresponds to calculating residues

$$\omega \quad \mapsto \quad \frac{1}{2\pi\mathrm{i}} \int_{C_S} \omega \;=\; \sum_{i=1}^{K} \mathrm{res}_{P_i}(\omega) \;=\; -\sum_{l=1}^{M} \mathrm{res}_{Q_l}(\omega). \tag{41}$$

Definition 5.2 The pairing

$$\mathcal{F}^\lambda \times \mathcal{F}^{1-\lambda} \to \mathbb{C}, \quad (f,g) \mapsto \langle f, g \rangle := \frac{1}{2\pi\mathrm{i}} \int_{C_S} f \cdot g, \tag{42}$$

between λ and $1-\lambda$ forms is called *Krichever-Novikov (KN) pairing* .

Note that the pairing depends not only on A (as the $\mathcal{F}^\lambda$ depend on it) but also critically on the splitting of A into I and O as the integration path will depend on it. Once the splitting is fixed the pairing will be fixed too.

Below we will see that the pairing is non-degenerate.

5.3 The homogeneous subspaces

Given the vector spaces $\mathcal{F}^\lambda$ of forms of weight λ we will now single out subspaces $\mathcal{F}^\lambda_m$ of degree m by giving a basis of these subspaces. This has been done in the 2-point case by Krichever and Novikov [33] and in the multi-point case by the author [41], [42], [43], [44], see also Sadov [40]. See in particular [52, Chapters 3,4,5] for a complete treatment. All proofs of the statements to come can be found there.

Depending on whether the weight λ is integer or half-integer we set $\mathbb{J}_\lambda = \mathbb{Z}$ or $\mathbb{J}_\lambda = \mathbb{Z} + 1/2$. For $\mathcal{F}^\lambda$ we introduce for $m \in \mathbb{J}_\lambda$ subspaces $\mathcal{F}^\lambda_m$ of dimension K, where $K = \#I$, by exhibiting certain elements $f^\lambda_{m,p} \in \mathcal{F}^\lambda$, $p = 1, \ldots, K$ which constitute a basis of $\mathcal{F}^\lambda_m$. The elements are the elements of degree m. As explained in the following, the degree is in an essential way related to the zero orders of the elements at the points in I.

Let $I = (P_1, P_2, \ldots, P_K)$ then we require for the zero-order at the point $P_i \in I$ of the element $f^\lambda_{n,p}$

$$\mathrm{ord}_{P_i}(f^\lambda_{n,p}) = (n + 1 - \lambda) - \delta_i^p, \quad i = 1, \ldots, K\ . \tag{43}$$

The prescription at the points in O is made in such a way that the element $f^\lambda_{m,p}$ is essentially uniquely given. Essentially unique means up to multiplication with a constant[4]. After fixing as additional geometric data a system of coordinates z_l

[4] Strictly speaking, there are some special cases where some constants have to be added such that the Krichever-Novikov duality (47) is valid.

centered at P_l for $l = 1, \ldots, K$ and requiring that

$$f^\lambda_{n,p}(z_p) = z_p^{n-\lambda}(1 + O(z_p))(dz_p)^\lambda \tag{44}$$

the element $f^\lambda_{n,p}$ is uniquely fixed. In fact, the element $f^\lambda_{n,p}$ only depends on the first order jet of the coordinate z_p.

The element $f^\lambda_{n,p}$ has outside of A exactly g zeros (counted with multiplicities).

Example 5.3 Here we will not give the general recipe for the prescription at the points in O. Just to give an example which is also an important special case, assume $O = \{Q\}$ is a one-element set. If either the genus $g = 0$, or $g \ge 2$, $\lambda \ne 0,\ 1/2,\ 1$ and the points in A are in generic position then we require

$$\mathrm{ord}_Q(f^\lambda_{n,p}) = -K \cdot (n + 1 - \lambda) + (2\lambda - 1)(g - 1). \tag{45}$$

In the other cases (e.g. for $g = 1$) there are some modifications at the point in O necessary for finitely many n.

Theorem 5.4 *[52, Thm. 3.6] Set*

$$\mathcal{B}^\lambda := \{\, f^\lambda_{n,p} \mid n \in \mathbb{J}_\lambda,\ p = 1, \ldots, K \,\}. \tag{46}$$

Then (a) $\mathcal{B}^\lambda$ is a basis of the vector space $\mathcal{F}^\lambda$.

(b) The introduced basis $\mathcal{B}^\lambda$ of $\mathcal{F}^\lambda$ and $\mathcal{B}^{1-\lambda}$ of $\mathcal{F}^{1-\lambda}$ are dual to each other with respect to the Krichever-Novikov pairing (42), i.e.

$$\langle f^\lambda_{n,p}, f^{1-\lambda}_{-m,r}\rangle = \delta^r_p\ \delta^m_n, \quad \forall n, m \in \mathbb{J}_\lambda, \quad r, p = 1, \ldots, K. \tag{47}$$

In particular, from part (b) of the theorem it follows that the Krichever-Novikov pairing is non-degenerate.

An important consequence from the KN duality decomposition with respect to the basis given above is that we can write

$$v \in \mathcal{F}^\lambda, \quad v = \sum a_{n,p} f^\lambda_{n,p} \tag{48}$$

with

$$a_{n,p} = \langle v, f^{1-\lambda}_{-n,p}\rangle = \sum_{i=1}^{K} \mathrm{res}_{P_i}(v \cdot f^{1-\lambda}_{-n,p}) = -\sum_{j=1}^{M} \mathrm{res}_{Q_j}(v \cdot f^{1-\lambda}_{-n,p})\,. \tag{49}$$

It is quite convenient to use special notations for elements of some important weights:

$$\begin{gathered} e_{n,p} := f^{-1}_{n,p}, \quad \varphi_{n,p} := f^{-1/2}_{n,p}, \quad A_{n,p} := f^0_{n,p}, \\ \omega^{n,p} := f^1_{-n,p}, \quad \Omega^{n,p} := f^2_{-n,p}. \end{gathered} \tag{50}$$

In view of (47) for the forms of weight 1 and 2 we invert the index n and write it as a superscript.

Remark 5.5 It is also possible (and for certain applications necessary) to write explicitely down the basis elements $f^\lambda_{n,p}$ in terms of "usual" objects defined on the Riemann surface Σ. For genus zero they can be given with the help of rational functions in the quasi-global variable z. Indeed the second half of this survey will deal with it. For genus one (i.e. the torus case) representations with the help of Weierstraß σ and Weierstraß $\wp$ functions exists. For genus ≥ 1 there exists expressions in terms of theta functions (with characteristics) and prime forms. Here the Riemann surface has first to be embedded into its Jacobian via the Jacobi map. See [52, Chapter 5], [42], [45] for more details.

5.4 The algebras

Theorem 5.6 *[52, Thm. 3.8]*

There exists constants R_1 and R_2 (depending on the number and splitting of the points in A and on the genus g) independent of λ and ν and independent of $n, m \in \mathbb{J}$ such that for the basis elements

$$
\begin{aligned}
f^\lambda_{n,p} \cdot f^\nu_{m,r} = \quad & f^{\lambda+\nu}_{n+m,r}\delta^r_p \\
& + \sum_{h=n+m+1}^{n+m+R_1} \sum_{s=1}^{K} a^{(h,s)}_{(n,p)(m,r)} f^{\lambda+\nu}_{h,s}, \qquad a^{(h,s)}_{(n,p)(m,r)} \in \mathbb{C}, \\
[f^\lambda_{n,p}, f^\nu_{m,r}] = \quad & (-\lambda m + \nu n)\, f^{\lambda+\nu+1}_{n+m,r}\,\delta^r_p \\
& + \sum_{h=n+m+1}^{n+m+R_2} \sum_{s=1}^{K} b^{(h,s)}_{(n,p)(m,r)} f^{\lambda+\nu+1}_{h,s}, \quad b^{(h,s)}_{(n,p)(m,r)} \in \mathbb{C}.
\end{aligned}
\tag{51}
$$

For $N = 2$ points and in generic situations one obtains $R_1 = g$ and $R_2 = 3g$.

The theorem says in particular that with respect to both the associative and Lie structure the algebra $\mathcal{F}$ is weakly almost-graded. The reason why we only have weakly almost-gradedness is that

$$
\mathcal{F}^\lambda = \bigoplus_{m\in\mathbb{J}_\lambda} \mathcal{F}^\lambda_m, \qquad \text{with} \quad \dim \mathcal{F}^\lambda_m = K, \tag{52}
$$

and if we add up for a fixed m all λ we get that our homogeneous spaces are infinite dimensional.

In the proof of the theorem the KN duality property of the basis plays a crucial role. By (48) we obtain e.g.

$$
a^{(h,s)}_{(n,p),(m,r)} = \langle f^\lambda_{n,p} \cdot f^\nu_{m,r}, f^{1-(\lambda+\nu)}_{-h,s} \rangle\ . \tag{53}
$$

By considering the orders of the 3 basis elements at the points in I and O separately we can give finite ranges for the index (h, s) for which there are poles both at the points in I and at the points in O. Only in those ranges there will be a residue. Recall that for a compact Riemann surface the total residue has to be zero. Hence, there can only be a non-vanishing residue if summed over I if it compensated by a non-vanishing residue summed over O and vice versa.

In the definition of our KN type algebra only finitely many λs are involved, hence the following is immediate

Theorem 5.7 *The Krichever-Novikov type vector field algebras $\mathcal{L}$, function algebras $\mathcal{A}$, differential operator algebras $\mathcal{D}^1$, Lie superalgebras $\mathcal{S}$, and Jordan superalgebras $\mathcal{J}$ are (strongly) almost-graded algebras and the corresponding modules $\mathcal{F}^\lambda$ are almost-graded modules.*

We obtain with $n \in \mathbb{J}_\lambda$

$$\begin{gathered}\dim \mathcal{L}_n = \dim \mathcal{A}_n = \dim \mathcal{F}^\lambda_n = K, \\ \dim \mathcal{S}_n = \dim \mathcal{J}_n = 2K, \quad \dim \mathcal{D}^1_n = 3K\ .\end{gathered} \tag{54}$$

If $\mathcal{U}$ is any of these algebras, with product denoted by $[\,,\,]$ then

$$[\mathcal{U}_n, \mathcal{U}_m] \subseteq \bigoplus_{h=n+m}^{n+m+R_i} \mathcal{U}_h, \tag{55}$$

with $R_i = R_1$ for $\mathcal{U} = \mathcal{A}$ and $R_i = R_2$ otherwise.

