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Functional inequalities on path space of sub-Riemannian manifolds and applications

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ABSTRACT

We consider the path space of a manifold with a measure induced by a stochastic flow with an infinitesimal generator that is hypoelliptic, but not elliptic. These generators can be seen as sub-Laplacians of a sub-Riemannian structure with a chosen complement. We introduce a concept of gradient for cylindrical functionals on path space in such a way that the gradient operators are closable in L^2. With this structure in place, we show that a bound on horizontal Ricci curvature is equivalent to several inequalities for functions on path space, such as a gradient inequality, log-Sobolev inequality and Poincaré inequality. As a consequence, we also obtain a bound for the spectral gap of the Ornstein–Uhlenbeck operator.

1. Introduction

Stochastic analysis on the path space over a complete Riemannian manifold has been well developed ever since B. K. Driver [13] proved the quasi-invariance theorem for the Brownian motion on compact Riemannian manifolds in 1992. A key point of the study is to first establish an integration by parts formula for the associated gradient operator induced by the quasi-invariant flows, then prove functional inequalities for the corresponding Dirichlet form (see e.g. [16,26] and references within). For more analysis on Riemannian path spaces we refer to [14,28,31] and references within. Recently, there has been an extensive study by A. Naber [30] on the equivalence of bounded Ricci curvature and certain inequalities on path space. R. Haslhofer and A. Naber [23] extended these results to characterize solutions of the Ricci flow. For further results in this direction see [9,10,24].

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In the present article, we develop this formalism in the framework of hypoelliptic operators and diffusions in sub-Riemannian geometry. Let (M, H, g) be a sub-Riemannian manifold, meaning that H is a subbundle of TM with a metric tensor g. Let ∇ be an affine connection on TM compatible with (H, g) in the sense that it preserves H and its metric g under parallel transport. We define an operator

$$L = \operatorname{tr}_H \nabla^2_{\times,\times}, \tag{1.1}$$

as the trace of the Hessian ∇^2 over H with respect to the inner product g. We assume that the subbundle H is bracket-generating, meaning that its sections and their iterated brackets span the entire tangent bundle. This makes L into a hypoelliptic operator on functions by Hörmander's theorem [25]. Let B_t^x be a standard Brownian motion in the inner product space H_x. Then the solution of the SDE,

$$dX_t^x = /\!/_t \circ dB_t^x, \quad X_0^x = x$$

is a diffusion on M with $\frac{1}{2}L$ as infinitesimal generator, where $/\!/_t \colon T_xM \to T_{X_t^x}M$ denotes ∇-parallel transport along X_t^x. For the case when $H = TM$ and ∇ is the Levi-Civita connection, the operator L is the Laplacian and X_t^x is the Brownian motion in M.

The analysis of path space of sub-Riemannian manifolds has been earlier considered in [2,3] for the case where the sub-Riemannian structure (H, g) is the restriction to the transverse bundle of a foliation that is Riemannian, totally geodesic and of Yang–Mills type. In this present paper, we will extend the approach in [3] to arbitrary sub-Riemannian manifolds with a metric preserving complement, which include sub-Riemannian manifolds coming from Riemannian foliations, but does not require anything of the metric along the foliation or even any extension of the sub-Riemannian metric.

An immediate difficulty for analysis on path space over a sub-Riemannian manifold is that if differentiation is only allowed in directions of the horizontal subbundle, then it is a priori not clear how to define the gradient of functionals on path space in a way that makes the operator closable. We introduce a gradient on path space in terms of a connection ∇ compatible with the sub-Riemannian structure which is canonical in the sense that any choice of complement V to the subbundle H determines it uniquely. To motivate the appropriateness of the definition, we first review the smooth path space and the development map with respect to an arbitrary connection. The underlying idea is that if we have a variation of curves $\{\gamma^s\}$ that are all tangent to H, then the corresponding variational vector field $Y = \partial_s \gamma^s|_{s=0}$ will not be in H in general. Yet it cannot be arbitrary in the sense that it is determined by $\operatorname{pr}_H Y$ for any choice of projection $\operatorname{pr}_H \colon TM \to H$, a fact that was also observed in [3]. We will construct the sub-Riemannian gradient on path space to reflect this property. The most straightforward advantage of our gradient is that it admits an integration by parts formula from where closability of the gradient operator in L^2 immediately follows.

We also develop a concept of a damped gradient analogous to the definition in Riemannian geometry, but with the adjoint connection $\hat{\nabla}$ of ∇. In spite of the fact that this adjoint will not be compatible with the sub-Riemannian structure, we show that the gradient and the damped gradient are related by the Ricci operator.

Having set up this formalism, we extend the approach of Naber to the sub-Riemannian case in our main result in Theorem 4.1. We establish functional inequalities on the path space of the stochastic flow $x \mapsto X_t^x$ including gradient inequalities, log-Sobolev inequalities and Poincaré inequalities. These inequalities are shown to be equivalent to bounds on *the horizontal Ricci operator* $\operatorname{Ric}_H \colon H \to H$ which is defined taking the trace of the curvature tensor only over H. We also show that bounds on Ric_H can equivalently be described by functional inequalities for functions on M. This equivalence result could also open the door to the study of an analogue of Ricci flow in the sub-Riemannian setting, which has already been considered in the case of the sub-Riemannian Heisenberg group in [15] using the formalism of metric measure spaces.

We want to emphasize that having similar relations between bounded Ricci curvature and functional inequalities in both the Riemannian and the sub-Riemannian case is quite surprising. By contrast, the relationship between lower Ricci curvature bounds and functional inequalities for the heat semigroup is much more complicated in the sub-Riemannian case compared to the Riemannian one, see e.g. [1,4,5,20,21] for details.

The structure of the paper is as follows. In Section 2 we first consider the smooth path space and development with respect to an arbitrary connection. We review the basic definitions of sub-Riemannian manifolds and connections compatible with such structures. In contrast to the Riemannian case, we do not have torsion-free compatible connections on such spaces, however, we give analogues of the Levi-Civita connection by defining a canonical connection with minimal torsion relative to a chosen complement V to the horizontal bundle H. We finally use these connections to define corresponding vector fields on smooth path space.

We generalize the definition of these vector fields in Section 3 in order to define a gradient and a damped gradient for functions on path space. We relate these concepts and look at their properties in Theorems 3.1, 3.3 and 3.4. In particular, we establish integration by parts formulas for both the gradient and the damped gradient, generalizing the Riemannian case and the case treated in [3]. Finally, in Section 4 we show that several functional inequalities related to functions on path space are equivalent to the analogue of bounded Ricci curvature. We state our main result in Theorem 4.1. From this result, we also obtain a spectral gap estimate in Corollary 4.5 for the Ornstein–Uhlenbeck operator corresponding to the gradient.

In Section 5, we look closer at how such results can be interpreted geometrically. Intuitively, we show that if one uses the canonical connection ∇ corresponding to a metric preserving complement V, then the sub-Riemannian path space has geometry "similar to M/V". This latter concept is well defined in the case when V is an integrable submanifold corresponding to a regular foliation Φ in which M/Φ has an induced Riemannian structure, but our formalism is valid for non-integrable choices of complements as well.

For the main results of this paper, we need to choose a complement which is metric preserving. To explain the reason behind this assumption and for later references, we include some formulas related to a general choice of connection and complement in the Appendix.

2. Smooth path space and sub-Riemannian geometry

2.1. Smooth path space and development

An affine manifold is a pair (M, ∇) where ∇ is an affine connection on TM. Let $\mathbf{T}$ denote the torsion of ∇, i.e.

$$\mathbf{T}(Y,Z) = \nabla_Y Z - \nabla_Z Y - [Y,Z], \quad Y, Z \in \Gamma(TM),$$

and let $\mathbf{R}$ denote its curvature

$$\mathbf{R}(Y_1,Y_2)Z = \left(\nabla_{Y_1}\nabla_{Y_2} - \nabla_{Y_2}\nabla_{Y_1} - \nabla_{[Y_1,Y_2]}\right) Z, \quad Y_1, Y_2, Z \in \Gamma(TM).$$

We define its *adjoint* $\hat{\nabla}$ as the connection

$$\hat{\nabla}_Y Z = \nabla_Y Z - \mathbf{T}(Y,Z). \tag{2.1}$$

Observe that the torsion of $\hat{\nabla}$ is $-\mathbf{T}$ and hence ∇ is the adjoint of $\hat{\nabla}$. We remark that if $(s,t) \mapsto \omega_t^s$ is a two-parameter function with values in M, then

$$D_s \frac{\partial}{\partial t}\omega_t^s = \hat{D}_t \frac{\partial}{\partial s}\omega_t^s,$$

where D_s and $\hat{D}_t$ denote covariant derivatives of respectively ∇ in the direction of s and $\hat{\nabla}$ in the direction of t.

Let $W_x^\infty(M)$ denote the space of smooth curves $[0,\infty) \to M$, $t \mapsto \omega_t$ satisfying $\omega_0 = x$. When $//_t\colon T_xM \to T_{\omega_t}M$ denotes parallel transport with respect to ∇ along a given path $\omega \in W_x^\infty(M)$, we say that $u \in W_0^\infty(T_xM)$ is *the anti-development* of ω_t if it is the unique solution of

$$\dot{u}_t = //_t^{-1}\dot{\omega}_t, \quad u_0 = 0,$$

with $\dot{u}_t = \frac{d}{dt}u_t$. Conversely, we say that ω is *the development* of u. We write $\operatorname{Dev}(u) = \omega$ and $\operatorname{Dev}^{-1}(\omega) = u$. We note that Dev^{-1} is defined for any element in $W_x^\infty(M)$, however, for a general $u \in W_0^\infty(T_xM)$, $t \mapsto \operatorname{Dev}(u)_t$ might be only defined for short time. If $\omega_t = \operatorname{Dev}(u)_t$ is defined for all time for any $u \in W_0^\infty(T_xM)$, $x \in M$, then ∇ is called *complete*. For the rest of this subsection, we assume that ∇ is complete. For the general case, see Remark 2.2. The next lemma describes the derivative of the map Dev.

Lemma 2.1. *Let $\omega \in W_x^\infty(M)$ be an arbitrary smooth curve with $\operatorname{Dev}^{-1}(\omega) = u$. Consider $\omega_t^s = \operatorname{Dev}(u+sk)_t$ for $k \in W_0^\infty(T_xM)$ and define*

$$Y_t = \frac{\partial}{\partial s}\omega_t^s|_{s=0}. \tag{2.2}$$

Write $//_t, \hat{//}_t\colon T_xM \to T_{\omega_t}M$ for parallel transport along ω relative to respectively ∇ and $\hat{\nabla}$. If we write

$$Y_t = //_t y_t = \hat{//}_t \hat{y}_t$$

with $\hat{y}_t = \hat{//}_t^{-1}//_t y_t$, then y_t and $\hat{y}_t$ are the unique solutions of

$$\begin{aligned} k_t &= y_t + \int_0^t \mathbf{T}_{//_s}(y_s, du_s) - \int_0^t \int_0^s \mathbf{R}_{//_r}(du_r, y_r) du_s \\ &= \int_0^t //_s^{-1}\hat{//}_s \dot{\hat{y}}_s \, ds - \int_0^t \int_0^s \mathbf{R}_{//_r}(du_r, y_r) du_s. \end{aligned}$$

We remark that in the above statement, we used the notation

$$\begin{aligned} \mathbf{T}_{//_t}(w_1, w_2) &= //_t^{-1}\mathbf{T}(//_t w_1, //_t w_2), \\ \mathbf{R}_{//_t}(w_1, w_2)w_3 &= //_t^{-1}\mathbf{R}(//_t w_1, //_t w_2)//_t w_3. \end{aligned}$$

We will use this notation for tensors in general throughout the paper.

Proof. Let D and $\hat{D}$ be the covariant derivative of respectively ∇ and $\hat{\nabla}$. Write $e_{1,s}(t), \dots, e_{n,s}(t)$ for an orthonormal ∇-parallel basis along $t \mapsto \omega_t^s$ with $e_{j,0}(t) = e_j(t)$ and $e_{j,s}(0) = e_j(0)$, and use the same basis to define $u_t + sk_t = \sum_{j=1}^n (u_j(t) + sk_j(t))e_j(0)$. Then

$$\hat{D}_t \partial_s \omega = D_s \partial_t \omega = \sum_{j=1}^n \dot{k}_j e_{j,s} + \sum_{j=1}^n (\dot{u}_j + s\dot{k}_j) D_s e_{j,s}.$$

By definition, we have $D_s e_{j,s}(0) = 0$. Furthermore, we have

$$D_t D_s e_{j,s} = (D_t D_s - D_s D_t) e_{j,s} = \mathbf{R}\left(\frac{\partial}{\partial t}\omega, \frac{\partial}{\partial s}\omega\right) e_{j,s}.$$

It follows that at $s = 0$,

$$\hat{D}_t Y = \hat{//}_t \dot{\hat{y}}_t = D_t Y - \mathbf{T}(\dot{\omega}_t, Y_t) = \sum_{j=1}^n \dot{k}_j e_j(t) + \sum_{j=1}^n \dot{u}_j (D_s e_{j,s}(t)|_{s=0}),$$

and hence,

$$//_t^{-1}\hat{//}_t \dot{\hat{y}}_t = \dot{y}_t - \mathbf{T}_{//_t}(\dot{u}_t, y_t) = \dot{k}_t + \sum_{j=1}^n \dot{u}_j(t) \int_0^t \mathbf{R}_{//_s}(\dot{u}_s, y_s) e_j(0) ds. \quad \square$$

Remark 2.2 (*Non-complete Connections*)**.** Let $\omega \in W_x^\infty(M)$ be any given curve with $u = \mathrm{Dev}^{-1}(\omega)$. Then for arbitrary $k \in W_0^\infty(T_xM)$ and any $T > 0$, there is some $\varepsilon > 0$ such that $t \mapsto \mathrm{Dev}(u + sk)_t$ has a solution on $[0, T]$ for $|s| < \varepsilon$. Hence, we have that $t \mapsto Y_t$ can still be defined as a derivative of a two-parameter family as in (2.2) for any $t \geq 0$.

2.2. Sub-Riemannian manifolds

We consider a sub-Riemannian manifold as a triple (M, H, g) where M is a connected manifold, $H \subseteq TM$ is a subbundle of the tangent bundle and $g = \langle \cdot\,, \cdot \rangle_g$ is a metric tensor on H. The sub-Riemannian structure (H, g) induces a map $\sharp : T^*M \to H \subseteq TM$ defined by

$$\langle \alpha, v \rangle = \langle \sharp\alpha, v \rangle_g, \quad \alpha \in T_x^*M,\ v \in H_x,\ x \in M.$$

We can then define a (degenerate) sub-Riemannian cometric g^* by

$$g^*(\alpha, \beta) = \langle \alpha, \beta \rangle_{g^*} = \langle \sharp\alpha, \sharp\beta \rangle_g.$$

We remark that in what follows, we use g, the map $\sharp$ as well as the cometric g^* to state our results. For $v \in H$ and $\alpha \in T^*M$, we also use the notation $|v|_g = \langle v, v \rangle_g^{1/2}$ and $|\alpha|_{g^*} = \langle \alpha, \alpha \rangle_{g^*}^{1/2}$ and ask the reader to keep in mind that $|\alpha|_{g^*}$ may vanish for non-zero covectors.

