

CONSTANT GAUSSIAN CURVATURE FOLIATIONS AND SCHLÄFLI FORMULAS OF HYPERBOLIC 3-MANIFOLDS

FILIPPO MAZZOLI

ABSTRACT. We study the geometry of the foliation by constant Gaussian curvature surfaces $(\Sigma_k)_k$ of a hyperbolic end, and how it relates to the structures of its boundary at infinity and of its pleated boundary. First, we show that the Thurston and the Schwarzian parametrizations are the limits of two families of parametrizations of the space of hyperbolic ends, defined by Labourie in [Lab92] in terms of the geometry of the leaves Σ_k . We give a new description of the renormalized volume using the constant curvature foliation. We prove a generalization of McMullen's Kleinian reciprocity theorem, which replaces the role of the Schwarzian parametrization with Labourie's parametrizations. Finally, we describe the constant curvature foliation of a hyperbolic end as the integral curve of a time-dependent Hamiltonian vector field on the cotangent space to Teichmüller space, in analogy to the Moncrief flow for constant mean curvature foliations in Lorentzian spacetimes.

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INTRODUCTION

Let Σ be an oriented closed surface of genus larger or equal than 2. We will denote by \mathcal{T}_Σ^c the Teichmüller space of Σ defined as the space of isotopy classes of conformal structures of Σ , while \mathcal{T}_Σ^h will stand for the space of isotopy classes of hyperbolic metrics of Σ . A

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hyperbolic end E of topological type $\Sigma \times (0, \infty)$ is a (non-complete) hyperbolic 3-manifold homeomorphic to $\Sigma \times (0, \infty)$, whose metric completion is obtained by adding to E a locally concave pleated surface homeomorphic to $\Sigma \times \{0\} \subset \Sigma \times [0, \infty)$.

It is well known by the work of Thurston that the deformation space of $(\mathrm{PSL}_2 \mathbb{C}, \mathbb{C}\mathbb{P}^1)$ -structures (also called *complex projective structures*) of Σ is in 1-to-1 correspondence with the space of hyperbolic ends homeomorphic to $\Sigma \times [0, \infty)$, modulo isometry isotopic to the identity. Through this correspondence, Thurston proved that a complex projective structure σ on Σ is uniquely determined by a pair $(m, \mu) \in \mathcal{T}_\Sigma^h$, where m is the isotopy class of the metric on ∂E , the pleated boundary of the hyperbolic end E associated to σ , and μ is the measured lamination along which ∂E is bent. Moreover, every such pair (m, μ) can be realized as the data associated to some complex projective structure. This defines a homeomorphism Th , which we call the *Thurston parametrization*, from the space of hyperbolic ends $\mathcal{E}(\Sigma)$ to $\mathcal{T}_\Sigma^h \times \mathcal{ML}_\Sigma$, where \mathcal{ML}_Σ denotes the space of measured laminations of Σ .

Using a purely complex analytic approach, we can also characterize the complex projective structure σ by its induced conformal structure c , together with a holomorphic quadratic differential, which measures how far is the complex projective structure σ from the Fuchsian uniformization of c . In this way, we obtain another map Sch , which we call the *Schwarzian parametrization*, from the space of hyperbolic ends $\mathcal{E}(\Sigma)$ to the bundle of holomorphic quadratic differentials over \mathcal{T}_Σ^c , which can be canonically identified with the cotangent bundle of the Teichmüller space \mathcal{T}_Σ^c . Similarly to Th , the function Sch is an homeomorphism between $\mathcal{E}(\Sigma)$ and $T^*\mathcal{T}_\Sigma^c$.

By the work of Labourie [Lab91], every hyperbolic end admits a unique foliation by convex constant Gaussian curvature surfaces, with curvature k varying in $(-1, 0)$. For simplicity, a surface with constant Gaussian curvature equal to k will be called a k -surface. Using the foliation by k -surfaces, Labourie introduced in [Lab92] two new families of parametrizations of $\mathcal{E}(\Sigma)$, indexed by $k \in (-1, 0)$. As already announced by Labourie in [Lab92], these families of maps exhibit relations with the classical Thurston and Schwarzian parametrizations. The aim of this paper is to clarify this connection, and to present a series of results that relate the k -surfaces of an hyperbolic end with the geometry of its conformal boundary at infinity, on one side, and of its locally concave pleated boundary, on the other.

In order to be more precise, we need to introduce some notation and to recall the properties that k -surfaces satisfy. Given a hyperbolic end E , we will denote by Σ_k its k -surface, and by I_k and \mathbb{I}_k the first and second fundamental forms of Σ_k , respectively. Since the determinant of the shape operator B_k (i. e. the extrinsic curvature of Σ_k) is equal to $k + 1 > 0$ (as a consequence of the Gauss equation), we can choose the normal vector field to Σ_k so that the second fundamental form $\mathbb{I}_k = I_k(B_k \cdot, \cdot)$ is positive definite. We define also its third fundamental form to be $\mathbb{I}_k := I_k(B_k \cdot, B_k \cdot)$, where B_k is the shape operator of Σ_k . In this way, every k -surface comes with the data of three Riemannian metrics I_k , \mathbb{I}_k and \mathbb{I}_k , which satisfy the following conditions:

- i) the Riemannian metrics $h_k := (-k)I_k$ and $h_k^* := (-\frac{k}{k+1})\mathbb{I}_k$ are hyperbolic, i. e. they have constant Gaussian curvature equal to -1 ;
- ii) if c_k denotes the conformal class of \mathbb{I}_k , then the identity maps $id: (\Sigma_k, c_k) \rightarrow (\Sigma_k, h_k)$ and $id: (\Sigma_k, c_k) \rightarrow (\Sigma_k, h_k^*)$ are harmonic, with opposite Hopf differentials (as observed by Labourie [Lab92]).

Based on these remarks, we can define Labourie's parametrizations $(\hat{\Phi}_k)_k$ and $(\hat{\Psi}_k)_k$. The map $\hat{\Phi}_k$ associates to the hyperbolic end E the point of the cotangent space to Teichmüller space given by (the isotopy class of) the conformal structure c_k , together with the Hopf differential of $id: (\Sigma_k, c_k) \rightarrow (\Sigma_k, h_k)$, while the map $\hat{\Psi}_k$ sends E to the pair $(m_k, m_k^*) \in (\mathcal{T}_\Sigma^h)^2$ of isotopy classes of h_k and h_k^* , respectively.

For convenience here we will consider a "normalization" Φ_k of the function $\hat{\Phi}_k$, which differ from the original map simply by the multiplication by $-\frac{2\sqrt{k+1}}{k}$ in the fibers of $T^*\mathcal{T}_\Sigma^c$. To normalize the parametrization $\hat{\Psi}_k$, we can proceed following the construction of Bonsante, Mondello, and Schlenker [BMS15]. First we say that two hyperbolic metrics h and h^* are normalized if there exists a tensor $b: T\Sigma \rightarrow T\Sigma$ satisfying the following properties:

- a) $h^*(\cdot, \cdot) = h(b\cdot, b\cdot)$;
- b) b is h -symmetric and it has determinant 1;
- c) b is Codazzi with respect to the Levi-Civita connection ∇ of h . In other words, for every tangent vector fields X and Y , we have $(\nabla_X b)Y = (\nabla_Y b)X$.

Observe that the hyperbolic metrics h_k and h_k^* previously defined satisfy these conditions, with $b_k := \frac{1}{\sqrt{k+1}}B_k$. By a result of Schoen [Sch93], given any pair of points m, m^* in \mathcal{T}_Σ^h , there exists a normalized pair of hyperbolic metrics h and h^* representing m and m^* , respectively. Then we define

$$j: \begin{array}{ccc} \mathcal{T}_\Sigma^h \times \mathcal{T}_\Sigma^h & \longrightarrow & \mathbb{R} \\ (m, m^*) & \longmapsto & \int_\Sigma \text{tr}(b) da_h, \end{array}$$

where $b: T\Sigma \rightarrow T\Sigma$ is the operator satisfying the conditions above with respect to h and h^* , and da_h is the area form of h . The function j is well defined and it is symmetric in its arguments. Given any isotopy class of metrics m^* , the function L_{m^*} , which associates to each $m \in \mathcal{T}_\Sigma^h$ the value $j(m, m^*)$, will be called the *hyperbolic length function of m^** (the reason for this name will be explained in Section 2.2). Finally, we define $\Psi_k(E)$ to be the point of the cotangent space $T^*\mathcal{T}_\Sigma^h$ given by $(m_k, -\frac{\sqrt{k+1}}{k}d(L_{m_k^*}))$, where $\hat{\Psi}_k(E) = (m_k, m_k^*)$.

Our first result relates the maps Φ_k and Ψ_k to the Schwarzian and Thurston parametrizations:

Theorem A. *The maps Φ_k converge to the Schwarzian parametrization as k goes to 0, and the maps Ψ_k converge to $dL \circ \text{Th}$ as k goes to -1 , where dL is defined as*

$$dL: \begin{array}{ccc} \mathcal{T}_\Sigma^h \times \mathcal{ML}_\Sigma & \longrightarrow & T^*\mathcal{T}_\Sigma^h \\ (m, \mu) & \longmapsto & (m, d(L_\mu)_m), \end{array}$$

and L_μ denotes hyperbolic length function of the measured lamination μ .

The proof of this fact is based on the works of Quinn [Qui18] and Belraoui [Bel17], which describe the limits of the geometric quantities associated to the k -surfaces Σ_k as they approach the conformal boundary at infinity, and the locally concave pleated boundary, respectively.

Now, let M be a convex co-compact hyperbolic 3-manifold, and let W_k and V_k^* be the functions

$$W_k(M) := V(M_k) - \frac{1}{4} \int_{\partial M_k} H_k da_k, \quad V_k^*(M) := V(M_k) - \frac{1}{2} \int_{\partial M_k} H_k da_k,$$

where M_k is the region of M that is contained between the k -surfaces sitting in the ends of M . These volume functions share interesting properties with the dual volume of the convex core $V_C^*(M)$ and the renormalized volume $V_R(M)$, respectively. First of all, they satisfy two Schläfli-type variation formulas that are the exact analogues of the ones of V_C^* and V_R , as shown by the following result:

Theorem B. *The first order variations of the functions W_k and V_k^* can be expressed as follows:*

$$\delta W_k = -\frac{1}{2} d(\text{ext}_{\mathcal{F}_k})(\delta c_k), \quad \delta V_k^* = -\frac{1}{2} d(L_{\mathbb{M}_k})(\delta I_k),$$

where \mathcal{F}_k is the horizontal foliation of the quadratic differential $-\frac{\sqrt{k+1}}{k} \text{Hopf}(id: (\Sigma, c_k) \rightarrow (\Sigma, h_k))$, $\text{ext}_{\mathcal{F}_k}$ is the extremal length function of \mathcal{F}_k , and $L_{\mathbb{M}_k}(I_k) := -\frac{\sqrt{k+1}}{k} L_{h_k^*}(h_k)$.

It is not difficult to see that the functions V_k^* approximate the dual volume V_C^* as k goes to -1 , simply because the k -surfaces ∂M_k converge to the boundary of the convex core of M . A similar property is satisfied by the W_k -volumes and the renormalized volume V_R . Indeed, a simple corollary of the Theorems A and B is the following fact:

Theorem C. *The renormalized volume of a quasi-Fuchsian manifold M satisfies*

$$V_R(M) = \lim_{k \rightarrow 0^-} \left(W_k(M) - \pi |\chi(\partial M)| \operatorname{arctanh} \sqrt{k+1} \right).$$

This result gives a simple description of the renormalized volume of a quasi-Fuchsian manifold M in terms of the limit of the W -volumes of M_k . We underline the fact that, with this characterization, we can define the renormalized volume with a fairly simple expression in terms of the k -surface foliations of M , instead of considering the original procedure, which passes through the study of equidistant foliations of the ends of M .

As highlighted by the work Krasnov and Schlenker [KS09], the Schläfli-type variation formulas of the dual volume V_C^* and the renormalized volume V_R have strong implications with respect to the symplectic geometry of the spaces $T^*\mathcal{T}_\Sigma^c$ and $T^*\mathcal{T}_\Sigma^h$, endowed with the symplectic structures ω^c and ω^h of cotangent manifolds, respectively. Here we develop the same ideas applied to the volumes V_k^* and W_k^* , and the Labourie parametrizations Φ_k and Ψ_k through the variation formulas of Theorem B. In particular, we will prove:

Theorem D. *For every $k, k' \in (-1, 0)$, the function $\Phi_k \circ \Psi_{k'}^{-1} : (T^*\mathcal{T}_\Sigma^h, \omega^h) \rightarrow (T^*\mathcal{T}_\Sigma^c, 2\omega^c)$ is a symplectomorphism.*

We observe that this result generalizes the previous works of Krasnov and Schlenker [KS09, Theorem 1.2] and Bonsante, Mondello, and Schlenker [BMS15, Theorem 1.11], concerning the maps $\operatorname{Sch} \circ (\operatorname{dL} \circ \operatorname{Th})^{-1}$ and $\operatorname{Sch} \circ \Psi_k^{-1}$, respectively. Another surprisingly simple consequence of the variation formulas of the volumes W_k and V_k^* is the following generalization of (Krasnov and Schlenker's reformulation from [KS09] of) McMullen's Kleinian reciprocity Theorem:

Theorem E. *Let M be a 3-manifold with boundary whose interior admits a complete convex co-compact hyperbolic structure, and denote by $\mathcal{G}(M)$ the space of isotopy classes of such structures of M . We set*

$$\phi_k : \mathcal{G}(M) \longrightarrow T^*\mathcal{T}_{\partial M}^c, \quad \psi_k : \mathcal{G}(M) \longrightarrow T^*\mathcal{T}_{\partial M}^h$$

to be the maps that associate, to a convex co-compact hyperbolic structure of M , the points of $T^\mathcal{T}_{\partial M}^c$ and $T^*\mathcal{T}_{\partial M}^h$ given by the vectors $(\Phi_k(E_i))_i$ and $(\Psi_k(E_i))_i$, respectively, where E_i varies among the set of hyperbolic ends of M . Then, for every $k \in (-1, 0)$, the images $\phi_k(\mathcal{G}(M))$ and $\psi_k(\mathcal{G}(M))$ are Lagrangian submanifolds of $(T^*\mathcal{T}_\Sigma^c, \omega^c)$ and $(T^*\mathcal{T}_\Sigma^h, \omega^h)$, respectively.*

In Section 5.1 we will discuss the relations between the original McMullen's formulation of the quasi-Fuchsian reciprocity (in terms of adjoint maps) and the statement we have presented here. Theorem E generalizes [KS09, Theorems 1.4, 1.5], which state that $\operatorname{Sch}(\mathcal{G}(M))$ and $(\operatorname{dL} \circ \operatorname{Th})(\mathcal{G}(M))$ are Lagrangian submanifolds of $T^*\mathcal{T}_{\partial M}^c$ and $T^*\mathcal{T}_{\partial M}^h$, respectively.

