Chapter 15
Georges-Louis Leclerc de Buffon's 'Essays on Moral Arithmetic'

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We offer a translation into English of the original French version published in 1777 of Buffon's *Essai d'Arithmetique Morale*. In this classic work, Buffon discusses degrees of certainty, probability, the moral value of money, the different evaluations of gains and of losses; moreover, he proposes repeated experiments to determine the moral value of a game. Our hope is that this remarkable work, which anticipates and relates to many aspects discussed in more recent literature, reaches a broader audience than the French original reaches today. Our belief is that we are first to translate this classic work completely. Remaining errors are ours.

Buffon (1707–1788) was acknowledged, and is mainly remembered, for his opus *Natural History*, of which he finalized thirty-six volumes during his lifetime. Another eight volumes were published posthumously.

His scientific work was generally based on the methods of empirical observation and experiment. Based on his evidence, he suggested that the origin of organisms resulted by spontaneous generation from smallest particles and the development and diversity from climatic changes. Differing from the common view at his time, he believed that the first life developed in the sea, and that the stepwise development of the species took long periods. Preceding Charles Darwin (1809–1882), he advanced the view of common descent of the species, discussing the relation between apes and man. By experimental evidence he showed that the earth was older than the 6,000 years calculated by theologians based on biblical data. This evidence led to a conflict with the church including the burning of his writings. He had to officially revoke his statement, but raised the point again in the *Essays*.

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Before dedicating himself to natural history, Buffon studied and contributed to mathematics during the 1720s and 1730s. He communicated, and maintained correspondence with, Gabriel Cramer (1704–1752), professor of analysis and geometry at the University of Geneva, from 1727 (Weil 1961). So, while it is likely that he started editing the *Essays* in the 1760s, his interest in the topic and many of his considered ideas date back to the early 1730s.

The *Essays on Moral Arithmetic* contains 25 articles, the first two being introductory. As pointed out in them, the work is dedicated to the measurement of, and, more generally, to the valuation of uncertainty. Buffon defines levels of uncertainty, or contrariwise, levels of certainty, by available evidence observed in nature, or more generally if the causes of a certain effect are unknown; if effects repeat constantly, he asserts, then the effect becomes certain; after a long sequence of repeated observations effects become physically certain. Conversely, if an effect has constantly failed to occur, he suggests, it will come to be refuted. He also points out that a change in an assumed constant effect surprises us; such effects have been brought to the attention of a great audience through the recent literature (Taleb 2007). Buffon illustrates the concept of certainty levels by his example of the experience of the sunrise. He reflects on how an ignorant man sees the sunrise, the sunset and learns to reinforce his belief and simultaneously to decrease his doubt by repeated experience, finally reaching certainty about the return of the sunrise (see articles III–VI).[1] He refers to physical certainty as an "almost infinite" probability level to which he assigns a relationship as of one to $2^{2,189,999}$.[2] Note the power term represents the number of days following the first day in 6,000 years, as this number of years represented the accepted time of existence of man and the earth during Buffon's lifetime; the base term, 2, represents the two possible effects either the sun returns on the next day or it does not return.

If one has only a limited number of observations of a constant effect so that physical certainty cannot be inferred, Buffon argues that uncertainty can be so much removed if only the number of experiences is sufficiently large; one can achieve moral certainty about the effects. Moral certainty can be considered as a boundedly-rational judgment level, sufficiently great to draw conclusions on the certainty of constant effects. Given the limitations of time and resources on the number of experiences (observations), moral certainty is a substitute for physical certainty that enables one to make a boundedly-rational judgment about the effects of nature even without understanding its causes. In Buffon's thinking, moral certainty implies a

[1]Reinforcement learning is an algorithm that tries to maximize payoffs under uncertainty by successively increasing the weight of better experiences (for further references see Gigerenzer and Selten 2001). In contrast, Buffon's belief learning focuses only on judgment and not on strategy.

[2]So, we should warn readers at the outset that Buffon uses the word certainty not in the absolute sense that we currently use. We think of a certain event as one that is bound to happen, that is, an event that has probability 1 of happening. However, and rather schizophrenically, we also say that one event is more certain than another. This latter sense is that of Buffon. He refers to certainties of different orders and hence uses the word certainty as a synonym for probability or likelihood.

lower degree of certainty than physical certainty, and can be determined either by evidence following a constant sequence, or by analogical reasoning based on testimonies of a constant sequence. The probability level that Buffon assigns to a complete level of moral certainty is one ten-thousandth (see articles VII–IX). This level is based on the argument that a 56 year old man is fearless about dying during any given day, an event which according to his mortality tables, reported elsewhere (Buffon 1777a), occurred with the corresponding relative frequency.

Buffon refers also to different levels of probability for the case when the underlying effect is not constant. Discussing such chance effect in more detail, Buffon focuses on the 50-50 game, where an effect occurs as often as it fails to occur. He suggests that, even in these games, the observation of a large number of results from the risk device can give us an advantage in the game, if the risk device is biased (see articles X–XI). Thus he indicates that underlying probabilities can also be learned from long series of observations, and can deviate from *a priori* probabilities.

Generally, Buffon condemns participation in games of chance, judging them as generally harmful to overall well-being in society (article XII). In particular, he refers to the contemporary popular game of *Pharaoh*, apparently a forerunner of *Poker*. He argues that people behave dishonestly and irrationally with respect to these games of chance. His judgment is based on: (1) a value function approach; and (2) a classification of individual income as either *necessary* or *superfluous* (articles XIII–XIV).

1. His value function exhibits *loss aversion*, as it punishes losses more than it rewards gains. Gains are valued relatively to the *ex post* income including the gains; losses are valued relatively to the *a priori* income position, that is, before subtracting the losses. Thus since losses loom larger than gains, overall well-being is reduced even in fair games. Compared to the logarithmic utility function proposed by Daniel Bernoulli (1700–1782) which was published in 1738, losses are, at the margin, similarly valued, however, the Buffonian value function values gains less than losses. Loss aversion is quite accepted among modern researchers. For instance, Selten and Chmura (2008) assign a double weight to a loss than to a gain in their application of the impulse balance theory to experimental data. Such a double weighting of losses relative to gains was suggested by experimental evidence in Tversky and Kahneman (1992). We should mention here, even without further discussion, that modern loss aversion decisively differs from the Buffonian version in several respects. However, Buffon appears to have been the first to propose loss aversion within a utility approach. This utility approach, however and unfortunately, was not embedded in a fully-fledged and fully-developed theory.
2. He nevertheless suggests the existence of individual utility based on the requirements of personal needs according to one's position in society. He defines *necessary income* (that necessary to sustain the social status of an individual) and *superfluous income* (that over and above that which is necessary). This classification suggests that utility is structured in a way that necessary income represents a safety-first element, similar to Lopez's aspiration theory (1987).

> Because of this classification of income, losses lead to a greater loss of overall well-being (caused by loss aversion), if the loss in the game is out of the necessary income and if the gain increases only the superfluous income, because necessary income must be valued higher than superfluous income. Finally, he assumes bounded utility in the sense of Cramer, that is, beyond a certain threshold superfluous income gives no extra utility.

Buffon dedicates a large part of the *Essays* to a presentation and discussion of the Petersburg gamble (articles XV–XXII). His discussion is so broad that it includes almost all currently-known 'solutions' to the Petersburg paradox.[3] In a footnote he quotes at length the letter he wrote to Gabriel Cramer in 1730. Thus he proves that some of his ideas preceded those of Daniel Bernoulli (1738). He concludes that the Petersburg paradox – that is, the discrepancy between the intuitive value and the mathematical expectation – arises from two causes; first, the small probabilities of the exorbitantly high payoffs are estimated as zero,[4] and second, because of the decreasing marginal utility of money the exorbitantly high payoffs lead to very low increased values.[5] Buffon next raises the solvency problem, according to which the payoff in the gamble can only be a finite amount, so that the value of the gamble must be based on a finite, rather short, gamble length. A remarkable contribution to the Petersburg paradox is his determination of the game value by repeated experiment.[6] A child played out 2^{11} Petersburg gambles to yield an average payoff of

[3]A concise survey of the literature and the solution concepts surrounding the Petersburg paradox accompanied by experimental-economics evidence is offered in Neugebauer (2010).

[4]The treatment of very small probabilities remains a very controversial issue today. As Selten (1998, p. 51) points out: "... In general, it is very difficult to judge how small a very small probability should be. Usually there will be no good theoretical reasons to specify a probability as 10^{-5} or rather than 10^{-10}. ... The value judgment ... that small differences between small probabilities should be taken very seriously and that wrongly describing something extremely improbable as having zero probability is an unforgivable sin... is unacceptable."

[5]One may criticise Buffon for not referring to the correspondences of Nicholas Bernoulli (1687–1759). Nicholas Bernoulli (1728) had referred to moral certainty in a letter written to Gabriel Cramer so that it is very likely that Buffon was aware of the fact that Nicholas Bernoulli had given this explanation earlier. Nevertheless, the value of presenting this explanation to a greater audience is an important contribution given that the statement of Nicholas Bernoulli had not been widely heard.

[6]This experiment seems to be the first conducted statistical experiment ever reported. It was replicated later by various researchers. Though the experimental approach is used to elicit the money value of the game, it is not an economic experiment, since the economic behaviour of subjects is not the purpose of study. According to the definition by Sauermann and Selten (1967, p. 8, our translation) this condition is crucial: "It is advisable to count only such experiments to experimental economic research in which the economic behavior of experimental subjects is observed. So-called simulation experiments, which only consist in the computation of numerical examples for theoretical models on electronic computers, do not belong, in this sense, to experimental economic research." An economic experiment on valuation of lotteries would have to elicit the willingness to pay from experimental subjects. A frequently used elicitation procedure in this context is the BDM approach as used in Selten et al. (1999). For a more general introduction to economic experiments see Hey (1991).

about 5 Ecu per game. From the outcomes of this experiment, Buffon motivated the geometric payoff distribution that results by application of the law of large numbers. Table 15.1 shows the outcomes of the repeated experiment and the statistically expected outcomes. By reflecting on the theoretical distribution, Buffon comes to the conclusion that the greater the number of repetitions the greater is the expected outcome. He, thus, anticipates partly the mathematical solution to the repeated Petersburg gamble as proposed by Feller (1945). In contrast to Bernoulli's utility function, his arguments stand the test of the Super-Petersburg paradox (Menger 1934), again by the cancelling of small probabilities and the zero marginal value of the superfluous income beyond a certain bound.

The contribution of the last two articles in the *Essays* preceding the concluding one (article XXV) is more to mathematics than to moral behaviour. Buffon presents his games of the *franc-carreau* and *Buffon's needle* (article XXIII); the latter is the reason why his name still remains in the mathematical sciences. The contribution is important since it introduces geometrical probability to the literature. According to available information on the internet, Buffon conducted experiments by throwing a stick over his shoulder into a tiled room. In similar random experiments on Buffon's needle, the number π was approximated in the nineteenth century.