As usual, we assume that H is bracket-generating, meaning that sections of H and their iterated brackets span the entire tangent bundle. A curve ω_t is called horizontal if it is absolutely continuous and satisfies $\dot\omega_t \in H_{\omega_t}$ for almost every t. The bracket-generating condition implies that any pair of points can be connected by a horizontal curve. We hence have a well defined distance on M given by

$$d_g(x, y) = \inf \left\{ \int_0^T \langle \dot\omega_t, \dot\omega_t \rangle_g^{1/2}\, dt \colon\ \omega_0 = x,\ \omega_T = y,\ \omega_t \text{ is horizontal} \right\}. \tag{2.3}$$

The topology induced by the metric d_g coincides with the manifold topology. We say that (M, H, g) is complete if (M, d_g) is a complete metric space. For more details on sub-Riemannian geometry, see e.g. [29].

2.3. Compatible connections and metric preserving complements

Let ∇ be an affine connection on TM for a sub-Riemannian manifold (M, H, g). We are interested in the following types on connections.

Definition 2.3. A connection ∇ is called compatible with the sub-Riemannian structure if for any $Z \in \Gamma(TM)$, $Y, Y_2 \in \Gamma(H)$,

(i) $\nabla_Z Y|_x \in H_x$ for any $x \in M$,
(ii) $Z\langle Y, Y_2 \rangle_g = \langle \nabla_Z Y, Y_2 \rangle_g + \langle Y, \nabla_Z Y_2 \rangle_g$.

Unlike what holds in Riemannian geometry, there exists no affine connection that is both compatible with the sub-Riemannian structure and also torsion free when H is bracket-generating and a proper subbundle of TM, see e.g. [22]. Let $t \mapsto \omega_t$ be any smooth horizontal curve with $\omega_0 = x$. If ∇ is a compatible connection then anti-development $u = \mathrm{Dev}^{-1}(\omega)$ is a smooth curve H_x, and the converse is also true for any curve $u \in W_x^\infty(H_x)$ if only for short time in general. If (M, H, g) is complete as a metric space, then the solution will exist for all time, see Proposition A.2, Appendix.

One way of obtaining a preferred connection is to choose a complement V to H, that is a subbundle $TM = H \oplus V$. We will always assume that our complement satisfies the following definition, introduced in [20].

Definition 2.4. For any $Z \in \Gamma(TM)$, let $\mathcal{L}_Z$ denote the corresponding Lie derivative. Let V be a complement to a sub-Riemannian manifold (M,H,g) with corresponding projections pr_H and pr_V. The complement V is then called metric preserving if for any $Z \in \Gamma(V)$ and $X \in \Gamma(H)$, we have

$$(\mathcal{L}_Z \, \mathrm{pr}_H^* \, g)(X,X) = 0.$$

For discussion of the general complement, see the Appendix. For such a complement, we have the following result.

Proposition 2.5. *Assume that V is metric preserving. There is a connection ∇ with torsion $\mathbf{T}$ satisfies the following properties:*

(a) *Both H and V are parallel with respect to ∇;*
(b) *∇ is compatible with (H,g);*
(c) *$\mathbf{T}(H,H) \subseteq V$;*
(d) *$\mathbf{T}(H,V) = 0$.*

Furthermore, if ∇' is another connection satisfying (a)–(d), then $\nabla_Y Z - \nabla'_Y Z = \nabla_{\mathrm{pr}_V Y} \mathrm{pr}_V Z - \nabla'_{\mathrm{pr}_V Y} \mathrm{pr}_V Z$.

This result is a special case of Proposition A.4, Appendix. Since the restriction $\nabla_{|V}|V$ will not have any influence of the next computation, we will write $\nabla = \nabla^{g,V}$ for any connection ∇ satisfying (a)–(d) in Proposition 2.5. We will now study the path space using such connections.

Let (M,H,g) be a complete sub-Riemannian manifold with a chosen metric-preserving complement V, and let $\nabla = \nabla^{g,V}$. Let ∇ have torsion $\mathbf{T}$ and curvature $\mathbf{R}$ and define $\mathrm{Dev} = \mathrm{Dev}^{\nabla}$ relative to this connection. Then the following result holds.

Proposition 2.6. *Consider $\omega_t^s = \mathrm{Dev}(u+sk)_t$ for $u,k \in W_0^\infty(H_x)$. Write $\omega = \mathrm{Dev}(u)$ and introduce a linear map $A_t = A_t^\omega : T_xM \to T_xM$ by*

$$A_t w = \int_0^t \mathbf{T}_{//_s}(du_s, w).$$

If $Y_t = \frac{\partial}{\partial s}\omega_t^s|_{s=0}$, then $Y_t = //_t y_t = \hat{//}_t \hat{y}_t$ with

$$y_t = h_t + \int_0^t dA_s h_s, \quad \hat{y}_t = h_t - \int_0^t A_s dh_s,$$

where $h_t = \mathrm{pr}_H \, y_t$ is the unique solution of

$$k_t = h_t - \int_0^t \int_0^s \mathbf{R}_{//_r}(du_r, h_r)\, du_s. \tag{2.4}$$

The statement is a special case of Lemma A.6, Appendix. Based on this result, we make the following definition.

Definition 2.7. Let (M,H,g) be a sub-Riemannian manifold with a metric preserving complement V and define $\nabla = \nabla^{g,V}$. For any $h \in W_0^\infty(H_x)$, we define a vector field D_h on $W_x^\infty(M)$ by

$$D_h|_\omega = //_t^{-1}\left(h_t + \textstyle\int_0^t \mathbf{T}_{//_s}(du_s, h_s)\right) = //_t^{-1}\left(h_t + \textstyle\int_0^t dA_s h_s\right),$$

where $u = \mathrm{Dev}^{-1}(\omega)$ and $//_t$ denotes parallel transport along ω with respect to ∇.

We note the following immediate consequence of Proposition 2.6.

Corollary 2.8. *For any horizontal curve ω with $u = \mathrm{Dev}^{-1}(\omega)$, we have*

$$\left\{\tfrac{d}{ds}\mathrm{Dev}(u+sk)|_{s=0} \colon\ k \in W_0^\infty(H)\right\} = \{D_h|_\omega \colon\ h \in W_0^\infty(H)\}.$$

We note that $\frac{d}{ds}\mathrm{Dev}(u+sk)|_{s=0} = D_h|_\omega$ where k and h are related by (2.4). In the case when ∇ is a flat connection, i.e. if $\mathbf{R} \equiv 0$, then $k = h$. We will generalize such vector fields to functions h with values in the Cameron–Martin space in the next section.

Remark 2.9. Let (M, H, g) be a sub-Riemannian manifold. We say that a Riemannian metric $\bar{g}$ *tames* g if $\bar{g}|_{H\times H} = g$. Assume that we have chosen a taming metric $\bar{g}$ for which $V = H^\perp$ is the orthogonal complement of H. Assume further that V is integrable with corresponding foliation Φ. Then the assumption of V being metric-preserving is equivalent to assuming that the metric $\bar{g}$ is bundle-like, or, in a different terminology, assuming that Φ is a Riemannian foliation. We refer to [20] for details. We emphasize that none of these properties depend on $\bar{g}|_{V\times V}$.

3. Diffusions and gradients on path space

Throughout this section, we assume that any complement V to a horizontal bundle H chosen is metric preserving. We also assume that M is compact for a simpler presentation. We hence have that all tensors are bounded and that all local martingales are indeed martingales. The same results hold in the non-compact case under some additional assumptions, see Section 3.7 for details.

3.1. Sub-Riemannian diffusions and notation

Let M be a compact manifold and let $W_x = W_x(M)$ be the space of continuous maps $\omega\colon [0,\infty) \to M$ with $\omega_0 = x$. Let (H, g) be a sub-Riemannian structure on M and let ∇ be a compatible connection. Recall that the Hessian of ∇ is defined as

$$\nabla^2_{Y_1,Y_2} = \nabla_{Y_1}\nabla_{Y_2} - \nabla_{\nabla_{Y_1}Y_2}, \quad Y_1, Y_2 \in \Gamma(TM).$$

We write $L = \mathrm{tr}_H \nabla^2_{\times,\times}$ for the connection sub-Laplacian of ∇ and let $x \mapsto X_t^x \in W_x$ be the stochastic flow with generator $\frac{1}{2}L$ and $X_0^x = x$ defined on the filtered probability space $(W_x, \mathscr{F}_\cdot, \mathbb{P}_x)$.

For $0 \le s \le t < \infty$, let $/\!/_{s,t} : T_{X_s^x}M \to T_{X_t^x}M$ denote the parallel transport along X_t^x with respect to ∇ and write $/\!/_{0,t} = /\!/_t$. Note that $/\!/_{s,t} = /\!/_t /\!/_s^{-1}$. The solution B_t^x of

$$dB_t^x = /\!/_t^{-1} \circ dX_t^x, \quad B_0^x = 0 \in H_x,$$

is a standard Brownian motion in H_x. Hence, X_t^x can be considered as the development of the Brownian motion in H_x.

For any $T > 0$, we define W_x^T as the curves in W_x restricted to $[0,T]$. We write the induced structure of a filtered probability space as $(W_x^T, \mathscr{F}_\cdot^T, \mathbb{P}_x^T)$ and the corresponding stochastic process as $X_{[0,T]}^x$. Introduce the Cameron–Martin space $\mathbb{H}_x^T := \mathbb{H}^T(H_x)$ as the Hilbert space of absolutely continuous functions $h\colon [0,T] \to H_x$ with $\int_0^T |\dot{h}_t|_g^2\, dt < \infty$ and with inner product

$$\langle h, k\rangle_{\mathbb{H}} = \int_0^T \langle \dot{h}_t, \dot{k}_t\rangle_g\, dt, \quad h, k \in \mathbb{H}_x^T.$$

More generally, we define

$$\begin{aligned}\mathbb{H}^T_{W,x} &= L^2(W^T_x \to \mathbb{H}^T_x; \mathscr{F}^T_\cdot, \mathbb{P}^T_x)\\ &= \left\{h \in L^2(W^T_x \to \mathbb{H}^T_x; \mathbb{P}^T_x)\colon\ h_t \text{ is } \mathscr{F}_t\text{-measurable, } t \in [0,T]\right\},\end{aligned}$$

as a Hilbert space with inner product $\langle h,k\rangle_{L^2} = \mathbb{E}\langle h,k\rangle_{\mathbb{H}}$. As usual, we write $\langle h, B^x\rangle_{\mathbb{H}} = \int_0^T \langle \dot{h}_s, dB^x_s\rangle_g$.

3.2. Gradient on path space

Let V be a metric preserving complement to H and define $\nabla = \nabla^{g,V}$. Let $x \mapsto X^x_\cdot$ be the corresponding stochastic flow. For any $h \in W^\infty_0(H_x)$, recall the definition of D_h on $W^\infty_x(M)$ from Definition 2.7. As parallel transport is well defined along a path in W_x almost surely, we can consider D_h as a $\mathbb{P}$-almost surely defined vector field on W_x. We want to make this definition more precise and valid for functions h in the Cameron–Martin space.

Inspired by Proposition 2.6, we define the following endomorphism $A_{x,t} = A_t : T_xM \to T_xM$ by

$$A_t(\,\cdot\,) = \int_0^t \mathbf{T}_{//s}(\circ dB^x_s, \,\cdot\,) = \int_0^t \mathbf{T}_{//s}(dB^x_s, \,\cdot\,) - \frac{1}{2}\int_0^t (\delta_H \mathbf{T})_{//s}(\,\cdot\,)ds,$$

where $\circ d$ denotes the Stratonovich differential and $\delta_H\mathbf{T} = -\operatorname{tr}_H(\nabla_\times \mathbf{T})(\times, \,\cdot\,)$. We remark that by the defining properties of ∇ in Proposition 2.5, we have that $A_t(H_x) \subseteq V_x$ and $A_t(V_x) = 0$. For fixed $T > 0$, consider the space of cylindrical functions

$$\mathscr{F}C^\infty = \left\{F\colon\ \omega \in W^T_x \mapsto f(\omega_{t_1}, \dots, \omega_{t_n}) \;\middle|\; \begin{array}{c} 0 \le t_1 < \cdots < t_n \le T, \\ n \ge 0,\ f \in C^\infty(M^n)\end{array}\right\}.$$

For $h \in \mathbb{H}^T_x$, we define D_h acting on a cylindrical function $F\colon \omega \mapsto f(\omega_{t_1}, \omega_{t_2}, \dots, \omega_{t_n})$ by

$$\begin{aligned}D_hF &= \sum_{i=1}^n \left\langle //_{t_i}^{-1} d_i f|_{(\omega_{t_1},\dots,\omega_{t_n})}, h_{t_i} + \textstyle\int_0^{t_i} dA_t h_t\right\rangle\\ &= \sum_{i=1}^n \left\langle //_{t_i}^{-1} d_i f|_{(\omega_{t_1},\dots,\omega_{t_n})}, \textstyle\int_0^{t_i} (\mathrm{id} + A_{t_i} - A_t) dh_t\right\rangle.\end{aligned}$$

Observe that this definition is a generalization of Definition 2.7. If we define $D_tF \in H_x$ by

$$D_tF := \sum_{i=1}^n 1_{t \le t_i} \sharp (\mathrm{id} + A_{t_i} - A_t)^* //_{t_i}^{-1} d_i f|_{(\omega_{t_1},\dots,\omega_{t_n})},$$

then for every $h \in \mathbb{H}^T_x$,

$$\int_0^T \langle D_tF, \dot{h}_t\rangle_g dt = D_hF.$$

Finally, we define *the gradient* $DF \in \mathbb{H}^T_{W,x}$ by the relation $\langle DF, h\rangle_{\mathbb{H}} = D_hF$.

We will show that the operator $D\colon F \to DF$ can be closed on path space by the following integration by parts formula. Recall that $\mathbf{R}$ is the curvature of ∇. Introduce the corresponding Ricci operator $\mathrm{Ric}\colon TM \to TM$ by

$$\mathrm{Ric}(v) = -\operatorname{tr}_H \mathbf{R}(\times, v)\times\,. \tag{3.1}$$

Theorem 3.1.