As last (but not least) application of the tools developed here, we prove that the k -surface foliations of hyperbolic ends correspond to integral curves of k -dependent Hamiltonian vector fields on the spaces $T^*\mathcal{T}_\Sigma^c$ and $T^*\mathcal{T}_\Sigma^h$. This phenomenon can be interpreted as the analogous of what observed by Moncrief [Mon89] for constant mean curvature foliations in 3-dimensional Lorenzian space-times. If $\dot{\Phi}_k$ and $\dot{\Psi}_k$ denote the vector fields $\frac{d}{dk}\Phi_k$ and $\frac{d}{dk}\Psi_k$, respectively, then we will prove:

Theorem F. *The k -dependent vector field $\dot{\Phi}_k \circ \Phi_k^{-1}$ (resp. $\dot{\Psi}_k \circ \Psi_k^{-1}$) is Hamiltonian with respect to the real cotangent symplectic structure of $T^*\mathcal{T}_\Sigma^c$ (resp. $T^*\mathcal{T}_\Sigma^h$), with Hamiltonian*

function $-\frac{1}{8(k+1)}m_k \circ \Phi_k^{-1}$ (resp. $-\frac{1}{2k}m_k \circ \Psi_k^{-1}$), where $m_k: \mathcal{E}(\Sigma) \rightarrow \mathbb{R}$ sends the hyperbolic end E into the integral of the mean curvature $\int_{\Sigma_k} H_k da_k$ of its k -surface Σ_k .

We observe that the role of the area functional in [Mon89] as Hamiltonian function here is replaced by the integral of the mean curvature, which coincides with the hyperbolic length $L_{\mathbb{I}_k}(I_k)$ considered above.

As a final remark, we summarize the transitional properties that k -surfaces possess in relation to the boundary of the convex core and the conformal boundary at infinity of convex co-compact hyperbolic manifolds in the following table:

On ∂CM	On ∂M_k	On $\partial_\infty M$
	Conformal class $c_k = [\mathbb{I}_k]$	Conformal structure c
Induced metric m	First fund. form I_k	
	Measured foliation \mathcal{F}_k	Measured foliation \mathcal{F}
Bending measure μ	Third fund. form \mathbb{I}_k	
	Extremal length $\text{ext}_{\mathcal{F}_k}(c_k)$	Extremal length $\text{ext}_{\mathcal{F}}(c)$
Hyperbolic length $L_\mu(m)$	$L_{\mathbb{I}_k}(I_k) = \int_{\partial M_k} H_k da_k$	
	Param. Φ_k (Cor 2.4)	Schwarzian param. Sch
Thurston param. Th	Param. $\check{\Psi}_k$ (Thm 2.5)	
	Volume W_k	Renorm. volume V_R
Dual volume V_C^*	Volume V_k^*	
	Thm 3.3 $\delta W_k = \frac{1}{2} d(\text{ext}_{\mathcal{F}_k})(\delta c_k)$	[Sch17, Thm 1.2] $\delta V_R = -\frac{1}{2} d(\text{ext}_{\mathcal{F}})(\delta c)$
[KS09, Lemma 2.2], [Maz18] $\delta V_C^* = -\frac{1}{2} d(L_\mu)(\delta m)$	Thm 3.7 $\delta V_k^* = -\frac{1}{2} d(L_{\mathbb{I}_k})(\delta I_k)$	
	$\phi_k(\mathcal{G}(M))$ is Lagrangian (Thm E)	McMullen's Kleinian reciprocity [McM98]
$(dL \circ \text{Th})(\mathcal{G}(M))$ is Lagrangian [KS09, Thm 1.4]	$\psi_k(\mathcal{G}(M))$ is Lagrangian (Thm E)	

Outline of the paper. In the first Section we recall the necessary background about harmonic and minimal Lagrangian maps between surfaces, the properties of k -surfaces and classical parametrizations of the space of hyperbolic ends, namely the Schwarzian and Thurston parametrizations.

The second Section is dedicated to the definition of the Labourie parametrizations Φ_k and Ψ_k , and to the proof of Theorem A, which is divided in two parts, Corollary 2.4 and Theorem 2.5.

Section 3 focuses on the Schläfli formulas of Theorem B (see Theorems 3.3 and 3.7). While the proof of the variation of formula of V_k^* is essentially a combination of results extracted from the works of Bonsante, Mondello, and Schlenker [BMS13], [BMS15], the variation formula of the volumes W_k will require a bit more care. The main technical ingredients of our analysis will be a new way to express the variation of the W -volume (see Proposition A.3 in the Appendix) and Gardiner's formula [Gar84, Theorem 8] for the differential of the extremal length function. In Section 3.2 we will combine the results from Section 2 with Theorem B and with the Schläfli formula of the renormalized volume V_R (from [Sch17, Theorem 1.2]) to deduce a new description of the renormalized volume of a convex co-compact hyperbolic 3-manifold in terms of the limit of the volumes W_k (Theorem C).

The remaining sections are dedicated to the proofs of Theorems D, E and F. As we will see, these results will follow as fairly elementary applications of what described in

Sections 2 and 3. In Section 4 we use two "relative versions" w_k and v_k^* of the volumes W_k and V_k^* to describe the pullback of the Liouville forms of $T^*\mathcal{T}_\Sigma^c$ and $T^*\mathcal{T}_\Sigma^h$ (Lemma 4.1) under the Labourie parametrizations Φ_k and Ψ_k , respectively. As immediate consequence, we will deduce Theorem D. In Section 5 we will prove Theorem E, whose proof is a simple application of the Schläfli formulas of Theorem B and of the dual Bonahon-Schläfli formula (see [KS09], [Maz18]). Finally, Section 6 is devoted to the proof of Theorem F, which is based on the formulas of Lemma 4.1 and on a elementary application of Cartan formula (see Lemma 6.2).

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1. PRELIMINARIES

In our presentation, Σ will always denote a closed orientable surface of genus $g \geq 2$.

Definition 1.1. Let Σ be a surface. Two Riemannian metrics g, g' on Σ are *conformally equivalent* if there exists a smooth function $\alpha \in \mathcal{C}^\infty(\Sigma)$ such that $g' = e^{2\alpha}g$. A *conformal structure* c on Σ is an equivalence class of Riemannian metrics with respect to the relation above, together with a choice of a orientation of Σ . A *hyperbolic metric* h on Σ is a Riemannian metric with Gaussian curvature constantly equal to -1 .

Given any surface Σ , we will denote by \mathcal{T}_Σ the *Teichmüller space* of Σ . By the uniformization theorem, \mathcal{T}_Σ can be interpreted either as the space of isotopy classes of hyperbolic metrics (complete Riemannian metrics of constant curvature -1), or of conformal structures on Σ . We will write \mathcal{T}_Σ^h (h for *hyperbolic*) when we want to emphasize the first interpretation, and \mathcal{T}_Σ^c (c for *conformal*) in latter case.

Given a conformal structure c on Σ , we denote by $Q(\Sigma, c)$ the space of holomorphic quadratic differentials of (Σ, c) . By the Riemann-Roch theorem, $Q(\Sigma, c)$ is a vector space of complex dimension $3g - 3$. It is well known that the cotangent space to Teichmüller space \mathcal{T}_Σ^c at the isotopy class of c can be naturally identified with the vector space $Q(\Sigma, c)$.

1.1. Harmonic and minimal Lagrangian maps. In what follows, we briefly recall the definitions of harmonic and minimal Lagrangian maps between hyperbolic surfaces, and the relative results that we will need in our presentation.

Definition 1.2. Let c and g be a conformal structure and a Riemannian metric on Σ , respectively. A smooth map $u: (\Sigma, c) \rightarrow (\Sigma, g)$ is *harmonic* if the $(2, 0)$ -part of u^*g with respect to the conformal structure c is a holomorphic quadratic differential. In such case, we call $(u^*g)^{(2,0)}$ the *Hopf differential* of u .

Equivalently, u is harmonic if there exists a (and, consequently, for any) Riemannian metric g' in the conformal class c , such that the g' -traceless part of u^*g is a g' -divergence free tensor (see [Tro92, p. 45-46] for the equivalence of these definitions).

Remark 1.3. If $f: (\Sigma, [g']) \rightarrow (\Sigma, g)$ is harmonic with Hopf differential q , then the g' -traceless part of f^*g is equal to $2 \operatorname{Re} q$ ($[g']$ is the conformal class of g').

Theorem 1.4 (See e. g. [Sam78]). *Let c be a conformal structure on Σ . Then, for any hyperbolic metric h on Σ , there exists a unique holomorphic quadratic differential $q(c, h) \in Q(\Sigma, c)$, and a unique diffeomorphism $u(c, h): (\Sigma, c) \rightarrow (\Sigma, h)$ isotopic to the identity, such that $u(c, h)$ is harmonic with Hopf differential $q(c, h)$.*

Theorem 1.5 ([Wol89, Theorem 3.1]). *For every $c \in \mathcal{T}_\Sigma^c$, the function*

$$\begin{aligned} \varphi_c : \mathcal{T}_\Sigma^h &\longrightarrow Q(\Sigma, c) \\ [h] &\longmapsto q(c, h), \end{aligned}$$

is a diffeomorphism.

Definition 1.6 (See [BMS13, Proposition 1.3]). Let h and h' be two hyperbolic metrics on Σ . A diffeomorphism $f: (\Sigma, h) \rightarrow (\Sigma, h')$ is *minimal Lagrangian* if it is area-preserving, and its graph is a minimal surface inside $(\Sigma^2, h \oplus h')$.

Equivalently, $f: (\Sigma, h) \rightarrow (\Sigma, h')$ is minimal Lagrangian if there exists a conformal structure c on Σ such that $f = u(c, h') \circ u(c, h)^{-1}$ and $q(c, h') = -q(c, h)$, with the notation introduced in Definition 1.2.

Remark 1.7. Using the first description of minimal Lagrangian maps, the conformal structure c , appearing in the second definition, can be recovered as the conformal class of the induced metric on the graph of f from the metric $h \oplus h'$ (by identifying the graph of f with Σ using one of the projections onto Σ). Moreover, the projections of the graph of f onto (Σ, h) and (Σ, h') are harmonic with respect to c .

Theorem 1.8 ([Lab92], [Sch93]). *For every hyperbolic metric h and for every isotopy class $m' \in \mathcal{T}_\Sigma^h$, there exists a unique hyperbolic metric $h' \in m'$ and a unique operator $b: T\Sigma \rightarrow T\Sigma$ such that:*

- i) $h' = h(b \cdot, b \cdot)$;
- ii) b is h -self-adjoint and positive definite,
- iii) $\det b = 1$;
- iv) b is Codazzi with respect to the Levi-Civita connection ∇ of h , i. e. $(\nabla_X b)Y = (\nabla_Y b)X$ for every X and Y .

Definition 1.9. Whenever we have a pair of hyperbolic metrics h, h' and an operator b as in the statement above, we say that the pair h, h' is *normalized*, and that b is the *Labourie operator* of the couple h, h' .

It turns out that, if h and h' are a normalized pair of hyperbolic metrics with Labourie operator b , then the conformal class c of the Riemannian metric $h(b \cdot, \cdot)$ is such that the maps

$$(\Sigma, h) \xleftarrow{id} (\Sigma, c) \xrightarrow{id} (\Sigma, h')$$

are harmonic, with opposite Hopf differentials. Therefore, Theorem 1.8 can be reformulated in the following way:

Theorem 1.10 ([Lab92], [Sch93]). *The function*

$$\mathcal{H}: T^*\mathcal{T}_\Sigma^c \longrightarrow \mathcal{T}_\Sigma^h \times \mathcal{T}_\Sigma^h \\ (c, q) \longmapsto (\varphi_c^{-1}(q), \varphi_c^{-1}(-q)),$$

is a diffeomorphism (here φ_c denotes the harmonic parametrization of Theorem 1.5).

1.2. Constant extrinsic curvature surfaces. Let Σ be a (space-like) surface immersed in a Riemannian (Lorentzian) 3-manifold M of constant sectional curvature $\sec(M)$, with first and second fundamental forms I and \mathbb{I} , and shape operator B . We denote by K_e its *extrinsic curvature*, i. e. $K_e = \det B$, and by K_i its *intrinsic curvature*, i. e. the Gauss curvature of the Riemannian metric I . For convenience, we define $\text{sgn}(M)$ to be $+1$ if M is a Riemannian manifold, and -1 if M is Lorentzian. Then, the Gauss-Codazzi equations of (Σ, I, \mathbb{I}) can be expressed as follows:

$$(1) \quad \begin{aligned} K_i &= \text{sgn}(M) K_e + \sec(M), \\ (\nabla_U B)V &= (\nabla_V B)U \quad \forall U, V, \end{aligned}$$

where U and V are tangent vector fields to Σ , and ∇ is the Levi-Civita connection of the metric I . The *third fundamental form* of Σ is the symmetric 2-tensor $I(B \cdot, B \cdot)$.

Definition 1.11. Let Σ be an immersed (space-like) surface of a Riemannian (Lorentzian) 3-manifold M . We say that Σ is *strictly convex* if its second fundamental form \mathbb{I} is positive definite.

Remark 1.12. Observe that the notion of strict convexity implicitly depends on the choice of a normal vector field of Σ . Moreover, if Σ is a strictly convex surface, then its third fundamental form is a Riemannian metric too.

Let Σ be a surface immersed in a hyperbolic 3-manifold M . The Gauss equation in this case has the following form:

$$(2) \quad K_i = K_e - 1.$$

Given $k \in (-1, 0)$, we say that Σ is a *k-surface* of M if its intrinsic curvature is constantly equal to k . If we define the shape operator of Σ using the normal vector field of Σ that points to the convex side of Σ , then the second fundamental form \mathbb{I} of Σ has strictly positive principal curvatures, since $\det B = K_e = k + 1 > 0$. Therefore \mathbb{I} is a positive definite symmetric bilinear form; in other words, Σ is strictly convex.