Finally, Buffon discusses the meaning of infinity (article XXIV) by highlighting its value as a mathematical tool which allows the generalisation of results and by pointing out that it is not a 'real' number.[7]

Summing up, the *Essays* present a collection of articles of Buffon dedicated to judgment and behaviour, including experimental methods and mathematics. Regardless of a number of issues that one can take with his presentation from a modern-day perspective (as it lacks a fully-fledged and fully-developed approach, and is not always convincing), it appears remarkable to see how much thought,

Table 15.1 Buffon's results

Number of "tails"	Buffon's observations	Payoff[a]	Buffon's geometric approximation
0	1,060	1	$2^{10} = 1{,}024 + 1$
1	494	2	$2^9 = 512$
2	232	4	$2^8 = 256$
3	137	8	$2^7 = 128$
4	56	16	$2^6 = 64$
5	29	32	$2^5 = 32$
6	25	64	$2^4 = 16$
7	8	128	$2^3 = 8$
8	6	256	$2^2 = 4$
9	–	512	$2^1 = 2$
10	–	1,024	$2^0 = 1$

[a]The total payoff was 10,057 Ecu, an average of 4.91 Ecu

[7]Following the 25 described articles, Buffon dedicates another ten articles to "arithmetic and geometric measures". These articles look to us less relevant to the study of human sciences and are therefore omitted in the translation.

discussed today in the human sciences literature, was already expressed during the eighteenth century. Buffon's *Essays* is outstanding from the standpoint of today when compared to many other works of that age because of its many and remarkable contributions: the introduction of the valuation of a game by experimental method, thus highlighting also the importance of empirical evidence; the discussion based on philosophical arguments of levels of significant and negligible probabilities; the valuation of losses and its distinction to the valuation of gains; the distinguishing of necessary income from superfluous income; and, finally, the introduction of geometric probability.

Before concluding, one remaining question must evidently be addressed. Since this paper has been prepared for publication in honour of Reinhard Selten, the reader may ask what is the relationship between Buffon, his *Essays*, and Reinhard Selten and his work?-Thereto we reply that although we do not know of any direct link, we see relationships and analogies in (1) spirit, (2) presentation, (3) methodology, (4) thought, and (5) the grandeur of scientific contribution.

1. Although Reinhard Selten is evidently a modern researcher, in our view he shares the mind-set of the great savants of the age of the enlightenment, that is, the dedication to research driven only by the desire to find and communicate the truth about the nature of things. Owed to the conservativeness of academic reviewers who were not always open to Selten's unorthodox theories, and he being reluctant to making any changes to his work that could bias his vision of the truth, he would publish his famous papers unchanged in unknown academic journals rather than having the changed version published in a prestigious journal. Even though the prestige of publishing in such journals would have helped his career at that time, he accepted the facts in a humorous way. We remember him saying that by publishing in an unknown journal "you can make a journal famous." He surely made several journals famous as he was frequently quoted for his publications in journals that were quite unknown or nonexistent before his publication. This achievement is evidently more exceptional than publishing in a famous journal, but it is a rocky road to making any impact in the human sciences, even for an exceptionally brilliant mind. He also pointed to the fact that in earlier times, as in the Buffon's, researchers had to start writing an essay on scientific questions by first apologizing for daring to raise these questions at all (see article I).
2. *Selten's archer* (Selten and Buchta 1994) who represents direction learning theory is as original and as illustrative as the great Buffonian metaphor of the blind man who learns by experience about the return of the sunrise. Nevertheless both presentations share a similar description about the learning of information about unknown things and the adjustment in hindsight. The difference, indeed, is in the adjustment itself; while the blind man adjusts his belief, the archer adjusts his strategy. The archer learns hitting a target with an arrow, similarly to a blind man, only by repeated experience through feedback information. If he is informed that the arrow missed the target to the left, on the next trial, he is going to adjust the direction of the arrow to the right rather than to the left, and

vice versa. This direction learning theory has shown to explain the behaviour of most experimental subjects in economics laboratory experiments in very different scenarios (see Selten 2004 for a review). Impulse balance theory is based on this direction learning story; it allows point-predictions of behaviour by balancing potential positive and negative impulses in games of strategy (for a simple application of this concept, see Selten and Neugebauer 2006).

3. Selten has used mainly mathematical and experimental approaches in his research. Similarly to Buffon, he was fascinated by mathematics from his youth, received a degree in mathematics, before he turned to applied mathematics and experimental research. During his career, his main interests have involved the measurement and valuation of games, both theoretically and experimentally, and in the construction of solution concepts based on (levels of) rationality.
4. Selten sees limits to the benefit of mathematics in the understanding of human behaviour. In the language of Buffon (article II), we can imagine although we did not witness, Selten could have said: "maths involves the truth of definitions. This truth is only helpful if one understands the problem well, that is, if the analysis of the problem is based on the right definitions." The Reinhard Selten School believes in the merit of experimental research to uncover the fundamental definitions of human behaviour, and to support the building of a boundedly rational system in form of a toolbox if a fully-fledged theory and fully-developed approach is not available. Buffon also seems to favour a boundedly rational system to human beings over pure rationality, as he states that rationality is cold and does not make man really happy. In particular, he accepts that less rare people gamble from time to time because hope makes them happy. In the language of Selten, many people tend to avoid risk taking as much as they can, but on some occasions they enjoy and permit themselves to take risks; "today I take a chance."

   In relation to Buffon's discussion of the Petersburg gamble, Selten accepts, at least from the behavioural viewpoint, a treatment of very small probabilities as zero. Discussing the outcomes of valuations of the Petersburg gamble by experimental subjects, Reinhard Selten was not at all surprised that according to the data people behave as if they neglect very small probabilities in the Petersburg gamble; "of cause they do that."

5. Originality and the greatness of ideas which lead to path-breaking approaches must be used as a description of the work of both Buffon and Selten.[8] While we do not know about Buffon, Selten had a reason for being so original. He argued that he was slower than other researchers and therefore had to take greater steps.

[8]In fact, we do not intend to give here a fair and thorough appraisal of Reinhard Selten's contributions. Many important articles have been republished in Selten (1988, 1999).

Both had a great impact on the development of research in games and experiments; Buffon as the founding father of experimental statistics; Reinhard Selten as the founding father of experimental economics in Europe. Finally, both have influenced thinking, being thought provoking and created great interests in the subject of their research.

Reinhard Selten is very active with his current research. Recently, at the celebration of the twenty-fifth anniversary of the Bonn Laboratory of Experimental Economics, he said that he was the "money most-intensive researcher" at the economics department at Bonn. So looking forward, we expect many new researches, advancements of the field bounded rationality and enlightening ideas of and by Reinhard Selten.

Essays on Moral Arithmetic

I. I do not attempt to present general essays on morality here; that would demand more enlightenment than I can presume, and more art than I recognize. The first and most sensible part of morality is rather an application of the maxims of our divine religion than of a human science; and I cannot even dare to try matters where the law of God is our principle, and Faith our rationale. The respectful gratitude or rather the adoration, man has to his Creator; brotherly charity, or rather the love he has to his fellow man, are natural feelings and virtues written on an ingenuous mind; all that stems from this pure source bears the character of the truth; the light is so bright that the existence of error cannot obscure it, the evidence so great that it admits no argument or discussion, or doubt, and no other measures than conviction.

The measurement of uncertain things is my object here. I will try to give some rules to estimate likelihood ratios, degrees of probability, weights of testimonies, influence of risks, inconvenience of perils; and judge at the same time the real value of our fears and of our hopes.

II. There are truths of different kinds, certainties of different orders, probabilities of different degrees. The purely intellectual truths like those of Geometry all reduce themselves to truths of definition; it is a matter of resolving the most difficult problem only to understand it well, and there are no other difficulties in calculation and in the other purely theoretical sciences than to untangle what we put in, and to untie the knots that the human mind has created in the study of the implications of the definitions and the assumptions that are used as the foundation and framework of these sciences. All their propositions always can be proven evidently, because you can always go back from each of these propositions to other preceding propositions which are identical to them, and from these to others back to the definitions. It is for this reason that evidence[9] itself belongs to the mathematical sciences and only to them; because one must distinguish the evidence of reasoning from the evidence that comes through the senses, that is, the intellectual evidence from the physical intuition; the latter is only a clear apprehension of objects or of images; while reasoning is a comparison of similar or identical ideas, or rather it is the immediate perception of their identity.

III. In the physical sciences, evidence is replaced by certainty; evidence is not susceptible to measure, because it has only one absolute property, that is the clear negation or the affirmation of the matter it shows; but certainty is never a positive absolute one, it requires several relationships that we must compare and from which we can estimate the measure. Physical certainty, that is, the most certain of all certainties, is nevertheless only the almost infinite probability that an effect, an event that never failed to happen, will happen again; for example, because the sun has always risen, it is thenceforth physically certain that it will rise tomorrow; a reason for being is to have been, but a reason for ceasing to be is to have come into being; and consequently one cannot say that it is equally certain that the sun will always rise, at least one must assume a preceding eternity, equal to the subsequent perpetuity, otherwise it will end as it has begun. For we must judge the future by the evidence from the past, whenever something has always been, or if it always behaved the same way, we must be assured that it will be or will behave always in the very same way: by *always*, I mean a very long time, and not an absolute eternity, the always of the future never being equal to the always of the past. The absolute of any kind, whatsoever, is not the responsibility of Nature or of the human mind. Men have considered as ordinary, natural effects all the events that have this kind of physical certainty; an always occurring effect ceases surprising us; in contrast a phenomenon that has never occurred, or one that has always occurred in the same way but ceases to occur or occurs in a different way would surprise us with reason and would be an event that appears to us so extra-ordinary that we would consider it as supernatural.

IV. These natural effects that do not surprise us, however, have everything necessary to surprise us; what circumstances of causes, what collection of principles is not necessary to produce a single insect, a single plant! What a prodigious combination of elements, motions and springs in the animal machine! The smallest works of Nature are subjects of the greatest admiration. The reason why we are not at all surprised by all these wonders is that we were born into this world of wonders, that we have always seen them, that our understanding and our eyes are equally accustomed to them; finally, all were before and will be still after us. If we had been born in a

[9][comment: Buffon uses the word *évidence*, which we translate throughout by *evidence*, also in the sense of something obvious or evident].

different world with a different physical shape and different senses, we would have had other relationships with exterior objects, we would have seen different wonders and we would not have been more surprised; the wonders of this world and of the other are based on the ignorance of the causes, and on the impossibility of knowing the reality of the things, of which we are only permitted to see are the relationships they have with ourselves.

There are therefore two ways of looking at the natural effects; the first is to see them as they present themselves to us without paying attention to their causes, or rather without looking for their causes; the second is to examine the effects in the view to relating them to the principles and causes; and these two points of view are strongly different and produce different reasons for surprise, the one causes the sensation of surprise, and the other creates the feeling of admiration.

V. We will speak here only of this first way to look at the effects of Nature; as incomprehensible, as complicated as they present themselves to us, we will judge them as the most evident and the simplest ones, and judge them only by their results; for example, if we cannot conceive or even imagine why matter attracts each other, we will certainly be satisfied that it actually attracts each other, and we will judge from that time that it always has attracted each other and that it always will continue to attract each other: it is the same with other phenomena of all kinds, as unbelievable as they may appear to us, we will believe them if we are sure that they have occurred very often, we will doubt them if they have failed to happen as often as they occurred, finally, we will deny them if we believe for sure that they never occurred; in a word, according to having seen and recognized them, or to having seen and recognized the opposite.

