UNRAMIFIEDNESS OF WEIGHT 1 HILBERT HECKE ALGEBRAS

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ABSTRACT. We prove that the Galois pseudo-representation valued in the mod p^n cuspidal Hecke algebra for $\mathrm{GL}(2)$ over a totally real number field F, of parallel weight 1 and level prime to p, is unramified at any place above p. The same is true for the non-cuspidal Hecke algebra at places above p whose ramification index is not divisible by $p - 1$. A novel geometric ingredient, which is also of independent interest, is the construction and study, in the case when p ramifies in F, of generalised Θ-operators using Reduzzi–Xiao's generalised Hasse invariants, including especially an injectivity criterion in terms of minimal weights.

CONTENTS

INTRODUCTION

Starting with Wiles [24] and Taylor–Wiles [21], $R = \mathbb{T}$ theorems have been developed and taken a role as cornerstones in number theory. They provide both the existence of Galois representations with values in Hecke algebras satisfying prescribed local properties and modularity lifting theorems. The state of $R = \mathbb{T}$ theorems for 2-dimensional representations in residual characteristic p of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ and Hecke algebras acting on elliptic modular forms is quite satisfactory. In particular, the notoriously difficult case of Galois representations that are unramified at an odd prime p has been settled by ground-breaking work of Calegari and Geraghty [3], in which they show that those correspond to modular forms of weight 1. More precisely, given an odd irreducible representation $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ unramified outside a finite set of places S not containing p, they show that

$$R^S_{\mathbb{Q},\bar{\rho}} \xrightarrow{\sim} \mathbb{T}^{(1)}_{\bar{\rho}},$$

where $R^S_{\mathbb{Q},\bar{\rho}}$ is the universal deformation ring parametrising deformations of $\bar{\rho}$ which are unramified outside S and $\mathbb{T}^{(1)}_{\bar{\rho}}$ is the local component at $\bar{\rho}$ of a weight 1 Hecke algebra of a certain level prime to p.

In this article, we address the corresponding question for parallel weight 1 Hilbert modular forms over a totally real field F of degree $d = [F : \mathbb{Q}] \geqslant 2$ and ring of integers $\mathfrak{o}$. We focus on the construction of the Galois (pseudo-)representation with values in the parallel weight 1 Hecke algebra with p-power torsion coefficients and proving its local ramification properties. In particular, given a finite set S of places in F relatively prime to p and a totally odd irreducible representation $\bar{\rho} : G_F \to GL_2(\overline{\mathbb{F}}_p)$ unramified outside S we show that there exists a surjective homomorphism

$$R^S_{F,\bar{\rho}} \twoheadrightarrow \mathbb{T}^{(1)}_{\bar{\rho}},$$

where $R^S_{F,\bar{\rho}}$ is the universal deformation ring parametrising deformations of $\bar{\rho}$ which are unramified outside S and $\mathbb{T}^{(1)}_{\bar{\rho}}$ is the local component at $\bar{\rho}$ of a weight 1 Hecke algebra of a certain level prime to p (see Corollary 3.10 for a precise statement).

Let $M_\kappa(\mathfrak{n}, R)$ be the R-module of Hilbert modular forms of parallel weight $\kappa \geqslant 1$ and prime to p level $\mathfrak{n}$ over a $\mathbb{Z}_p$-algebra R, as in Definition 2.1. This R-module is equipped with a commuting family of Hecke operators $T_{\mathfrak{q}}$ as well as with diamond operators $\langle \mathfrak{q} \rangle$ for all primes $\mathfrak{q}$ of F not dividing $\mathfrak{n}$. Let $K/\mathbb{Q}_p$ be a finite extension containing the images of all embeddings of F in $\overline{\mathbb{Q}}_p$, and let $\mathcal{O}$ be its valuation ring, ϖ a uniformiser and $\mathbb{F} = \mathcal{O}/\varpi$ its residue field. We put $M_\kappa(\mathfrak{n}, K/\mathcal{O}) = \varinjlim_n M_\kappa(\mathfrak{n}, \mathcal{O}/\varpi^n)$ and define the parallel weight 1 Hecke algebra

$$\mathbb{T}^{(1)} = \mathrm{im}\left(\mathcal{O}[T_{\mathfrak{q}}, \langle \mathfrak{q} \rangle]_{\mathfrak{q} \nmid \mathfrak{n} p} \to \mathrm{End}_{\mathcal{O}}(M_1(\mathfrak{n}, K/\mathcal{O}))\right),$$

as well as its cuspidal quotient $\mathbb{T}^{(1)}_{\mathrm{cusp}}$ acting faithfully on the submodule of parallel weight 1 cuspforms. We can now state the main results of this article. Let $p\mathfrak{o} = \prod_{\mathfrak{p}|p} \mathfrak{p}^{e_{\mathfrak{p}}}$ with $e_{\mathfrak{p}} \geqslant 1$. We emphasise that there is no restriction on the ramification of p in F.

Theorem 0.1. *There exists a $\mathbb{T}^{(1)}$-valued pseudo-representation $P^{(1)}$ of G_F of degree 2 which is unramified at all primes $\mathfrak{q}$ not dividing $\mathfrak{n}p$ and satisfies $P^{(1)}(\mathrm{Frob}_\mathfrak{q}) = (T_\mathfrak{q}, \langle \mathfrak{q} \rangle)$.*

Moreover, if $p - 1$ does not divide $e_\mathfrak{p}$ for some $\mathfrak{p} \mid p$, then $P^{(1)}$ is also unramified at $\mathfrak{p}$ and satisfies $P^{(1)}(\mathrm{Frob}_\mathfrak{p}) = (T_\mathfrak{p}, \langle \mathfrak{p} \rangle)$, in particular $T_\mathfrak{p} \in \mathbb{T}^{(1)}$.

Finally, the pseudo-representation $P^{(1)}_{\mathrm{cusp}}$ obtained after composing $P^{(1)}$ with the natural surjection $\mathbb{T}^{(1)} \to \mathbb{T}^{(1)}_{\mathrm{cusp}}$ is unramified at all $\mathfrak{p} \mid p$ and satisfies $P^{(1)}_{\mathrm{cusp}}(\mathrm{Frob}_\mathfrak{p}) = (T_\mathfrak{p}, \langle \mathfrak{p} \rangle)$.

The strategy of the proof is based on the *doubling* method developed in [23], further simplified and conceptualised in [14] and [3]. The parallel weight 1 Hilbert modular forms over $\mathcal{O}/\varpi^n$ can be mapped into some higher weight in two ways, per prime $\mathfrak{p}$ dividing p, either by multiplication by a suitable power of the total Hasse invariant, or by applying a V-operator. That *doubling* map is used by Calegari–Specter in [4] to prove an analogue of Theorem 0.1 when $F = \mathbb{Q}$, for which they successfully develop the notion of a *p-ordinary* pseudo-representation. In that case, one knows by a result of Katz that the doubling map is injective. Furthermore, the existence of the Hecke operator T_p acting on weight 1 modular forms and the knowledge of its precise effect on the q-expansion (both due to Gross) allow one to show that the image of the doubling map is contained in the p-ordinary part of the higher weight space.

The existence of an optimally integral Hecke operator $T_\mathfrak{p}$ acting on parallel weight 1 Hilbert modular forms with arbitrary coefficients having the desired effect on their q-expansions (see [6] improving on and correcting previous works such as [15] and [14]) allows us to adapt the overall Calegari–Specter strategy to the Hilbert modular setting, while slightly generalising and clarifying some aspects of their arguments (see §3), the main challenge being to prove the injectivity of the doubling map. Note that the simple calculation in [14] showing that injectivity after restriction to an eigenspace is insufficient as the Hecke algebra modulo p need not be semi-simple. Instead, we observe that the injectivity of the doubling map would follow from the injectivity of a certain generalised Θ-operator, introduced in the foundational work [2] of Andreatta–Goren for Hilbert modular forms in characteristic p defined over the Deligne–Pappas moduli space. When p is unramified in F, the theory of partial Θ-operators was also developed by Diamond–Sasaki [10] in a more general setting using a slightly different approach from that of [2]. However, when p is ramified in F, the results of [10] do not apply, while those of [2] are not sufficiently precise for our purposes, as the Hilbert modular forms defined over the Deligne–Pappas model 'miss' some weights, and as a consequence the injectivity result of the latter paper is not optimal. In order to tackle this problem, we go back to the root of the problem and work with the Pappas–Rapoport moduli space, which does not miss any weight.

Capitalising on the theory of generalised Hasse invariants developed by Reduzzi–Xiao [17] in this context, we carefully revisit [2] and develop in §1.5 the needed theory of generalised Θ-operators over the Pappas–Rapoport moduli space and prove a refined injectivity criterion in terms of the minimal weights. In particular, we show that the generalised Θ-operators are indeed injective on parallel

weight 1 Hilbert modular forms provided their weight is minimal at $\mathfrak{p}$. By the recent works [8, 9] of Diamond and Kassaei (see §1.3) weight 1 Hilbert modular forms having 'non-minimal' weight at $\mathfrak{p}$ could only possibly exist when $p - 1$ divides $e_{\mathfrak{p}}$, and are products of forms of partial weight 0 at $\mathfrak{p}$ with generalised Hasse invariants.

In order to show the vanishing of the space of Katz cuspforms of partial weight 0 at $\mathfrak{p}$, and thus complete the proof of the second and third parts of the Theorem, in §1.4 we construct a partial Frobenius endomorphism $\Phi_{\mathfrak{p}^e}$ of this space and show that it is simultaneously injective and nilpotent. Our construction is inspired by the one in Diamond–Sasaki [10, §9.8] in the case when p is unramified in F. We also compute its effect on q-expansions, which is crucially used in our proof and, in order to avoid having to switch between different cusps, we only study the partial Frobenius operator of an appropriate power of $\mathfrak{p}$, rather than of $\mathfrak{p}$ itself.

In the language of linear representations, we prove the following result, which can be seen as a first step towards an $R = \mathbb{T}$ theorem.

Corollary (Corollary 3.10). *For every non-Eisenstein maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{(1)}$ (see Definition 3.8) there exists a representation*

$$\rho_{\mathfrak{m}} : \mathrm{G}_F \to \mathrm{GL}_2(\mathbb{T}_{\mathfrak{m}}^{(1)}),$$

unramified at all primes $\mathfrak{q}$ not dividing $\mathfrak{n}$ such that $\mathrm{tr}(\rho_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{q}})) = T_{\mathfrak{q}}$ and $\det(\rho_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{q}})) = \langle \mathfrak{q} \rangle$.

We believe that our modest contribution to the theory of generalised Θ-operators in the setting of the Pappas–Rapoport splitting model is worthwhile on its own, beyond the application to our main theorem. On our way to the injectivity criterion, we also explore some related themes, such as the relation between Hilbert modular forms defined over the Pappas–Rapoport model with those defined over the Deligne–Pappas model, and the q-expansion and vanishing loci of the generalised Hasse invariants defined by Reduzzi–Xiao. We hope that it bridges the gap between many existing references in the literature and also clarifies some important aspects of the theory of mod p Hilbert modular forms. In the meantime, motivated by geometric Serre weight conjectures, Diamond [7] extended the techniques of [10] to also construct partial Θ-operators which have an optimal effect on weights in the case where p ramifies in F. Moreover, Diamond generalised in [7] the construction of the partial Frobenius operators (our partial Frobenius operator $\Phi_{\mathfrak{p}^e}$ is essentially Diamond's $V_{\mathfrak{p}}^e$). Note that Diamond also describes kernels of partial Θ-operators in terms of images of his partial Frobenius maps $V_{\mathfrak{p}}$ (see [7, Thm. 9.1.1]). However, our construction is less technical because we restrict to the Rapoport locus and we only consider the case of weights 0 at $\mathfrak{p}$.

Notation. Throughout the paper, we will use the following notation. We let F be a totally real number field of degree $d \geqslant 2$ and ring of integers $\mathfrak{o}$. We denote by $\overline{\mathbb{Q}} \subset \mathbb{C}$ the subfield of algebraic numbers and denote by $G_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ the absolute Galois group of F. For every prime $\mathfrak{q}$ of F we denote by $\mathrm{Frob}_\mathfrak{q} \in G_F$ a fixed choice of an arithmetic Frobenius at $\mathfrak{q}$. Let $\mathfrak{p}$ be a prime of F dividing p. Fixing an embedding ι_p of $\overline{\mathbb{Q}}$ into a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ allows one to see the absolute Galois group $G_{F_\mathfrak{p}} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/F_\mathfrak{p})$ of $F_\mathfrak{p}$ as a decomposition subgroup of G_F at $\mathfrak{p}$, and we let $I_\mathfrak{p}$ denote its inertia subgroup. Furthermore, we fix a finite extension $K/\mathbb{Q}_p$ containing the images of all embeddings of F in $\overline{\mathbb{Q}}_p$, and let $\mathcal{O}$ be its valuation ring, ϖ a uniformiser and $\mathbb{F} = \mathcal{O}/(\varpi)$ its residue field.

For a prime $\mathfrak{p}$ of F dividing p, denote the residue field of $F_\mathfrak{p}$ by $\mathbb{F}_\mathfrak{p}$ and the ring of Witt vectors of $\mathbb{F}_\mathfrak{p}$ by $W(\mathbb{F}_\mathfrak{p})$. We also let $f_\mathfrak{p}$ and $e_\mathfrak{p}$ denote the residue and the ramification index of $\mathfrak{p}$, respectively. Let Σ be the set of infinite places of F, which we view as embeddings $F \hookrightarrow \overline{\mathbb{Q}}_p$ via ι_p. We have a natural partitioning $\Sigma = \coprod_{\mathfrak{p}|p} \Sigma_\mathfrak{p}$ where $\Sigma_\mathfrak{p}$ contains exactly those embeddings inducing the place $\mathfrak{p}$. For $\sigma \in \Sigma_\mathfrak{p}$, we denote by $\overline{\sigma}$ its restriction to the maximal unramified subfield of $F_\mathfrak{p}$ or, equivalently, the induced embedding of $\mathbb{F}_{\mathfrak{p}(\tau)}$ into $\overline{\mathbb{F}}_p$. Furthermore, we let $\overline{\Sigma}_\mathfrak{p} = \{\overline{\sigma} \mid \sigma \in \Sigma_\mathfrak{p}\}$ and $\overline{\Sigma} = \{\overline{\sigma} \mid \sigma \in \Sigma\} = \coprod_{\mathfrak{p}|p} \overline{\Sigma}_\mathfrak{p}$. As a general rule, elements of Σ will be called σ whereas τ usually designates an element of $\overline{\Sigma}$. In both cases, $\mathfrak{p}(\sigma)$ and $\mathfrak{p}(\tau)$ denotes the underlying prime ideal. When either σ or τ is clear from the context, we will just denote this prime ideal by $\mathfrak{p}$. In particular, an element $\tau \in \overline{\Sigma}_\mathfrak{p}$ denotes both an embedding $\mathbb{F}_{\mathfrak{p}(\tau)} \hookrightarrow \mathbb{F}$ and the corresponding p-adic one $W(\mathbb{F}_{\mathfrak{p}(\tau)}) \hookrightarrow \mathcal{O}$. Denoting the absolute arithmetic Frobenius on $\mathbb{F}$ by ϕ, we have $\overline{\Sigma}_\mathfrak{p} = \{\phi^j \circ \tau \mid j \in \mathbb{Z}\} \simeq \mathbb{Z}/f_\mathfrak{p}\mathbb{Z} \simeq \mathrm{Gal}(\mathbb{F}_\mathfrak{p}/\mathbb{F}_p)$ for any choice $\tau \in \overline{\Sigma}_\mathfrak{p}$. For any $\tau \in \overline{\Sigma}_\mathfrak{p}$, we let $\Sigma_\tau = \{\sigma \in \Sigma_\mathfrak{p} \mid \overline{\sigma} = \tau\} = \{\sigma_{\tau,i} \mid 1 \leqslant i \leqslant e_\mathfrak{p}\}$, where the numbering is chosen in an arbitrary, but fixed way. As an abbreviation, we write $\widetilde{\tau} = \sigma_{\tau,e_\mathfrak{p}}$.

Let $\mathcal{C}$ be a fixed set of representatives, all relatively prime to p, for the narrow class group of F.

1. HILBERT MODULAR FORMS IN FINITE CHARACTERISTIC

This section refines the theory of Θ-operators developed by Andreatta–Goren in [2], when p ramifies in F, in the setting of Hilbert modular forms defined over the Pappas–Rapoport splitting models for Hilbert modular varieties with the aim of proving the injectivity of the doubling map in §2. Along the way, we will need the generalised Hasse invariants of Reduzzi and Xiao [17], results of Diamond and Kassaei [8, 9] about minimal weights as well as a partial Frobenius operator generalised from [10].

Throughout this section we fix an ideal $\mathfrak{n}$ of $\mathfrak{o}$ relatively prime to p and having a prime factor which does not divide $6\mathfrak{d}$, where $\mathfrak{d}$ denotes the different of F.

1.1. **Pappas–Rapoport splitting models for Hilbert modular varieties.** Since we allow our base field F to ramify at p, we have to be careful with the model we choose for our Hilbert modular variety.

Fix $\mathfrak{c} \in \mathcal{C}$. We first consider the functor from the category of locally Noetherian $\mathbb{Z}_p$-schemes to the category of sets which assigns to a scheme S the set of isomorphism classes of tuples (A, λ, μ) where:

(i) A is a *Hilbert-Blumenthal Abelian Variety (HBAV)* over S, *i.e.*, an abelian S-scheme of relative dimension d, together with a ring embedding $\mathfrak{o} \hookrightarrow \mathrm{End}_S(A)$.

(ii) λ is a $\mathfrak{c}$-polarisation of A/S, *i.e.*, an isomorphism $\lambda : A^\vee \to A \otimes_\mathfrak{o} \mathfrak{c}$ of HBAV's over S such that the induced isomorphism $\mathrm{Hom}_\mathfrak{o}(A, A \otimes_\mathfrak{o} \mathfrak{c}) \simeq \mathrm{Hom}_\mathfrak{o}(A, A^\vee)$ sends elements of $\mathfrak{c}$ (resp. of the cone $\mathfrak{c}_+$ of its totally positive elements) to symmetric elements (resp. to polarisations),

(iii) μ is a $\mu_\mathfrak{n}$-level structure on A, *i.e.*, an $\mathfrak{o}$-linear closed embedding of S-schemes $\mu : \mu_\mathfrak{n} \to A$, where $\mu_\mathfrak{n}$ denotes the Cartier dual of the constant group scheme $\mathfrak{o}/\mathfrak{n}$ over S.

Under our assumption on $\mathfrak{n}$ above, this functor is representable by a $\mathbb{Z}_p$-scheme $\mathcal{X}^{\mathrm{DP}}$ of finite type, called the *Deligne–Pappas moduli space* (see [2, Rem. 3.3] and [13, Lem. 1.4]).

Suppose now that A is an HBAV over a locally Noetherian $\mathcal{O}$-scheme S with structure map $s : A \to S$ and let $\Omega^1_{A/S}$ be the sheaf of relative differentials of A over S. Define

$$\omega_S = s_* \Omega^1_{A/S},$$

i.e., ω_S is the sheaf of invariant differentials of A over S. Consider the decomposition

$$(1.1) \qquad \mathfrak{o} \otimes_\mathbb{Z} \mathcal{O}_S = (\mathfrak{o} \otimes_\mathbb{Z} \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_S = \prod_{\mathfrak{p}|p} \mathfrak{o}_\mathfrak{p} \otimes_{\mathbb{Z}_p} \mathcal{O}_S = \prod_{\tau \in \overline{\Sigma}} \mathfrak{o}_{\mathfrak{p}(\tau)} \otimes_{W(\mathbb{F}_{\mathfrak{p}(\tau)}),\tau} \mathcal{O}_S.$$

It implies that we have a corresponding decomposition

$$(1.2) \qquad \omega_S = \bigoplus_{\tau \in \overline{\Sigma}} \omega_{S,\tau}.$$

The sheaf $\omega_{S,\tau}$ is locally free over S of rank $e_{\mathfrak{p}(\tau)}$ (see [17, §2.2]). Note that on $\omega_{S,\tau}$, the action of $W(\mathbb{F}_{\mathfrak{p}(\tau)}) \subset \mathfrak{o}_{\mathfrak{p}(\tau)}$ is via τ. Fix a uniformiser $\varpi_{\mathfrak{p}(\tau)}$ of $\mathfrak{o}_{\mathfrak{p}(\tau)}$. From the product decomposition above, we get an action of $\mathfrak{o}_{\mathfrak{p}(\tau)}$ on $\omega_{S,\tau}$. Denote the action of $\varpi_{\mathfrak{p}(\tau)}$ on $\omega_{S,\tau}$ by $[\varpi_{\mathfrak{p}(\tau)}]$.

We are now ready to present the Pappas–Rapoport model. Consider the functor from the category of locally Noetherian $\mathcal{O}$-schemes to the category of sets which assigns to a scheme S the set of isomorphism classes of tuples $(A, \lambda, \mu, (\mathcal{F}_\mathfrak{p})_{\mathfrak{p}|p})$ where (A, λ, μ) is as above and for all $\mathfrak{p} \mid p$, $\mathcal{F}_\mathfrak{p}$ is a collection $(\mathcal{F}^i_\tau)_{\tau \in \overline{\Sigma}_\mathfrak{p}, 0 \leqslant i \leqslant e_\mathfrak{p}}$ of $\mathfrak{o} \otimes \mathcal{O}_S$-modules, which are locally free as $\mathcal{O}_S$-modules, such that:

- $0 = \mathcal{F}^0_\tau \subset \cdots \subset \mathcal{F}^{e_\mathfrak{p}}_\tau = \omega_{S,\tau}$,
- for any $\sigma = \sigma_{\tau,i} \in \Sigma_\tau$, the $\mathcal{O}_S$-module $\omega_{S,\tau,i} = \omega_{S,\sigma} = \mathcal{F}^i_\tau/\mathcal{F}^{i-1}_\tau$ is locally free of rank 1 and annihilated by $[\varpi_\mathfrak{p}] - \sigma(\varpi_\mathfrak{p})$. Note that the numbering here depends on the one for Σ_τ.

