THE DUAL VOLUME OF QUASI-FUCHSIAN MANIFOLDS AND THE WEIL-PETERSSON DISTANCE

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ABSTRACT. Making use of the dual Bonahon-Schläfli formula, we prove that the dual volume of the convex core of a quasi-Fuchsian manifold M is bounded by an explicit constant, depending only on the topology of M, times the Weil-Petersson distance between the hyperbolic structures on the upper and lower boundary components of the convex core of M.

INTRODUCTION

Let Σ be a closed oriented surface of genus $g \geq 2$. The Teichmüller space of Σ, denoted by $\mathcal{T}(\Sigma)$, can be interpreted as the space of isotopy classes of either conformal structures or hyperbolic metrics on Σ, thanks to the uniformization theorem. The Weil-Petersson Kähler structure, with its induced distance d_{WP}, naturally arises from the interplay of these two interpretations. As described by Brock [Bro03], the coarse geometry of the Weil-Petersson distance turns out to be related to the growth of the volume of the convex core of quasi-Fuchsian manifolds. More precisely, in [Bro03] the author proved the existence of two constants $K_1 > 1$ and $K_2 > 0$, depending only on the topology of Σ, such that every quasi-Fuchsian manifold M satisfies

$$K_1^{-1} d_{WP}(c^+(M), c^-(M)) - K_2 \leq \mathrm{Vol}(CM) \leq K_1\, d_{WP}(c^+(M), c^-(M)) + K_2. \tag{1}$$

Inspired by this phenomenon, the aim of this paper is to determine an explicit control from above of the *dual volume* of the convex core of a quasi-Fuchsian manifold M in terms of the Weil-Petersson distance between the *hyperbolic metrics* on the boundary of its convex core, in analogy to what has been done by Schlenker [Sch13] for the *renormalized volume* of M and the Weil-Petersson distance between its *conformal structures* at infinity.

In order to be more precise, we need to introduce some notation. If M is a quasi-Fuchsian manifold homeomorphic to $\Sigma \times \mathbb{R}$, then CM will denote its convex core. When M is not Fuchsian, the subset CM is homeomorphic to the product of Σ with a compact interval of $\mathbb{R}$ with non-empty interior. Its boundary components $\partial^\pm CM$, are locally convex pleated surfaces with hyperbolic metrics $m^+(M)$, $m^-(M) \in \mathcal{T}(\Sigma)$ and bending measured laminations μ^+, μ^-. The manifold M can be extended at infinity by adding two surfaces $\partial^\pm_\infty M$, so that $M \cup \partial^\pm_\infty M$ is homeomorphic to $\Sigma \times [-\infty, +\infty]$. The surfaces $\partial^\pm_\infty M$ are endowed with natural complex structures $c^\pm(M)$, coming from the conformal action of the fundamental group of M on the boundary at infinity of the hyperbolic space $\mathbb{H}^3$. By a classical result of Bers [Ber60], the data of $c^\pm(M)$ uniquely determine the hyperbolic manifold M, and any couple of conformal structures can be realized in this way.

The notion of *dual volume* arises from the polar correspondence between the *hyperbolic* 3*-space* $\mathbb{H}^3$ and the *de Sitter* 3*-space* dS^3 (see [Riv86], [Sch02] for details), In general, if N is a convex subset with regular boundary of a quasi-Fuchsian (or, in general, co-compact hyperbolic) manifold M, we set the *dual volume* of N to be

$$\mathrm{Vol}^*(N) := \mathrm{Vol}(N) - \frac{1}{2}\int_{\partial N} H \,\mathrm{d}a\,,$$

Date: July 10, 2019.
Supported by the Luxembourg National Research Fund PRIDE15/10949314/GSM/Wiese.

where H is the trace of the shape operator of ∂N with respect the interior normal vector field. Even if the convex core does not have regular boundary, a simple approximation argument shows that it makes sense to define the dual volume of the convex core of M as $\mathrm{Vol}^*(CM) := \mathrm{Vol}(CM) - \frac{1}{2}L_\mu(m)$, where μ is the bending measured lamination of ∂CM and $m \in \mathcal{T}(\partial CM)$ is its hyperbolic metric. As we deform the quasi-Fuchsian structure $(M_t)_t$, the variation of the dual volume of the convex core is described by the *dual Bonahon-Schläfli formula*, which asserts that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Vol}^*(CM_t)|_{t=0} = -\frac{1}{2}\,\mathrm{d}(L_\mu)\,(\dot{m}).$$

Here $\dot{m}$ denotes the derivative of the hyperbolic metrics m_t on the boundary of the convex cores CM_t (a simple proof of this relation, originally showed in [KS09], can be found in [Maz18]).

The dual Bonahon-Schläfli formula, together with the properties of the bending measured lamination, allows us to bound uniformly the variation of $\mathrm{Vol}^*(CM)$ with respect to $\dot{m}$. The aim of this paper is to prove the following statement:

Theorem A. *There exists a universal constant $C > 0$ such that, for every quasi-Fuchsian manifold M homeomorphic to $\Sigma \times \mathbb{R}$, we have*

$$|\mathrm{Vol}^*(CM)| \leq C\,(g-1)^{1/2}\,d_{WP}(m^-(M), m^+(M)),$$

with $C \approx 7.3459$.

The dual volume and the hyperbolic volume of the convex core differ by the term $\frac{1}{2}L_\mu(m)$, which is bounded by $6\pi|\chi(\Sigma)|$ (see [BBB19]). Moreover, the structures $c^\pm(M)$ and $m^\pm(M)$ are at bounded Weil-Petersson distance from each other, by the works of Linch [Lin74] and Sullivan [Sul81] (see also Epstein and Marden [CEM06, Part II]). Therefore, Theorem A can be used to give an alternative proof of Brock's upper bound in (1) and to exhibit *explicit constants* satisfying the inequality, with a fairly simple argument.

A similar strategy has been developed by Schlenker [Sch13] using the notion of *renormalized volume*, first introduced in [KS08]. The key ingredients in the work [Sch13] are the variation formula of the renormalized volume $\mathrm{RVol}(M)$ and the Nehari's bound of the norm of the Schwarzian derivative of the complex projective structures at infinity of $\partial_\infty M$. In particular, the author showed that, for every quasi-Fuchsian manifold M, we have:

$$\mathrm{RVol}(M) \leq 3\sqrt{\pi}(g-1)^{1/2}\,d_{WP}(c^+(M), c^-(M)). \tag{2}$$

We remark that the multiplicative constant C appearing in our statement is larger than the one obtained using the renormalized volume, $3\sqrt{\pi} \approx 5.3174 < 7.3459 \approx C$. Therefore, the inequality (2) is more efficient in terms of coarse estimates.

Nevertheless, Theorem A carries more information than its implications concerning the coarse Weil-Petersson geometry, in particular when we consider quasi-Fuchsian structures that are close to the Fuchsian locus. In this case, Theorem A and the inequality (2) furnish complementary insights, since they involve the Weil-Petersson distance between the hyperbolic structures, on one side, and the conformal structures at infinity on the other. Moreover, Proposition 2.4 and its application for the bound of the dual volume show that the multiplicative constant in Theorem A can be improved if we have a better control of $L_\mu(m)$ than $L_\mu(m) \leq 6\pi|\chi(\Sigma)|$ (from [BBB19]), exactly as the inequality (2) can be improved if we have a better control of the L^∞-norm of the Schwarzian at infinity than the Nehari's bound.

We finally mention that, carrying on the analogy between the picture "at the convex core" and "at infinity" by Schlenker [Sch17], our result fits well into the comparison of the two descriptions of the space of quasi-Fuchsian structures, as summarized in the following table:

On ∂CM	On $\partial_\infty M$
Induced metrics $m^\pm$	Conformal structures $c^\pm$
Thurston's conjecture on prescribing $m^\pm$	Bers' Simultaneous Uniformization Theorem
Bending measured lamination μ	Measured foliation $\mathcal{F}$
Hyperbolic length $L_\mu(m)$	Extremal length $\mathrm{ext}_\mathcal{F}(c)$
Dual volume $\mathrm{Vol}^*(CM)$	Renormalized volume $\mathrm{RVol}(M)$
Dual Bonahon-Schläfli formula $\delta\,\mathrm{Vol}^* = -\frac{1}{2}\,\mathrm{d}(L_\mu)\,(\dot{m})$	[Sch17, Theorem 1.2] $\delta\,\mathrm{RVol} = -\frac{1}{2}\,\mathrm{d}(\mathrm{ext}_\mathcal{F})\,(\dot{c})$
Bound on $L_\mu(m)$ [BBB19] $L_{m_\pm}(l_\pm) \leq 6\pi\lvert\chi(S)\rvert$	[Sch17, Theorem 1.4] $\mathrm{ext}_\mathcal{F}(c) \leq 3\pi\lvert\chi(S)\rvert$
Bound of Vol^* with $d_{WP}(m^+,m^-)$ Theorem A	Bound of RVol with $d_{WP}(c^+,c^-)$ Inequality (2)

Outline of the paper. In Section 1 we recall the definition of Teichmüller space $\mathcal{T}(\Sigma)$ as deformation space of Riemann surface structures, and of its tangent and cotangent bundles via Beltrami differentials and holomorphic quadratic differentials. In particular we remind the notion of Weil-Petersson metric and of other similarly defined Finsler metrics on $\mathcal{T}(\Sigma)$, using L^p-norms of Beltrami differentials. Then, following [Tro92], we introduce the description of $\mathcal{T}(\Sigma)$ as the space of isotopy classes of hyperbolic metrics, and of its tangent bundle using traceless and divergence free (also called transverse traceless) symmetric tensors. The Section ends with a simple Lemma describing the relation between the two equivalent interpretations and between their norms.

Section 2 is devoted to the proof of Proposition 2.4, in which we produce a uniform bound of the L^p-norm of the differential of $L_\mu : \mathcal{T}(\Sigma) \to \mathbb{R}$, the hyperbolic length function of a measured lamination over the Teichmüller space. This is the main "quantitative" ingredient for the proof of Theorem A. The proof uses Tromba's description of $T\mathcal{T}(\Sigma)$ via transverse traceless tensors: we represent a variation of hyperbolic metrics $\dot{m}$ as the real part of a holomorphic quadratic differential Φ. Using standard properties of holomorphic functions, the pointwise norm of Φ at x can be bounded by the L^p-norm of Φ over some embedded geodesic ball in Σ centered at x. The variation of L_μ can be expressed as an integral over the support of μ of the product of the variation of the length measure of $\dot{m}$ times the tranverse measure of μ. Then the result will follow by using the pointwise estimation and a Fubini's exchange of integration over a suitable finite cover of Σ.

In Section 3 we obtain a uniform control of the differential of Vol^* in terms of the L^p-norm of the variation of the hyperbolic metrics on ∂CM (here we denote, with abuse, by Vol^* the function over the space $\mathcal{QF}(\Sigma)$ of quasi-Fuchsian structures on $\Sigma \times \mathbb{R}$, which associates to a manifold M the volume of its convex core $\mathrm{Vol}^*(CM)$). To do so, we will apply the works of Bridgeman, Canary, and Yarmola [BCY16] and Bridgeman, Brock, and Bromberg [BBB19], which give universal controls of the bending measure of the convex core. These results are to the dual volume as the Nehari's bound of norm of the Schwarzian derivative is to the renormalized volume (the bounds obtained in [BBB19] are actually proved *using* Nehari's bound). The dual Bonahon-Schläfli formula relates the variation of Vol^* with the differential of the length of the bending measured lamination, and the mentioned universal bounds combined with Proposition 2.4 will produce the desired control of $\mathrm{d}\,\mathrm{Vol}^*$ (see Corollary 3.8).

In Section 4 we will finally give a proof of Theorem A. Contrary to what happens for the conformal structures at infinity, the hyperbolic structures on ∂CM are only conjecturally thought to give a parametrization of the space of quasi-Fuchsian manifolds. Because of this, proving Theorem A from Corollary 3.8 is not as immediate as it is for the renormalized volume using its variation formula. Our procedure to overcome to this difficulty passes through the foliation of hyperbolic ends by constant Gaussian curvature surfaces Σ_k, with $k \in (-1,0)$, and the notion of landslide, which is a "smoother" analogue of earthquakes between hyperbolic metrics on Σ introduced by Bonsante, Mondello, and Schlenker [BMS13] (see also [BMS15]). By the work of Schlenker [Sch06] and Labourie [Lab91], the data of the metrics on the surfaces Σ_k parametrize the space of quasi-Fuchsian manifolds. Therefore, the strategy will roughly be to:

i) approximate the dual volume of the convex core CM by the dual volume Vol_k^* of the region enclosed by the k-surfaces of M;
ii) prove that the differentials of the functions Vol_k^* converge to the differential of Vol^* as k goes to -1, i. e. as the surfaces Σ_k get closer to the convex core CM;
iii) use the parametrization result for the metrics of Σ_k to deduce the statement of Theorem A via an approximation argument.