In order to give a geometric interpretation of the third fundamental form \mathbb{III} of Σ , we need to clarify the connection between the hyperbolic 3-space \mathbb{H}^3 and the *de Sitter* 3-space dS^3 , which is a Lorentzian analogue of the 3-dimensional unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4$. Let $\mathbb{R}^{3,1}$ denote the 4-dimensional Minkowski space, i. e. the vector space \mathbb{R}^4 endowed with a Lorentzian scalar product of signature $(3, 1)$. Then the hyperbolic space \mathbb{H}^3 can be viewed as (a connected component of) the set of vectors $x \in \mathbb{R}^{3,1}$ satisfying $\langle x, x \rangle = -1$. Similarly, the de Sitter space is defined as the set of vectors y satisfying $\langle y, y \rangle = 1$. The projection of \mathbb{R}^4 onto the projective space $\mathbb{R}P^3$ sends the light cone into the quadric $P\{x \mid \langle x, x \rangle = 0\}$ of $\mathbb{R}P^3$. The polarity correspondence determined by this quadric allows to construct, starting from a strictly convex surface $\tilde{\Sigma}$ in \mathbb{H}^3 , an associated strictly convex surface $\tilde{\Sigma}^*$ in dS^3 , whose points are the polar-duals of the tangent spaces to the surface $\tilde{\Sigma}$ in \mathbb{H}^3 . Moreover, if $\tilde{\Sigma}$ is the lift to \mathbb{H}^3 of a surface Σ sitting inside some hyperbolic end E (see Definition 1.14), then $\tilde{\Sigma}$ determines a space-like surface Σ^* inside a *maximal global hyperbolic spatially compact de Sitter spacetime* E^* (see e. g. [Mes07] for details). Finally, in this description, the first fundamental form I^* of Σ^* coincides with the tensor \mathbb{III} , and the second fundamental forms \mathbb{II}^* and \mathbb{II} are essentially the same ($\mathbb{II} = \pm \mathbb{II}^*$, depending on the conventions). We refer to [Sch02] for more detailed description of this correspondence.

Now, the Gauss equation of the dual surface $(\Sigma^*, I^*, \mathbb{II}^*)$ is

$$(3) \quad K_i^* = -K_e^* + 1.$$

Since the shape operator B^* of Σ^* coincides with $\pm B^{-1}$, the surface (Σ, I, \mathbb{II}) has extrinsic curvature K_e if and only if $(\Sigma^*, I^*, \mathbb{II}^*)$ has extrinsic curvature $K_e^* = K_e^{-1}$. Combining this fact with the Gauss equations (2) and (3), we see that, if Σ is a *k-surface*, then the tensors

$$h := -kI \quad \text{and} \quad h^* := -\frac{k}{k+1} \mathbb{III}$$

are Riemannian metrics of constant curvature -1 . We also set c to be the conformal class of \mathbb{II} .

Lemma 1.13. *Let Σ be a strictly convex surface immersed in a Riemannian (or Lorentzian) 3-manifold M . The following are equivalent:*

- the surface Σ has constant extrinsic curvature;
- the identity map $id: (\Sigma, c) \rightarrow (\Sigma, I)$ is harmonic.

Proof. The Levi-Civita connection $\nabla^{\mathbb{II}}$ of the Riemannian metric \mathbb{II} satisfies

$$\nabla_U^{\mathbb{II}} V = \nabla_U V + \frac{1}{2} B^{-1} (\nabla_U B) V,$$

where U and V are tangent vector fields to Σ , and ∇ is the Levi-Civita connection of I . This relation can be easily proved by showing that the right-hand side, as a function of U and V , defines a connection which is torsion-free and compatible with \mathbb{II} . The first property follows from the fact that ∇ is torsion-free, and from the Codazzi equation (1) satisfied

by B . The compatibility with respect to \mathbb{I} can be derived by the compatibility of ∇ with respect to I , and by the fact that B is I -self-adjoint.

The map $id: (\Sigma, \mathbb{I}) \rightarrow (\Sigma, I)$ is harmonic if and only if the \mathbb{I} -traceless part of I is the real part of a holomorphic quadratic differential. By the results of [Tro92, Chapter 2], this is equivalent to saying that $I - \frac{H}{2K_e} \mathbb{I}$ is divergence free with respect to $\nabla^{\mathbb{I}}$ and traceless with respect to \mathbb{I} (which is true by definition). Using the expression of $\nabla^{\mathbb{I}}$ above, we can prove that

$$\operatorname{div}_{\mathbb{I}} \left(I - \frac{H}{2K_e} \mathbb{I} \right) = -\frac{1}{2} d(\ln K_e).$$

From this equation the statement is clear. \square

This fact proves that both the maps

$$(\Sigma, h) \xleftarrow{id} (\Sigma, c) \xrightarrow{id} (\Sigma, h^*)$$

are harmonic. If \hat{T} denotes the \mathbb{I} -traceless part of the symmetric tensor T , then we have

$$\hat{h} = -k \left(I - \frac{H}{k+1} \mathbb{I} \right), \quad \hat{h}^* = -\frac{k}{k+1} (\mathbb{I} - H\mathbb{I}).$$

Using the relation $B^2 - HB + K_e \mathbb{1} = 0$, we see that $\hat{h} = -\hat{h}^*$. This shows that the two identity maps above have opposite Hopf differentials or, equivalently, that the map $id: (\Sigma, h) \rightarrow (\Sigma, h^*)$ is minimal Lagrangian (see Definition 1.6).

1.3. The space of hyperbolic ends.

Definition 1.14. Given Σ a closed surface, a *hyperbolic end* E of topological type $\Sigma \times [0, \infty)$ is a hyperbolic 3-manifold with underlying topological space $\Sigma \times (0, \infty)$ and whose metric completion $\bar{E} \cong \Sigma \times [0, \infty)$ is obtained by adding to E a locally concave pleated surface $\Sigma \times \{0\} \subset \Sigma \times [0, \infty)$. We will denote by ∂E the locally concave pleated boundary of E .

Two hyperbolic ends $E = (\Sigma \times (0, \infty), g)$ and $E' = (\Sigma \times (0, \infty), g')$ are equivalent if there exists an isometry between them that is isotopic to $id_{\Sigma \times (0, \infty)}$. We set $\mathcal{E}(\Sigma)$ to be the space of equivalence classes of hyperbolic ends of topological type $\Sigma \times (0, \infty)$.

Let E be a hyperbolic end. The manifold $\bar{E} \cong \Sigma \times [0, \infty)$ can be compactified by adding a topological surface "at infinity" $\partial_{\infty} E := \Sigma \times \{\infty\}$. The $(\operatorname{Iso}^+(\mathbb{H}^3), \mathbb{H}^3)$ -structure on E naturally determines a $(\operatorname{PSL}_2 \mathbb{C}, \mathbb{C}P^1)$ -structure (also called *complex projective structure*) σ_E on $\partial_{\infty} E$, coming from the action of $\operatorname{Iso}^+(\mathbb{H}^3) \cong \operatorname{PSL}_2 \mathbb{C}$ on the boundary at infinity $\partial_{\infty} \mathbb{H}^3 \cong \mathbb{C}P^1$.

By a classical construction due to Thurston, it is possible to invert this process: given a complex projective structure σ on a surface Σ , there exists a hyperbolic end E of topological type $\Sigma \times (0, \infty)$ whose induced complex projective structure on $\partial_{\infty} E$ coincides with σ . The universal cover \tilde{E} of E can be locally described as the envelope of those half-spaces H of \mathbb{H}^3 satisfying $\bar{H} \cap \partial_{\infty} \mathbb{H}^3 = D$, where D varies over the developed maximal discs of $(\tilde{\Sigma}, \tilde{\sigma})$ in $\partial_{\infty} \mathbb{H}^3 = \mathbb{C}P^1$. This construction establishes a one-to-one correspondence between the space of hyperbolic ends $\mathcal{E}(\Sigma)$ and the deformation space of complex projective structures on Σ . We refer to [KT92] for a more detailed exposition of Thurston's construction.

The Schwarzian parametrization. Let E be a hyperbolic end. Following the notation introduced above, we denote by c_0 the underlying conformal structure of σ_E , and by σ_0 the "Fuchsian structure" of c_0 , i. e. the complex projective structure on $\Sigma = \partial_{\infty} E$ determined by the uniformization map of $(\tilde{\Sigma}, \tilde{c}_0)$. The set of complex projective structures with underlying conformal structure c_0 can be interpreted as an affine space over the space of holomorphic quadratic differentials of (Σ, c_0) , and the correspondence sends each element $\sigma - \sigma_0$ into the *Schwarzian derivative* of σ with respect to σ_0 (see [Dum09] for details). In particular,

the element $\sigma_E - \sigma_0$ determines a unique holomorphic quadratic differential q_0 of (Σ, c_0) , called the *Schwarzian at infinity* of E . The resulting map

$$\begin{aligned} \text{Sch} : \mathcal{E}(\Sigma) &\longrightarrow T^*\mathcal{T}_\Sigma^c \\ [E] &\longmapsto (c_0, q_0), \end{aligned}$$

gives a parametrization of the space of hyperbolic ends $\mathcal{E}(\Sigma)$, which we will call the *Schwarzian parametrization*.

The Thurston parametrization. The Schwarzian parametrization of the space of hyperbolic ends $\mathcal{E}(\Sigma)$ uses the geometric structure of the boundary at infinity $\partial_\infty E$ of E . In the following we will describe a analogous construction, due to Thurston, involving the shape of the convex pleated boundary ∂E , instead of $\partial_\infty E$.

The surface ∂E is a topologically embedded surface in E , which is almost everywhere totally geodesic. The set of points where ∂E is not locally shaped as an open set of \mathbb{H}^2 is a closed subset λ that is disjoint union of simple (not necessarily closed) complete geodesics. The path metric of ∂E is an actual hyperbolic metric $h \in \mathcal{T}_\Sigma^h$, and the structure of the singular locus λ can be described using the notion of *transverse measured lamination*. In the simple case of λ composed by disjoint simple closed geodesics, each leaf γ_i of λ has an associated exterior dihedral angle ϑ_i , which measures the bending between the totally geodesic portions of ∂E meeting along γ_i . Given any geodesic arc α transverse to λ , we can define the transverse measure $\mu := \sum_i \vartheta_i \gamma_i$ along a geodesic segment α to be the sum $\sum_i \vartheta_i i(\gamma_i, \alpha)$, where $i(\gamma_i, \alpha)$ is the geometric intersection between α and γ_i . Using an approximation procedure, we can generalize the construction above to a generic support λ , obtaining a measured lamination $\mu \in \mathcal{ML}_\Sigma$, which measures the amount of bending that occurs transversely to λ . The datum of the hyperbolic metric h and the measured lamination μ is actually sufficient to describe the entire hyperbolic end. In other words, the map

$$\begin{aligned} \text{Th} : \mathcal{E}(\Sigma) &\longrightarrow \mathcal{T}_\Sigma^h \times \mathcal{ML}_\Sigma \\ [E] &\longmapsto (h, \mu) \end{aligned}$$

parametrizes the space of hyperbolic ends (for a detailed proof of this result, see [KT92, Section 2]). We will call Th the *Thurston parametrization* of $\mathcal{E}(\Sigma)$.

2. FOLIATIONS BY k -SURFACES

This Section is mainly devoted to the description of two families of parametrizations of the space of hyperbolic ends $\mathcal{E}(\Sigma)$, denoted by $(\Phi_k)_k$ and $(\Psi_k)_k$, firstly introduced by Labourie [Lab92], and further investigated by Bonsante, Mondello and Schlenker in [BMS13] and [BMS15]. After having recalled the necessary background, we will establish a connection between the asymptotic of these maps and the classical Schwarzian and Thurston parametrizations, applying the recent works of Quinn [Qui18] and Belraoui [Bel17], respectively.

Theorem 2.1 ([Lab91, Théorème 2]). *Every hyperbolic end has a unique foliation by k -surfaces Σ_k , with k varying in $(-1, 0)$. As k goes to -1 , the k -surface Σ_k approaches the concave pleated boundary of E , and as k goes to 0, Σ_k approaches the conformal boundary at infinity of E .*

Before describing the maps $(\Phi_k)_k$ and $(\Psi_k)_k$, we need to introduce some notation that we will use (and used) in the rest of the paper. Given E a hyperbolic end, with k -surface foliation $(\Sigma_k)_k$, we let I_k , \mathbb{I}_k and \mathbb{I}_k^* denote the first, second and third fundamental forms of Σ_k . Moreover, we set h_k and h_k^* to be the hyperbolic metrics $-k I_k$ and $-\frac{k}{k+1} \mathbb{I}_k$, respectively, and c_k to be the conformal class of \mathbb{I}_k . Finally, we will denote by q_k the holomorphic quadratic differential

$$-\frac{2\sqrt{k+1}}{k} \text{Hopf}((\Sigma_k, c_k) \rightarrow (\Sigma_k, h_k)) = \frac{2\sqrt{k+1}}{k} \text{Hopf}((\Sigma_k, c_k) \rightarrow (\Sigma_k, h_k^*)).$$

The choice of the multiplicative constant in the definition of q_k may look arbitrary at this point of the exposition, but it will be crucial in the following (see for instance Corollary 2.4 and Remark 6.3). The holomorphic quadratic differential q_k satisfies

$$(4) \quad 2\operatorname{Re} q_k = 2\sqrt{k+1} \left(I_k - \frac{H_k}{2(k+1)} \mathbb{I}_k \right) = -\frac{2}{\sqrt{k+1}} \left(\mathbb{I}_k - \frac{H_k}{2} \mathbb{I}_k \right).$$

For future references, we also observe that the area forms with respect to I_k and \mathbb{I}_k differ by a multiplicative constant, as follows:

$$(5) \quad da_{I_k} = \frac{1}{\sqrt{\det B_k}} da_{\mathbb{I}_k} = \frac{1}{\sqrt{k+1}} da_{\mathbb{I}_k}.$$

2.1. The parametrizations Φ_k . The first class of parametrizations described by Labourie [Lab92] is given by the following maps: for every $k \in (-1, 0)$ we define the function

$$\Phi_k : \begin{array}{ccc} \mathcal{E}(\Sigma) & \longrightarrow & T^*\mathcal{T}_\Sigma^c \\ [E] & \longmapsto & (c_k, q_k), \end{array}$$

which associates, to every hyperbolic end E , the point of the cotangent space to Teichmüller space (c_k, q_k) determined by the unique k -surface Σ_k contained in E , as above. We have:

Theorem 2.2 ([Lab92, Théorème 3.1]). *The function Φ_k is a diffeomorphism for every $k \in (-1, 0)$.*

In the following we will see how the maps Φ_k relate to the Schwarzian parametrization Sch. Using the hyperbolic Gauss map (see e. g. [Lab91]), we can think about the families $(I_k)_k$, $(\mathbb{I}_k)_k$ and $(\mathbb{I}_k)_k$ as paths in the space of $(2, 0)$ -symmetric tensors over the surface $\partial_\infty E$, which does not depend on k . In this way we can study the asymptotic of these geometric quantities as k goes to 0.