This functor is representable by a smooth $\mathcal{O}$-scheme $\mathcal{X}$ of finite type called the *Pappas–Rapoport moduli space* (see [17, Prop. 2.4] and [13, Lem. 1.4]).

In order to better understand the relation between the Deligne–Pappas and the Pappas–Rapoport moduli spaces, we recall that the *Rapoport locus* $\mathcal{X}^{\mathrm{Ra}}$ is the open subscheme of $\mathcal{X}^{\mathrm{DP}}$ classifying HBAV's $s : A \to S$ satisfying the following condition introduced by Rapoport: $s_*\Omega^1_{A/S}$ is a locally free $\mathfrak{o} \otimes_{\mathbb{Z}} \mathcal{O}_S$-module of rank 1. Then $\mathcal{X}^{\mathrm{Ra}}$ is the smooth locus of $\mathcal{X}^{\mathrm{DP}}$ and its complement is supported in the special fibre and has codimension at least 2 in it. The forgetful map $\mathcal{X} \to \mathcal{X}^{\mathrm{DP}}_{\mathcal{O}}$ induces an isomorphism on the open subscheme $\mathcal{X}^{\mathrm{Ra}}_{\mathcal{O}}$ (see [17, Prop. 2.4]). If p is unramified in F, the different schemes agree: $\mathcal{X} = \mathcal{X}^{\mathrm{Ra}}_{\mathcal{O}} = \mathcal{X}^{\mathrm{DP}}_{\mathcal{O}}$ (see [17, §1]).

Let $\mathcal{A}$ be the universal abelian scheme over $\mathcal{X}$ with structure morphism $s : \mathcal{A} \to \mathcal{X}$. Let $\omega_{\mathcal{X}} = s_*\Omega^1_{\mathcal{A}/\mathcal{X}}$. Note that the restriction of $\omega_{\mathcal{X}}$ to $\mathcal{X}^{\mathrm{Ra}}_{\mathcal{O}}$ is a locally free sheaf of rank 1 over $\mathfrak{o} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{X}^{\mathrm{Ra}}_{\mathcal{O}}}$. As abbreviation we write $\omega, \omega_\tau, \omega_{\tau,i}, \omega_\sigma$ for $\omega_{\mathcal{X}}, \omega_{\mathcal{X},\tau}, \omega_{\mathcal{X},\tau,i}, \omega_{\mathcal{X},\sigma}$. In particular, for each $\tau \in \overline{\Sigma}$, the sheaf ω_τ is equipped with a filtration the graded pieces of which are the invertible sheaves ω_σ for $\sigma \in \Sigma_\tau$. In [17] this is referred to as the *universal filtration*. We point out explicitly that the last graded piece $\omega_{\widetilde{\tau}}$ is a quotient of ω_τ.

Next we give, following Katz, a geometric definition of the space of Hilbert modular forms.

Definition 1.1. *A* Katz Hilbert modular form *of weight* $k = \sum_{\sigma \in \Sigma} k_\sigma \sigma \in \mathbb{Z}[\Sigma]$, *level* $\mathfrak{n}$ *and coefficients in an $\mathcal{O}$-algebra R is a global section of the line bundle* $\omega^{\otimes k} = \bigotimes_{\sigma \in \Sigma} \omega_\sigma^{\otimes k_\sigma}$ *over* $\mathcal{X} \times_{\mathcal{O}} R$. *We will denote by* $M_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; R)$ *the corresponding R-module.*

Its R-submodule of cuspforms $S_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; R)$ consists of those Katz Hilbert modular forms that vanish along the cuspidal divisor of any toroidal compactification of $\mathcal{X} \times_{\mathcal{O}} R$ (see [17, §2.11]).

As $\mathcal{X}$ admits toroidal compactifications (see [17, §2.11]) which are smooth and proper over $\mathcal{O}$ and to which ω_σ extends for all $\sigma \in \Sigma$, the Koecher principle implies, in view of [22, Tag 02O5], that $M_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; R)$ is a finitely generated R-module.

Remark 1.2. *When the weight $k \in \mathbb{Z}[\Sigma]$ is parallel, i.e., $k_\sigma = \kappa \in \mathbb{Z}$ for all $\sigma \in \Sigma$, one also could define a* Katz Hilbert modular form *of parallel weight $\kappa \in \mathbb{Z}$, level $\mathfrak{n}$ and coefficients in a $\mathbb{Z}_p$-algebra R as a global section of the line bundle $\left(\bigwedge^d s_*\Omega^1_{\mathcal{A}/\mathcal{X}^{\mathrm{DP}}}\right)^{\otimes \kappa}$ over $\mathcal{X}^{\mathrm{DP}} \times_{\mathbb{Z}_p} R$. By Zariski's Main Theorem applied to the proper birational map $\mathcal{X} \to \mathcal{X}^{\mathrm{DP}}_{\mathcal{O}}$ between normal varieties, this would lead to the same space as in Definition 1.1.*

1.2. **Generalised Hasse invariants.** From this point onwards we will work over $\mathbb{F}$. Let X be the Pappas–Rapoport moduli space over $\mathbb{F}$, *i.e.*, the special fibre $\mathcal{X} \times_{\mathcal{O}} \mathbb{F}$ of $\mathcal{X}$. There is a natural morphism $X \to \mathcal{X}^{\mathrm{DP}} \times_{\mathbb{Z}_p} \mathbb{F}$ obtained by forgetting the filtrations. Let $X^{\mathrm{Ra}} = \mathcal{X}^{\mathrm{Ra}} \times_{\mathbb{Z}_p} \mathbb{F}$. We have the equality

$$(1.3) \qquad \mathfrak{o} \otimes_{\mathbb{Z}} \mathbb{F} = \prod_{\mathfrak{p}|p} \mathfrak{o}_{\mathfrak{p}} \otimes_{\mathbb{Z}_p} \mathbb{F} \simeq \prod_{\tau \in \overline{\Sigma}} \mathfrak{o}_{\mathfrak{p}(\tau)} \otimes_{W(\mathbb{F}_{\mathfrak{p}(\tau)}),\tau} \mathbb{F} = \prod_{\tau \in \overline{\Sigma}} \mathbb{F}[x]/(x^{e_{\mathfrak{p}(\tau)}}),$$

coming from (1.1). Note that the last equality of (1.3) depends on the choice of the uniformiser $\varpi_{\mathfrak{p}(\tau)}$ of $\mathfrak{o}_{\mathfrak{p}(\tau)}$, made in the previous subsection for every $\tau \in \overline{\Sigma}$, and allows us to view ω_τ as an $\mathcal{O}_X[x]/(x^{e_{\mathfrak{p}}})$-module. If S is a locally Noetherian $\mathbb{F}$-scheme and A is an HBAV over S satisfying the Rapoport condition, then $\omega_{S,\tau}$ is a locally free $\mathcal{O}_S[x]/(x^{e_{\mathfrak{p}(\tau)}})$-module of rank 1. Hence, there is a unique filtration on $\omega_{S,\tau}$ satisfying the Pappas–Rapoport conditions given by $x^{e_{\mathfrak{p}(\tau)}-i}\omega_{S,\tau}$ for $0 \leqslant i \leqslant e_{\mathfrak{p}(\tau)}$. We point out again that the definition of $\mathcal{X}$ depends on the numbering of the embeddings in Σ_τ fixed above, but that X is independent of any such choice (see also [17, Rem. 2.3]).

If $\mathfrak{p} \mid p$ and $\tau \in \overline{\Sigma}_{\mathfrak{p}}$, then suppose the universal filtration on ω_τ is given by $(\mathcal{F}_\tau^i)_{1 \leqslant i \leqslant e_{\mathfrak{p}}}$. We now recall Reduzzi–Xiao's constructions of generalised Hasse invariants h_σ given in [17]. Let $\mathfrak{p} \mid p$ and $\tau \in \overline{\Sigma}_{\mathfrak{p}}$ and assume first that $2 \leqslant i \leqslant e_{\mathfrak{p}}$. There is a map $\mathcal{F}_\tau^i \to \mathcal{F}_\tau^{i-1}$ which sends a local section z of $\mathcal{F}_\tau^i$ to the section $x \cdot z$ of $\mathcal{F}_\tau^{i-1}$, where the action of x is given by $[\varpi_{\mathfrak{p}(\tau)}]$. Hence, we get a map $\mathcal{F}_\tau^i/\mathcal{F}_\tau^{i-1} \to \mathcal{F}_\tau^{i-1}/\mathcal{F}_\tau^{i-2}$ inducing a section $h_{\tau,i} = h_{\sigma_{\tau,i}}$ of $\omega_{\tau,i-1} \otimes \omega_{\tau,i}^{-1}$ over X. This h_σ is the *generalised Hasse invariant at* $\sigma = \sigma_{\tau,i}$ (see [17, Construction 3.3] and [15, §2.11] for more details). As $(\omega_\tau)_{|X^{\mathrm{Ra}}}$ is a locally free sheaf over $\mathcal{O}_{X^{\mathrm{Ra}}}[x]/(x^{e_{\mathfrak{p}}})$ of rank 1, we have $(\mathcal{F}_\tau^i)_{|X^{\mathrm{Ra}}} = (x^{e_{\mathfrak{p}}-i}\omega_\tau)_{|X^{\mathrm{Ra}}}$. It follows that $h_{\tau,i}$ is a nowhere vanishing section over X^{Ra} and multiplication by $h_{\tau,i}$ induces an isomorphism between $(\omega_{\tau,i})_{|X^{\mathrm{Ra}}}$ and $(\omega_{\tau,i-1})_{|X^{\mathrm{Ra}}}$.

For the case $i = 1$, the *generalised Hasse invariant* $h_{\tau,1}$ is defined as a global section over X of $\omega_{\phi^{-1}\circ\tau,e_{\mathfrak{p}}}^{\otimes p} \otimes \omega_{\tau,1}^{\otimes-1}$ (see [17, Construction 3.6] for more details). We let $h_\tau = \prod_{\sigma \in \Sigma_\tau} h_\sigma = \prod_{i=1}^{e_{\mathfrak{p}}} h_{\tau,i}$. It is a modular form of weight $p \cdot \widetilde{\phi^{-1} \circ \tau} - \widetilde{\tau}$.

Remark 1.3. *Let A be the universal abelian scheme over X and* $\mathrm{Ver} : A^{(p)} \to A$ *be the Verschiebung morphism, where* $A^{(p)} = A \times_{\mathbb{F},\phi} \mathbb{F}$. *It induces maps* $\omega_\tau \to \omega_{\phi^{-1}\circ\tau}^{(p)}$ *and further* $\mathcal{F}_\tau^{e_{\mathfrak{p}}}/\mathcal{F}_\tau^{e_{\mathfrak{p}}-1} \to (\mathcal{F}_{\phi^{-1}\circ\tau}^{e_{\mathfrak{p}}}/\mathcal{F}_{\phi^{-1}\circ\tau}^{e_{\mathfrak{p}}-1})^{(p)}$. *Note that* $\mathcal{F}_\tau^{e_{\mathfrak{p}}}/\mathcal{F}_\tau^{e_{\mathfrak{p}}-1} = \omega_{\tau,e_{\mathfrak{p}}}$, $(\mathcal{F}_{\phi^{-1}\circ\tau}^{e_{\mathfrak{p}}}/\mathcal{F}_{\phi^{-1}\circ\tau}^{e_{\mathfrak{p}}-1})^{(p)} = \omega_{\phi^{-1}\circ\tau,e_{\mathfrak{p}}}^{\otimes p}$ *and the resulting section of* $\omega_{\phi^{-1}\circ\tau,e_{\mathfrak{p}}}^{\otimes p} \otimes \omega_{\tau,e_{\mathfrak{p}}}^{\otimes-1}$ *over X is precisely given by h_τ (see [17, Lem. 3.8]). Moreover, its restriction to $X_{\mathbb{F}}^{\mathrm{Ra}}$ coincides with Andreatta–Goren's* partial Hasse invariant *constructed in [2, Def. 7.12]. In particular, when p is unramified in F, the generalised Hasse invariants constructed by Reduzzi–Xiao are the same as the partial Hasse invariants constructed by Andreatta–Goren.*

We will now determine the geometric q-expansions of these generalised Hasse invariants. We will mostly follow conventions of [11, §8]. Let $\infty_{\mathfrak{c}}$ be the standard infinity cusp whose Tate object is given by $(\mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{c}^*)/q^{\mathfrak{o}}$ (see [14, §2.3]). Here $\mathfrak{c}^* = \mathfrak{c}^{-1}\mathfrak{d}^{-1}$. Let $X^\wedge$ be the formal completion of a toroidal compactification of X along the divisor at the cusp $\infty_{\mathfrak{c}}$ (see [11, Thm. 8.6]). By *loc. cit.*, the pull back of ω to $X^\wedge$ is canonically isomorphic to $\mathcal{O}_{X^\wedge} \otimes \mathfrak{c}$. Choosing an identification

$$(1.4) \qquad\qquad \mathbb{F} \otimes \mathfrak{c} \xrightarrow{\sim} \mathbb{F} \otimes \mathfrak{o}$$

one can canonically identify $\omega_{\tau|X^\wedge}$ with $\tau(\mathcal{O}_{X^\wedge} \otimes \mathfrak{o}) = \mathcal{O}_{X^\wedge}[x]/(x^{e_{\mathfrak{p}(\tau)}})$ (see (1.3)). A global section of ω_τ over $X^\wedge$ is an element of

$$\left\{ \sum_{\xi \in \mathfrak{c}_+ \cup \{0\}} a_\xi q^\xi \ \middle|\ a_\xi \in \mathbb{F}[x]/(x^{e_{\mathfrak{p}(\tau)}}) \text{ and } a_{u^2\xi} = \tau(u)a_\xi, \forall u \in \mathfrak{o}^\times, u - 1 \in \mathfrak{n} \right\},$$

whereas a section z of $\omega_{\tau,i}$ over $X^\wedge$ is an element of

$$\left\{ x^{e_{\mathfrak{p}(\tau)}-i} \cdot \sum_{\xi \in \mathfrak{c}_+ \cup \{0\}} b_\xi q^\xi \ \middle|\ b_\xi \in \mathbb{F} \text{ and } b_{u^2\xi} = \tau(u)b_\xi, \forall u \in \mathfrak{o}^\times, u - 1 \in \mathfrak{n} \right\}$$

whose q-expansion is given by $\sum_{\xi \in \mathfrak{c}_+ \cup \{0\}} b_\xi q^\xi$ with respect to the choice of basis of $\omega_{\tau,i|X^\wedge}$ corresponding to $x^{e_{\mathfrak{p}(\tau)}-i}$.

Lemma 1.4. *Let $\mathfrak{p}|p$, $\tau \in \overline{\Sigma}_\mathfrak{p}$. Then for every $1 \leqslant i \leqslant e_\mathfrak{p}$, the geometric q-expansion of the generalised Hasse invariant $h_{\tau,i}$ at $\infty_\mathfrak{c}$ is 1. In particular, it does not vanish at any cusp.*

Proof. When $i > 1$, as $x \cdot z$ is a section of $\omega_{\tau,i-1}$ having by definition the same q-expansion, one concludes that $h_{\tau,i}$ has q-expansion 1, thus proving the claim in that case. In the remaining case of $i = 1$, we observe that the q-expansion of $h_\tau = \prod_{i=1}^{e_\mathfrak{p}(\tau)} h_{\tau,i}$ at $\infty_\mathfrak{c}$ is 1 by Remark 1.3 and [2, Prop. 7.14]. Hence the q-expansion of $h_{\tau,1}$ at $\infty_\mathfrak{c}$ is 1. Finally, since the $h_{\tau,i}$ can be defined in any level, we deduce their non-vanishing at all cusps from the non-vanishing at $\infty_\mathfrak{c}$. $\qquad\square$

We now collect some properties of the generalised Hasse invariants that will be used in the sequel. Let $Z_\sigma \subset X$ be the divisor of h_σ and, in order to shorten the notation, we let $Z_{\tau,i} = Z_{\sigma_{\tau,i}}$.

Lemma 1.5. *The complement of X^{Ra} in X coincides with $\bigcup_{\tau \in \overline{\Sigma}} \bigcup_{i=2}^{e_\mathfrak{p}(\tau)} Z_{\tau,i}$. Moreover, for any $I \subseteq \Sigma$, the intersection $\bigcap_{\sigma \in I} Z_\sigma$ is, either empty, or equidimensional of dimension $d - |I|$. In particular, the zero loci of two different generalised Hasse invariants do not have a common divisor.*

Proof. The first claim has been established in [15, Prop. 2.13 (2)]. For the second, if $\cap_{\sigma \in I} Z_\sigma$ is nonempty, then the tangent space computation in [17, Thm. 3.10] ensures the correct dimension. $\qquad\square$

Remark 1.6. *Diamond and Kassaei also prove Lemma 1.5 and obtain in addition the non-emptiness of the intersection (see [9, Prop. 5.8]). Here we sketch a constructive proof, following ideas of Andreatta and Goren [1], if $e_{\mathfrak{p}(\tau)}$ is odd for all $\tau \in \overline{\Sigma}$.*

Let $A = E \otimes_{\mathbb{Z}} \mathfrak{o}^*$, where E is a supersingular elliptic curve over $\mathbb{F}$. We see, as in [1, Proof of Thm. 10.1], that $\omega_{A,\tau} \simeq \mathbb{F}[x]/(x^{e_\mathfrak{p}(\tau)})$ for all $\tau \in \overline{\Sigma}$. Let $\mathrm{Frob}_A : A \to A^{(p)}$ be the Frobenius map and $H = \ker(\mathrm{Frob}_A)[\prod_{\mathfrak{p}|p} \mathfrak{p}^{[e_\mathfrak{p}/2]}]$. By imitating the calculations of [1, §8] (more specifically [1, Prop. 6.5, Lemmas 8.6, 8.9, Prop. 8.10]), one sees that if $A^{(1)} = A/H$, then

$$(1.5) \qquad \omega_{A^{(1)},\tau} \simeq x^{[e_\mathfrak{p}(\tau)/2]} \cdot \mathbb{F}[x]/(x^{e_\mathfrak{p}(\tau)}) \bigoplus x^{e_\mathfrak{p}(\tau)-[e_\mathfrak{p}(\tau)/2]} \cdot \mathbb{F}[x]/(x^{e_\mathfrak{p}(\tau)}) \text{ for all } \tau \in \overline{\Sigma}.$$

Note that $A^{(1)}$ is a $\mathfrak{c}'$-polarised HBAV over $\mathbb{F}$ for some $\mathfrak{c}' \in \mathcal{C}$. Let $\mathfrak{a} \subset \mathfrak{o}$ be an ideal relatively prime to p such that $\mathfrak{a}\mathfrak{c}'$ and $\mathfrak{c}$ represent the same element in the narrow class group of F. Let $H^{(1)}$

be an $\mathfrak{o}$-invariant subgroup scheme of $A^{(1)}[\mathfrak{a}]$ isomorphic to $\mathfrak{o}/\mathfrak{a}$ and let $A^{(2)} = A^{(1)}/H^{(1)}$. By [16, §1.9], $A^{(2)}$ is a $\mathfrak{c}$-polarised HBAV over $\mathbb{F}$ and since $\mathfrak{a}$ is relatively prime to p, we have $\omega_{A^{(2)}} = \omega_{A^{(1)}}$. Endowing each $\omega_{A^{(2)},\tau}$ with the 'alternating' filtration between the two summands in (1.5) yields a point in $\bigcap_{\tau \in \overline{\Sigma}} \bigcap_{i=2}^{e_{\mathfrak{p}(\tau)}} Z_{\tau,i}$, showing that the latter is non-empty.

If $e_{\mathfrak{p}(\tau)}$ is odd, then the filtration on $\omega_{A^{(2)},\tau}$ described above is unique. Moreover, as $A^{(2)}$ is supersingular (i.e. its p-torsion subgroup has no étale component), the map $\omega_{A^{(2)},\tau} \to \omega_{A^{(2)},\phi^{-1}\circ\tau}$ induced by the Verschiebung morphism is the zero map. Hence, we conclude, using the structure of $\omega_{A^{(2)},\tau}$ and the definition of the Hasse invariant $h_{\tau,1}$, that any such point also lies in $Z_{\tau,1}$. Thus, if $e_{\mathfrak{p}(\tau)}$ is odd for all $\tau \in \overline{\Sigma}$, then we get a point in $\bigcap_{\tau \in \overline{\Sigma}} \bigcap_{i=1}^{e_{\mathfrak{p}(\tau)}} Z_{\tau,i}$.

We illustrate the weights of the generalised and partial Hasse invariants in Table 1.2, where we let $\tau \in \overline{\Sigma}$ and write $e = e_{\mathfrak{p}(\tau)}$ as abbreviation.

FIGURE 1. Weights of Hasse invariants.