For point (*ii*), which is the most delicate part of our argument, we will highlight a connection between the differential of the functions Vol_k^* and the infinitesimal smooth grafting, introduced in [BMS13]. As described by McMullen [McM98], the earthquake map can be complexified using the notion of grafting along a measured lamination. In the same way the landslide admits a complex extension via the *smooth grafting map*. Moreover, the complex earthquake can be actually recovered by a suitable limit of complex landslides. Using this convergence procedure, we are able to show that $\mathrm{d}\,\mathrm{Vol}^*$ is the limit of the differentials $\mathrm{d}\,\mathrm{Vol}_k^*$, in the sense described by Proposition 4.3. The rest of the proof of Theorem A will be an elementary application of the results from the previous section, similarly to what done in [Sch13] with the renormalized volume.

Acknowledgments. I would like to thank my advisor Jean-Marc Schlenker for introducing me to this problem, and for his help and support all along this work.

1. Preliminaries

Let Σ be an oriented closed surface of genus $g \geq 2$. Two Riemannian metrics g, g' on Σ are said to be *conformally equivalent* if there exists a smooth function $u \colon \Sigma \to \mathbb{R}$ such that $g = e^{2u} g'$. A *Riemann surface structure* on Σ is a couple $X = (\Sigma, c)$, where c is a conformal class of Riemannian metrics of Σ. A *hyperbolic structure* on Σ is the datum of a Riemannian metric h with constant Gaussian curvature equal to -1.

The *Teichmüller space* of Σ, denoted by $\mathcal{T}(\Sigma)$, is the space of isotopy classes of conformal structures over the surface Σ. Thanks to the uniformization theorem, the Teichmüller space can be considered equivalently as the space of isotopy classes of hyperbolic metrics on Σ. We will write $\mathcal{T}_c(\Sigma)$ (c for *conformal*) when we want to emphasize the first interpretation, and $\mathcal{T}_h(\Sigma)$ (h for *hyperbolic*) in latter case.

In the following, we will recall the definition of the Weil-Petersson Riemannian metric on the Teichmüller space, and of other similarly defined Finsler metrics. Since the literature usually agrees on these definitions only up to multiplicative constant, we will spend some time in describing the setting we will work with, mainly because we will be interested in producing explicit bounds of our geometric quantities.

Let X be a Riemann structure on Σ. A *Beltrami differential* on X is a $(1,1)$-tensor ν that can be expressed in local coordinates as $\nu = n \, \partial_z \otimes \mathrm{d}\bar{z}$, where n is a measurable complex-valued function. If $h = \rho |\mathrm{d}z|^2$ is the unique hyperbolic metric in the conformal class c, then for any $q \in [1,\infty)$ we define the L^q-norm of the Beltrami differential $\nu = n \, \partial_z \otimes \mathrm{d}\bar{z}$ to be

$$\|\nu\|_{B,q} := \left(\int_X |n|^q \rho \, \mathrm{d}x \, \mathrm{d}y \right)^{1/q} .$$

When $q = \infty$, we set $\|\nu\|_{B,\infty} := \text{ess sup}_\Sigma |n|$. $B(X)$ will denote the space of Beltrami differentials of X with finite L^∞-norm.

A holomorphic quadratic differential on X is a symmetric 2-covariant tensor that can be locally written as $\Phi = \phi \, \mathrm{d}z^2$, where ϕ is holomorphic. In analogy to what was done above, for every $p \in [1, \infty)$ we define the L^p-norm of Φ to be

$$\|\Phi\|_{Q,p} := \left(\int_\Sigma \frac{|\phi|^p}{\rho^{p-1}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p} .$$

When $p = \infty$, we set $\|\Phi\|_{Q,\infty} := \text{ess sup}_\Sigma |\phi|/\rho$. When $p = 2$, the norm $\|\cdot\|_{Q,2}$ is induced by a scalar product, defined as follows:

$$\langle \Phi, \Psi \rangle_{Q,2} := \int_\Sigma \frac{\phi \overline{\psi}}{\rho} \, \mathrm{d}x \, \mathrm{d}y .$$

There is a natural pairing between the space of bounded Beltrami differentials $B(X)$ and the space of holomorphic quadratic differentials $Q(X)$: given a Beltrami differential $\nu = n \, \partial_z \otimes \mathrm{d}\bar{z}$ and a holomorphic quadratic differential $\Phi = \phi \, \mathrm{d}z^2$, we define

$$(\Phi, \nu) := \int_X \phi \bar{n} \, \mathrm{d}x \, \mathrm{d}y .$$

The Hölder inequality implies that, for every $\Phi \in Q(X)$ and $\nu \in B(X)$, we have

$$|(\Phi, \nu)| \leq \|\Phi\|_{Q,p} \|\nu\|_{B,q},$$

where p and q are conjugate exponents, i. e. $\frac{1}{p} + \frac{1}{q} = 1$. Therefore $(\cdot, \cdot)$ induces a continuous (and injective, in fact) linear operator

$$\begin{array}{ccc} Q(X) & \longrightarrow & B(X)^* \\ \Phi & \longmapsto & (\Phi, \cdot). \end{array}$$

Consequently, we can endow $Q(X)$ with the dual norm of $\|\cdot\|_{B,q}$, defined as

$$\|\Phi\|^*_{B,q} := \sup_{\nu \neq 0} \frac{|(\Phi, \nu)|}{\|\nu\|_{B,q}} .$$

An elementary argument proves the following:

Lemma 1.1. *For any couple of conjugate exponents p, q and for any $\Phi \in Q(X)$, we have $\|\Phi\|_{Q,p} = \|\Phi\|^*_{B,q}$.*

A Beltrami differential $\nu \in B(X)$ is *harmonic* if there exists a holomorphic quadratic differential $\Phi = \phi \, \mathrm{d}z^2$ such that $\nu = \bar{\phi}/\rho \; \partial_z \otimes \mathrm{d}\bar{z}$. We denote by $B_h(X)$ the space of harmonic Beltrami differentials on X.

Let $N(X)$ be the subspace of $B(X)$ of those Beltrami differentials ν verifying $(\Phi, \nu) = 0$ for every $\Phi \in Q(X)$. As described in [GL00], the space $B_h(\Sigma)$ and $N(X)$ are in direct sum, and the quotient of $(B(X), \|\cdot\|_{B,\infty})$ by the subspace $N(X)$ identifies with the tangent space to the Teichmüller space $T_X \mathcal{T}_c(\Sigma)$ (here we denote by X the isotopy class of the conformal structure, with abuse). Moreover, the pairing $(\cdot, \cdot)$ determines a natural isomorphism between the dual of $T_X \mathcal{T}(\Sigma)$ and the space of holomorphic quadratic differentials $(Q(X), \|\cdot\|_{Q,1})$, which is consequently identified with the cotangent space $T^*_X \mathcal{T}_c(\Sigma)$.

The inner product $\langle \cdot, \cdot \rangle_{Q,2}$ on $Q(X)$ induces a Hermitian scalar product on the cotangent space $T^*_X \mathcal{T}_c(\Sigma)$, and consequently on $T_X \mathcal{T}_c(\Sigma)$. Its real part is the *Weil-Petersson metric* of $\mathcal{T}_c(\Sigma)$, and it will be denoted by $\langle \cdot, \cdot \rangle_{WP}$.

We recall now the Riemannian description of the Teichmüller space as developed in [Tro92]. Let $S^{2,0}\Sigma$ be the bundle of 2-covariant symmetric tensors on Σ, and let $\Gamma(S^{2,0}\Sigma)$ denote the space of its smooth sections, which is an infinite dimensional vector space. The space $\mathcal{M}$ of smooth Riemannian metrics on Σ identifies with an open cone inside $\Gamma(S^{2,0}\Sigma)$.

Therefore, given any Riemannian metric g on Σ, the tangent space $T_g\mathcal{M}$ is canonically isomorphic to $\Gamma(S^{2,0}\Sigma)$. The metric g determines a scalar product on $T_g\mathcal{M}$, which can be expressed as $(h,k)_g := g^{ik}g^{jh}h_{ij}k_{kh}$, for h, k in $\Gamma(T^{2,0}\Sigma)$. The norm induced by this scalar product will be denoted by $\|h\|_g^2 := \langle h,h\rangle_g$. Given $h \in \Gamma(T^{2,0}\Sigma)$, we define the g-divergence of h to be the 1-form $\delta_g h(V) := \operatorname{tr}_g(\nabla_* h)(*,V)$, for any V tangent vector field to Σ. Now we set

$$S_{tt}(\Sigma,g) := \{h \in \Gamma(T^{2,0}\Sigma) \mid h \text{ is symmetric, } g\text{-traceless and } \delta_g h = 0\}.$$

An element of $S_{tt}(\Sigma,g)$ is usually called a *tranverse traceless* tensor (with respect to the metric g). As shown in [Tro92], every element of $S_{tt}(\Sigma,g)$ can be written (uniquely) as the real part of a holomorphic quadratic differential $\Phi \in Q(\Sigma,[g])$, and vice versa for every Φ, the tensor $\operatorname{Re}\Phi$ belongs to $S_{tt}(\Sigma,g)$. In particular, the space $S_{tt}(\Sigma,g)$ depends only on the conformal class of the metric g. If g is a hyperbolic metric, then $S_{tt}(\Sigma,g)$ is tangent to the space $\mathcal{M}^{-1}$ of hyperbolic metrics on Σ, and it is transverse to the orbit of g by the action of the group of diffeomorphisms isotopic to the identity. Therefore, the tangent space of the Teichmüller space at the isotopy class of g can be identified with $S(\Sigma,g)$.

For any open set $\Omega \subseteq \Sigma$ and any $p \in [1,\infty)$, the Fischer-Tromba p-norm of $h \in S_{tt}(\Sigma,g)$ is defined as

$$\|h\|_{FT,L^p(\Omega)} := \left(\int_\Omega \|h\|_g^p \,\mathrm{dvol}_g\right)^{1/p},$$

where dvol_g is the area form induced by g. When $p=\infty$, we set $\|h\|_{FT,L^\infty(\Omega)} := \sup_\Omega \|h\|_g$. If $\Omega = \Sigma$, we simply write $\|\cdot\|_{FT,p}$.

Let now m be a point of the Teichmüller space, and let g be a hyperbolic metric in the equivalence class m, with associated Riemann surface structure X.

Lemma 1.2. *The vector spaces $B_h(X)$ and $S_{tt}(\Sigma,g)$ are identified to $T_m\mathcal{T}(\Sigma)$ through the linear isomorphism*

$$\begin{array}{ccc} B_h(X) & \longrightarrow & S_{tt}(\Sigma,g) \\ \nu_\Phi & \longmapsto & 2\operatorname{Re}\Phi. \end{array}$$

Moreover, for every $\Phi \in Q(X)$ we have

$$\|\nu_\Phi\|_{B,q} = \frac{1}{2\sqrt{2}}\|2\operatorname{Re}\Phi\|_{FT,q}.$$

Proof. Let $g_t = \rho_t|\mathrm{d}z_t|^2$ be a smooth 1-parameter family of Riemannian metrics on Σ, with $g_0 = g$, and let $\Phi = \phi\,\mathrm{d}z_0{}^2$ be a holomorphic quadratic differential on the Riemann surface $X_0 = (\Sigma,[g_0])$. If we require the identity map $(\Sigma,g_0) \to (\Sigma,g_t)$ to be quasi-conformal with harmonic Beltrami differential

$$\nu^0_{t\Phi} := \frac{t\bar\phi}{\rho_0}\,\partial_{z_0} \otimes \mathrm{d}\bar z_0\,,$$

then the Riemannian metric g_t can be expressed as

$$g_t = \rho_t\left|\frac{\partial z_t}{\partial z_0}\right|^2 |\mathrm{d}z_0|^2 + 2t\rho_t\left|\frac{\partial z_t}{\partial z_0}\right|^2 \operatorname{Re}\left(\frac{\phi}{\rho_0}\,\mathrm{d}z_0{}^2\right) + O(t^2).$$

Therefore the first order variation of g_t at $t=0$ coincides with

$$\dot g_0 = \left(\frac{\mathrm{d}}{\mathrm{d}t}\,\rho_t\left|\frac{\partial z_t}{\partial z_0}\right|^2\Bigg|_{t=0} |\mathrm{d}z_0|^2\right) + 2\operatorname{Re}\Phi.$$

The quantity $\dot g_0$ identifies with a tangent vector to the space $\mathcal{M}$ of Riemannian metrics over Σ at the point g_0. The first term in the expression above is conformal to the Riemannian metric g_0, hence it is tangent to the conformal class $[g_0] \subset \mathcal{M}$. The remaining term $2\operatorname{Re}\Phi$ is a symmetric, g_0-traceless and divergence-free tensor, so it lies in the subspace $S_{tt}(\Sigma,g_0)$ of $T_{g_0}\mathcal{M}$.