In a recent work [Qui18], Quinn introduced the notion of *asymptotically Poincaré families of surfaces* inside a hyperbolic end E , and he determined a connection between their geometric properties and the complex projective structure at infinity of E . The foliation by k -surfaces is an example of such families and the asymptotic of their fundamental forms is understood. In order to do not introduce more notions, we specialize the results of [Qui18] in the form that we will need:

Theorem 2.3 ([Qui18]). *For every hyperbolic end $E \in \mathcal{E}(\Sigma)$ we have*

$$\lim_{k \rightarrow 0^-} h_k = \lim_{k \rightarrow 0^-} (-k)\mathbb{I}_k = h_0,$$

where h_0 is the hyperbolic metric in the conformal class at infinity c_0 . Moreover

$$\dot{h}_0 = -\frac{1}{2}h_0 - \operatorname{Re} q_0, \quad \frac{d}{dk} (-k)\mathbb{I}_k|_{k=0} = 0,$$

where q_0 is the Schwarzian at infinity of E .

Corollary 2.4. *The maps $(\Phi_k)_k$ converge to Sch \mathcal{C}^1 -uniformly over compact subsets, as k goes to 0.*

Proof. First we prove the pointwise convergence. Let E be a hyperbolic end, and consider the path $(\Phi_k(E))_k$ in $T^*\mathcal{T}_\Sigma^c$. We define $g_k := (-k)\mathbb{I}_k$. Then, the relations of Theorem 2.3 can be rewritten as follows:

$$g_0 := \lim_{k \rightarrow 0} g_k = h_0, \quad \dot{h}_0 = -\frac{1}{2}h_0 - \operatorname{Re} q_0, \quad \dot{g}_0 = 0.$$

The first relation proves that the conformal classes c_k converge to the conformal structure of $\partial_\infty E$. We need to show that the holomorphic quadratic differentials q_k converge to the

Schwarzian differential q_0 . This is a simple application of the relations above, we briefly summarize the steps in the following. First we observe that

$$\begin{aligned} \lim_{k \rightarrow 0} 2 \operatorname{Re} q_k &= \lim_{k \rightarrow 0} -\frac{2\sqrt{k+1}}{k} \left(h_k - \frac{\operatorname{tr}_{g_k}(h_k)}{2} g_k \right) \\ &= \lim_{k \rightarrow 0} -2\sqrt{k+1} \frac{h_k - h_0}{k} + \sqrt{k+1} \frac{\operatorname{tr}_{g_k}(h_k) g_k - 2h_0}{k} \\ &= -2\dot{h}_0 + \frac{d}{dk} \operatorname{tr}_{g_k}(h_k) g_k \Big|_{k=0}, \end{aligned}$$

where, in the last step, we are using that $\lim_{k \rightarrow 0} \operatorname{tr}_{g_k}(h_k) g_k = 2h_0$. A simple computation shows that $\frac{d}{dk} \operatorname{tr}_{g_k}(h_k) \Big|_{k=0} = -1$. Combining this with the relation above we obtain

$$\lim_{k \rightarrow 0} 2 \operatorname{Re} q_k = -2 \left(-\frac{1}{2} h_0 - \operatorname{Re} q_0 \right) - h_0 + 2\dot{g}_0 = 2 \operatorname{Re} q_0,$$

which was our claim.

In [Qui18], the author gave an alternative proof of the existence of the k -surface foliation, for k close to 0. The strategy of his proof is to apply the Banach implicit function theorem to a function

$$F : (-1, 0] \times \operatorname{Conf}^s(\Sigma, c) \longrightarrow \operatorname{Conf}^s(\Sigma, c),$$

which satisfies $F(k, \tau) = 0$ if and only if τ is (a proper multiple of) the metric at infinity associated to the k -surface. Here $\operatorname{Conf}^s(\Sigma, c)$ denotes the space of Sobolev metrics in the conformal class c (see [Qui18, Theorem 5.1] for details). The map F depends smoothly on k and also on the complex projective structure at infinity (c, q) . In particular, the implicit function theorem guarantees the smooth regularity of the metric at infinity τ_k , associated to the k -surface Σ_k , with respect to $k \in (-1, 0]$ and $(c, q) \in T^*\mathcal{T}_\Sigma^c$. Since the tensors I_k and \mathbb{I}_k are smooth functions of τ_k and (c, q) , the function $\Phi(k; c, q) := \Phi_k \circ \operatorname{Sch}^{-1}(c, q)$ is smooth in all its arguments. This properties imply the higher order convergence. \square

2.2. The parametrizations Ψ_k . The diffeomorphism \mathcal{H} from Theorem 1.10 allows us to convert the family of parametrizations $(\Phi_k)_k$, which take values in $T^*\mathcal{T}_\Sigma^c$, into a family of parametrizations $(\hat{\Psi}_k)_k$ with values in $\mathcal{T}_\Sigma^h \times \mathcal{T}_\Sigma^h$. Indeed, the functions

$$\begin{aligned} \hat{\Psi}_k := \mathcal{H} \circ \Phi_k : \quad \mathcal{E}(\Sigma) &\longrightarrow \mathcal{T}_\Sigma^h \times \mathcal{T}_\Sigma^h \\ [E] &\longmapsto (h_k, h_k^*), \end{aligned}$$

associate to each hyperbolic end E , the pair of hyperbolic metrics $h_k = (-k)I_k$ and $h_k^* = -\frac{k}{k+1}\mathbb{I}_k$ coming from the first and third fundamental forms of the k -surface Σ_k of E , as we described in Section 1.2.

The maps $\hat{\Psi}_k$ have been the main object of study of Bonsante, Mondello and Schlenker in [BMS13], [BMS15]. In these works, the authors introduced the notions of *landslide flow* and of *smooth grafting* SGr'_s , and studied their convergence to the classical earthquake flow and grafting map Gr . Our functions $\hat{\Psi}_k$ are actually the inverses of the maps SGr'_s (the relation between k and s is $k = -\frac{1}{\cosh^2(s/2)}$).

As the Schwarzian parametrization can be recovered from the limit of the maps Φ_k when $k \rightarrow 0$, the Thurston parametrization can be recovered from the limit of the maps $\hat{\Psi}_k$ when $k \rightarrow -1$. Indeed, we have:

Theorem 2.5. *The maps $\hat{\Psi}_k$ converge to Th , as k goes to -1 , in the following sense: if E is a hyperbolic end, then the length spectrum of \mathbb{I}_k converges to $\mathfrak{l}(\cdot, \mu)$, where $\mathfrak{l}(\cdot, \cdot)$ denotes the geometric intersection of currents. Moreover, the first fundamental forms I_k converge to the hyperbolic metric of the locally concave pleated boundary ∂E .*

Proof. Let E be a fixed hyperbolic end. The convergence of the first fundamental forms I_k is a direct consequence of Theorem 2.1.

By the correspondence between hyperbolic ends and maximal global hyperbolic spatially compact (MGHC) de Sitter spacetimes (see e. g. [Mes07] and the duality described in Section 1.2), the foliation by k -surfaces of E determines a constant curvature surfaces foliation of the MGHC de Sitter spacetime E^* dual of E . Through this correspondence, the third fundamental form \mathbb{I}_k of the leaf Σ_k in E can be interpreted as the first fundamental form of its dual surface Σ_k^* in E^* , which has constant intrinsic curvature equal to $\frac{k}{k+1}$. Moreover, the initial singularity of E^* is dual of the bending measured lamination μ of the pleated boundary ∂E , as shown by Benedetti and Bonsante [BB09, Chapter 3].

In [Bel17], the author studied the intrinsic metrics of families of surfaces which foliate a neighborhood of the initial singularity in E^* . In particular, Belraoui [Bel17, Theorem 2.10] proved that, for a wide class of such foliations, the intrinsic metrics of the surfaces converge, with respect to the Gromov equivariant topology, to the real tree dual of the measured lamination μ , as the surfaces approach the initial singularity of E^* . By applying this result to the constant curvature foliation of E^* , and interpreting \mathbb{I}_k as the first fundamental forms of its leaves, we deduce the convergence of the length spectrum of \mathbb{I}_k to $\iota(\cdot, \mu)$. \square

Hyperbolic length functions. Following [BMS15], we define

$$j: \mathcal{T}_\Sigma^h \times \mathcal{T}_\Sigma^h \longrightarrow \mathbb{R} \\ (h, h^*) \longmapsto \int_\Sigma \text{tr} b \, da_h,$$

which associates, to a normalized pair of hyperbolic metrics h, h' with Labourie operator $b: T\Sigma \rightarrow T\Sigma$ (see Definition 1.9), the integral of the trace of b with respect to the area measure of h (here we are identifying, with abuse, the hyperbolic metrics h and h' with their isotopy classes). The quantity $j(h, h')$ satisfies

$$j(h, h') = 2 E(\text{id}: (\Sigma, c) \rightarrow (\Sigma, h)) = 2 E(\text{id}: (\Sigma, c) \rightarrow (\Sigma, h')),$$

where c is the conformal class of $h(b, \cdot)$, and $E(\cdot)$ denotes the energy functional (see [BMS15, Section 1.2]). This shows in particular that j is symmetric, i. e. $j(h, h') = j(h', h)$.

For any hyperbolic metric h' , we define $L_{h'}: \mathcal{T}_\Sigma^h \rightarrow \mathbb{R}$ to be $L_{h'}(h) := j(h, h')$. The functions $L_{h'}$, which are real analytic by [BMS15, Proposition 1.2], can be interpreted as generalizations of length functions, in light of the following fact:

Proposition 2.6. *Let $(h_n)_n, (h_n^*)_n$ be two sequences of hyperbolic metrics. Suppose that $(h_n)_n$ converges to $h \in \mathcal{T}_\Sigma^h$, and that there exists a sequence of positive numbers $(\varepsilon_n)_n$ such that the length spectrum of $\varepsilon_n^2 h_n^*$ converges to $\iota(\cdot, \mu)$, for some measured lamination $\mu \in \mathcal{ML}_\Sigma$. Then*

$$\lim_{n \rightarrow \infty} \varepsilon_n L_{h_n^*}(h_n) = L_\mu(h).$$

Proof. Using the interpretation via k -surfaces, we can easily prove this statement, which is purely 2-dimensional, using 3-dimensional hyperbolic geometry.

First we observe that, since the injectivity radius of h_n^* is going to 0, the sequence ε_n must converge to 0. In particular, the limit of $k_n := -(\cosh^2 \varepsilon_n)^{-1}$ is equal to -1 , as n goes to infinity. In [BMS13, Proposition 6.2], the authors proved that, under our hypotheses, the sequence of hyperbolic ends $(E_n)_n$ given by $E_n := \hat{\Psi}_{k_n}^{-1}(h_n, h_n^*)$ (which, in the notation of [BMS13], coincides with $\text{SGR}'_{2\varepsilon_n}(h_n, h_n^*)$), converges to $E := \text{Gr}_\mu(h)$. Recalling the definitions of h_n, h_n^* , we see that

$$L_{h_n^*}(h_n) = -\frac{k_n}{\sqrt{k_n + 1}} \int_{\Sigma_{k_n}} H_{k_n} \, da_{I_{k_n}},$$

where Σ_{k_n} is the k_n -surface inside E_n , and I_{k_n} and H_{k_n} are its first fundamental form and mean curvature, respectively.

Since E_n goes to $E = \text{Gr}_\mu(h)$, and k_n goes to -1 , the intrinsic metrics of the surfaces Σ_{k_n} converge to the hyperbolic metric h of the pleated boundary ∂E , and the bending measures of ∂E_n converge to μ . In particular, the integral of the mean curvature of Σ_{k_n} converges to $L_\mu(h)$, the length of the bending measure of ∂E (see for instance [Maz18, Section 2]). From the relation between k_n and ε_n , we see that

$$\lim_{n \rightarrow \infty} \varepsilon_n \left(-\frac{k_n}{\sqrt{k_n + 1}} \right) = 1.$$

The combination of these two facts implies the statement. \square

As done in [BMS15], instead of working directly with $\hat{\Psi}_k$, we will introduce a family of maps $(\Psi_k)_k$ that have the advantage of taking values in the cotangent space $T^*\mathcal{J}_\Sigma^h$. This will be more convenient for the rest of our paper, since we investigate the properties of these parametrizations with respect to the cotangent symplectic structure of $T^*\mathcal{J}_\Sigma^h$ and $T^*\mathcal{J}_\Sigma^c$. The functions Ψ_k are defined as follows:

$$\begin{aligned} \Psi_k : \mathcal{E}(\Sigma) &\longrightarrow T^*\mathcal{J}_\Sigma^h \\ [E] &\longmapsto \left(h_k, -\frac{\sqrt{k+1}}{k} d(L_{h_k^*})_{h_k} \right), \end{aligned}$$

where $d(L_{h_k^*})_{h_k}$ denotes the differential of the function $L_{h_k^*}$, defined as before, at the point h_k . We also consider the function

$$\begin{aligned} dL : \mathcal{J}_\Sigma^h \times \mathcal{ML}_\Sigma &\longrightarrow T^*\mathcal{J}_\Sigma^h \\ (h, \mu) &\longmapsto (h, d(L_\mu)_h). \end{aligned}$$

Proposition 2.7. *The functions*

$$dL \circ \text{Th} : \mathcal{E}(\Sigma) \longrightarrow T^*\mathcal{J}_\Sigma^h \quad \text{and} \quad \Psi_k : \mathcal{E}(\Sigma) \longrightarrow T^*\mathcal{J}_\Sigma^h$$

are \mathcal{C}^1 diffeomorphisms, for every $k \in (-1, 0)$. Moreover, the functions Ψ_k converge pointwisely to $dL \circ \text{Th}$ as k goes to -1 .