	Weights										
	$\phi^{-1} \circ \tau$			τ					$\phi \circ \tau$		
	$\cdots$	$e-1$	e	1	2	$\cdots$	$e-1$	e	1	2	$\cdots$
$h_{\phi^{-1}\circ\tau,e}$		1	-1								
$h_{\tau,1}$			p	-1							
$h_{\tau,2}$				1	-1						
$\vdots$						$\ddots$	$\ddots$				
$h_{\tau,e-1}$						1	-1				
$h_{\tau,e}$							1	-1			
$h_{\phi\circ\tau,1}$								p	-1		
$h_{\phi\circ\tau,2}$									1	-1	
h_τ			p					-1			

One of the advantages of Definition 1.1 is that it allows us to define mod p Hilbert modular forms in *any* weight $k = \sum_{\sigma \in \Sigma} k_\sigma \sigma \in \mathbb{Z}[\Sigma]$, while the definition in [2] was *missing* some weights when p ramifies in F, namely theirs are indexed by $\overline{\Sigma}$, instead of Σ. Indeed, the space of modular forms introduced by Andreatta and Goren [2, Prop. 5.5] is

$$(1.6) \qquad M_{\bar{k}}^{\mathrm{AG}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F}) = \mathrm{H}^0(X^{\mathrm{Ra}}, \bigotimes_{\tau \in \overline{\Sigma}} \omega_{\widetilde{\tau}}^{k_\tau}), \text{ where } \bar{k} = \sum_{\tau \in \overline{\Sigma}} k_\tau \tau \in \mathbb{Z}[\overline{\Sigma}].$$

We will denote by $S_{\bar{k}}^{\mathrm{AG}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ the subspace of $M_{\bar{k}}^{\mathrm{AG}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ consisting of cuspforms, which are defined as modular forms such that the constant coefficient of the q-expansion at every cusp vanishes. If $k = \sum_{\sigma \in \Sigma} k_\sigma \sigma \in \mathbb{Z}[\Sigma]$, then for every $\tau \in \overline{\Sigma}$, let $k_\tau = \sum_{\sigma \in \Sigma_\tau} k_\sigma$ and define $\bar{k} := \sum_{\tau \in \overline{\Sigma}} k_\tau \tau \in$

$\mathbb{Z}[\overline{\Sigma}]$. We let

$$(1.7) \qquad H_k^{\mathrm{RX}} = \prod_{\tau \in \overline{\Sigma}} \prod_{i=2}^{e_{\mathfrak{p}(\tau)}} h_{\tau,i}^{\sum_{j=1}^{i-1} k_{\tau,j}},$$

where $k_{\tau,j} = k_{\sigma_\tau,j}$. In view of the table of weights of the generalised Hasse invariants, for every $\tau \in \overline{\Sigma}$, the (τ, i)-component of the weight of f/H_k^{RX} is 0 if $1 \leqslant i \leqslant e_{\mathfrak{p}(\tau)} - 1$ and the $\widetilde{\tau} = (\tau, e_{\mathfrak{p}(\tau)})$-component is k_τ. Since H_k^{RX} is invertible on X^{Ra}, we obtain the following result.

Lemma 1.7. *The restriction from X to X^{Ra} yields an injection of $M_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ into $M_{\bar{k}}^{\mathrm{AG}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ sending f to f/H_k^{RX}.*

A converse is described in Lemma 1.12 below.

1.3. **Minimal weights.** We recall the notion of *minimal weight* of a mod p Hilbert modular form.

Definition 1.8. *We define the* minimal weight *of $0 \neq f \in M_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ to be the unique weight k' such that $f = g \cdot \prod_{\sigma \in \Sigma} h_\sigma^{n_\sigma}$, where $g \in M_{k'}^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ and the integers $(n_\sigma)_{\sigma \in \Sigma}$ are as large as possible.*

Lemma 1.9. *The notion of minimal weight is well defined.*

Proof. First note that Z_σ is non-empty for every $\sigma \in \Sigma$. Indeed, this follows from [9, Cor. 5.7]. Alternatively, we have shown in Remark 1.6 that $Z_{\tau,i}$ is non-empty for every $\tau \in \overline{\Sigma}$ and $2 \leq i \leq e_{\mathfrak{p}(\tau)}$. Moreover, it is well known that the zero locus of $h_\tau = \prod_{i=1}^{e_{\mathfrak{p}(\tau)}} h_{\tau,i}$ in X^{Ra} is non-empty for every $\tau \in \overline{\Sigma}$ (see [2, Cor. 8.18]). By Lemma 1.5, the Hasse invariants $h_{\tau,i}$ with $\tau \in \overline{\Sigma}$ and $2 \leq i \leq e_{\mathfrak{p}(\tau)}$ are invertible on X^{Ra}. Therefore, it follows that the divisor $Z_{\tau,1}$ is non-empty for every $\tau \in \overline{\Sigma}$.

Recall from Lemma 1.5 that the zero loci of two different generalised Hasse invariants do not have a common divisor. Let j_σ be the order of vanishing of a Hilbert modular form $f \neq 0$ on Z_σ. So, if we divide f by $\prod_{\sigma \in \Sigma} h_\sigma^{j_\sigma}$, we get the modular form g needed in Definition 1.8. Hence, it follows that the notion of minimal weight is indeed well defined (see also the proof of [2, Thm. 8.19] and [9, §8]). $\square$

Remark 1.10. *When p is unramified in F (i.e., $\Sigma = \overline{\Sigma}$), the notion of minimal weights from Definition 1.8 is the same as the one introduced by Andreatta and Goren [2, §8.20]. On the other hand, when $\mathfrak{p}$ is ramified, multiplying $0 \neq f \in M_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ having minimal weight k with arbitrary powers of generalised Hasse invariants ($h_{\tau,i}$ with $2 \leqslant i \leqslant e_{\mathfrak{p}}$) yields forms sharing the same $\bar{k}$ but whose weights are not minimal anymore.*

In [8, 9], Diamond and Kassaei define the *minimal cone* by

$$C^{\mathrm{min}} = \left\{ \sum_{\tau \in \overline{\Sigma}} \sum_{i=1}^{e_{\mathfrak{p}(\tau)}} k_{\tau,i} \sigma_{\tau,i} \in \mathbb{Q}[\Sigma] \;\middle|\; \forall\, \tau \in \overline{\Sigma}, \forall\, 1 \leqslant i < e_{\mathfrak{p}(\tau)}, k_{\tau,i+1} \geqslant k_{\tau,i}, pk_{\tau,1} \geqslant k_{\phi^{-1} \circ \tau, e_{\mathfrak{p}(\tau)}} \right\}.$$

Regarding the minimal weights for Hilbert modular forms, Diamond and Kassaei prove the following result in [8, Cor. 5.3], when p is unramified in F, and in [9, Cor. 8.2], when p is ramified in F.

Proposition 1.11 (Diamond–Kassaei). *The minimal weight of $0 \neq f \in M_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ belongs to C^{min}.*

The minimal weights allow us to further elaborate on the relation between the modular forms defined by Andreatta and Goren [2] and those of Definition 1.1.

Lemma 1.12. *Let $\bar{k} \in \mathbb{Z}[\overline{\Sigma}]$. There is a finite subset $K \subset C^{\mathrm{min}}$ such that for every $f \in M_{\bar{k}}^{\mathrm{AG}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$, there is $k' \in K$, a modular form $g \in M_{k'}^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ and a product of generalised Hasse invariants $H = \prod_{\tau \in \overline{\Sigma}} \prod_{i=1}^{e_{\mathfrak{p}(\tau)}} h_{\tau,i}^{j_{\tau,i}}$ with $j_{\tau,i} \in \mathbb{Z}$ and $j_{\tau,1} \geqslant 0$, such that the restriction to X^{Ra} of $g \cdot H$ equals f. In particular, f and g have the same geometric q-expansion at the cusp $\infty_{\mathfrak{c}}$.*

Proof. The result is trivial for $f = 0$. Seeing $0 \neq f \in M_{\bar{k}}^{\mathrm{AG}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ as a meromorphic section of the line bundle $\bigotimes_{\tau \in \overline{\Sigma}} \omega_{\tilde{\tau}}^{\otimes k_\tau}$ over X, we let $j_{\tau,i} \in \mathbb{Z}$ be the order of vanishing of f along the divisor $Z_{\tau,i}$ defined by the Hasse invariant $h_{\tau,i}$ for $\tau \in \overline{\Sigma}$ and $1 \leqslant i \leqslant e_{\mathfrak{p}(\tau)}$. As f is holomorphic on X^{Ra}, which intersects every irreducible component of $Z_{\tau,1}$ non-trivially by Lemma 1.5, we deduce that $j_{\tau,1} \geqslant 0$. Dividing f by $H = \prod_{\tau \in \overline{\Sigma}} \prod_{i=1}^{e_{\mathfrak{p}(\tau)}} h_{\tau,i}^{j_{\tau,i}}$ yields a holomorphic section on all of X, *i.e.*, a Katz modular form g in a weight k' which is by construction minimal, hence belongs to C^{min} by Proposition 1.11. As the q-expansions of all generalised Hasse invariants at the cusp $\infty_{\mathfrak{c}}$ equal 1 by Lemma 1.4, both f and g have the same q-expansion.

We next prove that given $\bar{k}$, there are only finitely many $k' \in C^{\mathrm{min}}$ that can appear for non-zero modular forms in $M_{\bar{k}}^{\mathrm{AG}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ via the method in the previous paragraph. Since dividing by $h_{\tau,1}$ (for any $\tau \in \overline{\Sigma}$) subtracts $(p-1)$ from the sum of the weights, whereas multiplying or dividing by $h_{\tau,i}$ for $\tau \in \overline{\Sigma}$ and $2 \leqslant i \leqslant e_{\mathfrak{p}(\tau)}$ leaves that sum unchanged, we deduce that $\sum_{\sigma \in \Sigma} k'_\sigma \leqslant \sum_{\tau \in \overline{\Sigma}} k_\tau$. As in the language of [9], the minimal cone is contained in the standard cone, we have $k'_\sigma \geqslant 0$ for all $\sigma \in \Sigma$ and the claimed finiteness follows. $\quad\square$

The finiteness of K in Lemma 1.12 yields the following result.

Corollary 1.13. *The $\mathbb{F}$-vector space $M_{\bar{k}}^{\mathrm{AG}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ is finite dimensional.*

We now further use the work of Diamond and Kassaei to study the minimality of the weight for modular forms of parallel weight one.

Corollary 1.14. *Suppose $f \in M_1^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ is a non-zero Hilbert modular form and k is its minimal weight. Then, for any prime $\mathfrak{p} \mid p$, either $k_\sigma = 1$ for all $\sigma \in \Sigma_{\mathfrak{p}}$ (in that case, we say that the weight is* minimal at $\mathfrak{p}$*), or $k_\sigma = 0$ for all $\sigma \in \Sigma_{\mathfrak{p}}$, the latter case being possible only if $(p-1)$ divides $e_{\mathfrak{p}}$.*

Proof. By Proposition 1.11, we know that $k \in C^{\mathrm{min}}$. By definition of C^{min} one has $k_\sigma \geqslant 0$ for all $\sigma \in \Sigma$ and, moreover, if $k_\sigma = 0$ with $\sigma \in \Sigma_{\mathfrak{p}}$ for some $\mathfrak{p} \mid p$, then $k_\sigma = 0$ for all $\sigma \in \Sigma_{\mathfrak{p}}$.

We assume for the rest of this proof that $k_\sigma = 0$ for all $\sigma \in \Sigma_{\mathfrak{p}}$. Denote the weight of the Hasse invariant $h_{\tau,i}$ by $w_{\tau,i}$. By the definition of the minimal weight, there exist integers $n_{\tau,i} \geqslant 0$ such that

$$(1.8) \qquad \sum_{\sigma \in \Sigma_{\mathfrak{p}}} \sigma = \sum_{\tau \in \overline{\Sigma}_{\mathfrak{p}}} \sum_{i=1}^{e_{\mathfrak{p}}} n_{\tau,i} w_{\tau,i}.$$

From the description of $w_{\tau,i}$ (see Table 1.2), it follows that for all $i \geqslant 2$ one has $n_{\tau,i} = n_{\tau,i-1} + 1$ and furthermore $p n_{\tau,1} = n_{\phi^{-1}\circ\tau,1} + e_{\mathfrak{p}}$. It is then easy to find that $n_{\tau,1} = \frac{e_{\mathfrak{p}}}{p-1}$ for all $\tau \in \overline{\Sigma}_{\mathfrak{p}}$, showing that $p-1$ divides $e_{\mathfrak{p}}$. $\qquad \square$

The following result, the proof of which will be completed in the next subsection, shows that one can be more precise when restricting to cuspforms.

Proposition 1.15. *Let $\mathfrak{p}$ be a prime of F dividing p. Let $k = \sum_{\sigma \in \Sigma} k_\sigma \sigma \in \mathbb{Z}[\Sigma]$ be a weight such that $k_\sigma = 0$ for all $\sigma \in \Sigma_{\mathfrak{p}}$. Then $S_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F}) = 0$.*

Proof. By Lemma 1.7, $S_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ injects into $S_{\bar{k}}^{\mathrm{AG}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$, which is zero by Proposition 1.22. Alternatively, if there is a unique prime $\mathfrak{p}$ of F dividing p, then $k = 0$ and Koecher's principle applied to an embedding of the connected scheme X in a toroidal compactification implies that $\mathrm{H}^0(X, \mathcal{O}_X)$ consists only of forms which are constant, thus it does not contain any non-zero cuspforms. $\qquad \square$

Corollary 1.16. *The weight of any non-zero parallel weight 1 cuspform is minimal.*

Proof. Let f be a non-zero cuspform of parallel weight 1 and let k be its minimal weight. Suppose the minimal weight k is not $\sum_{\sigma \in \Sigma} \sigma$. Then by Corollary 1.14 we already know that there exists $\mathfrak{p} \mid p$ such that $k_\sigma = 0$ for all $\sigma \in \Sigma_{\mathfrak{p}}$. Moreover, as the generalised Hasse invariants do not vanish at any cusp (see Lemma 1.4), we have constructed a *non-zero cuspform* of weight k, contradicting Proposition 1.15. This proves the corollary. $\qquad \square$

Remark 1.17. *Confusion may arise from the fact that parallel weight 1 forms in our sense have weight exponents $e_{\mathfrak{p}(\tau)}$ when seen as a modular form in $M_{\bar{k}}^{\mathrm{AG}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ as $\bigwedge^{e_{\mathfrak{p}(\tau)}} \omega_\tau \simeq \omega_{\tilde{\tau}}^{\otimes e_{\mathfrak{p}(\tau)}}$ for $\tau \in \overline{\Sigma}$ over the Rapoport locus (see Lemma 1.7).*

1.4. Partial Frobenius operator. Fix $\mathfrak{c} \in \mathcal{C}$ and let $e \in \mathbb{N}$ be such that $\mathfrak{p}^{e e_{\mathfrak{p}}} = (\alpha)$ with $\alpha \in \mathfrak{o}_+$ and $\alpha \equiv p^e \equiv 1 \pmod{\mathfrak{n}}$. Also let $\beta \in \mathfrak{o}_+$ such that $p^e = \alpha \cdot \beta$. In order to lighten notation, we let $Y = X^{\mathrm{Ra}}$ denote the Rapoport locus and let $s : \mathcal{A} \to Y$ be the universal $\mathfrak{c}$-polarised HBAV endowed with $\mu_{\mathfrak{n}}$-level structure. Let $\mathcal{A}^{(p^e)} = \mathcal{A} \times_{Y, \mathrm{Fr}^e} Y$ be the base change by the e-th power of absolute Frobenius $\mathrm{Fr} : Y \to Y$. The e-th power of Verschiebung then defines an isogeny over Y

$$\mathcal{A}^{(p^e)} \xrightarrow{\mathrm{Ver}^e} \mathcal{A},$$

the kernel of which we denote by H. It is a finite group scheme with an $\mathfrak{o}/(p^e)$-action. Hence we can apply the Chinese remainder theorem to obtain the direct product decomposition $H = H_{\mathfrak{p}} \times H'_{\mathfrak{p}}$,

where $H_{\mathfrak{p}} = H[\alpha]$ is the $\mathfrak{p}$-component of H and $H'_{\mathfrak{p}} = H[\beta]$ is the product of $\mathfrak{p}'$-components of H for all $\mathfrak{p}' \neq \mathfrak{p}$ dividing p. We now define the abelian variety

$$\mathcal{B} = \mathcal{A}^{(p^e)}/H'_{\mathfrak{p}},$$

through which Ver^e factors, leading to an isogeny over Y

(1.9)
$$\mathcal{B} \xrightarrow{\ \ V_{\mathcal{A}}\ \ } \mathcal{A}.$$
$$\mathcal{B} \searrow_{t} \quad \swarrow_{s} \mathcal{A}$$
$$Y$$

Lemma 1.18. *The abelian variety $\mathcal{B}$ inherits a $\mu_{\mathfrak{n}}$-level structure and an $\alpha^{-1}\mathfrak{c}$-polarisation λ_{α}.*

Proof. As $\mathcal{A}^{(p^e)} \to \mathcal{B}$ is a p-primary isogeny, the $\mu_{\mathfrak{n}}$-level structure on $\mathcal{A}^{(p^e)}$ yields one on $\mathcal{B}$.

Regarding the polarisation, following a suggestion of the referee (see also [16, §1.9]), we claim that the kernel of the composed isogeny

$$\delta : \mathcal{B} \otimes_{\mathfrak{o}} \mathfrak{c} \xrightarrow{V_{\mathcal{A}} \otimes 1} \mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{c} \xrightarrow{\lambda^{-1}} \mathcal{A}^{\vee} \xrightarrow{V_{\mathcal{A}}^{\vee}} \mathcal{B}^{\vee}$$

equals the α-torsion of $\mathcal{B} \otimes_{\mathfrak{o}} \mathfrak{c}$, *i.e.*, $\ker(\delta)$ is α-torsion and has the same order as $(\mathcal{B} \otimes_{\mathfrak{o}} \mathfrak{c})[\alpha]$. As the order of finite flat group schemes is locally constant, it suffices to check this point-wise on the ordinary locus of Y which is dense.

As Ver^e is étale at an ordinary closed point $y \in Y$, its kernel is isomorphic to the constant group scheme given by $\mathfrak{o}/p^e\mathfrak{o}$, whence $\ker(V_{\mathcal{A}_y}) \simeq \mathfrak{o}/\alpha\mathfrak{o}$. Consequently, the kernel of the dual isogeny $V_{\mathcal{A}_y}^{\vee}$ is isomorphic to the Cartier dual $\mu_{\alpha\mathfrak{o}}$ of $\mathfrak{o}/\alpha\mathfrak{o}$. This gives us a short exact sequence of finite flat commutative group schemes

$$0 \to \mathfrak{c}/\alpha\mathfrak{c} \simeq \ker(V_{\mathcal{A}_y}) \otimes_{\mathfrak{o}} \mathfrak{c} \hookrightarrow \ker(\delta_y) \xrightarrow{\lambda^{-1} \circ (V_{\mathcal{A}_y} \otimes 1)} \ker(V_{\mathcal{A}_y}^{\vee}) \simeq \mu_{\alpha\mathfrak{o}} \to 0.$$

As the connected-étale sequence of any finite flat group scheme over a perfect field splits (see [20, §3.7]), we deduce that $\ker(\delta_y)$ is isomorphic to the group scheme $(\mathfrak{c}/\alpha\mathfrak{c}) \times \mu_{\alpha\mathfrak{o}}$. In particular $\ker(\delta_y)$ is α-torsion and has the same order as $(\mathcal{B}_y \otimes_{\mathfrak{o}} \mathfrak{c})[\alpha]$. Therefore, it follows that $\ker(\delta) = (\mathcal{B} \otimes_{\mathfrak{o}} \mathfrak{c})[\alpha]$.

Since $(\mathcal{B} \otimes_{\mathfrak{o}} \mathfrak{c})/(\mathcal{B} \otimes_{\mathfrak{o}} \mathfrak{c})[\alpha]$ is canonically isomorphic to $\mathcal{B} \otimes_{\mathfrak{o}} (\alpha^{-1}\mathfrak{c})$, we deduce an isomorphism $\mathcal{B} \otimes_{\mathfrak{o}} (\alpha^{-1}\mathfrak{c}) \xrightarrow{\sim} \mathcal{B}^{\vee}$ the inverse of which is the desired $\alpha^{-1}\mathfrak{c}$-polarisation λ_{α}. $\square$

We now verify that the HBAV $\mathcal{B}/Y$ satisfies the Rapoport condition. Recall that $\omega_{\mathcal{A}/Y} = s_*\Omega^1_{\mathcal{A}/Y}$ and $\omega_{\mathcal{B}/Y} := t_*\Omega^1_{\mathcal{B}/Y}$.

