BOUNDS FOR THE MOD 2 COHOMOLOGY OF $\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right)$

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Abstract. We attempt to calculate the mod 2 cohomology of $\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right)$ from the mod 2 cohomology of $\mathrm{SL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right)$, following the method in Nicolas Weiss' PhD thesis. In degree 1, we show that it is 2-dimensional, but in higher degrees, we can only provide non-coinciding lower and upper bounds.

Introduction

In the authors' paper submitted to a journal, the following ring structure on $\mathrm{H}^*(\mathrm{SL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right); \mathbb{F}_2)$ has been calculated.

Theorem 1. *The cohomology ring* $\mathrm{H}^*(\mathrm{SL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right); \mathbb{F}_2)$ *is isomorphic to the free module*

$$\mathbb{F}_2[e_4](x_2, x_3, y_3, z_3, s_3, x_4, s_4, s_5, s_6)$$

over $\mathbb{F}_2[e_4]$ *(the image of* $\mathrm{H}^*_{\mathrm{cts}}(\mathrm{SL}_2(\mathbb{C}); \mathbb{F}_2)$*), where the subscript of the classes specifies their degree,* e_4 *is the image of the second Chern class of the natural representation of* $\mathrm{SL}_2(\mathbb{C})$*, and all other classes are exterior classes.*

In the present report that is not intended for journal publication, we attempt to deduce the mod 2 cohomology of $\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right)$ from it, following the method in Nicolas Weiss' PhD thesis [4].

The Lyndon–Hochschild–Serre spectral sequence of the extension

$$1 \to \mathrm{SL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right) \to \mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right) \to^{\det} \left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right)^\times \to 1$$

yields upper bounds for $\mathrm{H}^q(\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right); \mathbb{F}_2)$, in terms of the sums $\bigoplus_{k=0}^{q} E_2^{k,q-k}$ over its E_2 terms. However, in the degrees $q-k$ where the dimension of $\mathrm{H}^{q-k}(\mathrm{SL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right); \mathbb{F}_2)$ is greater than one, we can compute the action of $\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right)^\times$ on it only indirectly, and this is why we have just an upper bound for the dimension of

$$E_2^{k,q-k} = \mathrm{H}^k\left(\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right)^\times; \mathrm{H}^{q-k}(\mathrm{SL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right); \mathbb{F}_2)\right).$$

We make the upper bound precise in Proposition 5, the lower bound in Proposition 2; we combine them to a frame in Corollary 7; and we conclude in Corollary 8 that $\dim_{\mathbb{F}_2} \mathrm{H}^1(\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right); \mathbb{F}_2) = 2$.

1. Lower bounds

In Nicolas Weiss' PhD thesis, lower bounds for $\mathrm{H}^q(\mathrm{GL}_2(\mathcal{O}); \mathbb{F}_2)$, in his case $\mathcal{O} = \mathbb{Z}[\sqrt{-1}\,]\left[\frac{1}{2}\right]$, in our case $\mathcal{O} = \mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]$, are obtained via the commutative diagram

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$$\begin{tikzcd}
\mathrm{H}^q(\mathrm{D}_2\left(\mathbb{Z}[\sqrt{-2}]\left[\tfrac{1}{2}\right]\right);\mathbb{F}_2) & \mathrm{H}^q(\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}]\left[\tfrac{1}{2}\right]\right);\mathbb{F}_2) \arrow[l, "\mathrm{Res}"'] \\
\bigotimes_2 \mathrm{H}^q(\mathrm{D}_2(\mathbb{F}_\ell);\mathbb{F}_2) \arrow[u, "\pi_D^*"] & \bigotimes_2 \mathrm{H}^q(\mathrm{GL}_2(\mathbb{F}_\ell);\mathbb{F}_2) \arrow[u, "\pi^*"'] \arrow[l, "\bigotimes_2 \mathrm{Res}^{\mathrm{GL}_2(\mathbb{F}_\ell)}_{\mathrm{D}_2(\mathbb{F}_\ell)}"']
\end{tikzcd} \quad (1)$$

where D_2 stands for taking the subgroup of diagonal 2×2 matrices, and the maps π^* and π_D^* are constructed as follows. We start with two non-trivial ring homomorphisms $\pi_1, \pi_2 : \mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right] \to \mathbb{F}_\ell$, which exist as soon as 2 is invertible in $\mathbb{F}_\ell$ and -2 is a square $r^2 \in \mathbb{F}_\ell$. In that case, we get π_1 by sending $\sqrt{-2}$ to r and π_2 by sending $\sqrt{-2}$ to $-r \in \mathbb{F}_\ell$.