But if the experience is the basis of our physical and moral knowledge, then analogy is the first instrument: when we observe that a thing occurs constantly in a particular way, we are assured by our experience that it will occur again in the same way; and when someone reports that a thing occurred in this or that way, if these facts have an analogy with the other facts that we know by ourselves, then we believe them; on the contrary, if the fact has no analogy with ordinary effects, that is, with the things that are known to us, we must be in doubt; and if it is directly opposed to what we know, we do not hesitate to deny it.

VI. Experience and analogy may give us different certainties, sometimes almost equal and sometimes of the same kind; for example, I am almost as certain of the existence of the city of Constantinople, which I never have seen, as I am of the existence of the Moon, that I have seen so often, and that because the testimonies in large number can produce a certainty almost equal to the physical certainty, when they concern things that have a full analogy with those that we know. The physical certainty must be measured by an immense number of probabilities, because such certainty is produced by a constant sequence of observations, which are what we call the *experience of all the times*. The moral certainty must be measured by a smaller number of probabilities, since it presupposes only a number of analogies with what is known to us.

Assuming a man that had never seen anything, heard anything, we investigate how the belief and the doubt generate itself in his mind; assume him struck for the first time by the appearance of the sun; he sees it shine from the top of the skies, then decline and finally disappear; what can he conclude? Nothing, except that he saw the sun, that he saw it follow a certain route, and that he no longer sees it; but this star reappears and disappears again on the next day; this second sight is a first experience, that must produce in him the hope to see the sun again, and he begins to believe that it could return, nevertheless he is very much in doubt; the sun reappears again; this third sight is a second experience which diminishes the doubt as much as it increases the probability of a third return; a third experience increases it to the point that he no longer doubts that the sun returns a fourth time; and finally when he will have seen this light-star appear and disappear regularly ten, twenty, hundred times again, he will believe to be certain that he will see it always appear, disappear and to move the same way; the more similar observations he will have, the greater will be the certainty to see the sun rise the next day; every observation, that is, every day, produces a probability, and the sum of these probabilities together, as it is very great, gives the physical certainty; one will therefore always be able to express this certainty by the numbers, dating back to the origin of the time of our experience and it will be the same for all the others effects of Nature; for example, if one wants to reduce here the seniority of the world and of our experience to 6,000 years, the sun has risen for us[10] only 2 million 190 thousand times, and as to date back to the second day that it rose, the probabilities to rise the next day increase, as the sequence 1, 2, 4, 8, 16, 32, 64. . . . or 2^{n-1}. One will have (where the natural sequence of the numbers, n is equal to 2,190,000), one will have, I say, $2^{n-1} = 2^{2,189,999}$; this already is such a prodigious number that we ourselves cannot form an idea, and it is by this reason that one must look at the physical certainty as composed from an immensity of probabilities; since by moving back the creation date by only 2,000 years, this immensity of probabilities becomes $2^{2,000}$ times more than $2^{2,189,999}$.[11]

VII. But it is not so easy to do the estimation of the value of analogy, nor consequently, to find the measure of moral certainty; it is in truth the degree

[10] I say for us, or rather for our climate, because it would not be exactly true for the climate of the poles.

[11] [comment: Buffon obviously means days rather than years].

of probability that gives analogical reasoning its power; and in itself the analogy is only the aggregate of the relationships with known things; nevertheless according to that aggregate or that relationship in general being more or less great, the consequence of the analogical reasoning will be more or less certain, but without ever being absolutely certain; for example, if a witness whom I suppose being of common sense, tells me that a child has just been born in this city, I will believe him without hesitating, as the fact of a child's birth is nothing other than very ordinary, but having on the contrary an infinity of relationships with the known things, that is, with the births of all other children, I will believe this fact therefore, however, without being absolutely certain about it; if the same man tells me that this child was born with two heads, I would believe it again, but more weakly, as a child with two heads has less relationships with known things; if he added that the newborn has not only two heads, but that it has also six arms and eight legs, I should have good reason to hardly believe it, but however weak my belief was, I could not refuse it in entirety; this monster, although very special, nevertheless being composed only of parts that all have some relationships with known things, and only having their assembly and their very extraordinary number. The power of the analogical reasoning will therefore be always proportional to the analogy itself, that is, to the number of the relationships with known things, and it is not a matter of making a good analogical reasoning, but to adjust oneself well to the fact of all the circumstances, to compare them with the analogous circumstances, to aggregate the number of these, to take next a model of comparison to which one will relate this found value, and one will have exactly the probability, that is, the degree of power of the analogical reasoning.

VIII. There is therefore a prodigious distance between the physical certainty and the certainty of the kind that one can deduce from most of the analogies; the first one is an immense sum of probabilities that forces us to believe; the other is only a smaller or greater probability, and often so small that it leaves us in the perplexity. The doubt is always inversely proportional to the probability, that is, it is always the greater the smaller the probability. In the order of the certainties produced by the analogy, one must place the moral certainty, which seems to even take the centre between doubt and physical certainty; and this centre is not a point, but a very extensive line, for which it is quite difficult to determine the limits: one can feel that it is a certain number of probabilities that equals the moral certainty, but what number is it? And can we hope to determine it as precisely as the one by which we have just represented the physical certainty?

After having reflected on it, I have thought that of all the possible moral probabilities, the one that most affects man in general is the fear of death, and I felt from that time that any fear or any hope, whose probability would be equal to the one that produces the fear of death, can morally be taken as the unit to which one must relate the measure of the other fears; and I relate to the same even the one of hopes, since there is no difference between hope and fear, other than from positive to negative; and the probabilities of both must be measured in the same way. I seek therefore for what is actually the probability that a man who is doing well, and consequently has no fear of death, dies nevertheless in the 24 h: consulting the Mortality Tables, I see one can deduce that there are only ten thousand one hundred eighty-nine to bet against one, that a 56 year old man will live more than a day.[12] Now as any man of that age, when reason has attained its full maturity and the experience all its force, nevertheless has no fear of death in the 24 h, although there is only ten thousand one hundred eighty-nine to bet against one that he will die in this short interval of time; from this I conclude that any equal or smaller probability must be regarded as zero, since any fear or any hope below ten thousand must not affect us or even occupy for a single moment the heart or the mind.[13]

To make me better understood, suppose that in a lottery where there is only a single prize and ten thousand blanks, a man takes only one ticket, I say that the probability to obtain the prize is only as one against ten thousand, his hope is zero, since there is no more probability, that is, reason to hope for the prize, than there are fears of death within 24 h; and that this fear not affecting him in any way, the hope for the prize must not affect him more, and even again much less, since the intensity of the fear of death is much greater than the intensity of any other fear or of any other hope. If, despite the evidence of this demonstration, this man insists on wanting to hope, and if a similar lottery is played every day, and he persists in buying a new ticket every day, always hoping to win the prize, one could, to disabuse him, bet with him at equal odds that he was dead before he won the prize.

[12]See hereafter the result of the mortality Tables [comment: the tables are not included in the translation].

[13]Having communicated this idea to Mr. Daniel Bernoulli, one of the greatest geometricians of our century, and most experienced in all of the science of probabilities, here is the response that he gave me by his letter, dated from Bâle March 19, 1762.

"I strongly approve, Sir, your way to estimate the limits of moral probabilities; you consult the nature of man by his actions, and you suppose indeed, that no one worries in the morning to die that day; hence, he will die according to you, with probability one in ten thousand; you conclude that one ten-thousandth of probability should not make any impression in the mind of man, and consequently this one ten-thousandth has to be regarded as an absolute nothing. This is doubtless the reasoning of a Mathematician-Philosopher; but this ingenious principle seems to lead to a smaller quantity, because the absence of fear is certainly not in those who are already ill. I do not fight your principle, but it seems rather to lead to $1/100,000$ rather than to $1/10,000$."

I confess to Mr. Bernoulli that as the one ten-thousandth is taken according to the mortality Tables that never represent the *average man*, that is, men in general, well or sick, healthy or ill, strong or weak, there is maybe a little more than ten thousand to bet against one, that a man, healthy and strong will not die in the 24 h; but it is hardly necessary to increase this probability to one hundred thousand. Moreover, this difference, although very large, changes nothing of the main implications that I draw from my principle.

It is the same in all the games, the bets, the perils, the risks; in all cases, in a word, where the probability is smaller than $1/10,000$, it must be, and it is in fact for us, absolutely zero; and by the same reason in all cases where this probability is greater than 10,000, it makes for us the moral certainty most complete.

IX. From it we can conclude that the physical certainty relates to the moral certainty as $2^{2,189,999} : 10,000$; and that whenever an event, of which we absolutely ignore the cause, occurs in the same way thirteen or fourteen times in a row, we are morally certain that it will occur again even a fifteenth time, for $2^{13} = 8,192$, and $2^{14} = 16,384$, and consequently when this event has occurred thirteen times, there is 8,192 to bet against 1 that it will occur a fourteenth time; and when it has occurred fourteen times, there is 16,384 to bet against 1 that it will occur even a fifteenth time, which is a greater probability than the one of 10,000 against 1, that is, greater than the probability that makes moral certainty.

One will perhaps be able to tell me, that although we do not have the fear or the worry of sudden death, it must be that the probability of sudden death is zero, and that its influence on our conduct is morally zero. A man whose mind is beautiful, when he loves someone, would he not reproach himself to delay a day the measures that must assure the happiness of the loved person? If a friend entrusts us with a considerable deposit, would we not put the same day a written comment on this deposit? We act therefore in these cases, as if the probability of sudden death was something, and we have reason to act like this. Therefore one must not regard the probability of sudden death as zero in general.

This kind of objection will vanish if one considers that one often does more for others than one does for oneself! When one puts a written comment the very moment when one receives a deposit, it is only honesty to the owner of the deposit, for his tranquillity, and not at all the fear of our death in the next 24 h; it is the same attentiveness that makes the happiness of someone or of us, it is not the feeling of the fear of an approaching death that guides us, it is our own satisfaction that drives us, we seek to enjoy all at the earliest that is possible to us.

A reasoning that might appear more justified is that all men are inclined to flatter; that hope seems to arise from a lower degree of probability than fear; and that consequently one is not entitled to substitute the measure of the one with the measure of the other: fear and hope are feelings not concrete things; it is possible, it is even more likely, that these feelings do not measure the precise degree of probability; and in that case, must one give them an equal measure or even assign them no measure?

Thereto I reply that the measure in question is not on feelings, but on the reasons that must originate them, and that any wise man must estimate the value of these feelings of fear or of hope only by the degree of probability; because even when Nature, for the happiness of man, had given him more inclination towards hope than towards fear, it is not less true that probability is the true measure of the one and of the other. It is not even by the application of this measure that one can figure out one's false hopes, or to reassure on its unfounded fears.