The computation above proves that the harmonic Beltrami differential ν_Φ, seen as an element of $T_m\mathcal{T}_c(\Sigma)$, corresponds to $2\operatorname{Re}\Phi \in S_{tt}(\Sigma, g_0) \cong T_m\mathcal{T}_h(\Sigma)$. Finally, an explicit computation shows the relation between the norms $\|\cdot\|_{B,q}$ and $\|\cdot\|_{FT,q}$. □

2. A BOUND OF THE DIFFERENTIAL OF THE LENGTH

Let $\mathcal{ML}(\Sigma)$ denote the space of measured laminations of Σ. The aim of this section is to produce, given $\mu \in \mathcal{ML}(\Sigma)$, a quantitative upper bound of the L^p-norm of the differential of the length function $L_\mu \colon \mathcal{T}_h(\Sigma) \to \mathbb{R}$, which associates to every class of hyperbolic metrics $m \in \mathcal{T}_h(\Sigma)$ the length of the m-geodesic realization of μ. This estimate is the content of Proposition 2.4, which will be our main technical ingredient to produce the upper bound of the dual volume in terms of the Weil-Petersson distance between the hyperbolic metrics on the convex core of a quasi-Fuchsian manifold.

We briefly sketch the structure of this section: Lemma 2.1 describes a natural way to express the differential of L_μ applied to a first order variation of hyperbolic metrics $\dot g$. Lemma 2.2 uses the properties of holomorphic functions to bound the pointwise value of a holomorphic quadratic differential at $x \in \Sigma$ with its L^q-norm on the ball centered at x. Then Proposition 2.4 will follow by selecting a first order variation $\dot g$ in $S_{tt}(\Sigma, g)$ and then carefully applying the bound of Lemma 2.2 in the expression found in Lemma 2.1.

Let $m \in \mathcal{T}_h(\Sigma)$ and $\mu \in \mathcal{ML}(\Sigma)$. Given a hyperbolic metric g in the equivalence class m, we identify the measured lamination μ with its g-geodesic realization inside (Σ, g). If λ is a g-geodesic lamination of Σ containing the support of μ, we can cover λ by finitely many flow boxes $\sigma_j \colon I \times I \to B_j$, where $I = [0,1]$ and σ_j is a homeomorphism verifying $\sigma_j^{-1}(\lambda) = D_j \times I$, for some closed subset D_j of I. We select also a collection $\{\xi_j\}_j$ of smooth functions with supports contained in the interior of B_j for every j, and such that $\sum_j \xi_j = 1$ over λ. Since the arcs $\sigma_j(I \times \{s\})$ are transverse to λ, it makes sense to integrate the first component of σ_j with respect to the measure μ. We set the *length of μ with respect to m* to be the quantity

$$L_\mu(m) := \sum_j \int_{D_j} \int_0^1 \xi_j(\sigma_j(p,\cdot))\, \mathrm{d}\ell(\cdot)\, \mathrm{d}\mu\,(p),$$

where $\mathrm{d}\ell(s) = \left\|\partial_s \sigma_j(p,s)\right\|_g \mathrm{d}s$. More generally, given a measurable function f defined on a neighborhood of λ, we define

$$\iint_\lambda f \,\mathrm{d}\ell\, \mathrm{d}\mu := \sum_j \int_{D_j} \int_0^1 \xi_j(\sigma_j(p,\cdot)) f(\sigma_j(p,\cdot))\, \mathrm{d}\ell(\cdot)\, \mathrm{d}\mu\,(p).$$

The quantity $L_\mu(m)$ does not depend on the choices we made of σ_j, ξ_j and the hyperbolic metric g in the equivalence class $m \in \mathcal{T}(\Sigma)$ (see e. g. [Bon96]). Therefore, any measured lamination μ of Σ determines a positive function L_μ on the Teichmüller space $\mathcal{T}(\Sigma)$, which associates to any $m \in \mathcal{T}_h(\Sigma)$ the length of the geodesic realization of μ in m.

Similarly, if $(g_t)_t$ is a smooth 1-parameter family of hyperbolic metrics on Σ, with $g_0 = g$ and $\dot g_0 = \dot g$, we set

$$\iint_\lambda \mathrm{d}\dot\ell\, \mathrm{d}\mu := \frac{1}{2} \sum_j \int_{D_j} \int_0^1 \xi_j(\sigma_j(p,\cdot))\, \frac{\dot g\left(\partial_s\sigma_j(p,\cdot), \partial_s\sigma_j(p,\cdot)\right)}{g\left(\partial_s\sigma_j(p,\cdot), \partial_s\sigma_j(p,\cdot)\right)}\, \mathrm{d}\ell(\cdot)\, \mathrm{d}\mu\,(p).$$

Lemma 2.1. *Let μ be a measured lamination of Σ, and let $(m_t)_t$ be a smooth path in $\mathcal{T}_h(\Sigma)$ verifying $m_0 = m$ and $\dot m_0 = \dot m \in T_m\mathcal{T}_h(\Sigma)$. Then we have*

$$\mathrm{d}\left(L_\mu\right)_m(\dot m) = \iint_\lambda \mathrm{d}\dot\ell\, \mathrm{d}\mu\,,$$

where $\iint_\lambda \mathrm{d}\dot\ell\, \mathrm{d}\mu$ is defined as above by selecting a smooth path $t \mapsto g_t$ of hyperbolic metrics representing $t \mapsto m_t$.

Proof. First we prove the statement when μ is a weight 1 simple closed curve γ in Σ. Let $\gamma_t\colon [0,1] \to \Sigma$ denote a parametrization of the geodesic representative of γ with respect to the hyperbolic metric g_t, which can be chosen to depend differentiably in t. Then the length of γ_t with respect to the metric g_t can be expressed as

$$L_\gamma(m_t) = \int_0^1 \sqrt{g_t(\gamma_t'(s), \gamma_t'(s))}\,\mathrm{d}s\,.$$

Now, by taking the derivative of this expression in t and using the fact that $\nabla\dot{\gamma}_0 \equiv 0$ (with ∇ being the Levi-Civita connection of g_0), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} L_\gamma(m_t)\bigg|_{t=0} = \frac{1}{2}\int_0^1 \frac{\dot{g}_0(\gamma_0'(s), \gamma_0'(s))}{\sqrt{g_0(\gamma_0'(s), \gamma_0'(s))}}\,\mathrm{d}s\,,$$

which coincides with the quantity $\iint_\gamma \mathrm{d}\dot{\ell}\,\mathrm{d}\mu$. By linearity we deduce the statement for any rational lamination $\mu = \sum_i a_i \gamma_i$.

Now, if μ is a general measured lamination, we select a sequence of rational laminations $(\mu_n)_n$ converging to μ. As shown in [Ker85], the functions L_{μ_n} are real analytic over $\mathcal{T}_h(\Sigma)$ and they converge in the $\mathscr{C}^\infty$-topology on compact sets to L_μ. In particular the terms $\mathrm{d}\big(L_{\mu_n}\big)_m(\dot{m})$ converge to $\mathrm{d}\big(L_\mu\big)_m(\dot{m})$. Since the expression $\iint_\lambda \mathrm{d}\dot{\ell}\,\mathrm{d}\mu$ can be proved to be continuous in the measured lamination $\mu \in \mathcal{ML}(\Sigma)$, the statement follows by an approximation argument. □

Before stating Lemma 2.2, we define for convenience the following quantities: for every $q \in [1,\infty)$ and $r > 0$, we set

$$C(r,q) := \left(\frac{2q-1}{4\pi}\,\frac{(\cosh(r/2))^{4q-2}}{(\cosh(r/2))^{4q-2}-1}\right)^{1/q}. \tag{3}$$

When $q = \infty$, we define $C(r,\infty) := 1$ for every $r > 0$.

Lemma 2.2. *Let (Σ, g) be a hyperbolic surface. Given $x \in \Sigma$ and $r < \operatorname{injrad}_g(x)$, we denote by $B_r(x)$ the metric ball of radius r centered at $x \in \Sigma$. Then, for every $q \in [1,\infty]$ and for every holomorphic quadratic differential on $(\Sigma, [g])$, we have*

$$\|\mathrm{Re}\,\Phi_x\| \leq C(r,q)\,\|\mathrm{Re}\,\Phi\|_{FT,L^q(B_r(x))}.$$

where $\|\mathrm{Re}\,\Phi_x\|$ is the pointwise norm of the tensor $\mathrm{Re}\,\Phi$ at x.

Proof. If $q = \infty$, the statement is clear. Consider $q < \infty$. By passing to the universal cover, we can assume the surface to be $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ and x to be $0 \in \Delta$. The hyperbolic metric of Δ is of the form

$$g_\Delta = \left(\frac{2}{1-|z|^2}\right)^2 |\mathrm{d}z|^2,$$

where $z \in \Delta$ is the natural coordinate of $\Delta \subset \mathbb{C}$. In what follows, we will denote by $\|\cdot\|$ the norm induced by the hyperbolic metric, and by $\|\cdot\|_0$ the one induced by the standard Euclidean metric $|\mathrm{d}z|^2$.

If $\Phi = \phi\,\mathrm{d}z^2$ is a holomorphic quadratic representative, then for any $\rho \in (0,1)$ the residue theorem tells us that

$$\phi(0) = \frac{1}{2\pi i}\int_{\partial B^E_\rho} \frac{\phi(z)}{z}\,\mathrm{d}z\,,$$

where $B^E_\rho = B^E_\rho(0) = \{z \in \Delta \mid |z| < \rho\}$ (here E stands for "Euclidean"). In particular we have

$$|\phi(0)|^q \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |\phi(\rho e^{i\theta})|\,\mathrm{d}\theta\right)^q \leq \frac{1}{2\pi}\int_0^{2\pi} |\phi(\rho e^{i\theta})|^q\,\mathrm{d}\theta\,, \tag{4}$$

where in the last step we used the Hölder inequality. At $z = \rho e^{i\theta}$, the hyperbolic norm of $\operatorname{Re}\Phi(z)$ can be expressed as follows:

$$\|\operatorname{Re}\Phi(z)\| = \frac{1}{\sqrt{2}}|\phi(\rho e^{i\theta})| \left(\frac{2}{1-\rho^2}\right)^{-2} \left\|\mathrm{d}z^2\right\|_0.$$

It is easy to check that the metric ball B_r centered at 0 with respect to the hyperbolic distance coincides with $B^E_{\tanh(r/2)}$, and that the hyperbolic volume form dvol is given by $\rho(2/(1-\rho^2))^2 \mathrm{d}\rho\, \mathrm{d}\theta$. Combining all these facts, if we multiply the inequality (4) by $\rho(2/(1-\rho^2))^{2-2q}$ and we integrate in $\int_0^{\tanh r/2} \mathrm{d}\rho$, we deduce that

$$\begin{aligned}
\int_{B_r} \|\operatorname{Re}\Phi\|^q \,\mathrm{dvol} &= 2^{-q/2}\left\|\mathrm{d}z^2\right\|_0^q \int_0^{\tanh r/2} \rho \left(\frac{2}{1-\rho^2}\right)^{2-2q} \int_0^{2\pi} \left|\phi(\rho e^{i\theta})\right|^q \mathrm{d}\theta\, \mathrm{d}\rho \\
&\geq 2\pi|\phi(0)|^q\, 2^{-q/2-2(q-1)}\left\|\mathrm{d}z^2\right\|_0^q \int_0^{\tanh r/2} \rho(1-\rho^2)^{2(q-1)}\, \mathrm{d}\rho \\
&= 4\pi\|\operatorname{Re}\Phi(0)\|^q\, \frac{1}{2q-1}\left(1 - \frac{1}{(\cosh(r/2))^{4q-2}}\right) \\
&= C(r,q)^{-q}\|\operatorname{Re}\Phi(0)\|^q,
\end{aligned}$$

which proves the assertion. □

We state here another useful fact we will use in the proof of Proposition 2.4:

Lemma 2.3. *Let (Σ, g) be a hyperbolic surface and let μ be a measured lamination on Σ. Then, for every L^1-function $f\colon N_r(\mu) \to \mathbb{R}$ defined on the r-neighborhood of μ in Σ, we have*

$$\iint_\lambda \left(\int_{B_r(\cdot)} f\,\mathrm{dvol}_g\right) \mathrm{d}\ell\, \mathrm{d}\mu = \int_\Sigma \left(\iint_{\lambda\cap B_r(\cdot)} \mathrm{d}\ell\, \mathrm{d}\mu\right) f\,\mathrm{dvol}_g\,.$$

Proof. Assume that μ is a 1-weighted simple closed curve $\gamma\colon [0,1] \to \Sigma$, and let $\tilde{f}$ denote the extension of the function f to Σ verifying $\tilde{f}(x) = 0$ for all $x \in \Sigma \setminus N_r(\gamma)$. We set $\xi : \Sigma^2 \to \mathbb{R}$ to be the function taking value $\xi(x,y) = 1$ if the distance between x and y is less than r, and $\xi(x,y) = 0$ otherwise. Then the integral on the left can be expressed as

$$\int_0^1 \int_\Sigma \tilde{f}(x)\; \xi(x,\gamma(t))\,\mathrm{dvol}_g(x)\,\mathrm{d}\ell(t)\,.$$

Applying Fubini's theorem we obtain

$$\begin{aligned}
\int_0^1 \int_\Sigma \tilde{f}(x)\xi(x,\gamma(t))\,\mathrm{dvol}_g(x)\,\mathrm{d}\ell(t) &= \int_\Sigma \int_0^1 \xi(x,\gamma(t))\,\mathrm{d}\ell(t)\,\tilde{f}(x)\,\mathrm{dvol}_g(x) \\
&= \int_\Sigma \left(\int_{\gamma^{-1}(B_r(x))} \mathrm{d}\ell(t)\right) \tilde{f}(x)\,\mathrm{dvol}_g(x).
\end{aligned}$$

The last term coincides with the right term of the equality in the statement in the case $\mu = \gamma$. By linearity we deduce the statement when μ a rational lamination, and by continuity of the two integrals in the statement with respect with μ we obtain the result for any general measured lamination. □

Let $m \in \mathcal{T}_h(\Sigma)$ and $\mu \in \mathcal{ML}(\Sigma)$, and select a hyperbolic metric g in the equivalence class m. If $(\widetilde{\Sigma}, \tilde{g})$ denotes the universal cover of (Σ, g), we define

$$D(m,\mu,r) := \sup_{\tilde{x}\in\widetilde{\Sigma}} \iint_{\tilde{\lambda}\cap B_r(\tilde{x})} \mathrm{d}\tilde{\ell}\,\mathrm{d}\tilde{\mu} < \infty.$$

where $\tilde{\lambda}$ denotes the support of the measured lamination $\tilde{\mu}$. In other words, $D(m,\mu,r)$ is the supremum, over the points $\tilde{x}$ in the universal cover $\widetilde{\Sigma}$, of the length of the portion of $\tilde{\mu}$ contained in the ball centered at $\tilde{x}$ of radius r.

Proposition 2.4. *For any $r > 0$ and for any $p \in [1, \infty]$ we have*

$$\left\| \mathrm{d}\left(L_\mu\right)_m \right\|_{Q,p} \leq L_\mu(m)^{1/p}\, C(r,q)\, D(m,\mu,r)^{1/q},$$

where p and q are conjugate exponents, i. e. $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $\dot{m}$ be a tangent vector to the Teichmüller space at m. As described in [Tro92], there exists a unique symmetric transverse-traceless tensor $\varphi \in S_{tt}(\Sigma, g)$ identified with the tangent vector $\dot{m} \in T_m\mathcal{T}_h(\Sigma)$. Let Φ be the holomorphic quadratic differential on $(\Sigma, [g])$ such that $\operatorname{Re}\Phi = \varphi$. We start by making use of Lemma 2.1. From the definition of $\iint_\lambda \mathrm{d}\dot{\ell}\,\mathrm{d}\mu$ and the inequality $|\varphi(v,v)| \leq \frac{1}{\sqrt{2}} \|\varphi\|_g \|v\|_g^2$, we see that

$$\left|\mathrm{d}\left(L_\mu\right)_m(\dot{m})\right| = \left| \iint_\lambda \mathrm{d}\dot{\ell}\,\mathrm{d}\mu \right| \leq \frac{1}{2\sqrt{2}} \iint_\lambda \|\varphi\|_g \,\mathrm{d}\ell\,\mathrm{d}\mu\,.$$

By applying the Hölder inequality on the right-side integral, we get

$$\left|\mathrm{d}\left(L_\mu\right)_m(\dot{m})\right| \leq \frac{1}{2\sqrt{2}} \iint_\lambda \|\varphi\|_g \,\mathrm{d}\ell\,\mathrm{d}\mu \leq \frac{L_\mu(m)^{1/p}}{2\sqrt{2}} \left(\iint_\lambda \|\varphi\|_g^q \,\mathrm{d}\ell\,\mathrm{d}\mu \right)^{1/q}. \tag{5}$$

Now we estimate the integral $\iint_\lambda \|\varphi\|_g^q \,\mathrm{d}\ell\,\mathrm{d}\mu$ by lifting it to a suitable covering of Σ, and then applying Lemma 2.2. More precisely, let $(\widehat{\Sigma}, \hat{g}) \to (\Sigma, g)$ be a N-index covering so that $\operatorname{injrad}(\widehat{\Sigma}, \hat{g}) > r$, for some $N \in \mathbb{N}$. We denote by $\hat{\bullet}$ the lift of the object $\bullet$ on $\widehat{\Sigma}$. It is immediate to check that the following relation holds

$$\iint_\lambda \|\varphi\|_g^q \,\mathrm{d}\ell\,\mathrm{d}\mu = \frac{1}{N} \iint_{\hat{\lambda}} \|\hat{\varphi}\|_{\hat{g}}^q \,\mathrm{d}\hat{\ell}\,\mathrm{d}\hat{\mu}\,.$$

Then, by applying Lemma 2.2 on the surface $(\widehat{\Sigma}, \hat{g})$ and at each point $\hat{x} \in \hat{\lambda}$, we get

$$\begin{aligned}
\iint_\lambda \|\varphi\|_g^q \,\mathrm{d}\ell\,\mathrm{d}\mu &= \frac{1}{N} \iint_{\hat{\lambda}} \|\hat{\varphi}\|_{\hat{g}}^q \,\mathrm{d}\hat{\ell}\,\mathrm{d}\hat{\mu} \\
&\leq \frac{C(r,q)^q}{N} \iint_{\hat{\lambda}} \|\hat{\varphi}\|_{FT, L^q(B_r(\cdot))}^q \,\mathrm{d}\hat{\ell}\,\mathrm{d}\hat{\mu} \\
&= \frac{C(r,q)^q}{N} \iint_{\hat{\lambda}} \left(\int_{B_r(\cdot)} \|\hat{\varphi}\|_{\hat{g}}^q \,\mathrm{dvol}_{\hat{g}} \right) \mathrm{d}\hat{\ell}\,\mathrm{d}\hat{\mu}\,.
\end{aligned}$$

Using Lemma 2.3 and the definition of $D(m, \mu, r)$, we obtain

$$\begin{aligned}
\iint_{\hat{\lambda}} \left(\int_{B_r(\cdot)} \|\hat{\varphi}\|_{\hat{g}}^q \,\mathrm{dvol}_{\hat{g}} \right) \mathrm{d}\hat{\ell}\,\mathrm{d}\hat{\mu} &= \int_{\widehat{\Sigma}} \left(\iint_{\hat{\lambda} \cap B_r(\cdot)} \mathrm{d}\hat{\ell}\,\mathrm{d}\hat{\mu} \right) \|\hat{\varphi}\|_{\hat{g}}^q \,\mathrm{dvol}_{\hat{g}} \\
&\leq D(m,\mu,r) \int_{\widehat{\Sigma}} \|\hat{\varphi}\|_{\hat{g}}^q \,\mathrm{dvol}_{\hat{g}} \\
&= N\, D(m,\mu,r)\, \|\varphi\|_{FT,q}^q,
\end{aligned}$$

where, in the last step, we are using again the fact that $(\Sigma, g) \to (\widehat{\Sigma}, \hat{g})$ is a N-index covering. Combining the last two estimates, we obtain

$$\iint_\lambda \|\varphi\|_g^q \,\mathrm{d}\ell\,\mathrm{d}\mu \leq C(r,q)^q\, D(m,\mu,r)\, \|\varphi\|_{FT,q}^q. \tag{6}$$

Using the inequalities (5) and (6), we have shown that

$$\left|\mathrm{d}\left(L_\mu\right)_m(\dot{m})\right| \leq \frac{L_\mu(m)^{1/p}\, C(r,q)\, D(m,\mu,r)^{1/q}}{2\sqrt{2}} \|\varphi\|_{FT,q}.$$

Finally, by applying Lemma 1.2, we obtain

$$\left|\mathrm{d}\left(L_\mu\right)_m(\dot{m})\right| \leq L_\mu(m)^{1/p}\, C(r,q)\, D(m,\mu,r)^{1/q} \|\dot{m}\|_{B,q},$$

which implies the statement, in light of Lemma 1.1. $\square$

3. The differential of the dual volume

In this section we use Proposition 2.4 to bound the differential of the function Vol^*, which associates to each quasi-Fuchsian manifold M the dual volume of its convex core. The link between Vol^* and the differential of the length of the bending measured lamination is given by the dual Bonahon-Schläfli formula (see [KS09], [Maz18]). We will recall and make use of the results by Bridgeman, Brock, and Bromberg [BBB19], and Bridgeman, Canary, and Yarmola [BCY16], which will allow us to estimate uniformly the quantities $L_\mu(m)$ and $D(m,\mu,r)$ appearing in Proposition 2.4.

Let M be a complete 3-dimensional hyperbolic manifold, We say that a subset $C \subset M$ is convex if for any geodesic arc γ of M connecting two points x and y of C (possibly equal), the arc γ is fully contained in C. The manifold M is called *quasi-Fuchsian* if it is homeomorphic to $\Sigma \times \mathbb{R}$ and it contains a compact non-empty convex subset. In this case, the intersection of all non-empty compact convex subsets of M is a non-empty compact convex subset, called the *convex core* of M and denoted by CM, which is obviously minimal with respect to the inclusion.

The boundary of the convex core CM is almost everywhere totally geodesic and it is homeomorphic to two copies of Σ. If we identify the universal cover of M with $\mathbb{H}^3$, then the preimage of ∂CM in $\mathbb{H}^3$ is the union of two locally convex pleated planes $H^\pm$ bent along a measured lamination $\tilde{\mu}$. Since these pleated planes are invariant under the action of the fundamental group of M, they determine two hyperbolic metrics m^+, $m^- \in \mathcal{T}_h(\Sigma)$ and two measured laminations μ^+, $\mu^- \in \mathcal{ML}(\Sigma)$. We will denote the couple of metrics (m^+, m^-) by $m \in \mathcal{T}(\partial CM)$ and the pairs of measured laminations (μ^+, μ^-) by $\mu \in \mathcal{ML}(\partial CM)$.

The action of the fundamental group Γ of M naturally extends to $\partial \mathbb{H}^3 \cong \mathbb{C}\mathrm{P}^1$ by Möbius tranformations. Given any $x_0 \in \mathbb{H}^3$, the subset Λ of accumulation points of Γx_0 in $\partial \mathbb{H}^3$ is called the *limit set* of Γ. The action of Γ is free and properly discontinuous on $\partial \mathbb{H}^3 \setminus \Lambda$, and it determines a pair of Riemann surface structures c^+, c^- on Σ and $\overline{\Sigma}$ (the surface Σ endowed with the opposite orientation), called the *conformal structures at infinity* of M. A well-known result of [Ber60] states that the space of quasi-Fuchsian structures on $\Sigma \times \mathbb{R}$ is parametrized by the couple of conformal structures at infinity. In other words, the map

$$\begin{array}{rccc} B: & \mathcal{QF}(\Sigma) & \longrightarrow & \mathcal{T}_c(\Sigma) \times \mathcal{T}_c(\overline{\Sigma}) \\ & M & \longmapsto & (c^+, c^-) \end{array}$$

is a homeomorphism. In fact B is a biholomorphism if we endow $\mathcal{QF}(\Sigma)$ with the complex structure of subset of the character variety $\chi(\pi_1\Sigma, \mathrm{PSL}_2\mathbb{C})$, and the natural complex structure of $\mathcal{T}_c(\Sigma)$. Another natural map on $\mathcal{QF}(\Sigma)$ is

$$\begin{array}{rccc} \Psi: & \mathcal{QF}(\Sigma) & \longrightarrow & \mathcal{T}_h(\Sigma) \times \mathcal{T}_h(\Sigma) \\ & M & \longmapsto & (m^+, m^-), \end{array}$$

which Thurston conjectured to be another parametrization of the space of quasi-Fuchsian manifolds. Bonahon [Bon98b] proved that the map Ψ is $\mathscr{C}^1$ (and actually not $\mathscr{C}^2$), therefore a first order variation of quasi-Fuchsian structures $\dot{M}$ determines a first order variation of the induced hyperbolic structures $\dot{m}$ on the convex core.