Lemma 1.19. *For any $\tau \in \overline{\Sigma} \setminus \overline{\Sigma}_{\mathfrak{p}}$, the map $V^*_{\mathcal{A},\tau} : \omega_{\mathcal{A}/Y,\tau} \to \omega_{\mathcal{B}/Y,\tau}$ is an isomorphism.*
On the other hand, if $\tau \in \overline{\Sigma}_{\mathfrak{p}}$, then the isogeny $\mathcal{A}^{(p^e)} \to \mathcal{B}$ induces an isomorphism

$$\omega_{\mathcal{B}/Y,\tau} \simeq \omega_{\mathcal{A}^{(p^e)}/Y,\tau} \simeq (\mathrm{Fr}^e)^*\omega_{\mathcal{A}/Y,\phi^{-e}\circ\tau}.$$

Proof. Let $r_{\mathfrak{p}}$ be the projection of $\mathfrak{o}/(p^e)$ on its $\mathfrak{p}$-primary component and let $\gamma \in \mathfrak{o}$ be such that its image in $\mathfrak{o}/(p^e)$ represents $r_{\mathfrak{p}}$. The the image of $\gamma' = 1 - \gamma$ in $\mathfrak{o}/(p^e)$ represents the complementary

idempotent $r'_{\mathfrak{p}} = 1 - r_{\mathfrak{p}}$. As γ' kills $\ker(V_{\mathcal{A}})$, the isogeny $\mathcal{B} \xrightarrow{\gamma'\cdot} \mathcal{B}$ factors through $V_{\mathcal{A}}$, yielding a factorisation

$$\omega_{\mathcal{B}/Y} \xrightarrow{\hspace{2cm}} \omega_{\mathcal{A}/Y} \xrightarrow{V_{\mathcal{A}}^*} \omega_{\mathcal{B}/Y}$$
$$\gamma'\cdot$$

If $\mathfrak{p}' \mid p$ and $\mathfrak{p}' \neq \mathfrak{p}$, the projection of γ' on the $\mathfrak{p}'$-component of $\mathfrak{o}/(p^e)$ is 1. Hence, it induces the identity on the $\mathfrak{p}'$-component of $\omega_{\mathcal{B}/Y}$. So the map $V_{\mathcal{A}}^*$ is split on the $\mathfrak{p}'$-component and hence $\omega_{\mathcal{B}/Y,\tau}$ is isomorphic to a direct summand of $\omega_{\mathcal{A}/Y,\tau}$ for all $\tau \in \overline{\Sigma}_{\mathfrak{p}'}$. Recall that both $\omega_{\mathcal{B}/Y,\tau}$ and $\omega_{\mathcal{A}/Y,\tau}$ are locally free sheaves over Y of the same rank. Therefore, after passing to their stalks, we conclude that $V_{\mathcal{A},\tau}^*$ is an isomorphism for all $\tau \in \overline{\Sigma}_{\mathfrak{p}'}$.

Similarly, as γ annihilates the kernel of the isogeny $\mathcal{A}^{(p^e)} \to \mathcal{B}$, we obtain an isomorphism between $\omega_{\mathcal{A}^{(p^e)}/Y,\tau}$ and $\omega_{\mathcal{B}/Y,\tau}$ for all $\tau \in \overline{\Sigma}_{\mathfrak{p}}$. This proves the lemma. $\qquad\square$

We get a $\mathfrak{c}$-polarisation on $\mathcal{B}$ from the $\alpha^{-1}\mathfrak{c}$-polarisation λ_α (which is obtained in Lemma 1.18) after identifying $\alpha^{-1}\mathfrak{c}$ with $\mathfrak{c}$ by multiplication by α. Thus, using Lemma 1.19, the universal property of $\mathcal{A} \to Y$ yields a Cartesian diagram

$$(1.10) \qquad\qquad \begin{array}{ccc} \mathcal{B} & \xrightarrow{\hspace{2cm}} & \mathcal{A}, \\ {\scriptstyle t}\downarrow & \square & \downarrow{\scriptstyle s} \\ Y & \xrightarrow[\phi_\alpha]{} & Y, \end{array}$$

from which we deduce a natural isomorphism $\phi_\alpha^* \omega_{\mathcal{A}/Y} \xrightarrow{\sim} \omega_{\mathcal{B}/Y}$ of $\mathfrak{o} \otimes \mathcal{O}_Y$-modules.

Let $k = \sum_{\sigma \in \Sigma} k_\sigma \sigma \in \mathbb{Z}[\Sigma]$ be a weight such that $k_\sigma = 0$ for all $\sigma \in \Sigma_{\mathfrak{p}}$. By Lemma 1.19

$$(1.11) \qquad\qquad\qquad V_{\mathcal{A}}^* : \omega_{\mathcal{A}/Y}^{\otimes k} \xrightarrow{\sim} \omega_{\mathcal{B}/Y}^{\otimes k}.$$

Definition 1.20. *Let $k = \sum_{\sigma \in \Sigma} k_\sigma \sigma \in \mathbb{Z}[\Sigma]$ be a weight such that $k_\sigma = 0$ for all $\sigma \in \Sigma_{\mathfrak{p}}$. The partial Frobenius operator $\Phi_{\mathfrak{p}^e}$ is defined as the composition of the adjunction morphism coming from (1.10) with (1.11)*

$$\Phi_{\mathfrak{p}^e} : \mathrm{H}^0(Y, \omega_{\mathcal{A}/Y}^{\otimes k}) \xrightarrow{\phi_\alpha^*} \mathrm{H}^0(Y, \omega_{\mathcal{B}/Y}^{\otimes k}) \xrightarrow[\sim]{(V_{\mathcal{A}}^*)^{-1}} \mathrm{H}^0(Y, \omega_{\mathcal{A}/Y}^{\otimes k}).$$

We next study the effect of $\Phi_{\mathfrak{p}^e}$ on q-expansions. To this end, we recall the definition and properties of Tate objects. For fractional ideals $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ of $\mathfrak{o}$ such that $\mathfrak{a}\mathfrak{b} \subset \mathfrak{c}$ and a cone C in $\mathfrak{c}_+^*$, we let

$$T_{\mathfrak{a},\mathfrak{b}} = (\mathbb{G}_m \otimes \mathfrak{a}^*)/q^{\mathfrak{b}} \to \overline{S}_C^\circ = \mathrm{Spec}(R_C^\circ)$$

be the Tate HBAV over the Noetherian algebra $R_C^\circ \supset \mathbb{F}[[q^\xi, \xi \in \mathfrak{c}_+]]$ (for more details we refer to [11, §2], where R_C° is denoted by $R_C^\wedge \otimes_{R_C} R$). It is equipped with a $\mu_{\mathfrak{n}}$-level structure which depends on the choice of an isomorphism between $\mathfrak{a}/\mathfrak{n}\mathfrak{a}$ and $\mathfrak{o}/\mathfrak{n}$. Moreover, the natural isomorphism

$$\lambda_{\mathfrak{a},\mathfrak{b}} : T_{\mathfrak{a},\mathfrak{b}}^\vee = T_{\mathfrak{b},\mathfrak{a}} \to T_{\mathfrak{a},\mathfrak{b}} \otimes_{\mathfrak{o}} (\mathfrak{a}\mathfrak{b}^{-1})$$

endows $T_{\mathfrak{a},\mathfrak{b}}$ with a canonical $\mathfrak{a}\mathfrak{b}^{-1}$-polarisation. Note that $T_{\mathfrak{c},\mathfrak{o}} = (\mathbb{G}_m \otimes \mathfrak{c}^*)/q^{\mathfrak{o}}$ is a Tate HBAV at the standard cusp $\infty_{\mathfrak{c}}$ of Y fitting, by universality of $\mathcal{A}/Y$, into a Cartesian diagram

$$(1.12)$$

$$
\begin{array}{ccc}
T_{\mathfrak{c},\mathfrak{o}} & \longrightarrow & \mathcal{A} \\
\downarrow & \square & \downarrow{\scriptstyle s} \\
\overline{S}^{\circ}_C & \xrightarrow{\alpha_Y} & Y.
\end{array}
$$

This gives a natural isomorphism $\alpha_Y^* \omega_{\mathcal{A}/Y} \simeq \omega_{T_{\mathfrak{c},\mathfrak{o}}/\overline{S}^{\circ}_C}$ and further, by adjunction and choice of canonical trivialisations using (1.4), we obtain a q-expansion map at the cusp $\infty_{\mathfrak{c}}$:

$$
\mathrm{H}^0(Y, \omega_{\mathcal{A}/Y}^{\otimes k}) \xrightarrow{\alpha_Y^*} \mathrm{H}^0(\overline{S}^{\circ}_C, \omega_{T_{\mathfrak{c},\mathfrak{o}}/\overline{S}^{\circ}_C}^{\otimes k}) \simeq R^{\circ}_C.
$$

Next we describe Ver^e on Tate objects. Define $T_{\mathfrak{c},\mathfrak{o}}^{(p^e)} = T_{\mathfrak{c},\mathfrak{o}} \times_{\overline{S}^{\circ}_C, \mathrm{Fr}^e} \overline{S}^{\circ}_C$ as the base change by the e-th power of absolute Frobenius. Note that $T_{\mathfrak{c},\mathfrak{o}}^{(p^e)} = T_{p^e\mathfrak{c},\mathfrak{o}}$ and the relative Frobenius map $\mathrm{Frob}_{T_{\mathfrak{c},\mathfrak{o}}}^e : T_{\mathfrak{c},\mathfrak{o}} \to T_{p^e\mathfrak{c},\mathfrak{o}}$ is the map induced by the inclusion $p^e\mathfrak{c} \hookrightarrow \mathfrak{c}$. The p^e-th Verschiebung is the dual of the p^e-th relative Frobenius on $T_{\mathfrak{c},\mathfrak{o}}^{\vee}$, *i.e.*, $\mathrm{Ver}^e = (\mathrm{Frob}_{T_{\mathfrak{c},\mathfrak{o}}^{\vee}}^e)^{\vee}$. We do not identify $((T_{\mathfrak{c},\mathfrak{o}}^{\vee})^{(p^e)})^{\vee}$ with $T_{\mathfrak{c},\mathfrak{o}}^{(p^e)}$ (as is usually done while defining Verschiebung) in order to get the desired maps on Tate objects. In particular, $\mathrm{Frob}_{T_{\mathfrak{c},\mathfrak{o}}^{\vee}}^e : T_{\mathfrak{o},\mathfrak{c}} \to T_{p^e\mathfrak{o},\mathfrak{c}}$ is the map induced by the inclusion $p^e\mathfrak{o} \hookrightarrow \mathfrak{o}$. Therefore, its dual map $\mathrm{Ver}^e : T_{\mathfrak{c},p^e\mathfrak{o}} \to T_{\mathfrak{c},\mathfrak{o}}$ is the natural projection obtained by going modulo $q^{\mathfrak{o}}$.

Our next aim is to specialise ϕ_{α} to the Tate objects. It follows from the previous paragraph that the base change of (1.9) to $\overline{S}^{\circ}_C$ is given by the following commutative diagram:

$$
\begin{array}{ccc}
T_{\mathfrak{c},\alpha\mathfrak{o}} & \xrightarrow{V_T} & T_{\mathfrak{c},\mathfrak{o}} \\
& \searrow \quad \swarrow & \\
& \overline{S}^{\circ}_C &
\end{array}
$$

where the map V_T is the natural projection obtained by going modulo $q^{\mathfrak{o}}$. Combining with (1.10), we get the following Cartesian diagram:

$$(1.13)$$

$$
\begin{array}{ccccc}
T_{\mathfrak{c},\alpha\mathfrak{o}} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{A} \\
\downarrow & \square & \downarrow{\scriptstyle t} & \square & \downarrow{\scriptstyle s} \\
\overline{S}^{\circ}_C & \xrightarrow{\alpha_Y} & Y & \xrightarrow{\phi_{\alpha}} & Y.
\end{array}
$$

On the other hand, considering the $\mathfrak{c}$-polarized HBAV $T_{\mathfrak{c},\mathfrak{o}}$ over $\overline{S}^{\circ}_{\alpha C}$ gives a Cartesian diagram

$$(1.14)$$

$$
\begin{array}{ccccc}
T_{\mathfrak{c},\alpha\mathfrak{o}} & \longrightarrow & T_{\mathfrak{c},\mathfrak{o}} & \longrightarrow & \mathcal{A} \\
\downarrow & \square & \downarrow & \square & \downarrow{\scriptstyle s} \\
\overline{S}^{\circ}_C & \xrightarrow{f_{\alpha}} & \overline{S}^{\circ}_{\alpha C} & \xrightarrow{\alpha_Y'} & Y,
\end{array}
$$

where f_α is induced by the morphism $R^\circ_{\alpha C} \to R^\circ_C$ sending q^ξ to $q^{\alpha\xi}$. We would like to emphasise that αC is considered as a cone in $\mathfrak{c}^*_+$, hence the dual cone (used in the construction of $R^\circ_{\alpha C}$, see [11, §2]) is considered as a cone in $\mathfrak{c}$ (and not in $\alpha^{-1}\mathfrak{c}$). In particular, the morphism $R^\circ_{\alpha C} \to R^\circ_C, q^\xi \mapsto q^{\alpha\xi}$ is not étale.

Lemma 1.21. *Under the notation developed above, $\phi_\alpha \circ \alpha_Y = \alpha'_Y \circ f_\alpha$.*

Proof. The proof proceeds by showing that the $\mathfrak{c}$-polarisation and $\mu_\mathfrak{n}$-level structure on $T_{\mathfrak{c},\alpha\mathfrak{o}}$ obtained from the base change in (1.13) coincide with the ones obtained from the base change in (1.14). This, along with the universality of $\mathcal{A}/Y$, implies $\phi_\alpha \circ \alpha_Y = \alpha'_Y \circ f_\alpha$.

As Mumford's construction of Tate objects presented in [11, §2] is functorial in $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ and C, we deduce that the $\mathfrak{c}$-polarisation on $T_{\mathfrak{c},\alpha\mathfrak{o}}$ arising from (1.14) is obtained from $\lambda_{\mathfrak{c},\alpha\mathfrak{o}}$ after identifying $\alpha^{-1}\mathfrak{c}$ with $\mathfrak{c}$ by multiplication by α.

We now derive the $\mathfrak{c}$-polarisation on $T_{\mathfrak{c},\alpha\mathfrak{o}}$ via the base change in (1.13). To do this, we proceed as in the proof of Lemma 1.18 to first obtain a $\alpha^{-1}\mathfrak{c}$-polarisation on $T_{\mathfrak{c},\alpha\mathfrak{o}}$ from $\lambda_{\mathfrak{c},\mathfrak{o}}$. From the proof of Lemma 1.18, it follows that the kernel of the isogeny

$$T_{\mathfrak{c},\alpha\mathfrak{o}} \otimes_\mathfrak{o} \mathfrak{c} \to T_{\mathfrak{c},\mathfrak{o}} \otimes_\mathfrak{o} \mathfrak{c} \to T_{\mathfrak{o},\mathfrak{c}} \to T_{\alpha\mathfrak{o},\mathfrak{c}}.$$

is just the α-torsion of $T_{\mathfrak{c},\alpha\mathfrak{o}} \otimes_\mathfrak{o} \mathfrak{c}$. Here the first map is induced by V_T (the natural projection given by going modulo $q^\mathfrak{o}$), the second map is $\lambda^{-1}_{\mathfrak{c},\mathfrak{o}}$, and the final map is induced by the inclusion $\alpha\mathfrak{o} \subset \mathfrak{o}$. Therefore, this composition of maps induces an isomorphism

$$\lambda : T^\vee_{\mathfrak{c},\alpha\mathfrak{o}} = T_{\alpha\mathfrak{o},\mathfrak{c}} \xrightarrow{\sim} (T_{\mathfrak{c},\alpha\mathfrak{o}} \otimes_\mathfrak{o} \mathfrak{c}) \otimes_\mathfrak{o} \alpha^{-1}\mathfrak{o} = T_{\mathfrak{c},\alpha\mathfrak{o}} \otimes_\mathfrak{o} \alpha^{-1}\mathfrak{c},$$

which is the $\alpha^{-1}\mathfrak{c}$-polarisation on $T_{\mathfrak{c},\alpha\mathfrak{o}}$ induced from $\lambda_{\mathfrak{c},\mathfrak{o}}$. From the description of the maps above, it follows that $\lambda = \lambda_{\mathfrak{c},\alpha\mathfrak{o}}$. Hence, the $\mathfrak{c}$-polarisation on $T_{\mathfrak{c},\alpha\mathfrak{o}}$ via the base change in (1.13) is obtained from $\lambda_{\mathfrak{c},\alpha\mathfrak{o}}$ by identifying $\alpha^{-1}\mathfrak{c}$ with $\mathfrak{c}$ by multiplication by α. Therefore, it follows that these two $\mathfrak{c}$-polarisations on $T_{\mathfrak{c},\alpha\mathfrak{o}}$ coincide.

As $\alpha \equiv 1 \pmod{\mathfrak{n}}$, the multiplication by α map preserves the $\mu_\mathfrak{n}$-level structure of $T_{\mathfrak{c},\mathfrak{o}}$. Hence, the $\mu_\mathfrak{n}$-level structure on $T_{\mathfrak{c},\alpha\mathfrak{o}}$ induced by f_α is same as the one coming from the quotient map $T_{\mathfrak{c},p^e\mathfrak{o}} \to T_{\mathfrak{c},\alpha\mathfrak{o}}$. This concludes the proof of the lemma. $\square$

We are now ready to compute the effect of $\Phi_{\mathfrak{p}^e}$ on q-expansions at $\infty_\mathfrak{c}$.

Proposition 1.22. *Let $\bar{k} = \sum_{\tau \in \overline{\Sigma}} k_\tau \tau \in \mathbb{Z}[\overline{\Sigma}]$ such that $k_\tau = 0$ for all $\tau \in \overline{\Sigma}_\mathfrak{p}$. Then the map $\Phi_{\mathfrak{p}^e}$ defines an endomorphism of $M^{\mathrm{AG}}_{\bar{k}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$, sending $f = \sum_{\xi \in \mathfrak{c}_+} a_\xi q^\xi$ to $\Phi_{\mathfrak{p}^e}(f) = \sum_{\xi \in \mathfrak{c}_+} a_\xi q^{\alpha\xi}$. In particular, the restriction of $\Phi_{\mathfrak{p}^e}$ to $S^{\mathrm{AG}}_{\bar{k}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ is injective and nilpotent, hence $S^{\mathrm{AG}}_{\bar{k}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F}) = \{0\}$.*

Proof. By definition, the q-expansion of $\Phi_{\mathfrak{p}^e}(f)$ at $\infty_\mathfrak{c}$ is the image of f under the map $\mathrm{H}^0(Y, \omega^{\otimes k}_{\mathcal{A}/Y}) \to \mathrm{H}^0(\overline{S}^\circ_C, \omega^{\otimes k}_{T_{\mathfrak{c},\alpha\mathfrak{o}}/\overline{S}^\circ_C})$ coming from the Cartesian diagram (1.13), followed by $(V^*_T)^{-1}$. By Lemma 1.21,

one can use instead the Cartesian diagram (1.14). Hence, the q-expansion of $\Phi_{\mathfrak{p}^e}(f)$ at $\infty_{\mathfrak{c}}$ can be obtained as the image of f under the adjunction morphism

$$\mathrm{H}^0(Y, \omega_{\mathcal{A}/Y}^{\otimes k}) \xrightarrow{\alpha_Y'^*} \mathrm{H}^0(\overline{S}_{\alpha C}^\circ, \omega_{T_{\mathfrak{c},\mathfrak{o}}/\overline{S}_{\alpha C}^\circ}^{\otimes k}) \xrightarrow{f_\alpha^*} \mathrm{H}^0(\overline{S}_C^\circ, \omega_{T_{\mathfrak{c},\alpha\mathfrak{o}}/\overline{S}_C^\circ}^{\otimes k})$$

followed by the map $(V_T^*)^{-1} : \mathrm{H}^0(\overline{S}_C^\circ, \omega_{T_{\mathfrak{c},\alpha\mathfrak{o}}/\overline{S}_C^\circ}^{\otimes k}) \xrightarrow{\sim} \mathrm{H}^0(\overline{S}_C^\circ, \omega_{T_{\mathfrak{c},\mathfrak{o}}/\overline{S}_C^\circ}^{\otimes k})$. Since the q-expansion $\sum_{\xi\in\mathfrak{c}_+} a_\xi q^\xi$ of f at the cusp $\infty_{\mathfrak{c}}$ is independent of a particular choice of a cone, it is given by the image of f under the map $\alpha_Y'^* : \mathrm{H}^0(Y, \omega_{\mathcal{A}/Y}^{\otimes k}) \to \mathrm{H}^0(\overline{S}_{\alpha C}^\circ, \omega_{T_{\mathfrak{c},\mathfrak{o}}/\overline{S}_{\alpha C}^\circ}^{\otimes k})$. As f_α is induced by the map sending q^ξ to $q^{\alpha\xi}$ we deduce that $f_\alpha^*(\sum_{\xi\in\mathfrak{c}_+} a_\xi q^\xi) = \sum_{\xi\in\mathfrak{c}_+} a_\xi q^{\alpha\xi}$. Finally, as V_T is induced from the identity map on the torus $\mathbb{G}_m \otimes \mathfrak{c}^*$, the morphism V_T^* is the identity on the q-expansions, *i.e.*, $(V_T^*)^{-1}(\sum_{\xi\in\mathfrak{c}_+} a_\xi q^{\alpha\xi}) = \sum_{\xi\in\mathfrak{c}_+} a_\xi q^{\alpha\xi}$, yielding the desired formula.

The rest follows from the q-expansion principle and the finite dimensionality of $S_{\tilde{k}}^{\mathrm{AG}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$. $\square$

1.5. **Refined injectivity criterion for Θ-operators.**

The purpose of this section is to extend the definition of the Andreatta–Goren operators $\Theta_\tau^{\mathrm{AG}}$ for $\tau \in \overline{\Sigma}$ and prove an injectivity criterion refining [2, Prop. 15.10] when p ramifies in F. Given $f \in M_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$, by Lemma 1.7 and the discussion after it, $\Theta_\tau^{\mathrm{AG}}\left(\frac{f}{H_k^{\mathrm{RX}}}\right)$ defines a meromorphic section over X, whose poles lie outside X^{Ra}. A careful study of the order at these poles will first show that multiplication by H_k^{RX} leads to a holomorphic section and further allow us to establish Proposition 1.28 (injectivity criterion). If p is unramified in F, our Θ-operators coincide exactly with those of Andreatta–Goren, and in that case everything that we prove here has already been proved in [2] (see also [10]).

