RANK n SWAPPING ALGEBRA FOR PGL_n FOCK–GONCHAROV $\mathcal{X}$ MODULI SPACE

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To William Goldman on the occasion of his sixtieth birthday

Abstract. The *rank n swapping algebra* is a Poisson algebra defined on the set of ordered pairs of points of the circle using linking numbers, whose geometric model is given by a certain subspace of $(\mathbb{K}^n \times \mathbb{K}^{n*})^r / \mathrm{GL}(n, \mathbb{K})$. For any ideal triangulation of D_k—a disk with k points on its boundary, using determinants, we find an injective Poisson algebra homomorphism from the fraction algebra generated by the Fock–Goncharov coordinates for $\mathcal{X}_{\mathrm{PGL}_n, D_k}$ to the rank n swapping multifraction algebra for $r = k \cdot (n-1)$ with respect to the (Atiyah–Bott–)Goldman Poisson bracket and the swapping bracket. This is the building block of the general surface case. Two such injective Poisson algebra homomorphisms related to two ideal triangulations $\mathcal{T}$ and $\mathcal{T}'$ are compatible with each other under the flips.

1. Introduction

We study the *Goldman Poisson structure* using circle, linking numbers, determinants and ratios.

1.1. **Background and motivation.** Let $\mathcal{R}_{G,S}$ be the space of gauge equivalence classes of flat connections on a fixed principal G-bundle over S, where G is a reductive Lie group and S is a connected oriented closed Riemann surface of genus $g > 1$. In early 80s, Atiyah and Bott [AB83] constructed a symplectic structure ω on $\mathcal{R}_{G,S}$ by symplectic reduction from infinite dimensional symplectic manifold $\mathcal{M}_{G,S}$ of all such connections via the moment map given by the curvature. From another point of view, the space $\mathcal{R}_{G,S}$ is $\mathrm{Hom}(\pi_1(S), G)/G$ using the monodromy representation of $\pi_1(S)$ with respect to the connection, where G acts by conjugation. Then Goldman [G84] identified the tangent space of $\mathrm{Hom}(\pi_1(S), G)/G$ with the group cohomology $H^1(\pi_1(S), \mathfrak{g})$ and interpret the symplectic structure ω in terms of the intersection pairings on the surface S—the cup product. This construction has been extended to the case where the Riemann surface S has finitely many boundary components in [AM95] [GHJW97] and references therein, even with marked points on the boundary in [FR98].

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When S is a closed Riemann surface of genus $g > 1$, there is a special connected component $H_n(S)$ of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R}),S}$, containing all the n-Fuchsian representations, called *Hitchin component* [H92]. Here an n-Fuchsian representation is the composition of a discrete faithful representation from $\pi_1(S)$ to $\mathrm{PSL}(2,\mathbb{R})$ with an irreducible representation from $\mathrm{PSL}(2,\mathbb{R})$ to $\mathrm{PGL}(n,\mathbb{R})$. The Hitchin component $H_n(S)$ is so nice that the quotient in the sense of geometric invariant theory [MK94] is the same as its topological quotient.

Let E be a n-dimensional vector space and let Ω be a non-zero volume form of E. A *flag* F for PGL_n is a nested collection of vector subspaces in E

$$\{0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = E \mid \dim F_i = i\}$$

equipped with the volume form Ω. The *flag variety* $\mathcal{B}$ is the space of all flags. Labourie and Guichard [Gu08][L06] identified each element ρ in the Hitchin component $H_n(S)$ with a ρ-equivariant hyperconvex Frenet curve ξ_ρ from the boundary at infinity $\partial_\infty(\pi_1(S))$ of $\pi_1(S)$ to the flag variety $\mathcal{B}(\mathbb{R})$ up to diagonal action by projective transformations. This identification allows us to study the Goldman Poisson structure on $H_n(S)$ by studying the Goldman Poisson structure on the space $\mathcal{FR}_n$ of hyperconvex Frenet curves up to projective transformations. We write ξ_ρ as a $(n-1)$-tuple $(\xi_\rho^1, \cdots, \xi_\rho^{n-1})$, where ξ_ρ^i takes values in the Grassmannian $G_i(\mathbb{R}^n)$ for $i = 1, \cdots, n-1$. Let $\tilde{\xi}_\rho^1$ ($\tilde{\xi}_\rho^{n-1}$ resp.) be any lift of ξ_ρ (ξ_ρ^{n-1} resp.) with the values in $\mathbb{R}^n$ ($\mathbb{R}^{n*}$ resp.). For any four distinct points $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}$ in $\partial_\infty(\pi_1(S))$, Labourie [L07] defined a special function on the Hitchin component $H_n(S)$, called the *weak cross ratio*, defined as follows:

$$\mathbb{B}_\rho(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}) = \frac{\left\langle \tilde{\xi}_\rho^1(\mathbf{x}) \middle| \tilde{\xi}_\rho^{n-1}(\mathbf{z}) \right\rangle}{\left\langle \tilde{\xi}_\rho^1(\mathbf{x}) \middle| \tilde{\xi}_\rho^{n-1}(\mathbf{t}) \right\rangle} \cdot \frac{\left\langle \tilde{\xi}_\rho^1(\mathbf{y}) \middle| \tilde{\xi}_\rho^{n-1}(\mathbf{t}) \right\rangle}{\left\langle \tilde{\xi}_\rho^1(\mathbf{y}) \middle| \tilde{\xi}_\rho^{n-1}(\mathbf{z}) \right\rangle}.$$

Investigating the algebraic nature of weak cross ratios, Labourie [L18] introduced the swapping algebra to characterize the Goldman Poisson structure on $H_n(S)$ for any $n > 1$ and the second Adler–Gel'fand–Dickey Poisson structure [Ma78][Di97] (and references therein) via Drinfel'd–Sokolov reduction [DS81] on the space $Opers_n$ of $\mathrm{SL}(n,\mathbb{R})$-opers with trivial holonomy. Let us recall the swapping algebra as follows.

We represent an *ordered pair* (of points) (x, y) of a given set $\mathcal{P} \subseteq S^1$ by the expression xy, and we consider the commutative ring

$$\mathcal{Z}(\mathcal{P}) := \mathbb{K}[\{xy\}_{x,y\in\mathcal{P}}] / (\{xx\}_{x\in\mathcal{P}})$$

over a field $\mathbb{K}$ of characteristic zero. The ring $\mathcal{Z}(\mathcal{P})$ is equipped with the Poisson bracket $\{\cdot,\cdot\}$, called the *swapping bracket*, defined by extending to $\mathcal{Z}(\mathcal{P})$ the following formula on arbitrary generators rx, sy:

$$\{rx, sy\} = \mathcal{J}(rx, sy) \cdot ry \cdot sx, \tag{1}$$

using Leibniz's rule. We define the *linking number* $\mathcal{J}(rx, sy) \in \{0, \pm 1, \pm\frac{1}{2}\}$ on S^1 as in Figure 1 which only depends on the corresponding position of the four points. The *swapping algebra* of $\mathcal{P}$ is $(\mathcal{Z}(\mathcal{P}), \{\cdot,\cdot\})$. Then the *swapping multifraction algebra* $(\mathcal{B}(\mathcal{P}), \{\cdot,\cdot\})$ is the sub-fraction algebra of the swapping algebra $(\mathcal{Z}(\mathcal{P}), \{\cdot,\cdot\})$ generated by *cross fractions* like $\frac{xz}{xt} \cdot \frac{yt}{yz}$. By considering the homomorphism

$$\tau : \mathcal{B}(\mathcal{P}) \to C^\infty(H_n(S))$$

that sends $\frac{xz}{xt}\cdot\frac{yt}{yz}$ to $\mathbb{B}_\rho(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t})$, Labourie [L18] showed that τ is "asymptotically Poisson" with respect to the swapping bracket and the Goldman Poisson bracket.

However τ is not injective. To make the kernel of τ smaller, the *rank n swapping algebra* $(\mathcal{Z}_n(\mathcal{P}),\{\cdot,\cdot\})$ is introduced in [Su17]. Here $(\mathcal{Z}_n(\mathcal{P}),\{\cdot,\cdot\})$ is the quotient of the swapping algebra $(\mathcal{Z}(\mathcal{P}),\{\cdot,\cdot\})$ by the Poisson ideal $R_n(\mathcal{P})$ generated by

$$\left\{\det\left(x_iy_j\right)_{i,j=1}^{n+1}\in\mathcal{Z}(\mathcal{P})\mid x_1,\cdots,x_{n+1},y_1,\cdots,y_{n+1}\in\mathcal{P}\right\}.$$

The geometric model for $\mathcal{Z}_n(\mathcal{P})$ in [Su17, Section 4] arises naturally from the classical geometric invariant theory [W39]. When $\mathcal{P}=\{x_1,\cdots,x_r\}$, we associate a pair $(\mathfrak{a}_i,\mathfrak{b}_i)\in\mathbb{K}^n\times\mathbb{K}^{n*}$ to each x_i for $i=1,\cdots,r$. We consider the space $D_{n,r}=(\mathbb{K}^n\times\mathbb{K}^{n*})^r$ of r vectors $\mathfrak{a}_1,\cdots,\mathfrak{a}_r$ in $\mathbb{K}^n$ and r covectors $\mathfrak{b}_1,\cdots,\mathfrak{b}_r$ in $\mathbb{K}^{n*}$. For any $g\in\mathrm{GL}(n,\mathbb{K})$, the action of g on the vector $\mathfrak{a}_i$ is the left multiplication by g, the action of g on the covector $\mathfrak{b}_i$ is the right multiplication by g^{-1}. We define the product between a vector $\mathfrak{a}_i$ in $\mathbb{K}^n$ and a covector $\mathfrak{b}_j$ in $\mathbb{K}^{n*}$ by $\langle\mathfrak{a}_i|\mathfrak{b}_j\rangle:=\mathfrak{b}_j(\mathfrak{a}_i)$, which is $\mathrm{GL}(n,\mathbb{K})$ invariant. Then we associate each $\langle\mathfrak{a}_i|\mathfrak{b}_j\rangle$ to each pair $x_ix_j\in\mathcal{Z}_n(\mathcal{P})$. The geometric model for $(\mathcal{Z}_n(\mathcal{P}),\{\cdot,\cdot\})$ is

$$\mathcal{D}_{n,r}=\{(\mathfrak{a}_1,\mathfrak{b}_1,\cdots,\mathfrak{a}_r,\mathfrak{b}_r)\in D_{n,r}\mid\ <\mathfrak{a}_i|\mathfrak{b}_i>=0,i=1,\cdots,r\}/\,\mathrm{GL}(n,\mathbb{K}),\tag{2}$$

which is also equipped with the swapping bracket.

The *rank n swapping multifraction algebra* $(\mathcal{B}_n(\mathcal{P}),\{\cdot,\cdot\})$ is the sub-fraction algebra of the rank n swapping algebra $(\mathcal{Z}_n(\mathcal{P}),\{\cdot,\cdot\})$ generated by cross fractions. Then the homomorphism τ is changed into

$$\tau_n:\mathcal{B}_n(\mathcal{P})\to C^\infty\left(H_n(S)\right)$$

that sends $\frac{xz}{xt}\cdot\frac{yt}{yz}$ to $\mathbb{B}_\rho(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t})$. However τ_n is still not injective because of the $\pi_1(S)$ invariance of the weak cross ratios.

It motivates us to consider an injective Poisson homomorphism θ that sends a coordinate fraction ring of $H_n(S)$ to $\mathcal{B}_n(\mathcal{P})$ in Definition 5.1. Our crucial point for defining such homomorphism is that we characterize a pairing between a vector and a covector by a $(n\times n)$-determinant in $\mathcal{Z}_n(\mathcal{P})$ instead of an ordered pair in $\mathcal{Z}_n(\mathcal{P})$. It turns out that the Fock–Goncharov coordinates for $\mathcal{X}_{\mathrm{PGL}_n,\hat{S}}$ [FG06] work well, even for their corresponding quantized algebras [Su]. The Poisson homomorphism in [L18, Theorem 10.7.2] for $Opers_n$ is still Poisson after replacing an ordered pair in $\mathcal{Z}_n(\mathcal{P})$ by a $(n\times n)$-determinant as shown in Section 7.