Definition 3.1. Let $N \subset M$ be a compact convex subset of M with regular boundary. The *dual volume* of N is defined as

$$\mathrm{Vol}^*(N) := \mathrm{Vol}(N) - \frac{1}{2}\int_{\partial N} H \,\mathrm{d}a,$$

where H is trace of the shape operator of ∂N with respect to the normal vector pointing towards the convex side. The dual volume of the convex core of M is set to be:

$$\mathrm{Vol}^*(CM) := \mathrm{Vol}(CM) - \frac{1}{2} L_\mu(m).$$

Contrary to the usual hyperbolic volume of the convex core (see [Bon98a] for details), the dual volume $\mathrm{Vol}^*(CM)$ turns out to be a $\mathscr{C}^1$-function on the space of quasi-Fuchsian manifolds, and its variation is described by the following result:

Theorem 3.2 ([KS09]). *Let $(M_t)_{t\in(-\varepsilon,\varepsilon)}$ be a smooth 1-parameter family of quasi-Fuchsian structures. We denote by $\mu=\mu_0\in\mathcal{ML}(\partial CM)$ the bending measure of the convex core of $M=M_0$ and by $(m_t)_t$ the family of hyperbolic metrics on the boundary of the convex core CM_t. Then the derivative of the dual volume of CM_t exists and it verifies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}^*(CM_t)\bigg|_{t=0}=\frac{1}{2}\,\mathrm{d}\big(L_\mu\big)_m(\dot m),$$

where $m=m_0$ and $\dot m=\dot m_0\in T_m\mathcal{T}_h(\Sigma)$.

This fact has been initially proved by [KS09] making use of Bonahon's work on the variation of the hyperbolic volume in [Bon98a]. The author of this paper has recently described an alternative proof of this relation that does not require the results of [Bon98a], which can be found in [Maz18].

An immediate corollary of the variation formula of the dual volume and of our estimate in Proposition 2.4 is the following:

Proposition 3.3. *Let $\mathrm{Vol}^*\colon\mathcal{QF}(\Sigma)\to\mathbb{R}$ denote the function associating to each quasi-Fuchsian manifold M the dual volume of its convex core CM. Then for every $r>0$ and for every $p\in[1,\infty]$ we have*

$$\left|\mathrm{d}\,\mathrm{Vol}^*_M(\dot M)\right|\leq\frac{1}{2}\,L_\mu(m)^{1/p}\,C(r,q)\,D(m,\mu,r)^{1/q}\,\|\dot m\|_{B,q},$$

where m and μ are the bending measure and the hyperbolic metric of the convex core of M, respectively, $C(r,q)$ and $D(m,\mu,r)$ are the constants defined in the previous section, and p and q are conjugated exponents.

In the remaining part of this section we describe a procedure to obtain a multiplicative factor in the above statement depending only on p and the genus of Σ.

As we mentioned before, the lift of the boundary of the convex core is the union of two locally bent pleated planes $H^\pm$, which are embedded in $\mathbb{H}^3$. This property turns out to determine uniform upper bounds of the quantities $L_\mu(m)$ and $D(m,\mu,r)$ appearing in the statement of Proposition 2.4. The first results in this direction have been developed by Epstein and Marden in [CEM06, Part II]. In our exposition, we will recall and make use of the works of Bridgeman, Brock, and Bromberg [BBB19] and Bridgeman, Canary, and Yarmola [BCY16], which will give us separate bounds for $L_\mu(m)$ and $D(m,\mu,r)$, respectively. We will also require r to be less than $\ln(3)/2$. This restriction simplifies our argument in the proof of Corollary 3.6. However, we do not exclude the possibility that a joint study of the quantity $L_\mu(m)^{1/p}D(m,\mu,r)^{1/q}$ and a careful choice of r might improve the multiplicative constants obtained here.

First we focus on the term $D(m,\mu,r)$, which we defined before the statement of Proposition 2.4. Let $\tilde\lambda$ denote the geodesic lamination in $\widetilde\Sigma$ given by the lift of the support of the measured lamination μ. Let Q be a component of $\widetilde\Sigma\setminus\tilde\lambda$ and let l_1, l_2, l_3 be three boundary components of Q. We will use the following fact:

Lemma 3.4 ([CEM06, Corollary II.2.4.3]). *Let $r<\ln(3)/2=\operatorname{arcsinh}(1/\sqrt{3})$, and suppose we have a point $x\in Q$ which is at distance $\leq\operatorname{arcsinh}(e^{-r})$ from both l_2 and l_3. Then its distance from l_1 is $>r$.*

Following [BCY16], given $\tilde\mu$ a measured lamination on $\mathbb{H}^2$, we denote by $\|\tilde\mu\|_s$ the supremum over α of the transverse measure of $\tilde\mu$ along α, where α varies among the geodesic arcs in $\mathbb{H}^3$ of length $s>0$ which are transverse to the support of $\tilde\mu$.

Theorem 3.5 ([BCY16]). *Let $s \in (0, 2\operatorname{arcsinh} 1)$ and let $\tilde{\mu}$ be a measured lamination of $\mathbb{H}^2$ so that the pleated plane with bending measure $\tilde{\mu}$ is embedded inside $\mathbb{H}^3$. Then*

$$\|\tilde{\mu}\|_s \leq 2\arccos\left(-\sinh(s/2)\right).$$

Corollary 3.6. *Let $\mu \in \mathcal{ML}(\Sigma)$ and $m \in \mathcal{T}_h(\Sigma)$ be the bending measure and the hyperbolic metric, respectively, of the boundary of an incompressible hyperbolic end inside a hyperbolic convex co-compact 3-manifold. Then for every $r < \ln(3)/2$ we have*

$$D(m, \mu, r) \leq 4r \arccos\left(-\sinh r\right).$$

Moreover, for every $\varepsilon > 0$ there exists $m_\varepsilon \in \mathcal{T}_h(\Sigma)$ and $\mu_\varepsilon \in \mathcal{ML}(\Sigma)$ as above verifying

$$D(m_\varepsilon, \mu_\varepsilon, r) \geq 2(\pi - \varepsilon) r \qquad \forall r > 0.$$

Proof. Let g be a hyperbolic metric in the equivalence class $m \in \mathcal{T}_h(\Sigma)$. We denote by $(\widetilde{\Sigma}, \tilde{g}) \to (\Sigma, g)$ the Riemannian universal cover of (Σ, g) and by $\tilde{\lambda}$ the support of the lift $\tilde{\mu}$ of the measured lamination μ to $\widetilde{\Sigma}$. Given a point $\tilde{x}$ in $\widetilde{\Sigma}$ and a positive $r < \ln(3)/2$, we are looking for an upper bound of the length of $\tilde{\mu} \cap B_r(\tilde{x})$, where $B_r(\tilde{x})$ denotes the metric ball of radius r at $\tilde{x}$.

The convenience of considering $r < \ln(3)/2$ comes from Lemma 3.4: under this hypothesis, any plaque Q of $\tilde{\lambda}$ at distance less than r from x has at most two components of its boundary intersecting $B_r(x)$. A simple argument proves that, if this happens, we can find a geodesic path α of length $< 2r$ that intersects all the leaves of $\tilde{\lambda} \cap B_r(\tilde{x})$. Each leaf of $\tilde{\lambda} \cap B_r(\tilde{x})$ has length $< 2r$, therefore the length of $\tilde{\mu} \cap B_r(\tilde{x})$ is bounded by $2r$ (the length of each leaf) times the total mass $\tilde{\mu}(\alpha)$, which can be estimated applying Theorem 3.5 with $s = 2r < \ln 3 < 2\operatorname{arcsinh} 1$. This proves the first part of the statement[1].

For what concerns the last part of the assertion, we fix a simple closed curve γ and we assign it the weight $\pi - \varepsilon$. By the work of Bonahon and Otal [BO04], we can find a quasi-Fuchsian manifold M_ε realizing $(\pi - \varepsilon)\gamma$ as the bending lamination of the upper component of the boundary of the convex core $\partial^+ CM_\varepsilon$. It is immediate to check that, if m_ε is the hyperbolic metric of $\partial^+ CM_\varepsilon$, then $D(m_\varepsilon, \mu_\varepsilon, r) \geq 2(\pi - \varepsilon) r$ for all $r > 0$. □

For the bound of the term $L_\mu(m)$, we will apply the following result:

Theorem 3.7 ([BBB19, Theorem 2.16]). *Let $\mu \in \mathcal{ML}(\Sigma)$ and $m \in \mathcal{T}_h(\Sigma)$ be the bending measure and the hyperbolic metric, respectively, of the boundary of an incompressible hyperbolic end inside a hyperbolic convex co-compact 3-manifold. Then*

$$L_\mu(m) \leq 6\pi |\chi(\Sigma)|.$$

Finally, given $p \in (1, \infty)$ and $r < \ln(3)/2$, we set

$$\begin{aligned} K(r,p) &:= \frac{1}{2}(24\pi)^{1/p}\, C(r,q)\, \left(4r \arccos(-\sinh r)\right)^{1/q} \\ &= \frac{1}{2}(24\pi)^{1/p} \left(\frac{2q-1}{\pi} \frac{(\cosh(r/2))^{4q-2}}{(\cosh(r/2))^{4q-2} - 1}\, r \arccos(-\sinh r)\right)^{1/q}, \end{aligned}$$

where $C(r,q)$ was defined in equation 3. We define also

$$K(r,1) = 12\pi, \qquad K(r,\infty) = \frac{r \arccos(-\sinh r)}{2\pi \tanh^2(r/2)}.$$

Corollary 3.8. *In the same notations of Proposition 3.3, for every $p \in [1, \infty]$ we have*

$$\left| \mathrm{d}\, \mathrm{Vol}_M^*(\dot{M}) \right| \leq K(p)(g-1)^{1/p} \|\dot{m}\|_{B,q},$$

where $K(p) := K(\ln(3)/2, p)$ and $\dot{m}$ denotes the variation of the hyperbolic metrics on the boundary of the convex core ∂CM of M. We have:

[1]See Remark 3.10.

- $K(1) = 12\pi$;
- $K(2) \approx 10.3887$;
- $K(\infty) \approx 2.66216$.

Proof. We combine Proposition 3.3, Corollary 3.6 and Theorem 3.7 on the upper and lower components of $\partial CM = \partial CM_0$, and then we take the limit as r goes to $\ln(3)/2$. □

We can compare this statement with the analogous bound for the differential of the renormalized volume:

Theorem 3.9 ([Sch13]). *Let* $\mathrm{RVol}\colon \mathcal{QF}(\Sigma) \to \mathbb{R}$ *denote the function associating to each quasi-Fuchsian manifold M its renormalized volume. Then for every $p \in [1,\infty]$ we have*

$$\mathrm{d}\,\mathrm{RVol}_M(\dot{M}) \leq H(p)(g-1)^{1/p}\|\dot{c}\|_{B,q},$$

where $\dot{c}$ denotes the variation of the conformal structures at infinity of M, and where $H(p) := \frac{3}{2}(8\pi)^{1/p}$.

Remark 3.10. From the first part of the proof of Corollary 3.6 is clear that our estimate of the constant $D(m,\mu,r)$ is far from being optimal. However, using the second part of the assertion, it is easy to see that the possible improvement of the constant $K(2)$ is not enough to make the multiplicative constant in Theorem A to be less than $3\sqrt{\pi}$, which is the one appearing in the analogous statement for the renormalized volume. Because of this, we preferred to present a simpler but rougher argument.

4. Dual volume and Weil-Petersson distance

This section is dedicated to the proof of the linear upper bound of the dual volume of a quasi-Fuchsian manifold M in terms of the Weil-Petersson distance between the hyperbolic structures on the boundary of its convex core CM. As we mentioned in Section 3, the data of the hyperbolic metrics of ∂CM is only conjectured to give a parametrization of the space of quasi-Fuchsian manifolds, contrary to what happens with the conformal structures at infinity. In particular, the same strategy used in [Sch13] to bound the renormalized volume cannot be immediately applied. In order to overcome this problem, we will take advantage of the foliation by constant Gaussian curvature surfaces (k-surfaces) of $M \setminus CM$, whose existence has been proved by Labourie [Lab91] (see also Remark 4.12). The space of hyperbolic structures with strictly convex boundary on $\Sigma \times [0,1]$ is parametrized by the data of the metrics on its boundary, as proved in [Sch06]. In particular, the Teichmüller classes of the metrics of the upper and lower k-surfaces parametrize the space of quasi-Fuchsian structures of topological type $\Sigma \times \mathbb{R}$. Moreover, the first order variation of the dual volume of the region M_k encosed between the two k-surfaces is intimately related to the notion of landslide, which was first introduced and studied in [BMS13], [BMS15]. This connection will be very useful to relate the first order variation of $\mathrm{Vol}^*(CM)$ and of $\mathrm{Vol}^*(M_k)$, as k goes to -1, allowing us to prove Theorem A using an approximation argument, together with the bounds obtained in the previous Section.