The lowest degree term component in (51) calculates for certain special cases.

$$\begin{aligned} A_{n,p} \cdot A_{m,r} &= A_{n+m,r}\, \delta^p_r \ + \ \text{h.d.t.} \\ A_{n,p} \cdot f^\lambda_{m,r} &= f^\lambda_{n+m,r}\, \delta^p_r \ + \ \text{h.d.t.} \\ [e_{n,p}, e_{m,r}] &= (m-n) \cdot e_{n+m,r}\, \delta^p_r \ + \ \text{h.d.t.} \\ e_{n,p} \,.\, f^\lambda_{m,r} &= (m+\lambda n) \ \cdot f^\lambda_{n+m,r}\, \delta^p_r \ + \ \text{h.d.t.} \end{aligned} \tag{56}$$

Here h.d.t. denote linear combinations of basis elements of degree between $n+m+1$ and $n+m+R_i$,

Finally, the almost-grading of $\mathcal{A}$ induces an almost-grading of the current algebra $\overline{\mathfrak{g}}$ by setting $\overline{\mathfrak{g}}_n = \mathfrak{g} \otimes \mathcal{A}_n$. We obtain

$$\overline{\mathfrak{g}} = \bigoplus_{n \in \mathbb{Z}} \overline{\mathfrak{g}}_n, \quad \dim \overline{\mathfrak{g}}_n = K \cdot \dim \mathfrak{g}. \tag{57}$$

5.5 Triangular decomposition and filtrations

Let $\mathcal{U}$ be one of the above introduced algebras (including the current algebra). On the basis of the almost-grading we obtain a triangular decomposition of the algebras

$$\mathcal{U} = \mathcal{U}_{[+]} \oplus \mathcal{U}_{[0]} \oplus \mathcal{U}_{[-]}, \tag{58}$$

where

$$\mathcal{U}_{[+]} := \bigoplus_{m>0} \mathcal{U}_m, \quad \mathcal{U}_{[0]} = \bigoplus_{m=-R_i}^{m=0} \mathcal{U}_m, \quad \mathcal{U}_{[-]} := \bigoplus_{m<-R_i} \mathcal{U}_m. \tag{59}$$

By the almost-gradedness the $[+]$ and $[-]$ subspaces are (infinite dimensional) subalgebras. The $[0]$ spaces in general not. Sometimes we call them *critical strips*.

With respect to the almost-grading of $\mathcal{F}^\lambda$ we introduce a filtration

$$\begin{gathered} \mathcal{F}^\lambda_{(n)} := \bigoplus_{m\ge n} \mathcal{F}^\lambda_m, \\ \dots \quad \supseteq \quad \mathcal{F}^\lambda_{(n-1)} \quad \supseteq \quad \mathcal{F}^\lambda_{(n)} \quad \supseteq \quad \mathcal{F}^\lambda_{(n+1)} \quad \dots \end{gathered} \tag{60}$$

Proposition 5.8 *[52, Prop. 3.15]*

$$\mathcal{F}^\lambda_{(n)} = \{\, f \in \mathcal{F}^\lambda \mid \operatorname{ord}_{P_i}(f) \ge n - \lambda, \forall i = 1, \dots, K \,\}. \tag{61}$$

6 Central Extensions

Central extensions of our algebras appear naturally in the context of quantization and regularization of actions. Of course they are also of independent mathematical interest. Recall that a projective action of a Lie algebra L defines a honest Lie action of a centrally extended algebra $\hat{L}$.

6.1 Central extensions and cocycles

For the convenience of the reader let us repeat the relation between central extensions and the second Lie algebra cohomology with values in the trivial module. A central extension of a Lie algebra W is a special Lie algebra structure on the vector space direct sum $\widehat{W} = \mathbb{C} \oplus W$. If we denote $\hat{x} := (0, x)$ and $t := (1, 0)$ then the Lie structure is given by

$$[\hat{x}, \hat{y}] = \widehat{[x, y]} + \psi(x, y) \cdot t, \quad [t, \widehat{W}] = 0, \quad x, y \in W, \tag{62}$$

with a bilinear form ψ. The map $x \mapsto \hat{x} = (0, x)$ is a linear splitting map. $\widehat{W}$ will be a Lie algebra, e.g. will fulfill the Jacobi identity, if and only if ψ is an antisymmetric bilinear form and fulfills the Lie algebra 2-cocycle condition

$$0 = d_2\psi(x, y, z) := \psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y). \tag{63}$$

Two central extensions are equivalent if they essentially correspond only to the choice of different splitting maps. A 2-cochain ψ is a coboundary if there exists a linear form $\varphi : W \to \mathbb{C}$ such that

$$\psi(x, y) = \varphi([x, y]). \tag{64}$$

Every coboundary is a cocycle. The second Lie algebra cohomology $\mathrm{H}^2(W, \mathbb{C})$ of W with values in the trivial module $\mathbb{C}$ is defined as the quotient of the space of 2-cocycles modulo coboundaries. Moreover, two central extensions are equivalent if and only if the difference of their defining 2-cocycles ψ and ψ' is a coboundary. In this way the second Lie algebra cohomology $\mathrm{H}^2(W, \mathbb{C})$ classifies equivalence classes of central extensions. The class [0] corresponds to the trivial central extension. In this case the splitting map is a Lie homomorphism. We construct central extensions of our algebras by exhibiting such Lie algebra 2-cocycles.

Clearly, equivalent central extensions are isomorphic. The opposite is not true. In our case we can always rescale the central element by multiplying it with a nonzero scalar. This is an isomorphism but not an equivalence of central extensions. Nevertheless, it is an irrelevant modification. Hence we will be mainly interested in central extensions modulo equivalence and rescaling. They are classified by [0] and the elements of the projectivized cohomology space $\mathbb{P}(\mathrm{H}^2(W, \mathbb{C}))$.

In the classical case we have $\dim \mathrm{H}^2(\mathcal{W}, \mathbb{C}) = 1$, hence there are only two essentially different central extensions, the splitting one given by the direct sum $\mathbb{C} \oplus \mathcal{W}$ of Lie algebras and the up to equivalence and rescaling unique non-trivial one, the Virasoro algebra $\mathcal{V}$.

6.2 Geometric cocycles

The cocycle of the Witt algebra $1/12\ (n^3 - n)\delta_n^{-m}$ used to define the Virasoro algebra is very special. Obviously, it does not make any sense in the higher genus and/or multi-point case. We need to find a geometric description. For this we have first to introduce connections.

6.2.1 Projective and affine connections

Let $(U_\alpha, z_\alpha)_{\alpha \in J}$ be a covering of the Riemann surface by holomorphic coordinates with transition functions $z_\beta = f_{\beta\alpha}(z_\alpha)$.

Definition 6.1 (a) A system of local (holomorphic, meromorphic) functions $R = (R_\alpha(z_\alpha))$ is called a (holomorphic, meromorphic) *projective connection* if it transforms as

$$R_\beta(z_\beta)\cdot(f'_{\beta,\alpha})^2 = R_\alpha(z_\alpha) + S(f_{\beta,\alpha}), \qquad \text{with} \quad S(h) = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2, \tag{65}$$

the Schwartzian derivative. Here $'$ denotes differentiation with respect to the coordinate z_α.

(b) A system of local (holomorphic, meromorphic) functions $T = (T_\alpha(z_\alpha))$ is called a (holomorphic, meromorphic) *affine connection* if it transforms as

$$T_\beta(z_\beta)\cdot(f'_{\beta,\alpha}) = T_\alpha(z_\alpha) + \frac{f''_{\beta,\alpha}}{f'_{\beta,\alpha}}. \tag{66}$$

Every Riemann surface admits a holomorphic projective connection [25],[23]. Given a point P then there exists always a meromorphic affine connection holomorphic outside of P and having maximally a pole of order one there [44].

From their very definition it follows that the difference of two affine (projective) connections will be a (quadratic) differential. Hence, after fixing one affine (projective) connection all others are obtained by adding (quadratic) differentials.

Next we introduce in a geometric way cocycles by integration of certain differentials, associated to pairs of Lie algebra elements, over arbitrary smooth curves. Such cocycles we call geometric cocycles. For the proofs that the following expressions are indeed 2-cocycles we refer to [44], [48] (and [52]).

6.2.2 The function algebra $\mathcal{A}$

We consider it as abelian Lie algebra. Let C be an arbitrary smooth but not necessarily connected curve. We set

$$\psi^1_C(g,h) := \frac{1}{2\pi \mathrm{i}}\int_C g\,dh, \quad g,h \in \mathcal{A}. \tag{67}$$

6.2.3 The current algebra $\overline{\mathfrak{g}}$

For $\overline{\mathfrak{g}} = \mathfrak{g}\otimes\mathcal{A}$ we fix a symmetric, invariant, bilinear form β (not necessarily non-degenerate) on the finite-dimensional Lie algebra $\mathfrak{g}$. Recall, that invariance means that we have $\beta([x,y],z) = \beta(x,[y,z])$ for all $x,y,z\in\mathfrak{g}$. Then the following expressions define a cocycle

$$\psi^2_{C,\beta}(x\otimes g, y\otimes h) := \beta(x,y)\cdot\frac{1}{2\pi \mathrm{i}}\int_C g\,dh, \quad x,y\in\mathfrak{g},\ g,h\in\mathcal{A}. \tag{68}$$

6.2.4 The vector field algebra $\mathcal{L}$

Here it is a little bit more delicate. First we have to choose a (holomorphic) projective connection R. We define

$$\psi^3_{C,R}(e,f) := \frac{1}{24\pi\mathrm{i}} \int_C \left(\frac{1}{2}(e'''f - ef''') - R\cdot(e'f - ef')\right) dz\,. \tag{69}$$

Only by the term coming with the projective connection it will be a well-defined differential, i.e. independent of the coordinate chosen. Another choice of a projective connection will result in a cohomologous one. Hence, the equivalence class of the central extension will be the same.

6.2.5 The differential operator algebra $\mathcal{D}^1$

For the differential operator algebra the cocycles of type (67) for $\mathcal{A}$ can be extended by zero on the subspace $\mathcal{L}$. The cocycles for $\mathcal{L}$ can be pulled back. In addition there is a third type of cocycles mixing $\mathcal{A}$ and $\mathcal{L}$:

$$\psi^4_{C,T}(e,g) := \frac{1}{24\pi\mathrm{i}} \int_C (eg'' + Teg')dz, \qquad e \in \mathcal{L}, g \in \mathcal{A}, \tag{70}$$

with an affine connection T, with at most a pole of order one at a fixed point in O. Again, a different choice of the connection will not change the cohomology class.

6.2.6 The Lie superalgebra $\mathcal{S}$

Here we have to take into account that it is not a Lie algebra. Hence, the Jacobi identity has to be replaced by the super-Jacobi identity. The conditions for being a cocycle for the superalgebra cohomology will change too. Recall the definition of the algebra from Section 4.4, in particular that the even elements (parity 0) are the vector fields and the odd elements (parity 1) are the half-forms. A bilinear form c is a cocycle if the following is true. The bilinear map c will be symmetric if both x and y are odd, otherwise it will be antisymmetric:

$$c(x,y) = -(-1)^{\bar{x}\bar{y}}c(x,y). \tag{71}$$

The super-cocycle condition reads as

$$(-1)^{\bar{x}\bar{z}}c(x,[y,z]) + (-1)^{\bar{y}\bar{x}}c(y,[z,x]) + (-1)^{\bar{z}\bar{y}}c(z,[x,y]) \ = \ 0. \tag{72}$$

With the help of c we can define central extensions in the Lie superalgebra sense. If we put the condition that the central element is even then the cocycle c has to be an even map and c vanishes for pairs of elements of different parity.

By convention we denote vector fields by $e, f, g, \ldots$ and -1/2-forms by $\varphi, \psi, \chi, ..$ and get

$$c(e, \varphi) = 0, \quad e \in \mathcal{L},\ \varphi \in \mathcal{F}^{-1/2}. \tag{73}$$

The super-cocycle conditions for the even elements is just the cocycle condition for the Lie subalgebra $\mathcal{L}$. The only other nonvanishing super-cocycle condition is for the *(even,odd,odd)* elements and reads as

$$c(e, [\varphi, \psi]) - c(\varphi, e\,.\,\psi) - c(\psi, e\,.\,\varphi) = 0. \tag{74}$$

Here the definition of the product $[e, \psi] := e\,.\,\psi$ was used.