Proof. A proof of the \mathcal{C}^1 -regularity of $dL \circ \text{Th}$ can be found in [KS09, Lemma 1.1]. The smoothness of the maps $\hat{\Psi}_k$ follows from the original work of Labourie [Lab92]. Up to scalar multiplication in the fiber, the functions Ψ_k are equal to the composition of the $\hat{\Psi}_k$'s with the map

$$\begin{aligned} \mathcal{J}_\Sigma^h \times \mathcal{J}_\Sigma^h &\longrightarrow T^*\mathcal{J}_\Sigma^h \\ (h, h') &\longrightarrow (h, d(L_{h'})_h). \end{aligned}$$

This function has been proved to be a diffeomorphism in [BMS15, Proposition 1.10]. This shows that Ψ_k is a diffeomorphism for every $k \in (-1, 0)$. The pointwise convergence of the functions Ψ_k follows from Theorem 2.5, Proposition 2.6 and the analyticity of the functions $L_{h'}$, established in [BMS15, Proposition 1.2]. \square

3. VOLUMES AND SCHLÄFLI FORMULAS

In this section we define two families of volume functions for convex co-compact hyperbolic 3-manifolds: the W_k -volumes, related to the notion of W -volume introduced in [KS08], and the V_k^* -volumes, related to the notion of *dual volume* introduced in [KS09]. For both these families we will prove a Schläfli-type variation formula, involving the extremal length, in the case of W_k , and the hyperbolic length functions L_{h^*} introduced in the previous section, in the case of V_k^* . We also describe a simple way to compute the *renormalized volume* V_R of a convex co-compact hyperbolic manifold using the volumes W_k .

3.1. **W_k -volumes.** Let M be a Kleinian manifold and let $\mathcal{G}(M)$ denote the space of convex co-compact hyperbolic structures of M . We define

$$W_k(M) := W(M_k) = V(M_k) - \frac{1}{4} \int_{\partial M_k} H_k da_{I_k},$$

where M_k denotes the compact region of M bounded by the union of the k -surfaces sitting inside the ends of M . The quantities $I_k, \mathbb{I}_k, \mathbb{I}_k, c_k$ and q_k of ∂M_k are defined using the conventions of the previous section.

Lemma 3.1. *The function $W_k: \mathcal{G}(M) \rightarrow \mathbb{R}$ satisfies*

$$d(W_k)_M(\delta M) = -\operatorname{Re}\langle q_k, \delta c_k \rangle.$$

Proof. We apply the variation formula of the W -volume, proved in Proposition A.3. Since the boundary of M_k is a k -surface for every convex co-compact structure M , the term involving δK_e vanishes. Therefore we have:

$$\begin{aligned} d(W_k)_M(\delta M) &= \frac{1}{4} \int_{\Sigma_k} \left(\delta \mathbb{I}_k, \mathbb{I}_k - \frac{H_k}{2} \mathbb{I}_k \right) da_{I_k} \\ \text{(eq. (5))} \quad &= \frac{1}{4\sqrt{k+1}} \int_{\Sigma_k} \left(\delta \mathbb{I}_k, \mathbb{I}_k - \frac{H_k}{2} \mathbb{I}_k \right) da_{\mathbb{I}_k} \\ \text{(eq. (4))} \quad &= -\frac{1}{8} \int_{\Sigma_k} (\delta \mathbb{I}_k, 2\operatorname{Re} q_k) da_{\mathbb{I}_k} \\ \text{(Lemma A.1)} \quad &= -\operatorname{Re}\langle q_k, \delta c_k \rangle. \end{aligned}$$

□

Starting from Lemma 3.1, the proof of the Schläfli formula for the volumes W_k proceeds in analogy to what done by Schlenker [Sch17] for the Schläfli formula for the renormalized volume, thanks to the following result:

Theorem 3.2 (Gardiner's formula, [Gar84, Theorem 8]). *Let (Σ, c) be a Riemann surface, and let \mathcal{F} denote the horizontal foliation of a homorphic quadratic differential q of (Σ, c) . Then the extremal length function $\operatorname{ext}_{\mathcal{F}}: \mathcal{T}_{\Sigma}^c \rightarrow \mathbb{R}$ satisfies*

$$d(\operatorname{ext}_{\mathcal{F}})_c(\delta c) = 2\operatorname{Re}\langle q, \delta c \rangle.$$

The combination of Lemma 3.1 and the Gardiner's formula immediately implies:

Theorem 3.3 (Schläfli formula for W_k). *The differential of the function $W_k: \mathcal{G}(M) \rightarrow \mathbb{R}$ can be expressed as follows:*

$$d(W_k)_M(\delta M) = -\frac{1}{2} d(\operatorname{ext}_{\mathcal{F}_k})_{c_k}(\delta c_k),$$

where \mathcal{F}_k denotes the horizontal foliation of the holomorphic quadratic differential q_k .

3.2. **The renormalized volume.** The definition of renormalized volume $V_R(M)$ of a conformally compact Einstein manifold M is motivated by the AdS/CFT correspondence of string theory [Wit98], [Gra00]. Krasnov and Schlenker [KS08] enlightened its geometrical meaning in the context of convex co-compact hyperbolic 3-manifolds, describing a regularization procedure based on equidistant foliations from convex subsets of M . In relation with the study of the geometry of the Teichmüller space, the renormalized volume furnishes a Kähler potential for the Weil-Petersson metric of the Teichmüller space, and it allows to give a remarkably simple proof of McMullen's Kleinian reciprocity (see [KS08] and Section 5). Moreover, its variation formula has been used by Schlenker [Sch13] to give a quantitative version of Brock's upper bound of the volume of the convex core of a quasi-Fuchsian manifold in terms of the Weil-Petersson distance between the hyperbolic metrics on the boundary of the convex core.

The aim of this Section is to describe a new and simpler way to define the renormalized volume of a quasi-Fuchsian manifold in terms of the asymptotic of its foliation by k -surfaces.

First we recall the Schläfli-type formula of the renormalized volume:

Theorem 3.4 ([KS08, Lemmas 8.3,8.5], [Sch17, Theorem 1.2]). *The differential of the renormalized volume $V_R: \mathcal{G}(M) \rightarrow \mathbb{R}$ can be expressed as follows:*

$$d(V_R)_M(\delta M) = -\operatorname{Re}\langle q_0, \delta c_0 \rangle = -\frac{1}{2} d(\operatorname{ext}_{\mathcal{F}_0})_{c_0}(\delta c_0).$$

The combination of Corollary 2.4 and Theorem 3.3 allows us to give the following description of the renormalized volume $V_R(M)$:

Theorem C. *The renormalized volume of a quasi-Fuchsian manifold M satisfies*

$$V_R(M) = \lim_{k \rightarrow 0^-} \left(W_k(M) - \pi |\chi(\partial M)| \operatorname{arctanh} \sqrt{k+1} \right).$$

Proof. Let $\tilde{W}_k(M) := W_k(M) - \pi |\chi(\partial M)| \operatorname{arctanh} \sqrt{k+1}$. We will prove the assertion by showing the following facts:

- i) the differentials of the functions \tilde{W}_k converge, uniformly over compact subsets of $\mathcal{QF}(\Sigma)$, to the differential of the renormalized volume V_R ;
- ii) the limit, as k goes to 0, of $\tilde{W}_k(M)$ coincides with $V_R(M)$ whenever M is Fuchsian.

Then the assertion will follow from the connectedness of the space $\mathcal{QF}(\Sigma)$.

The first step easily follows from our previous observations. By Corollary 2.4 and Theorem 3.3, $d\tilde{W}_k$ converges, uniformly over compact subsets of $\mathcal{QF}(\Sigma)$, to $-\frac{1}{2} d(\operatorname{ext}_{\mathcal{F}_0})(\delta c_0)$, where \mathcal{F}_0 is the horizontal foliation of the Schwarzian differential at infinity q_0 , and δc_0 is the variation of the conformal structure of $\partial_\infty M$. By Theorem 3.4, this coincides with dV_R .

It remains to prove the second part of the statement. Let M be a Fuchsian manifold. The equidistant surfaces from the convex core of M at distance $\varepsilon(k) := \operatorname{arctanh} \sqrt{k+1}$ are the two k -surfaces of M . Their fundamental forms can be expressed as follows:

$$I_k = -\frac{1}{k}h, \quad II_k = -\frac{k}{\sqrt{k+1}}h, \quad III_k = -\frac{k}{k+1}h,$$

where h is the hyperbolic metric on the totally geodesic surface sitting inside M . From here, we easily see that:

$$\int_{\Sigma_k} H_k da_{I_k} = 2\pi |\chi(\partial M)| \sinh 2\varepsilon(k), \quad V(M_k) = \pi |\chi(\partial M)| \left(\frac{\sinh 2\varepsilon(k)}{2} + \varepsilon(k) \right).$$

In particular, for every Fuchsian manifold M , we have

$$W_k(M) = V(M_k) - \frac{1}{4} \int_{\Sigma_k} H_k da_{I_k} = \pi |\chi(\partial M)| \operatorname{arctanh} \sqrt{k+1}.$$

Therefore the functions \tilde{W}_k vanish identically over the Fuchsian locus, and the same happens for $V_R(M)$. This concludes the proof of the second step, and therefore of the statement. \square

Remark 3.5. The quantity $\operatorname{arctanh} \sqrt{k+1}$ is equal to the distance of the k -surface from the convex core in the Fuchsian case. For a generic quasi-Fuchsian manifold M , the geometric maximum principle [Lab00, Lemme 2.5.1] shows that the k -surface is at distance less or equal than $\operatorname{arctanh} \sqrt{k+1}$ from the convex core CM .

In the proof that we gave above, we *assumed* the existence of the renormalized volume function V_R and we proved the convergence of the functions W_k to V_R . In fact, with some additional work, it is possible to show that the sequence of functions $(\tilde{W}_k)_k$ is convergent *without* assuming the existence of the function V_R . In other words, we can *define* the

renormalized volume $V_R(M)$ of a convex co-compact hyperbolic manifold M as the limit of the sequence $(\widetilde{W}_k(M))_k$.

3.3. **V_k^* -volumes.** In analogy to what done for the W_k -volumes, we define

$$V_k^*(M) := V^*(M_k) = V(M_k) - \frac{1}{2} \int_{\partial M_k} H_k \, da_{I_k}.$$

The Schläfli formula for V_k^* is a direct consequence of the variation formula for the dual volume (Proposition A.4) and the following expression for the variation of the length function L_{h^*} :

Lemma 3.6 ([BMS15, Lemma 7.9]).

$$d(L_{h^*})(\delta h) = -\frac{1}{2} \int_{\Sigma} (\delta h, h(b \cdot, \cdot) - \text{tr}(b)h)_h \, da_h.$$

In order to simplify the next statement, we extend the definition of the function j to constant curvature metrics, not necessarily hyperbolic. In particular, if g and g' are Riemannian metrics of with constant Gaussian curvatures K and K' , then we set $j(g, g')$ to be $(KK')^{-1/2} j((-K)g, (-K')g')$ (observe that $(-K)g$ and $(-K')g'$ are hyperbolic). In this way, the function j is 1/2-homogeneous in both its arguments, As before, L_g will denote the function $j(g, \cdot)$.

Theorem 3.7 (Schläfli formula for V_k^*). *The differential of the function $V_k^*: \mathcal{G}(M) \rightarrow \mathbb{R}$ can be expressed as follows:*

$$d(V_k^*)_M(\delta M) = -\frac{1}{2} d(L_{\mathbb{I}_k})(\delta I_k).$$

Proof. By Proposition A.4, the variation of V_k^* verifies

$$d(V_k^*)_M(\delta M) = \frac{1}{4} \int_{\Sigma_k} (\delta I_k, \mathbb{I}_k - H_k I_k)_{I_k} \, da_{I_k}.$$

Using the definitions of h_k, h_k^* , we can rephrase the expression above as follows:

$$d(V_k^*)_M(\delta M) = -\frac{\sqrt{k+1}}{4k} \int_{\Sigma_k} (\delta h_k, h_k(b_k \cdot, \cdot) - \text{tr}(b_k)h_k) \, da_{h_k},$$

where $b_k = \sqrt{k+1} B_k$ is the Labourie operator between h_k and h_k^* (see Definition 1.9). By Lemma 3.6, the expression above is equal to $\frac{\sqrt{k+1}}{2k} d(L_{h_k^*})_{h_k}(\delta h_k) = -\frac{1}{2} d(L_{\mathbb{I}_k})_{I_k}(\delta I_k)$. \square

Let M be a convex co-compact hyperbolic 3-manifold. By Theorem 2.1, the convex subsets M_k approximate, as k goes to -1 , the *convex core* of M , which is the smallest non-empty convex subset of M (here $C \subseteq M$ is convex if, for every $x, y \in C$, and for every geodesic arc γ starting at x and ending at y , γ is entirely contained in C). While the surfaces ∂M_k are smoothly embedded in M , the boundary of the convex core has the structure of a convex pleated surface, which is only topologically embedded in M . However, it is possible to reasonably extend the notion of dual volume to the convex core too, by setting:

$$V_C^*(M) := V_{-1}^*(M) = V(CM) - \frac{1}{2} L_\mu(h).$$

It is not difficult to see that this definition is continuous in k , i. e. the limit as k goes to 0 of the volumes $V_k^*(M)$ is equal to $V_C(M)$ (see e. g. [Maz18, Section 2]). It turns out that the variation formula of the dual volume of the convex core is, at least formally, the limit of the Schläfli formulas of V_k^* as k goes to -1 , in light of Proposition 2.7. Indeed, we have the following:

Theorem 3.8 (Dual Bonahon-Schläfli formula, [KS09], [Maz18]).

$$d(V_C^*)_M(\delta M) = -\frac{1}{2} d(L_\mu)(\delta h)$$

This result has been first established by Krasnov and Schlenker [KS09] applying the variation formula of the "standard" hyperbolic volume of the convex core, proved by Bonahon [Bon98]. In a recent work [Maz18], we gave a new proof of this result that does not involve the study of the first order variation of the bending measured lamination, which was a highly technical difficulty to handle in the original work [Bon98].

4. VOLUMES AND SYMPLECTOMORPHISMS

The aim of this section is to study the properties of the maps Φ_k and $\Psi_{k'}$. In particular, we will prove that the diffeomorphisms $\Phi_k \circ \Psi_{k'}^{-1} : T^*\mathcal{T}_\Sigma^{\mathfrak{h}} \rightarrow T^*\mathcal{T}_\Sigma^{\mathfrak{c}}$ are symplectic with respect to the cotangent symplectic structures of $T^*\mathcal{T}_\Sigma^{\mathfrak{h}}$ and $T^*\mathcal{T}_\Sigma^{\mathfrak{c}}$, up to a multiplicative factor. This fact extends the results of Krasnov and Schlenker [KS09] and Bonsante, Mondello, and Schlenker [BMS15] concerning the grafting map Gr and the smooth grafting map SGr, respectively.