(a) *For any $F \in \mathscr{F}C^\infty$, we have*

$$d_x\mathbb{E}_x[F] = \mathbb{E}_x[D_0F] - \frac{1}{2}\int_0^T \mathbb{E}_x[(\mathrm{Ric}_{//_s} Q_s)^* D_sF]\, ds,$$

where Q_t is the solution to the following equation:

$$dQ_t = -\frac{1}{2}\,\mathrm{Ric}_{//_t} Q_t\, dt, \qquad Q_0 = \mathrm{id}_{T_xM}\,. \tag{3.2}$$

(b) *For any $F \in \mathscr{F}C^\infty$ and $h \in \mathbb{H}_x^T$, we have*

$$\mathbb{E}_x[\langle DF, h\rangle_{\mathbb{H}}] = \mathbb{E}_x\left[F\int_0^T \langle \dot h_t + \frac{1}{2}\,\mathrm{Ric}_{//_t} h_t, dB_t\rangle_g\right].$$

In particular, for $F(X_{[0,T]}) = f(X_t)$, our result reduces to the following form, see the end of Section 3.5 for more details.

Corollary 3.2. *Assume that V is metric preserving and write $P_tf = \mathbb{E}[f(X_t)]$.*

(a) *Let Q_t be the solution of (3.2). Then*

$$dP_tf(v) = \mathbb{E}_x\left[\left\langle //_t^{-1} df|_{X_t}, Q_tv + \int_0^t dA_sQ_sv\right\rangle\right], \quad v \in T_xM;$$

(b) *for any $k \in \mathbb{H}_x^T$ with $h_t = Q_t\int_0^t Q_s^{-1}dk_s$, we have*

$$\mathbb{E}_x\left[f(X_T)\langle k, B\rangle_{\mathbb{H}}\right] = \mathbb{E}_x\left[\left\langle //_T^{-1} df|_{X_T}, h_T + \int_0^T dA_th_t\right\rangle\right].$$

We note that the result in (a) has already appeared in [22]. We show this formula by proving the corresponding derivative formula and integration by parts formula in Theorem 3.4 for the damped gradient.

3.3. The damped gradient on path space

We define *the damped gradient* $\tilde DF$ similarly to the formula in Riemannian geometry, but using parallel transport of the adjoint connection. We use the connection $\nabla = \nabla^{g,V}$ and define Ric as in (3.1). Define $\hat{//}_{s,t} : T_{X_s}M \to T_{X_t}M$ as parallel transport along X_t with respect to $\hat\nabla$, the adjoint of ∇, and write $\hat{//}_t = \hat{//}_{0,t}$. We first introduce $\hat Q_{s,t} : T_{X_s}M \to T_{X_s}M$, $s \le t$,

$$\frac{d}{dt}\hat Q_{s,t} = -\frac{1}{2}\,\mathrm{Ric}_{\hat{//}_{s,t}}\hat Q_{s,t}, \quad \hat Q_{s,s} = \mathrm{id}_{T_{X_s}M}\,.$$

We note that if $\hat Q_t = \hat Q_{0,t}$, then $\hat Q_{s,t} = \hat{//}_s\hat Q_t\hat Q_s^{-1}\hat{//}_s^{-1}$ and for $s \le r \le t$,

$$\hat Q_{s,t} = \hat{//}_{s,r}^{-1}\hat Q_{r,t}\hat{//}_{s,r}\hat Q_{s,r}.$$

For $F \in \mathscr{F}C^\infty$ with $F(\omega) = f(\omega_{t_1}, \dots, \omega_{t_n})$, we define

$$\tilde D_tF(\omega) := \sum_{i=1}^n 1_{t\le t_i}\sharp //_t^{-1}\hat Q_{t,t_i}^*\hat{//}_{t,t_i}^{-1} d_if|_{(\omega_{t_1},\dots,\omega_{t_n})},$$

and furthermore, for any $k \in \mathbb{H}_x^T$,

$$\tilde D_kF := \langle \tilde DF, k\rangle_{\mathbb{H}} := \int_0^T \tilde D_tFdk_t.$$

The next result clarifies the relationship between DF and $\tilde DF$.

Theorem 3.3. *Let $Q_t : T_xM \to T_xM$ be the solution to*

$$Q_0 = \mathrm{id}_{T_xM}, \quad dQ_t = -\frac{1}{2}\operatorname{Ric}_{//_t} Q_t dt.$$

(a) *For any $k \in \mathbb{H}_x^T$ and $F \in \mathscr{F}C^\infty$, if $h_t = Q_t \int_0^t Q_s^{-1} dk_s$, then*

$$\tilde{D}_k F = D_h F,$$

(b) *For any $F \in \mathscr{F}C^\infty$,*

$$\tilde{D}_t F = D_t F - \frac{1}{2}\int_t^T (\operatorname{Ric}_{//_s} Q_s Q_t^{-1})^* D_s F ds.$$

Next, with respect to the damped gradient, we establish the gradient formula and integration by parts formula on path space as follows.

Theorem 3.4.

(a) *(Derivative formula) For any $F \in \mathscr{F}C^\infty$ and $t > 0$, we have*

$$D_t \mathbb{E}_x[F|\mathscr{F}_t] = \mathbb{E}_x[\tilde{D}_t F|\mathscr{F}_t].$$

(b) *(The Clark–Ocône formula) For any $F \in \mathscr{F}C^\infty$, we have*

$$F = \mathbb{E}_x[F] + \int_0^T \langle \mathbb{E}_x[\tilde{D}_s F|\mathscr{F}_s], dB_s^x\rangle.$$

(c) *(Integration by parts formula) For any $k \in \mathbb{H}_x^T$ and $F \in \mathscr{F}C^\infty$,*

$$\mathbb{E}_x[\langle \tilde{D}F, k\rangle_{\mathbb{H}}] = \mathbb{E}_x\left[F\langle k, B^x\rangle_{\mathbb{H}}\right].$$

In particular, for any $k \in \mathbb{H}_x^T$ with $U_t = //_t^{-1} \hat{//}_t$, we have

$$\mathbb{E}\left[f(X_T^x)\langle k, B^x\rangle_{\mathbb{H}}\right] = \mathbb{E}\left[\left\langle //_T^{-1} df|_{X_T^x}, U_T \hat{Q}_T \int_0^T \hat{Q}_s^{-1} U_s^{-1} dk_s \right\rangle\right]. \tag{3.3}$$

We will prove Theorems 3.3 and 3.4 in the next subsections. Now we show how Theorem 3.1 follows from these results.

Proof of Theorem 3.1. We can prove (a) directly by using Theorem 3.4(a) and Theorem 3.3(b). For (b),

$$\begin{aligned}\mathbb{E}_x D_h F &= \mathbb{E}_x \tilde{D}_k F = \mathbb{E}_x\left[F\langle k, B\rangle_{\mathbb{H}}\right] \\ &= \mathbb{E}_x\left[F\int_0^T \langle \dot{h}_t + \frac{1}{2}\operatorname{Ric}_{//_t} h_t, dB_t^x\rangle_g\right],\end{aligned}$$

where the last equation follows from $dk_t = dh_t + \frac{1}{2}\operatorname{Ric}_{//_t} h_t dt$. $\square$

3.4. Proof of Theorem 3.3

Let (M, H, g) be a sub-Riemannian manifold with a metric preserving complement V. In all the steps below, we will consider the connection $\nabla = \nabla^{g,V}$.

Define the tensor Ric relative to ∇ as in (3.1). To prove Theorem 3.3, we first observe that Ric vanishes outside the horizontal bundle H.

Lemma 3.5. *Write* $\operatorname{Ric}_H = \operatorname{Ric}|_H$. *Then*

$$\operatorname{Ric} = \operatorname{Ric}_H \operatorname{pr}_H = \operatorname{pr}_H \operatorname{Ric}_H \operatorname{pr}_H .$$

Proof. We note that since H is parallel with respect to ∇, we have $\mathbf{R}(\,\cdot\,,\,\cdot\,)v \in H_x$ for any $v \in H_x$, $x \in M$. It follows that $\operatorname{Ric}(TM) \subseteq H$. From the proof of Lemma A.6, we also have that $\langle \mathbf{R}(v_1, z)v_1, v_2\rangle_g = 0$ whenever V is metric preserving, giving us $\operatorname{Ric}(V) = 0$. □

Next, we relate the damped gradient and the gradient by the following conversion formula.

Lemma 3.6. *Define* $U_t = /\!/_t^{-1} \hat{/\!/}_t$. *For any element in* $k \in \mathbb{H}_x^T$, *write* $h_t = Q_t \int_0^t Q_s^{-1} dk_s$. *We then have that*

$$U_t \hat{Q}_t \int_0^t \hat{Q}_s^{-1} U_s^{-1} dk_s = h_t + \int_0^t dA_s h_s.$$

Proof. We note first that $dU_t = \mathbf{T}_{/\!/_t}(\circ dB_t, U_t\,\cdot\,)$ and $U_0 = \operatorname{id}$, giving us that $U_t = \operatorname{id} + A_t$. Since $A_t^2 = 0$, we have that $U_t^{-1} = \operatorname{id} - A_t$. We will use this to find a formula for $\hat{Q}_t$ by

$$\begin{aligned} d\hat{Q}_t &= -\frac{1}{2} \operatorname{Ric}_{\hat{/\!/}_t} \hat{Q}_t\, dt = -\frac{1}{2} (\operatorname{id} - A_t) \operatorname{Ric}_{/\!/_t} (\operatorname{id} + A_t) \hat{Q}_t\, dt \\ &= -\frac{1}{2} \operatorname{Ric}_{/\!/_t} \hat{Q}_t\, dt + \frac{1}{2} A_t \operatorname{Ric}_{/\!/_t} \hat{Q}_t\, dt. \end{aligned}$$

Hence, we have that $\hat{Q}_t = Q_t + \frac{1}{2}\int_0^t A_s \operatorname{Ric}_{/\!/_s} Q_s\, ds = Q_t - \int_0^t A_s dQ_s$. Since $Q_t w = w$ for any $w \in V$, the inverse of $\hat{Q}_t$ is $\hat{Q}_t^{-1} = (\operatorname{id} + \int_0^t A_s dQ_s) Q_t^{-1}$.

We use these identities to compute

$$U_t \hat{Q}_t = Q_t + \int_0^t dA_r Q_r, \quad (U_t \hat{Q}_t)^{-1} = Q_t^{-1} - \int_0^t dA_r Q_r Q_t^{-1}, \tag{3.4}$$

and hence,

$$\begin{aligned} (d(U_t \hat{Q}_t))(U_t \hat{Q}_t)^{-1} &= \left(-\frac{1}{2} \operatorname{Ric}_{/\!/_t} Q_t dt + dA_t Q_t\right) (U_t Q_t)^{-1} \\ &= -\frac{1}{2} \operatorname{Ric}_{/\!/_t} dt + dA_t. \end{aligned}$$

If we write $a_t = U_t \hat{Q}_t \int_0^t \hat{Q}_s^{-1} U_s^{-1} dk_s$, then

$$da_t = d(U_t \hat{Q}_t)(U_t \hat{Q}_t)^{-1} a_t + dk_t = -\frac{1}{2} \operatorname{Ric}_{/\!/_t} a_t dt + dA_t a_t + dk_t, \quad a_0 = 0.$$

It follows that $a_t = h_t + \int_0^t dA_s h_s$ since h_t is the solution of $dh_t + \frac{1}{2} \operatorname{Ric}_{/\!/_t} h_t dt = dk_t$. □

Using this lemma, we prove Theorem 3.3 as follows.

Proof of Theorem 3.3. We consider $F \in \mathscr{F}C^\infty$. Note that

$$\begin{aligned} \int_0^T \tilde{D}_t F(\omega) dk_t &= \int_0^T \sum_{i=1}^n 1_{t \le t_i} \langle /\!/_t^{-1} \hat{Q}_{t,t_i} \hat{/\!/}_{t,t_i}^{-1} d_i f|_{(\omega_{t_1}, \ldots, \omega_{t_n})}, dk_t \rangle_g \\ &= \int_0^T \sum_{i=1}^n 1_{t \le t_i} \langle /\!/_{t_i}^{-1} d_i f|_{(\omega_{t_1}, \ldots, \omega_{t_n})}, U_{t_i} \hat{Q}_{t_i} \hat{Q}_t^{-1} U_t^{-1} dk_t \rangle_g. \end{aligned}$$

Hence, we have that $\int_0^T \langle \tilde{D}_t F(\omega), \dot{k}\rangle_g\, ds = \int_0^T \langle D_t F, \dot{h}\rangle_g\, ds$ by Lemma 3.6, which then proves (a).

The relationship between D_tF and $\tilde{D}_tF$ can be observed from

$$\begin{aligned}\int_0^T (\tilde{D}_tF - D_tF)\dot{k}_t\,dt &= -\frac{1}{2}\int_0^T D_tF\,\mathrm{Ric}_{//_t}\,Q_t\int_0^t Q_s^{-1}dt\\ &= -\frac{1}{2}\int_0^T\int_s^T D_tF\,\mathrm{Ric}_{//_t}\,Q_tQ_s^{-1}\,dtdk_s,\end{aligned}$$

which then implies (b). □

3.5. Proof of Theorem 3.4

We first prove Theorem 3.4(a) for the case $n = 1$.