If we have a cocycle c for the algebra $\mathcal{S}$ we obtain by restriction a cocycle for the algebra $\mathcal{L}$. For the mixing term we know that $c(e, \psi) = 0$. A naive try to put just anything for $c(\varphi, \psi)$ (for example 0) will not work as (74) relates the restriction of the cocycle on $\mathcal{L}$ with its values on $\mathcal{F}^{-1/2}$.

Proposition 6.2 *[51] Let C be any closed (differentiable) curve on Σ not meeting the points in A, and let R be any (holomorphic) projective connection then the bilinear extension of*

$$\begin{aligned} \Phi_{C,R}(e, f) &:= \frac{1}{24\pi \mathrm{i}} \int_C \left(\frac{1}{2}(e''' f - e f''') - R \cdot (e' f - e f') \right) dz \\ \Phi_{C,R}(\varphi, \psi) &:= -\frac{1}{24\pi \mathrm{i}} \int_C (\varphi'' \cdot \psi + \varphi \cdot \psi'' - R \cdot \varphi \cdot \psi)\, dz \\ \Phi_{C,R}(e, \varphi) &:= 0 \end{aligned} \tag{75}$$

gives a Lie superalgebra cocycle for $\mathcal{S}$, hence defines a central extension of $\mathcal{S}$. A different projective connection will yield a cohomologous cocycle.

Note that the $\Phi_{C,R}$ restricted to $\mathcal{L}$ gives $\Psi^3_{C,R}$.

A similar formula was given by Bryant in [12]. By adding the projective connection in the second part of (75) he corrected some formula appearing in [6]. He only considered the two-point case and only the integration over a separating cycle. See also [32] for the multi-point case, where still only the integration over a separating cycle is considered.

In contrast to the differential operator algebra case the two parts cannot be prescribed independently. Only with the same integration path (more precisely, homology class) and the given factors in front of the integral it will work. The reason for this is that (74) relates both.

6.3 Uniqueness and classification of central extensions

The above introduced cocycles depend on the choice of the connections R and T. Different choices will not change the cohomology class. Hence, this ambiguity does not disturb us. What really matters is that they depend on the integration curve C chosen.

In contrast to the classical situation, for the higher genus and/or multi-point situation there are many essentially different closed curves and hence many non-equivalent central extensions defined by the integration.

But we should take into account that we want to extend the almost-grading from our algebras to the centrally extended ones. This means we take $\deg \hat{x} := \deg x$ and assign a degree $deg(t)$ to the central element t, and still we want to obtain almost-gradedness.

This is possible if and only if our defining cocycle ψ is *local*, or *almost-graded* in the following sense, There exists $M_1, M_2 \in \mathbb{Z}$ such that

$$\forall n, m : \quad \psi(W_n, W_m) \neq 0 \implies M_1 \leq n + m \leq M_2. \tag{76}$$

Here W stands for any of our algebras (including the supercase). The name "local" was introduced in the two point case by Krichever and Novikov in [33]).

Very important, "local" is defined in terms of the almost-grading, and the almost-grading itself depends on the splitting $A = I \cup O$. Hence what is "local" depends on the splitting too.

We will call a cocycle *bounded* (from above) if there exists $M \in \mathbb{Z}$ such that

$$\forall n, m : \quad \psi(W_n, W_m) \neq 0 \implies n + m \leq M. \tag{77}$$

Similarly bounded from below can be defined. Locality means bounded from above and from below.

Given a cocycle class we call it *bounded* (respectively *local*) if and only if it contains a representing cocycle which is bounded (respectively local). Not all cocycles in a bounded class have to be bounded. If we choose as integration path a separating cocycle C_S, or one of the C_i then the above introduced geometric cocycles are local, respectively bounded (from above). Recall that in this case integration can be done by calculating residues at the in-points or at the out-points. All these cocycles are cohomologically nontrivial. The theorems in the following concern the opposite direction. They were treated in my works [48], [47], [51]. See also [52] for a complete and common treatment.

The following result for the vector field algebra $\mathcal{L}$ gives the principal structure of the classification results.

Theorem 6.3 *[48], [52, Thm. 6.41] Let $\mathcal{L}$ be the Krichever–Novikov vector field algebra with a given almost-grading induced by the splitting $A = I \cup O$.*
(a) The space of bounded cohomology classes is K-dimensional ($K = \#I$). A basis is given by setting the integration path in (69) to C_i, $i = 1, \ldots, K$ the little (deformed) circles around the points $P_i \in I$.
(b) The space of local cohomology classes is one-dimensional. A generator is given by integrating (69) over a separating cocycle C_S, i.e.

$$\psi^3_{C_S,R}(e, f) = \frac{1}{24\pi \mathrm{i}} \int_{C_S} \left(\frac{1}{2}(e'''f - ef''') - R \cdot (e'f - ef') \right) dz \,. \tag{78}$$

(c) Up to equivalence and rescaling there is only one non-trivial one-dimensional central extension $\widehat{\mathcal{L}}$ of the vector field algebra $\mathcal{L}$ which allows an extension of the almost-grading.

Remark 6.4 In the classical situation, Part (c) shows also that the Virasoro algebra is the unique non-trivial central extension of the Witt algebra (up to equivalence and rescaling). This result extends to the more general situation under the condition that one fixes the almost-grading, hence the splitting $A = I \cup O$. Here I like to repeat the fact that for $\mathcal{L}$ depending on the set A and its possible splittings into two disjoint subsets there are different almost-gradings. Hence, the "unique" central extension finally obtained will also depend on the splitting. Only in the two point case there is only one splitting possible. In the case that the genus $g \geq 1$ there are even integration paths possible in the definition of (69) which are not homologous to a separating cycle of any splitting. Hence, there are other central extensions possible not corresponding to any almost-grading.

The above theorem is a model for all other classification results. We will always obtain a statement about the bounded (from above) cocycles and then for the local cocycles.

If we consider the function algebra $\mathcal{A}$ as an abelian Lie algebra then every skew-symmetric bilinear form will be a non-trivial cocycle. Hence, there is no hope of uniqueness. But if we add the condition of $\mathcal{L}$-invariance to the cocycle, which is given as

$$\psi(e.g,h) + \psi(g,e.h) = 0, \quad \forall e \in \mathcal{L},\ g,h \in \mathcal{A} \tag{79}$$

things will change. Note that the cocycle (67) is $\mathcal{L}$-invariant.

Let us denote the subspace of local cohomology classes by H^2_{loc}, and the subspace of local and $\mathcal{L}$-invariant cohomology classes by $\mathrm{H}^2_{\mathcal{L},loc}$. Note that the conditions are only required for at least one representative in the cohomology class. We collect a part of the results for the cocycle classes of the other algebras in the following theorem.

Theorem 6.5 *[52, Cor. 6.48]*

(a) $\dim \mathrm{H}^2_{\mathcal{L},loc}(\mathcal{A},\mathbb{C}) = 1$,

(b) $\dim \mathrm{H}^2_{loc}(\mathcal{L},\mathbb{C}) = 1$,

(c) $\dim \mathrm{H}^2_{loc}(\mathcal{D}^1,\mathbb{C}) = 3$,

(d) $\dim \mathrm{H}^2_{loc}(\overline{\mathfrak{g}},\mathbb{C}) = 1$ *for* $\mathfrak{g}$ *a simple finite-dimensional Lie algebra,*

(e) $\dim \mathrm{H}^2_{loc}(\mathcal{S},\mathbb{C}) = 1$,

A basis of the cohomology spaces are given by taking the cohomology classes of the cocycles (67), (69), (70), (68), (75) obtained by integration over a separating cycle C_S.

Consequently, we obtain also for these algebras the corresponding result about uniqueness of almost-graded central extensions. For the differential operator algebra we get three independent cocycles. This generalizes results obtained in [2] for the classical case.

For results on the bounded cocycle classes we have to multiply the dimensions above by $K = \#I$. For the supercase with odd central element the bounded cohomology vanishes.

For $\mathfrak{g}$ a reductive Lie algebra and if the cocycle is $\mathcal{L}$-invariant if restricted to the abelian part, a complete classification of local cocycle classes for both $\overline{\mathfrak{g}}$ and $\mathcal{D}^1_{\mathfrak{g}}$ can be found in [47], [52, Chapter 9].

I like to mention that in all the applications I know of, the cocycles coming from representations, regularizations, etc. are local. Hence, the uniqueness or classification result above can be applied.

Remark 6.6 (On $\mathcal{L}$-invariance) The $\mathcal{L}$-invariance is a natural condition. If we restrict a cocycle for the differential operator algebra $\mathcal{D}^1$ to the subalgebra $\mathcal{A}$ of functions we obtain automatically an $\mathcal{L}$-invariant cocycle for $\mathcal{A}$. Vice versa, every $\mathcal{L}$-invariant cocycle for A can be extended by zero on the complementary space $\mathcal{L}$ to a cocycle for $\mathcal{D}^1$.

The cocycle (67) has another interesting property. It is *multiplicative*, i.e.

$$\psi(a \cdot b, c) + \psi(b \cdot c, a) + \psi(c \cdot a, b) = 0, \qquad a, b, c \in \mathcal{A}. \tag{80}$$

It is this property which is needed that (68) is a cocycle for $\overline{\mathfrak{g}}$, see also [47]. It is shown in [48] that the classes of multiplicative bounded cocycles equals the classes of bounded $\mathcal{L}$-invariant cocycles.

7 The Genus Zero and Multi-Point Case

In the second half of this survey we consider the situation that our Riemann surface is the Riemann sphere (i.e. of genus zero) and that we have a finite number of marked points where poles are allowed. Again the classical situation is a special case.

There are many reasons to study this situation in detail. First, of course we give an illustration of the definition and construction done in arbitrary genus in an explicit way using explicit forms of generators. Second, we are able to deduce additional properties in the genus zero case. In particular, we will construct the universal central extensions of those algebras which admit an universal central extension.

The third reason is that recently there was a revived interest in the genus zero multipoint situation by a number of people coming from field theory and representation theory. In these communities the algebras are called *N-Virasoro algebras*.

But in their work is no almost-graded structure used. Now, in the case of infinite dimensional Lie algebra the existence of a grading respectively the weaker version of an almost-grading is crucial for understanding and constructions. It was my intention in [54] to show that these N-Virasoro algebras are nothing else as multi-point genus zero KN type algebras and to use the theory of KN type (in particular the almost-gradedness) to deduce faster and more conceptual their results and even more, giving explanations of observed results.

In the following I will review our results. For the proofs I mainly refer to [54].

7.1 The geometric set-up

Now let Σ_0 be the Riemann sphere S^2, or equivalently $\mathbb{P}^1(\mathbb{C})$ with the quasi-global coordinate z. We call it quasi-global, as it is not defined at the point ∞. Let us denote the set of points

$$A = \{P_1, P_2, \ldots, P_N\}, \quad P_i \neq P_j, \text{ for } i \neq j. \tag{81}$$

For notational simplicity we single out the point P_N as reference point. By an automorphism of $\mathbb{P}^1(\mathbb{C})$, i.e. a fractional linear transformation or equivalently an element of $\mathrm{PGL}(2, \mathbb{C})$, the point P_N can be brought to ∞. In fact two more points could be normalized to be 0 and 1. In this section we will not do so, but see Section 8.