Proposition 2. *We get lower bounds*

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right);\mathbb{F}_2) \geq \begin{cases} 6, & q = 4, \\ 4, & q = 3, \\ 1, & q = 2, \\ 2, & q = 1. \end{cases}$$

For $1 \leq k \in \mathbb{N}$, the highest lower bounds $b_q \geq \dim_{\mathbb{F}_2}(\mathrm{image}(\mathrm{Res} \circ \pi^*)_q)$ that we can possibly get for the dimension over $\mathbb{F}_2$ of $\mathrm{H}^q(\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right);\mathbb{F}_2)$ using diagram 1, are

$$b_q = \begin{cases} 12k + 13, & q = 4 + 4k, \\ 12k + 10, & q = 3 + 4k, \\ 12k + 7, & q = 2 + 4k, \\ 12k + 4, & q = 1 + 4k. \end{cases}$$

Proof. First, we note that 2 does not have a multiplicative inverse in $\mathbb{F}_2$, so for mapping the element $\frac{1}{2}$, ℓ must be odd.

On the other hand,

$$\mathrm{D}_2\left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right) \cong \left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right)^\times \times \left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right)^\times$$

and $\left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right)^\times \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so we have

$$\mathrm{H}^*(\mathrm{D}_2\left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right);\mathbb{F}_2) \cong \bigotimes_2 (\mathbb{F}_2[\gamma_1] \otimes \Lambda(z_1)) \cong \mathbb{F}_2[\gamma_1, \gamma_1'] \otimes \Lambda(z_1, z_1'),$$

where γ_1, γ_1' are polynomial classes and z_1, z_1' are exterior classes, all of degree 1.

Then, the codomain of $(\mathrm{Res} \circ \pi^*)_q$, namely $\mathrm{H}^q(\mathrm{D}_2\left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right);\mathbb{F}_2)$, considered as an $\mathbb{F}_2$-vector space, has dimensions $6m + 1$ for $q = 2m$ and $6m + 4$ for $q = 2m + 1$. From this, we read off the highest possible bounds b_q of the proposition.

Now we establish the specific bound in dimensions $1 \leq q \leq 4$ at the specific prime $\ell = 11$. By Quillen's result [3], for all odd prime numbers $\ell \equiv 3 \mod 4$ (the "exceptional case" in Quillen's paper), we have an isomorphism of algebras

$$\mathrm{H}^*(\mathrm{GL}_2(\mathbb{F}_\ell);\mathbb{F}_2) \cong \mathbb{F}_2[e_1, e_3, c_2]/(c_2 e_1^2 = e_3^2),$$

where e_1, e_3 and c_2 are polynomial classes of degrees 1, 3 and 4 respectively, modulo the relation $c_2 e_1^2 = e_3^2$.

For $\ell = 11$, the two reduction homomorphisms $\pi_1, \pi_2 : \mathbb{Z}[\sqrt{-2}, \frac{1}{2}] \to \mathbb{F}_{11}$ given by $\pi_1(\sqrt{-2}) = 3$, $\pi_2(\sqrt{-2}) = -3$, induce on mod 2 cohomology the following homomorphisms. On

$$\mathbb{F}_2[\zeta_1] \cong \mathrm{H}^*(\mathbb{F}_{11}^\times;\mathbb{F}_2) \to \mathrm{H}^*((\mathbb{Z}[\sqrt{-2},\tfrac{1}{2}])^\times;\mathbb{F}_2) \cong \mathbb{F}_2[\gamma_1]\otimes\Lambda(z_1),$$