Lemma 3.7. *For any $x \in M$, consider $\hat{Q}_t^x = \hat{Q}_t$ as the solution of*

$$\frac{d}{dt}\hat{Q}_t = -\frac{1}{2}\mathrm{Ric}_{\hat{//}_t}\hat{Q}_t\,,\qquad \hat{Q}_0 = \mathrm{id}_{T_xM},$$

and let $\hat{Q}_t^* : T_x^*M \to T_x^*M$ be its dual. Define $U_t = //_t^{-1}\hat{//}_t$. Then we have

$$dP_tf|_x = \mathbb{E}\left[\hat{Q}_t^*\hat{//}_t^{-1}df|_{X_t^x}\right] = \mathbb{E}\left[\hat{Q}_t^*U_t^{-1}//_t^{-1}df|_{X_t^x}\right].$$

Proof. For $t \in [0, T]$, consider the T_x^*M-valued process

$$\tilde{N}_s = \hat{Q}_s^*\hat{//}_s^{-1}dP_{t-s}f|_{X_s}.$$

From (A.4) and the fact that V is metric preserving, it follows that $\tilde{N}_s$ is a local martingale

$$d\tilde{N}_s = \hat{Q}_s^*\hat{//}_s^{-1}\hat{\nabla}_{//_sdB_s}dP_{t-s}f|_{X_s},$$

and from our compactness assumption, it is a true martingale. □

Proof of Theorem 3.4. For part (a), write $F(\omega) = f(\omega_{t_1},\ldots,\omega_{t_n})$. We first consider the case when $t = 0$. Write $\Phi(x) = \mathbb{E}_x[F]$. Then we need to prove

$$\sharp d\Phi = \mathbb{E}_x[\tilde{D}_0F].$$

By Lemma 3.7, the desired assertion holds for $n = 1$. We will use an induction argument [27, Section 8.4] and assume that it holds for $n \geq 1$. We will prove that the assertion also holds for $n+1$. Let

$$g(x) = \mathbb{E}[f(x, X_{t_2-t_1}^x, X_{t_3-t_1}^x, \ldots, X_{t_{n+1}-t_1}^x)],$$

which by our induction hypothesis satisfies.

$$dg(x) = \sum_{i=1}^{n+1}\mathbb{E}\left[\hat{Q}_{0,t_i-t_1}\hat{//}_{t_i-t_1}^{-1}d_if|_{(x,X_{t_2-t_1}^x,\ldots,X_{t_{n+1}-t_1}^x)}\right]. \tag{3.5}$$

From (3.5) and from the result at $n = 1$,

$$\begin{aligned} d\mathbb{E}f|_{X_{t_1}^x,X_{t_2}^x,\ldots,X_{t_{n+1}}^x} &= d\mathbb{E}[g(X_{t_1}^x)] = \mathbb{E}\left[\hat{Q}_{0,t_1}\hat{//}_{t_1}^{-1}dg|_{X_{t_1}^x}\right]\\ &= \sum_{i=1}^n\mathbb{E}\left[\hat{Q}_{0,t_1}\hat{//}_{t_1}^{-1}\hat{Q}_{t_1,t_i}\hat{//}_{t_1,t_i}^{-1}d_if|_{(X_{t_1}^x,X_{t_2}^x,\ldots,X_{t_{n+1}}^x)}\right]\\ &= \sum_{i=1}^n\mathbb{E}\left[\hat{Q}_{0,t_i}\hat{//}_{t_i}^{-1}d_if|_{(X_{t_1}^x,X_{t_2}^x,\ldots,X_{t_{n+1}}^x)}\right], \end{aligned} \tag{3.6}$$

which is the desired result for $t = 0$. For a general t, consider $G = \mathbb{E}_x[F|\mathscr{F}_t]$. If $t_{m-1} < t \le t_m$, then

$$G(\omega) = g(\omega_{t_1}, \dots \omega_{t_{m-1}}, \omega_t) = \mathbb{E}_x[f(\omega_{t_1}, \dots, \omega_{t_{m-1}}, X_{t_m}, \dots, X_{t_n})|X_t = \omega_t].$$

Using the formula (3.6), we obtain

$$\begin{aligned}
&d_{\omega_t} g|_{(\omega_{t_1},\dots,\omega_{t_{m-1}},\omega_t)} = d_{\omega_t}\mathbb{E}[f(\omega_{t_1}, \dots, \omega_{t_{m-1}}, X_{t_m}, \dots, X_{t_n})|X_t = \omega_t] \\
&= d_{\omega_t}\mathbb{E}[f(\omega_{t_1}, \dots, \omega_{t_{m-1}-t}, X_{t_m-t}, \dots, X_{t_n-t})|X_0 = \omega_t] \\
&= \mathbb{E}_{\omega_t}\Big[\sum_{i=m}^{n} \hat{Q}^*_{0,t_i-t} \hat{/\!/}^{-1}_{t_i-t} d_i f|_{(\omega_{t_1},\dots,\omega_{t_{m-1}},X_{t_m},\dots,X_{t_n})}\Big] \\
&= \mathbb{E}\Big[\sum_{i=m}^{n} \hat{Q}^*_{t,t_i} \hat{/\!/}^{-1}_{t,t_i} d_i f|_{(\omega_{t_1},\dots,\omega_{t_{m-1}},X_{t_m},\dots,X_{t_n})} \,\Big|\, X_t = \omega_t\Big] \\
&= /\!/_t \mathbb{E}[\tilde{D}_t F|\mathscr{F}_t]
\end{aligned}$$

from the strong Markov property of X_t.

For part (b), we first observe that $\mathbb{E}[F \mid \mathscr{F}_t]$ is a martingale according to the definition. By martingale representation, we have

$$d\mathbb{E}[F \mid \mathscr{F}_t] = \langle D_t \mathbb{E}[F \mid \mathscr{F}_t], dB^x_t\rangle = \langle \mathbb{E}[\tilde{D}_t F \mid \mathscr{F}_t], dB^x_t\rangle.$$

Integrating from 0 to T gives

$$F - \mathbb{E}[F] = \mathbb{E}[F \mid \mathscr{F}_T] - \mathbb{E}[F] = \int_0^T \langle \mathbb{E}[\tilde{D}_t F \mid \mathscr{F}_t], dB^x_t\rangle. \tag{3.7}$$

For part (c), we first use (3.7) inside the term $\mathbb{E}\left[F\langle k, B^x\rangle_{\mathbb{H}}\right]$. By (3.7),

$$\begin{aligned}
\mathbb{E}\left[F\langle k, B^x\rangle_{\mathbb{H}}\right] &= \mathbb{E}\left[\left(\mathbb{E}[F] + \int_0^T \langle \mathbb{E}[\tilde{D}_t F \mid \mathscr{F}_t], dB^x_t\rangle\right)\langle k, B^x\rangle_{\mathbb{H}}\right] \\
&= \mathbb{E}\left[\int_0^T \langle \mathbb{E}[\tilde{D}_t F \mid \mathscr{F}_t], \dot{k}_t\rangle dt\right],
\end{aligned}$$

giving the formula in (c). By letting $F(X_{[0,T]}) = f(X_T)$ we have the equality in (3.3). □

Proof of Corollary 3.2. For (a), we first observe from Lemma 3.7 that

$$\langle dP_t f, v\rangle = \mathbb{E}\langle \hat{Q}^*_t \hat{/\!/}^{-1}_t df|_{X_t}, v\rangle = \mathbb{E}\langle /\!/^{-1}_t df|_{X_t}, U_t \hat{Q}_t v\rangle.$$

Then by (3.4), we have

$$\langle dP_t f, v\rangle = \mathbb{E}\left[\left\langle /\!/^{-1}_t df, Q_t v + \int_0^t dA_s Q_s v\right\rangle\right], \quad v \in T_x M.$$

The formula in (b) follows from Theorem 3.1 (b) by taking $F(X_{[0,T]}) = f(X_T)$ directly. □

3.6. Quasi-invariance

We want to link $\langle \tilde{D}F, k\rangle_{\mathbb{H}}$ to the directional derivative induced by some quasi-invariant flow. We use techniques from [32, Chapter 4.2]. For $k \in \mathbb{H}^T_x$ and $s \in (-\varepsilon, \varepsilon)$, let $X^s = X^{x,s}$ solve the SDE

$$dX^s_t = /\!/^s_t \circ\, dB_t - s /\!/^s_t dk_t, \qquad X^s_0 = x, \tag{3.8}$$

where $//^s$ is the parallel transport along X^s with respect to ∇. This flow is quasi-invariant, i.e., the distribution of $X^s_{[0,T]}$ is absolutely continuous with respect to that of $X_{[0,T]} = X^x_{[0,T]}$. Let

$$R^s = \exp\left(s\langle k, B\rangle_{\mathbb{H}} - \frac{s^2}{2}\langle k, k\rangle_{\mathbb{H}}\right) = \exp\left(s\int_0^T \langle \dot{k}, B\rangle_g - \frac{s^2}{2}\int_0^T |\dot{k}_t|_g^2\, dt\right).$$

If $\mathbb{P}$ denotes the Wiener measure on the path space of H_x, $d = \operatorname{rank} H_x$, then by the Girsanov theorem,

$$B^s_t := B_t - sk_t$$

is a d-dimensional Brownian motion in H_x under the probability measure $\mathbb{P}^s = R^s\mathbb{P} = (\xi_{s,k})_*\mathbb{P}$ with

$$\xi_{s,k}(u) = u + sk, \qquad u \in W_0(H_x).$$

Proposition 3.8. *For any $x \in M$ and $F \in \mathscr{F}C^\infty$,*

$$\mathbb{E}_x[\langle \tilde{D}F, k\rangle_{\mathbb{H}}] = \lim_{s\to 0} \mathbb{E}\frac{F(X^s_{[0,T]}) - F(X_{[0,T]})}{s}$$

holds for all $k \in \mathbb{H}^T$.

Proof. Let $B^s_t = B_t - sk_t$, which is the d-dimensional Brownian motion under $\mathbb{P}^s$. By the weak uniqueness of (3.8), we conclude that the distribution of X under $\mathbb{P}^s$ is consistent with that of X^s under $\mathbb{P}$. In particular,

$$\mathbb{E}[F(X^s_{[0,T]})] = \mathbb{E}\left[R^s F(X_{[0,T]})\right].$$

Thus, we have

$$\begin{aligned}\lim_{s\to 0}\mathbb{E}\frac{F(X^s_{[0,T]}) - F(X_{[0,T]})}{s} &= \lim_{s\to 0}\mathbb{E}\left[F(X_{[0,T]})\frac{R^s - 1}{s}\right] \\ = \mathbb{E}\left[F(X_{[0,T]})\int_0^T \langle \dot{k}_t, dB_t\rangle_g\right] &= \mathbb{E}_x[\langle \tilde{D}F, k\rangle_{\mathbb{H}}]. \quad \square\end{aligned}$$

3.7. Comments for the non-compact case

In order to have a simple exposition, we have assumed throughout this section that we are working over a compact manifold. This has the advantage that we can be assured that processes such as X^x_t, A_t and Q_t have infinite lifetime, all tensors are bounded and all local martingales that appear in our proofs are indeed true martingales. If one can find alternative ways to show that the same properties hold, our results hold without the compactness assumption. One way to ensure this on a non-compact manifold, is to verify that the following assumptions hold.

(A) Assume that X^x_t has infinite lifetime, i.e. assume that $P_t 1 = 1$.
(B) We must be able to pick a taming Riemannian metric $\bar{g}$ such that $\mathbf{T}$ and Ric are bounded and such that $\sup_{0\le t\le T} |//_t|_{\bar{g}\otimes\bar{g}^*}$ is finite for every T. We note that since $\hat{//}_t = //_t(\mathrm{id} + A_t)$, such assumptions are sufficient to bound parallel transport with respect to $\hat{\nabla}$ as well.
(C) Our cylindrical functions will now be defined as $\mathscr{F}C^\infty_0$ consisting of functions $F(\omega) = f(\omega_{t_1}, \dots, \omega_{t_n})$ where $f \in C^\infty_0(M^n)$ is of compact support. In order to differentiate expectations of such functions and in order that they remain bounded, we need to assume that $\sup_{t\in[0,T]} |dP_t f|_{\bar{g}^*}$ is bounded for every finite $T > 0$ and every compactly supported function $f \in C^\infty_0(M)$.

For sufficient conditions for these assumptions to be satisfied on non-compact manifolds, more specifically on complete sub-Riemannian manifolds coming from totally geodesic Riemannian foliations, see [22].

For alternate approaches of dealing with path space analysis on non-compact spaces in the Riemannian case, we also refer to [8]. For the rest of the paper, we will assume that properties (A) to (C) are satisfied or some other conditions are satisfied in order to ensure that the results of this section hold.

4. Bounded curvature and functional inequalities on path space

4.1. Inequalities equivalent to bounded curvature

Inspired by Naber's work [30], we have the following characterization formulae for the boundedness of Ric_H. Let V be a metric preserving complement with $\nabla = \nabla^{g,V}$ the corresponding connection. Recall that $\mathrm{Ric}_H := \mathrm{Ric}|_H$ where Ric is defined as in (3.1). We state the main result of the paper with proof given in the next section.

Theorem 4.1 (*Characterization of Bounded Horizontal Ricci Curvature by Functional Inequalities*)**.** *Let K be some fixed non-negative constant. Consider the following bound for the horizontal Ricci curvature* Ric_H

$$-K \le \mathrm{Ric}_H \le K. \tag{4.1}$$

I. *The following functional inequalities for functions on path space are equivalent to curvature bound (4.1):*

(i) *for any $F \in \mathscr{F}C_0^\infty$,*

$$|D_0\mathbb{E}_x[F]|_g \le \mathbb{E}_x\Big[|D_0F|_g + \frac{K}{2}\int_0^T \mathrm{e}^{\frac{K}{2}s}\,|D_sF|_g\,ds\Big];$$

(ii) *for any $F \in \mathscr{F}C_0^\infty$,*

$$|D_0\mathbb{E}_x[F]|_g^2 \le \mathrm{e}^{\frac{K}{2}T}\,\mathbb{E}_x\Big[|D_0F|_g^2 + \frac{K}{2}\int_0^T \mathrm{e}^{\frac{K}{2}s}\,|D_sF|_g^2\,ds\Big];$$

(iii) (Log-Sobolev inequality) *for any $F \in \mathscr{F}C_0^\infty$ and $t > 0$ in $[0,T]$,*

$$\begin{aligned}
&\mathbb{E}_x\Big[\mathbb{E}_x[F^2|\mathscr{F}_t]\log\mathbb{E}_x[F^2|\mathscr{F}_t]\Big] - \mathbb{E}_x[F^2]\log\mathbb{E}_x[F^2]\\
&\le 2\int_0^t \mathrm{e}^{\frac{K}{2}(T-r)}\left(\mathbb{E}_x|D_rF|_g^2 + \frac{K}{2}\int_r^T \mathrm{e}^{\frac{K}{2}(s-r)}\,\mathbb{E}_x|D_sF|_g^2\,ds\right)dr;
\end{aligned}$$

(iv) (Poincaré inequality) *for any $F \in \mathscr{F}C_0^\infty$ and $t > 0$ in $[0,T]$,*

$$\begin{aligned}
&\mathbb{E}_x\Big[\mathbb{E}_x[F|\mathscr{F}_t]^2\Big] - \mathbb{E}_x[F]^2\\
&\le \int_0^t \mathrm{e}^{\frac{K}{2}(T-r)}\left(\mathbb{E}_x|D_rF|_g^2 + \frac{K}{2}\int_r^T \mathrm{e}^{\frac{K}{2}(s-r)}\,\mathbb{E}_x|D_sF|_g^2\,ds\right)dr.
\end{aligned}$$

II. *The following functional inequalities on the manifold M are equivalent to the curvature bound (4.1):*

(v) *for $f \in C_0^\infty(M)$,*

$$\begin{aligned}
&|dP_tf(x)|_{g^*}^2 - \mathrm{e}^{\frac{K}{2}t}\,\mathbb{E}_x\left[\left|(\mathrm{id}+A_t)^*/\!/_t^{-1}df(X_t)\right|_{g^*}^2\right]\\
&\le \mathbb{E}_x\left[\frac{K}{2}\int_0^t \mathrm{e}^{\frac{K}{2}(t+s)}\,|(\mathrm{id}+A_t-A_s)^*/\!/_t^{-1}df(X_t)|_{g^*}^2\,ds\right];
\end{aligned}$$

and

$$\begin{aligned}&\left|2df - dP_tf\right|^2_{g^*}(x) - \mathrm{e}^{\frac{K}{2}t}\,\mathbb{E}_x\left[\left|2df(x) - (\mathrm{id}+A_t)^*/\!/_t^{-1}df(X_t)\right|^2_{g^*}\right]\\ &\quad\le \mathbb{E}_x\left[\frac{K}{2}\int_0^t \mathrm{e}^{\frac{K}{2}(t+s)}\left|(\mathrm{id}+A_t-A_s)^*/\!/_t^{-1}df(X_t)\right|^2_{g^*}ds\right].\end{aligned}$$

Remark 4.2. Theorem 4.1 describes equivalent statements for a symmetric bound of Ric_H. For non-symmetric bounds, i.e. if $K_1 \le \mathrm{Ric}_H \le K_2$ for some constants $K_1 \le K_2$, we can also give corresponding equivalent functional inequalities as in [10] by modifying the gradient operator on the path space to:

$$\bar{D}_tF := \sum_{i=1}^n 1_{t\le t_i}\,\mathrm{e}^{-\frac{K_1+K_2}{4}(t_i-t)}\,\sharp(\mathrm{id}+A_{t_i}-A_t)^*/\!/_{t_i}^{-1}d_iF.$$

The functional inequalities will then be written in terms of $\bar{D}F$. We refer to [10, Proposition 2.2] for a more detailed discussion.