Our points are given by their global coordinates

$$P_i = a_i, \quad a_i \in \mathbb{C}, \ i = 1, \ldots, N-1, \qquad P_N = \infty. \tag{82}$$

At these points we have the local coordinates

$$z - a_i, \ i = 1, \ldots, N-1, \qquad w = 1/z. \tag{83}$$

Recall that the canonical line bundle $\mathcal{K}$ of Σ_0 is the holomorphic line bundle whose local sections are the local holomorphic differentials. For $\mathbb{P}^1(\mathbb{C})$, in the language of algebraic geometry, we have that $\mathcal{K} = \mathcal{O}(-2)$. This bundle has a unique square root $L = \mathcal{O}(-1)$, which is the tautological bundle, respectively the dual of the hyperplane section bundle. We denote this bundle also by $\mathcal{K}^{1/2}$ [5]. We introduced meromorphic forms of weight λ as sections of the bundle $\mathcal{K}^\lambda$. In our genus zero situation we can describe the form by a meromorphic function on the affine part $\mathbb{C}$ with respect to the coordinate z. By this description its behaviour at the point ∞ is uniquely fixed by the fundamental transformation $dz = -w^{-2}dw$. Moreover, by the fixing $P_N = \infty$ the set of meromorphic forms f of weight λ on $\mathbb{P}^1(\mathbb{C})$ holomorphic outside of A correspond 1:1 to meromorphic functions $a(z)$ holomorphic outside of A via $f(z) = a(z)dz^\lambda$. Both a and f will have the same orders at the points in $\mathbb{C}$. For the order at the point ∞ we have

$$\mathrm{ord}_\infty(f) = \mathrm{ord}_\infty(a) - 2\lambda. \tag{84}$$

Recall that on a compact Riemann surface the sum of the orders (summed over all points) of a meromorphic function $f \not\equiv 0$ equals zero. Hence, more generally

Proposition 7.1 *Let $f \in \mathcal{F}^\lambda$, $f \not\equiv 0$ then*

$$\sum_{P \in \Sigma_0} \mathrm{ord}_P(f) = -2\lambda. \tag{85}$$

For this and related results see e.g. [49]. Also recall that the meromorphic functions in our case are nothing else as rational functions with respect to the variable z.

Recall from Section 4 that

[5] There is no ambiguity in choosing the square root.

$$\mathcal{F} := \bigoplus_{\lambda \in \frac{1}{2}\mathbb{Z}} \mathcal{F}^\lambda. \tag{86}$$

We specialize from Section 5 the almost-grading for genus zero as follows. We numerate the points in the splitting like

$$A = I \cup O, \qquad I = (P_1, P_2, \ldots, P_K), \quad O = (P_{K+1}, \ldots, P_N = \infty). \tag{87}$$

As $P_N = \infty \in O$ it is enough to construct a basis $\{A_{n,p} \mid n \in \mathbb{Z}, p = 1, \ldots, K\}$ for $\mathcal{A} = \mathcal{F}^0$. The decomposition of $\mathcal{A}$ into homogeneous subspaces induces a decomposition of $\mathcal{F}^\lambda$ by

$$\mathcal{F}_n^\lambda = \mathcal{A}_{n-\lambda} dz^\lambda, \quad \text{respectively} \quad f_{n,p}^\lambda = A_{n-\lambda,p} dz^\lambda. \tag{88}$$

The shift by $-\lambda$ is quite convenient and is beside other things related to the duality property. The recipe for constructing the $A_{n,p}$ is given in [43], [42], see also [52]. As a principal property we have (like in the arbitrary genus case)

$$\mathrm{ord}_{P_i}(A_{n,p}) = (n+1) - \delta_i^p, \ i = 1, \ldots, K\,. \tag{89}$$

At the points in O corresponding orders are set to make the element unique up to multiplication by a non-zero scalar.

Example 7.2 We call the splitting

$$I = (P_1, P_2, \ldots, P_K), \quad O = (P_N = \infty), \quad K = N - 1 \tag{90}$$

the *standard splitting*. Recall that P_i corresponds to the point given by the coordinate $z = a_i$. We set

$$\alpha(p) := \Big(\prod_{\substack{i=1\\ i\neq p}}^{K} (a_p - a_i)\Big)^{-1}. \tag{91}$$

and define

$$A_{n,p}(z) := (z - a_p)^n \cdot \prod_{\substack{i=1\\ i\neq p}}^{K} (z - a_i)^{n+1} \cdot \alpha(p)^{n+1}\,. \tag{92}$$

The last factor is a normalization factor yielding

$$A_{n,p}(z) = (z - a_p)^n (1 + O(z - a_p)). \tag{93}$$

With this description the order at ∞ is fixed as

$$-(Kn + K - 1)\,. \tag{94}$$

In particular, we obtain for $\mathcal{L}$ the basis

$$e_{n,p} = f_{n,p}^{-1} = A_{n+1,p} \frac{d}{dz} = (z - a_p)^{n+1} \cdot \prod_{\substack{i=1 \\ i \neq p}}^{K} (z - a_i)^{n+2} \cdot \alpha(p)^{n+2} \frac{d}{dz}. \tag{95}$$

7.2 The Algebras for the Standard Splitting

Theorem 5.6 is of course also valid in our special genus zero situation. We will have a closer look on the result for the standard splitting. Recall that in case of the standard splitting the set O consists only of the point ∞.

For illustration we give the bounds R_1 and R_2 in Theorem 5.6

Proposition 7.3

$$R_1 = \begin{cases} 0, & K = 1, \\ 1, & K > 1, \end{cases} \qquad R_2 = \begin{cases} 0, & K = 1, \\ 1, & K = 2, \\ 2, & K > 2\,. \end{cases} \tag{96}$$

Proof For the upper bound we have to check the order at ∞ with respect to the variable w. We use for the individual factors in the expression (53) the value (94) and sum over all factors and do not forget to decrease the order by 2 coming from dz. If we do this for the algebra $\mathcal{A}$ we get as order at ∞ for $\omega := A_{n,p} \cdot A_{m,r} \cdot A_{-(n+m+k)-1,s} dz$ the value $-2K + K \cdot k + 1$. A pole is only possible if this value is ≤ -1. Hence only for

$$k \leq -\frac{2}{K} + 2\,. \tag{97}$$

This yields the claimed value for R_1. For the Lie algebra $\mathcal{L}$, respectively for the Lie module we have to consider $\omega := [A_{n+1,p}, A_{m-\lambda,r}] \cdot A_{-(n+m+k)-(1-\lambda)}\, dz$. Here the As should be taken as the local expressions of the vector fields, resp. of the forms. For the order at ∞ we obtain $-3K + K \cdot k + 2$. Which yields that a pole is only possible for

$$k \leq -\frac{3}{K} + 3\,, \tag{98}$$

and hence the claimed value for R_2. □

Indeed these bounds are also valid for $K = M$.

For the algebra $\mathcal{A}$ we obtain by using (96) and (56)

$$A_{n,i} \cdot A_{m,j} = \delta_i^j\, A_{n+m,j} + \sum_{s=1}^{K} \alpha_{(n,i)(m,j)}^{(n+m+1,s)} A_{n+m+1,s}\ . \tag{99}$$

The unknown structure coefficients in (99) of the algebras can be directly calculated by calculating the residues of the rational expression for ω.

Example 7.4 $N = 2$**.** By a $\mathrm{PGL}(2,\mathbb{C})$ action the two points can be transported to 0 and ∞. This is the classical situation and there is only one splitting. Hence, everything is fixed. The above basis gives back the conventional one.

Example 7.5 $N = 3$**.** Here by a $\mathrm{PGL}(2,\mathbb{C})$ action the three points can be normalized to $\{0, 1, \infty\}$. Hence, up to isomorphy there are only one three-point algebra (for each type of algebras). If we fix such an algebra we obtain three different splittings of the set $\{0, 1, \infty\}$ and consequently also 3 essentially different almost-gradings, triangular decompositions, etc. The three-point case is in a certain sense special as there are still the automorphism of $\mathbb{P}^1(\mathbb{C})$ permuting these three points. They induce automorphisms of the algebras which permute the almost-gradings. We will consider this situation in detail in Section 8.

Example 7.6 $N = 4$**.** This is the first case where we have a moduli parameter for the geometric situation. We normalize our A to

$$\{0, 1, a, \infty\}, \qquad a \in \mathbb{C},\ a \neq 0, 1\,. \tag{100}$$

We have 2 different types of splittings, i.e. the type $4 = 3 + 1$ and the type $4 = 2 + 2$. For example

$$\{0, 1, a\} \cup \{\infty\}, \quad \text{and} \quad \{0, 1\} \cup \{a, \infty\}\,. \tag{101}$$

The first type is the standard splitting for which we gave the basis above (92). For the second splitting a basis of $\mathcal{A}$ and hence of all $\mathcal{F}^\lambda$ is

$$\begin{aligned} A_{n,1}(z) &= z^n (z-1)^{n+1} (z-a)^{-(n+1)} a^{(n+1)}, \\ A_{n,2}(z) &= z^{n+1} (z-1)^n (z-a)^{-(n+1)} (1-a)^{(n+1)}, \end{aligned} \tag{102}$$

where $n \in \mathbb{Z}$. The last factor is again a normalization constant. This basis defines an almost-grading for the four-point algebra which is not equivalent to the standard almost-grading. Again the algebra coefficients can be calculated easily via residues. Using (96) for $K = M = 2$ the general expression is given also by (99).

7.3 Central Extensions

Recall that in Section 6 we gave expressions for the central extensions by integrating differential forms $\widehat{\gamma}$ over cycles on the Riemann surface yielding cocycles. Such cocycles we called geometric cocycles. The relevant space is the homology space $\mathrm{H}_1(\Sigma \setminus A, \mathbb{C})$.

For genus zero and $N \geq 1$ we have

$$\dim \mathrm{H}_1(\Sigma_0 \setminus A, \mathbb{C}) = (N - 1). \tag{103}$$

A basis of this cohomology space is given by circles C_i around the points P_i where we leave out one of them. For example we can take $[C_i]$, $i = 1, \ldots, N - 1$. We have the relation

$$\sum_{i=1}^{N-1}[C_i] = -[C_N]. \tag{104}$$

But there is another choice. As explained above after choosing a splitting with separating cycle $[C_S]$ we take it as one of the basis elements and $N-2$ other $[C_i]s$ which are linearly independent. For the standard splitting with $P_N = \{\infty\}$ we have

$$[C_S] = -[C_\infty], \qquad [C_i],\ i = 1,\ldots,N-2. \tag{105}$$

There is a crucial difference to the case of higher genus. In genus zero our cycles are always build from circles around our points where poles are allowed. This is not the case in higher genus. And this makes a big difference. Integration around the C_i can be done via calculations of residues. Hence, we always get for our geometric cocycles (for the standard splitting)

$$\gamma_{[C_S]} = \sum_{i=1}^{N-1} \operatorname{res}_{P_i}(\widehat{\gamma}) = -\operatorname{res}_\infty(\widehat{\gamma}), \qquad \gamma_{[C_i]} = \operatorname{res}_{P_i}(\widehat{\gamma}),\ i = 1,\ldots,N-2. \tag{106}$$

Another difference to higher genus is that that in genus zero and our system of coordinates z_i the coordinate change maps are projective transformation and the Schwartzian derivative vanishes, see (65). Hence, the projective connection R appearing in the definition of the cocycle for the vector field algebra can be set to $R = 0$ on all our coordinate patches. In the mixing cocycle for the differential operator algebra there appears the affine connection. We cannot take $T = 0$ globally, but from (66) it follows that we may take $T_i = 0$ on the affine part and $T(w) = -2/w$ around the point ∞.