where ζ_1 and γ_1 are polynomial classes of degree 1, and z_1 is an exterior class of degree 1, Hans-Werner Henn has determined that $\pi_1^*(\zeta_1) = \gamma_1$ and $\pi_2^*(\zeta_1) = \gamma_1 + z_1$, because -1 is not a square in $\mathbb{F}_{11}$ and -3 neither, but 3 is a square. Then at $\ell = 11$, Diagram 1 becomes:

$$\begin{tikzcd}
\mathbb{F}_2[\gamma_1,\gamma_1']\otimes\Lambda(z_1,z_1') & \mathrm{H}^q\left(\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right);\mathbb{F}_2\right) \arrow[l, "\mathrm{Res}"'] \\
\bigotimes_2 \mathbb{F}_2[\zeta_1,\zeta_1'] \arrow[u, "\pi_1^*\otimes\pi_2^*"] & \bigotimes_2 \mathbb{F}_2[e_1,e_3,c_2]/(c_2e_1^2 = e_3^2) \arrow[u, "\pi^*"'] \arrow[l, "\bigotimes_2 \mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}"]
\end{tikzcd}$$

(2)

with $\pi_1^*\otimes\pi_2^*$ being a surjection. The analogue of [4, Théorème 116] tells us that

$$\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(e_1) = \zeta_1+\zeta_1',\ \mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(e_3) = \zeta_1(\zeta_1')^2+\zeta_1'\zeta_1^2 \text{ and } \mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(c_2) = \zeta_1^2\cdot(\zeta_1')^2.$$

This entails

$$\begin{aligned}
\pi_1^*\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(e_1) = \gamma_1+\gamma_1', &\quad \pi_2^*\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(e_1) = \gamma_1+z_1+\gamma_1'+z_1',\\
\pi_1^*\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(e_3) = \gamma_1(\gamma_1')^2+\gamma_1^2\gamma_1', &\quad \pi_2^*\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(e_3) = (\gamma_1+z_1)(\gamma_1')^2+(\gamma_1'+z_1')\gamma_1^2,\\
\pi_1^*\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(c_2) = \gamma_1^2\cdot(\gamma_1')^2, &\quad \pi_2^*\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(c_2) = (\gamma_1+z_1)^2\cdot(\gamma_1'+z_1')^2.
\end{aligned}$$

As $(\gamma_1+z_1)^2 = \gamma_1^2$ and $(\gamma_1'+z_1')^2 = (\gamma_1')^2$ in $\mathbb{F}_2[\gamma_1,\gamma_1']\otimes\Lambda(z_1,z_1')$, we get $\pi_1^*\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(c_2) = \pi_2^*\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(c_2)$; and as

$$\begin{gathered}
\pi_1^*\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(e_1e_3) = (\gamma_1+\gamma_1')(\gamma_1(\gamma_1')^2+\gamma_1^2\gamma_1') = \gamma_1(\gamma_1')^3+\gamma_1^3\gamma_1',\\
\pi_2^*\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(e_1e_3) = (\gamma_1+z_1+\gamma_1'+z_1')\left((\gamma_1+z_1)(\gamma_1')^2+(\gamma_1'+z_1')\gamma_1^2\right) =\\
(\gamma_1+z_1)(\gamma_1')^3+(\gamma_1z'+zz')(\gamma_1')^2+(\gamma_1^3+\gamma_1^2z)\gamma_1'+\gamma_1^3z+\gamma_1^2zz'\\
\pi_1^*\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(e_1)\pi_2^*\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(e_3) = (\gamma_1+\gamma_1')\left((\gamma_1+z_1)(\gamma_1')^2+(\gamma_1'+z_1')\gamma_1^2\right) =\\
(\gamma_1+z_1)(\gamma_1')^3+\gamma_1z_1(\gamma_1')^2+(\gamma_1^3+\gamma_1^2z')\gamma_1'+\gamma_1^3z',\\
\pi_2^*\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(e_1)\pi_1^*\mathrm{Res}_{\mathrm{D}_2(\mathbb{F}_\ell)}^{\mathrm{GL}_2(\mathbb{F}_\ell)}(e_3) = (\gamma_1+z_1+\gamma_1'+z_1')(\gamma_1(\gamma_1')^2+\gamma_1^2\gamma_1') =\\
\gamma_1(\gamma_1')^3+(z_1+z_1')\gamma_1(\gamma_1')^2+(\gamma_1+z_1+z_1')\gamma_1^2\gamma_1',
\end{gathered}$$

we conclude that the image of $(\mathrm{Res}\circ\pi^*)_q$ is spanned by the following vectors over $\mathbb{F}_2$.