Relative volumes. Let E be a hyperbolic end. We denote by E_k the portion of E that is in between the concave pleated boundary ∂E and the k -surface Σ_k of E . Now we define

$$w_k(E) := V(E_k) - \frac{1}{4} \int_{\Sigma_k} H_k da_k + \frac{1}{2} L_\mu(m),$$

where H_k and da_k are the mean curvature and the area form of Σ_k , and $L_\mu(m)$ is the length of the bending measure μ with respect to the hyperbolic metric m of ∂E . Similarly, we define

$$v_k^*(E) := V(E_k) - \frac{1}{2} \int_{\Sigma_k} H_k da_k + \frac{1}{2} L_\mu(m).$$

The functions w_k and v_k^* are relative versions of the W_k -volume and V_k^* -volume, respectively.

Cotangent symplectic structures. Let M be a smooth n -manifold, with cotangent bundle $\pi : T^*M \rightarrow M$. The Liouville form λ of T^*M is the 1-form defined by:

$$\lambda_{(p,\alpha)}(v) := \alpha(d\pi_{(p,\alpha)}(v))$$

for every $(p, \alpha) \in T^*M$ and $v \in T_{(p,\alpha)}T^*M$. The 2-form $\omega := d\lambda$ is non-degenerate and it defines a natural symplectic structure on the total space T^*M .

In the following, $\lambda^{\mathfrak{h}}$, $\lambda^{\mathfrak{c}}$ will denote the Liouville forms of $T^*\mathcal{T}_\Sigma^{\mathfrak{h}}$, $T^*\mathcal{T}_\Sigma^{\mathfrak{c}}$, respectively, and $\omega^{\mathfrak{h}}$, $\omega^{\mathfrak{c}}$ their associated symplectic forms. As before, Th stands for the Thurston parametrization, which we briefly recalled in the end of Section 1. The reader can find the necessary notation concerning the geometry of k -surfaces at the beginning of Section 2, and the definitions of the parametrizations Φ_k and Ψ_k in Sections 2.1 and 2.2, respectively.

The first step of our analysis will be to describe the pullback of the Liouville forms $\lambda^{\mathfrak{c}}$ and $\lambda^{\mathfrak{h}}$ by the maps Φ_k and $dL \circ \text{Th}$, Ψ_k , respectively. In particular, we have:

Lemma 4.1. *The following relations hold:*

$$(6) \quad (\Phi_k^* \lambda^{\mathfrak{c}})_E(\delta E) = \frac{1}{4} \int_{\Sigma_k} (\delta \mathbb{I}_k, \text{Re } q_k)_{\mathbb{I}_k} da_{\mathbb{I}_k},$$

$$(7) \quad ((dL \circ \text{Th})^* \lambda^{\mathfrak{h}})_E(\delta E) = d(L_\mu)_m(\delta m),$$

$$(8) \quad (\Psi_k^* \lambda^{\mathfrak{h}})_E(\delta E) = -\frac{1}{2} \int_{\Sigma_k} (\delta I_k, \mathbb{I}_k - H_k I_k)_{I_k} da_{I_k},$$

where δI_k and $\delta \mathbb{I}_k$ represent the variations of the first and second fundamental forms of the k -surface, respectively.

Proof. The Liouville form λ^c of $T^*\mathcal{J}_\Sigma^c$ satisfies

$$(\Phi_k^* \lambda^c)_E(\delta E) = \lambda_{(c_k, q_k)}^c(d(\Phi_k)_E(\delta E)) = \operatorname{Re}\langle q_k, \delta c_k \rangle,$$

where δc_k is the Beltrami differential representing the variation of c_k as we deform the hyperbolic end along the direction δE . Then relation (6) follows from Lemma A.1.

Relation (7) has been originally shown by Krasnov and Schlenker in the proof of [KS09, Theorem 1.2]. First observe that the 1-form $(dL \circ \operatorname{Th})^* \lambda^h$ is well defined since the function $dL \circ \operatorname{Th}$ is \mathcal{C}^1 (see Proposition 2.7). Similarly to what done above, we see that

$$((dL \circ \operatorname{Th})^* \lambda^h)_E(\delta E) = \lambda_{(m, d(L_\mu)_m)}^h(d(dL \circ \operatorname{Th})_E(\delta E)) = d(L_\mu)_m(\delta m),$$

where δm denotes the first order variation of the hyperbolic metric of the concave pleated surface ∂E along the direction δE .

Finally, the Liouville form λ^h satisfies

$$(\Psi_k^* \lambda^h)_E(\delta E) = -\frac{\sqrt{k+1}}{k} d(L_{h_k^*})_{h_k}(\delta h_k).$$

Therefore, relation (8) follows from Lemma 3.6 by backtracking the multiplicative factors involved in the definitions of all the quantities. \square

Similarly to what done in the previous Section, we can describe the first order variation of the relative volume functions w_k and v_k^* as follows:

Lemma 4.2. *The relative volumes w_k and v_k^* satisfy:*

$$\begin{aligned} d(w_k)_E(\delta E) &= \frac{1}{4} \int_{\Sigma_k} \left(\delta \mathbb{I}_k, \mathbb{I}_k - \frac{H_k}{2} \mathbb{I}_k \right) da_{I_k} + \frac{1}{2} d(L_\mu)_m(\delta m), \\ d(v_k^*)_E(\delta E) &= \frac{1}{4} \int_{\Sigma_k} (\delta I_k, I_k - H_k I_k) da_{I_k} + \frac{1}{2} d(L_\mu)_m(\delta m). \end{aligned}$$

Proof. Both the relations can be proved by applying the same strategy of [KS09, Proposition 4.3]. Let $(g_t)_t$ be a differentiable 1-parameter family of hyperbolic metrics on $\Sigma \times (0, \infty)$ so that the first order variation of $E_t = (\Sigma \times (0, \infty), g_t)$ coincides with δE . For any t , we choose an embedded surface S in $\Sigma \times (0, \infty)$ that lies below the k -surface of E_t (i. e. it is contained in the interior of the region $(E_t)_k$ for all small values of t). Now we decompose the quantity $w_k(E)$ in two terms:

$$w_k(E) = \left(V(N(S, \Sigma_{t,k})) - \frac{1}{4} \int_{\Sigma_{t,k}} H_{t,k} da_{k,t} \right) + \left(V(N(\partial E_t, S)) + \frac{1}{2} L_{\mu_t}(m_t) \right)$$

where $\Sigma_{t,k}$ is the k -surface of E_t , and $N(S', S'')$ denotes the region of E bounded by S' from below and S'' from above.

Following step by step the proof of Proposition A.3, we see that the variation of the first term equals

$$\frac{1}{4} \int_{\Sigma_k} \left(\delta \mathbb{I}_k, \mathbb{I}_k - \frac{H_k}{2} \mathbb{I}_k \right) da_{I_k} + \frac{1}{2} \int_S \left(\delta H + \frac{1}{2} (\delta I, \mathbb{I}) \right) da,$$

where the mean curvature H and the second fundamental form \mathbb{I} of S are defined with respect to the normal vector field of S pointing *towards* the concave boundary ∂E .

The variation formula of the right term can be computed with the exact same argument of [Maz18], the only difference is that we are looking at a region bounded by a smooth surface and a locally *concave* pleated surface, while in [Maz18] we were considering the convex core, which is a region bounded by convex pleated surfaces. This leads to the following variation:

$$\frac{1}{2} d(L_\mu)_m(\delta m) + \frac{1}{2} \int_S \left(\delta(-H) + \frac{1}{2} (\delta I, -\mathbb{I}) \right) da.$$

The signs multiplying H and \mathbb{I} are due to the fact that we need to consider the mean curvature and the second fundamental form defined with the normal vector field pointing *outside* of $N(\partial E, S)$, which is the opposite of the one considered above. In particular, when we look at the sum of the two terms, the integrals over S simplify, and we are left with the first relation of our statement.

The second relation follows by an analogous argument, replacing the use of Proposition A.3 with Proposition A.4. \square

Lemma 4.3. *For every $k \in (-1, 0)$, we have*

$$\begin{aligned} dw_k &= -\Phi_k^* \lambda^c + \frac{1}{2} (dL \circ \text{Th})^* \lambda^h, \\ dv_k^* &= -\frac{1}{2} \Psi_k^* \lambda^h + \frac{1}{2} (dL \circ \text{Th})^* \lambda^h. \end{aligned}$$

Proof. The statement is a direct consequence of Lemma 4.1 and Lemma 4.2. \square

Taking the differential of the identities in Lemma 4.3, and remembering that $d^2 = 0$, we immediately conclude the following:

Theorem 4.4. *For every $k \in (-1, 0)$, the maps*

$$\begin{aligned} \Phi_k \circ (dL \circ \text{Th})^{-1} &: (T^* \mathcal{T}_\Sigma^h, \omega^h) \longrightarrow (T^* \mathcal{T}_\Sigma^c, 2\omega^c), \\ \Psi_k \circ (dL \circ \text{Th})^{-1} &: (T^* \mathcal{T}_\Sigma^h, \omega^h) \longrightarrow (T^* \mathcal{T}_\Sigma^h, \omega^h) \end{aligned}$$

are symplectomorphisms.

Observe that Theorem 4.4 is a direct consequence of what we just observed.

Remark 4.5. Theorem 4.4, combined with Corollary 2.4, implies that the map

$$\text{Sch} \circ (dL \circ \text{Th})^{-1} : (T^* \mathcal{T}_\Sigma^h, \omega^h) \longrightarrow (T^* \mathcal{T}_\Sigma^c, 2\omega^c)$$

is a symplectomorphism, which has been originally shown in [KS09, Theorem 1.2]. In addition, [KS09, Theorem 1.2] and Theorem D imply also [BMS15, Theorem 1.11], which states that the function

$$\text{Sch} \circ \Psi_k^{-1} : (T^* \mathcal{T}_\Sigma^h, \omega^h) \longrightarrow (T^* \mathcal{T}_\Sigma^c, 2\omega^c)$$

is a symplectomorphism for every $k \in (-1, 0)$. Finally, by applying Theorem D to the case $k = k'$, and taking care of the multiplicative factors involved in the definitions of Φ_k and Ψ_k , we deduce that the function

$$\begin{aligned} \hat{\mathcal{H}} : (T^* \mathcal{T}_\Sigma^c, \omega^c) &\longrightarrow (T^* \mathcal{T}_\Sigma^h, \omega^h) \\ (c, q) &\longrightarrow (h(c, q), d(L_{h(c, -q)})) \end{aligned}$$

is a symplectomorphism, where $h(c, \pm q) = \varphi_c^{-1}(\pm q)$ is the hyperbolic metric of Σ for which the identity map $(\Sigma, c) \rightarrow (\Sigma, h(c, \pm q))$ is harmonic with Hopf differential equal to $\pm q$ (see Theorem 1.5).

5. KLEINIAN RECIPROCITIES

Let M be a Kleinian manifold and let $\mathcal{G}(M)$ denote the space of convex co-compact hyperbolic structures of M . Any isotopy class of hyperbolic metrics $[g] \in \mathcal{G}(M)$ has a collection of k -surfaces, each one sitting inside a hyperbolic end E_i of (M, g) . In this way, we can define a function

$$\phi_k : \mathcal{G}(M) \longrightarrow T^* \mathcal{T}_{\partial M}^c,$$

which associates to any class $[g]$ the data $(\Phi_k(E_i))_i$ of its k -surfaces. Similarly, we define the function $\psi_k : \mathcal{G}(M) \rightarrow T^* \mathcal{T}_{\partial M}^h$, sending $[g]$ into the data $(\Psi_k(E_i))_i$.

Theorem D. *For every $k \in (-1, 0)$, the image $\phi_k(\mathcal{G}(M))$ (resp. $\psi_k(\mathcal{G}(M))$) is a Lagrangian submanifold of $T^* \mathcal{T}_\Sigma^c$ (resp. $T^* \mathcal{T}_\Sigma^h$).*

Proof. The statement is a consequence of Lemma 4.3 and of the variation formula of the dual volume of a convex co-compact hyperbolic manifold. To see this, first we apply Lemma 4.3 to each end of M :

$$dw_{k,i} = -\Phi_{k,i}^* \lambda_i^c + \frac{1}{2} (dL_i \circ \text{Th}_i)^* \lambda_i^h.$$

By the dual Bonahon-Schläfli formula (Theorem 3.8), we have that

$$dV^*(\delta E) = -\frac{1}{2} \sum_i d(L_{\mu_i})(\delta m_i) = -\frac{1}{2} \sum_i ((dL_i \circ \text{Th}_i)^* \lambda_i^h)(\delta E).$$

Therefore we deduce that

$$d\left(\sum_i w_{k,i} + V^*\right) = \sum_i dw_{k,i} + dV^* = -\phi_k^* \left(\sum_i \lambda_i^c\right) = -\phi_k^* \lambda^c.$$

The function $\sum_i w_{k,i} + V^*$ turns out to be equal to the W -volume of M_k , the portion of M contained in the union of the k -surfaces of the ends $(E_i)_i$. Indeed:

$$\begin{aligned} \sum_i w_{k,i}(E_i) + V^*(M) &= \sum_i \left(V(E_{k,i}) - \frac{1}{4} \int_{\Sigma_{k,i}} H_{k,i} da_{k,i} + \frac{1}{2} L_{\mu_i}(m_i) \right) + V(CM) - \frac{1}{2} L_{\mu}(m) \\ &= V(CM) + \sum_i V(E_{k,i}) - \frac{1}{4} \sum_i \int_{\Sigma_{k,i}} H_{k,i} da_{k,i} + \frac{1}{2} L_{\mu}(m) - \frac{1}{2} L_{\mu}(m) \\ &= V(M_k) - \frac{1}{4} \int_{\partial M_k} H_k da_k \\ &= W_k(M). \end{aligned}$$

Therefore we have proved that $dW_k = -\phi_k^* \lambda^c$. Taking the differential of this identity we obtain that $\phi_k^* \omega^c = 0$. This implies the statement, since ϕ_k is an embedding and $2 \dim \mathcal{G}(M) = \dim T^* \mathcal{T}_{\partial M}^c$.