The construction of $\Theta_\tau^{\mathrm{AG}}$ goes via the Kummer cover. By definition, the ordinary locus X^{ord} of X^{Ra} is endowed with a Galois cover $X^{\mathrm{ord}}(\mu_{(p)}) \to X^{\mathrm{ord}}$ with group $(\mathfrak{o}/(p))^\times$, where $X(\mu_{(p)})$ is the Deligne–Pappas moduli space of level $p\mathfrak{n}$. Taking the quotient by the p-Sylow subgroup yields a cover $\pi : X^{\mathrm{Kum}} \to X^{\mathrm{ord}}$ with group $\prod_{\mathfrak{p}|p}(\mathfrak{o}/\mathfrak{p})^\times$, called the *Kummer cover*. Let $\widetilde{\pi} : \widetilde{X} \to X$ be the normal closure of X in X^{Kum}. It can be described explicitly using the generalised Hasse invariants as follows. For $\tau \in \overline{\Sigma}$, we write $\mathfrak{p} = \mathfrak{p}(\tau)$, $f = f_\mathfrak{p}$ and we let

$$(1.15) \qquad H_\tau = \prod_{j=0}^{f-1} (h_{\phi^{-j}\circ\tau})^{p^j}.$$

It is a modular form of weight $(p^f - 1)\widetilde{\tau}$ and X^{Kum} is obtained by adjoining a $(p^f - 1)$-th root s_τ of it for all $\tau \in \overline{\Sigma}$. The nowhere vanishing section s_τ provides a trivialisation of the line bundle $\pi^*\omega_{\widetilde{\tau}}$ over X^{Kum} (see [2, Def. 7.4]), and by definition of the normalisation, it also defines a section over $\widetilde{X}$ (see [2, Prop. 7.9]). As

$$(1.16) \qquad H_{\phi^{-1}\circ\tau}^p = H_\tau \cdot (h_\tau)^{p^f-1},$$

$H_{\phi^{-1}\circ\tau}^p/H_\tau$ is a $(p^f - 1)$-th power, this construction does not depend on the choice of $\tau \in \overline{\Sigma}_\mathfrak{p}$.

Next we describe the Kodaira–Spencer maps. By [17, Thm. 2.9] there is a decomposition

$$\Omega^1_{X/\mathbb{F}} = \bigoplus_{\tau \in \overline{\Sigma}} \Omega^1_{X/\mathbb{F},\tau} \tag{1.17}$$

where each $\Omega^1_{X/\mathbb{F},\tau}$ is endowed with a filtration whose successive subquotients are naturally isomorphic to $\omega^{\otimes 2}_{\tau,i}$ with $1 \leqslant i \leqslant e_{\mathfrak{p}}$ for $\mathfrak{p} = \mathfrak{p}(\tau)$ in descending order, $i.e.$, $\omega^{\otimes 2}_{\widetilde{\tau}} = \omega^{\otimes 2}_{\tau,e_{\mathfrak{p}}}$ is naturally a quotient. As the map $\pi : X^{\mathrm{Kum}} \to X^{\mathrm{ord}}$ is étale, we have $\Omega^1_{X^{\mathrm{Kum}}/\mathbb{F}} = \pi^* \Omega^1_{X^{\mathrm{ord}}/\mathbb{F}}$, the elements of which we view as meromorphic sections of the sheaf $\widetilde{\pi}^* \Omega^1_{X/\mathbb{F}}$ over $\widetilde{X}$. Given a section of $\widetilde{\pi}^* \Omega^1_{X/\mathbb{F}}$, we denote by a subscript $\tau \in \overline{\Sigma}$ its projection onto the τ-component via (1.17). Consider the surjective map

$$\mathrm{KS}_\tau : \widetilde{\pi}^* \Omega^1_{X/\mathbb{F},\tau} \to \widetilde{\pi}^* \omega^{\otimes 2}_{\widetilde{\tau}}.$$

Definition 1.23. *Let* $k = \sum_{\sigma \in \Sigma} k_\sigma \in \mathbb{Z}[\Sigma]$ *and* $k_\tau = \sum_{\sigma \in \Sigma_\tau} k_\sigma$ *for* $\tau \in \overline{\Sigma}$. *Let* $f \in M^{\mathrm{Katz}}_k(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$. *Recall that* $f/H^{\mathrm{RX}}_k \in M^{\mathrm{AG}}_{\bar{k}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ *(see Lemma 1.7). We put*

$$H^{\mathrm{AG}}_k = \prod_{\tau \in \overline{\Sigma}} s^{k_\tau}_\tau, \text{ and } H_k = H^{\mathrm{AG}}_k \cdot \pi^*(H^{\mathrm{RX}}_k).$$

Similarly to [2, Def. 7.19]*, we further put*

$$r(f) = \pi^*\big(f/H^{\mathrm{RX}}_k\big)/H^{\mathrm{AG}}_k = \pi^*(f)/H_k \in \mathrm{H}^0(X^{\mathrm{Kum}}, \mathcal{O}_{X^{\mathrm{Kum}}})$$

where we restricted f *and* H^{RX}_k *to* X^{ord}.

Definition 1.24. *For* $\tau \in \overline{\Sigma}$*, we define the* generalised Θ-operator *acting on* $f \in M^{\mathrm{Katz}}_k(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ *as*

$$\Theta_\tau(f) = \mathrm{KS}_\tau\big(d(r(f))_\tau\big) \cdot H_k \cdot \pi^*(h_\tau) = H^{\mathrm{RX}}_k \cdot \Theta^{\mathrm{AG}}_\tau\left(\frac{f}{H^{\mathrm{RX}}_k}\right),$$

viewed as an element of $\mathrm{H}^0(X^{\mathrm{Kum}}, \pi^* \omega^{\otimes k'})$ *where*

(1) If $\tau \neq \phi^{-1} \circ \tau$*, then* $k'_\sigma = \begin{cases} k_\sigma + 1 & \text{if } \sigma = \sigma_{\tau, e_{\mathfrak{p}(\tau)}} = \widetilde{\tau}, \\ k_\sigma + p & \text{if } \sigma = \sigma_{\phi^{-1} \circ \tau, e_{\mathfrak{p}(\tau)}} = \widetilde{\phi^{-1} \circ \tau}, \\ k_\sigma & \text{otherwise.} \end{cases}$

(2) If $\tau = \phi^{-1} \circ \tau$*, then* $k'_\sigma = \begin{cases} k_\sigma + p + 1 & \text{if } \sigma = \sigma_{\tau, e_{\mathfrak{p}(\tau)}} = \widetilde{\tau}, \\ k_\sigma & \text{otherwise.} \end{cases}$

We will now prove that $\Theta_\tau(f)$ yields an element of $M^{\mathrm{Katz}}_{k'}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$. In order to prove this, we proceed as in [2] to calculate the poles of $d(r(f))_\tau$ along the divisors of the generalised Hasse invariants.

Using the trivialisation of the line bundles $\pi^* \omega_{\widetilde{\tau}}$ given by the sections s_τ and the generalised Hasse invariants $h_{\tau,i}$'s for $i > 1$, we get trivialisations of $\pi^* \omega_{\tau,i}$ for all $\tau \in \overline{\Sigma}$ and $1 \leqslant i \leqslant e_{\mathfrak{p}(\tau)}$. Using these trivialisations we can view the pullbacks $\widetilde{\pi}^* h_{\tau,i}$ and $\widetilde{\pi}^* h_\tau$ of Hasse invariants as functions over $\widetilde{X}$

(see [2, §12.32] for more details), whose differentials are denoted by $d(h_{\tau,i})$ and $d(h_\tau)$, respectively (viewed as meromorphic sections of $\widetilde{\pi}^*\Omega^1_{X/\mathbb{F}}$).

For $\tau' \in \overline{\Sigma}$, we let $\widetilde{Z}$ be an irreducible component of the effective Weil divisor of $\widetilde{X}$ associated to $\widetilde{\pi}^*(h_{\tau'})$ (see §1.2). From the construction of $\widetilde{X}$ and [2, §9.3, Prop. 9.4] (see also [2, §12.32]), we can choose a uniformiser δ at the generic point of $\widetilde{Z}$ such that $\delta^{p^{f_{\mathfrak{p}(\tau')}}-1} = H_{\tau'}$ (see (1.15) for the definition of $H_{\tau'}$). We fix this choice from now on and let v_δ be the corresponding normalised discrete valuation. For the sake of readability, we will often drop $\widetilde{\pi}^*$ from the notation when pulling back Hilbert modular forms, especially generalised Hasse invariants; for instance, we usually write $v_\delta(h_\tau)$ for $v_\delta(\widetilde{\pi}^*(h_\tau))$.

We will first calculate $v_\delta((d\delta)_\tau)$, where $(d\delta)_\tau$ is viewed as a meromorphic section of $\widetilde{\pi}^*\Omega^1_{X/\mathbb{F},\tau}$ over $\widetilde{X}$. We also prove some complementary results which will be used in the proof of the injectivity criterion.

Lemma 1.25. *(i) Let $\tau \in \overline{\Sigma}$ different from τ'. Then $(d\delta)_\tau = 0$.*

(ii) There is a unique $1 \leqslant i_0 \leqslant e_{\mathfrak{p}(\tau')}$ (depending on $\widetilde{Z}$) such that $v_\delta(h_{\tau',i_0}) = p^{f_{\mathfrak{p}(\tau')}} - 1$ and $v_\delta(h_{\tau',i}) = 0$ for all $i \neq i_0$. Moreover, $v_\delta(h_{\tau'}) = p^{f_{\mathfrak{p}(\tau')}} - 1$ and $v_\delta(h_\tau) = 0$ if $\tau \neq \tau'$.

(iii) $v_\delta(d(h_{\tau',i})) \geqslant 0$ for all i and for i_0 found in (ii), $v_\delta(d(h_{\tau',i_0})) = 0$.

(iv) $v_\delta(s_{\tau'}) = 1$, $v_\delta(s_\tau) = p^j$ if $\tau = \phi^j \circ \tau'$ and $v_\delta(s_\tau) = 0$ if $\tau \neq \phi^j \circ \tau'$ for any integer j.

(v) $v_\delta((d\delta)_{\tau'}) = 2 - p^{f_{\mathfrak{p}(\tau')}}$.

(vi) $(d\delta)_{\tau'} = D + g \cdot (d(h_{\tau',i_0}))_{\tau'}$ where $g = -\delta^{2-p^{f_{\mathfrak{p}(\tau')}}} \cdot \left(\prod_{j=1}^{f_{\mathfrak{p}(\tau')}-1}(h_{\phi^{-j}\circ\tau'})^{p^j} \right) \cdot \left(\prod_{j\neq i_0} h_{\tau',j} \right)$ and D is a meromorphic section of $\widetilde{\pi}^*\Omega^1_{X/\mathbb{F},\tau'}$ such that $v_\delta(D) \geqslant 0$.

(vii) $\mathrm{KS}_{\tau'}(d(h_{\tau',e_{\mathfrak{p}(\tau')}})_{\tau'}) \neq 0$ and if $\widetilde{Z}$ is an irreducible component of the effective Weil divisor of $\widetilde{X}$ associated with $\widetilde{\pi}^*(h_{\tau',e_{\mathfrak{p}(\tau')}})$, then $v_\delta(\mathrm{KS}_{\tau'}(d(h_{\tau',e_{\mathfrak{p}(\tau')}})_{\tau'})s_{\tau'}^{-2}) = -2$.

Proof. Recall that we have chosen δ such that $\delta^{p^{f_{\mathfrak{p}(\tau')}}-1} = H_{\tau'}$. Hence

$$-\delta^{p^{f_{\mathfrak{p}(\tau')}}-2}d\delta = \left(\prod_{j=1}^{f_{\mathfrak{p}(\tau')}-1}(h_{\phi^{-j}\circ\tau'})^{p^j} \right) \cdot d(h_{\tau'}).$$

Since X^{Ra} is Zariski dense in X, it follows from [2, Lem. 12.34] that $(d(h_{\tau'}))_\tau = 0$ if $\tau \neq \tau'$. Hence, we get $d\delta = (d\delta)_{\tau'}$, which implies that $(d\delta)_\tau = 0$ if $\tau \neq \tau'$. Now

$$d(h_{\tau'}) = d\left(\prod_{i=1}^{e_{\mathfrak{p}(\tau')}} h_{\tau',i} \right) = \sum_{i=1}^{e_{\mathfrak{p}(\tau')}} \left(\prod_{j\neq i} h_{\tau',j} \right) \cdot d(h_{\tau',i}).$$

Since δ is a uniformiser at the generic point of $\widetilde{Z}$, there is a unique i_0 such that $v_\delta(h_{\tau',i_0}) > 0$. Note that $v_\delta(h_{\tau',i}) = 0$ for $i \neq i_0$ and $v_\delta(d(h_{\tau',i})) \geqslant 0$ for all i. Moreover, it follows from [17, Thm. 3.10] that $v_\delta(d(h_{\tau',i_0})) = 0$. So (ii) and (iii) follow from the discussion above. Combining this with (1.16) gives (iv). Hence, $v_\delta(d(h_{\tau'})) = v_\delta(\prod_{i\neq i_0} h_{\tau',i}(d(h_{\tau',i_0}))) = 0$ from which (v) follows and combining this with (ii) and (iii) gives us (vi).

We will now prove Statement (vii). Recall that, by [17, Thm 2.9], $\Omega^1_{X/\mathbb{F},\tau'}$ admits a canonical filtration whose successive subquotients are (isomorphic to) $\omega^{\otimes 2}_{\tau',i}$ with $1 \leqslant i \leqslant e_{\mathfrak{p}(\tau')}$. Recall that $\mathrm{KS}_{\tau'}$ is the surjective map from $\widetilde{\pi}^*\Omega^1_{X/\mathbb{F},\tau'}$ onto its first subquotient $\widetilde{\pi}^*\omega^{\otimes 2}_{\tau',e_{\mathfrak{p}(\tau')}}$. On the other hand, by [17, Thm. 3.10], $\Omega^1_{Z_{\tau',e_{\mathfrak{p}(\tau')}}/\mathbb{F},\tau'}$ admits a canonical filtration whose successive subquotients are (isomorphic to) $\omega^{\otimes 2}_{\tau',i}$ with $1 \leqslant i < e_{\mathfrak{p}(\tau')}$. Here $Z_{\tau',e_{\mathfrak{p}(\tau')}} \subset X$ is the divisor of $h_{\tau',e_{\mathfrak{p}(\tau')}}$. Therefore, we conclude that $\mathrm{KS}_{\tau'}(d(h_{\tau',e_{\mathfrak{p}(\tau')}})_{\tau'}) \neq 0$. Since $s_{\tau'}$ gives a trivialisation of the line bundle $\pi^*\omega_{\widetilde{\tau'}}$, $v_\delta(\mathrm{KS}_{\tau'}(d(h_{\tau',e_{\mathfrak{p}(\tau')}})_{\tau'})s^{-2}_{\tau'})$ is well defined. We conclude $v_\delta(\mathrm{KS}_{\tau'}(d(h_{\tau',e_{\mathfrak{p}(\tau')}})_{\tau'})s^{-2}_{\tau'}) = -2$ by combining [17, Thm. 3.10] with (iii) and (iv) (see also [2, Prop. 12.37]). This concludes the proof of the lemma. $\qquad\square$

In order to compute $v_\delta(d(r(f))_\tau)$, it is sufficient to work in the discrete valuation ring obtained by localising at the generic point of $\widetilde{Z}$. Letting $r(f) = \frac{u}{\delta^n}$, with $v_\delta(u) = 0$, we have

$$(1.18) \qquad d(r(f))_\tau = \frac{(du)_\tau}{\delta^n} - \frac{nu(d\delta)_\tau}{\delta^{n+1}}.$$

Lemma 1.26. *Let $\tau \in \overline{\Sigma}$ and let $f \in M^{\mathrm{Katz}}_k(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$. Then*

(i) $v_\delta((du)_\tau) \geqslant \inf\{0, v_\delta((d\delta)_\tau)\}$,

(ii) $v_\delta(d(r(f))_\tau) \geqslant v_\delta(r(f))$, *if $\tau \neq \tau'$,*

(iii) $v_\delta(d(r(f))_{\tau'}) \geqslant v_\delta(r(f)) - (p^{f_{\mathfrak{p}(\tau')}} - 2)$, *if $p | v_\delta(r(f))$,*

(iv) $v_\delta(d(r(f))_{\tau'}) = v_\delta(r(f)) - (p^{f_{\mathfrak{p}(\tau')}} - 1)$, *if $p \nmid v_\delta(r(f))$.*

Proof. Let $\tau \in \overline{\Sigma}$. The proof of (i) is similar to the proof of [2, Prop. 12.35] and we reproduce parts of it here. Let B (resp. $\widetilde{B}$) be the local ring of the generic point of $\widetilde{\pi}(\widetilde{Z})$ (resp. of $\widetilde{Z}$). As in [2, Cor. 9.6], we have $B \subset B^{\mathrm{et}} \subset \widetilde{B}$ where B^{et} is étale over B and $\widetilde{B} = B^{\mathrm{et}}[\delta]$. Writing $u = \sum^{p^{f_{\mathfrak{p}(\tau')}}-2}_{h=0} u_h\delta^h$ with $u_h \in B^{\mathrm{et}}$, we have

$$(du)_\tau = \sum^{p^{f_{\mathfrak{p}(\tau')}}-2}_{h=0} \left(\delta^h(du_h)_\tau + hu_h\delta^{h-1}(d\delta)_\tau\right).$$

Now $\delta^h(du_h)_\tau$ lies in $\widetilde{B} \otimes_{B^{\mathrm{et}}} \Omega^1_{B^{\mathrm{et}}/\mathbb{F}}$, which is the same as $\widetilde{\pi}^*\Omega^1_{B/\mathbb{F}}$ since B^{et} is étale over B. Hence $\delta^h(du_h)_\tau$ has no poles and (i) follows. Combining this inequality with (1.18) and Lemma 1.25(i), (v) gives us the other parts of the lemma. $\qquad\square$

Finally we are ready to prove that $\Theta_\tau(f)$ is also a mod p Hilbert modular form.

Proposition 1.27. *Let $\tau \in \overline{\Sigma}$, $f \in M^{\mathrm{Katz}}_k(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ and k' be as in Definition 1.24. Then $\Theta_\tau(f)$ descends to a global section of the line bundle $\omega^{\otimes k'}$ over X^{ord}, and further extends to a section over X, yielding an element $\Theta_\tau(f) \in M^{\mathrm{Katz}}_{k'}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$.*

Proof. The descent follows by applying [2, Thm. 12.39] to f/H^{RX}_k.

As KS_τ is a surjective map of locally free sheaves with a locally free kernel over the normal scheme $\widetilde{X}$, the orders of the poles of $\mathrm{KS}_\tau(d(r(f))_\tau)$ are less than or equal to the orders of the

poles of $d(r(f))_\tau$, i.e., $v_\delta(\mathrm{KS}_\tau(d(r(f))_\tau)) \geqslant v_\delta(d(r(f))_\tau)$ (see the proof of [2, Prop. 12.37] for more details). Note that $r(f) \cdot H_k = \pi^*(f)$ has no poles on $\widetilde{X}$, i.e., $v_\delta(\pi^* f) \geqslant 0$. Combining Lemma 1.25(ii) and Lemma 1.26, we get that $\Theta_\tau(f)$ has no poles over $\widetilde{X}$. Hence, the section obtained by descending from X^{Kum} to X^{ord} extends to all of X and is thus a Hilbert modular form. $\qquad\square$

The effect of Θ_τ on the geometric q-expansions of Hilbert modular forms will be used in §2.2 and can be described as follows. The identification (1.4), used in defining the geometric q-expansion $\sum_{\xi \in \mathfrak{c}_+ \cup \{0\}} a_\xi(f) q^\xi$ of f at the cusp $\infty_\mathfrak{c}$, allows one to consider the map

$$\bar{\tau}_\mathfrak{c} : \mathbb{F} \otimes \mathfrak{c} \xrightarrow{\sim} \mathbb{F} \otimes \mathfrak{o} \twoheadrightarrow \mathbb{F}[x]/(x^{e_{\mathfrak{p}(\tau)}}) \twoheadrightarrow \mathbb{F},$$

where the middle map is given by the idempotent at τ. By [2, Cor. 12.40] we obtain the following q-expansion at the cusp $\infty_\mathfrak{c}$:

$$(1.19) \qquad\qquad \Theta_\tau(f) = \sum_{\xi \in \mathfrak{c}_+} \bar{\tau}_\mathfrak{c}(1 \otimes \xi) a_\xi q^\xi.$$

The proof of our main theorem uses the injectivity of Θ_τ on certain mod p Hilbert modular forms.

Proposition 1.28. *Let $f \in M_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ and let $\tau \in \overline{\Sigma}_\mathfrak{p}$. Suppose $p \nmid k_{\tau,e_\mathfrak{p}}$ and $h_{\tau,e_\mathfrak{p}}$ does not divide f. Then $\Theta_\tau(f) \neq 0$.*

In particular, if the weight of f is minimal at $\mathfrak{p}$, and $p \nmid k_{\tau,e_\mathfrak{p}}$, then $\Theta_\tau(f) \neq 0$.