q	in image of $(\mathrm{Res}\circ\pi^*)_q$	in cokernel of $(\mathrm{Res}\circ\pi^*)_q$	b
1	$\gamma_1+\gamma_1', \gamma_1+z_1+\gamma_1'+z_1',$	z_1, γ_1	2
2	$\gamma_1^2+(\gamma_1')^2$	$\gamma_1^2, \gamma_1\gamma_1', \gamma_1z_1, \gamma_1z_1', \gamma_1'z_1, \gamma_1'z_1'$	1
3	$\gamma_1^3+(\gamma_1')^3, (\gamma_1^2+(\gamma_1')^2)(\gamma_1+z_1+\gamma_1'+z_1'),$		
	$\gamma_1(\gamma_1')^2+\gamma_1^2\gamma_1', (\gamma_1+z_1)(\gamma_1')^2+(\gamma_1'+z_1')\gamma_1^2$	six classes	4
4	$\gamma_1^4+(\gamma_1')^4, \gamma_1^2\cdot(\gamma_1')^2, \gamma_1(\gamma_1')^3+\gamma_1^3\gamma_1'$		
	$(\gamma_1+z_1)(\gamma_1')^3+(\gamma_1z'+zz')(\gamma_1')^2+(\gamma_1^3+\gamma_1^2z)\gamma_1'+\gamma_1^3z+\gamma_1^2zz',$		
	$(\gamma_1+z_1)(\gamma_1')^3+\gamma_1z_1(\gamma_1')^2+(\gamma_1^3+\gamma_1^2z')\gamma_1'+\gamma_1^3z'$		
	$\gamma_1(\gamma_1')^3+(z_1+z_1')\gamma_1(\gamma_1')^2+(\gamma_1+z_1+z_1')\gamma_1^2\gamma_1'$	seven classes	6

□

Remark 3. If we use primes of a different shape, this makes us end up with $\pi_1^* = \pi_2^*$. In fact, numerical evidence (for all primes $3 \le \ell < 2$millions) suggests that

If $\ell \ge 3$ prime, -2 a square in $\mathbb{F}_\ell$ and -1 not a square in $\mathbb{F}_\ell$, then $\ell = 8m+3$.

Example 4. By Quillen's result [3] (as recalled in [2, p. 5]), for all odd prime numbers $\ell \equiv 1 \mod 4$ (the "typical case" in Quillen's paper), we have an isomorphism of algebras

$$\mathrm{H}^*(\mathrm{GL}_2(\mathbb{F}_\ell);\, \mathbb{F}_2) \cong \mathbb{F}_2[c_1, c_2] \otimes \Lambda(e_1, e_3),$$

where the degree of the Chern class c_m is $2m$ and the degrees of the exterior classes are $\deg(e_1) = 1$, $\deg(e_3) = 3$. For $\ell = 17$, the two reduction homomorphisms $\pi_1, \pi_2 : \mathbb{Z}[\sqrt{-2}, \frac{1}{2}] \to \mathbb{F}_{17}$ given by $\pi_1(\sqrt{-2}) = 7$, $\pi_2(\sqrt{-2}) = -7$, induce the same homomorphism on mod 2 cohomology. On

$$\mathbb{F}_2[y] \otimes \Lambda(x) \cong \mathrm{H}^*\left((\mathbb{F}_{17})^\times; \mathbb{F}_2\right) \to \mathrm{H}^*((\mathbb{Z}[\sqrt{-2}, \tfrac{1}{2}])^\times; \mathbb{F}_2) \cong \mathbb{F}_2[\gamma_1] \otimes \Lambda(z_1),$$