Instead of understanding the Goldman Poisson structure as the cup product on the surface, the rank n swapping algebra provides another natural description using circle, linking numbers, determinants, ratios and classical geometric invariant ring.

1.2. **The main result.** *We use $\boldsymbol{x}$ for a vertex on the surface to distinguish it from an element x of $\mathcal{P}$ throughout this paper.*

Let D_k be a disk D with $k\geq 3$ points $m_b=\{\mathbf{s}\prec\mathbf{w}\prec\cdots\prec\mathbf{t}\prec\mathbf{s}\}$ on ∂D, where $\prec$ defines a cyclic order with respect to the anticlockwise orientation on the circle. In this case $\mathcal{X}_{\mathrm{PGL}_n,D_k}\cong\mathcal{B}^k/\,\mathrm{PGL}_n$ with respect to the diagonal action of PGL_n. Given an ideal triangulation $\mathcal{T}$ of D_k, the *n-triangulation* $\mathcal{T}_n$ is a subdivision of $\mathcal{T}$ such that each triangle of $\mathcal{T}$ is divided into n^2 triangles as in Figure 3. Fock and Goncharov (Definition 3.5) parameterize $\mathcal{X}_{\mathrm{PGL}_n,D_k}$ by assigning the coordinate X_V to each vertex V of certain subset of vertices of $\mathcal{T}_n$. Let $\mathcal{FX}(\mathcal{T}_n)$ be the fraction ring generated by these $\{X_V\}$ over the field $\mathbb{K}$. The *rank n Fock–Goncharov Poisson* *bracket* is given by

$$\{X_V,\ X_W\}_n = \varepsilon_{V,W} \cdot X_V \cdot X_W,$$

where by Figure 3

$$\varepsilon_{VW} = \#\{\ arrows\ from\ V\ to\ W\ \} - \#\{\ arrows\ from\ W\ to\ V\ \}. \tag{3}$$

The *rank n Fock–Goncharov algebra* is $(\mathcal{FX}(\mathcal{T}_n), \{\cdot,\cdot\}_n)$.

Given $\xi \in \mathcal{B}^k$, for any $\mathbf{r} \in m_b$, we choose a basis $\{\mathbf{r}_1, \cdots, \mathbf{r}_n\}$ of the flag $\xi(\mathbf{r})$ such that $\mathbf{r}_1, \cdots, \mathbf{r}_i$ span the i dimensional subspace $\xi^i(\mathbf{r})$ of $\xi(\mathbf{r})$. Then we choose a covector $\mathbf{r}_i^c$ such that $\langle \mathbf{r}_i | \mathbf{r}_i^c \rangle = 0$ for $i = 1, \cdots, n-1$. For $\mathcal{X}_{\mathrm{PGL}_n, D_k}$, the key observation for relating the rank n Fock–Goncharov algebra to the rank n swapping multifraction algebra is the following:

Any $\boldsymbol{r} \in m_b$ and $i \in \{1, \cdots, n-1\}$ provide us a pair

$$(\boldsymbol{r}_i, \boldsymbol{r}_i^c) \in E \times E^* \ \ such\ \ that\ \ \langle \boldsymbol{r}_i | \boldsymbol{r}_i^c \rangle = 0$$

which embeds $\mathcal{X}_{\mathrm{PGL}_n, D_k}$ into the subspace of $(E \times E^*)^{k(n-1)} / \mathrm{GL}_n$ subject to $\langle \boldsymbol{r}_i | \boldsymbol{r}_i^c \rangle = 0$. The induced Poisson structure on $\mathcal{X}_{\mathrm{PGL}_n, D_k}$ from the swapping bracket does not depend on the choice of bases of the flags.

Here each $\mathbf{r} \in m_b$ is related to $n-1$ elements in $E \times E^*$. Therefore we define

$$\mathcal{P} = \{s_{n-1} \prec \cdots \prec s_1 \prec w_{n-1} \prec \cdots \prec w_1 \prec \cdots \prec t_{n-1} \prec \cdots \prec t_1 \prec s_{n-1}\}$$

on S^1, where each $\mathbf{r} \in m_b$ corresponds to $n-1$ anticlockwise ordered points $r_{n-1}, \ldots, r_1$ nearby in $\mathcal{P}$ as in Figure 4.

Suppose that V is a vertex of the n-triangulation $\mathcal{T}_n$ related to the marked triangle $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ of the ideal triangulation $\mathcal{T}$ and a triple of non-negative integers (m, l, p) with $m + l + p = n$. Choose some bases

$$\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}, \ \ \{\mathbf{y}_1, \cdots, \mathbf{y}_n\}, \ \ \{\mathbf{z}_1, \cdots, \mathbf{z}_n\}.$$

for the flags $\xi(\mathbf{x})$, $\xi(\mathbf{y})$, $\xi(\mathbf{z})$ respectively. Fix a volume form Ω of E. Let

$$\Delta_V = \Omega\left(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_m \wedge \mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_l \wedge \mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_p\right).$$

Let $\mathcal{FA}_n$ be the fraction ring generated by all these determinants with the fixed bases of flags.

For any $d > 1$ and any $x_1, \cdots, x_d, y_1, \cdots, y_d \in \mathcal{P}$, let us adopt the notation

$$\Delta\left((x_1, \cdots, x_d), (y_1, \cdots, y_d)\right) := \det \begin{pmatrix} x_1 y_1 & \cdots & x_1 y_d \\ \cdots & \cdots & \cdots \\ x_d y_1 & \cdots & x_d y_d \end{pmatrix} \in \mathcal{Z}_n(\mathcal{P}). \tag{4}$$

Fixed a choice of distinct $u_1, \cdots, u_n \in \mathcal{P}$, the homomorphism χ_n from $\mathcal{FA}_n$ to $\mathcal{Q}_n(\mathcal{P})$ is defined by extending the following formula on the generators to $\mathcal{FA}_n$ using Leibniz's rule

$$\chi_n(\Delta_V) = \Delta\left((x_1, \cdots, x_m, y_1, \cdots, y_l, z_1, \cdots, z_p), (u_1, \cdots u_n)\right). \tag{5}$$

We define the homomorphism $\theta_{\mathcal{T}_n}$ from $\mathcal{FX}(\mathcal{T}_n)$ to $\mathcal{B}_n(\mathcal{P})$ by restricting the homomorphism χ_n to the fraction ring $\mathcal{FX}(\mathcal{T}_n)$. Then the homomorphism $\theta_{\mathcal{T}_n}$ does not depend on the choice of bases of flags and the choice of distinct $u_1, \cdots, u_n \in \mathcal{P}$. More explicitly, we have

$$\theta_{\mathcal{T}_n}(X_V) = \chi_n(X_V) = \chi_n(\prod_W \Delta_W^{\varepsilon_{VW}}).$$

Theorem 1.1. [MAIN RESULT THEOREM 5.5] *Given an ideal triangulation $\mathcal{T}$ of D_k, there is an injective Poisson homomorphism $\theta_{\mathcal{T}_n}$ from the rank n Fock–Goncharov algebra for the moduli space $\mathcal{X}_{\mathrm{PGL}_n,D_k}$ to the rank n swapping multifraction algebra $(\mathcal{B}_n(\mathcal{P}),\{\cdot,\cdot\})$, with respect to the Goldman Poisson bracket and the swapping bracket.*

The above theorem generalizes the result for $n=2$ in [Su17], for $n=3$ in Chapter 3 of [Su14]. Combining this theorem with the main result in [L18], we again relate the Fock–Goncharov Poisson structure with the Goldman Poisson structure.

To prove the main result, we introduce the quotient of two $(n\times n)$-determinants

$$\frac{\Delta\left((x_1,\cdots,x_n),(u_1,\cdots,u_n)\right)}{\Delta\left((y_1,\cdots,y_n),(u_1,\cdots,u_n)\right)}$$

with the same right side n-tuple of distinct points $(u_1,\cdots,u_n)$, called the $(n\times n)$-*determinant ratio* in the field of fractions $\mathcal{Q}_n(\mathcal{P})$ of $\mathcal{Z}_n(\mathcal{P})$. It has the nice property that it does not depend the choice of $(u_1,\cdots,u_n)$, due to the $R_n(\mathcal{P})$ relations. Then we realize that any $\theta_{\mathcal{T}_n}(X_V)$ is a product of two or three $(n\times n)$-determinant ratios which can be represented by two or three oriented edges of $\mathcal{T}_n$ as in Figure 7. Such $(n\times n)$-determinant ratio is called *oriented edge ratio*. As a consequence $\theta_{\mathcal{T}_n}$ does not depend on the choice of $u_1,\cdots,u_n\in\mathcal{P}$.

By Lemma 2.6, the swapping bracket

$$\{ab,\Delta((x_1,\cdots,x_n),(y_1,\cdots,y_n))\}=\Delta^R(ab)=\Delta^L(ab)$$

can be expressed in two different ways regarding to the right or left side of $\overrightarrow{ab}$ in Figure 2. Then we compute the swapping bracket between two $(n\times n)$-determinants in our main proposition 5.7, which is the most technical part for proving the main theorem. Let us fix some notations

$$[A,B]:=\frac{\{A,B\}}{AB},\quad w^i:=w_1,\cdots,w_i. \tag{6}$$

We stress the fact that the formula

$$\begin{aligned}&\left[\Delta\left(\left(x^m,y^l,z^p\right),(v^n)\right),\Delta\left(\left(x^{m'},y^{l'},z^{p'}\right),(u^n)\right)\right]\\ =\;&\frac{1}{2}\cdot\min\{m,m'\}-\frac{1}{2}\cdot\min\{l,l'\}-\frac{1}{2}\cdot\min\{p,p'\}\end{aligned}$$

in Proposition 5.7 strictly depends on the cyclic order in Figure 5 and the condition (*) $l\geq l'$ or $p\leq p'$ is strict. Essentially, the $+$ and $-$ sign before $\frac{1}{2}\cdot\min$ is due to our cyclic order. Then we obtain Proposition 5.12, which shows that the $[\cdot,\cdot]$ bracket between any two oriented edge ratios belongs to $\{-\frac{1}{2},0,\frac{1}{2}\}$ and only depends the corresponding positions of two oriented edge ratios as in Figure 8, 9. Our oriented edge ratios correspond to the generalized Kashaev coordinates [K98] [Ki16], but their Poisson bracket is different from the swapping bracket for two oriented edges lying on two different ideal triangles. In the proof of Proposition 5.12, we choose the right side n-tuples wisely for the $(n\times n)$-determinant ratios in each case in order to use Proposition 5.7 under the condition (*). Finally by checking all the cases, we finish the proof of the main theorem.

1.3. **Compatible.** For D_k, we can transform any ideal triangulation $\mathcal{T}$ to any other ideal triangulation $\mathcal{T}'$ by a finite sequence of flips. In Proposition 6.1, we prove that the corresponding two injective Poisson homomorphisms $\theta_{\mathcal{T}_n}$ and $\theta_{\mathcal{T}'_n}$ are compatible by a generalized Plücker relation in $\mathcal{Z}_n(\mathcal{P})$. It is realted to the result for $n=2$ in

[Su17, Lemma 5.5] where the cross fractions are used to define the homomorphism $\theta_{\mathcal{T}_n}$. As a corollary, the rank n Fock–Goncharov Poisson bracket does not depend on the ideal triangulation $\mathcal{T}$. Note that these properties only become possible after dividing $(\mathcal{Z}(\mathcal{P}), \{\cdot,\cdot\})$ by $R_n(\mathcal{P})$.

