On idempotent n-ary uninorms

Jimmy Devillet[1], Gergely Kiss[2], and Jean-Luc Marichal[1]

[1] University of Luxembourg, Mathematics Research Unit
Esch-sur-Alzette, Luxembourg
{jimmy.devillet,jean-luc.marichal}@uni.lu
[2] Hungarian Academy of Sciences, Alfréd Rényi Institute of Mathematics
Budapest, Hungary
kigergo57@gmail.com

Abstract. In this paper we describe the class of idempotent n-ary uninorms on a given chain. When the chain is finite, we axiomatize the latter class by means of the following conditions: associativity, quasitriviality, symmetry, and nondecreasing monotonicity. Also, we show that associativity can be replaced with bisymmetry in this new axiomatization.

1 Introduction

Let X be a nonempty set and let $n \geq 2$ be an integer. For a few decades, many classes of binary aggregation functions have been investigated due to their great importance in data fusion (see, e.g. [8] and the references therein). Among these classes, the class of binary uninorms plays an important role in fuzzy logic. Recently, the study of the class of n-ary uninorms gained an increasing interest (see, e.g. [9]).

This paper, which is a shorter version of [6][3], focuses on characterizations of the class of idempotent n-ary uninorms (Definition 3). In Section 2, we provide a characterization of these operations and show that they only depend on the extreme values of the variables (Proposition 1). We also provide a description of these operations as well as an alternative axiomatization when the underlying set is finite (Theorem 1). In particular, we extend characterizations of the class of idempotent binary uninorms obtained by Couceiro et al. [4, Theorems 12 and 17] to the class of idempotent n-ary uninorms. In Section 3, we investigate some subclasses of bisymmetric n-ary operations and derive several equivalences involving associativity and bisymmetry. More precisely, we show that if an n-ary operation has a neutral element, then it is associative and symmetric if and only if it is bisymmetric (Corollary 1). Also, we show that if an n-ary operation is quasitrivial and symmetric, then it is associative if and only if it is bisymmetric (Corollary 1). These observations enable us to replace associativity with bisymmetry in our axiomatization (Theorem 2).

We adopt the following notation throughout. We use the symbol X_k if X contains $k \geq 1$ elements, in which case we assume without loss of generality that $X_k = \{1, \ldots, k\}$. Finally, for any integer $k \geq 1$ and any $x \in X$, we set $k \cdot x = x, \ldots, x$ (k times). For instance, we have $F(3 \cdot x, 2 \cdot y) = F(x, x, x, y, y)$.

Recall that a binary relation R on X is said to be

[3] This paper is also an extended version of [7].

- *total* if $\forall x, y$: xRy or yRx;
- *transitive* if $\forall x, y, z$: xRy and yRz implies xRz;
- *antisymmetric* if $\forall x, y$: xRy and yRx implies $x = y$.

Recall also that a *total ordering on* X is a binary relation $\leq$ on X that is total, transitive, and antisymmetric. The ordered pair $(X, \leq)$ is then called a *chain*.

Definition 1. *An operation* $F\colon X^n \to X$ *is said to be*

- idempotent *if* $F(n \cdot x) = x$ *for all* $x \in X$*;*
- quasitrivial *(or* conservative*) if* $F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$ *for all* $x_1, \ldots, x_n \in X$*;*
- symmetric *if* $F(x_1, \ldots, x_n)$ *is invariant under any permutation of* $x_1, \ldots, x_n$*;*
- associative *if*

$$
\begin{aligned}
&F(x_1, \ldots, x_{i-1}, F(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{2n-1}) \\
&= F(x_1, \ldots, x_i, F(x_{i+1}, \ldots, x_{i+n}), x_{i+n+1}, \ldots, x_{2n-1})
\end{aligned}
$$

  for all $x_1, \ldots, x_{2n-1} \in X$ *and all* $i \in \{1, \ldots, n-1\}$*;*
- bisymmetric *if*

$$F(F(\mathbf{r}_1), \ldots, F(\mathbf{r}_n)) \ = \ F(F(\mathbf{c}_1), \ldots, F(\mathbf{c}_n))$$

  for all $n \times n$ *matrices* $[\mathbf{c}_1 \ \cdots \ \mathbf{c}_n] = [\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T \in X^{n \times n}$.
- nondecreasing *for some total ordering* $\leq$ *on* X *if* $F(x_1, \ldots, x_n) \leq F(x_1', \ldots, x_n')$ *whenever* $x_i \leq x_i'$ *for all* $i \in \{1, \ldots, n\}$.