4.1. Constant Gaussian curvature surfaces.

The existence of the foliation by constant Gaussian curvature surfaces is guaranteed by the following result:

Theorem 4.1 ([Lab91, Théorème 2]). *Every geometrically finite 3-dimensional hyperbolic end E is foliated by a family of strictly convex surfaces $(\Sigma_k)_k$ with constant curvature $k \in (-1,0)$. As k goes to -1, the surface Σ_k converges to the locally concave pleated boundary of E, and as k goes to 0, Σ_k approaches the conformal boundary at infinity $\partial_\infty E$.*

A surface of constant Gaussian curvature k embedded in some hyperbolic 3-manifold is called a *k-surface*. From the Gauss equations we see that the extrinsic curvature of a k-surface is equal to $k+1$. Therefore, if k is in $(-1,0)$, the principal curvatures have the same sign and never vanish. In particular the leaves of the foliation of Theorem 4.1 are all convex surfaces.

Given a quasi-Fuchsian manifold M, we denote by $m_k^{\pm}(M) \in \mathcal{T}_h(\Sigma)$ the isotopy classes of the hyperbolic metrics $-k\, I_k^{\pm}$, where $I_k^{\pm}$ is the first fundamental form of the upper/lower k-surface $\Sigma_k^{\pm}$ of M. Then for every $k \in (-1,0)$ we have maps

$$\begin{array}{rccc} \Psi_k : & \mathcal{QF}(\Sigma) & \longrightarrow & \mathcal{T}_h(\Sigma)\times\mathcal{T}_h(\Sigma) \\ & M & \longmapsto & (m_k^-(M), m_k^+(M)). \end{array}$$

The family of functions $(\Psi_k)_k$ is clearly related to the maps Ψ and B we considered in Section 3. As k goes to -1, $\Psi_k(M)$ converges to $\Psi(M)$, and as k goes to 0, $\Psi_k(M)$ converges to $B(M)$. The convenience in considering the foliation by k-surfaces relies in the following result, based on the works of Labourie [Lab91] and Schlenker [Sch06]:

Theorem 4.2. *The map Ψ_k is a $\mathscr{C}^1$-diffeomorphism for every $k \in (-1,0)$.*

Proof. Let $(N,\partial N)$ be a compact connected 3-manifold admitting a hyperbolic structure with convex boundary. Schlenker [Sch06] proved that any Riemannian metric with Gaussian curvature > -1 on ∂N is uniquely realized as the restriction to the boundary of a hyperbolic metric on N with smooth strictly convex boundary. In other words, if $\mathscr{G}$ and $\mathscr{H}$ denote the spaces of isotopy classes of metrics on N with strictly convex boundary and of metrics on ∂N with Gaussian curvature > -1, respectively, then the restriction map

$$\begin{array}{rccc} r : & \mathscr{G} & \longrightarrow & \mathscr{H} \\ & [g] & \longmapsto & [g|_{\partial N}] \end{array}$$

is a homeomorphism. The surjectivity was already been showed by Labourie in [Lab91], therefore the proof proceeds by showing the local injectivity of r. To do so, the strategy in [Sch06] is to apply the Nash-Moser implicit function theorem.

Let us fix now a $k \in (-1,0)$, and consider $N = \Sigma \times I$. If $\mathscr{G}_k$ is the space of hyperbolic structures on N with boundary having constant Gaussian curvature equal to k, then $\mathcal{G}_k$ identifies with the space of quasi-Fuchsian manifolds $\mathcal{QF}(\Sigma)$, thanks to Theorem 4.1 and the fact that any hyperbolic structure with convex boundary on N uniquely extends to a quasi-Fuchsian structure (see e. g. [CEM06, Theorem I.2.4.1]). In addition, the space $\mathscr{H}_k$ of constant k Gaussian curvature structures on ∂N clearly identifies with the product of two copies of the Teichmüller space $\mathcal{T}_h(\Sigma)$, one for each component of ∂N. Therefore the function r restricts to $r_k : \mathscr{G}_k \to \mathscr{H}_k$, which can be identified with Ψ_k thanks to what we just observed. The map r_k is now a function between finite dimensional differential manifolds. The fact that r verifies the hypotheses to apply the Nash-Moser inverse function theorem implies in particular that r_k verifies the hypotheses to apply the ordinary inverse function theorem between finite dimensional manifolds. In particular, this shows that r_k is a $\mathscr{C}^1$-diffeomorphism, for any $k \in (-1,0)$, as desired. □

4.2. **The proof of Theorem A.** In the following we outline the proof of Theorem A. Let $\mathrm{Vol}_k^*(M)$ denote the dual volume of the convex subset enclosed by the two k-surfaces in M. We define $V_k^* : \mathcal{T}_h(\Sigma)\times\mathcal{T}_h(\Sigma) \to \mathbb{R}$ to be the composition $\mathrm{Vol}_k^* \circ \Psi_k^{-1}$. An immediate corollary of Theorem 4.2 is that the function V_k^* is $\mathscr{C}^1$ for every $k \in (-1,0)$.

Let now M be a fixed quasi-Fuchsian manifold. Since the Teichmuller space endowed with the Weil-Petersson metric is a unique geodesic space [Wol87], there exists a unique Weil-Petersson geodesic $\beta_k : [0,1] \to \mathcal{T}_h(\Sigma)$ verifying $\beta_k(0) = m_k^-$ and $\beta_k(1) = m_k^+$, where $m_k^{\pm} = m_k^{\pm}(M)$. We set γ_k to be the path in $\mathcal{T}_h(\Sigma)^2$ given by $\gamma_k(t) = (\beta_k(t), m_k^-)$. By construction $\Psi_k^{-1}(\gamma_k(0))$ is a Fuchsian manifold for every $k \in (-1,0)$ and $\Psi_k^{-1}(\gamma_k(1)) = M$. We decompose the differential of the function V_k^* as follows

$$\mathrm{d}V_k^* = \mathrm{d}V_k^{*,+} + \mathrm{d}V_k^{*,-} \in T^*\mathcal{T}_h(\Sigma) \oplus T^*\mathcal{T}_h(\Sigma).$$

Now we observe that

$$\begin{aligned}|V_k^*(\gamma_k(1)) - V_k^*(\gamma_k(0))| &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} V_k^*(\gamma_k(t))\,\mathrm{d}t \right| \\ &\leq \int_0^1 \|\mathrm{d}V_k^{*,+}\|_{\gamma_k(t)} \left\|\beta_k'(t)\right\| \mathrm{d}t \\ &\leq \max_{t\in[0,1]} \|\mathrm{d}V_k^{*,+}\|_{\gamma_k(t)}\, \ell_{WP}(\beta_k) \\ &= \max_{t\in[0,1]} \|\mathrm{d}V_k^{*,+}\|_{\gamma_k(t)}\, d_{WP}(m_k^+, m_k^-),\end{aligned}$$

where $\|\cdot\|_p$ denotes the Weil-Petersson norm on $T_p^*\mathcal{T}_h(\Sigma)$. The step from the first to the second line follows from the fact that the second component of the curve γ_k does not depend on t, and in the last step we used that β_{WP} is a Weil-Petersson geodesic. Since the dual volume of the convex core of a Fuchsian manifold vanishes, we have that

$$\lim_{k\to -1} V_k^*(\gamma_k(1)) - V_k^*(\gamma_k(0)) = \mathrm{Vol}^*(CM).$$

By Theorem 4.1 we have

$$\lim_{k\to -1} d_{WP}(m_k^+, m_k^-) = d_{WP}(m^+, m^-)$$

where m^+, m^- are the hyperbolic metrics of the upper and lower components of ∂CM, respectively. Therefore, taking the limit as k goes to -1 of the inequality above we obtain

$$|\mathrm{Vol}^*(CM)| \leq \liminf_{k\to -1} \max_{t\in[0,1]} \|\mathrm{d}V_k^{*,+}\|_{\gamma_k(t)}\, d_{WP}(m^+, m^-). \tag{7}$$

If $\pi^+ : \mathcal{T}_h(\Sigma)^2 \to \mathcal{T}_h(\Sigma)$ denotes the projection onto the first component, then the functions $\mathrm{d}V_k^{*,+} \circ \Psi_k$ are sections of the bundles $(\pi^+ \circ \Psi_k)^*(T^*\mathcal{T}(\Sigma))$. In order to simplify the notation, we will set $\mathrm{d}L_{\mu^+}$ to be the map

$$\mathcal{QF}(\Sigma) \ni M \longmapsto \mathrm{d}(L_{\mu^+(M)})_{\pi^+\circ\Psi(M)} \in T^*\mathcal{T}_h(\Sigma).$$

Assuming that the functions $(\mathrm{d}V_k^{*,+} \circ \Psi_k)_k$ converge to $\mathrm{d}L_{\mu^+}$ uniformly over compact sets of $\mathcal{QF}(\Sigma)$ as k goes to -1, then Theorem A easily follows:

Proof of Theorem A. The paths $\Psi_k^{-1}(\gamma_k)$ considered above lie inside a common compact subset of $\mathcal{QF}(\Sigma)$. Following the proof of Corollary 3.8, we observe that $\|\mathrm{d}L_{\mu^+}\|$ is bounded by $K(2)/\sqrt{2}$ (the factor $1/\sqrt{2}$ appears because we consider only the upper component of the bending measure). Therefore, by uniform convergence we have

$$\liminf_{k\to -1} \max_{t\in[0,1]} \|\mathrm{d}V_k^{*,+}\|_{\gamma_k(t)} \leq K(2)/\sqrt{2} \approx 7.3459,$$

which, combined with the inequality (7), implies the statement. □

Therefore, the last step left is to prove:

Proposition 4.3. *The functions $(\mathrm{d}V_k^{*,+} \circ \Psi_k)_k$ converge uniformly to $\mathrm{d}L_{\mu^+}$ over compact sets of $\mathcal{QF}(\Sigma)$ as k goes to -1*

We will deduce this fact from the so called *dual differential Schläfli formula*, stated in Theorem 4.4, and from the connection between the first order variation of the volumes V_k^* and the notion of landslides introduced in [BMS13], [BMS15].

Theorem 4.4 ([RS99])**.** *Let N be a compact manifold with boundary, and assume that there exists a smooth 1-parameter family $(g_t)_t$ of hyperbolic metrics with strictly convex boundary on N. Then there exists the derivative of $t \mapsto \mathrm{Vol}^*(N, g_t)$ and it satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Vol}^*(N, g_t)|_{t=0} = \frac{1}{4} \int_{\partial N} (\delta g|_{\partial N}, HI - I\!I)\, \mathrm{d}a_I\,,$$

where $\delta g = \frac{\mathrm{d}}{\mathrm{d}t} g_t|_{t=0}$.

Proof. This relation is a corollary of [RS99, Theorem 8]. It is enough to apply this result to the definition of dual volume $\mathrm{Vol}^*(N, g_t)$, together with the relation

$$\delta\left(\int_{\partial N} H \,\mathrm{d}a\right) = \int_{\partial N}\left(\delta H + \frac{H}{2}(\delta I, I)\right)\mathrm{d}a_I\,,$$

which follows by differentiating the expression $H\,\mathrm{d}a = H\sqrt{\det I}\,\mathrm{d}x\wedge\mathrm{d}y$ in local coordinates. □

4.3. **Earthquakes and landslides.** We briefly recall the definition of landslide flow, introduced in Bonsante, Mondello, and Schlenker [BMS13], and the properties that we will need for the proof of Proposition 4.3.