Before finishing this article, I must observe that one must beware of mistaking what I called effects of unknown cause; because I understand only the effects of which the causes, although unknown, must be supposed constant, such as the natural effects; any new discovery in physics noted by thirteen or fourteen experiences, all of which are confirmed, already has a degree of equal certainty to the one of moral certainty, and this degree of certainty doubles with each new experience; so that by multiplying, one approaches more and more the physical certainty. But we must not conclude from this reasoning that the effects of chance follow the same law; it is true that in a sense these effects are among those for which we ignore the immediate causes; but we know that in general these causes are quite far from being able to be supposed constant; on the contrary, they are necessarily variable and volatile as much as it is possible. Thus even by the notion of chance itself, it is evident that there is no connection, no dependence, between its effects; that consequently the past cannot have any influence on the future, and one would be very much and even completely mistaken, if one wanted to infer from previous events, any reason for or against the posterior events. If a card, for example, has won three times in a row, it is not less probable that it will win a fourth time, and one can bet equally that it will win or that it will lose, no matter how many times it has won or lost, as long as the laws of the game are such that the chances are equal. Presuming or believing the opposite, as some players do, is to go against the principle of chance itself, or not to remember that by the conventions of the game it is always equally distributed.

X. In the effects for which we see the causes, a single test is sufficient to cause the physical certainty; for example, I see that in a clock the weight makes the wheels turn, and that the wheels make the pendulum go; I am certain from that time on without the need of reiterated experiences, that the pendulum will always go the same as long as the weight makes the wheels turn; this is a necessary consequence of an arrangement that we made ourselves while constructing the machine; but when we see a new phenomenon, an effect in the still unknown Nature, as we are ignorant about its causes, and they can be constant or variable, permanent or intermittent, natural or accidental, we do not have other means to obtain the certainty than by repeating the experience as often as necessary; nothing here depends on us and we only know as much as we experiment; we are only assured by the effect itself and by the repetition of the effect. As soon as it has occurred thirteen or fourteen times in the same way, we already have a degree of probability equal to the moral certainty that it will occur for even a fifteenth time, and from this point we soon can cover an immense interval, and conclude by analogy that this effect depends on the general laws of Nature; that it is consequently as old as all the other effects and that there is physical certainty that it always will occur as it always has occurred and it only needed having been observed.

In the risks that we arranged, balanced and calculated ourselves, one must not say that we are ignorant of the causes of the effects: we are ignorant of the true immediate cause of each effect in particular; but we clearly see the first and general cause of all the effects. I do not know, for example, and I even cannot imagine in any way, what is the difference of the movements of the hand, to pass or not to pass ten with three dice, which nevertheless is the immediate cause of the event, but I see evidently by the number and the make of the dice, which are here the main and general causes the chances are absolutely equal, you are indifferent to betting on passing or not passing ten; I see moreover that these same events, if they happen, have no connection, since to every throw of the dice the risk always is the same, and nevertheless always new; that the past throw cannot have any influence on the throw to come; that one always can equally bet for or against; finally that the longer one plays, the greater the number of effects for and the number of effects against, they will approach equality. Making sure that every experience here gives a outcome exactly opposite to that of the experience on the natural effects, I want to say, the certainty of the inconstancy instead of the one of the constancy of the causes; in this one, each test leads to the doubling of the probability of the replication of the effect, that is, the certainty of the constancy of the cause; in the effects of risk, on the contrary, each test increases the certainty of the inconstancy of the cause by showing us always more and more that it is absolutely volatile and totally indifferent to produce the one or the other of these effects.
When a gamble is in its nature perfectly fair, the player has no reason to choose this or that side; since finally, from the supposed fairness of this game, it necessarily follows that there are no good reasons to prefer the one or the other side; and consequently if one deliberated, one could only be determined by the wrong reasons; also the logic of the players has seemed to me completely wrong, and even the good minds that allow themselves to play fall as players into absurdities at which they soon blush as reasonable men.

XI. Besides, all this supposes that after having balanced the risks and having them rendered equal, as in the game *passing ten* with three dice, these same dice which are the devices of the risk are as perfect as possible, that is, they are exactly cubic, the material is homogenous, the numbers are painted and not marked hollow, so that the weight on one face is not more than on any other; but as it is not given to man to make anything perfect, and there is no such dice made with this strict precision, it is often possible to recognize by observation on which side the imperfection of the devices of chance tips the risk. Thus, it is only necessary to attentively observe for a long time the sequence of events, to count them exactly, to compare their relative numbers; and if one of these two numbers exceeds by far the other, one will be able to conclude, with great reason, that the imperfection of the devices of chance destroys the perfect equality of the risk, and gives it actually a stronger inclination to one side than to the other. For example, I suppose that before playing *passing ten*, one of the players was subtle enough, or rather, rogue enough to having thrown in advance thousand times the three dice to be used, and to having recognized that in these thousand trials there were six hundred that passed ten, he will have thenceforth a great advantage over his opponent by betting on passing, since by experience the probability of passing ten with these same dice relates to the probability of not passing-ten as 600:400 or 3:2. This difference that stems from the imperfection of the devices can therefore be recognized by observation, and it is for this reason that the players often change dice and cards, when their luck is against them.

Thus, however obscure the destinies may be, however opaque the future may appear to us, we may nevertheless by reiterated experiences, become, in some cases, also enlightened about future events, as if we were beings or rather superior natures who immediately deduce the effects from their causes. And among the very things that seem to be pure risk, as games and lotteries, one can again recognize the inclination of the risk. For example, in a lottery drawn every fortnight, and in which one publishes the winning numbers, if one observes those that most often won during one year, two years, three consecutive years, one can deduct, with reason, that the same numbers will win again more often than the others; because in any way one can vary the motion and the position of the device of chance, it is impossible to render them perfect enough to maintain the absolute equality of the chance; there is a certain routine to do, to place, to mix the tickets, which even in the midst of confusion produces a certain order, and makes that certain tickets must come out more often than the others; it is the same with the arrangement of cards to play; they have a kind of sequence of which one can grasp some terms by force of observations; because while assembling them with the hand one follows a certain routine, the player himself by mixing them has his routine; the whole is done in a certain way more often than in another, and thenceforth the attentive observer to results collected in large number will always bet with great advantage that such card, for example, will follow such other card. I say that this observer will have a great advantage, because the *a priori* risks must be absolutely equal, the least inequality, that is, the least degree of higher probability has very great influences on the game, which is in itself only a multiplied and always repeated bet. If this difference recognized by the experience of the inclination of the risk was only one hundredth, it is evident that in a hundred throws the observer would gain his stake, that is, the sum that he risks every time; so that a player equipped with these dishonest observations, cannot fail to ruin in the long run all his opponents. But we will give a powerful antidote against the epidemic evil of the passion of play, and at the same time some preventives against the illusion of this dangerous art.

XII. It is generally known that the game is a greedy passion, where the habit is ruinous, but this truth has perhaps never been shown unless through a sad experience on which one did not reflect enough to correct it by conviction. A player whose wealth is exposed every day to the draws of chance, exhausts himself little by little and finally finds himself necessarily ruined; he attributes his losses only to the same risk that he blames for unfairness; he equally regrets what he lost as what he did not win; the greed and the false hope gave him claims on the goods of others; so humbled to find himself in necessity as afflicted to no longer having the means to satisfy his cupidity; in his despair he blames his ill-fated star, he does not imagine that this blind power, the fortune of the game, marches in truth by an indifferent and uncertain pace, but that to every step it nevertheless tends to a goal, and pulls to a certain end that is the ruin of those who try it; he does not see that the apparent indifference it has for the good or for the evil produces with time the necessity of evil, that a long random sequence is a fatal chain whose extension causes the misfortune; he does not feel that regardless of the hard tax of the cards and of the even harder tribute paid to the roguery of some opponents, he has passed his life making ruinous conventions; that finally the game by its very nature is a vicious contract in its principle, a harmful contract to each contractor in particular, and contrary to the good of any society.

This is not at all a speech of vague morals, these are precise truths of metaphysics that I subject to calculation or rather to the strength of reason; truths that I pretend to mathematically show to all those who have their minds clear enough and the imagination strong enough to combine without geometry and to calculate without algebra.

I will not speak about these games invented by the artifice and worked out by the avarice, where the chance loses a part of its rights, where the fortune can never balance, because it is invincibly entailed and always obligated to lean to one side, I want to say all these games where the chances unequally divide up offer at once an assured as dishonest gain, and leave the other only with a sure and shameful loss, as in *Pharaoh*, where the banker is only a roguish solicitor and the punter a fool whom one agreed not to mock.

It is in games in general, in the most equal game, and consequently the most honest one that I find a vicious essence, I include even in the name of game all the conventions, all the bets where one puts at risk a part of his goods to obtain a similar part of the goods of others; and I say that in general the game is a misunderstood pact, a disadvantageous contract to both parties, whose effect is to make the loss always greater than the gain; and to remove the good to add to the evil. The demonstration of it is as easy as evident.

XIII. Take two men of equal fortune, who, for example, each have one hundred thousand pounds of goods, and suppose that these two men play in one or more throws of the dice fifty thousand pounds, that is, half of their goods; it is certain that whoever wins increases his goods only by a third, and that whoever loses diminishes his by half; each of them had one hundred thousand pounds before the game, but after the event of the game, one will have one hundred fifty thousand pounds, that is, a third more than he had, and the other has only fifty thousand pounds, that is, half less of what he had; so the loss is a sixth part bigger than the gain; since there is this difference between the third and the half, the agreement to play the game is detrimental to both, and consequently essentially vicious.

This reasoning is not fallacious, it is true and exact, although one player only lost precisely what the other won; this numerical equality of the sum does not prevent the true inequality of the loss and gain; the equality is only apparent, and the inequality very real. The agreement both men make when betting half of their goods is equal to the effect of another agreement that nobody has ever decided to make, that would be the agreement to throw each the twelfth part of his goods into the sea. For one can show them, before they risk half of their goods, that the loss necessarily being a sixth greater than the gain, this sixth must be considered as a real loss, that can indifferently fall on the one or on the other, consequently it has to be divided equally.

If two men decided to play all their goods, what would be the effect of this agreement? One would only double his fortune, and the other would reduce his to zero; now which proportion is here between the loss and the gain? the same as between all and nothing; the gain of the one is only equal to a rather modest sum, and the loss of the other is numerically infinite, and morally so great that the work of a lifetime would maybe not suffice to regain his goods.

The loss is therefore infinitely greater than any gain when one plays all his goods; it is greater by a sixth part when one plays half of his good, it is greater by a twentieth part when one plays the quarter of his good; in one word, however small the portion of his fortune that one risks in the game, there is always more to lose than to gain; the agreement to play the game is thus a vicious contract that tends to ruin the two contract parties. This is a new truth, but a very useful one, that I wish to be known to all of those who, for greed or for laziness, spend their life gambling.

One has often asked why one is more sensitive to loss than to gain; one could not give to this question a fully satisfactory answer, unless one does not doubt the truth I have just presented; now the response is easy: one is more sensitive to loss than to gain, because in fact, while supposing them numerically equal, the loss is nevertheless always and necessarily greater than the gain; the feeling is generally an implicit reasoning only less clear, but often brighter, and always surer than the direct product of the reason. One feels that the gain does not give us as much pleasure as the loss causes us pain; this feeling is only the implicit result of the reasoning that I have just presented.