Proof. We follow the proof of [2, Prop. 15.10]. Let δ be the uniformiser at the generic point of an irreducible component of the Weil divisor of $\widetilde{X}$ attached to $\widetilde{\pi}^* h_{\tau,e_\mathfrak{p}}$ chosen just before Lemma 1.25. As $v_\delta(\widetilde{\pi}^*(f)) = 0$, using Lemma 1.25(ii),(iv) (see also [2, Prop. 15.9]) we deduce

$$n := v_\delta(r(f)) = -\sum_{j=0}^{f-1} p^j k_{\phi^j \circ \tau} - (p^{f_\mathfrak{p}} - 1) \sum_{j=1}^{e_\mathfrak{p}-1} k_{\tau,j} \equiv -k_{\tau,e_\mathfrak{p}} \pmod{p}.$$

Hence $p \nmid n$ and Lemma 1.26 (i) shows that the right most term in (1.18) has a strictly lower valuation than the other term on the right hand side. Thus, Lemma 1.25 (vi) shows that

$$d(r(f))_\tau = D' - \frac{nu\delta^{2-p^{f_{\mathfrak{p}(\tau)}}} \cdot \big(\prod_{j=1}^{f_{\mathfrak{p}(\tau)}-1}(h_{\phi^j \circ \tau})^{p^j}\big) \cdot \big(\prod_{j \neq e_\mathfrak{p}} h_{\tau,j}\big)}{\delta^{n+1}}(dh_{\tau,e_\mathfrak{p}})_\tau,$$

where D' is a meromorphic section of $(\widetilde{\pi}^* \Omega^1_{X/\mathbb{F}})_\tau$ and the right most term has a strictly smaller valuation than D'. Combining this with Lemma 1.25 (vii), we get that $\mathrm{KS}_\tau(d(r(f))_\tau) \neq 0$. This implies that $\Theta_\tau(f) \neq 0$. $\qquad\square$

Remark 1.29. *When p is unramified in F, Proposition 1.28 can also be deduced from [10, Thm. 8.2.2] whose proof is different. Furthermore, in [10, Thm. 9.8.2], Diamond and Sasaki also determine the kernel of Θ_τ in terms of the partial Frobenius operator at τ that they define. Meanwhile, the case when p is ramified in F has been treated in [7]. Proposition 1.28 follows from [7, Thm. 5.2.1] and the kernel of Θ_τ is described in terms of partial Frobenius operators in [7, Cor. 9.1.2].*

2. DOUBLING AND HECKE ALGEBRAS

2.1. **Hilbert modular forms of parallel weight** 1. It is important to distinguish between Katz Hilbert modular forms defined on the fine moduli space and those on the coarse quotient by the action of the totally positive units of $\mathfrak{o}$. The latter enjoy the good Hecke theory for $\mathrm{GL}(2)$ and are the natural objects to study in relation with two dimensional Galois representations (see [14]). In this section, we will define Hilbert modular forms of parallel weight building on Definition 1.1. Even though we give a definition valid in all levels $\mathfrak{n}$ that are prime to p, we nevertheless need to consider the following condition (which is stronger than the one we imposed in §1) expressing that $\mathfrak{n}$ is sufficiently divisible:

(2.20)
$\mathfrak{n}$ is divisible by a prime above a prime number q splitting completely in $F(\sqrt{\epsilon}\,|\,\epsilon \in \mathfrak{o}_+^\times)$, and such that $q \equiv -1 \pmod{4\ell}$ for all prime numbers ℓ such that $[F(\mu_\ell) : F] = 2$.

This condition ensures that $\mathcal{X}^{\mathrm{DP}}$ is a scheme on which $[\epsilon] \in E = \mathfrak{o}_+^\times/\{\epsilon \in \mathfrak{o}^\times|\epsilon - 1 \in \mathfrak{n}\}^2$ acts properly and discontinuously by sending (A, λ, μ) to $(A, \epsilon\lambda, \mu)$ (see [12, Lem. 2.1(iii)]). For any $\mathfrak{c} \in \mathcal{C}$, any $\mathbb{Z}_p$-algebra R, and any parallel weight k, this induces an action of E on $M_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; R)$, whose invariants are denoted by $M_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; R)^E$. The following definition is equivalent to the one used in [14, §2.2].

Definition 2.1. *If $\mathfrak{n}$ satisfies* (2.20)*, then the space of Hilbert modular forms over a $\mathbb{Z}_p$-algebra R of (parallel) weight $\kappa \in \mathbb{Z}$ and level $\mathfrak{n}$ is given by*

$$M_\kappa(\mathfrak{n}, R) = \bigoplus_{\mathfrak{c}\in\mathcal{C}} M_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; R)^E,$$

where $k = \sum_{\sigma\in\Sigma} \kappa\sigma$. For a general level $\mathfrak{n}$, let $\mathfrak{q}_1 \neq \mathfrak{q}_2$ be primes such that both $\mathfrak{n}\mathfrak{q}_1$ and $\mathfrak{n}\mathfrak{q}_2$ satisfy (2.20) *and define*

$$M_\kappa(\mathfrak{n}, R) = M_\kappa(\mathfrak{n}\mathfrak{q}_1, R) \cap M_\kappa(\mathfrak{n}\mathfrak{q}_2, R),$$

where the intersection can be taken in $M_\kappa(\mathfrak{n}\mathfrak{q}_1\mathfrak{q}_2, R)$. Note that the primes $\mathfrak{q}_1$ and $\mathfrak{q}_2$ can be chosen from a set of primes of positive density and that the definition does not depend on this choice.

For $f \in M_\kappa(\mathfrak{n}, R)$, we let $\sum_{\mathfrak{b}\in\mathcal{I}\cup\{(0)\}} a(\mathfrak{b}, f)q^{\mathfrak{b}}$ be the adelic q-expansion of f, where $\mathcal{I}$ denotes the group of fractional ideals of F (see [14, §2.6]).

We denote by $S_\kappa(\mathfrak{n}, R)$ the R-submodule of $M_\kappa(\mathfrak{n}, R)$ consisting of Hilbert modular cuspforms.

For $f \in M_\kappa(\mathfrak{n}, R)$ and $\mathfrak{c} \in \mathcal{C}$, we will let $f_{\mathfrak{c}}$ denote the corresponding E-invariant element of $M_k^{\mathrm{Katz}}(\mathfrak{c}, \mathfrak{n}; \mathbb{F})$ or, equivalently, its geometric q-expansion at the cusp $\infty_{\mathfrak{c}}$ (see [14, §2.5]). Recall that when $\mathfrak{n}$ satisfies (2.20), $M_\kappa(\mathfrak{n}, R)$ is endowed with Hecke and diamond operators (see [14, §3.1-3.3]). When $\mathfrak{n}$ is not sufficiently divisible, Hecke and diamond operators exist nonetheless because they stabilise the intersection $M_\kappa(\mathfrak{n}\mathfrak{q}_1, R) \cap M_\kappa(\mathfrak{n}\mathfrak{q}_2, R)$, where the auxiliary primes $\mathfrak{q}_1, \mathfrak{q}_2$ may be chosen appropriately. When it is not clear from the context, a superscript between brackets indicates the

weight of the space of Hilbert modular forms on which an operator acts, e.g. $T_{\mathfrak{p}}^{(1)}$. Since we are interested in torsion coefficients, we let $M_\kappa(\mathfrak{n}, K/\mathcal{O}) = \varinjlim_n M_\kappa(\mathfrak{n}, \mathcal{O}/\varpi^n)$, where the inductive limit is taken by identifying $M_\kappa(\mathfrak{n}, \mathcal{O}/\varpi^n)$ with the subspace $M_\kappa(\mathfrak{n}, \mathcal{O}/\varpi^n) \otimes_\mathcal{O} (\varpi\mathcal{O})$ of $M_\kappa(\mathfrak{n}, \mathcal{O}/\varpi^{n+1})$.

2.2. **Doubling.** We shall rely on the following lifting result.

Lemma 2.2. *Suppose that* $\mathfrak{n}$ *satisfies* (2.20)*. There exists a* $\kappa_0 \in \mathbb{Z}$ *such that for all* $\kappa \geqslant \kappa_0$ *and all* $n \in \mathbb{N}$*, the natural map*

$$M_\kappa(\mathfrak{n}, \mathcal{O}) \otimes_\mathcal{O} \mathcal{O}/\varpi^n \to M_\kappa(\mathfrak{n}, \mathcal{O}/\varpi^n)$$

is a Hecke equivariant isomorphism.

Proof. The proof of [14, Lem. 2.2] works unchanged after replacing $\mathbb{Z}_p$ by $\mathcal{O}$ and p by ϖ^n. $\square$

We also need a generalisation of the total Hasse invariant modulo ϖ^n.

Lemma 2.3. *For every* $n \in \mathbb{N}$*, there is a* $\kappa_n \in \mathbb{N}$ *such that* $(\kappa_n - 1)$ *is a multiple of* $(p-1)p^{n-1}$*, and a modular form* $h_n \in M_{\kappa_n-1}(\mathfrak{o}, \mathcal{O}/\varpi^n)$ *having* q-*expansion equal to* 1 *at* $\infty_\mathfrak{c}$ *for all* $\mathfrak{c} \in \mathcal{C}$*. In particular, it does not vanish at any cusp.*

Proof. Let $h \in M_{p-1}(\mathfrak{o}, \mathbb{F})$ be the usual Hasse invariant (see [14, §3.4]). Note that since it exists for every level satisfying (2.20), it exists in level $\mathfrak{o}$. For r such that $r(p-1)$ is big enough to apply Lemma 2.2, the modular form $h^r \in M_{r(p-1)}(\mathfrak{o}, \mathcal{O})$ has q-expansion congruent to 1 modulo ϖ at each cusp $\infty_\mathfrak{c}$, $\mathfrak{c} \in \mathcal{C}$. A big enough power of it satisfies the required congruence relation and condition on the weight. $\square$

The theory of generalised Θ-operators presented in §1.5 allows us to prove the following result.

Lemma 2.4. *Assume that* $\mathfrak{n}$ *satisfies* (2.20)*. Then there does not exist any* $0 \neq f \in M_1(\mathfrak{n}, \mathbb{F})$ *such that* $f_\mathfrak{c}$ *has minimal weight at a fixed prime* $\mathfrak{p}$ *dividing* p *(see Corollary 1.14) for all* $\mathfrak{c} \in \mathcal{C}$ *and such that* $a(\mathfrak{b}, f) = 0$ *for all ideals* $\mathfrak{b} \subset \mathfrak{o}$ *not divisible by* $\mathfrak{p}$*.*

Proof. The minimality of the weight at $\mathfrak{p}$ implies that there exists a $\tau \in \overline{\Sigma}_\mathfrak{p}$ such that $h_{\widetilde{\tau}}$ does not divide $f_\mathfrak{c}$ (the proof of Corollary 1.14 implies that this is true for all $\tau \in \overline{\Sigma}_\mathfrak{p}$). Let $\mathfrak{b} = (\xi)\mathfrak{c}^{-1}$. Then, by definition, $a_\xi = a(\mathfrak{b}, f)$ and this is zero unless $\mathfrak{p} \mid \mathfrak{b}$, in which case $\mathfrak{p} \mid (\xi)$. Thus, it follows that $\overline{\tau}_\mathfrak{c}(1 \otimes \xi) = 0$. By (1.19), this shows that the geometric q-expansion of $\Theta_\tau(f_\mathfrak{c})$ vanishes at $\infty_\mathfrak{c}$ for all $\mathfrak{c} \in \mathcal{C}$, *i.e.*, $\Theta_\tau(f_\mathfrak{c}) = 0$, contradicting the injectivity criterion from Proposition 1.28. $\square$

For $\mathfrak{p} \mid p$ and $n \in \mathbb{N}$, we define the $V_\mathfrak{p}$-operator by (see [6], improving on and correcting previous works such as [15] and [14], for the definition of $T_\mathfrak{p}^{(1)}$)

$$V_{\mathfrak{p},n} = \langle \mathfrak{p} \rangle^{-1}(h_n T_\mathfrak{p}^{(1)} - T_\mathfrak{p}^{(\kappa_n)} h_n)$$

with h_n and κ_n from Lemma 2.3. A simple computation on q-expansions (see [14, Prop. 3.6]) shows that $V_{\mathfrak{p},n}$ has the following effect on adelic q-expansions:

$$a((0), V_{\mathfrak{p},n}f) = a((0), f)[\mathfrak{p}^{-1}],$$
$$a(\mathfrak{r}, V_{\mathfrak{p},n}f) = a(\mathfrak{p}^{-1}\mathfrak{r}, f)$$

for non-zero ideals $\mathfrak{r} \subseteq \mathfrak{o}$.

Proposition 2.5. *Let $\mathfrak{p} \mid p$ be a prime and assume that $\mathfrak{n}$ satisfies (2.20).*

(i) If $f \in S_1(\mathfrak{n}, K/\mathcal{O})$ and $a(\mathfrak{b}, f) = 0$ for all ideals $\mathfrak{b} \subset \mathfrak{o}$ not divisible by $\mathfrak{p}$, then $f = 0$.

(ii) For all $n \in \mathbb{N}$, the 'doubling map'

$$(h_n, V_{\mathfrak{p},n}) : S_1(\mathfrak{n}, \mathcal{O}/\varpi^n)^{\oplus 2} \xrightarrow{(f,g) \mapsto h_n f + V_{\mathfrak{p},n} g} M_{\kappa_n}(\mathfrak{n}, \mathcal{O}/\varpi^n)$$

is injective and compatible with the Hecke operators $T_{\mathfrak{q}}$ for $\mathfrak{q} \nmid \mathfrak{n}p$. The Hecke operator $T_{\mathfrak{p}}^{(\kappa_n)}$ acts on the image by the formula $T_{\mathfrak{p}}^{(\kappa_n)} \circ (h_n, V_{\mathfrak{p},n}) = (h_n, V_{\mathfrak{p},n}) \circ \begin{pmatrix} T_{\mathfrak{p}}^{(1)} & 1 \\ -\langle \mathfrak{p} \rangle & 0 \end{pmatrix}$. In particular, the image $W_{\mathfrak{p},n}$ of $(h_n, V_{\mathfrak{p},n})$ lies in the $\mathfrak{p}$-ordinary part of $M_{\kappa_n}(\mathfrak{n}, \mathcal{O}/\varpi^n)$ and is stable under all Hecke operators $T_{\mathfrak{q}}$ for $\mathfrak{q} \nmid \mathfrak{n}p$.

If $(p-1)$ does not divide $e_{\mathfrak{p}}$, then the same statements hold after replacing the spaces $S_1(\mathfrak{n}, K/\mathcal{O})$ and $S_1(\mathfrak{n}, \mathcal{O}/\varpi^n)$ by $M_1(\mathfrak{n}, K/\mathcal{O})$ and $M_1(\mathfrak{n}, \mathcal{O}/\varpi^n)$, respectively.

Proof. (i) For $f \in S_1(\mathfrak{n}, \mathcal{O}/\varpi)$, the claim is precisely the content of Lemma 2.4, in view of Corollaries 1.14 and 1.16. The induction step from $n-1$ to n follows from the q-expansion principle and the exact sequence

$$0 \to S_1(\mathfrak{n}, \mathcal{O}/\varpi) \otimes_{\mathcal{O}} \varpi^{n-1}\mathcal{O} \to S_1(\mathfrak{n}, \mathcal{O}/\varpi^n) \to S_1(\mathfrak{n}, \mathcal{O}/\varpi^{n-1}).$$

(ii) The injectivity follows from (i) applied to the first component of an element in the kernel. The matrix is obtained from a calculation as in [14, Lem. 3.7]. $\qquad\square$

2.3. **Hecke algebras.** For $\kappa \geqslant 1$ and $n \in \mathbb{N}$, we consider the following complete Artinian (resp. Noetherian) semi-local $\mathcal{O}$-algebras

$$\mathbb{T}_n^{(\kappa)} = \mathrm{im}\left(\mathcal{O}[T_{\mathfrak{q}}, \langle \mathfrak{q} \rangle]_{\mathfrak{q} \nmid \mathfrak{n}p} \to \mathrm{End}_{\mathcal{O}}(M_\kappa(\mathfrak{n}, \mathcal{O}/\varpi^n))\right),$$

$$\mathbb{T}_{\mathrm{cusp},n}^{(\kappa)} = \mathrm{im}\left(\mathcal{O}[T_{\mathfrak{q}}, \langle \mathfrak{q} \rangle]_{\mathfrak{q} \nmid \mathfrak{n}p} \to \mathrm{End}_{\mathcal{O}}(S_\kappa(\mathfrak{n}, \mathcal{O}/\varpi^n))\right), \text{ resp.,}$$

(2.21) $$\mathbb{T}^{(\kappa)} = \mathrm{im}\left(\mathcal{O}[T_{\mathfrak{q}}, \langle \mathfrak{q} \rangle]_{\mathfrak{q} \nmid \mathfrak{n}p} \to \mathrm{End}_{\mathcal{O}}(M_\kappa(\mathfrak{n}, K/\mathcal{O}))\right) = \varprojlim_n \mathbb{T}_n^{(\kappa)},$$

$$\mathbb{T}_{\mathrm{cusp}}^{(\kappa)} = \mathrm{im}\left(\mathcal{O}[T_{\mathfrak{q}}, \langle \mathfrak{q} \rangle]_{\mathfrak{q} \nmid \mathfrak{n}p} \to \mathrm{End}_{\mathcal{O}}(S_\kappa(\mathfrak{n}, K/\mathcal{O}))\right) = \varprojlim_n \mathbb{T}_{\mathrm{cusp},n}^{(\kappa)}.$$

Note that they all contain $\langle \mathfrak{p} \rangle$ for $\mathfrak{p} \mid p$ since p is relatively prime to $\mathfrak{n}$. Moreover, the restriction to the cusp space gives surjective morphisms $\mathbb{T}_n^{(\kappa)} \twoheadrightarrow \mathbb{T}_{\mathrm{cusp},n}^{(\kappa)}$ and $\mathbb{T}^{(\kappa)} \twoheadrightarrow \mathbb{T}_{\mathrm{cusp}}^{(\kappa)}$. We also consider the torsion free Hecke $\mathcal{O}$-algebra:

$$\mathbb{T}_{\mathcal{O}}^{(\kappa)} = \mathrm{im}\left(\mathcal{O}[T_{\mathfrak{q}}, \langle \mathfrak{q} \rangle]_{\mathfrak{q} \nmid \mathfrak{n}p} \to \mathrm{End}_{\mathcal{O}}(M_\kappa(\mathfrak{n}, \mathcal{O}))\right).$$

Let I_n be the annihilator of $\mathbb{T}_{\mathcal{O}}^{(\kappa)}$ acting on $M_\kappa(\mathfrak{n}, \mathcal{O}) \otimes_{\mathcal{O}} (\mathcal{O}/\varpi^n)$. Then we have natural surjective ring homomorphisms

$$\mathbb{T}_n^{(\kappa)} \twoheadrightarrow \mathbb{T}_{\mathcal{O}}^{(\kappa)}/I_n \text{ and } \mathbb{T}^{(\kappa)} \twoheadrightarrow \mathbb{T}_{\mathcal{O}}^{(\kappa)},$$

the latter coming from the fact that the intersection $\bigcap_n I_n$ is zero. For sufficiently large κ, both homomorphisms are isomorphisms due to Lemma 2.2. However, this need no longer be true in our principal case of interest $\kappa = 1$ since the inclusions

$$M_1(\mathfrak{n}, \mathcal{O}) \otimes_{\mathcal{O}} (\mathcal{O}/\varpi^n) \hookrightarrow M_1(\mathfrak{n}, \mathcal{O}/\varpi^n) \text{ and } M_1(\mathfrak{n}, \mathcal{O}) \otimes_{\mathcal{O}} (K/\mathcal{O}) \hookrightarrow M_1(\mathfrak{n}, K/\mathcal{O})$$

need not be isomorphisms, in general. The kernel of $\mathbb{T}^{(1)} \twoheadrightarrow \mathbb{T}_{\mathcal{O}}^{(1)}$ is a finitely generated torsion $\mathcal{O}$-module, which is isomorphic to the kernel of $\mathbb{T}_n^{(1)} \to \mathbb{T}_{\mathcal{O}}^{(1)}/I_n$ for $n \in \mathbb{N}$ sufficiently large. Recall that multiplication by the Hasse invariant h_n allows us to see $M_1(\mathfrak{n}, \mathcal{O}/\varpi^n)$ inside $M_{\kappa_n}(\mathfrak{n}, \mathcal{O}/\varpi^n)$ equivariantly for all Hecke operators $T_\mathfrak{q}$ and $\langle \mathfrak{q} \rangle$ for $\mathfrak{q} \nmid \mathfrak{n}p$, yielding a surjection $\mathbb{T}_n^{(\kappa_n)} \twoheadrightarrow \mathbb{T}_n^{(1)}$ (see Proposition 2.5). For a prime $\mathfrak{p} \mid p$, consider also the Hecke algebra:

$$(2.22) \qquad \widetilde{\mathbb{T}}_n^{(\kappa_n)} = \mathbb{T}_n^{(\kappa_n)}[T_\mathfrak{p}^{(\kappa_n)}] \subset \operatorname{End}_{\mathcal{O}}(M_{\kappa_n}(\mathfrak{n}, \mathcal{O}/\varpi^n))).$$

Corollary 2.6. *Let* $\mathfrak{p} \mid p$. *Then for any* $n \in \mathbb{N}$, *there is a surjection sending* $T_\mathfrak{p}^{(\kappa_n)}$ *to* U *(considered as a polynomial variable):*

$$\widetilde{\mathbb{T}}_n^{(\kappa_n)} \twoheadrightarrow \mathbb{T}_{\mathrm{cusp},n}^{(1)}[T_\mathfrak{p}^{(1)}, U]/\big(U^2 - T_\mathfrak{p}^{(1)}U + \langle \mathfrak{p} \rangle\big).$$

The same statement holds after replacing $\mathbb{T}_{\mathrm{cusp},n}^{(1)}$ *by* $\mathbb{T}_n^{(1)}$, *provided* $(p-1) \nmid e_\mathfrak{p}$.