Given a total ordering $\leq$ on X, the *maximum* (resp. *minimum*) *operation on* X *for* $\leq$ is the symmetric n-ary operation $\max_\leq$ (resp. $\min_\leq$) defined by $\max_\leq(x_1, \ldots, x_n) = x_i$ (resp. $\min_\leq(x_1, \ldots, x_n) = x_i$) where $i \in \{1, \ldots, n\}$ is such that $x_j \leq x_i$ (resp. $x_i \leq x_j$) for all $j \in \{1, \ldots, n\}$.

Definition 2. *Let* $F\colon X^n \to X$ *be an operation. An element* $e \in X$ *is said to be a* neutral element *of* F *if*

$$F((i-1) \cdot e, x, (n-i) \cdot e) \ = \ x$$

for all $x \in X$ *and all* $i \in \{1, \ldots, n\}$.

2 A first characterization

In this section we provide a characterization of the n-ary operations on X that are associative, quasitrivial, symmetric, and nondecreasing for some total ordering $\leq$ on X. We will also show that in the case where X is finite these operations are exactly the idempotent n-ary uninorms.

Recall that a *uninorm* on a chain $(X, \leq)$ is a binary operation $U\colon X^2 \to X$ that is associative, symmetric, nondecreasing for $\leq$, and has a neutral element (see [5, 11]). It is not difficult to see that any idempotent uninorm is quasitrivial.

The concept of uninorm can be easily extended to n-ary operations as follows.

Definition 3 (see [9]). *Let $\leq$ be a total ordering on X. An n-ary* uninorm *is an operation $F\colon X^n \to X$ that is associative, symmetric, nondecreasing for $\leq$, and has a neutral element.*

The next proposition provides a characterization of idempotent n-ary uninorms. In particular, since any idempotent uninorm is quasitrivial, it shows that an idempotent n-ary uninorm always outputs either the greatest or the smallest of its input values.

Proposition 1. *Let $\leq$ be a total ordering on X and let $F\colon X^n \to X$ be an operation. Then F is an idempotent n-ary uninorm if and only if there exists a unique idempotent uninorm $U\colon X^2 \to X$ such that*

$$F(x_1,\ldots,x_n) \ = \ U(\min_{\leq}(x_1,\ldots,x_n), \max_{\leq}(x_1,\ldots,x_n)), \qquad x_1,\ldots,x_n \in X.$$

In this case, the uninorm U is uniquely defined as $U(x,y) = F((n-1)\cdot x, y)$.

We now introduce the concept of single-peaked total ordering which first appeared for finite chains in social choice theory (see Black [2, 3]).

Definition 4. *Let $\leq$ and $\preceq$ be total orderings on X. We say that $\preceq$ is* single-peaked for $\leq$ *if for any $a, b, c \in X$ such that $a < b < c$ we have $b \prec a$ or $b \prec c$.*

When X is finite, the single-peakedness property of a total ordering $\preceq$ on X for some total ordering $\leq$ on X can be easily checked by plotting a function, say $f_{\preceq}$, in a rectangular coordinate system in the following way. Represent the reference totally ordered set $(X, \leq)$ on the horizontal axis and the reversed version of the totally ordered set $(X, \preceq)$, that is $(X, \preceq^{-1})$, on the vertical axis. The function $f_{\preceq}$ is defined by its graph $\{(x, x) : x \in X\}$.[4] We then see that the total ordering $\preceq$ is single-peaked for $\leq$ if and only if $f_{\preceq}$ has only one local maximum.

