On the single-peakedness property

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Single-peaked orderings

Motivating example (Romero, 1978)

Suppose you are asked to order the following six objects in decreasing preference:

$$\begin{aligned} a_1 &: \quad \text{0 sandwich} \\ a_2 &: \quad \text{1 sandwich} \\ a_3 &: \quad \text{2 sandwiches} \\ a_4 &: \quad \text{3 sandwiches} \\ a_5 &: \quad \text{4 sandwiches} \\ a_6 &: \quad \text{more than 4 sandwiches} \end{aligned}$$

We write $a_i \prec a_j$ if a_i is preferred to a_j

Single-peaked orderings

a_1 : 0 sandwich
a_2 : 1 sandwich
a_3 : 2 sandwiches
a_4 : 3 sandwiches
a_5 : 4 sandwiches
a_6 : more than 4 sandwiches

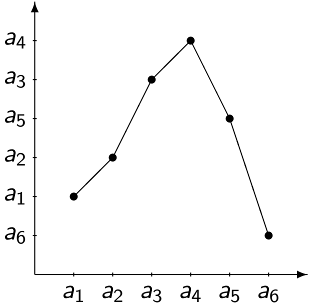
- after a good lunch: $a_1 \prec a_2 \prec a_3 \prec a_4 \prec a_5 \prec a_6$
- if you are starving: $a_6 \prec a_5 \prec a_4 \prec a_3 \prec a_2 \prec a_1$
- a possible intermediate situation: $a_4 \prec a_3 \prec a_5 \prec a_2 \prec a_1 \prec a_6$
- a quite unlikely preference: $a_6 \prec a_5 \prec a_2 \prec a_1 \prec a_3 \prec a_4$

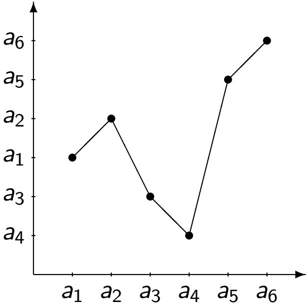
Single-peaked orderings

Let us represent graphically the latter two preferences with respect to the reference ordering $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$

$a_4 \prec a_3 \prec a_5 \prec a_2 \prec a_1 \prec a_6$

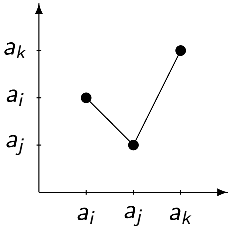
$a_6 \prec a_5 \prec a_2 \prec a_1 \prec a_3 \prec a_4$





Single-peaked orderings

Definition. (Black, 1948)
Let $\leq$ and $\preceq$ be total orderings on $X_n = \{a_1, \ldots, a_n\}$.
Then $\preceq$ is said to be *single-peaked for* $\leq$ if the following patterns are forbidden



Mathematically:

$$a_i < a_j < a_k \quad \Longrightarrow \quad a_j \prec a_i \quad \text{or} \quad a_j \prec a_k$$

Single-peaked orderings

$$a_i < a_j < a_k \quad \Longrightarrow \quad a_j \prec a_i \quad \text{or} \quad a_j \prec a_k$$

Let us assume that $X_n = \{a_1, \ldots, a_n\}$ is endowed with the ordering $a_1 < \cdots < a_n$

For $n = 4$

$a_1 \prec a_2 \prec a_3 \prec a_4$	$a_4 \prec a_3 \prec a_2 \prec a_1$
$a_2 \prec a_1 \prec a_3 \prec a_4$	$a_3 \prec a_2 \prec a_1 \prec a_4$
$a_2 \prec a_3 \prec a_1 \prec a_4$	$a_3 \prec a_2 \prec a_4 \prec a_1$
$a_2 \prec a_3 \prec a_4 \prec a_1$	$a_3 \prec a_4 \prec a_2 \prec a_1$

There are 2^{n-1} total orderings $\preceq$ on X_n that are single-peaked for $\leq$

Weak orderings

Recall that a *weak ordering* (or *total preordering*) on X_n is a binary relation $\precsim$ on X_n that is total and transitive.

Defining a weak ordering on X_n amounts to defining an ordered partition of X_n

$$C_1 \prec \cdots \prec C_k$$

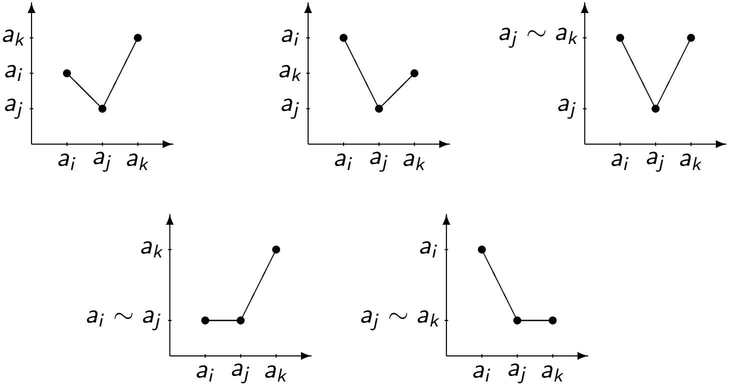
where $C_1, \ldots, C_k$ are the equivalence classes defined by $\sim$