1.4. **From disk to surface.** Let $\hat{S} = (S = S_{g,m}, \emptyset)$ with $2g-2+m>0$. In this case, we obtain a homotopy equivalent surface S' by shrinking holes on S to punctures. The ideal triangulation of $\hat{S}$ is the ideal triangulation of S' with vertices at the punctures. Let $\widetilde{\mathcal{T}_n}$ be all the lifts of the n-triangulation $\mathcal{T}_n$ into the universal cover of the surface S'. The *Farey set* $\mathcal{F}_\infty(S)$ is the countably infinite collection of vertices of $\widetilde{\mathcal{T}}$, equipped with a cyclic order on the boundary at infinity $\partial_\infty \pi_1(S')$. Let $\mathcal{P}$ be a cyclic subset of S^1 obtained by splitting each point of $\mathcal{F}_\infty(S)$ into $n-1$ points nearby in S^1 as we did for D_k. By our main theorem 5.5, the injective homomorphism $\theta_{\widetilde{\mathcal{T}_n}}$ from $\mathcal{FX}(\widetilde{\mathcal{T}_n})$ to $\mathcal{B}_n(\mathcal{P})$ is Poisson with respect to the rank n Fock–Goncharov Poisson bracket and the swapping bracket. Thus the swapping bracket identifies with the rank n Fock–Goncharov Poisson bracket on the universal cover $\mathrm{Conf}_{\mathcal{F}_\infty(S),n} \cong \mathcal{B}^{\mathcal{F}_\infty(S)} / \mathrm{PGL}_n$. Moreover, $\pi_1(S)$ acts on both $\mathcal{FX}(\widetilde{\mathcal{T}_n})$ and $\theta_{\widetilde{\mathcal{T}_n}}(\mathcal{FX}(\widetilde{\mathcal{T}_n}))$ through the $\pi_1(S)$ action on the Farey set $\mathcal{F}_\infty(S)$, thus $\theta_{\widetilde{\mathcal{T}_n}}$ is $\pi_1(S)$-equivariant with respect to these actions. By [FG06, Lemma 1.1], $\mathcal{X}_{\mathrm{PGL}_n,\hat{S}} \cong \mathrm{Conf}^{\pi_1(S)}_{\mathcal{F}_\infty(S),n}$. Then the rank n Fock–Goncharov Poisson bracket on $\mathcal{X}_{\mathrm{PGL}_n,\hat{S}}$ is induced by the $\pi_1(S)$-equivariant homomorphism $\theta_{\widetilde{\mathcal{T}_n}}$ from the swapping bracket. This construction also works for $\hat{S} = (S, m_b)$ where $m_b \subset \partial S$ is a finite set since the cyclic order on the boundary at infinity $\partial_\infty \pi_1(S')$ induces a cyclic order on every lift of a boundary component containing marked points in m_b.

1.5. **From cross fractions to $(n \times n)$-determinant ratios.** Instead of characterizing the weak cross ratio by the cross fraction in the homomorphism τ_n, we use a product of two $(n \times n)$-determinant ratios to characterize the weak cross ratio. By Theorem 7.3, such characterization is compatible with the former with respect to the swapping bracket.

1.6. **Summary, further development.** Using $(n \times n)$-determinant ratios instead of cross fractions, we provide the following understanding of "the space of all cross ratios" proposed by Labourie. Using the swapping bracket, we define the Poisson structure on a subspace of $(\mathbb{K}^n \times \mathbb{K}^{n*})^{\#\mathcal{P}} / \mathrm{GL}(n,\mathbb{K})$. It induces a Poisson bracket $\omega_{\mathcal{SW}}$ on the sub fraction ring $\mathcal{DR}(\mathcal{FR}_n)$ of functions of $\mathcal{FR}_n$ (space of hyperconvex Frenet curves up to projective transformations) generated by all elements corresponding to $(n \times n)$-determinant ratios. By our main theorem, the Fock–Goncharov coordinate fraction ring of $\mathcal{X}_{\mathrm{PGL}_n,\hat{S}}$ is Poisson embedded into $\mathcal{DR}(\mathcal{FR}_n)$ with respect to the Goldman Poisson structure and $\omega_{\mathcal{SW}}$. On the other hand, combing [L18, Theorem 10.7.2] and Theorem 7.3, the fraction ring of acceptable observables on the space $Opers_n$ of $\mathrm{SL}(n,\mathbb{R})$-opers with trivial holonomy is also Poisson embedded into $\mathcal{DR}(\mathcal{FR}_n)$ with respect to the second Gel'fand-Dickey Poisson structure and $\omega_{\mathcal{SW}}$.

In a forthcoming paper [Su], we define the quantized rank n swapping algebra $\mathcal{Z}^q_n(\mathcal{P})$ generated over $\mathbb{K}_q = \mathbb{K}[q, q^{-1}]$ by non commutative indeterminates. Given any ideal triangulation $\mathcal{T}$, we give an injective homomorphism $\theta^q_{\mathcal{T}_n}$ from the quantized Fock–Goncharov coordinate fraction algebra [FG09] for $\mathcal{X}_{\mathrm{PGL}_n,D_k}$ to the quantized rank n swapping multifraction algebra $\mathcal{B}_n^q(\mathcal{P})$. Moreover, we show that any two homomorphisms $\theta^q_{\mathcal{T}_n}$ and $\theta^q_{\mathcal{T}'_n}$ are compatible using Skein relations.

Moreover, we suggest the following research directions.

(1) In the conference organized by Goldman at Maryland University in 2016, Labourie [L] described a compactification of the Hitchin component $H_n(S)$ using a tropical version of cross ratio and the rank n swapping algebra. It is interesting to relate this compactification to the compactification of cluster $\mathcal{X}$ variety at infinity in [FG16] through the injective Poisson homomorphism $\theta_{\mathcal{T}_n}$.

(2) It is interesting to investigate the Fock–Goncharov Poisson structures for the surface S with bordered cusps as in [CM17], [CMR17] via the rank n swapping algebra, where they build very interesting links with Painlevé-type equations.

2. Rank n swapping algebra

In this section, we recall the swapping algebra [L18] and the rank n swapping algebra [Su17]. Lemma 2.6 ([Su17, Lemma 3.5, Remark 3.6]) is the key technical formula to use for proving our main proposition 5.7.

2.1. Swapping algebra.

Definition 2.1. [LINKING NUMBER] *Let (r, x, s, y) be a quadruple of points in $\mathcal{P} \subset S^1$. We represent an ordered pair (r, x) of $\mathcal{P}$ by the expression rx. Let o be any point different from $r, x, s, y \in S^1$. Let σ be a homeomorphism from $S^1 \backslash o$ to $\mathbb{R}$ with respect to the anticlockwise orientation of S^1. Let $\triangle(a) = -1; 0; 1$ whenever $a < 0$; $a = 0$; $a > 0$ respectively.*

The linking number *between rx and sy is*

$$\begin{aligned}\mathcal{J}(rx, sy) = &\frac{1}{2} \cdot \triangle(\sigma(r) - \sigma(x)) \cdot \triangle(\sigma(r) - \sigma(y)) \cdot \triangle(\sigma(y) - \sigma(x)) \\ &- \frac{1}{2} \cdot \triangle(\sigma(r) - \sigma(x)) \cdot \triangle(\sigma(r) - \sigma(s)) \cdot \triangle(\sigma(s) - \sigma(x)).\end{aligned} \tag{7}$$

In fact, the value of $\mathcal{J}(rx, sy)$ belongs to $\{0, \pm 1, \pm\frac{1}{2}\}$, and does not depend on the choice of the point o and depends only on the relative positions of r, x, s, y. In Figure 1, we describe five possible values of $\mathcal{J}(rx, sy)$.

For $\mathcal{P}$ a cyclic subset of S^1, we represent an ordered pair (r, x) of $\mathcal{P}$ by the expression rx. Then we consider the associative commutative ring

$$\mathcal{Z}(\mathcal{P}) := \mathbb{K}[\{xy\}_{\forall x, y \in \mathcal{P}}]/\{xx | \forall x \in \mathcal{P}\}$$

over a field $\mathbb{K}$ of characteristic zero, where $\{xy\}_{\forall x,y\in\mathcal{P}}$ are the set of variables. Then we equip $\mathcal{Z}(\mathcal{P})$ with a swapping bracket, defined as follows.

Definition 2.2. [SWAPPING BRACKET [L18, $\alpha = 0$ case]] *The swapping bracket over $\mathcal{Z}(\mathcal{P})$ is defined by extending the following formula on arbitrary generators rx, sy to $\mathcal{Z}(\mathcal{P})$ using* Leibniz's rule

$$\{rx, sy\} = \mathcal{J}(rx, sy) \cdot \cdot ry \cdot sx.$$

By direct computations, Labourie proved the following theorem.

Theorem 2.3. [LABOURIE [L18]] *The swapping bracket is Poisson.*

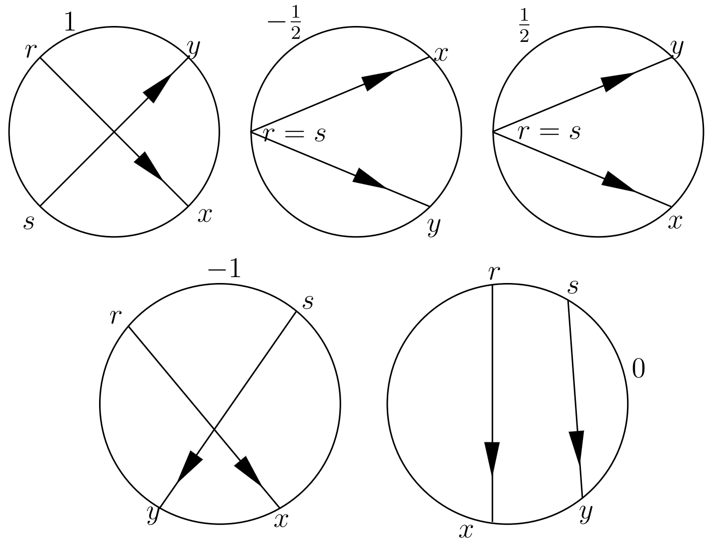

FIGURE 1. Linking number $\mathcal{J}(rx, sy)$ between rx and sy

Let $\mathcal{Q}(\mathcal{P})$ be the field of fractions of $\mathcal{Z}(\mathcal{P})$. We extend the swapping bracket to $\mathcal{Q}(\mathcal{P})$ by

$$\{rx, \frac{1}{sy}\} = -\frac{\{rx, sy\}}{sy^2}.$$

Definition 2.4. *The* cross fraction *determined by* (x, y, z, t) *is the element*

$$\frac{xz}{xt} \cdot \frac{yt}{yz}.$$

Let $\mathcal{B}(\mathcal{P})$ be the subring of $\mathcal{Q}(\mathcal{P})$ generated by cross fractions.

The swapping fraction (multifraction resp.) algebra of $\mathcal{P}$ *is the ring $\mathcal{Q}(\mathcal{P})$ ($\mathcal{B}(\mathcal{P})$ resp.) equipped with the swapping bracket, denoted by $(\mathcal{Q}(\mathcal{P}), \{\cdot,\cdot\})$ $((\mathcal{B}(\mathcal{P}), \{\cdot,\cdot\})$ resp.).*

By [Su17, Proposition 2.9], the ring $\mathcal{B}(\mathcal{P})$ is closed under $\{\cdot,\cdot\}$.