with y a polynomial class of degree 2, γ_1 are polynomial classes of degree 1, while x and z_1 are exterior classes of degree 1, the converse of Hans-Werner Henn's argument (-1 is a square in $\mathbb{F}_{2^n+1}$, n even, but 7 and -7 are not squares in $\mathbb{F}_{17}$) should imply $\pi_1^*(y) = \pi_2^*(y) = \gamma_1^2$ and $\pi_1^*(x) = \pi_2^*(x) = z_1$. Then at $\ell = 17$, we can simplify Diagram 1, replacing $\pi_1^* \otimes \pi_2^*$ by π_1^*:

$$\begin{tikzcd}
\mathbb{F}_2[\gamma_1, \gamma_1'] \otimes \Lambda(z_1, z_1') & \mathrm{H}^q(\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right); \mathbb{F}_2) \arrow[l, "\mathrm{Res}"'] \\
\mathbb{F}_2[y, y'] \otimes \Lambda(x, x') \arrow[u, "\pi_1^*"] & \mathbb{F}_2[c_1, c_2] \otimes \Lambda(e_1, e_3) \arrow[l, "\mathrm{Res}^{\mathrm{GL}_2(\mathbb{F}_\ell)}_{\mathrm{D}_2(\mathbb{F}_\ell)}"] \arrow[u, "\pi_1^*"']
\end{tikzcd} \tag{3}$$

with y' a copy of y and x' a copy of x, supported at the other corner of the diagonal matrices. The analogue of [4, Théorème 116] tells us that

$$\mathrm{Res}^{\mathrm{GL}_2(\mathbb{F}_\ell)}_{\mathrm{D}_2(\mathbb{F}_\ell)}(e_1) = x + x', \quad \mathrm{Res}^{\mathrm{GL}_2(\mathbb{F}_\ell)}_{\mathrm{D}_2(\mathbb{F}_\ell)}(c_1) = y + y',$$
$$\mathrm{Res}^{\mathrm{GL}_2(\mathbb{F}_\ell)}_{\mathrm{D}_2(\mathbb{F}_\ell)}(e_3) = x \cdot y' + x' \cdot y \text{ and } \mathrm{Res}^{\mathrm{GL}_2(\mathbb{F}_\ell)}_{\mathrm{D}_2(\mathbb{F}_\ell)}(c_2) = y \cdot y'.$$

This entails

$$\pi_1^*\mathrm{Res}^{\mathrm{GL}_2(\mathbb{F}_\ell)}_{\mathrm{D}_2(\mathbb{F}_\ell)}(e_1) = z_1 + z_1', \quad \pi_1^*\mathrm{Res}^{\mathrm{GL}_2(\mathbb{F}_\ell)}_{\mathrm{D}_2(\mathbb{F}_\ell)}(c_1) = \gamma_1^2 + (\gamma_1')^2,$$
$$\pi_1^*\mathrm{Res}^{\mathrm{GL}_2(\mathbb{F}_\ell)}_{\mathrm{D}_2(\mathbb{F}_\ell)}(e_3) = z_1 \cdot (\gamma_1')^2 + z_1' \cdot \gamma_1^2 \text{ and } \pi_1^*\mathrm{Res}^{\mathrm{GL}_2(\mathbb{F}_\ell)}_{\mathrm{D}_2(\mathbb{F}_\ell)}(c_2) = \gamma_1^2 \cdot (\gamma_1')^2.$$

We conclude that the image of $(\mathrm{Res} \circ \pi^*)_q$ is spanned by the following basis vectors over $\mathbb{F}_2$.