Landslides are described by a map

$$\begin{array}{rccc}\mathcal{L}: & S^1\times\mathcal{T}\times\mathcal{T} & \longrightarrow & \mathcal{T}\times\mathcal{T}\\ & (e^{i\theta}, m, m') & \longmapsto & \mathcal{L}_{e^{i\theta}}(m,m'),\end{array}$$

where $\mathcal{T}$ stands for $\mathcal{T}(\Sigma)$. The first component of $\mathcal{L}_{e^{i\theta}}(m,m')$, which we will denote by $\mathcal{L}^1_{e^{i\theta}}(m,m')$, is called the *landslide of m with respect to m' with parameter $e^{i\theta}$*. The map $\mathcal{L}$ is defined via the following result:

Theorem 4.5 ([Lab92],[Sch93])**.** *Let $m, m' \in \mathcal{T}$. Then, for any representative $h \in m$, there exists a unique $h' \in m'$ and a unique $b\colon T\Sigma \to T\Sigma$ such that:*

- *b is h-self-adjoint;*
- *b has determinant* 1*;*
- *b is Codazzi with respect to the Levi-Civita connection ∇ of h, i. e. $(\nabla_X b)Y = (\nabla_Y b)X$ for all X, Y.*

The operator b is also called the *Labourie operator* of the couple h, h'. In the following, we will identify, with abuse, a pair of isotopy classes m, $m' \in \mathcal{T}$ with a pair of hyperbolic metrics h, h' satisfying the conclusions of the Theorem above. Given $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and two metrics h, h' with Labourie operator b, we denote by b^θ the endomorphism $\cos(\theta/2)\mathbb{1} + \sin(\theta/2)Jb$, where J is the almost complex structure of h, and we set $h^\theta := h(b^\theta\cdot, b^\theta\cdot)$. Then the function $\mathcal{L}$ is defined as:

$$\mathcal{L}_{e^{i\theta}}(h,h') := (h^\theta, h^{\pi+\theta}).$$

It turns out that, for any θ, the metric h^θ is hyperbolic, and $\mathcal{L}$ actually defines a flow, in the sense that it satisfies $\mathcal{L}_{e^{i\theta}}\circ\mathcal{L}_{e^{i\theta'}} = \mathcal{L}_{e^{i(\theta+\theta')}}$ for all θ, θ'.

As earthquakes extend to *complex earthquakes* (see [McM98]), similarly happens for landslides. Fixed $h, h' \in \mathcal{T}$, the map $\mathcal{L}^1_\bullet(h,h')$ extends to a holomorphic function $C_\bullet(h,h')$ defined on a open neighborhood of the closure of the unit disc Δ in $\mathbb{C}$. If $\zeta = \exp(s+i\theta) \in \overline{\Delta}$, then C_ζ can be written as

$$C_\zeta(h,h') = \mathrm{sgr}_s \circ \mathcal{L}_{e^{i\theta}}(h,h'),$$

where $\mathrm{sgr}_s\colon \mathcal{T}^2 \to \mathcal{T}$ is called the *smooth grafting map*. If $s = 0$, then $\mathrm{sgr}_0\circ\mathcal{L}_{e^{i\theta}} = \mathcal{L}^1_{e^{i\theta}}$.

Constant Gaussian curvature surfaces are a natural example in which pairs of metrics as in Theorem 4.5 arise. Let Σ_k be a k-surface in a hyperbolic 3-manifold, with first fundamental form I_k and shape operator B_k. The *third fundamental form* of Σ_k is defined as $I\!I\!I_k := I_k(B_k\cdot, B_k\cdot)$. Either by direct computation or by using the duality correspondence between hypersurfaces of $\mathbb{H}^3$ and dS^3 (see [Riv86], [Sch02]), we see that the third fundamental form is a constant Gaussian curvature metric too, with curvature $\frac{k}{k+1}$. Moreover, if we set

$$h_k := -k\, I_k, \qquad h'_k := -\frac{k}{k+1}\, I\!I\!I_k, \qquad b_k := \frac{1}{\sqrt{k+1}}B_k, \tag{8}$$

then h_k and $h'_k = h_k(b_k\cdot, b_k\cdot)$ are hyperbolic metrics satisfying the properties of Theorem 4.5. We refer to [BMS13] and [BMS15] for a more detailed exposition about landslides, and to Labourie [Lab92] for what concerns k-surfaces.

Fixed h', we set $l^1(h,h')$ to be the infinitesimal generator of the landslide flow with respect to the hyperbolic metric h' at the point $h \in \mathcal{T}$. In other words,

$$l^1(h,h') := \frac{\mathrm{d}}{\mathrm{d}\theta} \mathcal{L}^1_{e^{i\theta}}(h,h')\big|_{\theta=0} \in T_h\mathcal{T}.$$

Landslides extend the notion of earthquake in the sense explained by the following Theorem:

Theorem 4.6 ([BMS13, Proposition 6.8]). *Let $(h_n)_n$ and $(h'_n)_n$ be two sequences of hyperbolic metrics on Σ such that $(h_n)_n$ converges to $h \in \mathcal{T}$, and $(h'_n)_n$ converges to a projective class of measured lamination $[\mu]$ in the Thurston boundary of Teichmüller space. If $(\theta_n)_n$ is a sequence of positive numbers such that $\theta_n \ell_{h'_n}$ converges to $\iota(\mu,\cdot)$, then $\mathcal{L}^1_{e^{i\theta_n}}(h_n,h'_n)$ converges to the left earthquake $\mathcal{E}_{\mu/2}(h)$, and $\theta_n \cdot l^1(h_n,h'_n)|_{h_n}$ converges to $\frac{1}{2}e_\mu|_h = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{t\mu/2}(h)$.*

Remark 4.7. The last part of the assertion follows from the fact that the functions $e^{i\theta} \mapsto \mathcal{L}^1_{e^{i\theta}}(h,h')$ extend to holomorphic functions $\zeta \mapsto C_\zeta(h,h')$, where ζ varies in a neighborhood of $\overline{\Delta}$. In particular, the uniform convergence of the complex landslides $C_\bullet(h_n,h'_n)$ to the complex earthquake map implies uniform convergence in the $\mathscr{C}^\infty$-topology with respect to the complex parameter ζ.

In order to prove the relation between the differential of V^*_k and the landslide flow, it will be useful to have an explicit expression to compute the variation of the hyperbolic length of a simple closed curve α of Σ along the infinitesimal landslide $l^1(h,h')$.

Lemma 4.8. *Let α be a simple closed curve in Σ. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}\theta} L_\alpha(\mathcal{L}^1_{e^{i\theta}}(h,h'))\big|_{\theta=0} = -\int_\alpha \frac{h(b\alpha', J\alpha')}{2\|\alpha'\|^2_h} \mathrm{d}\ell_h,$$

where J is the complex structure of h and b is the Labourie operator of the couple h, h'.

Proof. With abuse, we denote the h-geodesic realization of α by α itself. By definition of landslide we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \mathcal{L}^1_{e^{i\theta}}(h,h')(\alpha',\alpha')\big|_{t=0} = \dot{h}(\alpha',\alpha') = h(\alpha', Jb\alpha').$$

Since J is h-self-adjoint and $J^2 = -id$, we deduce that $\dot{h}(\alpha',\alpha') = -h(b\alpha', J\alpha')$. Combining this relation with Proposition 2.1 we obtain the statement. □

In order to simplify the notation, we will write $\mathcal{T}^2$ for the Teichmüller space of the surface $\Sigma = \Sigma^+ \sqcup \Sigma^-$, and by I_k, $I\!I_k$ and $I\!I\!I_k$ the fundamental forms of the surface $\Sigma_k = \Sigma^+_k \sqcup \Sigma^-_k$. The relation between landslides and the dual volume of the region enclosed by the two k-surfaces is described by the following fact:

Proposition 4.9. *Let M be a quasi-Fuchsian manifold and let h_k, $h'_k \in \mathcal{T}^2$ denote the hyperbolic metrics $-k\, I_k$ and $-k(k+1)^{-1}\, I\!I\!I_k$. Then we have*

$$\mathrm{d}V^*_k \circ \Psi_k(M) = \sqrt{-\frac{k+1}{k}}\, \hat{\omega}_{WP}(l^1(h_k,h'_k),\cdot) \in T^*_{\Psi_k(M)}\mathcal{T}^2,$$

where $\hat{\omega}_{WP} = \omega_{WP} \oplus \omega_{WP}$.

Proof. Given a simple closed curve α in Σ_k, we denote by e_α the infinitesimal generator of the left earthquake flow along α on $\mathcal{T}^2$. We will prove the statement by showing that, for every simple closed curve α, we have:

$$\mathrm{d}(V^*_k)_{\Psi_k(M)}(e_\alpha) = \sqrt{-\frac{k+1}{k}}\, \hat{\omega}_{WP}(l^1(h_k,h'_k),e_\alpha). \tag{9}$$

Since the constant k will be fixed, we will not write the dependence on k in the objects involved in the argument, in order to simplify the notation. Given any first order variation of metrics $\delta I = \delta I_k$ on $\Sigma = \Sigma_k$, we can find a variation δg of hyperbolic metrics on M satisfying $\delta g|_\Sigma = \delta I$. Our first step will be to construct an explicit variation δI corresponding to the vector field e_α, and then to apply Proposition 4.4 to compute $\mathrm{d}V_k^*(e_\alpha)$.

We will identify the curve α with its I-geodesic parametrization of length L_α and at speed 1. Let J denote the almost complex structure of I, and set V to be the vector field along α given by $-J\alpha'$. We can find a $\varepsilon > 0$ so that the map

$$\begin{array}{rccc} \xi : & \mathbb{R}/L_\alpha\mathbb{Z} \times [0,\varepsilon] & \longrightarrow & \Sigma \\ & (s,r) & \longmapsto & \exp_{\alpha(s)}(rV(s)) \end{array}$$

is a diffeomorphism onto its image (here exp is the exponential map with respect to I). The image of ξ is a closed cylinder in Σ having α as left boundary component. Observe that the metric I equals $\mathrm{d}r^2 + \cosh^2 r\,\mathrm{d}s^2$ in the coordinates defined by ξ^{-1}. We also choose a smooth function $\eta : [0,\varepsilon] \to [0,1]$ that coincides with 1 in a neighborhood of 0, and with 0 in a neighborhood of ε. Now define

$$\begin{array}{rccc} f_t : & \mathbb{R}/L_\alpha\mathbb{Z} \times [0,\varepsilon] & \longrightarrow & \mathbb{R}/L_\alpha\mathbb{Z} \times [0,\varepsilon] \\ & (s,r) & \longmapsto & (s + t\eta(r), r). \end{array}$$

The maps $u_t := \xi \circ f_t \circ \xi^{-1}$ give a smooth isotopy of the strip $\mathrm{Im}\,\xi$ adjacent to α, with $u_0 = id$. Finally we set

$$\delta I := \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u_t^* I\big|_{t=0} = 2\eta'(r)\cosh^2 r \;\mathrm{d}r\mathrm{d}s & \text{inside } \mathrm{Im}\,\xi, \\ 0 & \text{elsewhere,} \end{cases}$$

where here $2\,\mathrm{d}s\,\mathrm{d}r = \mathrm{d}s \otimes \mathrm{d}r + \mathrm{d}r \otimes \mathrm{d}s$. Thanks to our choice of the function η, δI is a smooth symmetric tensor of Σ_k that represents the first order variation of I along the infinitesimal left earthquake e_α. By Proposition 4.4, we have that

$$\mathrm{d}V_k^*(\delta g) = \frac{1}{4}\int_{\Sigma_k} (\delta g|_{\Sigma_k}, HI - I\!I)\,\mathrm{d}a = -\frac{1}{4}\int_0^{L_\alpha}\int_0^{\varepsilon} (\delta I, I\!I)\cosh r\,\mathrm{d}r\,\mathrm{d}s,$$

where the last step follows from the fact that δI is I-traceless. Let ∇ denote the Levi-Civita connection of I. Then the coordinate vector fields of ξ^{-1} satisfy:

$$\nabla_{\partial_r}\partial_r = 0, \qquad \nabla_{\partial_s}\partial_r = \nabla_{\partial_r}\partial_s = \tanh r\;\partial_s, \qquad \nabla_{\partial_s}\partial_s = -\sinh r\cosh r\;\partial_r.$$

By definition, $(\delta I, I\!I) = 2\,I^{rr}I^{ss}\,\delta I_{rs}\,I\!I_{rs} = 2\eta'\,I\!I_{rs}$. If we set $f(r) := \int_0^{L_\alpha} I\!I_{rs}\,\mathrm{d}s$, then, integrating by parts and recalling that $\eta(\varepsilon) = 0$, we get

$$\begin{aligned} \mathrm{d}V_k^*(\delta g) &= -\frac{1}{2}\int_0^{\varepsilon} \eta'(r) f(r)\cosh r\,\mathrm{d}r \\ &= \frac{1}{2}f(0) + \frac{1}{2}\int_0^{\varepsilon} \eta(r)(f'(r)\cosh r + f(r)\sinh r)\,\mathrm{d}r \end{aligned} \tag{$\star$}$$

Being the second fundamental form a Codazzi tensor, we have $(\nabla_{\partial_r} I\!I)_{rs} = (\nabla_{\partial_s} I\!I)_{rr}$. Using the expressions of the connection given above, this relation can be rephrased as $\partial_r I\!I_{rs} = \partial_s I\!I_{rr} - \tanh r\; I\!I_{sr}$. Hence we deduce

$$f'(r) = \int_0^{L_\alpha} (\partial_s I\!I_{rr} - \tanh r\; I\!I_{sr})\,\mathrm{d}s = -\tanh r\; f(r),$$

where the first summand vanishes because α is a closed curve. Therefore the integral in the relation ($\star$) equals 0, and we end up with the equation

$$\mathrm{d}V_k^*(\delta g) = \frac{1}{2}\int_0^{L_\alpha} I\!I_{rs}\,\mathrm{d}s = -\frac{1}{2}\int_0^{L_\alpha} I(B\alpha', J\alpha')\,\mathrm{d}s \tag{10}$$

since $\partial_r|_{r=0} = V = -J\alpha'$ and $\partial_s|_{r=0} = \alpha'$.