2.2. Rank n swapping algebra.

Definition 2.5. [THE RANK n SWAPPING RING $\mathcal{Z}_n(\mathcal{P})$] *Recall the notation in Equation (4). For $n \geq 2$, let $R_n(\mathcal{P})$ be the ideal of $\mathcal{Z}(\mathcal{P})$ generated by*
$\{D \in \mathcal{Z}(\mathcal{P}) \mid D = \Delta\left((x_1, \cdots, x_{n+1}), (y_1, \cdots, y_{n+1})\right), \forall x_1, , x_{n+1}, y_1, \cdots, y_{n+1} \in \mathcal{P}\}.$

The rank n swapping ring *$\mathcal{Z}_n(\mathcal{P})$ is the quotient ring $\mathcal{Z}(\mathcal{P})/R_n(\mathcal{P})$.*

Lemma 2.6. [[Su17, Lemma 3.5, Remark 3.6]] *For any integer $m \geq 2$, suppose $x_1, \cdots, x_m$ ($y_1, \cdots, y_m$ resp.) in $\mathcal{P}$ are mutually distinct and anticlockwise ordered (m, x_i, y_i used here do not involve m, x_i, y_i in any other places). Assume that a, b belong to $\mathcal{P}$ and $x_1, \cdots, x_l, y_1, \cdots, y_k$ are on the* ***right*** *side of the oriented edge $\overrightarrow{ab}$ (include coinciding with a or b) as illustrated in Figure 2. Let u (v resp.) be strictly*

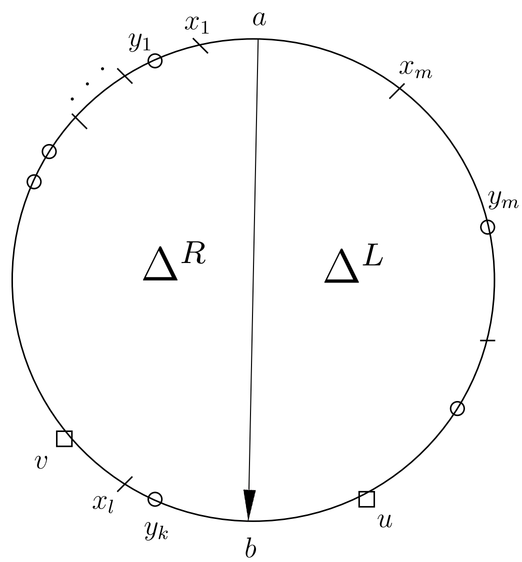

FIGURE 2. $\{ab, \Delta((x_1, \cdots, x_m), (y_1, \cdots, y_m))\}$

on the left (right resp.) side of $\overrightarrow{ab}$. Let

(8)

$$\Delta^R(ab) = \sum_{d=1}^{l} \mathcal{J}(ab, x_d u) \cdot x_d b \cdot \Delta((x_1, \cdots, x_{d-1}, a, x_{d+1}, \cdots, x_m), (y_1, \cdots, y_m))$$

$$+ \sum_{d=1}^{k} \mathcal{J}(ab, u y_d) \cdot a y_d \cdot \Delta((x_1, \cdots, x_m), (y_1, \cdots, y_{d-1}, b, y_{d+1}, \cdots, y_m)),$$

(9)

$$\Delta^L(ab) = \sum_{d=k+1}^{m} \mathcal{J}(ab, x_d v) \cdot x_d b \cdot \Delta((x_1, \cdots, x_{d-1}, a, x_{d+1}, \cdots, x_m), (y_1, \cdots, y_m))$$

$$+ \sum_{d=l+1}^{m} \mathcal{J}(ab, v y_d) \cdot a y_d \cdot \Delta((x_1, \cdots, x_m), (y_1, \cdots, y_{d-1}, b, y_{d+1}, \cdots, y_m)),$$

then we have

$$\{ab, \Delta((x_1, \cdots, x_m), (y_1, \cdots, y_m))\} = \Delta^R(ab) = \Delta^L(ab).$$

The following proposition is a consequence of the above lemma.

Proposition 2.7. *The ideal $R_n(\mathcal{P})$ is a Poisson ideal of $\mathcal{Z}(\mathcal{P})$ with respect to the swapping bracket.*

Definition 2.8. [RANK n SWAPPING ALGEBRA OF $\mathcal{P}$] *The* rank n swapping algebra of $\mathcal{P}$ *is the ring $\mathcal{Z}_n(\mathcal{P}) = \mathcal{Z}(\mathcal{P})/R_n(\mathcal{P})$ equipped with the swapping bracket, denoted by $(\mathcal{Z}_n(\mathcal{P}), \{\cdot, \cdot\})$.*

By Theorem 4.7 in [Su17], $\mathcal{Z}_n(\mathcal{P})$ is an integral domain. Generators of $\mathcal{Z}_n(\mathcal{P})$ are non-zero divisors, so the cross fraction is well defined in the field of fractions of $\mathcal{Z}_n(\mathcal{P})$.

Let $\mathcal{Q}_n(\mathcal{P})$ be the field of fractions of $\mathcal{Z}_n(\mathcal{P})$. Let $\mathcal{B}_n(\mathcal{P})$ be the sub-fraction ring of $\mathcal{Z}_n(\mathcal{P})$ generated by cross fractions.

Definition 2.9. *Then, the* rank n swapping fraction (multifraction resp.) algebra of $\mathcal{P}$ *is* $\mathcal{Q}_n(\mathcal{P})$ *(*$\mathcal{B}_n(\mathcal{P})$ *resp.) equipped with the swapping bracket, denoted by* $(\mathcal{Q}_n(\mathcal{P}), \{\cdot,\cdot\})$ *(*$(\mathcal{B}_n(\mathcal{P}), \{\cdot,\cdot\})$ *resp.).*

3. Fock–Goncharov coordinates

In this subsection, we explain explicitly the Fock–Goncharov coordinates for $\mathcal{X}_{\mathrm{PGL}_n,\hat{S}}$ in [FG06].

Let $\hat{S} = (S, m_b)$ that admits an ideal triangulation, where S is a compact oriented surface and m_b is a finite collection of marked points on ∂S considered modulo isotopy. Let m_p be the set of punctures of S. An *ideal triangulation* $\mathcal{T}$ of $\hat{S}$ is a maximal collection of non-homotopic essential arcs joining points in $m_b \cup m_p$ which are pairwise disjoint on the interior parts.

Definition 3.1. [$\mathcal{X}_{\mathrm{PGL}_n,\hat{S}}$ [FG06, Definition 2.1]] *A* PGL_n-framed local system *is a pair* (ρ, ξ) *where*

(1) $\rho \in \mathrm{Hom}(\pi_1(S), \mathrm{PGL}_n)/\,\mathrm{PGL}_n$,
(2) ξ *is a monodromy invariant map from* $m_b \cup m_p$ *to* $\mathcal{B}$.

The moduli space $\mathcal{X}_{\mathrm{PGL}_n,\hat{S}}$ *is the collection of equivalent classes of the pairs with the equivalence relation* $(\rho, \xi) \sim (g \circ \rho \circ g^{-1}, g \circ \xi)$ *for any* $g \in \mathrm{PGL}_n$.

Definition 3.2. [n-TRIANGULATION] *For any triangulation* $\mathcal{T}$*, we denote its vertices by* $V_{\mathcal{T}}$ *and its edges by* $E_{\mathcal{T}}$*. Given an ideal triangulation* $\mathcal{T}$ *of* $\hat{S}$*, we define the* n-triangulation $\mathcal{T}_n$ *of* $\mathcal{T}$ *to be: we subdivide each triangle of* $\mathcal{T}$ *into* n^2 *triangles as shown in Figure 3. Let*

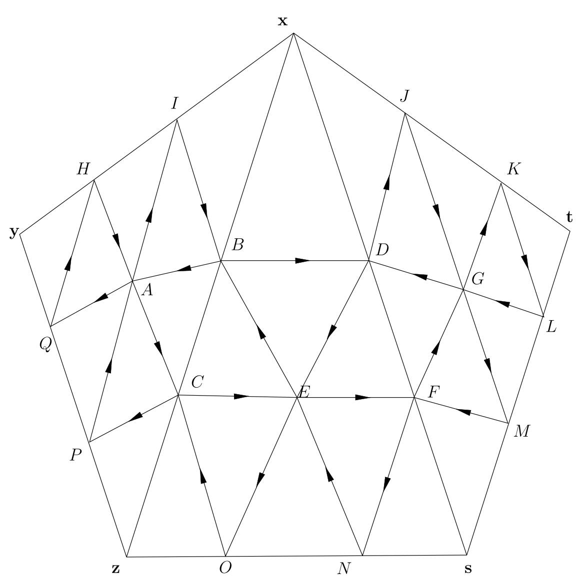

Figure 3. 3-triangulation of D_5

$$\mathcal{I}_n = \left(V_{\mathcal{T}_n} \backslash V_{\mathcal{T}}\right) \cap \left(\bigcup_{e \in E_{\mathcal{T}}} e\right),$$

and

$$\mathcal{I}'_n = \{V \in \mathcal{I}_n \mid V \text{ is not on the boundary of } \mathcal{T}\}.$$

For the case in Figure 3, $\mathcal{I}'_n = \{B, C, D, F\}$. Any element $W \in \mathcal{I}_n$ is specified by an oriented edge $\overrightarrow{\boldsymbol{u}\boldsymbol{v}}$ and a pair of positive numbers (m, l) with $m + l = n$ where m is the least number of edges between W and $\boldsymbol{v}$ in $E_{\mathcal{T}_n}$. We also denote W by $v^{m,l}_{\boldsymbol{u},\boldsymbol{v}}$. For example, the vertex $B = v^{2,1}_{\boldsymbol{x},\boldsymbol{z}}$ is specified by $\overrightarrow{\boldsymbol{x}\boldsymbol{z}}$ and $(2, 1)$.

A marked triangle $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ *is a triangle $\overline{\boldsymbol{u}\boldsymbol{v}\boldsymbol{w}}$ with a mark on each vertex.*

Let

$$\mathcal{J}_n = V_{\mathcal{T}_n} \backslash \left(\mathcal{I}_n \cup V_{\mathcal{T}}\right).$$

For the case in Figure 3, $\mathcal{J}_n = \{A, E, G\}$. Any element $U \in \mathcal{J}_n$ is specified by a marked triangle $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ and a triple of positive numbers (m, l, p) with $m+l+p = n$ where m (l resp.) is the least number of edges between U and $\overline{\boldsymbol{v}\boldsymbol{w}}$ ($\overline{\boldsymbol{u}\boldsymbol{w}}$ resp.) in $E_{\mathcal{T}_n}$. We also denote U by $v^{m,l,p}_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}}$. For example, the vertex $A = v^{1,1,1}_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}$ is specified by $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ and $(1, 1, 1)$.

With respect to the orientation of S, we define a quiver $\Gamma_{\mathcal{T}_n}$ *with vertices $\mathcal{I}_n \cup \mathcal{J}_n$ and oriented edges as in Figure 3.*

Definition 3.3. [FLAGS] *Let E be a n-dimensional vector space and let Ω be a volume form of E. A* flag *is a nested sequence of vector subspaces in E*

$$F := F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = E, \quad \dim F_i = i$$

equipped with the volume form Ω.

A basis of a flag *F is a basis $\{f_1, \cdots, f_n\}$ of E such that the vectors $f_1, \cdots, f_i$ span the vector space F_i for $i = 1, \cdots, n$.*

The flag variety *$\mathcal{B}$ is the collection of the flags.*

Notation 3.4. *When X is $\mathcal{T}$ or $\mathcal{T}_n$ or $m_b \cup m_p$ or $\mathcal{I}_n \cup \mathcal{J}_n$, we use $\widetilde{X}$ for denoting all its lifts in the universal cover $\widetilde{S}$.*

The construction of Fock–Goncharov $\mathcal{X}$ coordinates are based on Lusztig's theory of total positivity [Lu94][Lu98].

Definition 3.5. [FOCK–GONCHAROV COORDINATES [FG06, Section 9]] *Fix an ideal triangulation $\mathcal{T}$ of $\hat{S}$ and its n-triangulation $\mathcal{T}_n$. Given a framed local system (ρ, ξ) of $\mathcal{X}_{\mathrm{PGL}_n, \hat{S}}$, by Deck transformation, we get a ρ-equivariant map ξ_ρ from $\widetilde{m_b \cup m_p}$ in the universal cover $\widetilde{S}$ to the flag variety $\mathcal{B}$. For any vertex $\widetilde{V} \in \widetilde{\mathcal{I}_n \cup \mathcal{J}_n}$ of $\widetilde{\mathcal{T}_n}$, suppose that $\widetilde{V}$ is specified by a marked triangle $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ of $\widetilde{\mathcal{T}}$ and a triple of non-negative integers (m, l, p) with $m + l + p = n$. Let*

$$\{\boldsymbol{f}_1, \cdots, \boldsymbol{f}_n\}, \ \{\boldsymbol{g}_1, \cdots, \boldsymbol{g}_n\}, \ \{\boldsymbol{h}_1, \cdots, \boldsymbol{h}_n\}$$

be the bases of the three flags $\xi_\rho(\boldsymbol{f})$, $\xi_\rho(\boldsymbol{g})$, $\xi_\rho(\boldsymbol{h})$ respectively. Recall Ω is the fixed the volume form of E. Fix the notation

$$\boldsymbol{v}^i := \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_i.$$

We define

$$\Delta_{\widetilde{V}} = \Omega\left(\boldsymbol{f}^m \wedge \boldsymbol{g}^l \wedge \boldsymbol{h}^p\right).$$

Let $\mathcal{FA}_n$ be the fraction ring generated by these determinants over the field $\mathbb{K}$ of characteristic zero with respect to a choice of bases of flags.