q	in image of $(\mathrm{Res} \circ \pi^*)_q$	in cokernel of $(\mathrm{Res} \circ \pi^*)_q$	b
1	$z_1 + z_1'$	z_1, γ_1, γ_1'	1
2	$\gamma_1^2 + (\gamma_1')^2$	$\gamma_1^2, \gamma_1\gamma_1', \gamma_1 z_1, \gamma_1 z_1', \gamma_1' z_1, \gamma_1' z_1'$	1
3	$(\gamma_1^2 + (\gamma_1')^2)(z_1 + z_1'), z_1 \cdot (\gamma_1')^2 + z_1' \cdot \gamma_1^2$	eight classes	2
$4m$	$(\gamma_1^4 + (\gamma_1')^4)^m, (\gamma_1^4 + (\gamma_1')^4)^{m-1} \cdot \gamma_1^2 \cdot (\gamma_1')^2, \ldots, \gamma_1^{2m} \cdot (\gamma_1')^{2m},$		
	$(z_1 z_1' \cdot (\gamma_1')^2 + z_1 z_1' \cdot \gamma_1^2)\left(\gamma_1^4 + (\gamma_1')^4\right)^{m-1}, \ldots,$		
	$(z_1 z_1' \cdot (\gamma_1')^2 + z_1 z_1' \cdot \gamma_1^2)(\gamma_1^{2(m-1)} \cdot (\gamma_1')^{2(m-1)})$	$10m$ classes	$2m+1$
$4m+1$	$(z_1 + z_1')\left(\gamma_1^4 + (\gamma_1')^4\right)^m, \ldots,$		
	$(z_1 + z_1')(\gamma_1^{2m} \cdot (\gamma_1')^{2m})$	$11m+3$ classes	$m+1$
$4m+2$	The above degree $4m$ classes times $(\gamma_1^2 + (\gamma_1')^2)$	$10m+6$ classes	$2m+1$
$4m+3$	The above degree $4m+1$ classes times $(\gamma_1^2 + (\gamma_1')^2)$ and		
	$(z_1 \cdot (\gamma_1')^2 + z_1' \cdot \gamma_1^2)(\gamma_1^4 + (\gamma_1')^4)^m, \ldots,$		
	$(z_1 \cdot (\gamma_1')^2 + z_1' \cdot \gamma_1^2)(\gamma_1^{2m} \cdot (\gamma_1')^{2m})$	$10m+8$ classes	$2m+2$

2. Upper bounds

Proposition 5. *We compute the following upper bounds for the dimensions of* $E_2^{p,q} = \mathrm{H}^p\left(\left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right)^\times ; \mathrm{H}^q(\mathrm{SL}_2\left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right) ; \mathbb{F}_2)\right)$ *over* $\mathbb{F}_2$.

$q \equiv 6 \mod 4$	2	4	4	4	4	4	4	...
$q \equiv 5 \mod 4$	1	2	2	2	2	2	2	...
$q \equiv 4 \mod 4$	3	6	6	6	6	6	6	...
$q \equiv 3 \mod 4$	4	8	8	8	8	8	8	...
$q = 2$	1	2	2	2	2	2	2	...
$q = 1$	0	0	0	0	0	0	0	...
$q = 0$	1	2	2	2	2	2	2	...
p	0	1	2	3	4	5	6	...

Proof. From Theorem 1, we read off the dimensions

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{SL}_2\left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right) ; \mathbb{F}_2) = \begin{cases} 2, & q \equiv 6 \mod 4, \\ 1, & q \equiv 5 \mod 4, \\ 3, & q \equiv 4 \mod 4, \\ 4, & q \equiv 3 \mod 4, \\ 1, & q = 2, \\ 0, & q = 1. \end{cases}$$

The units group $\left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right)^\times$ has generators $\sqrt{-2}$ (of infinite order) and -1 (of order 2), so its structure is $\left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right)^\times \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and its mod 2 cohomology ring is $\mathrm{H}^*\left(\left(\mathbb{Z}[\sqrt{-2}]\left[\frac{1}{2}\right]\right)^\times ; \mathbb{F}_2\right) \cong \mathbb{F}_2[\gamma_1] \otimes \Lambda(z_1)$, where γ_1 is a polynomial class and z_1 is an exterior class, both of degree 1. Hence

$$\dim_{\mathbb{F}_2} \mathrm{H}^p\left(\left(\mathbb{Z}[\sqrt{-2}]\left[\tfrac{1}{2}\right]\right)^\times ; \mathbb{F}_2\right) = 2 \text{ for all } p \geq 1.$$