If we consider the standard splitting and specialize the cocycles γ introduced in Section 6 then the theorems presented there show that $\gamma_{[C_S]}$ will be local, whereas the $\gamma_{[C_i]}$ are bounded. We also know that the $\gamma_{[C_i]}$, $i = 1,\ldots,N-1$ gives a basis of the space of bounded cocycle classes. Here bounded means bounded with respect to the standard splitting where the only point in O is the point ∞. For simplicity we will sometimes use γ_i for $\gamma_{[C_i]}$ and γ_S for $\gamma_{[C_S]}$.

7.4 Universal central extensions

In the following we will show that in the genus zero case all cocycle classes will be bounded. Moreover, we will describe with this result the universal central extensions in case that they exist.

Theorem 7.7 *In the case of genus zero all cocycle classes for the algebras in Section 4 are bounded classes with respect to the standard splitting. In particular, a basis of all cocycle classes is given by* $\gamma_i,\ i = 1,\ldots,N-1$. *This is also true for the function algebra* $\mathcal{A}$ *if we assume that the cocycle is* $\mathcal{L}$*-invariant or multiplicative.*

Note that we do not make any reference on any splitting at the beginning. Also we remark that this theorem is only true in genus zero. Below we will make some comments on the proof. But first we draw some consequences.

First we take a look on the function algebra $\mathcal{A}$.

Theorem 7.8 *In the N-point genus zero situation the space of $\mathcal{L}$-invariant (or multiplicative) cocycles is $(N-1)$ dimensional and is isomorphic to* $\mathrm{H}_1(\Sigma_0 \setminus A, \mathbb{C})$ *via*

$$[C] \mapsto \gamma_C^{\mathcal{A}}; \qquad \gamma_C^{\mathcal{A}}(f,g) = \frac{1}{2\pi \mathrm{i}} \int_C f dg. \tag{107}$$

In particular, every $\mathcal{L}$-invariant cocycle is multiplicative and vice versa. These cocycles are geometric.

Proposition 7.9 *Up to rescaling the cocycle*

$$\gamma_\infty^{\mathcal{A}} = -\sum_{i=1}^{N-1} \gamma_{C_i}^{\mathcal{A}}(f,g) = \mathrm{res}_\infty(f dg) \tag{108}$$

is the unique $\mathcal{L}$-invariant (and equivalently multiplicative) cocycle which is local with respect to the standard splitting.

Remark 7.10 (Heisenberg algebra.) We consider the central extensions of $\mathcal{A}$ with central terms coming from the $\mathcal{L}$-invariant or equivalently multiplicative cocycles. The corresponding central extension $\widehat{\mathcal{A}}$ of $\mathcal{A}$ will have a $(N-1)$-dimensional center and will be given as

$$[\widehat{f}, \widehat{g}] = \sum_{i=1}^{N-1} \alpha_i \cdot \gamma_{C_i}^{\mathcal{A}}(f,g) \cdot t_i, \quad \alpha_i \in \mathbb{C}, \quad [t_i, \widehat{\mathcal{A}}] = 0\,. \tag{109}$$

The local cocycle $\gamma_{C_S}^{\mathcal{A}}$ will yield a one-dimensional central extension which is almost-graded. One might either call the $(N-1)$-dimensional central extension or the almost-graded one-dimensional central extension *(multi-point) Heisenberg algebra*.

Now we turn to the other algebras. A Lie algebra is called perfect if the commutator ideal of L equals L, i.e. $L = [L, L]$. Recall that a Lie algebra L has a universal central extension if and only if it is perfect. In this case the (higher-dimensional) center can be given by the space of cocycle classes $\mathrm{H}^2(L, \mathbb{C})$.

Proposition 7.11 *The genus zero KN type vector field algebras, differential operator algebras, the super algebras, and current algebras for semi-simple finite Lie algebras are perfect Lie algebras and consequently admit central extensions.*

For the proofs see e.g. [54], resp. the Remark 7.13 below. For the general genus vector field case see also [60] and for the general current algebra case see also [31], [30]. Obviously, the function algebra as abelian Lie algebra is not perfect.

Combining Proposition 7.11 and Theorem 7.7 we obtain

Theorem 7.12
(a) In genus zero with N marked points the vector field algebra $\mathcal{L}$ has a universal central extension with $(N-1)$-dimensional center.

(b) The differential operator algebra $\mathcal{D}^1$ has a universal central extension with $3(N-1)$-dimensional center.

(c) The Lie superalgebra $\mathcal{S}$ has a universal central extension with $(N-1)$-dimensional center.

(d) The current algebra $\overline{\mathfrak{g}}$ associated to a simple Lie algebra $\mathfrak{g}$ has a universal central extension with $(N-1)$ dimensional center.

The corresponding cocycles and hence the universal central extension can be explicitly given by the cocycles introduced in Section 6, in Equations (69), (67), (70), (75), (68), where the integration runs over C_i, $i = 1, \ldots, N-1$. Equivalently they can be calculated via residues at the point P_i, $i = 1, \ldots, N-1$.

Proof (of Theorem 7.7) We only give some rough ideas of the proof. The complete proof needs several pages and can be found in the original publication [54]. Also we will only deal with the vector field algebra as a typical example.

The following elements

$$e_n^{(i)} = (z-a_i)^{n+1}\frac{d}{dz}, \quad n \in \mathbb{Z},\ i = 1, \ldots, N-1. \tag{110}$$

give a generating set of $\mathcal{L}$. For each i separately, we get the usual Witt algebra as subalgebra of $\mathcal{L}$. Now we fix one of the superscripts, e.g. $i = 1$, then

$$e_n^{(1)},\ n \in \mathbb{Z}, \qquad e_m^{(i)}, \quad m \le -2,\ i = 2, \ldots, N-1\ . \tag{111}$$

is a basis [41]. This basis does not define any almost-grading. But we can relate it to the filtration (60) defined with respect to the almost-grading given by the standard splitting. With respect to this splitting we denote the basis elements as usual by $e_{n,p}$ We repeat the definition and the property of the filtration for $\lambda = -1$

$$\begin{aligned} \mathcal{L}_{(n)} &= \bigoplus_{m \ge n} \mathcal{L}_m, \qquad \mathcal{L}_m := \langle e_{m,p} \mid p = 1, \ldots, K \rangle\ , \\ \mathcal{L}_{(n)} &= \{ f \in \mathcal{L} \mid \mathrm{ord}_{P_i}(f) \ge n+1,\ i = 1,\ .,N-1\ \}\ . \end{aligned} \tag{112}$$

Now we take the alternate basis into account and get

$$\begin{aligned} \mathcal{L}_{(n)} &= \bigcap_{i=1,\ldots,N-1} \langle e_k^{(i)} \mid k \ge n \rangle_{\mathbb{C}}, \qquad \text{for } n \ge -1 \\ \mathcal{L}_{(n)} &= \sum_{i=1}^{N-1} \langle e_k^{(i)} \mid k \ge n \rangle_{\mathbb{C}}, \qquad \text{for } n < -1\ . \end{aligned} \tag{113}$$

Attention, the sum in the second line is not a direct sum. Also note that $\mathcal{L}_{-1}$ consists of those vector fields which are holomorphic on the affine part. In particular for $\mathcal{L}_{-1}$ all sets in the intersection are equal, hence it is enough to consider just one i.

We make a cohomological change via the linear form $\phi : \mathcal{L} \to \mathbb{C}$ with

$$\begin{gathered} \phi(e_n^{(1)}) := \frac{1}{n}\gamma(e_0^{(1)}, e_n^{(1)}), \quad n \in \mathbb{Z}, n \neq 0 \qquad \phi(e_0^{(1)}) := \frac{1}{2}\gamma(e_{-1}^{(1)}, e_1^{(1)}), \\ \phi(e_n^{(i)}) := \frac{1}{n}\gamma(e_0^{(i)}, e_n^{(i)}), \quad n \leq -2\,. \end{gathered} \tag{114}$$

The cohomologous cocycle is now

$$\gamma'(e, f) = \gamma(e, f) - \phi([e, f]). \tag{115}$$

With the help of the filtration property and (113) we are able to show that

$$\gamma'(e, f) = 0 \qquad e \in \mathcal{L}_k, f \in \mathcal{L}_l \quad \text{if } k + l > 0. \tag{116}$$

Hence, $[\gamma]$ is bounded class class for the standard splitting and we can use Theorem 6.3 to conclude the proof. □

Remark 7.13 The alternative generating set introduced above shows also that $\mathcal{L}$ is a perfect algebra. As $[e_o^{(i)}, e_n^{(i)}] = n \cdot e_n^{(i)}$ and $[e_{-1}^{(i)}, e_1^{(i)}] = 2 \cdot e_0^{(i)}$ all generators can be written as commutators, i.e. $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$.

Remark 7.14 Let me give here some references to other works on multi-point genus zero work: [1], [5], [7], [8], [9], [13], [14], [15], [16], [17], [18], [24], [26], [27], [45].

Remark 7.15 In the genus one case, i.e. the elliptic curve case, also interesting algebras show up, see e.g. [45], [19], [20] [21] [39], [11] [10]. In joint work with Alice Fialowski the current author examined deformations of the elliptic algebras to genus zero algebras, respectively subalgebras of them.

8 The three-point and Genus Zero Case

8.1 Symmetries

The case of genus zero with only three points where poles are allowed is to a certain extend special as we have additional symmetries. These symmetries can be used to simplify the calculations of structure constants even further.

Additionally, the three-point cases play a role in quite a number of applications. See e.g. the tetrahedron algebra appearing in statistical mechanics, in particular the work of Terwilliger and collaborators [24], [5].

By a complex automorphism of the Riemann sphere, i.e. by a fractional linear transformation, respectively by an PGL(2) action the three points can be brought to the points $0, 1$ and ∞. The corresponding automorphism will yield an isomorphism of the involved algebras. Even after this is done there are still automorphisms of $\mathbb{P}^1(\mathbb{C})$ permuting the 3 points $\{0, 1, \infty\}$. Hence, we still have the action of the symmetric group S_3 of 3 elements. The corresponding algebraic maps are now automorphisms of the algebras.

Recall that in the previous section we discussed for every splitting of the set A, (here $\{0, 1, \infty\}$), into two disjoint non-empty subset I and O a distinguished basis which yields an almost-graded structure for the algebras. Essentially different splittings will yield essentially different basis elements respectively essentially different almost-gradings.

Here the only type of splitting is into a subset consisting of two points and a subset consisting of one point. After having fixed $A = \{0, 1, \infty\}$ we apply an automorphism from the remaining group S_3 such that

$$A = I \cup O, \qquad I := \{0, 1\}, \quad \text{and} \quad O := \{\infty\}. \tag{117}$$

This is exactly the situation which we will consider here.

I like to stress the fact, that this does not say, that there is only one possible choice of an almost-grading. In fact, given the set A and hence a unique algebra, we have 3 essentially different splitting, hence also 3 essentially different set of basis elements and consequently 3 almost-gradings. But in the three-point situation there will be always an automorphism of our algebra mapping the different almost-gradings to each other.