For any $V \in \mathcal{I}_n' \cup \mathcal{J}_n$, we choose one of its lift $\widetilde{V}$ in the universal cover $\widetilde{S}$. For all the lifts of the quiver $\Gamma_{\mathcal{T}_n}$ into $\widetilde{S}$, we define $\widetilde{\varepsilon}_{\widetilde{V}\widetilde{W}}$ for any $\widetilde{V}, \widetilde{W} \in \widetilde{\mathcal{I}_n \cup \mathcal{J}_n}$ by

$$\widetilde{\varepsilon}_{\widetilde{V}\widetilde{W}} = \#\{\text{ arrows from } \widetilde{V} \text{ to } \widetilde{W}\ \} - \#\{\text{ arrows from } \widetilde{W} \text{ to } \widetilde{V}\ \} \tag{10}$$

as in Figure 3. We define

$$X_V = \prod_{\widetilde{W}} \Delta_{\widetilde{W}}^{\widetilde{\varepsilon}_{\widetilde{V}\widetilde{W}}},$$

which does not depend on the lift and the bases of flags that we choose. The collection of $\{X_V\}_{V \in \mathcal{I}_n' \cup \mathcal{J}_n}$ parametrizes $\mathcal{X}_{\mathrm{PGL}_n, \hat{S}}$.

Let $\mathcal{F}\mathcal{X}(\mathcal{T}_n)$ be the fraction ring generated by $\{X_V\}_{V \in \mathcal{I}_n' \cup \mathcal{J}_n}$ over the field $\mathbb{K}$ of characteristic zero. Then $\mathcal{F}\mathcal{X}(\mathcal{T}_n) \subset \mathcal{F}\mathcal{A}_n$.

Remark 3.6. *More explicitly, for $V \in \mathcal{J}_n$ corresponding to a marked triangle $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ of $\widetilde{\mathcal{T}}$ and a triple of positive integers (m, l, p) with $m + l + p = n$, the Fock–Goncharov $\mathcal{X}$ coordinate at V, also called the* triple ratio, *is*

$$X_V = \frac{\Omega\left(\boldsymbol{f}^{m+1} \wedge \boldsymbol{g}^{l} \wedge \boldsymbol{h}^{p-1}\right)}{\Omega\left(\boldsymbol{f}^{m+1} \wedge \boldsymbol{g}^{l-1} \wedge \boldsymbol{h}^{p}\right)} \cdot \frac{\Omega\left(\boldsymbol{f}^{m-1} \wedge \boldsymbol{g}^{l+1} \wedge \boldsymbol{h}^{p}\right)}{\Omega\left(\boldsymbol{f}^{m} \wedge \boldsymbol{g}^{l+1} \wedge \boldsymbol{h}^{p-1}\right)} \cdot \frac{\Omega\left(\boldsymbol{f}^{m} \wedge \boldsymbol{g}^{l-1} \wedge \boldsymbol{h}^{p+1}\right)}{\Omega\left(\boldsymbol{f}^{m-1} \wedge \boldsymbol{g}^{l} \wedge \boldsymbol{h}^{p+1}\right)}.$$

For $V \in \mathcal{I}_n'$ corresponding to an oriented edge $\overrightarrow{\boldsymbol{x}\boldsymbol{z}}$ with two adjacent anticlockwise oriented ideal triangles $\overrightarrow{\boldsymbol{x}\boldsymbol{y}\boldsymbol{z}}$ and $\overrightarrow{\boldsymbol{x}\boldsymbol{z}\boldsymbol{t}}$, and a pair of positive integers $(m, n-m)$, the Fock–Goncharov $\mathcal{X}$ coordinate at V, also called the edge function, *is*

$$X_V = \frac{\Omega\left(\boldsymbol{x}^{m} \wedge \boldsymbol{z}^{n-m-1} \wedge \boldsymbol{t}_1\right) \cdot \Omega\left(\boldsymbol{x}^{m-1} \wedge \boldsymbol{y}_1 \wedge \boldsymbol{z}^{n-m}\right)}{\Omega\left(\boldsymbol{x}^{m} \wedge \boldsymbol{y}_1 \wedge \boldsymbol{z}^{n-m-1}\right) \cdot \Omega\left(\boldsymbol{x}^{m-1} \wedge \boldsymbol{z}^{n-m} \wedge \boldsymbol{t}_1\right)},$$

Edge functions generalize Thurston's shear coordinates [T86].

In both cases, X_V does not depend on the bases that we choose since each term like $\boldsymbol{x}^m$ appears once in the numerator, once in the denominator. Moreover, because Ω does not change under the projective transformations, X_V is invariant by the projective transformations. So X_V is a well-defined function on $\mathcal{X}_{\mathrm{PGL}_n, \hat{S}}$.

Remark 3.7. *When the Riemann surface S is closed, with the help of Lie group G invariant functions, Goldman [G86] studied the Hamiltonian flows on $\mathcal{R}_{G,S}$ where the twist flows are described explicitly. For the Hitchin component $H_3(S)$, the flows related to the Fock–Goncharov parameters are studied in [G13] [WZ17]. For the Hitchin component $H_n(S)$, the Hamiltonian flows related to these parameters are studied in [SWZ17] [SZ17].*

The Fuchsian rigidity with respect to triple ratios (edge functions resp.) can be found in [HS19].

Definition 3.8. [Rank n Fock–Goncharov algebra] *Let $\mathcal{F}\mathcal{X}(\mathcal{T}_n)$ be the fraction ring generated by $\{X_V\}_{V \in \mathcal{I}_n' \cup \mathcal{J}_n}$ over the field $\mathbb{K}$ of characteristic zero. The* rank n Fock–Goncharov Poisson bracket *$\{\cdot,\cdot\}_n$ is defined by extending to $\mathcal{F}\mathcal{X}(\mathcal{T}_n)$ the following formula for any $V, W \in \mathcal{I}_n' \cup \mathcal{J}_n$ using Leibniz's rule:*

$$\{X_V,\ X_W\}_n = \varepsilon_{V,W} \cdot X_V \cdot X_W,$$

where ε is defined in Equation (3).

The rank n Fock–Goncharov algebra *of $\mathcal{T}_n$ is the ring $\mathcal{F}\mathcal{X}(\mathcal{T}_n)$ equipped with the rank n Fock–Goncharov Poisson bracket, denoted by $(\mathcal{F}\mathcal{X}(\mathcal{T}_n), \{\cdot,\cdot\}_n)$.*

Remark 3.9. (1) *As shown in* [FG06, Section 15], *the rank n Fock–Goncharov Poisson bracket arises from a special K_2 class in $\mathcal{A}_{\mathrm{SL}_n,\hat{S}}$. It can be understood as a canonical Poisson bracket defined for a cluster $\mathcal{X}$ variety* [GSV03].

(2) *Actually, the rank n Fock–Goncharov Poisson bracket identifies with the Goldman Poisson structure through a different way of symplectic reduction. V. Fock and A. Rosly* [FR98] *observed that the Goldman Poisson structure on $\mathcal{X}_{G,\hat{S}}$ can be obtained as a quotient of the space of graph connections by the Poisson action of a lattice gauge group endowed with a Poisson-Lie structure. When $G = \mathrm{PGL}(n,\mathbb{R})$, we can calculate Fock–Rosly Poisson bracket between any two Fock–Goncharov coordinates explicitly, which results in the rank n Fock–Goncharov Poisson bracket. In* [N13, part I Theorem 3.23], *Nie uses an approach—the quasi-Poisson structure* [AMM98] [AKM02] *that is equivalent to that of Fock and Rosly, to explicitly identify the Goldman Poisson structure with the Fock–Goncharov Poisson structure on $\mathcal{X}_{G,\hat{S}}$.*

Theorem 3.10. [V. V . Fock, A. B. Goncharov [FG06, Theorem 1.11], [FG04, Theorem 2.5] for $n = 3$] *Given an ideal triangulation $\mathcal{T}$ and its n-triangulation $\mathcal{T}_n$ of $\hat{S}$, the Fock–Goncharov $\mathcal{X}$ coordinates $\{X_V\}_{V\in\mathcal{I}'_n\cup\mathcal{J}_n}$ provide a positive regular atlas on $\mathcal{X}_{\mathrm{PGL}_n,\hat{S}}$.*

4. $(n \times n)$-determinant ratio

In this section, we construct $(n \times n)$-determinant ratios in $\mathcal{Q}_n(\mathcal{P})$ and relate them with the rank n Fock–Goncharov algebra.

Let us recall the geometric model for $\mathcal{Z}_n(\mathcal{P})$ from [Su17, Section 4], which should always be kept in mind while we do the computations in the rank n swapping algebra. Let $\mathcal{P} = \{x_1, \cdots, x_r\}$. We associate a pair $(\mathfrak{a}_i, \mathfrak{b}_i) \in \mathbb{K}^n \times \mathbb{K}^{n*}$ to x_i for $i = 1, \cdots, r$. We consider the space $D_{n,r} = (\mathbb{K}^n \times \mathbb{K}^{n*})^r$ of r vectors $\mathfrak{a}_1, \cdots, \mathfrak{a}_r$ in $\mathbb{K}^n$ and r covectors $\mathfrak{b}_1, \cdots, \mathfrak{b}_r$ in $\mathbb{K}^{n*}$. For any $g \in \mathrm{GL}(n,\mathbb{K})$, the action of g on the vector $\mathfrak{a}_i$ is the left multiplication by g, the action of g on the covector $\mathfrak{b}_i$ is the right multiplication by g^{-1}. We define the product between a vector $\mathfrak{a}_i$ in $\mathbb{K}^n$ and a covector $\mathfrak{b}_j$ in $\mathbb{K}^{n*}$ by $\langle \mathfrak{a}_i | \mathfrak{b}_j \rangle := \mathfrak{b}_j(\mathfrak{a}_i)$, which is $\mathrm{GL}(n,\mathbb{K})$ invariant. Let us associate each $\langle \mathfrak{a}_i | \mathfrak{b}_j \rangle$ to each ordered pair $x_i x_j \in \mathcal{Z}_n(\mathcal{P})$ as follows. Let $B_{n\mathbb{K}}$ be the subring of $\mathbb{K}[D_{n,r}]$ generated by $\{\langle \mathfrak{a}_i | \mathfrak{b}_j \rangle\}_{i=1,j=1}^r$. C. D. Concini and C. Procesi [CP76] proved that $B_{n\mathbb{K}} = \mathbb{K}[D_{n,r}]^{\mathrm{GL}(n,\mathbb{K})}$.

Let W be the polynomial ring $\mathbb{K}[\{\mathsf{z}_{i,j}\}_{i,j=1}^r]$,

$$R = \{f \in W \mid f = \det \begin{pmatrix} \mathsf{z}_{i_1,j_1} & \cdots & \mathsf{z}_{i_1,j_{n+1}} \\ \cdots & \cdots & \cdots \\ \mathsf{z}_{i_{n+1},j_1} & \cdots & \mathsf{z}_{i_{n+1},j_{n+1}} \end{pmatrix}, \forall i_k, j_l = 1, \cdots, r\}.$$

Let T be the ideal of W generated by R. Then Weyl [W39] show that $B_{n\mathbb{K}} \cong W/T$. Recall $\mathcal{P} = \{x_1, \cdots, x_r\} \subset S^1$. Let $S_{n\mathbb{K}}$ be the ideal of $B_{n\mathbb{K}}$ generated by $\{\langle a_i | b_i \rangle\}_{i=1}^r$. Taking quotient by $S_{n\mathbb{K}}$, we identify $\langle \mathfrak{a}_i | \mathfrak{b}_j \rangle$ with $x_i x_j$ through $\mathsf{z}_{i,j}$, where we identify $\mathfrak{a}_i$ with x_i on the left and $\mathfrak{b}_j$ with x_j on the right of ordered pair $x_i x_j$ in $\mathcal{Z}_n(\mathcal{P})$.