4.2. Proof of Theorem 4.1

We will show equivalence of the properties in Theorem 4.1 by proving the relations

$$\begin{tikzcd}
 & (4.1) \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & (iii) \arrow[d, Rightarrow] \\
(v) \arrow[ur, Rightarrow] & (i) \arrow[d, Rightarrow] & (iv) \arrow[dl, Rightarrow] \\
 & (ii) \arrow[ul, Rightarrow] &
\end{tikzcd}$$

We divide the proof into two parts.

Proof of Theorem 4.1, Part I. In this part, we prove the implications "(4.1) ⇒ (i) ⇒ (ii) ⇒ (v)", "(4.1) ⇒ (iii)" and "(4.1) ⇒ (iv)".

"(4.1) ⇒ (i)" By Theorem 3.1(a), we have

$$d_x\mathbb{E}_x[F] = \mathbb{E}_x[D_0F] - \frac{1}{2}\int_0^T \mathbb{E}_x[(\mathrm{Ric}_{/\!/_s}\,Q_s)^*D_sF]\,ds.$$

Then using condition (4.1), we can prove (i) by controlling Q_t and Ric_H:

$$\left|d_x\mathbb{E}_x[F]\right|_g \le \left|\mathbb{E}_x[D_0F]\right|_g + \frac{K}{2}\int_0^T \mathrm{e}^{\frac{1}{2}Ks}\,\mathbb{E}_x[|D_sF|_g]\,ds.$$

"(i) ⇒ (ii)" It is easily observed that

$$\begin{aligned}\mathbb{E}_x\left[|\tilde{D}_tF|^2_g\middle|\mathscr{F}_t\right] &= \mathbb{E}_x\left[\left(|D_tF|_g + \frac{K}{2}\int_t^T \mathrm{e}^{\frac{K}{2}(s-t)}\,|D_sF|_g ds\right)^2\middle|\mathscr{F}_t\right]\\ &\le \mathrm{e}^{\frac{K}{2}(T-t)}\,\mathbb{E}_x\left[\left(|D_tF|^2_g + \frac{K}{2}\int_t^T \mathrm{e}^{\frac{K}{2}(s-t)}\,|D_sF|^2_g\,ds\right)\middle|\mathscr{F}_t\right],\end{aligned} \tag{4.2}$$

which implies (ii) with $t=0$.

"(ii) ⇒ (v)" If $F(\omega)=f(\omega_t)$, then

$$D_0F(X_t)=\sharp(\mathrm{id}+A_t)^*//_t^{-1}df(X_t),\quad D_sF(X_t)=\sharp(\mathrm{id}+A_t-A_s)^*//_t^{-1}df(X_t).$$

We obtain from (ii) that

$$\begin{aligned}|dP_tf|_{g^*}^2\le \mathrm{e}^{\frac{k}{2}t}\,\mathbb{E}_x\left[|(\mathrm{id}+A_t)^*//_t^{-1}df(X_t)|_{g^*}^2\right]\\ +\frac{K}{2}\mathbb{E}_x\left[\int_0^t \mathrm{e}^{\frac{K}{2}s}\,|(\mathrm{id}+A_t-A_s)^*//_t^{-1}df(X_t)|_{g^*}^2ds\right].\end{aligned}$$

Moreover, if $F(\omega)=2f(x)-f(\omega_t)$, then

$$\begin{aligned}&|2df-dP_tf|_{g^*}^2(x)-\mathrm{e}^{\frac{K}{2}t}\,\mathbb{E}_x\left[\left|2df(x)-(\mathrm{id}+A_t)^*//_t^{-1}df(X_t)\right|_{g^*}^2\right]\\ &\le \mathbb{E}_x\left[\frac{K}{2}\int_0^t \mathrm{e}^{\frac{K}{2}(t+s)}\,|(\mathrm{id}+A_t-A_s)^*//_t^{-1}df(X_t)|_{g^*}^2\,ds\right].\end{aligned}$$

"(4.1) ⇒(iii)(iv)" We now prove (iii) and (iv) by using estimate (4.2) above and the Itô formula,

$$\begin{aligned}d(\mathbb{E}_x[F^2|\mathscr{F}_t]\log\mathbb{E}_x[F^2|\mathscr{F}_t])&=dM_t+\frac{\left|\mathbb{E}_x[\tilde D_tF^2|\mathscr{F}_t]\right|_g^2}{2\mathbb{E}_x[F^2|\mathscr{F}_t]}dt\\ &\le dM_t+2\mathbb{E}_x\left[\left|\tilde D_tF\right|_g^2|\mathscr{F}_t\right]dt,\end{aligned}$$

where M_t is a local martingale such that

$$dM_t=(1+\log\mathbb{E}_x[F^2|\mathscr{F}_t])\langle\mathbb{E}_x(\tilde D_tF^2|\mathscr{F}_t),dB_t\rangle.$$

Integrating from 0 to t and taking expectation of both sides, we prove the inequality (iii). Similarly, we can prove (iv) by taking into consideration of the process $\mathbb{E}_x[F|\mathscr{F}_t]^2$ and using the following Itô formula:

$$d\mathbb{E}_x[F|\mathscr{F}_t]^2=d\tilde M_t+\mathbb{E}_x\left[\left|\tilde D_tF\right|_g^2|\mathscr{F}_t\right]dt,$$

where $\tilde M_t$ is a local martingale such that

$$d\tilde M_t=2\mathbb{E}_x[F|\mathscr{F}_t]\langle\mathbb{E}_x(\tilde D_tF|\mathscr{F}_t),dB_t\rangle.$$

Integrating from 0 to t and taking expectation of both sides, we prove inequality (iv).

To give the second part of the proof, we first need to include the following lemmas. For the Riemannian manifold case, the corresponding results can be found for instance in [10,32].

Lemma 4.3 (*Bochner–Weitzenböck Formula*). *Let $L=\mathrm{tr}_H\nabla^2_{\times,\times}$ be the connection sub-Laplacian on tensors. For any $f\in C^\infty(M)$, we then have*

$$Ldf(Z)-dLf(Z)=-2\,\mathrm{tr}_H\nabla_\times df(\mathbf{T}(\times,Z))+df(\mathrm{Ric}(Z)+\delta_H\mathbf{T}(Z)).$$

In particular,

$$\begin{aligned}\frac{1}{2}L|df|_{g^*}^2=&\langle dLf,df\rangle_{g^*}+\langle(\mathrm{Ric}+\delta_H\mathbf{T})^*(df),df\rangle_{g^*}\\ &+|\nabla df|_{g^*\otimes g^*}^2-2\,\mathrm{tr}_H\nabla_\times df(\mathbf{T}(\times,\sharp df)).\end{aligned}$$

Proof. The result follows from Lemma A.5, Appendix, for the case of a metric preserving complement and from the property of the torsion for our choice of connection. □

Lemma 4.4. *For $x \in M$, let $f \in C_0^\infty(M)$ be such that $\nabla df|_x = 0$ and $df(V_x) = 0$. Then the following limits hold:*

(a) $\displaystyle \frac{1}{2}\langle \sharp df, \operatorname{Ric} \sharp df\rangle_g(x) = \lim_{t\downarrow 0} \frac{\left\langle df, \mathbb{E}_x[(\mathrm{id} + A_t)^* //_t^{-1} df|_{X_t}]\right\rangle_{g^*}(x) - \langle df, dP_t f\rangle_{g^*}(x)}{t}$;

(b) $\displaystyle \lim_{t\downarrow 0} \frac{1}{t^2}\mathbb{E}_x\left[|A_t^* //_t^{-1} df|_{g^*}^2\right] = 0$;

(c) $\displaystyle \frac{1}{2}\langle \sharp df, \operatorname{Ric} \sharp df\rangle_g(x) = \lim_{t\downarrow 0} \frac{\mathbb{E}_x\left[\left|(\mathrm{id} + A_t)^* //_t^{-1} df|_{X_t}\right|_{g^*}^2\right] - |dP_t f|_{g^*}^2(x)}{t}$.

Proof. Let $\nabla^H f = \sharp df$ denote the horizontal gradient of a function f. Choose a taming Riemannian metric $\bar g$, i.e. a Riemannian metric $\bar g$ such that $\bar g|_H = g$. We can always choose $\bar g$ so that H and V are orthogonal. We will use this taming metric to construct a relatively compact neighborhood of t where we have reasonable estimates for the first exit time. If $\bar\nabla f$ is the gradient with respect to $\bar g$, we have that $\mathrm{pr}_H \bar\nabla f = \nabla^H f$. Let $d_{\bar g}$ be the Riemannian distance of $\bar g$. Choose sufficient small r such that the ball $B_{\bar g}(x, r)$ of $\bar g$-radius r centered at x is outside the cut-locus of x. Let $\rho_t := d_{\bar g}(x, X_t)$. Then there exists a constant $c_1 > 0$ such that $L d_{\bar g}^2(x, \cdot) \le 2c_1$. By Itô's formula,

$$\begin{aligned} d\rho_t^2 &= 2\rho_t\langle \nabla^H \rho_t, //_t dB_t\rangle_{g^*} + \frac{1}{2} L\rho_t^2 dt \\ &\le 2\rho_t\langle \nabla^H \rho_t, //_t dB_t\rangle_{g^*} + c_1 dt, \quad t < \sigma_r, \end{aligned}$$

where $\sigma_r = \inf\{t \ge 0 : X_t \notin B_{\bar g}(x, r)\}$. For fixed $t > 0$ and $\delta > 0$, define

$$Z_s := \exp\left(\frac{\delta}{t}\rho_s^2 - \frac{\delta}{t}c_1 s - \frac{2\delta^2}{t^2}\int_0^s \rho_u^2\, du\right), \quad s \le \sigma_r.$$

Then as $|\nabla^H(d_{\bar g}(x, \cdot))|_g \le |\bar\nabla(d_{\bar g}(x, \cdot))|_{\bar g} = 1$, we have

$$\begin{aligned} dZ_s &= \exp\left(\frac{\delta}{t}\rho_s^2 - \frac{\delta}{t}c_1 s - \frac{2\delta^2}{t^2}\int_0^s \rho_u^2\, du\right) \\ &\quad \times \left(\frac{2\delta}{t}\rho_s\langle \nabla^H \rho_s, //_s dB_s\rangle_{g^*} - \frac{2\delta^2}{t^2}\rho_s^2\, ds + \frac{2\delta^2}{t^2}\rho_s^2 |\nabla^H \rho_s|_{g^*}^2 ds\right) \\ &\le \exp\left(\frac{\delta}{t}\rho_s^2 - \frac{\delta}{t}c_1 s - \frac{2\delta^2}{t^2}\int_0^s \rho_u^2\, du\right)\left(\frac{2\delta}{t}\rho_s\langle \nabla^H \rho_s, //_s dB_s\rangle_{g^*}\right), \end{aligned}$$

and hence Z_s is a supermartingale. Therefore,

$$\begin{aligned} \mathbb{P}(\sigma_r \le t) &= \mathbb{P}\left(\max_{s\in[0,t]} \rho_{s\wedge\sigma_r} \ge r\right) \\ &\le \mathbb{P}\left(\max_{s\in[0,t]} Z_{s\wedge\sigma_r} \ge \mathrm{e}^{\delta r^2/t - \delta c_1 - 2\delta^2 r^2/t}\right) \\ &\le \exp\left(c_1\delta - \frac{1}{t}(\delta r^2 - 2\delta^2 r^2)\right). \end{aligned}$$

If we take $\delta = 1/4$, then

$$\mathbb{P}(\sigma_r \le t) \le \exp\left(\frac{1}{4}c_1 - \frac{r^2}{8t}\right).$$

(a) By Itô's formula and the estimate for σ_r, for small t and $f \in C_0^\infty(M)$ such that $\nabla df(x) = 0$, we get

$$\begin{aligned}\mathbb{E}_x\left[//_t^{-1} df|_{X_t}\right] &= df(x) + \mathbb{E}_x\left[\int_0^{t\wedge\sigma_r} //_s^{-1} L(df)|_{X_s}\, ds\right] + \mathrm{o}(t)\\ &= df|_x + L(df)|_x\, t + \mathrm{o}(t).\end{aligned}$$

From this equality, the result is easily derived using Taylor expansion:

$$\begin{aligned}&\left\langle df, \mathbb{E}_x //_t^{-1} df|_{X_t}\right\rangle_{g^*}(x) - \langle df, dP_t f\rangle_{g^*}(x)\\ &\quad= \Big(\langle df, Ldf\rangle_{g^*} - \langle df, dLf\rangle_{g^*}\Big)(x)t + \mathrm{o}(t)\\ &\quad= \frac{1}{2}\langle \mathrm{Ric}^*\, df, df\rangle_g(x)\, t + \mathrm{o}(t),\end{aligned}$$

where again we used the Weitzenböck identity in Lemma 4.3 and the fact that $(\delta_H \mathbf{T})^* df|_x = 0$ since $df(V_x) = 0$.

Finally, from Lemma 4.3, we see that $Ldf|V = dLf|V$. Hence, the process N_s in T_x^*M given by

$$N_s v = \langle //_s^{-1} dP_{t-s} f|_{X_t}, v\rangle$$

where $v \in V_x$, is a martingale. As a consequence, for any $v \in T_x M$,

$$\begin{aligned}&\mathbb{E}_x[\langle A_t^* //_t^{-1} df|_{X_t}, v\rangle] = \mathbb{E}_x[\langle N_t, A_t v\rangle]\\ &= \mathbb{E}_x\left[\int_0^{t\wedge\sigma_r} \mathrm{tr}_H \langle \nabla_{//_s \times} dP_{t-s} f, \mathbf{T}(//_s \times, //_s v)\rangle ds\right] + \mathrm{o}(t),\end{aligned}$$

which is of order $\mathrm{o}(t)$ since $\nabla df(x) = 0$.