Remark 8.1 In [45] the situation $A = \{\alpha, -\alpha, \infty\}$ was considered for $\alpha \in \mathbb{C}$, $\alpha \neq 0$. This changes nothing. The corresponding algebras are isomorphic to the algebras considered here. Even our basis elements can be identified (up to some rescaling and re-indexing) and the structure equations remain the same (again up to scaling factors). The reason for the choice there was that we wanted to introduce a free parameter α which can be used to study degeneration process, respectively deformation families. For $\alpha \to 0$ we "degenerate" to the two-point situation. In this respect see also our joint work with Fialowski [19], [20], [21]. Clearly, everything that will be done in this section could be formulated also for $\{\alpha, -\alpha, \infty\}$, $\alpha \neq 0$.

8.2 The associative algebra

The basis elements of degree n of the algebra $\mathcal{A}$ with respect to our standard splitting are (see Section 7)

$$A_{n,1}(z) = z^n(z-1)^{n+1}, \qquad A_{n,2}(z) = z^{n+1}(z-1)^n, \quad n \in \mathbb{Z}, \tag{118}$$

where we ignore the scaling factors. We "symmetrize" and "anti-symmetrize" them for each degree separately by taking

$$\begin{aligned} A_n(z) = A_{n,2}(z) - A_{n,1}(z) \quad &= z^n(z-1)^n, \\ B_n(z) = A_{n,2}(z) + A_{n,1}(z) \quad &= z^n(z-1)^n(2z-1), \end{aligned} \tag{119}$$

Proposition 8.2 *The associative algebra $\mathcal{A}$ of meromorphic functions on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ holomorphic outside of* $0,\ 1,$ *and* ∞ *has as vector space basis*

$$\{A_n, B_n \mid n \in \mathbb{Z}\}, \tag{120}$$

with the structure equations

$$\begin{aligned} A_n \cdot A_m &= A_{n+m}, \\ A_n \cdot B_m &= B_{n+m}, \\ B_n \cdot B_m &= A_{n+m} + 4A_{n+m+1}. \end{aligned} \tag{121}$$

Proof That the elements $A_{n,1}, A_{n,2}$ with $n \in \mathbb{Z}$ are a basis of $\mathcal{A}$ is a general fact by its very construction as Krichever–Novikov multi-point basis in [42] corresponding to our splitting of A. The transformation (119) is obviously a base change which even respects the homogeneous subspaces. By direct calculations the structure equations follow. □

Our splitting introduces a (strong) almost-grading for the algebra $\mathcal{A}$

$$\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_n, \qquad \mathcal{A}_n = \langle A_n, B_n \rangle_{\mathbb{C}}, \quad \dim \mathcal{A}_n = 2. \tag{122}$$

This is clear from the general construction. But it can be easily illustrated by (121) as

$$\mathcal{A}_n \cdot \mathcal{A}_m \quad \subseteq \quad \mathcal{A}_{n+m} \oplus \mathcal{A}_{n+m+1}. \tag{123}$$

Next we want to study central extensions of $\mathcal{A}$ (considered as Lie algebras) which are given by geometric cocycles as introduced in Section 6 and further discussed in Section 7. We showed that the 2 following cocycles $\gamma_0^{\mathcal{A}}$, $\gamma_\infty^{\mathcal{A}}$ defined by

$$\gamma_0^{\mathcal{A}}(f,g) = \operatorname{res}_0(f\,dg), \qquad \gamma_\infty^{\mathcal{A}}(f,g) = \operatorname{res}_\infty(f\,dg), \tag{124}$$

constitute a basis of the geometric cocycles. Recall that the set of geometric cocycles coincide with the $\mathcal{L}$-invariant respectively multiplicative cocycles. Note also that

$$\gamma_1^{\mathcal{A}}(f,g) = \operatorname{res}_1(f\,dg) = -\gamma_0^{\mathcal{A}}(f,g) - \gamma_\infty^{\mathcal{A}}(f,g) \tag{125}$$

gives the linearly dependent 3. cocycle. Recall also that $\operatorname{res}_a(f\,dg) = -\operatorname{res}_a(g\,df)$, $a \in \Sigma_0$.

To calculate the cocycles it is enough to do the calculation for all type of pairs of basis elements A_n and B_m. See [54] for details of the calculation. We start with the result

Proposition 8.3

$$\begin{aligned}
\gamma_\infty^{\mathcal{A}}(A_n, A_m) &= 2n\,\delta_m^{-n},\\
\gamma_\infty^{\mathcal{A}}(A_n, B_m) &= 0,\\
\gamma_\infty^{\mathcal{A}}(B_n, B_m) &= 2n\delta_m^{-n} + 4(2n+1)\,\delta_m^{-n-1}\ .
\end{aligned}$$

Proposition 8.4

$$\begin{aligned}
\gamma_0^{\mathcal{A}}(A_n, A_m) &= -n\,\delta_m^{-n},\\
\gamma_0^{\mathcal{A}}(A_n, B_m) &= n\,\delta_m^{-n} + 2n\,\delta_m^{-n-1} + \sum_{k=2}^{\infty} n\,(-1)^{k-1}2^k\,\frac{(2k-3)!!}{k!}\delta_m^{-n-k},\\
\gamma_0^{\mathcal{A}}(B_n, B_m) &= -n\delta_m^{-n} - 2(2n+1)\,\delta_m^{-n-1}\ .
\end{aligned}$$

Here $(2k-1)!! = 1\cdot 3\cdots(2k-1)$ is the double factorial.

Note that the second relation in the above proposition is expressed as a formal infinite sum. But for given n and m maximally one term will be non-zero. Hence, it has a well-defined value.

In accordance with the general results [48] about local cocycles only the cocycle $\gamma_\infty^{\mathcal{A}}$ is local. Here it will vanish for n, m outside of $-1 \le n+m \le 0$. Consequently, only the central extension, the Heisenberg algebra, defined via $\gamma_\infty^{\mathcal{A}}$ will admit an extension of the almost-grading to the central extension. Consequently, we obtain only in this case a triangular decomposition which is of importance for the representations appearing in field theory. This is not possible for the central extension defined by $\gamma_0^{\mathcal{A}}$ nor by $\gamma_1^{\mathcal{A}}$.

8.3 Current and affine algebra

Recall that for a finite-dimensional Lie algebra $\mathfrak{g}$ the current algebra of KN-type is defined by $\overline{\mathfrak{g}} = \mathfrak{g}\otimes\mathcal{A}$. In particular every choice of a basis in $\mathfrak{g}$ and a basis in $\mathcal{A}$ will yield a basis of $\overline{\mathfrak{g}}$. For $\mathcal{A}$ we choose the basis A_n, B_n, $n\in\mathbb{Z}$ introduced above. Automatically we get an almost-graded structure of $\overline{\mathfrak{g}}$ induced by the splitting $\{0,1\}\cup\{\infty\}$. For its algebraic structure we obtain (via Proposition 8.2)

Proposition 8.5

$$\begin{aligned}
[x\otimes A_n, y\otimes A_m] &= [x,y]\otimes A_{n+m},\\
[x\otimes A_n, y\otimes B_m &= [x,y]\otimes B_{n+m},\\
[x\otimes B_n, y\otimes B_m &= [x,y]\otimes (A_{n+m} + 4A_{n+m+1}).
\end{aligned}$$

Its central extensions are given by geometric cocycles

$$\gamma_{\beta}^{\overline{\mathfrak{g}}}(x \otimes f, y \otimes g) = \beta(x, y) \cdot \gamma^{\mathcal{A}}(f, g), \tag{126}$$

with $\beta(.,.)$ a symmetric, invariant bilinear form for $\mathfrak{g}$ and $\gamma^{\mathcal{A}}$ a multiplicative 2-cocycle for the algebra $\mathcal{A}$. Recall that if $\mathfrak{g}$ is a simple Lie there exists a universal central extension $\widehat{\mathfrak{g}}$. In our case it has a two-dimensional center and will be given by

$$\begin{aligned}[x \otimes f, y \otimes g] = [x, y] \otimes (f \cdot g) + \alpha_0 \cdot \beta(x, y) \cdot \gamma_0^{\mathcal{A}}(f, g) \cdot t_0 \\ + \alpha_\infty \cdot \beta(x, y) \cdot \gamma_\infty^{\mathcal{A}}(f, g) \cdot t_\infty\end{aligned} \tag{127}$$

with $\alpha_0, \alpha_\infty \in \mathbb{C}$, t_0, t_∞ central in $\widehat{\mathfrak{g}}$, and β the Cartan-Killing form. The values of the cocycles for the introduced basis elements have been calculated above and will not be repeated here. In accordance with the general results [47] the centrally extended current algebra will be an almost-graded extension of the current algebra with respect to this basis if and only if $\alpha_0 = 0$. It is an easy task to write everything explicitely for special cases of the Lie algebra $\mathfrak{g}$.

8.4 Three-point $\mathfrak{sl}(2, \mathbb{C})$-current algebra for genus 0

As an example we will give the universal central extension of the 3-point $\mathfrak{sl}(2, \mathbb{C})$-current algebra. The general theory has been developed above. The 3-point $\overline{\mathfrak{sl}}(2, \mathbb{C})$ algebra is of relevance in quite a number of applications, we only name statistical mechanics [24], [26]. Hence, the explicite knowledge of the structure equations with respect to some basis might be of some interest. We take $\mathfrak{sl}(2, \mathbb{C})$ with the standard matrix generators

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{128}$$

and induced relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = 2X\,. \tag{129}$$

As basis elements for the current algebra $\overline{\mathfrak{sl}}(2, \mathbb{C})$ with respect to the almost-grading introduced above we take the elements

$$Z^{(s)} = Z \otimes A_n, \quad Z^{(a)} = Z \otimes B_n, \qquad Z \in \{H, X, Y\}. \tag{130}$$

The structure of the current algebra comes via (121) from $\mathcal{A}$ (and of course from $\mathfrak{g}$). We need the Cartan-Killing form. Up to a normalization it is given by

$$\beta(A, B) = \mathrm{tr}(A \cdot B). \tag{131}$$

Hence,

$$\beta(X,Y) = \beta(Y,X) = 1, \quad \beta(H,H) = 2,$$
$$\beta(H,X) = \beta(H,Y) = \beta(X,H) = \beta(Y,H) = 0. \quad (132)$$

The universal central extension will have a two-dimensional center and will be given by

$$[Z\otimes f, W\otimes g] = [Z,W]\otimes f\cdot g + \alpha_\infty\cdot\beta(Z,W)\cdot\gamma_\infty^{\mathcal{A}}(f,g)\cdot t_\infty + \alpha_0\cdot\beta(Z,W)\cdot\gamma_0^{\mathcal{A}}(f,g)\cdot t_0,$$

with t_∞, t_0 central elements, $\alpha_\infty, \alpha_0 \in \mathbb{C}$. Recall that $\gamma_a^{\mathcal{A}}(f,g)$ can be calculated as $\mathrm{res}_a(f dg)$.

From the general theory we know that the central extension will be almost-graded with respect to the standard splitting if and only if $\alpha_0 = 0$.