XIV. Money must not be estimated by its numerical quantity: if the metal, that is merely the sign of wealth, was wealth itself, that is, if the happiness or the benefits that result from wealth were proportional to the quantity of money, men would have reason to estimate it numerically and by its quantity, but it is barely necessary that the benefits that one derives from money are in just proportion with its quantity; a rich man of one hundred thousand Ecus income is not ten times happier than the man of only ten thousand Ecus; there is more than that what money is, as soon as one passes certain limits it has almost no real value, and cannot increase the well-being of its possessor; a man that discovered a mountain of gold would not be richer than the one that found only one cubic fathom.[14]

Money has two values both arbitrary, both conventions, where one is the measure of the benefits to the individual, and where the other determines the rate of well-being of the society; the first of these values has never been estimated other than in a very vague way; the second is suitable for a just estimation by the comparison of the quantity of money with the proceeds of the land and the labour of men.

To succeed in giving some precise rules on the value of money, I will examine special cases of which the mind grasps easily the combinations, and that, as examples, drive us by induction to the general estimation of the value of the money for the poor, for the rich, and even for the more or less wise.

For the man who in his budget, whatever it is, has only what is necessary, money has an infinite value; for the man who in his budget abounds in superfluous, money has almost no value anymore. But what is necessary, what is superfluous? I understand by necessary the *income which one is obliged to spend to live as one has always lived*; with this necessary income, one can have its comforts and even pleasures; but soon the habit creates needs; so, in the definition of the superfluous I will not account for any of the pleasures we are used to, and I say that the superfluous is the *income that can bring us new pleasures*; the loss of the necessary income is a loss that one feels infinitely, and when one risks a considerable part of this necessary income, the risk cannot be offset by any hope, however great one may suppose; on the contrary the loss of the superfluous income has limited

[14][comment: old measure; one cubic fathom ≈6.12 m^3].

effects; and if even superfluous income is still more sensitive to the loss than to the gain, it is because in fact the loss is generally always greater than the gain, this feeling is based on the principle that reasoning had not developed, because the ordinary feelings are based on common concepts or on simple inductions; but the delicate feelings depend on exquisite and elevating ideas, and are in fact only the results of several combinations often too subtle to be clearly noticed, and almost always too complicated to be reduced to a reasoning that can prove them.

XV. The mathematicians who have evaluated games of chance, and whose research in this field deserves praise, have considered money only as a quantity susceptible to growth and diminution, without other value than the number; they have estimated by the numerical quantity of money the relations of the gain and loss; they have calculated the risk and hope relatively to this very numerical quantity. We consider here the value of the money from a different point of view, and through our principles we will give the solution to some embarrassing cases for ordinary calculation. The problem, for example, of the game of heads and tails, where one supposes that two men (Peter and Paul) play against each other at these conditions that Peter will throw a coin in the air as many times as it will be necessary till tails shows up, and if that occurs at the first throw, Paul will give him one Ecu; if that occurs only on the second throw, Paul will give him two Ecus; if that occurs only on the third throw, he will give him four Ecus; if that occurs only on the fourth throw, Paul will give eight Ecus; if that occurs only on the fifth throw, he will give sixteen Ecus, and so on always doubling the number of the Ecus: it is obvious that in this condition Peter can only win, and that his gain will be at least an Ecu, maybe two Ecus, maybe four Ecus, maybe eight Ecus, maybe sixteen Ecus, maybe thirty-two Ecus, etc. maybe five hundred twelve Ecus, etc. maybe sixteen thousand three hundred eighty-four Ecus, etc. maybe five hundred twenty-four thousand four hundred forty-eight Ecus, etc. maybe even ten million, hundred million, hundred thousand millions of Ecus, maybe finally an infinity of Ecus. Because it is not impossible to throw five times, ten times, fifteen times, twenty times, thousand times, hundred thousand times the coin without tails showing up. One asks therefore how much Peter must give to Paul to compensate him, or what amounts to the same, what sum is equivalent to the hope of Peter who can only win.

This problem was proposed to me for the first time by the blessed Mr. Cramer, the famous professor of mathematics at Geneva, on a trip that I made to this city in the year 1730; he told me that it was previously proposed by Mr. Nicolas Bernoulli to Mr. de Montmort, as in fact it is found on the pages 402 and 407 of the *Analysis of the games of chance* by this author: I dreamed this problem some time without finding the knot; I could not see that it was possible to make agreed mathematical calculations with common sense without introducing some moral considerations; and having expressed my ideas to Mr. Cramer,[15] he told me that I was right, and that he had also resolved this question by a similar approach; he showed me then his solution almost identical to the one printed later in 1738 in the

[15]It follows what I left then in writing to Mr. Cramer, and of what I kept the original copy. "Mr. de Montmort is satisfied to reply to Mr. Nic. Bernoulli that the equivalent is equal to the sum of the sequence 1/2, 1/2, 1/2, 1/2, etc. Ecus continued to infinity, i.e., $\infty/2$, and I do not believe in fact that one can dispute his mathematical calculation; but far from giving an infinite equivalent, there is no man of common sense who would want to give twenty pounds, or even ten.

The reason of this contradiction between the mathematical calculation and the common sense seems to me to consist in the relation between money and its resulting advantage. A mathematician in his calculation estimates the money only by its quantity, i.e., by its numerical value; but the moral man must estimate it otherwise and only by the advantages or the pleasure that it can obtain; it is certain that he must behave in this view, and to estimate the money only in proportion to the resulting advantages, and not relatively to the quantity that, beyond certain limits, could not at all increase his happiness; he would, for example, hardly be happier with thousand million than he would be with hundred, or with hundred thousand million more than with thousand million; thus beyond certain limits, he would make a very big mistake to risk his money. If, for example, ten thousand Ecus were all his goods, he would make an infinite mistake to risk them, and the more these ten thousand pounds will be an object in relation to it, the bigger will be the mistake; I believe therefore that his mistake would be infinite, when these ten thousand pounds were a part of his necessary income, that is, when these ten thousand pounds will be absolutely necessary for him to live, as he was raised and as he has always lived; if these ten thousand pounds are of his superfluous income, his mistake diminishes, and the more they will be a small part of his superfluous income the more will diminish his mistake; but it will never be zero, unless he can consider this part of his superfluous income with indifference, or that he only considers the expected sum necessary to succeed in a purpose that will give him in proportion as much of pleasure as this very sum is larger than the one that he risks, and it is by this way of foreseeing a future happiness, that one can not at all give rules, there are people for whom hope itself is a greater pleasure than they could obtain by the enjoyment of their stake; to reason therefore more certainly on all these things, one must establish some principles; I would say, for example, that necessary income is equal to the sum that one is obliged to spend to continue to live as one always lived; the necessary income of a King will be, for example, ten millions of revenues (because a king that had less would be a poor king); the necessary income of a nobleman, will be ten thousand pounds of revenue (because a decent man who has less would be a poor nobleman); the necessary income of a peasant will be five hundred pounds, because with less he would be in misery, he cannot spend less to live and to nourish his family. I would suppose that the necessary income cannot give us new pleasures, or to speak more exactly, I would not account for any of the pleasures or advantages that we always had, and according to that, I would define the superfluous income as what could bring us other pleasures or new advantages; I would further say, that the loss of the necessary income makes itself feel infinitely; thus that it cannot be compensated by any hope, that on the contrary the feeling of the loss of the superfluous income is limited, and that consequently it can be compensated; I believe that oneself feels this truth when one plays, because the loss, even if small, gives us always more pain than an equal gain gives us pleasure, and is that which leads to wounded pride, since I suppose the game is entirely one of pure chance. I would also say that the quantity of the money within the necessary income is proportional to what it returns to us, but that within the superfluous income this proportion begins diminishing, and diminishes the more the greater the superfluous income becomes.

I leave you, Mister, to judge these ideas, etc. Geneva, October 3 1730. *Signed*, Le Clerc de Buffon.

Memories of the Academy of Petersburg following an excellent work of Mr. Daniel Bernoulli on the measure of chances where I saw that most of the ideas of Mr. Dan. Bernoulli agreed with mine, which gave me great pleasure since I always have, in addition to his great talents in geometry, considered and acknowledged Mr. Dan. Bernoulli as one of the best minds of this century. I found also the idea of Mr. Cramer very justified, and worthy of a man who has given us proofs of his skill in all the mathematical sciences, and to whose memory I do justice with so much more pleasure than there is to the company and to the friendship of this scholar whom I owe part of the first knowledge that I acquired in this field. Mr. de Montmort gives the solution to this problem by the ordinary rules, and he says, that the equivalent sum to the hope of the one who can only win, is equal to the sum of the sequence 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2 Ecu, etc. continued to infinity, and that consequently this equivalent sum is an infinite sum of money. The reason on which this calculation is based is that there is one half of probability that Peter who can only win, will have one Ecu; one quarter of probability that he will have two; one eight of probability that he will have four; one sixteenth of probability that he will have eight; one thirty-second of probability that he will have sixteen, etc. to infinity; and that consequently his hope for the first case is one half Ecu, because the hope measures itself by the probability multiplied by the amount to be obtained; now the probability is one half, and the sum to be obtained for the first throw is one Ecu; therefore hope is one half Ecu: equally his hope for the second case is again one half Ecu, because the probability is one quarter, and the sum to be obtained is two Ecus; now a quarter multiplied by two Ecus gives again one half Ecu. One will find likewise that his hope for the third case is again one half Ecu; for the fourth case one half Ecu, in a word for all the cases to infinity always one half Ecu each, since the number of Ecus grows in the same proportion as the number of the probabilities diminishes; therefore the sum of all these hopes is an infinite sum of money, and consequently it is necessary that Peter gives to Paul the equivalent half of an infinity of Ecus.

That is mathematically true and one cannot dispute this calculation; also Mr. de Montmort and the other geometricians considered this problem as well resolved; but this solution is so far from being the true one that instead of giving an infinite sum, or even a very large sum, which already is quite different, there is no man of common sense who would give twenty Ecus or even ten to buy this hope of putting himself in the place of the one who can only win.

XVI. The reason for this extraordinary contradiction to common sense and calculation comes from two causes, the first one is that the probability must be considered as zero as soon as it is very small, that is, below 1/10,000; the second cause is the relation between the quantity of money and its resulting benefits; the mathematician in his calculation estimates the money by its quantity, but the moral man must estimate it otherwise; for example, if one proposes to a man with a mediocre fortune to put one hundred thousand pounds in a lottery, because there is only one hundred thousand to bet against one that he will win hundred thousand times one hundred thousand pounds; it is certain that the probability to obtain hundred thousand times one hundred thousand pounds, being one against hundred thousand, it is certain, I say, mathematically speaking, that his hope will be worth his stake of one hundred thousand pounds; but this man would make a very big mistake to risk this sum, and an even bigger mistake, since the probability of winning is very small, although the money to win increases in proportion, and that because with hundred thousand times one hundred thousand pounds he will not have double the benefits that he will have with fifty thousand times one hundred thousand pounds, or ten times as much benefit as he will have with ten thousand times hundred thousand pounds; and as the value of money, for the moral man, is not proportional to its quantity, but rather to the benefits that money can buy, it is obvious that this man must risk only in proportion to the hope of these benefits, which he must not calculate by the numerical quantity of the sums that he could obtain, since the quantity of money, beyond certain limits, could not further increase his happiness, and he would not be happier with hundred thousand millions of income than with one thousand million.