(b) Using that $\mathbb{E}_x[A_t^* //_t^{-1} df|_{X_t}] = \mathrm{o}(t)$, that df vanishes at V_x and that $\nabla df = 0$, we have

$$\begin{aligned}\mathbb{E}_x\left[|A_t^* //_t^{-1} df|^2\right] &= \mathbb{E}_x\left[\int_0^{t\wedge\sigma_r} \left|//_s^{-1} \nabla_{//_t \cdot} df|_{X_s}(A_s\, \cdot)\right|^2_{g^*\otimes g^*} ds\right]\\ &\quad+ \mathbb{E}_x\left[\int_0^{t\wedge\sigma_r} \left|df\mathbf{T}(//_s\, \cdot, //_s\, \cdot)\right|^2 ds\right]\\ &\quad+ 2\,\mathbb{E}_x\left[\int_0^{t\wedge\sigma_r} \Big\langle \nabla_{//_s \cdot} df(A_s\, \cdot), df\mathbf{T}(//_s\, \cdot, //_s\, \cdot)\Big\rangle\, ds\right] + \mathrm{o}(t^2)\\ &= \mathrm{o}(t^2).\end{aligned}$$

As a consequence,

$$\lim_{t\downarrow 0} \frac{1}{t^2}\mathbb{E}_x\left[|A_t^* //_t^{-1} df|^2_{g^*}\right] = 0.$$

(c) Since $\nabla df(x) = 0$, we have $|\nabla df|_{g^*\otimes g^*}(x) = 0$ and

$$\mathrm{tr}_H\, \nabla_\times df(\mathbf{T}(\times, \sharp df))(x) = 0.$$

By the Weitzenböck formula in Lemma 4.3,

$$\frac{1}{2}L|df|^2_{g^*}(x) - \langle df, dLf\rangle_{g^*}(x) = \frac{1}{2}\langle \mathrm{Ric}^*(df), df\rangle_{g^*}(x).$$

Thus, by the Taylor expansions at the point x (we drop x below for simplicity):

$$\mathbb{E}_x\left[|//_t^{-1} df(X_t)|^2_{g^*}\right] = |df|^2_{g^*} + \frac{1}{2}L|df|^2_{g^*}\, t + \mathrm{o}(t),$$

and

$$|dP_tf|_{g^*}^2 = |df|_{g^*}^2 + \langle dLf, df\rangle_{g^*}\ t + \mathrm{o}(t),$$

we obtain

$$\frac{1}{2}\langle \mathrm{Ric}^*\, df, df\rangle_{g^*}(x) = \lim_{t\downarrow 0} \frac{\mathbb{E}_x\left[|//_t^{-1}df|_{X_t}|_{g^*}^2\right] - |dP_tf|_{g^*}^2(x)}{t}. \tag{4.3}$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}_x\left[\left|(\mathrm{id}+A_t)^*//_t^{-1}df|_{X_t}\right|_{g^*}^2\right] = \mathbb{E}_x\left[\left|(\mathrm{id}+A_t^*)//_t^{-1}df|_{X_t}\right|_{g^*}^2\right]\\
&= \mathbb{E}_x\left[|//_t^{-1}df|_{X_t}|_{g^*}^2\right] + 2\mathbb{E}_x\langle //_t^{-1}df||_{X_t}, A_t^*//_t^{-1}df|_{X_t}\rangle + \mathbb{E}_x\left[|A_t^*//_t^{-1}df|_{X_t}|_{g^*}^2\right]\\
&= \mathbb{E}_x\left[|//_t^{-1}df(X_t)|_{g^*}^2\right] + 2\mathbb{E}_x\langle //_t^{-1}df(X_t) - dP_tf(x), A_t//_t^{-1}df(X_t)\rangle\\
&\quad + \mathbb{E}_x\left[|A_t^*//_t^{-1}df(X_t)|_{g^*}^2\right]\\
&= \mathbb{E}_x\left[|//_t^{-1}df|_{X_t}|_{g^*}^2\right] + 2\mathbb{E}_x\int_0^{t\wedge\sigma_r} \mathrm{tr}_H\langle \nabla_{//_s\times}dP_{t-s}f, \mathbf{T}(//_s\times, //_{t-s}^{-1}df|_{X_t})\rangle ds\\
&\quad + \mathbb{E}_x\left[|A_t^*//_t^{-1}df|_{X_t}|_{g^*}^2\right] + \mathrm{o}(t).
\end{aligned}$$

Since $\nabla df = 0$ and $df(V_x) = 0$, then

$$\lim_{t\downarrow 0}\left[\frac{2\mathbb{E}_x\int_0^{t\wedge\sigma_r}\mathrm{tr}_H\langle \nabla_{//_s\times}dP_{t-s}f, \mathbf{T}(//_s\times, //_{t-s}^{-1}df|_{X_t})\rangle ds}{t}\right] = 0. \tag{4.4}$$

Combining (4.4), (b) and (4.3), we finish the proof. □

Proof of Theorem 4.1, Part II. We finish the remaining implications "(iii)⇒(iv)⇒(ii)" and "(v) ⇒ (4.1)".

"(iv)⇒(ii)" By Itô's formula, we have

$$\begin{aligned}
\mathbb{E}_x[\mathbb{E}_x[F|\mathscr{F}_t]^2] - \mathbb{E}_x[F]^2 &= \int_0^t \mathbb{E}_x|D_s\mathbb{E}_x[F|\mathscr{F}_s]|^2\, ds\\
&\le \int_0^t \mathrm{e}^{\frac{K}{2}(T-s)}\left(\mathbb{E}_x|D_sF|_g^2 + \frac{K}{2}\int_s^T \mathrm{e}^{\frac{K}{2}(r-s)}\,\mathbb{E}_x|D_rF|_g^2\, dr\right) ds.
\end{aligned} \tag{4.5}$$

For any function $F(\omega) = f(\omega_{t_1}, \dots, \omega_{t_n})$ with $t_1 > 0$, the functions $r \mapsto \mathbb{E}_x|D_rF|_g^2$ and $s \mapsto \mathbb{E}_x|D_s\mathbb{E}_x[F|\mathscr{F}_s]|^2$ are continuous at time 0. Hence, we can divide both sides of (4.5) by t and take the limit as t goes to 0 to obtain (ii).

If $t_1 = 0$, i.e. if $F(\omega) = f(x, \omega_{t_2}, \dots, \omega_{t_n})$, then $r \mapsto \mathbb{E}_x|D_rF|^2$ is not continuous at time 0. We construct a family of functions

$$F_\varepsilon(\omega) = f(\omega_\varepsilon, \omega_{t_2}, \dots, \omega_{t_n})$$

for $0 < \varepsilon < t_2$. For this family of functions, $r \mapsto \mathbb{E}_x|D_rF_\varepsilon|^2$ is continuous at time 0 and

$$|d_x\mathbb{E}_x[F_\varepsilon]|_{g^*}^2 \le \mathrm{e}^{\frac{K}{2}T}\,\mathbb{E}_x\left[|D_0F_\varepsilon|_g^2 + \frac{K}{2}\int_0^T \mathrm{e}^{\frac{K}{2}s}\,|D_sF_\varepsilon|_g^2\, ds\right].$$

By considering the limit $\varepsilon \downarrow 0$, we prove (ii).

"(iii)⇒(iv)" Applying the log-Sobolev inequality (iii) for $F^2 = 1 + \varepsilon G$, we have

$$\begin{aligned}
&\mathbb{E}_x\Big[\mathbb{E}_x[(1+\varepsilon G)|\mathscr{F}_t]\log\mathbb{E}_x[(1+\varepsilon G)|\mathscr{F}_t]\Big] - \mathbb{E}_x[(1+\varepsilon G)]\log\mathbb{E}_x[(1+\varepsilon G)]\\
&\le 2\int_0^t \mathrm{e}^{\frac{K}{2}(T-r)}\left(\mathbb{E}_x|D_r\sqrt{1+\varepsilon G}|_g^2 + \frac{K}{2}\int_r^T \mathrm{e}^{\frac{K}{2}(s-r)}\,\mathbb{E}_x|D_s\sqrt{1+\varepsilon G}|_g^2\, ds\right) dr.
\end{aligned}$$

Using the Taylor expansion at $\varepsilon = 0$, we have

$$\begin{aligned}
&\mathbb{E}_x\left[\varepsilon\mathbb{E}[G|\mathscr{F}_t] + \frac{\varepsilon^2}{2}\mathbb{E}[G|\mathscr{F}_t]^2\right] - \left[\varepsilon\mathbb{E}_x(G) + \frac{\varepsilon^2}{2}(\mathbb{E}_x G)^2\right] + \mathrm{o}(\varepsilon^2)\\
&\le 2\int_0^t \mathrm{e}^{\frac{K}{2}(T-r)}\left(\mathbb{E}_x|D_r\sqrt{1+\varepsilon G}|_g^2 + \frac{K}{2}\int_r^T \mathrm{e}^{\frac{K}{2}(s-r)}\,\mathbb{E}_x|D_s\sqrt{1+\varepsilon G}|_g^2\,ds\right)dr\\
&= \frac{1}{2}\int_0^t \mathrm{e}^{\frac{K}{2}(T-r)}\left(\varepsilon^2\mathbb{E}_x|D_rG|_g^2 + \frac{K}{2}\int_r^T \mathrm{e}^{\frac{K}{2}(s-r)}\,\varepsilon^2\mathbb{E}_x|D_sG|_g^2\,ds\right)dr + \mathrm{o}(\varepsilon^2).
\end{aligned}$$

Dividing both sides with ε^2 and letting $\varepsilon \to 0$, we then obtain (iv).

"(v)⇒(4.1)" For any point $x \in M$ and any $\alpha \in H_x^*$ with $\alpha(V_x) = 0$, we choose a function $f : M \to \mathbb{R}$ such that

$$df(x) = \alpha, \quad \nabla df(x) = 0.$$

We note that then $\langle(\delta_H \mathbf{T})^*(df), df\rangle(x) = 0$. Observe also that the inequalities of (v) are equivalent to:

$$\begin{aligned}
&\frac{|dP_tf|_{g^*}^2 - \mathbb{E}_x\left[|(\mathrm{id}+A_t)^*/\!/_t^{-1}df(X_t)|_{g^*}^2\right]}{t}\\
&\le \frac{(\mathrm{e}^{Kt}-1)}{t}\mathbb{E}_x\left[|(\mathrm{id}+A_t)^*/\!/_t^{-1}df(X_t)|_{g^*}^2\right]\\
&\quad + \frac{K}{2t}\mathbb{E}_x\left[\int_0^t \mathrm{e}^{\frac{K}{2}s}\,|(\mathrm{id}+A_t-A_s)^*/\!/_t^{-1}df(X_t)|_{g^*}^2 ds\right],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{4(1-\mathrm{e}^{\frac{K}{2}t})}{t}|df|_{g^*}^2(x) - \frac{4(1-\mathrm{e}^{\frac{K}{2}t})}{t}\mathbb{E}_x\langle df(x), (\mathrm{id}+A_t)^*/\!/_t^{-1}df(X_t)\rangle\\
&- \frac{4\langle df(x), dP_tf(x)\rangle - 4\mathbb{E}_x\langle df(x), (\mathrm{id}+A_t)^*/\!/_t^{-1}df(X_t)\rangle}{t}\\
&+ \frac{|dP_tf|_{g^*}^2 - \mathrm{e}^{\frac{K}{2}t}\,\mathbb{E}_x|(\mathrm{id}+A_t)^*/\!/_t^{-1}df(X_t)|_{g^*}^2}{t}\\
&\le \frac{K}{2t}\mathbb{E}_x\left[\int_0^t \mathrm{e}^{\frac{K}{2}(t+s)}\,|(\mathrm{id}+A_t-A_s)^*/\!/_t^{-1}df(X_t)|_{g^*}^2\,ds\right].
\end{aligned}$$

Letting t tend to 0 and using Lemma 4.4, we obtain

$$-K|\sharp\alpha|_g^2(x) \le \langle\sharp\alpha, \mathrm{Ric}\,\sharp\alpha\rangle_g(x) \le K|\sharp\alpha|_g^2(x). \quad \square$$

4.3. The Ornstein–Uhlenbeck operator

For cylindrical functions $F, G \in \mathscr{F}C_0^\infty$ define a bilinear form

$$\mathscr{E}(F,G) = \mathbb{E}\langle DF, DG\rangle_{\mathbb{H}} = \mathbb{E}\int_0^T \langle D_tF, D_tG\rangle_g\,dt.$$

Then $(\mathscr{E}, \mathscr{F}C_0^\infty)$ is a positive bilinear form on $L^2(W_x^T; \mathbb{P}^{x,T})$. It is standard that the integration by parts formula in Theorem 3.1 implies closability of the form (see e.g. the argument in [32, Lemma 4.3.1.]). We shall use $(\mathscr{E}, \mathrm{Dom}(\mathscr{E}))$ to denote the closure of $(\mathscr{E}, \mathscr{F}C_0^\infty)$. Let $(\mathscr{L}, \mathrm{Dom}(\mathscr{E}))$ be the analogue of the Ornstein–Uhlenbeck operator as the generator of the Dirichlet form $\mathscr{E}$. Let $\mathrm{gap}(\mathscr{L})$ denote the spectral gap of the Ornstein–Uhlenbeck operator. The following is then a consequence of Theorem 4.1.