Definition 4.1. *For any $d > 1$ and any $x_1, \cdots, x_d, y_1, \cdots, y_d \in \mathcal{P}$, recall the notation*

$$\Delta\left((x_1, \cdots, x_d), (y_1, \cdots, y_d)\right) := \det \begin{pmatrix} x_1y_1 & \cdots & x_1y_d \\ \cdots & \cdots & \cdots \\ x_dy_1 & \cdots & x_dy_d \end{pmatrix} \in \mathcal{Z}_n(\mathcal{P}). \tag{11}$$

We call $(x_1, \cdots, x_d)$ $((y_1, \cdots, y_d)$ resp) the left (right resp.) side n-tuple of *the determinant* $\Delta\left((x_1, \cdots, x_d), (y_1, \cdots, y_d)\right)$.

Theorem 4.2. [[Su17] Theorem 4.6] *We have $B_{n\mathbb{K}}/S_{n\mathbb{K}} \cong \mathcal{Z}_n(\mathcal{P})$.*

Lemma 4.3. *For $n > 1$, let $x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathcal{P}$. Suppose that $x_1, \cdots, x_n$ (y_1, $\cdots$, y_n resp.) are mutually distinct, we have*

$$\Delta\left((x_1, \cdots, x_n), (y_1, \cdots, y_n)\right) \neq 0.$$

in $\mathcal{Z}_n(\mathcal{P})$.

Proof. For any $i = 1, \cdots, n$, let $x_i, y_i \in \mathcal{P}$ and let $(x_{i,v}, x_{i,c}), (y_{i,v}, y_{i,c}) \in \mathbb{K}^n \times \mathbb{K}^{n*}$. Under identification $B_{n\mathbb{K}}/S_{n\mathbb{K}} \cong \mathcal{Z}_n(\mathcal{P})$ of Theorem 4.2, we identify the vector $x_{i,v}$ in $\mathbb{K}^n$ with x_i in $\mathcal{P}$ on the left and the covector $y_{j,c}$ in $\mathbb{K}^{n*}$ with y_j in $\mathcal{P}$ on the right of ordered pair x_iy_j in $\mathcal{Z}_n(\mathcal{P})$. Then the determinant $\Delta\left((x_1, \cdots, x_n), (y_1, \cdots, y_n)\right)$ is not zero in $\mathcal{Z}_n(\mathcal{P})$ if and only if $\det_{1\leq i,j\leq n}(x_{i,v}, y_{j,c})$ is not always zero in $\mathbb{K}$ for any generic $\mathbb{K}$-point of $B_{n\mathbb{K}}/S_{n\mathbb{K}}$. Actually, for any generic $\mathbb{K}$-point of $B_{n\mathbb{K}}/S_{n\mathbb{K}}$, the value of $\det_{1\leq i,j\leq n}(x_{i,v}, y_{j,c})$ is interpreted as the volume of $x_{1,v}, \cdots, x_{n,v}$ with respect to the dual basis of $y_{1,c}, \cdots, y_{n,c}$. If $x_{1,v}, \cdots, x_{n,v}$ and $y_{1,c}, \cdots, y_{n,c}$ are both in general position, the volume $\det_{1\leq i,j\leq n}(x_{i,v}, y_{j,c})$ is not zero. We conclude that

$$\Delta\left((x_1, \cdots, x_n), (y_1, \cdots, y_n)\right) \neq 0$$

in $\mathcal{Z}_n(\mathcal{P})$. □

Proposition 4.4. *Let $x_1, \cdots, x_{n-1}, t, y, v_1, \cdots, v_n, u_1 \in \mathcal{P}$. If $x_1, \cdots, x_{n-1}, y$ ($v_1, \cdots, v_n, u_1$ resp.) are mutually distinct, we have*

$$\frac{\Delta\left((x_1, \cdots, x_{n-1}, t), (v_1, v_2, \cdots, v_n)\right)}{\Delta\left((x_1, \cdots, x_{n-1}, y), (v_1, v_2, \cdots, v_n)\right)} = \frac{\Delta\left((x_1, \cdots, x_{n-1}, t), (u_1, v_2, \cdots, v_n)\right)}{\Delta\left((x_1, \cdots, x_{n-1}, y), (u_1, v_2, \cdots, v_n)\right)}$$

in $\mathcal{Q}_n(\mathcal{P})$.

Proof. Consider the $(n+1) \times (n+1)$ matrix

$$M = \begin{pmatrix} x_1u_1 & x_1v_1 & \cdots & \cdots & x_1v_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1}u_1 & x_{n-1}v_1 & \cdots & \cdots & x_{n-1}v_n \\ tu_1 & tv_1 & \cdots & \cdots & tv_n \\ yu_1 & yv_1 & \cdots & \cdots & yv_n \end{pmatrix}$$

The adjugate of M is

$$M^\star = \begin{pmatrix} A_{1,1} & \cdots & (-1)^{n+2}A_{n+1,1} \\ \cdots & (-1)^{j+i}A_{j,i} & \cdots \\ (-1)^{n+2}A_{1,n+1} & \cdots & A_{n+1,n+1} \end{pmatrix}$$

whose (i,j) entry is $(-1)^{j+i}A_{j,i}$ and $A_{j,i}$ equals the determinant of the $(n \times n)$-matrix obtained from M by deleting the j-th row and the i-th column.
We already know that

$$\det M = 0$$

in $\mathcal{Z}_n(\mathcal{P})$, hence we obtain

$$M^\star \cdot M = 0_{(n+1)\times(n+1)}. \tag{12}$$

The entries of the matrices M, $M^\star$ and $M^\star \cdot M$ are polynomials in $\mathcal{Z}_n(\mathcal{P})$. Recall that by Theorem 4.2, we identify a vector with a point on the left and a covector with a point on the right of ordered pairs of points in $\mathcal{Z}_n(\mathcal{P})$. When we specify the values of vectors and covectors, under $B_{n\mathbb{K}}/S_{n\mathbb{K}} \cong \mathcal{Z}_n(\mathcal{P})$, we specify the values of all the polynomials in $\mathcal{Z}_n(\mathcal{P})$. Then the values of the matrices M, $M^\star$ and $M^\star\cdot M$ provides the linear endomorphisms of $\mathbb{K}^{n+1}$: f, g and $g\circ f$ respectively. Actually, any polynomial P is zero in the ring $\mathcal{Z}_n(\mathcal{P})$ over a field $\mathbb{K}$ of characteristic zero, if and only if P is zero in all of the generic $\mathbb{K}$-points. The following arguments are true for any generic $\mathbb{K}$-point of $B_{n\mathbb{K}}/S_{n\mathbb{K}}$, thus true for $\mathcal{Z}_n(\mathcal{P})$. By Equation (12), the rank of $g\circ f$ (the dimension of the image of $g \circ f$) is 0. By the above lemma, we have

$$\Delta\left((x_1,\cdots,x_{n-1},y),(v_1,\cdots,v_n)\right) \neq 0.$$

Thus, for any generic $\mathbb{K}$-point, the rank of f is at least n. Therefore, for any generic $\mathbb{K}$-point, we have the rank of g is at most 1 (If not so, we will get the rank of $g\circ f$ is not 0). By considering the top right corner 2×2 minor of $M^\star$, for any generic $\mathbb{K}$-point, we have

$$A_{n,1}\cdot A_{n+1,2} - A_{n,2}\cdot A_{n+1,1} = 0,$$

which implies that

$$\frac{\Delta\left((x_1,\cdots,x_{n-1},t),(v_1,v_2,\cdots,v_n)\right)}{\Delta\left((x_1,\cdots,x_{n-1},y),(v_1,v_2,\cdots,v_n)\right)} = \frac{\Delta\left((x_1,\cdots,x_{n-1},t),(u_1,v_2,\cdots,v_n)\right)}{\Delta\left((x_1,\cdots,x_{n-1},y),(u_1,v_2,\cdots,v_n)\right)}$$

in $\mathcal{Q}_n(\mathcal{P})$. □

Moreover, by applying Proposition 4.4 n times, we have

Corollary 4.5. *Let $x_1,\cdots,x_{n-1},t,y,v_1,\cdots,v_n,u_1,\cdots,u_n \in \mathcal{P}$. Suppose that $x_1,\cdots,x_{n-1}$, y ($v_1,\cdots,v_n$ and $u_1,\cdots,u_n$ resp.) are mutually distinct, in $\mathcal{Q}_n(\mathcal{P})$, we have*

$$\frac{\Delta\left((x_1,\cdots,x_{n-1},t),(v_1,\cdots,v_n)\right)}{\Delta\left((x_1,\cdots,x_{n-1},y),(v_1,\cdots,v_n)\right)} = \frac{\Delta\left((x_1,\cdots,x_{n-1},t),(u_1,\cdots,u_n)\right)}{\Delta\left((x_1,\cdots,x_{n-1},y),(u_1,\cdots,u_n)\right)}.$$

By the above corollary, we can define a ratio of two $(n\times n)$-determinants that does not depend on the right side n-tuple.

Definition 4.6. [$(n\times n)$-DETERMINANT RATIO] *Let $x_1,\cdots,x_{n-1},y\in\mathcal{P}$ be different from each other. The* $(n\times n)$-determinant ratio *of $x_1,\cdots,x_{n-1},t,y$:*

$$E(x_1,\cdots,x_{n-1}|t,y) := \frac{\Delta\left((x_1,\cdots,x_{n-1},t),(v_1,\cdots,v_n)\right)}{\Delta\left((x_1,\cdots,x_{n-1},y),(v_1,\cdots,v_n)\right)}$$

for any $v_1,\cdots,v_n\in\mathcal{P}$ different from each other.

The fraction ring $\mathcal{D}_n(\mathcal{P})$ generated by all the $(n\times n)$-determinant ratios is called $(n\times n)$-determinant ratio fraction ring.

Remark 4.7. *The fraction ring $\mathcal{D}_n(\mathcal{P})$ is also a fraction ring generated by all elements of the form $\frac{\Delta((x_1,\cdots,x_n),(v_1,\cdots,v_n))}{\Delta((y_1,\cdots,y_n),(v_1,\cdots,v_n))}$, since*

$$\frac{\Delta\left((x_1,\cdots,x_n),(v_1,\cdots,v_n)\right)}{\Delta\left((y_1,\cdots,y_n),(v_1,\cdots,v_n)\right)} = \prod_{i=1}^{n} E(x_1,\cdots,x_{i-1},y_{i+1}\cdots,y_n|x_i,y_i).$$

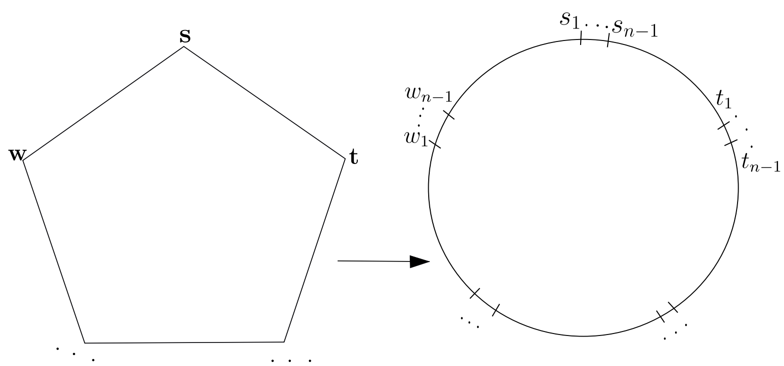

FIGURE 4. $\{\mathbf{s} \prec \mathbf{w} \prec \cdots \prec \mathbf{t} \prec \mathbf{s}\} \to \{s_{n-1} \prec \cdots \prec s_1 \prec w_{n-1} \prec \cdots \prec w_1 \prec \cdots \prec t_{n-1} \prec \cdots \prec t_1 \prec s_{n-1}\}$

By Corollary 4.5, we have

Corollary 4.8. *Let $a, b, x_1, \cdots, x_{n-1}, t, y \in \mathcal{P}$, $x_1, \cdots, x_{n-1}, y$ be different from each other. The value of*

$$\{ab, E(x_1, \cdots, x_{n-1}|t, y)\}$$

in $\mathcal{Q}_n(\mathcal{P})$ does not depend on the choice of right side n-tuple $(v_1, \cdots, v_n)$.