XVII. To understand the connection between and the truth of all that I have just advanced, we examine more closely than the geometricians did, the question that has just been proposed; since the ordinary calculation cannot resolve it, because of the morale which is causing difficulties with the mathematics, let us see if other rules enable us to reach a solution which does not violate common sense, and which is at the same time in accordance with experience; this research will not be useless, and we will furnish the means to estimate exactly the price of money and the value of hope in all cases. The first thing I remark is that in the mathematical calculation that gives as equivalent to the hope of Peter an infinite sum of money, this infinite sum of money is the sum of a sequence composed of an infinite number of terms each worth one half Ecu, and I notice that this sequence that mathematically must have an infinity of terms, cannot morally have more than thirty, since if the game takes until this thirtieth term, that is, if tails shows up only after twenty nine throws, Peter will be due a sum of 520 million 870 thousand 912 Ecus, that is, as much money as exists perhaps in the whole kingdom of France. An infinite sum of money is a thing that does not exist, and all hopes based on the terms to infinity beyond thirty do not exist either. There is here a moral impossibility that destroys the mathematical possibility; because mathematically it is possible and even physically to throw thirty times, fifty, hundred times in a row, etc. the coin without tails showing up; but it is impossible to satisfy the condition of the problem,[16] is to be said, to pay the number of Ecus that are due, if that occurs; because any money on the earth would not suffice to give the due sum, only up to the fortieth throw, since that would require thousand twenty-four times more money than exists only in the whole kingdom of France, and that it would be necessary that on the whole earth there are one thousand and twenty four kingdoms as rich as France.

Now the mathematician has only found this infinite sum of money equivalent to the hope of Peter, because the first case gives him one half Ecu, the second case one half Ecu, and every case to the infinite one always one half Ecu; therefore the moral man, also counting in the same way at the beginning, will find twenty Ecus instead of the infinite sum, since all the terms that are beyond the fortieth one yield such great amounts of money that do not exist; so that it is necessary to count only one half Ecu for the first case, one half Ecu for the second, one half Ecu for the third one, etc. until forty, which gives in total twenty Ecus as equivalent for Peter's hope, a sum already well reduced and well different from the infinite sum. This sum of twenty Ecus will be reduced again a lot considering that the thirty-first term would give more than one thousand million Ecus, that is, it would suppose that Peter had a lot more money than the richest kingdom of Europe, an impossible thing to suppose, and thenceforth the terms between thirty and forty are again imaginary, and hopes based on these terms must be looked at as zero, thus the equivalent of Peter's hope, already is reduced to fifteen Ecus.

One will reduce it again considering that the value of money should not be estimated by its quantity, Peter must not count thousand millions of Ecus as if they served him to the double of five hundred millions Ecus, or to the quadruple of two hundred fifty millions of Ecus, etc. and that consequently the hope of the thirtieth term is not one half Ecu, nor the hope of the twenty ninth, of the twenty eighth, etc. the value of this hope, that mathematically finds itself to be one half Ecu for every term, must be diminished from the second term, and always diminished until the last term of the sequence; because one must not estimate the value of the money by its numerical quantity.

XVIII. But how to estimate it, how to find this value case by case for different quantities? What are accordingly two millions of money, if this is not the double of one million of the same metal? Can we give precise and general rules for this estimation? It seems that everyone must judge his state, and next estimate his utility and the quantity of money proportional to this state and to the usage he can make from it; but this approach is still vague and too

[16]For this reason, one of our most skilful geometricians, the blessed M. Fontaine, introduced in the solution he gave us to this problem the declaration of Peter's goods, because indeed he can give for equivalent only the totality of goods he owns. See this solution in the mathematical works of M. Fontaine, in-4.° Paris, 1764.

special for it to serve as a principle, and I believe that one can find more general methods and safer ones to do this estimation; the first method that presents itself is to compare the mathematical calculation with experience; because in many cases, we can by repeated experiences, reach, as I have said, knowing the effect of chance as surely as if we deduced it immediately from the causes.

I have therefore made two thousand and forty eight experiences on this question, that is, I played two thousand and forty eight times this game by letting a child throw the coin in the air; the two thousand and forty eight trials produced ten thousand and fifty seven Ecus in all, thus the equivalent sum to the hope of the one who can only win is almost five Ecus for every trial. In this experiment, there were one thousand and sixty trials that produced only one Ecu, four hundred and ninety four trials that produced two Ecus, two hundred and thirty two trials that produced four Ecus, one hundred and thirty seven trials that produced eight Ecus, fifty six trials that produced sixteen Ecus, twenty nine that produced thirty-two Ecus, twenty five trials that produced sixty four Ecus, eight trials that produced some one hundred and twenty eight, and finally six trials that produced two hundred and fifty six. I regard this result generally as good, because it is based on a large number of experiences, and besides it agrees with another mathematical and indisputable reasoning by which one finds almost the same equivalent of five Ecus. Here is the reasoning. If one plays two thousand and forty eight trials, there must be naturally one thousand and twenty-four trials that will produce only one Ecu each, five hundred and twelve trials that will produce two, two hundred and fifty six trials that will produce four, one hundred and twenty eight trials that will produce eight, sixty-four trials that will produce sixteen, thirty-two trials that will produce thirty-two, sixteen trials that will produce sixty-four, eight trials that will produce one hundred and twenty eight, four trials that will produce two hundred and fifty six, two trials that will produce five hundred and twelve, one trial that will produce one thousand and twenty four; and finally one trial that one cannot estimate, but that one can neglect without appreciable error, because I can suppose without violating more than slightly the equality of chance, that there were one thousand and twenty five instead of one thousand and twenty four trials that produced only one Ecu, besides the equivalent on this trial being placed at the most extreme cannot be more than fifteen Ecus, since one has seen that for a trial of this game all the terms beyond the thirtieth term of the sequence give sums of such great money that they do not exist, and that consequently the greatest equivalent one can suppose is fifteen Ecus. Adding together all these Ecus which I naturally must expect by the indifference of risk, I have eleven thousand two hundred and sixty five Ecus for two thousand and forty eight trials. Thus this reasoning gives very roughly five Ecus and one half as the equivalent which agrees with the experience to within 1/11. I feel, although one will be able to criticise me, that this type of calculation that gives five Ecus and one half for equivalent when one plays two thousand and forty eight trials, would give a greater equivalent, if one added a much larger number of trials; because, for example, it turns out that if instead of playing two thousand and forty eight trials, one only plays one thousand and twenty four, the equivalent is very roughly five Ecus; that if one plays only five hundred and twelve trials, the equivalent is no more than very roughly four Ecus and one half; that if one plays only two hundred and fifty six, it is no more than four Ecus, and thus always diminishing; but the reason for concern is in the trial outcome that one cannot estimate which makes a considerable part of the totality, and it is even much more considerable, if one plays less than that, and consequently a large number of trials is necessary, like one thousand and twenty four or two thousand and forty eight which are so large that this trial outcome can be looked at as negligible value, or even as zero. While following the same argument, one will find that if one plays one million and forty eight thousand five hundred and seventy six trials, the equivalent by this reasoning itself would be found to be almost ten Ecus; but one must consider all in the morale, and from there one will see that it is not possible to play one million and forty eight thousand five hundred and seventy six trials of this game, because even if one suppose only 2 min of time for every trial, including the time that it is necessary to pay, etc. one would find that it was necessary to play during two million ninety seven thousand one hundred and fifty two minutes, that is, more than 13 years of sequence, 6 h a day, which is morally impossible. And if one pays attention, one will find that between playing only one trial and playing the largest morally possible number of trials, this reasoning that gives different equivalents for all the different numbers of trials, yields an average equivalent of five Ecus. Thus I still say that the equivalent sum to the hope of the one who can only win is five Ecus instead of the half of an infinite sum of Ecus, as said by the mathematicians, and as their calculation seems to demand.

XIX. Let us see if according to this estimation it would not be possible to derive the ratio of the money value that corresponds to the benefits resulting from it.

The progression of the probabilities is	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{2^\infty}$
The progression of the money amounts to win is	1	2	4	8	16	32	64	128	256	2^∞.

The sum of all these probabilities, multiplied by the one of the money at stake is $\frac{\infty}{2}$, that is the equivalent data to the mathematical calculation for the hope of the one who can only win. But we saw that this sum $\frac{\infty}{2}$ can, in reality, only be five Ecus; it is necessary therefore to look for a sequence, such that the sum multiplied by the sequence of probabilities equals five Ecus, and this sequence being geometric as that of the probabilities, one will find that it is 1, $\frac{9}{5}$, $\frac{81}{25}$, $\frac{729}{125}$, $\frac{6{,}561}{625}$, $\frac{59{,}049}{3{,}125}$, instead of 1, 2, 4, 8, 16, 32.

Now this sequence 1, 2, 4, 8, 16, 32, etc. represents the quantity of money, and consequently its numerical and mathematical value.

And the other sequence 1, $\frac{9}{5}$, $\frac{81}{25}$, $\frac{729}{125}$, $\frac{6{,}561}{625}$, $\frac{59{,}049}{3{,}125}$, represents the geometric quantity of money given by the experience, and consequently its moral and real value.

Here is therefore a general estimation sufficiently close to the value of money for all possible cases, and independent of any assumption. For example, one sees, comparing the two sequences, that two thousand pounds do not produce double the benefit of one thousand pounds, that $\frac{1}{5}$ is being discounted, and that two thousand pounds are in the morale and in reality only $\frac{9}{5}$ of two thousand pounds, that is, eighteen hundred pounds. A man who has twenty thousand pounds of goods must not estimate it as the double of the goods of another one who has ten thousand pounds, because he actually does only have eighteen thousand pounds of money of this same currency whose value is calculated by the resulting benefits; and a man who even has forty thousand pounds is not four times richer than the one who has ten thousand pounds, because in this real currency he has only 32 thousand and 400 pounds; a man who has 80 thousand pounds, has, by the same currency, only 58 thousand 300 pounds; the one who has 160 thousand pounds, must only count 104 thousand 900 pounds, that is, that although he has sixteen times more than the first one, he hardly has only ten times as much of our real currency. Or even again a man who has thirty-two times as much of money as another, for example 320 thousand pounds in comparison to a man who has 10 thousand pounds, his wealth in the real currency is only 188 thousand pounds, that is, he is eighteen or nineteen times richer, instead of thirty-two times, etc. The miser is like the mathematician; both of them esteem the money by its numerical quantity, the sensible man considers neither the mass nor the number, he sees only the benefits that he can derive from it; he reasons better than the miser, and discerns better than the mathematician. The Ecu that the poor has set aside to pay a necessary tax and the Ecu that completes the bags of the financier have for the miser and for the mathematician just the same value, the latter will count them by two equal quantities, the other will take it from them with equal pleasure, the sensible man instead will count the Ecu of the poor for a Louis, and the Ecu of the financier for a Liard.