All the data needed has been calculated already before. If we collect them we obtain the following results.

$$[H_n^{(s)}, H_m^{(s)}] = \alpha_\infty \cdot 4n \cdot \delta_m^{-n} \cdot t_\infty - \alpha_0 \cdot 2n \cdot \delta_m^{-n} \cdot t_0,$$
$$[H_n^{(s)}, H_m^{(a)}] = 2\alpha_0\left(n\,\delta_m^{-n} + 2n\,\delta_m^{-n-1} + \sum_{k=2}^{\infty} n\,(-1)^{k-1}2^k\,\frac{(2k-3)!!}{k!}\delta_m^{-n-k}\right)\cdot t_0.$$
$$[H_n^{(a)}, H_m^{(a)}] = \alpha_\infty(4n\delta_m^{-n} + 8(2n+1)\delta_m^{-n-1})\cdot t_\infty - \alpha_0(2n\delta_m^{-n} + 4(2n+1)\delta_m^{-n-1})\cdot t_0$$

$$[H_n^{(s)}, X_m^{(s)}] = 2X_{n+m}^{(s)}, \quad [H_n^{(s)}, X_m^{(a)}] = 2X_{n+m}^{(a)}, \quad [H_n^{(a)}, X_m^{(a)}] = 2X_{n+m}^{(s)} + 8X_{n+m}^{(s)},$$
$$[H_n^{(s)}, Y_m^{(s)}] = -2Y_{n+m}^{(s)}, \quad [H_n^{(s)}, Y_m^{(a)}] = -2Y_{n+m}^{(a)}, \quad [H_n^{(a)}, Y_m^{(a)}] = -2Y_{n+m}^{(s)} - 8Y_{n+m}^{(s)},$$

$$[X_n^{(s)}, Y_m^{(s)}] = H_{n+m}^{(s)} + \alpha_\infty \cdot 2n \cdot \delta_m^{-n} \cdot t_\infty - \alpha_0 \cdot n \cdot \delta_m^{-n} \cdot t_0,$$
$$[X_n^{(s)}, Y_m^{(a)}] = H_{n+m}^{(a)} + \alpha_0\left(n\,\delta_m^{-n} + 2n\,\delta_m^{-n-1} + \sum_{k=2}^{\infty} n\,(-1)^{k-1}2^k\,\frac{(2k-3)!!}{k!}\delta_m^{-n-k}\right)\cdot t_0.$$
$$[X_n^{(a)}, Y_m^{(a)}] = H_{n+m}^{(s)} + 4H_{n+m+1}^{(s)} + \alpha_\infty(2n\delta_m^{-n} + 4(2n+1)\delta_m^{-n-1})\cdot t_\infty$$
$$- \alpha_0(n\delta_m^{-n} + 2(2n+1)\delta_m^{-n-1})\cdot t_0\,.$$

Of course, the elements t_∞ and t_0 are central and we have anti-symmetry. The local cocycle, i.e. the cocycle coming with t_∞ was given in [50] and reproduced in [52, Equ. 12.75]. Unfortunately, there the central terms related to $[H_n^{(.)}, H_m^{(.)}]$ were forgotten.

Remark 8.6 By Cox and Jurisich [16, Thm.2.4] a different form of a universal central extension for the $\mathfrak{sl}(2,\mathbb{C})$ current algebra was proposed. This form was taken up in [17]. An inspection of the structure equation shows that in the proposed form two independent cocycles coming with the central elements ω_0 and ω_1 show up. Both would be local. But this contradicts the uniqueness of the local cocycle class (up to rescaling) as obtained in [47], which was also recalled in the current article. A closer examination shows that if "$\omega_1 \neq 0$" the proposed structure constants do not define a Lie algebra.

The principal structure, as far as the central extension is concerned, in particular also that we have one unique local cocycle class (up to rescaling) does not depend in an essential manner on the simple Lie algebra. See also Bremner [9] for the example of the 4-point case. Here the universal central extension is 3-dimensional. One of the classes will be local with respect to the standard splitting, the other two not. The latter two are "coupled" with ultraspherical (Gegenbauer) polynomials. I like also to mention that in [5] also the 3-point case was considered in another basis exhibiting another symmetry useful in the context of statistical mechanics.

Remark 8.7 As an additional example we like to give the case of the current algebra of $\mathfrak{gl}(n,\mathbb{C})$ for the N-point case. Of course, as $\mathfrak{gl}(n,\mathbb{C})$ is not perfect it does not admit a universal central extension. But by the classification results we can give the maximal central extension for which the cocycles are multiplicative (or $\mathcal{L}$-invariant)

$$\begin{aligned}[x\otimes f, y\otimes g] = [x,y]\otimes f\cdot g + \sum_{i=1}^{N-1}\alpha_i\cdot \mathrm{tr}(x\cdot y)\,\mathrm{res}_{a_i}(f\,dg)\cdot t_i \\ + \sum_{i=1}^{N-1}\beta_i\cdot \mathrm{tr}(x)\mathrm{tr}(y)\,\mathrm{res}_{a_i}(f\,dg)\cdot s_i\,.\end{aligned} \tag{133}$$

Here $\alpha_i,\beta_i\in\mathbb{C}$ and t_i and s_i are central.

8.5 Vector field algebra

Recall that in the genus $g=0$ case and $P_N=\infty$ the elements g for $\mathcal{F}^\lambda$ for $\lambda\in\frac{1}{2}\mathbb{Z}$ are given by

$$g(z)=a(z)dz^\lambda,\quad \text{with}\quad a(z)\in\mathcal{A}. \tag{134}$$

In particular, a basis of $\mathcal{A}$ induces a basis of $\mathcal{F}^\lambda$. We take as basis elements the elements

$$g_n^\lambda := A_{n-\lambda}dz^\lambda,\qquad h_n^\lambda := B_{n-\lambda}dz^\lambda,\quad n\in\mathbb{J}_\lambda. \tag{135}$$

The corresponding almost-grading reads as

$$\mathcal{F}^\lambda=\bigoplus_{n\in\mathbb{J}_\lambda}\mathcal{F}_n^\lambda,\qquad \mathcal{F}_n^\lambda=\langle g_n^\lambda, h_n^\lambda\rangle_{\mathbb{C}}. \tag{136}$$

For the vector field (i.e. forms of weight -1) we use

$$e_n := A_{n+1}\frac{d}{dz},\qquad f_n := B_{n+1}\frac{d}{dz},\quad n\in\mathbb{Z}. \tag{137}$$

The vector spaces $\mathcal{F}^\lambda$ are modules over $\mathcal{L}$. If $e=a\frac{d}{dz}$ and $g=bdz^\lambda$ then the module structure reads as

$$e\,.\,g=(a\cdot b'+\lambda\,b\cdot a')dz^\lambda. \tag{138}$$

With respect to our basis elements the structure equations are given by

Proposition 8.8

$$
\begin{aligned}
e_n \,.\, g_m^\lambda &= (m+\lambda n)\, h_{m+n}^\lambda, \\
e_n \,.\, h_m^\lambda &= (m+\lambda n)\, g_{m+n}^\lambda + (4(m+\lambda n)+2)\, g_{n+m+1}^\lambda, \\
f_n \,.\, g_m^\lambda &= (m+\lambda n)\, g_{m+n}^\lambda + (4(m+\lambda n)+2\lambda)\, g_{n+m+1}^\lambda, \\
f_n \,.\, h_m^\lambda &= (m+\lambda n)\, h_{m+n}^\lambda + (4(m+\lambda n)+2+2\lambda)\, h_{n+m+1}^\lambda .
\end{aligned}
$$

For $\lambda = -1$ we get the vector field algebra structure.

Proposition 8.9

$$
\begin{aligned}
[e_n, e_m] &= (m-n)\, f_{m+n}, \\
[e_n, f_m] &= (m-n)\, e_{m+n} + (4(m-n)+2)\, e_{n+m+1}, \\
[f_n, f_m] &= (m-n)\, f_{m+n} + 4(m-n)\, f_{n+m+1} .
\end{aligned}
$$

These expressions clearly exhibit the almost-graded structure. Observe that the algebra of global holomorphic vector fields is the subalgebra

$$
\langle e_{-1}, f_{-1}, e_0 \rangle_{\mathbb{C}}, \tag{139}
$$

which is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

Next we calculate the universal central extension. We know that it has a two-dimensional center and can be given as

$$
[\widehat{e}, \widehat{f}] = \widehat{[e,f]} + \alpha_0 \cdot \gamma_0^{\mathcal{L}}(e,f) \cdot t_o + \alpha_\infty \cdot \gamma_\infty^{\mathcal{L}}(e,f) \cdot t_\infty \tag{140}
$$

with

$$
\begin{aligned}
\gamma_0^{\mathcal{L}}(e,f) &= 1/2\, \mathrm{res}_0(e\cdot f''' - f\cdot e''')dz = \mathrm{res}_0(e\cdot f''')dz = -\,\mathrm{res}_0(f\cdot e''')dz \\
\gamma_\infty^{\mathcal{L}}(e,f) &= 1/2\, \mathrm{res}_\infty(e\cdot f''' - f\cdot e''') = \mathrm{res}_\infty(e\cdot f''')dz = -\,\mathrm{res}_\infty(f\cdot e''')dz \,.
\end{aligned} \tag{141}
$$

First we consider the point ∞ and obtain in this way the local cocycle.

Proposition 8.10

$$
\begin{aligned}
\gamma_\infty^{\mathcal{L}}(e_n, e_m) &= 2(n^3-n)\,\delta_m^{-n} + 4n(n+1)(2n+1)\delta_m^{-n-1} \\
\gamma_\infty^{\mathcal{L}}(e_n, f_m) &= 0, \\
\gamma_\infty^{\mathcal{L}}(f_n, f_m) &= 2(n^3-n)\,\delta_m^{-n} + 8n(n+1)(2n+1)\delta_m^{-n-1} + 8(n+1)(2n+1)(2n+3)\delta_m^{-n-2}
\end{aligned}
$$

Proposition 8.11

$$\gamma_0^{\mathcal{L}}(e_n, e_m) = -(n^3 - n)\,\delta_n^{-m} - 2n(n+1)(2n+1)\delta_m^{-n-1}$$
$$\gamma_0^{\mathcal{L}}(e_n, f_m) = (n^3 - n)\,\delta_m^{-n} + 6n^2(n+1)\delta_m^{-n-1} + 6n(n+1)^2\delta_m^{-n-2}$$
$$+ \sum_{k\geq 3} n(n+1)(n+k-1)(-1)^k 2^k \cdot 3 \cdot \frac{(2k-5)!!}{k!}\delta_m^{-n-k}$$
$$\gamma_0^{\mathcal{L}}(f_n, f_m) = -(n^3 - n)\,\delta_m^{-n} - 4n(n+1)(2n+1)\delta_m^{-n-1} - 4(n+1)(2n+1)(2n+3)\delta_m^{-n-2}\,.$$

See [27] for similar results.

Note also that we can express the term

$$\frac{(2k-5)!!}{k!}2^k \cdot 3 = \frac{12}{k(k-1)}\binom{2(k-2)}{k-2} \tag{142}$$

if this is more convenient. See Section 8.8 for some mathematical background on these calculations.

Again only the cocycle $\gamma_\infty^{\mathcal{L}}$ will be local with respect to the almost-grading introduced by our splitting, respectively by our basis. Hence, only for its corresponding central extension we have a triangular decomposition. Everything what was said for the function algebra case, remains true here.

8.6 Differential operator algebra $\mathcal{D}^1$

Recall that $\mathcal{D}^1$ is the (Lie algebra) semi-direct sum of $\mathcal{A}$ with $\mathcal{L}$ where $\mathcal{L}$ operates on $\mathcal{A}$ by taking the derivative. The homogeneous subspace of degree n is now

$$\langle A_n, B_n, e_n, f_n\rangle_{\mathbb{C}}. \tag{143}$$

The subalgebra $\mathcal{A}$ is abelian and Proposition 8.9 gives the structure equations for the vector fields. Proposition 8.8 specialized for $\lambda = 0$ yields the equations for the mixed terms.