Proof. The injection from Proposition 2.5 gives a morphism $\widetilde{\mathbb{T}}_n^{(\kappa_n)} \to \operatorname{End}_{\mathcal{O}}\big(S_1(\mathfrak{n}, \mathcal{O}/\varpi^n)^{\oplus 2}\big)$ of $\mathcal{O}$-algebras compatible with $T_\mathfrak{q}$ and $\langle \mathfrak{q} \rangle$ for all $\mathfrak{q} \nmid \mathfrak{n}p$ and, hence, we get a surjection $\mathbb{T}_n^{(\kappa_n)} \twoheadrightarrow \mathbb{T}_{\mathrm{cusp},n}^{(1)}$. Moreover, $T_\mathfrak{p}^{(\kappa_n)}$ acts on the image of $S_1(\mathfrak{n}, \mathcal{O}/\varpi^n)^{\oplus 2}$ via the matrix $\begin{pmatrix} T_\mathfrak{p}^{(1)} & 1 \\ -\langle \mathfrak{p} \rangle & 0 \end{pmatrix}$, whence it is annihilated by its characteristic polynomial $U^2 - T_\mathfrak{p}^{(1)}U + \langle \mathfrak{p} \rangle$ and does not satisfy any non-trivial linear relation over $\mathbb{T}_n^{(1)}[T_\mathfrak{p}^{(1)}]$, thus proving the existence of the desired homomorphism. For the surjectivity, let us observe that the image of $S_1(\mathfrak{n}, \mathcal{O}/\varpi^n)^{\oplus 2}$ is contained in the $T_\mathfrak{p}^{(\kappa_n)}$-ordinary subspace of $M_{\kappa_n}(\mathfrak{n}, \mathcal{O}/\varpi^n)$, and that the endomorphism $T_\mathfrak{p}^{(\kappa_n)} + \langle \mathfrak{p} \rangle (T_\mathfrak{p}^{(\kappa_n)})^{-1}$ of the latter space acts on the former as $\begin{pmatrix} T_\mathfrak{p}^{(1)} & 0 \\ 0 & T_\mathfrak{p}^{(1)} \end{pmatrix}$. Finally, assuming $(p-1) \nmid e_\mathfrak{p}$ allows us to apply Proposition 2.5 with $M_1(\mathfrak{n}, \mathcal{O}/\varpi^n)$ instead of $S_1(\mathfrak{n}, \mathcal{O}/\varpi^n)$, leading to the validity of the result with $\mathbb{T}_{\mathrm{cusp},n}^{(1)}$ replaced by $\mathbb{T}_n^{(1)}$. $\qquad\square$

3. PSEUDO-REPRESENTATIONS FOR WEIGHT 1 HECKE ALGEBRAS

3.1. **Pseudo-representations of degree** 2. In this section, we recall some definitions due to Chenevier [5] and Calegari–Specter [4].

Definition 3.1. *Let R be a complete Noetherian local $\mathcal{O}$-algebra with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = \mathbb{F}$ considered with its natural $\mathfrak{m}$-adic topology. An R-valued pseudo-representation of degree 2 of G_F is a tuple $P = (T, D)$ consisting of continuous maps $T, D : \mathrm{G}_F \to R$ such that*

(i) D is a group homomorphism $\mathrm{G}_F \to R^\times$,

(ii) $T(1) = 2$ and $T(gh) = T(hg) = T(g)T(h) - D(g)T(g^{-1}h)$ for all $g, h \in \mathrm{G}_F$.

We extend $T : \mathrm{G}_F \to R$ to an R-linear map $R[\mathrm{G}_F] \to R$ and we denote this map by T as well. Given $g \in \mathrm{G}_F$, we define $D(g - 1) := D(g) - T(g) + 1$.

The pseudo-representation $P = (T, D)$ is said to be unramified *at $\mathfrak{p}$ if $D(h-1) = T(g(h-1)) = 0$ for all $g \in \mathrm{G}_F$ and all $h \in \mathrm{I}_\mathfrak{p}$.*

Any continuous representation $\rho : \mathrm{G}_F \to \mathrm{GL}_2(R)$ yields a degree 2 pseudo-representation $P_\rho = (\mathrm{tr} \circ \rho, \det \circ \rho)$. The converse is true when the semi-simple representation $\bar{\rho} : \mathrm{G}_F \to \mathrm{GL}_2(\mathbb{F})$ corresponding to the residual pseudo-representation is absolutely irreducible (see [5, Thm. 2.22]). Further, if ρ is unramified outside a finite set of places S, then so is P_ρ. Again, the converse is true in the residually absolutely irreducible case. This can be seen by applying *loc. cit.* to the Galois group of the maximal extension of F unramified outside S over F.

We introduce a notion of ordinarity inspired from Calegari–Specter [4].

Definition 3.2. *Let $\widetilde{P} = (P, \alpha_\mathfrak{p})$ with $P = (T, D)$ a degree 2 pseudo-representation of G_F over R and $\alpha_\mathfrak{p} \in R$ a root of $X^2 - T(\mathrm{Frob}_\mathfrak{p})X + D(\mathrm{Frob}_\mathfrak{p}) \in R[X]$.*

We say that $\widetilde{P}$ is ordinary *at $\mathfrak{p}$ of weight $\kappa \geqslant 1$, if for all $h, h' \in \mathrm{I}_\mathfrak{p}$ and all $g \in \mathrm{G}_F$ we have*

(i) $D(h-1) = 0$ and $T(h-1) = \chi_p^{\kappa-1}(h) - 1$, where χ_p denotes the p-adic cyclotomic character,

(ii) $T\big(g(h - \chi_p^{\kappa-1}(h))(h' \mathrm{Frob}_\mathfrak{p} - \alpha_\mathfrak{p})\big) = 0$.

Remark 3.3. *Note that our notion of $\mathfrak{p}$-ordinary pseudo-representations implies the one of Calegari–Specter ([4, Def. 2.5]). Let $P = (T, D) : \mathrm{G}_F \to R^2$ be a degree 2 pseudo-representation and let $(\bar{T}, \bar{D}) : \mathrm{G}_F \to \mathbb{F}^2$ be its residual pseudo-representation. Suppose there exists a lift $\mathrm{Frob}_\mathfrak{p} \in \mathrm{G}_{F_\mathfrak{p}}$ of the arithmetic Frobenius at $\mathfrak{p}$ such that the polynomial $X^2 - \bar{T}(\mathrm{Frob}_\mathfrak{p})X + \bar{D}(\mathrm{Frob}_\mathfrak{p})$ has distinct roots in $\mathbb{F}$. Then $(P, \alpha_\mathfrak{p})$ is a $\mathfrak{p}$-ordinary pseudo-representation in the sense of Definition 3.2 if and only if it is $\mathfrak{p}$-ordinary in the sense of Calegari–Specter. However, if this hypothesis does not hold, then we expect that the two notions are not equivalent.*

Let $\overline{P} = (\overline{T}, \overline{D}) : \mathrm{G}_F \to \mathbb{F}^2$ be a fixed degree 2 pseudo-representation unramified outside $\mathfrak{n}p\infty$. Denote by $P^{\mathrm{ps}} = (T^{\mathrm{ps}}, D^{\mathrm{ps}}) : \mathrm{G}_F \to (R^{\mathrm{ps}})^2$ the universal deformation of $\overline{P}$ unramified outside $\mathfrak{n}p\infty$ in the category of complete Noetherian local $\mathcal{O}$-algebras with residue field $\mathbb{F}$ and consider the quotient

$$(3.23) \qquad R^{\mathrm{ps}}[X]/(X^2 - T^{\mathrm{ps}}(\mathrm{Frob}_\mathfrak{p})X + D^{\mathrm{ps}}(\mathrm{Frob}_\mathfrak{p})) \twoheadrightarrow R^{\mathrm{ord}}$$

which classifies pairs $(P, \alpha_\mathfrak{p})$ such that P is a deformation of $\overline{P}$ unramified outside $\mathfrak{n}p\infty$ and $(P, \alpha_\mathfrak{p})$ is ordinary at $\mathfrak{p}$ of weight κ. The universal ring R^{ord}, classifying deformations of $\overline{P}$ which are

unramified outside $\mathfrak{n}p\infty$ and are ordinary at $\mathfrak{p}$ of weight κ, is the quotient of the ring $R^{\mathrm{ps}}[X]/(X^2 - T^{\mathrm{ps}}(\mathrm{Frob}_{\mathfrak{p}})X + D^{\mathrm{ps}}(\mathrm{Frob}_{\mathfrak{p}}))$ by the ideal generated by the set

$$\left\{ D^{\mathrm{ps}}(h-1), T^{\mathrm{ps}}(h-1) - \chi_p^{\kappa-1}(h) + 1, T^{\mathrm{ps}}\big(g(h - \chi_p^{\kappa-1}(h))(h'\,\mathrm{Frob}_{\mathfrak{p}} - X)\big) \mid h, h' \in \mathrm{I}_{\mathfrak{p}}, g \in \mathrm{G}_F \right\}$$

and a direct computation shows that R^{ord} is independent of the choice of $\mathrm{Frob}_{\mathfrak{p}}$.

Note that R^{ord} is a finite R^{ps}-algebra. As R^{ps} is a local ring, it follows that R^{ord} is a semi-local ring and all of its maximal ideals contain the unique maximal ideal $\mathfrak{m}^{\mathrm{ps}}$ of R^{ps}. After going modulo $\mathfrak{m}^{\mathrm{ps}}$ in R^{ord}, it is easy to see, using the description of the ideal from the previous paragraph, that the number of maximal ideals of R^{ord} is the number of distinct $\alpha \in \mathbb{F}$ such that $(\overline{P}, \alpha)$ is a $\mathfrak{p}$-ordinary pseudo-representation of weight κ.

Now suppose $\overline{P}$ is unramified at $\mathfrak{p}$ and $\kappa \equiv 1 \pmod{p-1}$. Then we have

$$\overline{T}\big(g(h - \chi_p^{\kappa-1}(h))(h'\,\mathrm{Frob}_{\mathfrak{p}} - X)\big) = \overline{T}\big(g(h-1)h'\,\mathrm{Frob}_{\mathfrak{p}}\big) - X\overline{T}(g(h-1)) = \overline{T}(h'\,\mathrm{Frob}_{\mathfrak{p}}\,g(h-1)) = 0.$$

Here we are repeatedly using the fact that $\overline{T}(g(h-1)) = 0$ for all $g \in G_F$ and $h \in \mathrm{I}_{\mathfrak{p}}$, which is a consequence of the assumption that $\overline{P}$ is unramified at $\mathfrak{p}$. Thus, in this case, we see that $(\overline{P}, \alpha)$ is a $\mathfrak{p}$-ordinary pseudo-representation of weight κ if and only if α is a root of the polynomial $X^2 - \overline{T}(\mathrm{Frob}_{\mathfrak{p}})X + \overline{D}(\mathrm{Frob}_{\mathfrak{p}})$. Hence, in this case, R^{ord} is a semi-local Noetherian ring with two maximal ideals if the polynomial $X^2 - \overline{T}(\mathrm{Frob}_{\mathfrak{p}})X + \overline{D}(\mathrm{Frob}_{\mathfrak{p}})$ has two distinct roots and it is a local Noetherian ring otherwise.

3.2. Existence of an ordinary Hecke algebra-valued pseudo-representation.
We continue to use the notation from §2. Let $\mathfrak{m}$ be any maximal ideal of $\mathbb{T}^{(1)}$ (or equivalently of $\mathbb{T}_n^{(1)}$ for some n) and denote also by $\mathfrak{m}$ the maximal ideals of $\mathbb{T}^{(\kappa_n)}$ and $\mathbb{T}_n^{(\kappa_n)}$ defined as the pull-back of $\mathfrak{m} \subset \mathbb{T}_n^{(1)}$.

Lemma 3.4. *There exists a $\mathbb{T}_{n,\mathfrak{m}}^{(\kappa_n)}$-valued pseudo-representation $P_{n,\mathfrak{m}}^{(\kappa_n)}$ of G_F of degree 2 which is unramified at all primes $\mathfrak{q} \nmid \mathfrak{n}p$ and $P_{n,\mathfrak{m}}^{(\kappa_n)}(\mathrm{Frob}_{\mathfrak{q}}) = (T_{\mathfrak{q}}, \langle\mathfrak{q}\rangle)$. In particular, after replacing $\mathcal{O}$ by a finite unramified extension, there exists a unique semi-simple Galois representation*

$$\overline{\rho}_{\mathfrak{m}} : \mathrm{G}_F \to \mathrm{GL}_2(\mathbb{T}^{(1)}/\mathfrak{m})$$

unramified outside $\mathfrak{n}p\infty$ satisfying

$$\mathrm{tr}(\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{q}})) = T_{\mathfrak{q}} \pmod{\mathfrak{m}} \quad \text{and} \quad \det(\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{q}})) = \langle\mathfrak{q}\rangle \pmod{\mathfrak{m}}$$

for all primes $\mathfrak{q} \nmid \mathfrak{n}p$.

Proof. After enlarging K, we may assume that it contains all the eigenvalues of $\mathbb{T}^{(\kappa_n)}$ acting on $M_{\kappa_n}(\mathfrak{n}, \mathcal{O})$. The $\mathcal{O}$-algebra $\mathbb{T}^{(\kappa_n)}$ generated by the Hecke operators outside the level and p is torsion-free and reduced, hence $\mathbb{T}_{\mathfrak{m}}^{(\kappa_n)} \otimes_{\mathcal{O}} K = \prod_{g \in \mathcal{N}} K$ where $\mathcal{N}$ denotes the set of newforms occurring in $M_{\kappa_n}(\mathfrak{n}, \mathcal{O})_{\mathfrak{m}}$. As is well known, one can attach to each such eigenform g a G_F-pseudo-representation P_g of degree 2 unramified outside $\mathfrak{n}p\infty$ such that $P_g(\mathrm{Frob}_{\mathfrak{q}}) = (a(\mathfrak{q}, g), \psi_g(\mathfrak{q})\,\mathrm{N}(\mathfrak{q})^{\kappa_n-1})$ for all $\mathfrak{q} \nmid \mathfrak{n}p$, where $\langle\mathfrak{q}\rangle g = \psi_g(\mathfrak{q})g$. Since the natural homomorphism $\mathbb{T}_{\mathfrak{m}}^{(\kappa_n)} \to \mathbb{T}_{\mathfrak{m}}^{(\kappa_n)} \otimes_{\mathcal{O}} K$ is injective, in view of the Chebotarev Density Theorem, we obtain a $\mathbb{T}_{\mathfrak{m}}^{(\kappa_n)}$-valued G_F-pseudo-representation

$P_{\mathfrak{m}}^{(\kappa_n)}$ unramified outside $\mathfrak{n}p\infty$ such that $P_{\mathfrak{m}}^{(\kappa_n)}(\mathrm{Frob}_{\mathfrak{q}}) = (T_{\mathfrak{q}}, \langle \mathfrak{q} \rangle \, \mathrm{N}(\mathfrak{q})^{\kappa_n-1})$ for all $\mathfrak{q} \nmid \mathfrak{n}p$ (see [5, Cor. 1.14]).

Note that $\mathrm{N}(\mathfrak{q})^{\kappa_n-1} \equiv 1 \pmod{\varpi^n}$. Composing $P_{\mathfrak{m}}^{(\kappa_n)}$ with the surjection $\mathbb{T}_{\mathfrak{m}}^{(\kappa_n)} \twoheadrightarrow \mathbb{T}_{n,\mathfrak{m}}^{(\kappa_n)}$, we get the desired pseudo-representation. Finally $\mathbb{T}_n^{(\kappa_n)}/\mathfrak{m} = \mathbb{T}_n^{(1)}/\mathfrak{m}$ along with [5, Thm. A] finishes the proof of the lemma. $\qquad\square$

Let $R_{\mathfrak{m}}^{\mathrm{ps}}$ be the universal deformation ring of the corresponding degree 2 pseudo-representation $\overline{P}_{\mathfrak{m}} = (\mathrm{tr} \circ \overline{\rho}_{\mathfrak{m}}, \det \circ \overline{\rho}_{\mathfrak{m}})$ unramified outside $\mathfrak{n}p\infty$ in the category of complete Noetherian local $\mathcal{O}$-algebras with residue field $\mathbb{F}$ (chosen large enough in order to contain the residue field of $\mathbb{T}_{\mathfrak{m}}^{(1)}$). Using the surjection $\mathbb{T}_{\mathfrak{m}}^{(\kappa_n)} \twoheadrightarrow \mathbb{T}_{n,\mathfrak{m}}^{(\kappa_n)} \twoheadrightarrow \mathbb{T}_{n,\mathfrak{m}}^{(1)}$ and then passing to the projective limit $\mathbb{T}_{\mathfrak{m}}^{(1)} = \varprojlim_n \mathbb{T}_{n,\mathfrak{m}}^{(1)}$, we obtain the following result.

Corollary 3.5. *For any maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{(1)}$, there exists a $\mathbb{T}_{\mathfrak{m}}^{(1)}$-valued pseudo-representation $P_{\mathfrak{m}}^{(1)}$ of G_F of degree 2 which is unramified for all primes $\mathfrak{q} \nmid \mathfrak{n}p$ and $P_{\mathfrak{m}}^{(1)}(\mathrm{Frob}_{\mathfrak{q}}) = (T_{\mathfrak{q}}, \langle \mathfrak{q} \rangle)$. It yields a surjection $R_{\mathfrak{m}}^{\mathrm{ps}} \twoheadrightarrow \mathbb{T}_{\mathfrak{m}}^{(1)}$.*

Note that for a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_n^{(\kappa_n)}$, the algebra $\mathbb{T}_{n,\mathfrak{m}}^{(\kappa_n)}$ is in general only semi-local (see (2.22)). By the main result of [14], $\overline{\rho}_{\mathfrak{m}}$ is unramified at $\mathfrak{p}$, allowing us to consider the ideal

$$(3.24) \qquad \widetilde{\mathfrak{m}} = \left(\mathfrak{m}, (T_{\mathfrak{p}}^{(\kappa_n)})^2 - \mathrm{tr}(\widehat{\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{p}})})T_{\mathfrak{p}}^{(\kappa_n)} + \det(\widehat{\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{p}})}) \right) \subset \mathbb{T}_n^{(\kappa_n)},$$

where $\mathrm{tr}(\widehat{\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{p}})})$ and $\det(\widehat{\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{p}})})$ are some lifts of $\mathrm{tr}(\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{p}}))$ and $\det(\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{p}}))$, respectively in $\mathbb{T}_n^{(\kappa_n)}$. Note that the ideal $\widetilde{\mathfrak{m}}$ does not depend on the choices of these lifts.

Let $\widetilde{\mathbb{T}}_{n,\widetilde{\mathfrak{m}}}^{(\kappa_n)}$ be the completion of $\widetilde{\mathbb{T}}_n^{(\kappa_n)}$ with respect to $\widetilde{\mathfrak{m}}$. The algebra $\widetilde{\mathbb{T}}_{n,\widetilde{\mathfrak{m}}}^{(\kappa_n)}$ then has at most two local components. Let $R_{\mathfrak{m}}^{\mathrm{ord}}$ be the universal $\mathcal{O}$-algebra classifying deformations of $\overline{P}_{\mathfrak{m}}$ unramified outside primes dividing $\mathfrak{n}p\infty$ and ordinary at $\mathfrak{p}$ of weight 1 (see (3.23)).

Lemma 3.6. *There exists a $\mathfrak{p}$-ordinary $\widetilde{\mathbb{T}}_{n,\widetilde{\mathfrak{m}}}^{(\kappa_n)}$-valued pseudo-representation $\widetilde{P}_{n,\widetilde{\mathfrak{m}}}^{(\kappa_n)} = (P_{n,\mathfrak{m}}^{(\kappa_n)}, T_{\mathfrak{p}}^{(\kappa_n)})$ of degree 2 and weight 1 of G_F such that $P_{n,\mathfrak{m}}^{(\kappa_n)}(\mathrm{Frob}_{\mathfrak{q}}) = (T_{\mathfrak{q}}, \langle \mathfrak{q} \rangle)$ for all $\mathfrak{q} \nmid \mathfrak{n}p$. It yields a surjection $R_{\mathfrak{m}}^{\mathrm{ord}} \twoheadrightarrow \widetilde{\mathbb{T}}_{n,\widetilde{\mathfrak{m}}}^{(\kappa_n)}$.*

Proof. Let $\widetilde{\mathbb{T}}^{(\kappa_n)} = \mathbb{T}^{(\kappa_n)}[T_{\mathfrak{p}}^{(\kappa_n)}]$ and denote also by $\widetilde{\mathfrak{m}}$ the ideal of $\widetilde{\mathbb{T}}^{(\kappa_n)}$ defined as the pull-back of $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbb{T}}_n^{(\kappa_n)}$. Let $\widetilde{\mathbb{T}}_{\widetilde{\mathfrak{m}}}^{(\kappa_n)}$ be the completion of $\widetilde{\mathbb{T}}^{(\kappa_n)}$ with respect to $\widetilde{\mathfrak{m}}$.