Corollary 4.5. *Assume that there exists some non-negative constant K such that*

$$-K \le \mathrm{Ric}_H \le K.$$

Then

(a) *for any $F \in \mathrm{Dom}(\mathscr{E})$ with $\mathbb{E}_x[F] = 0$,*

$$\mathbb{E}_x(F^2) \le \frac{1}{2}(\mathrm{e}^{KT} + 1)\,\mathscr{E}_x(F, F);$$

(b) *for any $F \in \mathrm{Dom}(\mathscr{E})$ with $\mathbb{E}_x[F^2] = 1$,*

$$\mathbb{E}_x(F^2 \log F^2) \le (\mathrm{e}^{KT} + 1)\,\mathscr{E}_x(F, F);$$

(c) *the spectral gap has the following estimate:*

$$\mathrm{gap}(\mathscr{L})^{-1} \le \frac{1}{2}(\mathrm{e}^{KT} + 1).$$

Proof. The inequalities in (a) and (b) are derived by using Theorem 4.1 (iii) and (iv) with $t = T$:

$$\begin{aligned}
\mathbb{E}_x[F^2] - (\mathbb{E}_x[F])^2 &\le \int_0^T \mathrm{e}^{\frac{K}{2}(T-t)}\,\mathbb{E}_x\left[|D_tF|_g^2 + \frac{K}{2}\int_t^T \mathrm{e}^{\frac{K}{2}(s-t)}\,|D_sF|_g^2\,ds\right] dt\\
&\le \frac{1}{2}\,\mathbb{E}_x\left[\int_0^T (\mathrm{e}^{\frac{K}{2}(T+t)} + \mathrm{e}^{\frac{K}{2}(T-t)})|D_tF|_g^2\,dt\right]\\
&\le \frac{1}{2}(\mathrm{e}^{KT} + 1)\mathbb{E}_x\left[\int_0^T |D_tF|_g^2\,dt\right].
\end{aligned}$$

The estimate in (c) is derived according to the definition of the spectral gap. □

5. On the geometry of path space and complements

5.1. Integrable complements

Let (M, H, g) be a sub-Riemannian manifold and let V be a metric preserving complement that is also Frobenius integrable, i.e. $[V, V] \subseteq V$. Let Φ be the corresponding foliation of V. Then M/Φ locally has the structure of a Riemannian manifold. More precisely, any $x \in M$ has a neighborhood U such that $\pi_U : U \to U/\Phi|_U$ is a submersion of differentiable manifolds. Since V is metric preserving, there exists a Riemannian metric $\check{g}_U$ on U/Φ_U such that

$$\langle v, w\rangle_g = \langle (\pi_U)_* v, (\pi_U)_* w\rangle_{\check{g}_U}, \quad v, w \in H_x,\ x \in U.$$

Consider the special case when Φ is a regular foliation, i.e. when $\check{M} = M/\Phi$ is a differentiable manifold. Write the corresponding projection as $\pi : M \to \check{M}$. Let $\check{g}$ be the corresponding complete Riemannian metric on $\check{M}$. For the sake of simplicity, we assume that (M, H, g) is complete, which implies that $(\check{M}, \check{g})$ is complete, as this is a distance decreasing map. Write $\check{\nabla}$ for the Levi-Civita connection of $\check{g}$. Let $x \in M$ be a given point with $\check{x} = \pi(x)$. Let $W^\infty_{x,H}$ and $W^\infty_{\check{x}}$ be the space of smooth curves with domain $[0, \infty)$ starting at x and $\check{x}$, respectively, where the curves starting at x are required to be horizontal. Then since H is an Ehresmann connection on π, curves starting at $\check{x}$ have unique horizontal lifts to x. Hence, the map $W^\infty_{x,H} \to W^\infty_{\check{x}}$, $\omega \mapsto \pi(\omega)$, is a bijection.

Next, let hY denote the horizontal lift of a vector field Y on $\check{M}$, that is hY is the unique section of H satisfying $\pi_* hY = Y$. We can then describe the connection $\nabla = \nabla^{g,V}$ as

$$\nabla_{hY} hY_2 = h\check{\nabla}_Y Y_2, \quad \nabla_Z hY = 0, \quad \nabla_{hY} Z = [hY, Z], \tag{5.1}$$

for $Y, Y_2 \in \Gamma(T\check{M})$, $Z \in \Gamma(V)$. Hence, if $\mathrm{Dev}_{\check{x}} : T_{\check{x}}\check{M} \to \check{M}$ is the development map of $\check{\nabla}$, then we have the following commutative diagram:

$$\begin{array}{ccc} W_0^\infty(H_x) & \xrightarrow{\mathrm{Dev}_x} & W^\infty_{x,H} \\ \downarrow{\scriptstyle \pi_{*,x}} & & \downarrow{\scriptstyle \pi} \\ W_0^\infty(T_{\check{x}}\check{M}) & \xrightarrow[\mathrm{Dev}_{\check{x}}]{} & W^\infty_{\check{x}} \end{array}$$

with every map in the diagram being a bijection.

Going from smooth curves to continuous curves, the concept of horizontal curves will no longer be well defined. However, if $\check{B}_t^{\check{x}}$ is the standard Brownian motion in $T_{\check{x}}\check{M}$ and $\check{X}_t^{\check{x}}$ denotes the Brownian motion in $\check{M}$, then we can still make sense of the following diagram

$$\begin{array}{ccc} B_t^x & \xmapsto{\mathrm{Dev}_x} & X_t^x \\ \downarrow{\scriptstyle \pi_{*,x}} & & \downarrow{\scriptstyle \pi} \\ \check{B}_t^{\check{x}} & \xmapsto[\mathrm{Dev}_{\check{x}}]{} & \check{X}_t^{\check{x}} \end{array} ,$$

We finally note that by (5.1) we have $\mathrm{Ric}_H = \pi^* \check{\mathrm{Ric}}|_H$ where $\check{\mathrm{Ric}}$ denotes the Ricci operator on $\check{M}$. In summary, if we consider the path space $W_x(M)$ with the probability distribution given by the sub-Riemannian Brownian motion, then, viewed from the connection $\nabla = \nabla^{g,V}$, the path space has a geometry similar to that of the path space of $M/\exp(V)$. See [20] for more details.

5.2. An instructive example

Let $(M^{(1)}, g^{(1)})$ and $(M^{(2)}, g^{(2)})$ be two oriented Riemannian manifolds, both of dimension n. Let $\mathrm{SO}(TM^{(1)})$ and $\mathrm{SO}(TM^{(2)})$ be the oriented orthonormal frame bundles. With respect to the diagonal action of $\mathrm{SO}(n)$ on $\mathrm{SO}(TM^{(1)}) \times \mathrm{SO}(TM^{(2)})$, we define

$$\mathbf{M} = (\mathrm{SO}(TM^{(1)}) \times \mathrm{SO}(TM^{(2)}))/\,\mathrm{SO}(n).$$

We can consider elements $q \in \mathbf{M}$ as linear isometries $q : T_{x^{(1)}}M^{(1)} \to T_{x^{(2)}}M^{(1)}$ where $(\varphi^{(1)}, \varphi^{(2)}) \cdot \mathrm{SO}(n)$, $\varphi^{(1)} \in \mathrm{SO}(TM)_{x^{(1)}}$, $\varphi^{(2)} \in \mathrm{SO}(TM)_{x^{(2)}}$ can be identified with $q = \varphi^{(2)} \circ (\varphi^{(1)})^{-1}$. Define $\pi^{(1)} : \mathbf{M} \to M^{(1)}$ and $\pi^{(2)} : \mathbf{M} \to M^{(2)}$ such that $q : T_{x^{(1)}}M^{(1)} \to T_{x^{(2)}}M^{(2)}$ is mapped to $x^{(1)}$ and $x^{(2)}$, respectively. We can then define a subbundle $\mathbf{H} \subseteq T\mathbf{M}$ by

$$\mathbf{H} = \left\{ \dot{q}_t : \begin{array}{l} \pi_*^{(1)} \dot{q}_t = \pi_*^{(2)} \dot{q}_t, \\ q_t Y_t \text{ is a parallel vector field for every parallel } Y_t \text{ along } \pi(q_t) \end{array} \right\}.$$

Then $\mathbf{H}$ is an Ehresmann connection on both $\pi^{(1)}$ and $\pi^{(2)}$. Furthermore, for any element $v \in \mathbf{H}$, we have

$$|\pi_*^{(1)} v|^2_{g^{(1)}} = |\pi_*^{(2)} v|^2_{g^{(2)}} =: |v|^2_{\mathbf{g}}.$$

Consider the sub-Riemannian manifold $(\mathbf{M}, \mathbf{H}, \mathbf{g})$. This corresponds to the optimal control problem of rolling $M^{(1)}$ on $M^{(2)}$ without twisting or slipping along a minimizing curve. For more details and conditions for $\mathbf{H}$ being bracket-generating see e.g. [12,18,19].

Consider the choices of complement $V^{(1)} = \ker \pi_*^{(1)}$ and $V^{(2)} = \ker \pi_*^{(2)}$. Both of $V^{(1)}$ and $V^{(2)}$ are metric preserving complements by definition. If $\nabla^{(1)} = \nabla^{\mathbf{g},V^{(1)}}$ and $\nabla^{(2)} = \nabla^{\mathbf{g},V^{(2)}}$ are the corresponding compatible connections with horizontal Ricci operator $\mathrm{Ric}^{(1)}_{\mathbf{H}}$ and $\mathrm{Ric}^{(2)}_{\mathbf{H}}$ respectively, then we have that $\mathrm{Ric}^{(1)}_{\mathbf{H}} = (\pi^{(1)})^* \mathrm{Ric}_{g^{(1)}}$ and $\mathrm{Ric}^{(2)}_{\mathbf{H}} = (\pi^{(2)})^* \mathrm{Ric}_{g^{(2)}}$, where $\mathrm{Ric}_{g^{(1)}}$ and $\mathrm{Ric}_{g^{(2)}}$ are the respective Ricci curvatures of $g^{(1)}$ and $g^{(2)}$. This illustrates that our formalism for path space of sub-Riemannian manifolds really depends on the choice of complementary subbundle.

5.3. A non-integrable complement

We will include an example from [7]. Consider the Lie algebra $\mathfrak{so}(4)$. If $e_1, \dots, e_4$ is the standard basis of $\mathbb{R}^4$, we write $e_{ij} \in \mathfrak{so}(4)$ for the matrices satisfying $e_{ij}e_k = \delta_{ik}e_j - \delta_{jk}e_i$. Consider the inner product on $\mathfrak{so}(4)$ given by $\langle Y_1, Y_2 \rangle = -\frac{1}{2} \operatorname{tr} Y_1 Y_2$. Consider an orthogonal decomposition $\mathfrak{so}(4) = \mathfrak{h} \oplus \mathfrak{v}$ where $\mathfrak{v} = \operatorname{span}\{e_{12}, e_{23}\}$. On $\mathrm{SO}(4)$, define subbundles $T\,\mathrm{SO}(4) = H \oplus V$ where H and V are respective left translations of $\mathfrak{h}$ and $\mathfrak{v}$. We define a sub-Riemannian metric g by left translation of the restriction of inner product of $\mathfrak{so}(4)$ to $\mathfrak{h}$.

The subbundle V is not integrable, but it is metric preserving from the bi-invariance of the inner product on $\mathfrak{so}(4)$. Furthermore, if we define $\nabla = \nabla^{g,V}$, then

$$\frac{1}{2} \leq \mathrm{Ric}_H \leq 2.$$

See [7, Example 3.1] for detailed calculations.

5.4. How to understand the curvature bounds

Let (M, H, g) be a given sub-Riemannian manifold. As the above calculations show, our curvature bounds will in general depend on the choice of complement V which determines the connection $\nabla = \nabla^{g,V}$. This dependence can be understood in the following way. Firstly, our connection sub-Laplacian $L = \operatorname{tr}_H \nabla^2_{\times,\times}$ depends on the choice of complement, and hence the same is true for the underlying diffusion X_t^x. See e.g. [17,20] for more details relating connections and sub-Laplacians. Furthermore, even for complements that define the same sub-Laplacian L, the derivatives $\frac{d}{ds} \operatorname{Dev}(B^x_\cdot - sk.)|_{s=0}$ on cylindrical functions will differ. In this sense, the curvature Ric_H can be seen as a curvature of the development map.

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Appendix. General geometric formulas

In most of our previous result, we restricted ourselves to sub-Riemannian manifolds (M, H, g) equipped with a choice of metric preserving complement V. In this appendix, we include formulas without this assumption to point out the additional complications that exist in general and for the benefit of future research.

A.1. Horizontal compatibility and completeness

Let (M, H, g) be a sub-Riemannian manifold and ∇ an affine connection. We introduce the following weakening of the concept of compatibility

Definition A.1. A connection is called horizontally parallel if (i) and (ii) of Definition 2.3 holds for all horizontal vector fields $Z, Y, Y_2 \in \Gamma(H)$.

We note the following related to the notion of completeness with respect to the sub-Riemannian metric and horizontally compatible sub-Riemannian connections.

Proposition A.2. *Let (M, H, g) be a complete sub-Riemannian manifold and let ∇ be a connection that is horizontally compatible with the sub-Riemannian structure (H, g). Then for any smooth curve $u \in W_0^\infty(H_x)$, $\mathrm{Dev}(u)$ is defined for all time.*

In particular, this result holds for the compatible connections.

Proof. Let $u \in W_0^\infty(H_x)$ be fixed. For a given $T > 0$, let $\varphi(T) = \int_0^T |\dot{u}|^2_{g_x} dt$ denote the length of u up to time T. Let $[0, T]$ be some interval for which the solution of

$$\dot{u} = /\!/_t^{-1} \dot{\omega}_t, \quad \omega_0 = x$$

exists. Then since $/\!/_t^{-1}$ is a linear isometry by our assumptions, we have that $\omega_t, t \in [0, T]$ has to be contained in the ball $B_g(x, \varphi(T) + \varepsilon)$, $\varepsilon > 0$, centered at x with radius $\varphi(T) + \varepsilon$ defined relative to the sub-Riemannian distance d_g defined in (2.3). Since we are assuming that (M, H, g) is complete, all such balls have compact closures, see e.g. [6]. Hence, for any $T > 0$, we can solve the development equation in $B_g(x, \varphi(T) + \varepsilon)$. It follows that $\mathrm{Dev}(u)$ is well defined. $\square$

Note that the map Dev restricted to $W_0^\infty(H_x)$ only depends on parallel transport along horizontal curves, so it really only depends on a partial connection, which will be discussed in the next section.

A.2. Partial connections on sub-Riemannian manifolds

A partial connection $\boldsymbol{\nabla}$ *on* H *in the direction of* H is a map $\boldsymbol{\nabla}\colon \Gamma(H) \times \Gamma(H) \to \Gamma(H)$, $(Y, Z) \mapsto \boldsymbol{\nabla}_Y Z$ satisfying that for $f \in C^\infty(M)$,

$$\boldsymbol{\nabla}_{fY} Z = f \boldsymbol{\nabla}_Y Z \quad \text{and} \quad \boldsymbol{\nabla}_Y fZ = (Yf)Z + f\boldsymbol{\nabla}_Y Z.$$

In other words, covariant derivatives are only defined in the direction of H. A partial connection will give us a well defined parallel transport along H-horizontal curves. For more on partial connections, see [11].

Let (M, H, g) be a sub-Riemannian manifold. A partial connection on H in the direction of H is *compatible with* (H, g) if

$$Z\langle Y_1, Y_2\rangle_g = \langle \boldsymbol{\nabla}_Z Y_1, Y_2\rangle_g + \langle Y_1, \boldsymbol{\nabla}_Z Y_2\rangle \tag{A.1}$$

for any $Z, Y_1, Y_2 \in \Gamma(H)$. We define its torsion $\mathbf{t}\colon H \times H \to TM$ by

$$\mathbf{t}(X, Y) = \boldsymbol{\nabla}_X Y - \boldsymbol{\nabla}_Y X - [X, Y].$$

Then we have the following result.