In an analogous manner we can prove that $\psi_k^* \lambda^h = -2dV_k^*$. To see this, it is enough to replace the role of the relative W -volumes $w_{k,i}$ with the dual volumes $v_{k,i}^*$ and then proceed in the exact same way. Again, by taking the differential of the identity $\psi_k^* \lambda^h = -2dV_k^*$, we obtain the second part of the statement. \square

Theorem E is a generalization of Krasnov and Schlenker's reformulation of McMullen's Kleinian reciprocity Theorem [KS09, Theorem 1.5], and their result can be recovered by taking the limit of the identity $\phi_k^* \omega^c = 0$ and applying Corollary 2.4 to each hyperbolic end of M . Moreover, Krasnov and Schlenker [KS09, Theorem 1.4] proved that the image of the function $dL \circ \text{Th}$ is Lagrangian inside $(T^* \mathcal{T}_{\Sigma}^h, \omega^h)$. Since the map $dL \circ \text{Th}$ is the limit of the ψ_k 's, the part of the statement concerning the maps ψ_k can be similarly seen as an extension of Krasnov and Schlenker's original result.

5.1. Quasi-Fuchsian reciprocities. In this section we present a generalization of McMullen's quasi-Fuchsian reciprocity Theorem in its original formulation from [McM98]. First we will recall McMullen's original statement, and then we will see how to formulate Theorem E in a similar manner. We define the Bers' embeddings to be the maps:

$$\beta_X : \begin{array}{ccc} \mathcal{T}_{\bar{\Sigma}} & \longrightarrow & T_X^* \mathcal{T}_{\Sigma} \\ Y & \longmapsto & \text{Sch}(Q(X, Y))^+ \end{array} \quad \beta_Y : \begin{array}{ccc} \mathcal{T}_{\Sigma} & \longrightarrow & T_Y^* \mathcal{T}_{\bar{\Sigma}} \\ Y & \longmapsto & \text{Sch}(Q(X, Y))^- \end{array}$$

where $Q(X, Y)$ denotes the unique quasi-Fuchsian manifold with conformal classes at infinity (X, Y) , and $\text{Sch}(Q(X, Y))^{\pm}$ are the Schwarzian differentials at infinity on the upper and lower boundaries at infinity. McMullen's original formulation of the quasi-Fuchsian reciprocity Theorem is the following:

Theorem 5.1 ([McM98, Theorem 1.6]). *Given $(X, Y) \in \mathcal{T}_\Sigma \times \mathcal{T}_{\bar{\Sigma}}$, the differentials of the Bers' embeddings*

$$d(\beta_X)_Y : T_Y \mathcal{T}_{\bar{\Sigma}} \longrightarrow T_X^* \mathcal{T}_\Sigma, \quad d(\beta_Y)_X : T_X \mathcal{T}_\Sigma \longrightarrow T_Y^* \mathcal{T}_{\bar{\Sigma}}$$

are adjoint linear operators. In other words, $d(\beta_X)_Y = d(\beta_Y)_X^*$.

We want to analogously state statements in the case in which Sch is replaced by ϕ_k or ψ_k . For every $k \in (-1, 0)$, let B_k^c and B_k^h be the maps

$$\begin{array}{ccc} B_k^c : \mathcal{QF}_\Sigma & \longrightarrow & \mathcal{T}_\Sigma^c \times \mathcal{T}_{\bar{\Sigma}}^c \\ M & \longmapsto & (c_k^+, c_k^-) \end{array} \quad \begin{array}{ccc} B_k^h : \mathcal{QF}_\Sigma & \longrightarrow & \mathcal{T}_\Sigma^h \times \mathcal{T}_{\bar{\Sigma}}^h \\ M & \longmapsto & (h_k^+, h_k^-) \end{array}$$

where c_k^\pm are the conformal classes of the second fundamental forms of the upper and lower k -surface of M , respectively, and h_k^\pm are the hyperbolic metrics $(-k)I_k^\pm$ of the upper and lower k -surface of M , respectively.

A consequence of Labourie and Schlenker's works [Lab91], [Sch06] (see also [Maz19, Theorem 4.2] for details) is that the function B_k^h is a diffeomorphism for every $k \in (-1, 0)$. We do not know if the same is true for B_k^c , we will assume this to be true for the rest of this section. In analogy to Bers' embeddings, we define the following maps:

$$\begin{array}{ccc} \beta_{k,X}^c : \mathcal{T}_{\bar{\Sigma}}^c & \longrightarrow & T_X^* \mathcal{T}_\Sigma^c \\ Y & \longmapsto & \phi_k^+ \circ (B_k^c)^{-1}(X, Y) \end{array} \quad \begin{array}{ccc} \beta_{k,Y}^c : \mathcal{T}_\Sigma^c & \longrightarrow & T_Y^* \mathcal{T}_{\bar{\Sigma}}^c \\ Y & \longmapsto & \phi_k^- \circ (B_k^c)^{-1}(X, Y) \end{array}$$

$$\begin{array}{ccc} \beta_{k,X}^h : \mathcal{T}_{\bar{\Sigma}}^h & \longrightarrow & T_X^* \mathcal{T}_\Sigma^h \\ Y & \longmapsto & \psi_k^+ \circ (B_k^h)^{-1}(X, Y) \end{array} \quad \begin{array}{ccc} \beta_{k,Y}^h : \mathcal{T}_\Sigma^h & \longrightarrow & T_Y^* \mathcal{T}_{\bar{\Sigma}}^h \\ Y & \longmapsto & \psi_k^- \circ (B_k^h)^{-1}(X, Y) \end{array}$$

where:

- $(B_k^c)^{-1}(X, Y)$ is the conjecturally unique quasi-Fuchsian manifold whose upper and lower k -surfaces have X and Y as conformal classes of their second fundamental forms, respectively;
- $(B_k^h)^{-1}(X, Y)$ is the unique quasi-Fuchsian manifold whose upper and lower k -surfaces have X and Y as hyperbolic structures induced by their first fundamental forms, respectively;
- $\phi_k^\pm \circ (B_k^c)^{-1}(X, Y)$ are the holomorphic quadratic differentials q_k^\pm on the upper and lower k -surfaces (as defined in Section 2.1);
- $\psi_k^\pm \circ (B_k^h)^{-1}(X, Y)$ are the 1-forms $d(L_{\mathbb{M}_k^\pm})_{I_k^\pm}$ on the upper and lower k -surfaces (as defined in Section 2.2).

Now that we have introduced all the notation, we are ready to state the formulations of the quasi-Fuchsian reciprocity Theorems that follow from Theorem E:

Theorem 5.2. *For every $(X, Y) \in \mathcal{T}_\Sigma^h \times \mathcal{T}_{\bar{\Sigma}}^h$, the differentials of the maps*

$$d(\beta_{k,X}^h)_Y : T_Y \mathcal{T}_{\bar{\Sigma}}^h \longrightarrow T_X^* \mathcal{T}_\Sigma^h, \quad d(\beta_{k,Y}^h)_X : T_X \mathcal{T}_\Sigma^h \longrightarrow T_Y^* \mathcal{T}_{\bar{\Sigma}}^h$$

are adjoint linear operators.

Theorem 5.3. *If the map B_k^c is a diffeomorphism, then for every $(X, Y) \in \mathcal{T}_\Sigma^c \times \mathcal{T}_{\bar{\Sigma}}^c$, the differentials of the maps*

$$d(\beta_{k,X}^c)_Y : T_Y \mathcal{T}_{\bar{\Sigma}}^c \longrightarrow T_X^* \mathcal{T}_\Sigma^c, \quad d(\beta_{k,Y}^c)_X : T_X \mathcal{T}_\Sigma^c \longrightarrow T_Y^* \mathcal{T}_{\bar{\Sigma}}^c$$

are adjoint linear operators.

Proof of Theorems 5.2 and 5.3. Let $F : N^+ \times N^- \rightarrow T^*(N^+ \times N^-)$ be a smooth function satisfying $\pi \circ F = id$, where $\pi : T^*(N^+ \times N^-) \rightarrow N^+ \times N^-$ is the cotangent bundle projection. For every X in N^+ , we set $F_X^+ : N^- \rightarrow T_X^* N^+$ to be $F_X^+(Y) := F(X, Y)^+$, where $F(X, Y)^+$ is the component of $F(X, Y)$ in the fiber $T_X^* N^+$, and, for every $Y \in N^-$, we set

$F_Y^- : N^+ \rightarrow T_Y^* N^-$ to be $F_Y^-(X) := F(X, Y)^-$, where $F(X, Y)^-$ is the component of $F(X, Y)$ in the fiber $T_Y^* N^-$. Then, the following relation holds:

$$\langle d(F_Y^-)_X(u), v \rangle - \langle d(F_X^+)_Y(v), u \rangle = (F^* \omega)_{(X, Y)}((u, 0), (0, v)),$$

for all $(X, Y) \in N^+ \times N^-$, $u \in T_X N^+$, $v \in T_Y N^-$. A proof of this relation can be found in [] for the function $F = \text{Sch}$, the proof of the general case is formally identical. Now, using this relation for the maps $F = \phi_k \circ (B_k^c)^{-1}$ and $F = \psi_k \circ (B_k^h)^{-1}$, and applying Theorem E, we obtained the desired statement. \square

6. THE k -FLOWS ARE HAMILTONIAN

In this section we show that the k -surface foliation of a hyperbolic end can be described as the integral curve of a time-dependent Hamiltonian vector field with respect to the symplectic structure $2\Phi_k^* \omega^c = \Psi_k^* \omega^h$ on $\mathcal{E}(\Sigma)$, which does not depend on k in light of Theorem D. The vector fields we will look at are defined in terms of the diffeomorphisms $(\Phi_k)_k$ and $(\Psi_k)_k$ as follows:

$$\begin{aligned} X_k &:= \frac{d}{dh} \Phi_{k+h} \circ \Phi_k^{-1} \Big|_{h=0} \in \Gamma(T(T^* \mathcal{J}_\Sigma^c)), \\ Y_k &:= \frac{d}{dh} \Psi_{k+h} \circ \Psi_k^{-1} \Big|_{h=0} \in \Gamma(T(T^* \mathcal{J}_\Sigma^h)). \end{aligned}$$

In order to simplify the notation, whenever we have an object X that depends on the curvature k , we will denote by \dot{X} its derivative with respect to k . We denote by $m_k : \mathcal{E}(\Sigma) \rightarrow \mathbb{R}$ the function

$$m_k(E) := \int_{\Sigma_k} H_k da_{I_k}.$$

Lemma 6.1. *For every $k \in (-1, 0)$, we have*

$$(9) \quad \lambda^c(X_k) \circ \Phi_k = -\dot{w}_k + \frac{1}{8(k+1)} m_k,$$

$$(10) \quad \lambda^h(Y_k) \circ \Psi_k = -2\dot{v}_k^* + \frac{1}{2k} m_k.$$

Proof. Let E be a fixed hyperbolic end. If (c_k, q_k) denotes the point $\Phi_k(E) \in T^* \mathcal{J}_\Sigma^c$, then the Liouville form λ^c satisfies

$$\lambda^c(X_k) \circ \Phi_k(E) = (\lambda^c)_{\Phi_k(E)} \left(\frac{d}{dh} \Phi_{k+h}(E) \Big|_{h=0} \right) = \text{Re} \langle q_k, \dot{c}_k \rangle.$$

By Proposition A.3, we have

$$\begin{aligned} \dot{w}_k(E) &= \frac{1}{4} \int_{\Sigma_k} \left(\left(\dot{I}_k, \mathbb{I}_k - \frac{H_k}{2} \mathbb{I}_k \right)_{\mathbb{I}_k} + \frac{H_k}{2(k+1)} \right) da_{I_k} \\ &= -\text{Re} \langle q_k, \dot{c}_k \rangle + \frac{1}{8(k+1)} \int_{\Sigma_k} H_k da_{I_k} \\ &= -\text{Re} \langle q_k, \dot{c}_k \rangle + \frac{1}{8(k+1)} m_k(E). \end{aligned}$$

Combining these two relations we obtain the first part of the statement. Similarly, we see that

$$\lambda^h(Y_k) \circ \Psi_k(E) = -\frac{\sqrt{k+1}}{k} d(L_{h_k}^*)_{h_k}(\dot{h}_k).$$

By definition of h_k , we have $\dot{h}_k = -I_k - k \dot{I}_k$. Using Lemma 3.6, we obtain

$$\begin{aligned} \lambda^{\flat}(Y_k) \circ \Psi_k(E) &= \frac{1}{2k} \int_{\Sigma_k} (-I_k - k \dot{I}_k, \mathbb{I}_k - H_k I_k)_{I_k} \, da_{I_k} \\ &= -\frac{1}{2} \int_{\Sigma_k} (\dot{I}_k, \mathbb{I}_k - H_k I_k)_{I_k} \, da_{I_k} + \frac{1}{2k} \int_{\Sigma_k} H_k \, da_{I_k} \\ &= -2v_k^*(E) + \frac{1}{2k} m_k(E), \end{aligned}$$

where, in the last step, we used Proposition A.4. \square

Lemma 6.2. *Let M and N be a n - and a $2n$ -manifold, respectively, and let $\varphi_t: N \rightarrow T^*M$ be a 1-parameter family of diffeomorphisms, indexed by a variable t varying in an open interval J of \mathbb{R} . Denote by λ the Liouville form of T^*M , and set V_t to be the vector field of T^*M given by*

$$V_t := \frac{d}{dh} \varphi_{t+h} \circ \varphi_t^{-1} \Big|_{h=0},$$

for any $t \in J$. Then we have

$$(\varphi_t^{-1})^* \left(\frac{d}{dt} \varphi_t^* \lambda \right) = \iota_{V_t} \omega + d(\iota_{V_t} \lambda),$$

for every $t \in J$.