We have $\widetilde{\mathbb{T}}_{\widetilde{\mathfrak{m}}}^{(\kappa_n)} \otimes_{\mathcal{O}} K = \prod_{g \in \widetilde{\mathcal{N}}} K$, where $\widetilde{\mathcal{N}}$ denotes the subset of $\mathcal{N}$ (see the proof of Lemma 3.4) consisting of newforms occurring in $M_{\kappa_n}(\mathfrak{n}, \mathcal{O})_{\widetilde{\mathfrak{m}}}$. As $\mathfrak{p}$ does not divide $\mathfrak{n}$, any $g \in \widetilde{\mathcal{N}}$ is an eigenvector for $T_{\mathfrak{p}}^{(\kappa_n)}$ (resp. $\langle \mathfrak{p} \rangle$) whose eigenvalue $a(\mathfrak{p}, g)$ (resp. $\psi_g(\mathfrak{p})$) is necessarily a p-adic unit by (3.24), *i.e.*, g is $\mathfrak{p}$-ordinary. By a result due to Hida and Wiles, when g is ordinary at all places dividing p, and to Saito [18] and Skinner [19] in general, p-adic Galois representation ρ_g attached to g is ordinary at $\mathfrak{p}$, *i.e.*, its restriction to $\mathrm{G}_{F_{\mathfrak{p}}}$ has a one-dimensional unramified quotient on which $\mathrm{Frob}_{\mathfrak{p}}$ acts by the (unique) p-adic unit root $\alpha_{\mathfrak{p},g}$ of the Hecke polynomial $X^2 - a(\mathfrak{p}, g)X + \psi_g(\mathfrak{p}) \, \mathrm{N}(\mathfrak{p})^{\kappa_n-1}$. This implies that $\alpha_{\mathfrak{p},g}$ is also a root of $X^2 - \mathrm{tr}(\rho_g)(\mathrm{Frob}_{\mathfrak{p}})X + \det(\rho_g)(\mathrm{Frob}_{\mathfrak{p}})$, for *any* choice

of a Frobenius element $\mathrm{Frob}_{\mathfrak{p}} \in \mathrm{G}_{F_{\mathfrak{p}}}$. Thus, the pseudo-representation $P_g = (\mathrm{tr}(\rho_g), \det(\rho_g))$ is $\mathfrak{p}$-ordinary of weight κ_n with respect to $\alpha_{\mathfrak{p},g}$ in the sense of Definition 3.2.

Since $\widetilde{\mathbb{T}}_{\widetilde{\mathfrak{m}}}^{(\kappa_n)}$ is a semi-local finite $\mathcal{O}$-algebra, applying Hensel's lemma to each local component shows that the polynomial $X^2 - T_{\mathfrak{p}}^{(\kappa_n)} X + \langle \mathfrak{p} \rangle \mathrm{N}(\mathfrak{p})^{\kappa_n - 1}$ admits a unique unit root U in $\widetilde{\mathbb{T}}_{\widetilde{\mathfrak{m}}}^{(\kappa_n)}$. By the Chebotarev Density Theorem, gluing the $\mathfrak{p}$-ordinary pseudo-representations $\widetilde{P}_g = (P_g, \alpha_{\mathfrak{p},g})$ for all $g \in \widetilde{\mathcal{N}}$ gives us a $\widetilde{\mathbb{T}}_{\widetilde{\mathfrak{m}}}^{(\kappa_n)}$-valued $\mathfrak{p}$-ordinary pseudo-representation $(P_{\mathfrak{m}}^{(\kappa_n)}, U)$ of weight κ_n such that $P_{\mathfrak{m}}^{(\kappa_n)}(\mathrm{Frob}_{\mathfrak{q}}) = (T_{\mathfrak{q}}, \langle \mathfrak{q} \rangle \mathrm{N}(\mathfrak{q})^{\kappa_n - 1})$ for all $\mathfrak{q} \nmid \mathfrak{n}p$. We have $\chi_p^{\kappa_n - 1}(g) \equiv 1 \pmod{\varpi^n}$ for all $g \in G_F$. Hence, the reduction of $(P_{\mathfrak{m}}^{(\kappa_n)}, U)$ to $\widetilde{\mathbb{T}}_{n,\widetilde{\mathfrak{m}}}^{(\kappa_n)}$ is a $\mathfrak{p}$-ordinary pseudo-representation of weight 1. Note that by Hensel's Lemma, U reduces to $T_{\mathfrak{p}}^{(\kappa_n)}$ in $\widetilde{\mathbb{T}}_{n,\widetilde{\mathfrak{m}}}^{(\kappa_n)}$, since the former (resp. the latter) is the unique unit root of $X^2 - T_{\mathfrak{p}}^{(\kappa_n)} X + \langle \mathfrak{p} \rangle \mathrm{N}(\mathfrak{p})^{\kappa_n - 1}$ in $\widetilde{\mathbb{T}}_{\widetilde{\mathfrak{m}}}^{(\kappa_n)}$ (resp. in $\widetilde{\mathbb{T}}_{n,\widetilde{\mathfrak{m}}}^{(\kappa_n)}$). As $\mathrm{N}(\mathfrak{q})^{\kappa_n - 1} \equiv 1 \pmod{\varpi^n}$ for all $\mathfrak{q} \nmid \mathfrak{n}p$, this completes the proof of the lemma. $\qquad\square$

3.3. **Proof of the main theorem.** In the proof of Theorem 0.1 we can assume without loss of generality that $\mathfrak{n}$ satisfies (2.20), because given any prime q, the Hecke algebra in level $\mathfrak{n}$ is a quotient of the one in level $\mathfrak{n}q$. Moreover, since the algebra $\mathbb{T}^{(1)}$ is semi-local, equal to the product of $\mathbb{T}_{\mathfrak{m}}^{(1)}$ where $\mathfrak{m}$ runs over its maximal ideals, it is enough to prove the theorem after localisation at $\mathfrak{m}$.

Recall that in Corollary 3.5 we constructed a $\mathbb{T}_{\mathfrak{m}}^{(1)}$-valued pseudo-representation $P_{\mathfrak{m}}^{(1)} = (T, D)$ of G_F, whose image under the surjective homomorphism $\mathbb{T}_{\mathfrak{m}}^{(1)} \twoheadrightarrow \mathbb{T}_{\mathrm{cusp},\mathfrak{m}}^{(1)} \twoheadrightarrow \mathbb{T}_{\mathrm{cusp},n,\mathfrak{m}}^{(1)}$ will be denoted by $P_{n,\mathfrak{m}}^{(1)} = (T_n, D_n)$, for $n \in \mathbb{N}$. This gives the first row of the following commutative diagram:

$$(3.25) \qquad
\begin{array}{ccccc}
R_{\mathfrak{m}}^{\mathrm{ps}} & \xrightarrow{\ \mathrm{Cor.\ 3.5}\ } & \mathbb{T}_{\mathrm{cusp},\mathfrak{m}}^{(1)} & \longrightarrow\!\!\!\!\!\longrightarrow & \mathbb{T}_{\mathrm{cusp},n,\mathfrak{m}}^{(1)} \\
\downarrow & & & & \big\uparrow \\
R_{\mathfrak{m}}^{\mathrm{ord}} & \xrightarrow{\ \mathrm{Lemma\ 3.6}\ } & \widetilde{\mathbb{T}}_{n,\widetilde{\mathfrak{m}}}^{(\kappa_n)} & \xrightarrow{\ \mathrm{Cor.\ 2.6}\ } & \mathbb{T}_{\mathrm{cusp},n,\mathfrak{m}}^{(1)}[T_{\mathfrak{p}}^{(1)}, U]/(U^2 - T_{\mathfrak{p}}^{(1)}U + \langle \mathfrak{p} \rangle).
\end{array}$$

The morphisms in the second row come from Lemma 3.6 and Corollary 2.6. Combining them, we see that $\widetilde{P}_{n,\mathfrak{m}}^{(1)} = (P_{n,\mathfrak{m}}^{(1)}, U)$ is a $\mathfrak{p}$-ordinary pseudo-representation of weight 1.

We now perform the key 'doubling' step, as presented in [4, Prop. 2.10], and slightly improved upon, since the surjectivity of the composed map $R_{\mathfrak{m}}^{\mathrm{ord}} \to \mathbb{T}_{\mathrm{cusp},n,\mathfrak{m}}^{(1)}[T_{\mathfrak{p}}^{(1)}, U]/(U^2 - T_{\mathfrak{p}}^{(1)}U + \langle \mathfrak{p} \rangle)$ will not be used in the sequel. One has

$$\mathbb{T}_{\mathrm{cusp},n,\mathfrak{m}}^{(1)}[T_{\mathfrak{p}}^{(1)}, U]/(U^2 - T_{\mathfrak{p}}^{(1)}U + \langle \mathfrak{p} \rangle) = \mathbb{T}_{\mathrm{cusp},n,\mathfrak{m}}^{(1)}[T_{\mathfrak{p}}^{(1)}] \oplus U \cdot \mathbb{T}_{\mathrm{cusp},n,\mathfrak{m}}^{(1)}[T_{\mathfrak{p}}^{(1)}].$$

Since $\widetilde{P}_{n,\mathfrak{m}}^{(1)}$ is ordinary at $\mathfrak{p}$ of weight 1, for all $g \in G_F$ and $h \in I_{\mathfrak{p}}$ the following equality holds:

$$T_n(gh\,\mathrm{Frob}_{\mathfrak{p}}) - T_n(g\,\mathrm{Frob}_{\mathfrak{p}}) = U(T_n(gh) - T_n(g)) \in \mathbb{T}_{\mathrm{cusp},n,\mathfrak{m}}^{(1)}[T_{\mathfrak{p}}^{(1)}] \cap U\mathbb{T}_{\mathrm{cusp},n,\mathfrak{m}}^{(1)}[T_{\mathfrak{p}}^{(1)}] = \{0\},$$

hence $T_n(gh) = T_n(g)$, *i.e.*, $P_{n,\mathfrak{m}}^{(1)}$ is unramified at $\mathfrak{p}$.

Note that U satisfies the following relations

$$U^2 - T_{\mathfrak{p}}^{(1)}U + \langle \mathfrak{p} \rangle = 0 \text{ and } U^2 - T_n(\mathrm{Frob}_{\mathfrak{p}})U + D_n(\mathrm{Frob}_{\mathfrak{p}}) = 0$$

in the ring $\mathbb{T}^{(1)}_{\mathrm{cusp},n,\mathfrak{m}}[T^{(1)}_{\mathfrak{p}}, U]/(U^2 - T^{(1)}_{\mathfrak{p}}U + \langle \mathfrak{p} \rangle)$. Indeed, the second relation follows from the fact that $(P^{(1)}_{n,\mathfrak{m}}, U)$ is a $\mathfrak{p}$-ordinary pseudo-representation of weight 1. As the former polynomial is minimal, one obtains the desired equality $(T_n(\mathrm{Frob}_{\mathfrak{p}}), D_n(\mathrm{Frob}_{\mathfrak{p}})) = (T^{(1)}_{\mathfrak{p}}, \langle \mathfrak{p} \rangle)$, in particular $T^{(1)}_{\mathfrak{p}} \in \mathbb{T}^{(1)}_{\mathrm{cusp},n,\mathfrak{m}}$. Letting n vary finishes the proof of Theorem 0.1 for $\mathbb{T}^{(1)}_{\mathrm{cusp},\mathfrak{m}}$.

In order to obtain the theorem for $\mathbb{T}^{(1)}_{\mathfrak{m}}$, we replace $\mathbb{T}^{(1)}_{\mathrm{cusp},\mathfrak{m}}$ by $\mathbb{T}^{(1)}_{\mathfrak{m}}$, $\mathbb{T}^{(1)}_{\mathrm{cusp},n,\mathfrak{m}}$ by $\mathbb{T}^{(1)}_{n,\mathfrak{m}}$, and $S_1(\mathfrak{n}, \mathcal{O}/\varpi^n)$ by $M_1(\mathfrak{n}, \mathcal{O}/\varpi^n)$ throughout. The arguments continue to work if we assume that $p - 1$ does not divide $e_{\mathfrak{p}}$, which is used in Corollary 2.6.

Corollary 3.7. *Let $\mathfrak{p} \mid p$. Then $T^{(1)}_{\mathfrak{p}} \in \mathbb{T}^{(1)}_{\mathrm{cusp}}$, i.e., for all $n \in \mathbb{N}$, the Hecke operator $T^{(1)}_{\mathfrak{p}}$ acts on $S_1(\mathfrak{n}, \mathcal{O}/\varpi^n)$ by an element of $\mathbb{T}^{(1)}_{\mathrm{cusp},n}$. Moreover, if $(p-1) \nmid e_{\mathfrak{p}}$, then one also has $T^{(1)}_{\mathfrak{p}} \in \mathbb{T}^{(1)}$.*

3.4. **Non-Eisenstein ideals.**

Definition 3.8. *A maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{(\kappa)}$ (or of $\mathbb{T}^{(\kappa)}_n$) is called Eisenstein if the corresponding $(\mathbb{T}^{(\kappa)}_{\mathfrak{m}}/\mathfrak{m})$-valued pseudo-representatation of G_F is the sum of two $(\mathbb{T}^{(\kappa)}_{\mathfrak{m}}/\mathfrak{m})$-valued characters.*

We now prove that in the non-Eisenstein case it suffices to consider the cuspidal Hecke algebra.

Proposition 3.9. *The localisation of the natural surjection $\mathbb{T}^{(\kappa)}_n \twoheadrightarrow \mathbb{T}^{(\kappa)}_{n,\mathrm{cusp}}$ at any non-Eisenstein maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{(\kappa)}_n$ is an isomorphism.*

Proof. It suffices to prove that the localisation of $M_\kappa(\mathfrak{n}, \mathcal{O}/\varpi^n)/S_\kappa(\mathfrak{n}, \mathcal{O}/\varpi^n)$ at a non-Eisenstein ideal vanishes. By multiplication by a suitable power of h_n which does not vanish at any cusp (see Lemma 2.3), we can assume that κ is sufficiently large so that Lemma 2.2 applies yielding $M_\kappa(\mathfrak{n}, \mathcal{O}/\varpi^n) = M_\kappa(\mathfrak{n}, \mathcal{O}) \otimes_{\mathcal{O}} (\mathcal{O}/\varpi^n)$. Hence the natural Hecke equivariant morphism

$$M_\kappa(\mathfrak{n}, \mathcal{O})/S_\kappa(\mathfrak{n}, \mathcal{O}) \to M_\kappa(\mathfrak{n}, \mathcal{O}/\varpi^n)/S_\kappa(\mathfrak{n}, \mathcal{O}/\varpi^n)$$

is surjective. The former, however, can be Hecke equivariantly embedded into $M_\kappa(\mathfrak{n}, \mathbb{C})/S_\kappa(\mathfrak{n}, \mathbb{C})$ which is well known to be generated by Eisenstein series whose Galois representations are reducible. This proves the proposition. $\qquad\square$

Henceforth we assume $\mathfrak{m}$ to be a non-Eisenstein ideal of $\mathbb{T}^{(1)}$, so that the corresponding residual Galois representation $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible. Therefore, by combining Theorem 0.1 with a result of Chenevier ([5, Thm. 2.22]), we get a representation

$$\rho_{\mathfrak{m}} : \mathrm{G}_F \to \mathrm{GL}_2(\mathbb{T}^{(1)}_{\mathfrak{m}}),$$

unramified outside $\mathfrak{n}p\infty$, and uniquely characterised by the property that for all primes $\mathfrak{q} \nmid \mathfrak{n}p$ one has $\mathrm{tr}(\rho_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{q}})) = T_{\mathfrak{q}}$ and $\det(\rho_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{q}})) = \langle \mathfrak{q} \rangle$. By combining Theorem 0.1 with Proposition 3.9, we deduce that the pseudo-representation $P^{(1)}_{\mathfrak{m}}$ is unramified at all primes $\mathfrak{p} \mid p$ and, by the discussion after Definition 3.1, we conclude that $\rho_{\mathfrak{m}}$ is unramified at these primes as well. Let S be the set of places of F dividing $\mathfrak{n}\infty$ and let $R^S_{F,\overline{\rho}_{\mathfrak{m}}}$ be the universal deformation ring of $\overline{\rho}_{\mathfrak{m}}$ unramified outside S

in the category of complete Noetherian local $\mathcal{O}$-algebras with residue field $\mathbb{F}$. Hence $\rho_{\mathfrak{m}}$ induces an $\mathcal{O}$-algebra homomorphism $R^S_{F,\overline{\rho}_{\mathfrak{m}}} \to \mathbb{T}^{(1)}_{\mathfrak{m}}$.

As $\mathbb{T}^{(1)}$ is generated by $T_{\mathfrak{q}}$ and $\langle \mathfrak{q} \rangle$ for $\mathfrak{q} \nmid \mathfrak{n}p$ as an $\mathcal{O}$-algebra, we obtain the following result.

Corollary 3.10. *There exists a surjective homomorphism $R^S_{F,\overline{\rho}_{\mathfrak{m}}} \twoheadrightarrow \mathbb{T}^{(1)}_{\mathfrak{m}}$ of $\mathcal{O}$-algebras.*

REFERENCES

[1] F. ANDREATTA AND E. Z. GOREN, *Geometry of Hilbert modular varieties over totally ramified primes*, Int. Math. Res. Not., (2003), pp. 1786–1835.

[2] ———, *Hilbert modular forms: mod p and p-adic aspects*, Mem. Amer. Math. Soc., 173 (2005), pp. vi+100.

[3] F. CALEGARI AND D. GERAGHTY, *Modularity lifting beyond the Taylor-Wiles method*, Invent. Math., 211 (2018), pp. 297–433.

[4] F. CALEGARI AND J. SPECTER, *Pseudorepresentations of weight one are unramified*, Algebra Number Theory, 13 (2019), pp. 1583–1596.

[5] G. CHENEVIER, *The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings*, in Automorphic forms and Galois representations. Vol. 1, vol. 414 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 2014, pp. 221–285.

[6] F. DIAMOND, *Kodaira-Spencer isomorphisms and degeneracy maps on Iwahori-level Hilbert modular varieties: the saving trace*, 2021.

[7] ———, *Geometric weight-shifting operators on Hilbert modular forms in characteristic p*, J. Inst. Math. Jussieu, 22 (2023), pp. 1871–1930.

[8] F. DIAMOND AND P. KASSAEI, *Minimal weights of Hilbert modular forms in characteristic p*, Compos. Math., 153 (2017), pp. 1769–1778.

[9] ———, *The Cone of Minimal Weights for Mod p Hilbert Modular Forms*, International Mathematics Research Notices, (2023), pp. 12148–12171.

[10] F. DIAMOND AND S. SASAKI, *A Serre weight conjecture for geometric Hilbert modular forms in characteristic p*, J. Eur. Math. Soc. (JEMS), 25 (2023), pp. 3453–3536.

[11] M. DIMITROV, *Compactifications arithmétiques des variétés de Hilbert et formes modulaires de Hilbert pour $\Gamma_1(\mathfrak{c},\mathfrak{n})$*, in Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter, Berlin, 2004, pp. 527–554.

[12] ———, *On Ihara's lemma for Hilbert modular varieties*, Compos. Math., 145 (2009), pp. 1114–1146.

[13] M. DIMITROV AND J. TILOUINE, *Variétés et formes modulaires de Hilbert arithmétiques pour $\Gamma_1(\mathfrak{c},\mathfrak{n})$*, in Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter, Berlin, 2004, pp. 555–614.

[14] M. DIMITROV AND G. WIESE, *Unramifiedness of Galois representations attached to Hilbert modular forms mod p of weight 1*, J. Inst. Math. Jussieu, 19 (2020), pp. 281–306.

[15] M. EMERTON, D. REDUZZI, AND L. XIAO, *Unramifiedness of Galois representations arising from Hilbert modular surfaces*, Forum Math. Sigma, 5 (2017), pp. e29, 70.

[16] M. KISIN AND K. F. LAI, *Overconvergent Hilbert modular forms*, Amer. J. Math., 127 (2005), pp. 735–783.

[17] D. A. REDUZZI AND L. XIAO, *Partial Hasse invariants on splitting models of Hilbert modular varieties*, Ann. Sci. Éc. Norm. Supér. (4), 50 (2017), pp. 579–607.

[18] T. SAITO, *Hilbert modular forms and p-adic Hodge theory*, Compos. Math., 145 (2009), pp. 1081–1113.

[19] C. SKINNER, *A note on the p-adic Galois representations attached to Hilbert modular forms*, Doc. Math., 14 (2009), pp. 241–258.

[20] J. TATE, *Finite flat group schemes*, in Modular forms and Fermat's last theorem (Boston, MA, 1995), Springer, New York, 1997, pp. 121–154.

[21] R. TAYLOR AND A. WILES, *Ring-theoretic properties of certain Hecke algebras*, Ann. of Math. (2), 141 (1995), pp. 553–572.

[22] THE STACKS PROJECT AUTHORS, *The stacks project*. https://stacks.math.columbia.edu, 2023.

[23] G. WIESE, *On Galois representations of weight one*, Doc. Math., 19 (2014), pp. 689–707.

[24] A. WILES, *Modular elliptic curves and Fermat's last theorem*, Ann. of Math. (2), 141 (1995), pp. 443–551.

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