Now we apply Lemma 4.8 to α, the hyperbolic metrics $h = -k\, I$, $h' = -\frac{k}{k+1}\, I\!I\!I$ and the operator $b = \frac{1}{\sqrt{k+1}} B$ (here B is the shape operator of Σ_k), obtaining

$$\mathrm{d}(L_\alpha)_h\,(l^1(h,h')) = -\frac{1}{2}\sqrt{-\frac{k}{k+1}}\int_0^{L_\alpha} I(B\alpha', J\alpha')\,\mathrm{d}s\,.$$

This relation, combined with (10), proves that

$$\mathrm{d}V_k^*\,(\delta g) = \sqrt{-\frac{k+1}{k}}\;\mathrm{d}(L_\alpha)_h\,(l^1(h,h'))$$

By the work of Wolpert [Wol83], we have $\mathrm{d}L_\alpha = \hat{\omega}_{WP}(\cdot, e_\alpha)$, which proves relation (9), and therefore the statement. □

Since the complex landislide is holomorphic with respect to the complex structure of $\mathcal{T}^2$, an equivalent way to state Proposition 4.9 is the following:

Proposition 4.10. *Let M be a quasi-Fuchsian manifold and let h_k, $h'_k \in \mathcal{T}^2$ denote the hyperbolic metrics $-k\, I_k$ and $-k(k+1)^{-1}\, I\!I\!I_k$. Then the Weil-Petersson gradient of V_k^* coincides, up to a multiplicative factor, with the infinitesimal grafting with respect to the couple (h_k, h'_k). In other words,*

$$\mathrm{grad}_{WP}\, V_k^* = \sqrt{-\frac{k+1}{k}}\;\frac{\mathrm{d}}{\mathrm{d}s}\;\mathrm{sgr}_s(h_k, h'_k)\big|_{s=0}\,.$$

The behavior of the third fundamental forms $I\!I\!I_k$ of the k-surfaces, as k approaches -1, is well understood and described by the following Theorem:

Theorem 4.11. *Let $(E_n)_n$ be a sequence of hyperbolic ends converging to an hyperbolic end E homeomorphic to $\Sigma \times \mathbb{R}_{\geq 0}$, and let $(k_n)_n$ be any decreasing sequence of numbers converging to -1. Then $\ell_{I\!I\!I_n}$ converges to $\iota(\mu, \cdot)$, where $I\!I\!I_n$ denotes the third fundamental form of the k_n-surface of E_n, and μ is the bending measured lamination of the concave boundary of E.*

Remark 4.12. Theorem 4.11 is in fact a restatement of [Bel17, Theorem 2.10]. In [Bel17] the author works with *maximal global hyperbolic spatially compact* (MGHC) *de Sitter spacetimes*, which connect to the world of hyperbolic ends through the duality between the de Sitter and the hyperbolic space-forms, as observed by Mess [Mes07]. In particular, this phenomenon allowed Barbot, Béguin, and Zeghib [BBZ11] to give an alternative proof of the existence of the foliation by k-surfaces.

Proof of Proposition 4.3. Let $(M_n)_n$ be a sequence of quasi-Fuchsian manifolds converging to M, and let $(k_n)_n$ be a decreasing sequence converging to -1. We denote my m_n and m'_n the isotopy classes of the hyperbolic metrics

$$h_n := -k_n I_{k_n}, \qquad h'_n := -\frac{k_n}{1+k_n} I\!I\!I_{k_n},$$

where I_{k_n} and $I\!I\!I_{k_n}$ are the first and second fundamental forms of the k_n-surface $\Sigma^+_{k_n} \sqcup \Sigma^-_{k_n}$ sitting inside M_n. The k_n-surface is at distance $< \operatorname{arctanh}(\sqrt{k_n+1})$ from the convex core of M_n (apply the same argument of [BMS13, Lemma 6.14] in the hyperbolic setting), therefore the metrics m_n converge to the metric m on the boundary of the convex core of M. If we take

$$\theta_n := \sqrt{-\frac{1+k_n}{k_n}},$$

then, by Theorem 4.11, the length spectrum of $\theta_n \ell_{m'_n}$ converges to the bending measure μ of the boundary of the convex core of M. Therefore, applying Theorem 4.6 we obtain that

$l_1(m_n, m_n')|_{\Psi_{k_n}(M_n)}$ converges to $1/2\ e_\mu|_m$. Combining this with Proposition 4.9, we prove that

$$\lim_{n\to\infty} \mathrm{d}V^*_{k_n} \circ \Psi_{k_n}(M_n) = \frac{1}{2}\hat{\omega}_{WP}(e_\mu, \cdot) = -\frac{1}{2}\,\mathrm{d}(L_\mu)_m(\cdot),$$

where the last step follows from [Wol83]. This concludes the proof. □

REFERENCES

[BBB19] Martin Bridgeman, Jeffrey F. Brock, and Kenneth Bromberg. "Schwarzian derivatives, projective structures, and the Weil-Petersson gradient flow for renormalized volume". *Duke Math. J.* 168.5 (2019), pp. 867–896.

[BBZ11] Thierry Barbot, François Béguin, and Abdelghani Zeghib. "Prescribing Gauss curvature of surfaces in 3-dimensional spacetimes: application to the Minkowski problem in the Minkowski space". *Ann. Inst. Fourier (Grenoble)* 61.2 (2011), pp. 511–591.

[BCY16] Martin Bridgeman, Richard D. Canary, and Andrew Yarmola. "An improved bound for Sullivan's convex hull theorem". *Proc. Lond. Math. Soc. (3)* 112.1 (2016), pp. 146–168.

[Bel17] Mehdi Belraouti. "Asymptotic behavior of Cauchy hypersurfaces in constant curvature space-times". *Geom. Dedicata* 190 (2017), pp. 103–133.

[Ber60] Lipman Bers. "Simultaneous uniformization". *Bull. Amer. Math. Soc.* 66 (1960), pp. 94–97.

[BMS13] Francesco Bonsante, Gabriele Mondello, and Jean-Marc Schlenker. "A cyclic extension of the earthquake flow I". *Geom. Topol.* 17.1 (2013), pp. 157–234.

[BMS15] Francesco Bonsante, Gabriele Mondello, and Jean-Marc Schlenker. "A cyclic extension of the earthquake flow II". *Ann. Sci. Éc. Norm. Supér. (4)* 48.4 (2015), pp. 811–859.

[BO04] Francis Bonahon and Jean-Pierre Otal. "Laminations measurées de plissage des variétés hyperboliques de dimension 3". *Ann. of Math. (2)* 160.3 (2004), pp. 1013–1055.

[Bon96] Francis Bonahon. "Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form". *Ann. Fac. Sci. Toulouse Math. (6)* 5.2 (1996), pp. 233–297.

[Bon98a] Francis Bonahon. "A Schläfli-type formula for convex cores of hyperbolic 3-manifolds". *J. Differential Geom.* 50.1 (1998), pp. 25–58.

[Bon98b] Francis Bonahon. "Variations of the boundary geometry of 3-dimensional hyperbolic convex cores". *J. Differential Geom.* 50.1 (1998), pp. 1–24.

[Bro03] Jeffrey F. Brock. "The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores". *J. Amer. Math. Soc.* 16.3 (2003), pp. 495–535.

[CEM06] Richard D. Canary, David Epstein, and Albert Marden, eds. *Fundamentals of hyperbolic geometry: selected expositions*. Vol. 328. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2006, pp. xii+335.

[GL00] Frederick P. Gardiner and Nikola Lakic. *Quasiconformal Teichmüller theory*. Vol. 76. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2000, pp. xx+372.

[Ker85] Steven P. Kerckhoff. "Earthquakes are analytic". *Comment. Math. Helv.* 60.1 (1985), pp. 17–30.

[KS08] Kirill Krasnov and Jean-Marc Schlenker. "On the renormalized volume of hyperbolic 3-manifolds". *Comm. Math. Phys.* 279.3 (2008), pp. 637–668.

[KS09] Kirill Krasnov and Jean-Marc Schlenker. "A symplectic map between hyperbolic and complex Teichmüller theory". *Duke Math. J.* 150.2 (2009), pp. 331–356.

[Lab91] François Labourie. "Problème de Minkowski et surfaces à courbure constante dans les variétés hyperboliques". *Bull. Soc. Math. France* 119.3 (1991), pp. 307–325.

[Lab92] François Labourie. "Surfaces convexes dans l'espace hyperbolique et $\mathbf{C}P^1$-structures". *J. London Math. Soc. (2)* 45.3 (1992), pp. 549–565.

[Lin74] Michele Linch. "A comparison of metrics on Teichmüller space". *Proc. Amer. Math. Soc.* 43 (1974), pp. 349–352.

[Maz18] Filippo Mazzoli. "The dual Bonahon-Schl\"afli formula". *arXiv e-prints*, arXiv:1808.08936 (Aug. 2018), arXiv:1808.08936. arXiv: `1808 . 08936 [math.DG]`.

[McM98] Curtis T. McMullen. "Complex earthquakes and Teichmüller theory". *J. Amer. Math. Soc.* 11.2 (1998), pp. 283–320.

[Mes07] Geoffrey Mess. "Lorentz spacetimes of constant curvature". *Geom. Dedicata* 126 (2007), pp. 3–45.

[Riv86] Igor Rivin. *On Geometry of convex polyhedra in Hyperbolic 3-space*. Thesis (Ph.D.) – Princeton University. ProQuest LLC, Ann Arbor, MI, 1986, p. 51.

[RS99] Igor Rivin and Jean-Marc Schlenker. "The Schläfli formula in Einstein manifolds with boundary". *Electron. Res. Announc. Amer. Math. Soc.* 5 (1999), pp. 18–23.

[Sch02] J.-M. Schlenker. "Hypersurfaces in H^n and the space of its horospheres". *Geom. Funct. Anal.* 12.2 (2002), pp. 395–435.

[Sch06] Jean-Marc Schlenker. "Hyperbolic manifolds with convex boundary". *Invent. Math.* 163.1 (2006), pp. 109–169.

[Sch13] Jean-Marc Schlenker. "The renormalized volume and the volume of the convex core of quasifuchsian manifolds". *Math. Res. Lett.* 20.4 (2013), pp. 773–786.

[Sch17] Jean-Marc Schlenker. "Notes on the Schwarzian tensor and measured foliations at infinity of quasifuchsian manifolds". *ArXiv e-prints* (Aug. 2017). arXiv: `1708.01852 [math.GT]`.

[Sch93] Richard M. Schoen. "The role of harmonic mappings in rigidity and deformation problems". *Complex geometry (Osaka, 1990)*. Vol. 143. Lecture Notes in Pure and Appl. Math. Dekker, New York, 1993, pp. 179–200.

[Sul81] Dennis P. Sullivan. "Travaux de Thurston sur les groupes quasi-fuchsiens et les variétés hyperboliques de dimension 3 fibrées sur S^1". *Bourbaki Seminar, Vol. 1979/80*. Vol. 842. Lecture Notes in Math. Springer, Berlin-New York, 1981, pp. 196–214.

[Tro92] Anthony J. Tromba. *Teichmüller theory in Riemannian geometry*. Lectures in Mathematics ETH Zürich. Lecture notes prepared by Jochen Denzler. Birkhäuser Verlag, Basel, 1992, p. 220.

[Wol83] Scott A. Wolpert. "On the symplectic geometry of deformations of a hyperbolic surface". *Ann. of Math. (2)* 117.2 (1983), pp. 207–234.

[Wol87] Scott A. Wolpert. "Geodesic length functions and the Nielsen problem". *J. Differential Geom.* 25.2 (1987), pp. 275–296.

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