Proof. The statement is a consequence of Cartan formula. The time-dependent family of vector fields $(V_t)_t$ corresponds to an ordinary vector field \tilde{V} on the manifold $J \times T^*M$, by setting

$$\tilde{V}(t, \cdot) := \partial_t + V_t(\cdot) \in T_t J \times T(T^*M) \cong T_{(t, \cdot)}(J \times T^*M).$$

An integral curve $\gamma = \gamma(t)$ of $(V_t)_t$ in T^*M corresponds to the integral curve $t \mapsto (t, \gamma(t))$ of \tilde{V} in $J \times T^*M$. Let π denote the projection of $J \times T^*M$ onto its second component. We apply Cartan formula to the 1-form $\pi^* \lambda$ and the vector field \tilde{V} , obtaining

$$(11) \quad \mathcal{L}_{\tilde{V}} \pi^* \lambda = \iota_{\tilde{V}} d(\pi^* \lambda) + d(\iota_{\tilde{V}} \pi^* \lambda).$$

A straightforward computation proves the following relations:

$$\begin{aligned} \iota_{\tilde{V}} d(\pi^* \lambda) \Big|_{(t, \cdot)} &= \pi^* (\iota_{V_t} d\lambda) \Big|_{(t, \cdot)}, \\ (\iota_{\tilde{V}} \pi^* \lambda)(t, \cdot) &= (\iota_{V_t} \lambda \circ \pi)(t, \cdot), \\ (\varphi_t^{-1} \circ \pi)^* \left(\frac{d}{dt} \varphi_t^* \lambda \right) \Big|_{(t, \cdot)} &= \mathcal{L}_{\tilde{V}} \pi^* \lambda \Big|_{(t, \cdot)}. \end{aligned}$$

Replacing these expressions in the equation (11), we obtain that, for every $t \in J$

$$\pi^* \left((\varphi_t^{-1})^* \left(\frac{d}{dt} \varphi_t^* \lambda \right) - \iota_{V_t} \omega - d(\iota_{V_t} \lambda) \right) \Big|_{(t, \cdot)} = 0.$$

Since $d\pi_{(t, \cdot)}$ is surjective, the pullback by π at (t, \cdot) is injective on k -forms. In particular, for every $t \in J$ we must have

$$(\varphi_t^{-1})^* \left(\frac{d}{dt} \varphi_t^* \lambda \right) - \iota_{V_t} \omega - d(\iota_{V_t} \lambda) = 0,$$

which proves the statement. \square

Theorem E. *For every $k \in (-1, 0)$, the vector field X_k of $T^*\mathcal{J}_\Sigma^c$ is Hamiltonian with respect to the symplectic structure ω^c , with Hamiltonian function $-\frac{1}{8(k+1)} m_k \circ \Phi_k^{-1}$. Similarly, the vector field Y_k of $T^*\mathcal{J}_\Sigma^{\flat}$ is Hamiltonian with respect to the symplectic structure ω^{\flat} , with Hamiltonian function $-\frac{1}{2k} m_k \circ \Psi_k^{-1}$.*

Proof. From Lemma 4.3 we see that

$$(12) \quad \frac{d}{dk} \Phi_k^* \lambda^c = \frac{d}{dk} \left[-dw_k + \frac{1}{2} (dL \circ \text{Th})^* \lambda^b \right] = -d\dot{w}_k.$$

Applying Lemma 6.2 to $N = \mathcal{E}(\Sigma)$, $M = \mathcal{T}_\Sigma^c$ and $\varphi_t = \Phi_k$, we get

$$(13) \quad (\Phi_k^{-1})^* \left(\frac{d}{dk} \Phi_k^* \lambda^c \right) = \iota_{X_k} \omega^c + d(\iota_{X_k} \lambda^c).$$

Now, putting everything together, we obtain

$$\begin{aligned} \text{(eq. (13))} \quad \iota_{X_k} \omega^c &= (\Phi_k^{-1})^* \left(\frac{d}{dk} \Phi_k^* \lambda^c \right) - d(\iota_{X_k} \lambda^c) \\ \text{(eq (9) and (12))} \quad &= -(\Phi_k^{-1})^* d\dot{w}_k - d \left(-\dot{w}_k \circ \Phi_k^{-1} + \frac{1}{8(k+1)} m_k \circ \Phi_k^{-1} \right) \\ &= -d(\dot{w}_k \circ \Phi_k^{-1}) + d(\dot{w}_k \circ \Phi_k^{-1}) - \frac{1}{8(k+1)} d(m_k \circ \Phi_k^{-1}) \\ &= -\frac{1}{8(k+1)} d(m_k \circ \Phi_k^{-1}), \end{aligned}$$

which proves the first part of the statement. With the exact same strategy we can prove the assertion concerning the vector fields $(Y_k)_k$. \square

Remark 6.3. It can be easily checked that the choice of the multiplicative constant in the definition of q_k , and consequently of Φ_k , becomes relevant for Theorem F to hold. The same holds for the multiplicative constant in the definition of Ψ_k .

APPENDIX A.

Lemma A.1. *Let $(g_t)_t$ be a 1-parameter family of Riemannian metrics on Σ , with conformal classes $c_t = [g_t]$. If δc denotes the Beltrami differential representing the variation of the conformal classes $(c_t)_t$, and δg the variation of the Riemannian metrics $(g_t)_t$, then we have*

$$\text{Re}\langle q, \delta c \rangle = \frac{1}{4} \int_{\Sigma} (\delta g, \text{Re } q)_g da_g.$$

Proof. Let $(g_t)_t$ be a smooth 1-parameter family of metrics so that the conformal class of $g = g_0$ is equal to $c = c_0$, and the derivative at $t = 0$ of the conformal class c_t of g_t coincides with δc . If X_t is an object that depends on t , then δX will denote its derivative with respect to t at $t = 0$. Let J_t be the almost complex structure of g_t for every t . As shown in [BMS15, Section 2.1], the Beltrami differential ν_t of the map $id: (\Sigma, c) \rightarrow (\Sigma, c_t)$ satisfies

$$\nu_t = (\mathbb{1} - J_t J)^{-1} (1 + J_t J),$$

In particular its derivative $\delta \nu$ can be expressed as $\frac{1}{2} \delta J J$. The almost complex structure J_t of g_t is characterized by the relation $da_t(\cdot, \cdot) = g_t(J_t \cdot, \cdot)$, where da_t is the area form of the metric g_t . Taking the derivative of this identity, and using the fact that $da_g = \sqrt{\det(g_{ij})} dx^1 \wedge dx^2$ in local coordinates, we obtain

$$\frac{1}{2} (\delta g, g)_g da = \frac{1}{2} \text{tr}(g^{-1} \delta g) da = \delta(da_t) = \delta(g_t(J_t \cdot, \cdot)) = \delta g(J \cdot, \cdot) + g(\delta J \cdot, \cdot).$$

If $\delta g = g(A \cdot, \cdot)$, with A g -self-adjoint, then from the relation above we see that

$$\delta J J = A - \frac{1}{2} \text{tr}(A) \mathbb{1} = A_0,$$

where A_0 stands for the traceless part of A . In particular, this proves that $\delta \nu = \frac{1}{2} A_0$. The pairing between Beltrami differentials and holomorphic quadratic differentials can be described as follows:

$$\langle q, \mu \rangle := \int_{\Sigma} q \bullet \mu,$$

where $q \bullet \mu$ is the \mathbb{C} -valued 2-form given by

$$(q \bullet \mu)(u, w) := \frac{1}{2i}(q(\mu(u), w) - q(u, \mu(w))).$$

Again, we refer to [BMS15, Section 2.1] for a more detailed description. Let now B be the traceless and g -self-adjoint operator satisfying $\operatorname{Re} q(\cdot, \cdot) = g(B\cdot, \cdot)$. Given any unit vector u , the basis u, Ju is orthonormal and positive oriented. In particular, since $q \bullet \delta v$ is a multiple of the volume form da_g (Σ is a 2-manifold), we must have $q \bullet \delta v = (q \bullet \delta v)(u, Ju) da_g$. Now we observe:

$$\begin{aligned} \operatorname{Re}(q \bullet \delta v)(u, Ju) &= \operatorname{Re} \frac{1}{2i}(q(\delta v(u), Ju) - q(u, \delta v(Ju))) \\ (q \text{ } \mathbb{C}\text{-linear and } J^2 = -1) &= \operatorname{Re} \frac{1}{2i}(iq(\delta v(u), u) + q(J^2 u, \delta v(Ju))) \\ (q \text{ } \mathbb{C}\text{-linear}) &= \operatorname{Re} \frac{1}{2}(q(\delta v(u), u) + q(\delta v(Ju), Ju)) \\ (\text{def. of } B \text{ and } \delta v = \frac{1}{2}A_0) &= \frac{1}{4}(g(BA_0 u, u) + g(BA_0 Ju, Ju)) \\ (u, Ju \text{ orthon. basis}) &= \frac{1}{4} \operatorname{tr}(BA_0) \\ (B \text{ traceless}) &= \frac{1}{4} \operatorname{tr}(BA) \\ (\text{def.s of } A \text{ and } B) &= \frac{1}{4}(\operatorname{Re} q, \delta g) \end{aligned}$$

Combining what we have proved so far, we obtain that

$$\operatorname{Re}\langle q, \delta c \rangle = \int_{\Sigma} \operatorname{Re} q \bullet \delta v = \frac{1}{4} \int_{\Sigma} (\operatorname{Re} q, \delta g) da_g,$$

which is our desired relation. \square

In the following, we are going to recall the variation formula for the dual volume of hyperbolic 3-manifolds, and we will prove the expression of the variation formula of the W -volume that we apply in this paper. First, in order to simplify the notation, we are going to denote by da the area form of the *first* fundamental form I and by (A, B) the scalar product between $(1, 1)$ -tensors induced by the metric I .

In our paper we consider two kinds of variation. One possibility is to fix a smooth compact 3-manifold with boundary N and vary its hyperbolic metrics $(g_t)_t$. Each Riemannian metric g_t determines an induced metric I_t , a second fundamental form \mathbf{I}_t and a mean curvature H_t on ∂N . In this situation, the variations δI , $\delta \mathbf{I}$ and δH have to be understood as the derivatives in t of the families of tensors $(I_t)_t$, $(\mathbf{I}_t)_t$ and $(H_t)_t$ on the fixed manifold ∂N . The other possibility is to fix a hyperbolic metric g on a manifold N' , and to look at a 1-parameter family of subsets N_t of N' , whose boundaries are smooth and vary regularly in t . The relations that we are going to describe hold in both cases, so we will intentionally give "ambiguous" statements that can be applied in both the situations.

Theorem A.2 (Differential Schläfli formula, [RS99, Theorem 1, Theorem 2]). *The variation of the hyperbolic volume of N can be expressed as follows:*

$$\delta V(N) = \frac{1}{2} \int_{\partial N} \left(\delta H + \frac{1}{2}(\delta I, \mathbf{I}) \right) da.$$

The W -volume of a smooth convex subset N of a hyperbolic manifold is defined by the following expression:

$$W(N) := V(N) - \frac{1}{4} \int_{\partial N} H da.$$

Similarly, we define the dual volume of N to be:

$$V^*(N) := V(N) - \frac{1}{2} \int_{\partial N} H da.$$

The first relation expresses the variation of the W -volume:

Proposition A.3.

$$\delta W(N) = \frac{1}{4} \int_{\partial N} \left(\left(\delta \mathbb{I}, \mathbb{I} - \frac{H}{2} \mathbb{I} \right)_{\mathbb{I}} + \frac{\delta K^e}{2K^e} H \right) da.$$

Proof. Since $I = \mathbb{I} B^{-1}$, we have $\delta I = \delta \mathbb{I} B^{-1} - \mathbb{I} B^{-1} \delta B B^{-1}$. Therefore

$$\begin{aligned} (\delta I, \mathbb{I}) &= \text{tr}(I^{-1} \delta I I^{-1} \mathbb{I}) \\ &= \text{tr}(B \mathbb{I}^{-1} (\delta \mathbb{I} B^{-1} - \mathbb{I} B^{-1} \delta B B^{-1}) B \mathbb{I}^{-1} \mathbb{I}) \\ &= \text{tr}(B \mathbb{I}^{-1} \delta \mathbb{I}) - \text{tr}(\delta B) \\ &= \text{tr}(\mathbb{I}^{-1} \mathbb{I} \mathbb{I}^{-1} \delta \mathbb{I}) - \delta H \\ &= (\mathbb{I}, \delta \mathbb{I})_{\mathbb{I}} - \delta H. \end{aligned}$$

Using the fact that $da_g = \sqrt{\det(g_{ij})} dx^1 \wedge dx^2$, we find $\delta(da_g) = \frac{1}{2}(\delta g, g)_g da_g$. Hence we have:

$$\begin{aligned} \delta(da) &= \delta \left(\frac{da_{\mathbb{I}}}{\sqrt{K^e}} \right) \\ &= -\frac{\delta K^e}{2(K^e)^{3/2}} da_{\mathbb{I}} + \frac{1}{2\sqrt{K^e}} (\delta \mathbb{I}, \mathbb{I})_{\mathbb{I}} da_{\mathbb{I}} \\ &= \left(-\frac{\delta K^e}{2K^e} + \frac{1}{2} (\delta \mathbb{I}, \mathbb{I})_{\mathbb{I}} \right) da. \end{aligned}$$

Applying Theorem A.2 and using the relations found above, we obtain:

$$\begin{aligned} \delta W(N) &= \delta V(N) - \frac{1}{4} \delta \left(\int_{\partial N} H da \right) \\ &= \frac{1}{2} \int_{\partial N} \left(\delta H + \frac{1}{2} (\delta I, \mathbb{I}) \right) da - \frac{1}{4} \int_{\partial N} (\delta H da + H \delta(da)) \\ &= \frac{1}{4} \int_{\partial N} \left(2\delta H + (\mathbb{I}, \delta \mathbb{I})_{\mathbb{I}} - \delta H - \delta H - H \left(-\frac{\delta K^e}{2K^e} + \frac{1}{2} (\delta \mathbb{I}, \mathbb{I})_{\mathbb{I}} \right) \right) da \\ &= \frac{1}{4} \int_{\partial N} \left(\left(\delta \mathbb{I}, \mathbb{I} - \frac{H}{2} \mathbb{I} \right)_{\mathbb{I}} + \frac{\delta K^e}{2K^e} H \right) da, \end{aligned}$$

which proves the statement. \square

On the other hand, the variation formula of the dual volume satisfies:

Proposition A.4.

$$\delta V^*(N) = \frac{1}{4} \int_{\partial N} (\delta I, \mathbb{I} - HI) da$$

Proof. We need to apply the differential Schläfli formula and to express the variation of the term $\int H da$. We can proceed similarly to what done for the W -volume, obtaining

$$\begin{aligned} \delta V^*(N) &= \delta V(N) - \frac{1}{2} \delta \left(\int_{\partial N} H da \right) \\ &= \frac{1}{2} \int_{\partial N} \left(\delta H + \frac{1}{2} (\delta I, \mathbb{I}) \right) da - \frac{1}{2} \int_{\partial N} (\delta H da + H \delta(da)) \\ &= \frac{1}{4} \int_{\partial N} \left(2\delta H + (\delta I, \mathbb{I}) - 2\delta H - 2 \frac{H}{2} (\delta I, I) \right) da \\ &= \frac{1}{4} \int_{\partial N} (\delta I, \mathbb{I} - HI) da, \end{aligned}$$

where, in the second step we used the fact that $\delta(da) = \frac{1}{2} (\delta I, \mathbb{I}) da$. This concludes the proof of the statement. \square

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