Example 1. Consider $X = X_6$ endowed with the usual total ordering $\leq$ defined by $1 < 2 < 3 < 4 < 5 < 6$. Figure 1 gives the functions $f_{\preceq}$ and $f_{\preceq'}$ corresponding to the total orderings $3 \prec 4 \prec 2 \prec 5 \prec 1 \prec 6$ and $4 \prec' 2 \prec' 6 \prec' 1 \prec' 3 \prec' 5$, respectively, on X_6. We see that $\preceq$ is single-peaked for $\leq$ since $f_{\preceq}$ has only one local maximum while $\preceq'$ is not single-peaked for $\leq$ since $f_{\preceq'}$ has three local maxima.

It is known (see, e.g., [1]) that there are exactly 2^{k-1} single-peaked total orderings on X_k for the usual total ordering $\leq$ defined by $1 < \ldots < k$.

The following theorem provides several characterizations of the class of associative, quasitrivial, symmetric, and nondecreasing operations $F\colon X^n \to X$. In particular, it provides a new axiomatization as well as a description of idempotent n-ary uninorms when the underlying set X is finite. In the latter case, it also extends characterizations of the class of idempotent uninorms obtained by Couceiro et al. [4, Theorems 12 and 17] to the class of idempotent n-ary uninorms.

Theorem 1. *Let $\leq$ be a total ordering on X and let $F\colon X^n \to X$ be an operation. The following assertions are equivalent.*

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[4] When $X = X_k$ for some integer $k \geq 1$, the graphical representation of $f_{\preceq}$ is then obtained by joining the points $(1, 1), \ldots, (k, k)$ by line segments.

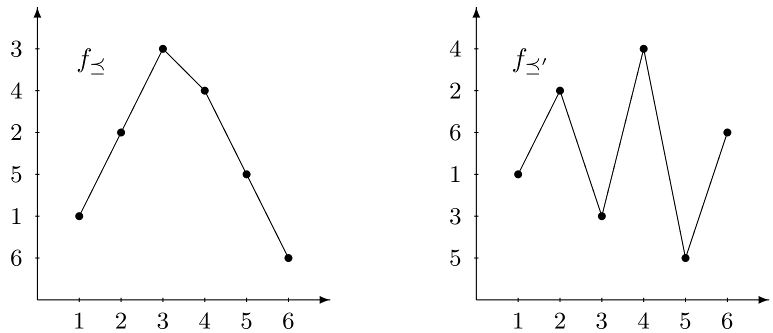

Fig. 1. $\preceq$ is single-peaked (left) while $\preceq'$ is not (right)

(i) *F is associative, quasitrivial, symmetric, and nondecreasing for $\leq$.*
(ii) *There exists a quasitrivial, symmetric, and nondecreasing operation $G\colon X^2 \to X$ such that*

$$F(x_1,\ldots,x_n) \;=\; G(\min_{\leq}(x_1,\ldots,x_n),\max_{\leq}(x_1,\ldots,x_n)), \qquad x_1,\ldots,x_n \in X.$$

(iii) *There exists a total ordering $\preceq$ on X that is single-peaked for $\leq$ and such that $F = \max_{\preceq}$.*

If $X = X_k$ for some integer $k \geq 1$, then any of the assertions (i) – (iii) above is equivalent to the following one.

(iv) *F is an idempotent n-ary uninorm.*

Moreover, there are exactly 2^{k-1} operations $F\colon X_k^n \to X_k$ satisfying any of the assertions (i) – (iv).

Now, let us illustrate Theorem 1 for binary operations. Recall that the contour plot of any operation $F\colon X_k^2 \to X_k$ is the undirected graph (X_k^2, E), where

$$E \;=\; \{\{(x,y),(u,v)\} \mid (x,y) \neq (u,v) \text{ and } F(x,y) = F(u,v)\}.$$

We can always represent the contour plot of any operation $F\colon X_k^2 \to X_k$ by fixing a total ordering on X_k. For instance, using the usual total ordering $\leq$ on X_6, in Figure 2 (left) we represent the contour plot of an operation $F\colon X_6^2 \to X_6$[5]. It is not difficult to see that F is quasitrivial and symmetric. To check whether F is associative and nondecreasing it suffices by Theorem 1 to find a total ordering $\preceq$ on X_6 that is single-peaked for $\leq$ and such that $F = \max_{\preceq}$. In Figure 2 (right) we represent the contour plot of F by using the total ordering $\preceq$ on X_6 defined by $3 \prec 4 \prec 2 \prec 5 \prec 6 \prec 1$. It is not difficult to see that $\preceq$ is single-peaked for $\leq$. Also, we have $F = \max_{\preceq}$ which shows by Theorem 1 that F is associative and nondecreasing for $\leq$. Thus, by Theorem 1 we conclude that F is an idempotent uninorm.