As a consequence,

Corollary 4.9. *Let $\mathcal{P}' = \{u_1, \cdots, u_n\} \cup \mathcal{P}$. The value of*

$$\left\{ab, \frac{\Delta\left((x_1, \cdots, x_{n-1}, t), (u_1, \cdots, u_n)\right)}{\Delta\left((x_1, \cdots, x_{n-1}, y), (u_1, \cdots, u_n)\right)}\right\}$$

in $\mathcal{Q}_n(\mathcal{P}')$ can be expressed in $\mathcal{Q}_n(\mathcal{P})$ by replacing $u_1, \cdots, u_n$ with any n different elements $v_1, \cdots, v_n$ in $\mathcal{P}$.

By the above corollary, we can calculate the swapping bracket between two $(n \times n)$-determinant ratios with the right side n-tuples in any preferred position.

5. Main theorem

5.1. **Homomorphism from rank n Fock–Goncharov algebra to rank n swapping multifraction algebra.** Let D_k be a disk D with k points $m_b = \{\mathbf{s} \prec \mathbf{w} \prec \cdots \prec \mathbf{t} \prec \mathbf{s}\}$ on ∂D, where $\prec$ is defined with respect to the anticlockwise cyclic order on a circle. In this case $\mathcal{X}_{\mathrm{PGL}_n, D_k} \cong \mathcal{B}^k / \mathrm{PGL}_n$ and

$$X_V = \prod_{W \in \mathcal{I}_n \cup \mathcal{J}_n} \Delta_W^{\varepsilon_{V,W}}.$$

Definition 5.1. *Given an ideal triangulation $\mathcal{T}$ of D_k and its n-triangulation $\mathcal{T}_n$, we have the fraction ring $\mathcal{FX}(\mathcal{T}_n) \subset \mathcal{FA}_n$ as defined in Definition 3.5.*

Let

$$\mathcal{P} = \{s_{n-1} \prec \cdots \prec s_1 \prec w_{n-1} \prec \cdots \prec w_1 \prec \cdots \prec t_{n-1} \prec \cdots \prec t_1 \prec s_{n-1}\}$$

on S^1 with $\#\mathcal{P} = k(n-1)$, where each $\boldsymbol{r} \in m_b$ corresponds to $n-1$ anticlockwise ordered points $r_{n-1}, \ldots, r_1$ nearby in $\mathcal{P}$ as in Figure 4.

Fix a choice of distinct $u_1, \cdots, u_n \in \mathcal{P}$, the homomorphism χ_n *(which depends on the choice) from $\mathcal{FA}_n$ to $\mathcal{Q}_n(\mathcal{P})$ is defined by extending the following formula on the generators to $\mathcal{FA}_n$ using Leibniz's rule*

$$\chi_n(\Delta_V) = \Delta\left((x_1, \cdots, x_m, y_1, \cdots, y_l, z_1, \cdots, z_p), (u_1, \cdots u_n)\right),$$

where $\Delta_V = \Omega\left(\boldsymbol{x}^m \wedge \boldsymbol{y}^l \wedge \boldsymbol{z}^p\right)$ and any vertex V of $\mathcal{T}_n$ is specified by a marked triangle $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ of $\mathcal{T}$ and a triple of non-negative integers (m, l, p) with $m+l+p = n$ (Here Ω is the volume form of E and the bases of the flags $\xi_\rho(\boldsymbol{x})$, $\xi_\rho(\boldsymbol{y})$, $\xi_\rho(\boldsymbol{z})$ are

$$\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n\}, \ \ \{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_n\}, \ \ \{\boldsymbol{z}_1, \cdots, \boldsymbol{z}_n\}$$

respectively. Recall the notation $\boldsymbol{v}^i := \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_i$).

We define the homomorphism $\theta_{\mathcal{T}_n}$ from $\mathcal{FX}(\mathcal{T}_n)$ to $\mathcal{B}_n(\mathcal{P})$ by restricting the homomorphism χ_n to the fraction ring $\mathcal{FX}(\mathcal{T}_n)$. We have

$$\theta_{\mathcal{T}_n}(X_V) = \chi_n(X_V) = \prod_W \chi_n(\Delta_W)^{\varepsilon_{VW}}.$$

Proposition 5.2. *The image of $\theta_{\mathcal{T}_n}$ lies in $\mathcal{B}_n(\mathcal{P})$.*

Proof. For any mutually distinct $v_1, \cdots, v_n \in \mathcal{P}$ ($u_1, \cdots, u_n \in \mathcal{P}$ resp.) and any permutation $\sigma \in S_n$, by [L18] Proposition 2, we have

$$\frac{v_1 u_{\sigma(1)} \cdots v_n u_{\sigma(n)}}{v_1 u_1 \cdots v_n u_n} \in \mathcal{B}_n(\mathcal{P}).$$

Thus we obtain

$$\frac{\Delta\big((v_1, \cdots, v_n), (u_1, \cdots, u_n)\big)}{v_1 u_1 \cdots v_n u_n} = \sum_{\sigma \in S_n} \epsilon_\sigma \cdot \frac{v_1 u_{\sigma(1)} \cdots v_n u_{\sigma(n)}}{v_1 u_1 \cdots v_n u_n} \in \mathcal{B}_n(\mathcal{P}).$$

Since $\theta_{\mathcal{T}_n}(X_V)$ can be written as fraction of four or six $\frac{\Delta\left((v_1, \cdots, v_n), (u_1, \cdots, u_n)\right)}{v_1 u_1 \cdots v_n u_n}$'s, we conclude that $\theta_{\mathcal{T}_n}(X_V)$ belongs to $\mathcal{B}_n(\mathcal{P})$. □

Proposition 5.3. *The homomorphism $\theta_{\mathcal{T}_n}$ is an injective homomorphism.*

Proof. The homomorphism χ_n sends $(n \times n)$-determinants for $\mathbb{K}^n$ to $(n \times n)$-determinants in $\mathcal{Z}(\mathcal{P})$. By Theorem 4.2, we have the ring isomorphism $B_{n\mathbb{K}}/S_{n\mathbb{K}} \cong \mathcal{Z}_n(\mathcal{P})$, thus a choice of distinct $u_1, \cdots, u_n \in \mathcal{P}$ for the right side n-tuple for $(n \times n)$-determinants in $\mathcal{Z}(\mathcal{P})$ corresponds to fix a basis for $\mathbb{K}^n$. So any relation among $(n \times n)$-determinants for $\mathbb{K}^n$ in $\mathcal{FA}_n$ corresponds to a relation among $(n \times n)$-determinants in $\mathcal{Z}(\mathcal{P})$. Hence it follows that the homomorphism χ_n is injective. Since the homomorphism $\theta_{\mathcal{T}_n}$ is the homomorphism χ_n restricted to $\mathcal{FX}(\mathcal{T}_n)$, we conclude that the homomorphism $\theta_{\mathcal{T}_n}$ is injective. □

Recall the notation $w^i := w_1, \cdots, w_i$.

Proposition 5.4. *For $V \in \mathcal{J}_n$ associating to $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ and a triple of positive integers (m, l, p) with $m + l + p = n$, we have*

$$\theta_{\mathcal{T}_n}(X_V) = E\left(x^{m+1}, y^{l-1}, z^{p-1} | y_l, z_p\right) \cdot E\left(x^{m-1}, y^{l+1}, z^{p-1} | z_p, x_m\right) \cdot E\left(x^{m-1}, y^{l-1}, z^{p+1} | x_m, y_l\right).$$

For $V \in \mathcal{I}'_n$ corresponding to $\overrightarrow{\boldsymbol{xz}}$ and $(i, n-i)$, suppose that two adjacent anticlockwise oriented ideal triangles $\overrightarrow{\boldsymbol{xyz}}$ and $\overrightarrow{\boldsymbol{xzt}}$ have a common edge $\overrightarrow{\boldsymbol{xz}}$, we have

$$\theta_{\mathcal{T}_n}(X_V) = -E\left(x_1, \cdots, x_i, z_1, \cdots, z_{n-i-1} | t_1, y_1\right) \cdot E\left(x_1, \cdots, x_{i-1}, z_1, \cdots, z_{n-i} | y_1, t_1\right).$$

Proof. We only prove the first case. The other case will follow in a similar way. We have

$$\frac{\Delta\big((x^{m+1},y^{l},z^{p-1}),(u^n)\big)}{\Delta\big((x^{m+1},y^{l-1},z^{p}),(u^n)\big)} = (-1)^{p-1}\frac{\Delta\big((x^{m+1},y^{l-1},z^{p-1},y_l),(u^n)\big)}{\Delta\big((x^{m+1},y^{l-1},z^{p-1},z_p),(u^n)\big)}$$
$$= \quad (-1)^{p-1}E\left(x^{m+1},y^{l-1},z^{p-1}|y_l,z_p\right),$$

$$\frac{\Delta\big((x^{m-1},y^{l+1},z^{p}),(u^n)\big)}{\Delta\big((x^{m},y^{l+1},z^{p-1}),(u^n)\big)} = (-1)^{l+p}\frac{\Delta\big((x^{m-1},y^{l+1},z^{p-1},z_p),(u^n)\big)}{\Delta\big((x^{m-1},y^{l+1},z^{p-1},x_m),(u^n)\big)}$$
$$= \quad (-1)^{l+p}E\left(x^{m-1},y^{l+1},z^{p-1}|z_p,x_m\right),$$

$$\frac{\Delta\big((x^{m},y^{l-1},z^{p+1}),(u^n)\big)}{\Delta\big((x^{m-1},y^{l},z^{p+1}),(u^n)\big)} = (-1)^{(l+p)-(p+1)}\frac{\Delta\big((x^{m-1},y^{l-1},z^{p+1},x_m),(u^n)\big)}{\Delta\big((x^{m-1},y^{l-1},z^{p+1},y_l),(u^n)\big)}$$
$$= \quad (-1)^{l-1}E\left(x^{m-1},y^{l-1},z^{p+1}|x_m,y_l\right),$$

Taking the product of the above three terms, we obtain that

$$\theta_{\mathcal{T}_n}(X_V) = E\left(x^{m+1},y^{l-1},z^{p-1}|y_l,z_p\right)\cdot E\left(x^{m-1},y^{l+1},z^{p-1}|z_p,x_m\right)\cdot$$
$$E\left(x^{m-1},y^{l-1},z^{p+1}|x_m,y_l\right).$$

□

5.2. **Proof of the main theorem.** The main technical part of the proof of the main theorem is contained in Proposition 5.7. Moreover, in Proposition 5.12 we show how to compute the swapping bracket between two oriented edge ratios. Finally, we give a proof of our main theorem by considering different cases.

Theorem 5.5. [MAIN RESULT] *Let D_k be a disk with k points on its boundary. For an integer $n>1$, given an ideal triangulation $\mathcal{T}$ of D_k and its n-triangulation $\mathcal{T}_n$, the homomorphism $\theta_{\mathcal{T}_n}$ from $\mathcal{FX}(\mathcal{T}_n)$ to $\mathcal{B}_n(\mathcal{P})$ is Poisson with respect to the rank n Fock–Goncharov Poisson bracket and the swapping bracket.*

As shown in Proposition 5.4, the image of one Fock–Goncharov $\mathcal{X}$ coordinate can be written as a product of two or three $(n\times n)$-determinant ratios. We start by computing the swapping bracket between two $(n\times n)$-determinants in our cases. Recall the notation in Equation (6) $[A,B] := \frac{\{A,B\}}{AB}$. We will use the following fact frequently, by the Leibniz's rule, $\forall A,B,C,D\in\mathcal{Z}(\mathcal{P})$

$$\left[\frac{A}{B},\frac{C}{D}\right] = [A,C]-[A,D]-[B,C]+[B,D]. \tag{13}$$

Lemma 5.6. *For $n\geq 2$, let $M=(c_sd_t)_{s,t=1}^n$ be a $(n\times n)$-matrix with $c_s,d_t\in\mathcal{P}$, let M_{st} be the determinant of the matrix obtained from M by deleting the s-th row and the t-th column. Let $B\in\mathcal{Q}_n(\mathcal{P})$, we have*

$$\{\det M, B\} = \sum_{s=1}^{n}\sum_{t=1}^{n}(-1)^{s+t}\cdot\det M_{st}\cdot\{c_sd_t,B\}$$

in $\mathcal{Q}_n(\mathcal{P})$.