Lemma A.3. *Let $\boldsymbol{\nabla}$ be a partial connection on H in the direction of H.*

(a) *The map $X, Y \mapsto \mathbf{t}(X,Y) \mod H$ does not depend on the choice of $\boldsymbol{\nabla}$. In particular, $\mathbf{t}$ cannot vanish when H is bracket-generating. Furthermore, if V is a choice of complement for H, that is $TM = H \oplus V$, with corresponding projection pr_V, then $\mathrm{pr}_V \mathbf{t}$ is independent of choice of partial connection.*
(b) *Assume that $\boldsymbol{\nabla}$ is compatible with the sub-Riemannian metric. Then it is uniquely determined by its torsion.*
(c) *Let V be a choice of complement to H. Then there is a unique partial connection $\boldsymbol{\nabla}$ compatible with the sub-Riemannian structure (H,g) and with $\mathbf{t}(H,H) \subseteq V$.*

Proof. The result in (a) follows from the fact that $\mathbf{t}(X,Y) = -[X,Y] \mod H$. To prove (b), choose an arbitrary complement V and a reference compatible partial connection $\boldsymbol{\nabla}'$. Write its torsion $\mathbf{t}' = \mathbf{t}'_H + \mathbf{t}'_V = \mathrm{pr}_H \mathbf{t}' + \mathrm{pr}_V \mathbf{t}'$. For any partial connection $\boldsymbol{\nabla}$, we write $\boldsymbol{\nabla}_X Y = \boldsymbol{\nabla}'_X Y + \kappa(X)Y$ with torsion $\mathbf{t} = \mathbf{t}_H + \mathbf{t}'_V$. We then have that

$$\kappa(X)Y - \kappa(Y)X = \mathbf{t}(X,Y) - \mathbf{t}'(X,Y) = (\mathbf{t}_H - \mathbf{t}'_H)(X,Y),$$

from the definition of torsion and using that (A.1) implies,

$$\langle \kappa(X)Y_1, Y_2\rangle_g + \langle Y_1, \kappa(X)Y_2\rangle_g = 0.$$

Hence, it follows that κ is determined by

$$\begin{aligned}\langle \kappa(X)Y_1, Y_2\rangle &= \frac{1}{2}\langle (\mathbf{t}_H - \mathbf{t}'_H)(X,Y_1), Y_2\rangle_g \\ &\quad - \frac{1}{2}\langle (\mathbf{t}_H - \mathbf{t}'_H)(Y_1,Y_2), X\rangle_g - \frac{1}{2}\langle (\mathbf{t}_H - \mathbf{t}'_H)(X,Y_2), Y_1\rangle_g.\end{aligned}$$

Hence κ is uniquely determined by $\mathbf{t}_H$. Furthermore, to prove (c), if we take

$$\langle \kappa(X)Y_1, Y_2\rangle = \frac{1}{2}\left(-\langle \mathbf{t}'_H(X,Y_1), Y_2\rangle_g + \langle \mathbf{t}'_H(Y_1,Y_2), X\rangle_g + \langle \mathbf{t}'_H(X,Y_2), Y_1\rangle_g\right),$$

then $\mathbf{t}_H = 0$ and this is the unique such choice. □

Given a sub-Riemannian manifold (M,H,g), let V be a choice of complement. Let pr_H and pr_V be the corresponding projections. We write $\boldsymbol{\nabla}^{g,V}$ for the unique compatible partial connection with $\mathbf{t}(H,H) \subseteq V$. We will also write $\nabla = \nabla^{g,V}$ for an affine connection of the following form

$$\nabla_X Y = \begin{cases} \boldsymbol{\nabla}^{g,V}_X Y, & \text{if } X, Y \in \Gamma(H); \\ \mathrm{pr}_H[X,Y], & \text{if } X \in \Gamma(V),\ Y \in \Gamma(H); \\ \mathrm{pr}_V[X,Y], & \text{if } X \in \Gamma(H),\ Y \in \Gamma(V), \end{cases}$$

and where $\nabla_{|V}|V$ can be an arbitrary partial connection on V in the direction of V. We note the following.

Proposition A.4. *The connection of the form $\nabla = \nabla^{g,V}$ with torsion $\mathbf{T}$ satisfies the following properties:*

(a) *Both H and V are parallel with respect to ∇;*
(b) *∇ is horizontally compatible with (H,g);*
(c) *$\mathbf{T}(H,H) \subseteq V$;*
(d) *$\mathbf{T}(H,V) = 0$.*

Conversely, any connection ∇ satisfying (a)–(d) is of the form $\nabla^{g,V}$.

We also note that $\nabla^{g,V}$ is always horizontally compatible and is compatible if and only if V is metric preserving.

A.3. Weizenböck formulas

Let (M, H, g) be a given sub-Riemannian manifold. Let ∇ be an arbitrary connection with torsion $\mathbf{T}$ and curvature $\mathbf{R}$. Write

$$\delta_H \mathbf{T}(Z) = -\operatorname{tr}_H(\nabla_\times \mathbf{T})(\times, Z), \quad \operatorname{Ric}(Z) = -\operatorname{tr}_H \mathbf{R}(\times, Z)\times .$$

Assume that H is preserved under parallel transport under ∇, and hence ∇g is well defined. For any vector field $Z \in \Gamma(TM)$, define $q_Z \colon H \to H$ by the formula

$$\langle q_Z v_1, v_2\rangle_g = \frac{1}{2}(\nabla_Z g)(v_1, v_2), \quad v_1, v_2 \in H. \tag{A.2}$$

We note that $Z \mapsto q_Z$ is a tensorial map, so we can consider $q \in \Gamma(T^*M \otimes \operatorname{End} H)$ as a tensor.

Lemma A.5 (*Weitzenböck Formula*). *Let $\hat{\nabla}$ denote the adjoint connection of ∇ as in (2.1). Write*

$$L = \operatorname{tr}_H \nabla^2_{\times,\times} \quad \text{and} \quad \hat{L} = \operatorname{tr}_H \hat{\nabla}^2_{\times,\times}$$

for the Laplacians on tensors. Then, for any function $f \in C^\infty(M)$, we have

$$\begin{aligned} Ldf(Z) - dLf(Z) &= -2\operatorname{tr}_H \nabla_\times df(\mathbf{T}(\times, Z) - q_Z\times) \\ &\quad + df(\operatorname{Ric}(Z) + \delta_H\mathbf{T}(Z) - \operatorname{tr}_H \mathbf{T}(\times, \mathbf{T}(\times, Z))); \end{aligned} \tag{A.3}$$

$$\hat{L}df(Z) - dLf(Z) = 2\operatorname{tr}_H \nabla_\times df(q_Z\times) + df(\operatorname{Ric}(Z)). \tag{A.4}$$

Proof. For a given point x and any elements $v \in H_x$ and $w \in T_xM$, choose arbitrary vector fields $Y \in \Gamma(H)$, $Z \in \Gamma(TM)$ such that $Y(x) = v$, $Z(x) = w$, $\nabla Y(x) = 0$ and $\nabla Z(x) = 0$. We remark that this is possible since we assumed that H was parallel with respect to ∇. Then at $x \in M$,

$$\begin{aligned} (\nabla^2_{Y,Y} df)(Z) &= Y\nabla_Y df(Z) = Y\nabla_Z df(Y) + Ydf(\mathbf{T}(Z,Y)) \\ &= (\nabla^2_{Y,Z} df)(Y) - (\nabla_Y df)(\mathbf{T}(Y,Z)) - df((\nabla_Y \mathbf{T})(Y,Z)) \\ &= (\nabla^2_{Z,Y} df)(Y) + (\mathbf{R}(Y,Z)df)(A) - (\nabla_{\mathbf{T}(Y,Z)} df)(Y) \\ &\qquad - (\nabla_Y df)(\mathbf{T}(Y,Z)) - df((\nabla_Y \mathbf{T})(Y,Z)) \\ &= (\nabla^2_{Z,Y} df)(Y) - df(\mathbf{R}(Y,Z)Y) \\ &\qquad - 2(\nabla_Y df)(\mathbf{T}(Y,Z)) - df((\nabla_Y \mathbf{T})(Y,Z) + \mathbf{T}(Y, \mathbf{T}(Y,Z))). \end{aligned}$$

Next, let us insert an orthonormal basis $Y_1, \ldots, Y_k$ of H. We can choose this orthonormal basis such that $\nabla_Z Y_i(x) = q_Z Y_i(x)$ for some given point x. Evaluated at $x \in M$, we have

$$(\nabla_{Z,Y_i} df)(Y_i) = Z(\nabla_{Y_i} df)(Y_i) - (\nabla_{q_Z Y_i} df)(Y_i) - (\nabla_{Y_i} df)(q_Z Y_i).$$

Summing over this basis and using the symmetry of q_Z gives us (A.3). The result in (A.4) then follows from the identity

$$\begin{aligned} \hat{\nabla}_{Y,Y} df(Z) &= (\nabla^2_{Y,Y} df)(Z) + 2(\nabla_Y df)(\mathbf{T}(Y,Z)) \\ &\qquad + df((\nabla_Y \mathbf{T})(Y,Z)) + df(\mathbf{T}(Y, \mathbf{T}(Y,Z))). \quad \square \end{aligned}$$

A.4. The smooth horizontal path space seen from an arbitrary complement

Let (M,H,g) be a complete sub-Riemannian manifold and let V be an arbitrary choice of complement. Let $\nabla = \nabla^{g,V}$ be the corresponding connection horizontally compatible with (H,g) and with torsion $\mathbf{T}$ and curvature $\mathbf{R}$. Define the development map Dev relative to this connection. For any $Z \in \Gamma(TM)$, define q_Z as in (A.2) and note that $q_Z = q_{\mathrm{pr}_V Z}$ since the connection is horizontally compatible. We note the following result.

Lemma A.6. *Let $t \mapsto \omega_t$ be a smooth horizontal curve with $u = \mathrm{Dev}^{-1}(\omega) \in W_0^\infty(H_x)$. Define $A_t = A_t^\omega \colon T_xM \to T_xM$ by $A_t = \int_0^t \mathbf{T}_{//_s}(du_s, \cdot)$. Consider $\omega_t^s = \mathrm{Dev}(u+sk)_t$ and define $Y_t = \frac{d}{ds}\omega_t^s|_{s=0}$. Then $Y_t = //_t y_t = \hat{//}_t \hat{y}_t$ with*

$$y_t = h_t + \int_0^t dA_s h_s, \quad \hat{y}_t = h_t - \int_0^t A_s dh_s.$$

where $h_t = \mathrm{pr}_H y_t$ is the solution of

$$\begin{aligned} k_t = h_t &- \int_0^t \int_0^s \mathbf{R}_{//_r}(du_r, h_r)\, du_s \\ &- \frac{1}{2}\int_0^t\int_0^s \left((\nabla_{du_r} q)_{//_r, \int_0^r dA_{r_2} h_{r_2}} du_s + (\nabla_{du_s} q)_{//_r, \int_0^r dA_{r_2} h_{r_2}} du_r \right) \\ &+ \frac{1}{2}\int_0^t\int_0^s \sharp \left\langle (\nabla_{\cdot} q)_{//_r, \int_0^r dA_{r_2} h_{r_2}} du_r, du_s \right\rangle_g . \end{aligned} \tag{A.5}$$

Proof. Write $Y_t = //_t y_t$, $\mathrm{pr}_H y_t = h_t$. Observe that from Lemma 2.1, we must have

$$0 = \mathrm{pr}_V k_t = \mathrm{pr}_V y_t - \int_0^t dA_s h_s.$$

Then

$$y_t = \mathrm{pr}_H y_t + \mathrm{pr}_V y_t = h_t + \int_0^t dA_s h_s.$$

Furthermore, we have that

$$d(//_t^{-1} \hat{//}_t) = //_t^{-1} \mathbf{T}(//_t du_t, \hat{//}_t \,\cdot\,).$$

The solution of this equation is $//_t^{-1}\hat{//}_t = \mathrm{id} + A_t$ and $\hat{//}_t^{-1} //_t = \mathrm{id} - A_t$, since A_t vanishes on V. As a consequence

$$\hat{y}_t = \hat{//}_t^{-1} //_t y_t = h_t + \int_0^t dA_s h_s - A_t h_t = h_t - \int_0^t A_s dh_s.$$

Finally, we will prove (A.5) by first observing that

$$\begin{aligned} k_t &= h_t - \int_0^t \int_0^s \mathbf{R}_{//_r}(du_r, y_r) du_s \\ &= h_t - \int_0^t \int_0^s \mathbf{R}_{//_r}(du_r, h_r) du_s - \int_0^t \int_0^s \mathbf{R}_{//_r}(du_r, \mathrm{pr}_V y_r) du_s. \end{aligned} \tag{A.6}$$

Note that for arbitrary $z \in V_x$ and $v_1, v_2, v_3 \in H_x$, we have that

$$(\mathbf{R}(v_1,z)g)(v_2,v_3) = 2\langle (\nabla_{v_1} q)_z v_2, v_3\rangle_g = \langle \mathbf{R}(v_1,z)v_2, v_3\rangle_g + \langle v_2, \mathbf{R}(v_1,z)v_3\rangle_g,$$

and hence from the first Bianchi identity

$$\begin{aligned} \langle \mathbf{R}(v_1,z)v_2, v_3\rangle_g &= \langle \circlearrowleft \mathbf{R}(v_1,z)v_2, v_2\rangle_g + \langle \mathbf{R}(v_2,z)v_1, v_3\rangle_g \\ &= \langle \circlearrowleft (\nabla_{v_1}\mathbf{T})(z,v_2) + \circlearrowleft \mathbf{T}(\mathbf{T}(v_1,z)v_2), v_3\rangle_g + \langle \mathbf{R}(v_2,z)v_1, v_3\rangle_g \\ &= \langle \mathbf{R}(v_2,z)v_1, v_3\rangle_g. \end{aligned}$$

Define $S_z(v_1, v_2, v_3) = \langle \mathbf{R}(v_1, z)v_2, v_3\rangle_g$. We conclude that

$$\begin{aligned}\langle(\nabla_{v_1} q)_z v_2, v_3\rangle_g &= S_z(v_1, v_2, v_3) + S_z(v_1, v_3, v_2),\\ 0 &= S_z(v_1, v_2, v_3) - S_z(v_2, v_1, v_3).\end{aligned}$$

Considering

$$\begin{aligned}S_z(v_1, v_2, v_3) - S_z(v_2, v_1, v_3) = 0,\\ S_z(v_2, v_3, v_1) - S_z(v_3, v_2, v_1) = 0,\\ S_z(v_3, v_1, v_2) - S_z(v_1, v_3, v_2) = 0,\end{aligned}$$

and subtracting the second line from the sum of the first and the third, we obtain

$$S_z(v_1, v_2, v_3) - S_z(v_1, v_3, v_2) - \langle(\nabla_{v_2} q_z)v_1, v_3\rangle + \langle(\nabla_{v_3} q)_z v_2, v_1\rangle_g = 0.$$

In conclusion

$$2S_z(v_1, v_2, v_3) = \langle(\nabla_{v_1} q)_z v_2, v_3\rangle_g + \langle(\nabla_{v_2} q)_z v_1, v_3\rangle_g - \langle(\nabla_{v_3} q)_z v_1, v_2\rangle_g.$$

Combining this with the formula (A.6), we prove (A.5). □

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