XX. Another consideration that comes to the support of this estimation of the moral value of money is that a probability must be regarded as zero as soon as it is only $\frac{1}{10{,}000}$, that is, as soon as it is as small as the unfelt fear of death in the 24 h. One can even say that given the intensity of this fear of death which is considerably larger than the intensity of all the other feelings of fear or of hope, one must consider as almost zero, a fear or a hope that has only $\frac{1}{1{,}000}$ of probability. The weakest man could draw the lots without any emotion, if the ticket of death was mixed with ten thousand tickets of life; and the strong man must draw without fear, if this ticket is mixed with a thousand; thus in any case where the probability is under a thousandth, one must look at it as almost zero. Now, in our question, the probability is found to be $\frac{1}{1{,}024}$ as early as the tenth term of the sequence $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, $\frac{1}{128}$, $\frac{1}{256}$, $\frac{1}{512}$, $\frac{1}{1{,}024}$, it follows that morally thinking, we must neglect all the following terms, and limit all our hopes at this tenth term; which again produces five Ecu as the desired equivalent, and consequently confirms the accuracy of our estimation. By rearranging and cancelling all the calculations where the probability becomes smaller than one thousandth, there will no longer remain any contradiction between the mathematical calculation and common sense. All difficulties of this kind disappear. The man pervaded by this truth will not dedicate more to vain hopes or to false fears; he will not gladly give his Ecu to obtain a thousand, unless he does clearly see that the probability is greater than one thousandth. Finally, he will correct his frivolous hope of making a great fortune with small means.

XXI. So far I have only reasoned and calculated for the truly wise man who is determined only by the weight of reason; but must we not give some attention to the great number of men whom illusion or passion deceive, and who are often quite comfortable with being deceived? Is there yet nothing to lose if things are always presented just as they are? Is hope, however small the probability, not a good for all men, and the only good for the unfortunate ones? After having calculated for the wise man, we therefore calculate also for the much less rare man who enjoys his mistakes often more than his reason. Regardless of cases where for lack of all means, a glimmer of hope is a supreme good; regardless of these circumstances where the restless heart can only rest on the objects of its illusion, and only enjoys its desires; are there not thousands and thousands of occasions where even wisdom must throw forward a volume of hope in the absence of real evidence? For example, the desire to do good, recognized in those who hold the reins of government, albeit without exercise, spreads on all the people a sum of happiness one cannot estimate; hope, albeit vain, is therefore a real good, the enjoyment is taken by anticipation of all other goods. I am forced to admit that full wisdom does not imply full happiness of man, that unfortunately the mere reason had at any time only a small number of cold listeners, and never created enthusiasts; that man stuffed with goods would not yet be happy if he did not hope for new ones; that the superfluous income becomes with time a very necessary thing, and that the mere difference between the wise and the non-wise is that the latter, at the very moment he reaches an overabundance of goods, converts this lovely superfluous income into the sad necessary income, and raises his state equal to that of his new fortune; while the wise man would use this overabundance only to spread the benefits and to obtain himself some new pleasures, sparing the consumption of the superfluous income at the same time as multiplying the enjoyment of it.

XXII. The display of hope is the business of all money swindlers. The big art of the lottery maker is to present large sums with very small probabilities, soon swollen by the spring of greed. These swindlers still enlarge this ideal product, dividing it and giving it for very little money, which everyone can lose, a hope that, though much smaller, seems to be a part of the grandeur of the total sum. One does not know that when the probability is below one thousandth, hope becomes zero however large the promised sum, since anything, however great it may be, is necessarily reduced to nothing as soon as it is multiplied by nothing, as is here the large sum of money multiplied by the probability of zero, as in general any number, multiplied by zero, is always zero. One is unaware again that independently of this reduction of the probabilities to nothing, as soon as they are below one thousandth, hope suffers a successive and proportional decline of the moral value of money, always less than its numerical value, so that hope which numerically appears to double from one to another, has nevertheless only $\frac{9}{5}$ of the real hope instead of 2; and correspondingly the one whose numerical hope is 4 does only have $3\frac{6}{25}$ of this moral hope, this outcome being the mere reality. Instead of 8, this product is only $5\frac{104}{125}$; instead of 16, it is only $10\frac{311}{625}$; instead of 32, $18\frac{2,799}{3,125}$; instead of 64,$34\frac{191}{15,625}$; instead of 128, $61\frac{17,342}{78,625}$; instead of 256, $110\frac{77,971}{390,625}$; instead of 512, $198\frac{701,739}{1,953,125}$; instead of 1,024,$357\frac{456,276}{9,765,625}$, etc. thus one sees how much the moral hope differs in each case from the numerical hope for the real outcome that results from it; the wise man must therefore reject as false all the propositions, though proven by calculation, where the very large quantity of money seems to compensate the very small probability; and if he wants to risk with less disadvantage, he must never allocate his funds to the large venture, it is necessary to divide them. To risk one hundred thousand francs on a single vessel or twenty-five thousand francs on four vessels is not the same thing, because one will have one hundred for the outcome of moral hope in the latter case, while one will have only eighty-one for the same outcome in the first case. It is by this same reason that the most surely profitable businesses are those where the mass of the debt is divided between a large number of creditors. The owner of the mass must suffer only light bankruptcies instead of the one that ruins him when this mass of his business can be placed only with a single hand or similarly only be divided between a small number of debtors. Playing for high stakes in the moral sense is to play a bad game; a punter in the game of pharaoh who gets into his head to push all his cards until fifteen would lose close to a quarter on the moral product of his hope, because while his numerical hope is to pull 16, his moral hope is only $13\frac{104}{125}$. It is the same for countless other examples that one could give; and in all it will always result that the wise man must put at risk the least possible, and that the prudent man who, through his position or his business, is forced to risk large funds, must divide them, and subtract from his speculations the hopes for which the probability is very small, albeit the sum to obtain is proportionally also large.

XXIII. Analysis is the only instrument that has been used up-to-now in the science of probabilities to determine and to fix the ratios of chance; Geometry appeared hardly appropriate for such a delicate matter; nevertheless if one looks at it closely, it will be easily recognized that this advantage of Analysis over Geometry is quite accidental, and that chance according to whether it is modified and conditioned is in the domain of geometry as well as in that of analysis; to be assured of this, it is enough to see that games and problems of conjecture ordinarily revolve only around the ratios of discrete quantities; the human mind, rather familiar with numbers than with measurements of size, has always preferred them; games are a proof of it because their laws are a continual arithmetic; to put therefore Geometry in possession of its rights on the science of chance is only a matter of inventing some games that revolve on size and on its ratios or to analyse the small number of those of this nature that are already found; the clean-tile [franc-carreau] game can serve us as example: here are its very simple terms.

In a room parquetted or paved with equal tiles of an unspecified shape one throws an Ecu in the air; one player bets that after its fall this Ecu will be located clean-tile, that is, on a single tile; the second bets that this Ecu will be located on two tiles, that is, it will cover one of the joints which separates them; a third player bets that the Ecu will be located on two joints; a fourth one bets that the Ecu will be located on three, four or six joints: one asks for the chances of each of these players.

To begin with, I seek the chances of the first and second player; to find them, I inscribe in one of the tiles a similar figure, distant from the tile borders, by the length of half the diameter of the Ecu; the chances of the first player will relate to the one of the second as the area of the circumscribed annulus relates to the area of the inscribed figure; that can be easily shown, because as long as the centre of the Ecu is in the inscribed figure, this Ecu can be located only on a single tile, since by construction this inscribed figure is everywhere distant from the contour of the tile, of a distance equal to the radius of the Ecu; and in contrast, as soon as the centre of the Ecu falls outside of the inscribed figure, the Ecu is necessarily located on two or several tiles, since then its radius is greater than the distance of the contour of this inscribed figure from the contour of the tile; and yet, all points where this centre of the Ecu may fall are represented in the first case by the area of the annulus which is the remainder of the tile; therefore the chances of the first player relate to the chances of the second, as this first area relates to the second; thus to render equal the chances of these two players, it is necessary that the area of the inscribed figure is equal to the one of the annulus, or equivalently, that it is the half of the total surface of the tile.

I enjoyed to do the calculation, and I found that to play a fair game on two square tiles, the border length of the tile must relate to the diameter of the Ecu as $1 : 1 - \sqrt{1/2};$ that is, almost three and half times greater than the diameter of the coin with which one plays.
To play on triangular equilateral tiles, the border length of the tile must relate to the diameter of the coin as $1 : \frac{(1/2)\sqrt{3}}{3+3\sqrt{1/2}},$ that is, almost six times greater than the diameter of the coin.
On lozenge tiles, the border length of the tile must relate to the diameter of the coin as $1 : \frac{(1/2)\sqrt{3}}{2+\sqrt{2}},$ that is, almost four times greater.
Finally, on hexagon tiles, the border length of the tile must relate to the diameter of the coin as $1 : \frac{(1/2)\sqrt{3}}{1+\sqrt{1/2}},$ that is, almost double.
I have not done the calculation for other figures, because these are the only ones by which one can fill a space without leaving some intervals for other figures; and I do not think it is necessary to tell that if the joints of the tiles have some width, they give advantage to the player who bets on the joint, and that consequently one will do well to render the game even more equal by giving to the square tile a little more than three and half times, to the triangular six times, to the lozenges four times, and to the hexagons two times the length of the diameter of the coin with which one plays.
Now I seek the chances of the third player who bets that the Ecu will be located on two joints; and to find it, I inscribe in one of the tiles a similar figure as I have already done, next I extend the inscribed borders of this figure until they meet those of the tile, the chances of the third player will relate to the one of his opponent as the sum of the spaces enclosed between the extension of these lines and the borders of the tile relates to the remainder of the surface of the tile. To be fully shown this requires only to be well understood.
I have also done the calculation for this case, and I found that to play a fair game on square tiles, the border length of the tile must relate to the diameter of the coin as $1 : 1/\sqrt{2},$ that is, greater than by a little less than a third.
On triangular equilateral tiles, the border length of the tile must relate to the diameter of the coin as $1 : 1/2,$ that is, double.
On lozenge tiles, the border length of the tile must relate to the diameter of the coin as $1 : \frac{(1/2)\sqrt{3}}{\sqrt{2}},$ that is, greater by about two-fifth.
On hexagon tiles, the border length of the tile must relate to the diameter of the coin as $1 : (1/2)\sqrt{3},$ that is, greater by a half-quarter.
Now, the fourth player bets that on triangular equilateral tiles the Ecu will be located on six joints, that on square or on lozenge tiles it will be located on four joints, and on hexagon tiles it will be located on three joints; to determine his chances, I describe around the vertex of the tile a circle equal to the Ecu, and I say that on triangular equilateral tiles, his chances will relate to the one of his opponent as the half of the area of this circle relates to the one of the rest of the tile; that on square or on lozenge tiles, his chances will relate to the one of the other as the entire area of the circle

relates to the one of the rest of the tile; and that on hexagon tiles, his chances will relate to the one of his opponent as the double of this area of the circle relates to the remainder of the tile. While supposing therefore that the circumference of the circle relates to the diameter as 22 relates to 7; one will find that to play a fair game on triangular equilateral tiles, the border length of the tile must relate to the diameter of the coin as $1 : \frac{\sqrt{7}\sqrt{3}}{22}$, that is, greater by a little more than a quarter.