Using these two dimension formulas, we can set up the table of the proposition. For those entries in the table of the proposition which do not follow immediately from the two dimension formulas, we specialize the Universal Coefficient Theorem to

$$\mathrm{H}^p\left(\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z};\, \mathbb{F}_2^n)\right) \cong \mathrm{Hom}\left(\mathrm{H}_p\left(\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z};\, \mathbb{Z}\right), \mathbb{F}_2^n\right) \oplus \mathrm{Ext}^1_{\mathbb{Z}}\left(\mathrm{H}_{p-1}\left(\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z};\, \mathbb{Z}\right), \mathbb{F}_2^n\right).$$

Using the action of $\mathbb{Z}$ on $\mathbb{R}$ by translations and the trivial action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{R}$, we get an equivariant spectral sequence that converges to

$$\mathrm{H}_p\left(\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z};\, \mathbb{Z}\right) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & p \geq 2, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & p = 1, \\ \mathbb{Z}, & p = 0. \end{cases}$$

Using $\mathrm{Hom}\left(\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{F}_2^n\right) \cong \mathbb{F}_2^{2n}$, $\quad \mathrm{Ext}^1_{\mathbb{Z}}\left(\mathbb{Z}/2\mathbb{Z}, \mathbb{F}_2^n\right) \cong \mathbb{F}_2^n/2\mathbb{F}_2^n \cong \mathbb{F}_2^n$ and $\mathrm{Ext}^1_{\mathbb{Z}}\left(\mathbb{Z}, \mathbb{F}_2^n\right) = 0$, we obtain that for trivial $\mathbb{F}_2^n$ coefficients,

$$\mathrm{H}^p\left(\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z};\, \mathbb{F}_2^n)\right) \cong \mathbb{F}_2^{2n} \text{ for all } p \geq 1.$$

$\square$

Corollary 6. *We get upper bounds for* $\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right); \mathbb{F}_2)$ *by summing over the* E_2 *terms, namely, letting* $k \in \mathbb{N} \cup \{0\}$*:*

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right); \mathbb{F}_2) \leq \begin{cases} 20k+22, & q = 6+4k, \\ 20k+19, & q = 5+4k, \\ 20k+15, & q = 4+4k, \\ 20k+8, & q = 3+4k, \\ 3, & q = 2, \\ 2, & q = 1. \end{cases}$$

3. Conclusion

Combining with the lower bound of Proposition 2, we arrive to the following conclusion.

Corollary 7. *The dimension* $d_q = \dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right); \mathbb{F}_2)$ *is framed as follows for* $1 \leq q \leq 4$*, and in the case that we arrive at the highest possible lower bounds in Proposition 2, it is framed as follows for* $1 \leq k \in \mathbb{N}$*:*

$$\begin{cases} 12k+13 \leq d_q \leq 20k+15, & q = 4+4k, \\ 12k+10 \leq d_q \leq 20k+8, & q = 3+4k, \\ 12k+7 \leq d_q \leq 20k+2, & q = 2+4k, \\ 12k+4 \leq d_q \leq 20k-1, & q = 1+4k, \\ 6 \leq d_4 \leq 15, & q = 4, \\ 4 \leq d_3 \leq 8, & q = 3, \\ 1 \leq d_2 \leq 3, & q = 2, \\ 2 \leq d_1 \leq 2, & q = 1. \end{cases}$$

Corollary 8. *In degree* 1*, we obtain* $\dim_{\mathbb{F}_2} \mathrm{H}^1(\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right); \mathbb{F}_2) = 2$.

Proof. Use the lower bound produced by the prime $\ell = 11$ in Proposition 2. □

This final result implies that the action of $\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right)^\times$ on $\mathrm{H}^{q-k}(\mathrm{SL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right); \mathbb{F}_2)$ is trivial in the involved degrees $q-k \in \{0,1\}$, but this was clear already from the dimensions of the module. Hence we do not gain any information here that would be useful for computing $\dim_{\mathbb{F}_2} \mathrm{H}^q(\mathrm{GL}_2\left(\mathbb{Z}[\sqrt{-2}\,]\left[\frac{1}{2}\right]\right); \mathbb{F}_2)$ precisely in higher degrees.

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