Proposition 8.12

$$[e_n, A_m] = -[A_m, e_n] = m\,B_{m+n},$$
$$[e_n, B_m] = -[B_m, e_n] = m\,A_{m+n} + (4m+2)\,A_{n+m+1},$$
$$[f_n, A_m] = -[A_m, f_n] = m\,A_{m+n} + 4m\,A_{n+m+1},$$
$$[f_n, B_m] = -[B_m, f_n] = m\,B_{m+n} + (4m+2)\,B_{m+n+1},$$

The geometric cocycles yield a 6-dimensional central extension. In addition to the 4 basis elements given by the pure cocycles given by the Propositions 8.3, 8.4, 8.10, 8.11 we have two additional basis cocycles corresponding to the mixing cocycle (70) ($e \in \mathcal{L}, g \in \mathcal{A}$)

$$\gamma_0^{(m)}(e, g) = \mathrm{res}_0(eg''dz), \qquad \gamma_\infty^{(m)}(e, g) = \mathrm{res}_\infty(eg''dz). \tag{144}$$

Evaluated for the basis elements we obtain

Proposition 8.13

$$\begin{aligned}
\gamma_\infty^{(m)}(e_n, A_m) &= 0,\\
\gamma_\infty^{(m)}(e_n, B_m) &= -2n(n+1)\delta_m^{-n} - 4(n+1)(2n+1)\delta_m^{-n-1},\\
\gamma_\infty^{(m)}(f_n, A_m) &= -2n(n+1)\delta_m^{-n} - 4(n+1)(2n+3)\delta_m^{-n-1},\\
\gamma_\infty^{(m)}(f_n, B_m) &= 0\,.
\end{aligned}$$

Proposition 8.14

$$\begin{aligned}
\gamma_0^{(m)}(e_n, A_m) &= -n(n+1)\delta_m^{-n} - 2(n+1)^2\delta_m^{-n-1}\\
&\qquad + \sum_{k\geq 2}(n+1)(n+k)(-1)^k 2^k \cdot \frac{(2k-3)!!}{k!}\delta_m^{-n-k}\,.\\
\gamma_0^{(m)}(e_n, B_m) &= n(n+1)\delta_m^{-n} + 2(n+1)(2n+1)\delta_m^{-n-1},\\
\gamma_0^{(m)}(f_n, A_m) &= n(n+1)\delta_m^{-n} + 2(n+1)(2n+3)\delta_m^{-n-1},\\
\gamma_0^{(m)}(f_n, B_m) &= -n(n+1)\delta_m^{-n} - 6(n+1)^2\delta_m^{-n-1} - 6(n+1)(n+2)\delta_m^{-n-2}\\
&\qquad + \sum_{k\geq 3}(n+1)(n+k)(-1)^{k-1}2^k \cdot 3 \cdot \frac{(2k-5)!!}{k!}\delta_m^{-n-k}\,.
\end{aligned}$$

By Theorem 7.12 the differential operator algebra admits a universal central extension and the introduced six geometric cocycles, each associated to a different central element will yield the universal central extension.

Proposition 8.15 *A cocycle class $[\gamma]$ for $\mathcal{D}^1$ will be local (with respect to the standard splitting) if and only if γ is cohomologous to a linear combination*

$$\gamma \quad \sim \quad \alpha_1 \cdot \gamma_\infty^{\mathcal{A}} + \alpha_2 \cdot \gamma_\infty^{\mathcal{L}} + \alpha_3 \cdot \gamma_\infty^{(m)}, \qquad \alpha_i \in \mathbb{C}. \tag{145}$$

In particular, the space of local cohomology classes is 3-dimensional.

This is a general result of [48], [52] which is for the 3-point illustrated by the above calculations. In the very general case (meaning arbitrary genus, arbitrary number of marked points, arbitrary splitting) the three cocycles in (145) are obtained by integrating over a separating cycle.

8.7 The Lie superalgebra

In addition to the basis elements e_n and f_n of $\mathcal{L}$ we take

$$\varphi_n = A_{n+1/2}(dz)^{-1/2}, \quad \psi_n = B_{n+1/2}(dz)^{-1/2}, \qquad n \in \mathbb{Z} + 1/2. \tag{146}$$

Additionally to the structure constants of Proposition 8.9 we have

Proposition 8.16

$$
\begin{aligned}
[\varphi_n, \varphi_m] &= e_{n+m} \\
[\varphi_n, \psi_m] &= f_{n+m} \\
[\psi_n, \psi_m] &= e_{n+m} + 4\, e_{n+m+1} \\
[e_n, \varphi_m] &= (m - n/2)\, \psi_{n+m} \\
[e_n, \psi_m] &= (m - n/2)\, \varphi_{n+m} + (4m - 2n + 2)\, \varphi_{n+m+1} \\
[f_n, \varphi_m] &= (m - n/2)\, \varphi_{n+m} + (4m - 2n - 1)\, \varphi_{n+m+1} \\
[f_n, \psi_m] &= (m - n/2)\, \psi_{n+m} + (4m - 2n + 1)\, \psi_{n+m+1}\ .
\end{aligned}
$$

Similar expressions are given by Leidwanger and Morier-Genoud [36, Prop. 3.8].

Next we consider 2-cocycles. We have a two-dimensional space of geometric cocycles generated by γ_0^S and γ_∞^S obtained by taking the residue from (75) at 0 and ∞. For pairs of pure vector field type arguments we have the result of Proposition 8.10 and Proposition 8.11. For mixing of pure types it is zero. It remains to consider pairs of $-1/2$-forms. The calculations (see [54]) yields

Proposition 8.17

$$
\begin{aligned}
\gamma_\infty^S(\varphi_n, \varphi_m) &= 0 \\
\gamma_\infty^S(\varphi_n, \psi_m) &= -4(n - 1/2)(n + 1/2)\delta_m^{-n} - 8n(2n + 1)\delta_m^{-n-1}. \\
\gamma_\infty^S(\psi_n, \psi_m) &= 0
\end{aligned}
$$

Proposition 8.18

$$
\begin{aligned}
\gamma_0^S(\varphi_n, \varphi_m) = {} & -2(n + 1/2)(n - 1/2)\delta_m^{-n} + 4(n + 1/2)^2\delta_m^{-n-1} \\
& + 2\sum_{k\ge 2}(n + 1/2)(n - 1/2 + k)(-1)^k 2^k\,\frac{(2k - 3)!!}{k!}\delta_m^{-n-k}
\end{aligned}
$$

$$
\gamma_0^S(\varphi_n, \psi_m) = 2(n - 1/2)(n + 1/2)\delta_m^{-n} + 4n(2n + 1)\delta_m^{-n-1}.
$$

$$
\begin{aligned}
\gamma_0^S(\psi_n, \psi_m) = {} & -2(n + 1/2)(n - 1/2)\delta_m^{-n} - 12(n + 1/2)^2\delta_m^{-n-1} \\
& - 12(n + 1/2)(n + 3/2)\delta_m^{-n-2} \\
& + 2\sum_{k\ge 3}(n + 1/2)(n - 1/2 + k)(-1)^{k-1}2^k \cdot 3 \cdot \frac{(2k - 5)!!}{k!}\delta_m^{-n-k}\ .
\end{aligned}
$$

We illustrated again that only the cocycle γ_∞^S is local with respect to the standard splitting. The γ_∞^S is the one which was considered by Kreusch [32] (up to a different indexing of the basis elements).

Remark 8.19 Here we considered the central element to be even. We could have even dropped this assumption. In [51] we show that the corresponding cocycles (with odd central element) are cohomologically trivial.

8.8 Some Remarks on the Calculations of the Residues

Just to indicate the kind of arguments used above we will calculate the residues of $A_n dz$ and $B_n dz$ at the points where poles might be. In fact, everything can be finally reduced to calculating their values for the points 0, 1 and ∞. The results in the whole section are essentially based on such kind of residue calculations for products of of linear factors and their inverse (yielding Laurent series).

As starting point we need the Laurent series expansion of $(z-1)^m$ around zero. We collect the following well-known facts about binomial series.

The expansion

$$(z-1)^m = \sum_{k=0}^{\infty} \binom{m}{k} z^k (-1)^{m-k}, \qquad z \in \mathbb{C},\ |z| < 1 \tag{147}$$

is valid for all $m \in \mathbb{Z}$.

For negative exponents an equivalent expression is

$$(z-1)^{-n} = (-1)^n \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} z^k, \qquad z \in \mathbb{C},\ |z| < 1, \tag{148}$$

where $n \in \mathbb{N}$. We have the easy relation

$$\binom{2k}{k} = \frac{(2k-1)!!}{k!}\, 2^k, \qquad k \in \mathbb{N}, \tag{149}$$

where $(2k-1)!! = 1 \cdot 3 \cdots (2k-1)$ is the double factorial.

Lemma 8.20 *For the residues at the point* 0 *we have*

$$\operatorname{res}_0(A_{-n} dz) = \begin{cases} 0, & n \le 0, \\ -1, & n = 1, \\ (-1)^n \dfrac{(2n-3)!!}{(n-1)!}\, 2^{n-1}, & n \ge -2. \end{cases}$$

Proof If $n < 0$ then there is no pole at $z = 0$. Hence let $n > 0$. We use (148) and calculate

$$A_{-n}(z) = (-1)^n \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} z^{k-n}. \tag{150}$$

The residue at $z = 0$ is given by the coefficient paired with z^{-1}. Hence it is given by the coefficient for $k = n - 1$

$$\operatorname{res}_0(A_{-n}dz) = (-1)^n \binom{2(n-1)}{n-1}. \tag{151}$$

For $n = 1$ we obtain the value -1. For $n > 1$ we use (149) and obtain

$$\operatorname{res}_0(A_{-n}dz) = (-1)^n \frac{(2n-3)!!}{(n-1)!} \, 2^{n-1}. \tag{152}$$

This was the claim. □

From this lemma the non-locality of the cocycle defined via the residue at 0 follows.

Lemma 8.21

$$\operatorname{res}_0(B_m dz) = \begin{cases} 1, & m = -1, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof We check that $B_m = (1/(m+1)A_{m+1})'$ if $m \neq -1$. Hence, for $m \neq -1$ the differential $B_m dz$ is an exact differential and hence does not have any residue. It remains $B_{-1}(z) = z^{-1}(z-1)^{-1}(2z-1)$ which obviously has as residue $+1$ at $z = 0$.□

Lemma 8.22

$$\operatorname{res}_1(A_m dz) = -\operatorname{res}_0(A_m dz)$$
$$\operatorname{res}_1(B_m dz) = \operatorname{res}_0(B_m dz)\ .$$

Proof We make a change of local coordinates $v = 1 - z$. and the point $z = 1$ corresponds to the point $v = 0$. □

Lemma 8.23

$$\operatorname{res}_\infty(A_m dz) = 0,$$
$$\operatorname{res}_\infty(B_m dz) = -2\operatorname{res}_0(B_m dz),$$
$$= \begin{cases} -2, & m = -1, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof By the residue theorem [49] for a compact Riemann surface the sum over all residues of a meromorphic differential is zero. As our differentials have only poles at $0, 1, \infty$ the claim follows from Lemmas 8.20, 8.21, and 8.22. □

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