For $n = 3$, we have 13 weak orderings

$$\begin{array}{lll}
a_1 \prec a_2 \prec a_3 & a_1 \sim a_2 \prec a_3 & a_1 \sim a_2 \sim a_3 \\
a_1 \prec a_3 \prec a_2 & a_1 \prec a_2 \sim a_3 & \\
a_2 \prec a_1 \prec a_3 & a_2 \prec a_1 \sim a_3 & \\
a_2 \prec a_3 \prec a_1 & a_3 \prec a_1 \sim a_2 & \\
a_3 \prec a_1 \prec a_2 & a_1 \sim a_3 \prec a_2 & \\
a_3 \prec a_2 \prec a_1 & a_2 \sim a_3 \prec a_1 &
\end{array}$$

Single-plateaued weak orderings

Definition. (Black, 1948)
Let $\leq$ be a total ordering on X_n and let $\precsim$ be a weak ordering on X_n. Then $\precsim$ is said to be *single-plateaued for* $\leq$ if the following patterns are forbidden



Single-plateaued weak orderings

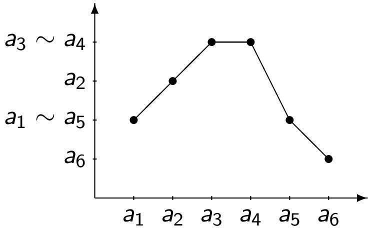
Mathematically:

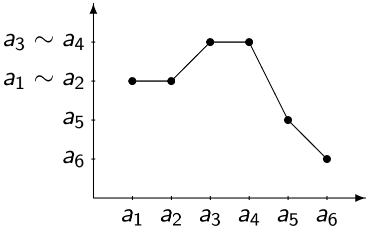
$$a_i < a_j < a_k \quad \Longrightarrow \quad a_j \prec a_i \quad \text{or} \quad a_j \prec a_k \quad \text{or} \quad a_i \sim a_j \sim a_k$$

Examples

$a_3 \sim a_4 \prec a_2 \prec a_1 \sim a_5 \prec a_6$

$a_3 \sim a_4 \prec a_2 \sim a_1 \prec a_5 \prec a_6$





Single-plateaued weak orderings

$n \in \mathbb{N}$

$u(n)$: number of weak orderings on X_n that are single-plateaued for $\leq$ (OEIS: A048739)

Proposition (Couceiro,D.,Marichal, 2019)

We have the closed-form expression

$$2\,u(n)+1 \;=\; \tfrac{1}{2}(1+\sqrt{2})^{n+1}+\tfrac{1}{2}(1-\sqrt{2})^{n+1} \;=\; \textstyle\sum_{k\geq 0}\binom{n+1}{2k}\,2^k$$

$u(0)=0,\; u(1)=1,\; u(2)=3,\; u(3)=8,\; u(4)=20,\; \ldots$

Example. $u(3)=8$

$a_1 \prec a_2 \prec a_3$	$a_1 \sim a_2 \prec a_3$	$a_1 \sim a_2 \sim a_3$
$a_2 \prec a_1 \prec a_3$	$a_2 \prec a_3 \prec a_1$	$a_2 \prec a_1 \sim a_3$
$a_3 \prec a_2 \prec a_1$	$a_3 \sim a_2 \prec a_1$	

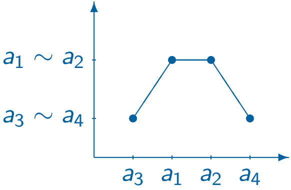
Single-plateaued weak orderings

Q: Given $\precsim$ is it possible to find $\leq$ for which $\precsim$ is single-plateaued?

Example: On $X_4 = \{a_1, a_2, a_3, a_4\}$ consider $\precsim$ and $\precsim'$ defined by

$$a_1 \sim a_2 \prec a_3 \sim a_4 \qquad \text{and} \qquad a_1 \prec' a_2 \sim' a_3 \sim' a_4$$

Yes! Consider $\leq$ defined by $a_3 < a_1 < a_2 < a_4$



No!

2-quasilinear weak orderings

Definition.
We say that $\precsim$ is *2-quasilinear* if

$$a \prec b \sim c \sim d \quad \Longrightarrow \quad a, b, c, d \text{ are not pairwise distinct}$$

Proposition (D., Marichal, Teheux)

We have

$$\precsim \text{ is 2-quasilinear} \quad \Longleftrightarrow \quad \exists \leq \text{ for which } \precsim \text{ is single-plateaued}$$

2-quasilinear weak orderings

$v(n)$: number of weak orderings on X_n that are 2-quasilinear (OEIS: A307005)

Proposition (D., Marichal, Teheux)

We have the closed-form expression

$$v(n) \; = \; \sum_{k=0}^{n} \frac{n!}{(n+1-k)!}\, G_k\, , \qquad n \geq 1,$$

where $G_n = \frac{\sqrt{3}}{3}\,(\frac{1+\sqrt{3}}{2})^n - \frac{\sqrt{3}}{3}\,(\frac{1-\sqrt{3}}{2})^n$.

$v(0) = 0,\ v(1) = 1,\ v(2) = 3,\ v(3) = 13,\ v(4) = 71,\ \ldots$

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