[5] To simplify the representation of the connected components, we omit edges that can be obtained by transitivity.

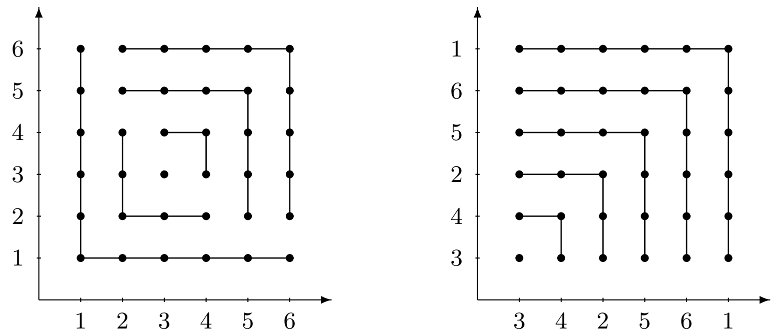

Fig. 2. An idempotent uninorm $F\colon X_6^2 \to X_6$

Remark 1. We observe that an alternative characterization of idempotent uninorms on chains was provided in [10]. Due to Proposition 1, we can extend this characterization to the class of idempotent n-ary uninorms.

3 An alternative characterization

In this section we investigate bisymmetric n-ary operations and derive several equivalences involving associativity and bisymmetry. More precisely, if an n-ary operation has a neutral element, then it is associative and symmetric if and only if it is bisymmetric. Also, if an n-ary operation is quasitrivial and symmetric, then it is associative if and only if it is bisymmetric. In particular, these observations enable us to replace associativity with bisymmetry in Theorem 1.

Definition 5. *We say that an operation $F\colon X^n \to X$ is* ultrabisymmetric *if*

$$F(F(\mathbf{r}_1),\ldots,F(\mathbf{r}_n)) \;=\; F(F(\mathbf{r}'_1),\ldots,F(\mathbf{r}'_n))$$

for all $n \times n$ matrices $[\mathbf{r}_1 \;\cdots\; \mathbf{r}_n]^T, [\mathbf{r}'_1 \;\cdots\; \mathbf{r}'_n]^T \in X^{n\times n}$, where $[\mathbf{r}'_1 \;\cdots\; \mathbf{r}'_n]^T$ is obtained from $[\mathbf{r}_1 \;\cdots\; \mathbf{r}_n]^T$ by exchanging two entries.

Ultrabisymmetry seems to be a rather strong property. However, as the next result shows, this property is satisfied by any operation that is bisymmetric and symmetric.

Proposition 2. *Let $F\colon X^n \to X$ be an operation. If F is ultrabisymmetric, then it is bisymmetric. The converse holds whenever F is symmetric.*

Proposition 3. *Let $F\colon X^n \to X$ be an operation. Then the following assertions hold.*

(a) If F is quasitrivial and ultrabisymmetric, then it is associative and symmetric.
(b) If F is associative and symmetric, then it is ultrabisymmetric.
(c) If F is bisymmetric and has a neutral element, then it is associative and symmetric.

Corollary 1. *Let $F\colon X^n \to X$ be an operation. Then the following assertions hold.*

(a) If F is quasitrivial and symmetric, then it is associative if and only if it is bisymmetric.
(b) If F has a neutral element, then it is associative and symmetric if and only if it is bisymmetric.

From Corollary 1 we immediately derive the following theorem, which is an important and surprising result.

Theorem 2. *In Theorem 1(i) we can replace associativity with bisymmetry. Also, in Theorem 1(iv) we can replace associativity and symmetry with bisymmetry.*

Acknowledgments

The first author is supported by the Luxembourg National Research Fund under the project PRIDE 15/10949314/GSM. The second author is also supported by the Hungarian National Foundation for Scientific Research, Grant No. K124749.

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