On lozenge tiles, the chances will be the same as on triangular equilateral tiles.

On square tiles, the side of the tile must relate to the diameter of the coin, as $1 : \frac{\sqrt{11}}{7}$, that is, greater by about one fifth. On hexagon tiles, the border length of the tile must relate to the diameter of the coin as $1 : \frac{\sqrt{21}\sqrt{3}}{44}$, that is, greater by about one thirteenth.

I omit here the solution of several other cases, like when one of the players bets that the Ecu will fall only on one joint or on two, on three, etc., these are not more difficult than the preceding ones; and besides, one plays rarely this game under different conditions than the mentioned ones.

But if instead of throwing in the air a round piece, as an Ecu, one would throw a piece of another shape, as a squared Spanish pistole, or a needle, a stick, etc. the problem demands a little more geometry, although in general it is always possible to give its solution by space comparisons, as we will show.

I suppose that in a room where the floor is simply divided by parallel joints one throws a stick in the air, and that one of the players bets that the stick will not cross any of the parallels on the floor, and that the other in contrast bets that the stick will cross some of these parallels; one asks for the chances of these two players. One can play this game on a checkerboard with a sewing needle or a headless pin.

To find it, I first draw between the two parallel joints on the floor, AB and CD,[17] two other parallel lines ab and cd, at a distance from the primary ones of half the length of the stick EF, and I evidently see that as long as the middle of the stick is between these second two parallels, it never will be able to cross the primary ones in whatever position EF, ef, it may be located; and as everything that can occur above ab alike occurs below cd, it is only necessary to determine the one or the other; that is why I notice that all the positions of the stick can be represented by one quarter of the circumference of the circle of which the stick length is the diameter; letting therefore $2a$ denote the distance CA of the floor-joints, c the quarter of the circumference of the circle of which the length of the stick is the diameter, letting $2b$ denote the length of the stick, and f the length AB of the joints,

[17][comment: see Fig. 15.1].

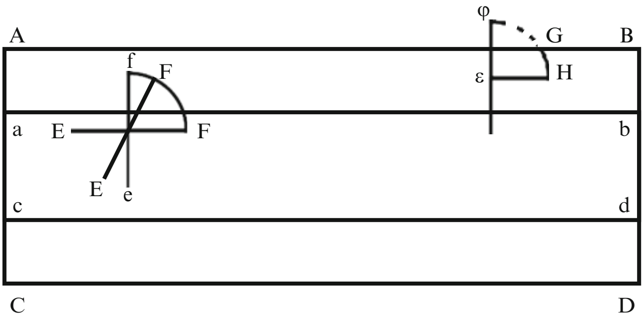

Fig. 15.1 Buffon's needle

I will have $f\overline{(a-b)}c$ as the expression that represents the probability of not crossing the floor-joint, or equivalently, as the expression of all cases where the middle of the stick falls below the line *ab* and above the line *cd*. But when the middle of the stick falls outside the space *abcd*, enclosed between the second parallels, it can, depending on its position, cross or not cross the joint; so that the middle of the stick being located, for example, in ε, the arch φG represents all the positions where it will cross the joint, and the arch *GH* all those where it will not cross, and as it will be the same for all the points on the line $\varepsilon\varphi$, I denote by *dx* the small parts of this line, and γ the arches of the circle φG, and I have $f(sin\gamma dx)$ as the expression of all the cases where the stick crosses, and $f(bc - sin\gamma dx$ as the expression of the cases where it does not cross; I add this last expression to the one noticed above $f\overline{(a-b)}c$ in order to have the entirety of the cases where the stick will not cross, and from that time I see that the chances of the first player relates to the one of the second as $ac - sin\gamma dx : sin\gamma dx$.

If one wants therefore that the game is fair, one will have $ac = 2sin\gamma dx$ or $a = \frac{sin\gamma dx}{(1/2)c}$, that is, equal to the area of a part of the cycloid whose generating circle has as diameter the stick-length $2b$; now, one knows that this cycloid-area is equal to the square of the radius, therefore $a = \frac{b^2}{(1/2)c}$, that is, the length of the stick must be almost three quarters of the distance of the floor-joints.

The solution of this first case drives us easily to the one of another which before would have appeared more difficult, that is to determine the chances of these two players in a paved room of square tiles, because by inscribing in one of the square tiles, a square distant from all borders of the tile by the length b, first one has $c(a-b)^2$ as the expression for one part of the cases where the stick does not cross the joint; next one finds $\overline{(2a-b)}sin\gamma dx$ for those of all the cases where it does cross, and finally $c\overline{b(2a-b)} - \overline{(2a-b)}sin\gamma dx$ for the rest of the cases where it does not

cross; thus the chances of the first player relates to the one of the second as $c\overline{(a-b)}^2 + cb\overline{(2a-b)} - \overline{(ca-b)}sin\gamma dx : \overline{(2a-b)}sin\gamma dx.$[18]
If one wants therefore that the game be fair, one has $c\overline{(a-b)}^2 + cb\overline{(2a-b)} = \overline{(2a-b)}^2 sin\gamma dx.$ or $\frac{(1/2)ca^2}{2a-b} = sin\gamma dx;$ but as we have seen above, $sin\gamma dx = b^2;$ therefore $\frac{(1/2)ca^2}{2a-b} = b^2;$ thus the border length of the tile must relate to the length of the stick, almost as $\frac{41}{22} : 1,$ that is, not completely double. If one played therefore on a checkerboard with a needle of half the length of the side-length of the squares on the checkerboard, it is advantageous to bet that the needle will cross the joints.
One finds by a similar calculation that if one plays with a squared money coin, the sum of the chances will relate to the chances of the player that bets on the joint as $a^2c : 4ab^2\sqrt{1/2} - b^3 - (1/2)Ab,$ where A denotes the excess of the area of the circle circumscribed around the square, and b the semi-diagonal of this square.
These examples suffice to give an idea of the games that one can imagine on the relationships of size; one could propose several other problems of this type, which do not cease to be interesting and even useful: if one asked, for example, how much one risks passing a river on a more or less narrow plank; what must be the fear one must have of lightning or of a bomb drop, and a number of other problems of conjecture where one must consider only the ratio of the size, and that consequently belong to geometry as much as to analysis.

XXIV. From the first steps that one takes in Geometry, one finds Infinity, and since the earliest times the Geometricians caught sight of it; the squaring of the parable and the treatise on *Numero Arenae* by Archimède prove that this great man gave thought to infinity, and even some of his thoughts we must share; we have extended his ideas, though handled in different ways, finally we have found the art of applying calculus to his ideas: but the basis for the metaphysics of the infinite has not changed at all, and it has only been recently that some Geometricians gave us views on infinity that are different from those Ancients, and so far from the nature of the things and truth, that they were even neglected in the Works of these great Mathematicians. Hence arose all the oppositions, all the contradictions that one suffered in calculus; hence arose the disputes between the Geometricians on how to make this calculation, and on the principles from which it derives; one was surprised by the kinds of miracles this calculation produced, this surprise was followed by confusion; it was believed that infinity produced all these wonders; it was imagined that the knowledge of infinity had been refused in

[18][comment: Buffon obviously means $c(a-b)^2 + cb(2a-b) - (2a-b)\sin\gamma dx : (2a-b)\sin\gamma dx$, and in consistence with his earlier notation, one should have the term $(2a-b)\sin\gamma\, dx$ without cap for the cases where it does cross].

all the centuries and has been reserved for ours; finally it was built on systems that have only served to obscure thought. Let us say therefore a few words on the nature of infinity, which while enlightening seems to have blinded men.

We have clear ideas about the magnitude, we see that things in general can be augmented or diminished, and the idea that a thing becomes larger or smaller is an idea as present and as familiar to us as the one about the thing itself; anything being thus presented to us or being only imagined, we see that it is possible to augment or diminish it; nothing stops, nothing destroys this possibility, one can always conceive the half of the smallest thing and the double of the largest thing; one even can conceive that it can become one hundred times, one thousand times, one hundred thousand times smaller or larger; and it is this possibility of growth without limits in what consists the true concept one must have on infinity; this concept is derived by us from the concept of the finite; a finite thing is a thing that has ends, limits; an infinite thing is only the same finite thing on which we remove these ends and limits; the concept of infinity is thus only a concept of deprivation, and has nothing of a real object. Here is not the place to show that space, time, duration, are no infinite realities; it will suffice to prove that there exists no number currently infinite or infinitely small, or larger or smaller than an infinite one, etc.

Numbers are only an assembly of units of the same kind; the unit is not at all a number, the unit designates a single thing in general; but the first number 2 denotes not only two things, but again two similar things, two things of same kind; it is the same for all the other numbers: yet these numbers are only representations, and never exist independently of the things that they represent; the characters that they designate do not give them any reality at all, they require a subject or rather an assembly of subjects to represent, to make their existence possible; I understand their intelligible existence, because they can only have real values; now an assembly of units or of subjects can never be other than finite, that is, that one always will be able to assign the parts of which it is composed; consequently numbers cannot be infinite whatever the growth one gives them.

But, one will say, the last term of the natural sequence 1, 2, 3, 4, etc. is it not infinite? Are there no last terms of other even more infinite sequences than the last term of the natural sequence? It seems that in general the numbers have to become infinite in the end, but can they still grow? Thereto I reply, that this growth to which they are susceptible proves evidently that they can not at all be infinite; I say further that in these sequences there is no last term; that only even to suppose them a last term, is to destroy the quintessence of the sequence that consists in the succession of the terms that can be followed by other terms, and these other terms again by others; but all are of the same nature as the preceding ones, this is to say, all finite, all composed of units; thus when one supposes that a sequence has a last term, and that the latter term is an infinite number, one contradicts to the definition of the number and to the general law of sequences.

Most of our errors in metaphysics come from the reality that we give to ideas of deprivation; we know the finite, we recognize its real properties, we examine it, and by considering it after this examination we do not recognize it anymore, and we believe having created a new being, while we did only destroy some part of what we formerly had known.

One must therefore consider infinity, in small, in large, only as a deprivation, an entrenchment of the concept of the finite, which can be used like an assumption that, in some case, can help to simplify the concepts, and must generalize their results in the practice of the Sciences; thus all the art is reduced to capitalizing on this assumption, trying to apply it to the subjects that one considers. The whole merit is therefore in the application, in a word in the use one makes of it.

XXV. All our knowledge is based on relationships and comparisons, everything is therefore relation in the Universe; and hence everything is subject to measure, even our ideas being all relative have nothing absolute. There are, as we have explained, different degrees of probabilities and of certainty. And even the evidence is more or less clear, more or less intense, according to the different aspects, that is, according to the relationships under which it is presented; the truth transmitted and compared by different minds appears under more or less large relationships, since the result of the affirmation or negation of a proposition by all men in general seems to still give weight to the truths that are best shown and most independent of any convention.

The properties of matter that appear to us evidently distinct from each other have no relation between them; size cannot be compared to gravity, inscrutability not to time, movement not to surface, etc. These properties have in common only the underlying subject, and that gives them their being; each of these properties considered separately asks therefore for a measure of its kind, that is, a measure different from all the others.

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