Proof. Firstly, we have

$$\det M = \sum_{\sigma\in S_n}\epsilon_\sigma\cdot c_1d_{\sigma(1)}\cdot\dots\cdot c_nd_{\sigma(n)}$$

where S_n is the permutation group of n elements, ϵ_σ is the sign of σ in S_n. By the Leibniz's rule, we have

$$\begin{aligned}
\{\det M, B\} &= \sum_{\sigma\in S_n} \epsilon_\sigma \cdot \{\prod_{i=1}^{n} c_i d_{\sigma(i)}, B\} \\
&= \sum_{\sigma\in S_n} \epsilon_\sigma \sum_{s=1}^{n} \prod_{i=1, i\neq s}^{n} c_i d_{\sigma(i)} \cdot \{c_s d_{\sigma(s)}, B\} \\
&= \sum_{s=1}^{n}\sum_{t=1}^{n} \left(\sum_{\sigma\in S_n, \sigma(s)=t} \epsilon_\sigma \cdot \prod_{i=1, i\neq s}^{n} c_i d_{\sigma(i)} \right) \cdot \{c_s d_t, B\} \\
&= \sum_{s=1}^{n}\sum_{t=1}^{n} (-1)^{s+t} \cdot \det M_{st} \cdot \{c_s d_t, B\}\,.
\end{aligned}$$

We conclude that

$$\{\det M, B\} = \sum_{s=1}^{n}\sum_{t=1}^{n} (-1)^{s+t} \cdot \det M_{st} \cdot \{c_s d_t, B\}\,.$$

□

Recall the notation $[A,B] := \frac{\{A,B\}}{AB}$. Before computing the swapping (Poisson) bracket, Let us recall some useful properties: for any $A, B \in \mathcal{Q}_n(\mathcal{P})$

$$\begin{gathered}
[A,B] = [-A,B] = [A,-B] = [-A,-B], \\
[1/A, B] = -[A,B], \quad [A,B] = -[B,A], \\
[A,A] = 0.
\end{gathered}$$

Recall the notation $w^i := w_1, \cdots, w_i$.

Proposition 5.7. [Main proposition] *Suppose that $x_{n-1} \prec \cdots \prec x_1 \prec v_1 \prec \cdots \prec v_n \prec y_{n-1} \prec \cdots \prec y_1 \prec z_{n-1} \prec \cdots \prec z_1 \prec u_1 \prec \cdots \prec u_n \prec x_{n-1}$ are ordered anticlockwise in $\mathcal{P}$ as shown in Figure 5, for non-negative integers m, l, p, m', l', p' with $m + l + p = n$ and $m' + l' + p' = n$.*

***If** $l \geq l'$ **or** $p \leq p'$ (*) as in Figure 6, we have*

$$\begin{aligned}
C &= \left[\Delta\left(\left(x^m, y^l, z^p\right), (v^n)\right), \Delta\left(\left(x^{m'}, y^{l'}, z^{p'}\right), (u^n)\right)\right] \\
&= \frac{1}{2}\cdot \min\{m, m'\} - \frac{1}{2}\cdot \min\{l, l'\} - \frac{1}{2}\cdot\min\{p, p'\}
\end{aligned}$$

in $\mathcal{Q}_n(\mathcal{P})$.

Remark 5.8. *Lemma 2.6 and Lemma 5.6 allows us to compute the swapping bracket between any two $(n\times n)$-determinants. The general result is complicated. But with respect to the cyclic order in Figure 5, the formula of Proposition 5.7 is simple under the condition (*) $l \geq l'$ or $p \leq p'$. Essentially, the $+$ and $-$ sign before $\frac{1}{2}\cdot\min$ in our formula is due to our cyclic order.*

The condition (*) is strict and is used in case 3 of the proof. This condition depends on the cyclic order of points and is crucial to the proof of the main theorem. Finding the proper cyclic order and the condition (*) for this proposition is not as direct as the proof of this proposition.

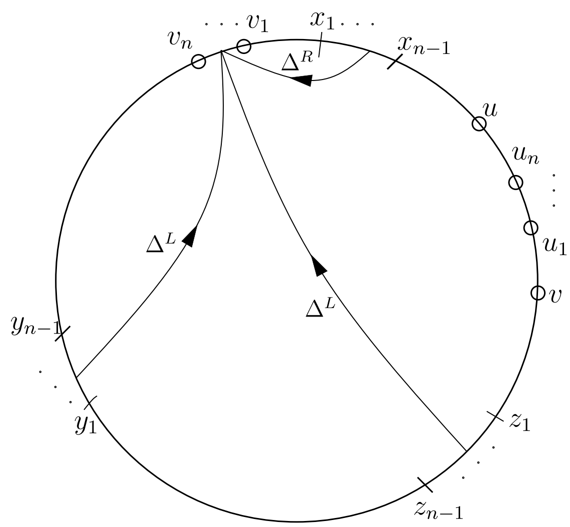

FIGURE 5. Computation of $\left[\Delta\left(\left(x^m, y^l, z^p\right), (v^n)\right), \Delta\left(\left(x^{m'}, y^{l'}, z^{p'}\right), (u^n)\right)\right]$

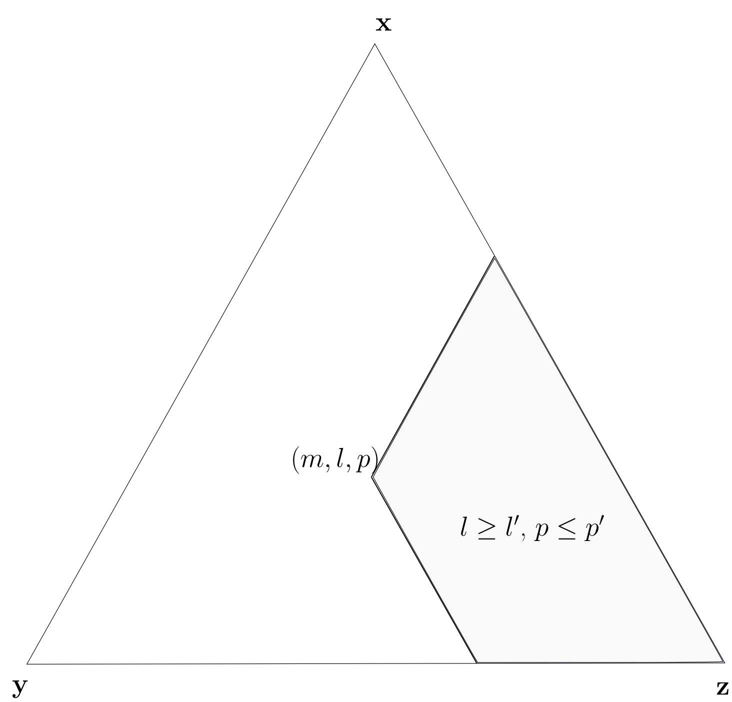

FIGURE 6. The region of (m', l', p') satisfies the condition (*).

Proof. Let $M = (c_s v_t)_{s,t=1}^n$ be a $(n \times n)$ matrix with $c_s = x_s$ for $s = 1, \cdots, m$, $c_s = y_{s-m}$ for $s = m+1, \cdots, m+l$ and $c_s = z_{s-m-l}$ for $s = m+l+1, \cdots, n$. Then $\det M = \Delta\left(\left(x^m, y^l, z^p\right), (v^n)\right)$. Let $B = \Delta\left(\left(x^{m'}, y^{l'}, z^{p'}\right), (u^n)\right)$.

By Lemma 5.6, we have

$$C=\sum_{s=1}^{n}\frac{1}{\det M\cdot B}\sum_{t=1}^{n}(-1)^{s+t}\cdot M_{st}\cdot\{c_sv_t,B\}.$$

Given $s=1,\cdots,n$, we compute the sum

$$\frac{1}{\det M\cdot B}\sum_{t=1}^{n}(-1)^{s+t}M_{st}\cdot\{c_sv_t,B\}$$

over t, where the summation is called the *sum over t for c_s* for short. Thus C equals to the sum of the above term over s. In the following three cases x_sv_t or $y_{s-m}v_t$ or $z_{s-m-l}v_t$ takes the place of c_sv_t.

1. For the sum over t for x_s where $1\le s\le m$:

Let us fix the notation

$$w^i\backslash w_j\cup x:=w_1,\cdots,w_{j-1},x,w_{j+1},\cdots,w_i.$$

Here $u\in S^1$ is strictly on the left side of $\overrightarrow{x_sv_t}$ for any possible s and t. By Lemma 2.6 Equation (8), and using $\Delta^R(x_sv_t)$ with respect to the right side of $\overrightarrow{x_sv_t}$, we have

$$\left\{x_sv_t,\Delta\left(\left(x^{m'},y^{l'},z^{p'}\right),(u^n)\right)\right\}=\Delta^R(x_sv_t)=\sum_{i=1}^{s}Q_{x_i},$$

where $Q_{x_i}=\mathcal{J}(x_sv_t,x_iu)\cdot x_iv_t\cdot\Delta\left(\left(x^{m'}\backslash x_i\cup x_s,y^{l'},z^{p'}\right),(u^n)\right)$.

(1) Suppose that $1\le s\le m'$. If $i\le m'$ and $i\neq s$, then x_s appears twice in the left side n-tuple $\left(x^{m'}\backslash x_i\cup x_s,y^{l'},z^{p'}\right)$. We obtain

$$Q_{x_i}=\begin{cases}\frac{1}{2}\cdot x_sv_t\cdot\Delta\left(\left(x^{m'},y^{l'},z^{p'}\right),(u^n)\right) & \text{if } i=s,\\ 0 & \text{if } i<s.\end{cases}$$

Thus we get

$$\left\{x_sv_t,\Delta\left(\left(x^{m'},y^{l'},z^{p'}\right),(u^n)\right)\right\}=\frac{1}{2}\cdot x_sv_t\cdot\Delta\left(\left(x^{m'},y^{l'},z^{p'}\right),(u^n)\right).$$

Hence for $1\le s\le\min\{m,m'\}$ the sum over t for x_s equals

$$\frac{1}{\det M\cdot B}\sum_{t=1}^{n}(-1)^{s+t}\cdot\det M_{st}\cdot\{c_sv_t,B\}=\frac{1}{\det M\cdot B}\sum_{t=1}^{n}(-1)^{s+t}M_{st}\cdot\frac{1}{2}\cdot c_sv_t\cdot B=\frac{1}{2}.$$

(2) Suppose that $m'<s\le m$, we have

$$\begin{aligned}&\left\{x_sv_t,\Delta\left(\left(x^{m'},y^{l'},z^{p'}\right),(u^n)\right)\right\}\\ =\ &\sum_{i=1}^{m'}x_iv_t\cdot\Delta\left(\left(x^{m'}\backslash x_i\cup x_s,y^{l'},z^{p'}\right),(u^n)\right).\end{aligned}$$

The sum over t for x_s equals

$$\frac{1}{\det M\cdot B}\sum_{i=1}^{m'}\left(\Delta((x^m\backslash x_s\cup x_i,y^l,z^p),(v^n))\cdot\Delta((x^{m'}\backslash x_i\cup x_s,y^{l'},z^{p'}),(u^n))\right).$$

Since $i \leq m' < s \leq m$, we have x_i appears twice in the left side n-tuple $\left(x^m \backslash x_s \cup x_i, y^l, z^p\right)$. So

$$\Delta\left(\left(x^m \backslash x_s \cup x_i, y^l, z^p\right), (v^n)\right) = 0.$$

Hence in this case the sum over t for x_s equals 0.

The computations of the other two cases follow the similar strategy as above.

2. For the sum over t for y_s where $1 \leq s \leq l$:

Here $v \in S^1$ is strictly on the right side of $\overrightarrow{y_s v_t}$ for any possible s and t. By Lemma 2.6 Equation (9), and using $\Delta^L(y_s v_t)$ with respect to the left side of $\overrightarrow{y_s v_t}$, we have

$$\left\{y_s v_t, \Delta\left(\left(x^{m'}, y^{l'}, z^{p'}\right), (u^n)\right)\right\} = \sum_{i=s}^{l'} Q_{y_i},$$

where $Q_{y_i} = \mathcal{J}(y_s v_t, y_i v) \cdot y_i v_t \cdot \Delta\left(\left(x^{m'}, y^{l'} \backslash y_i \cup y_s, z^{p'}\right), (u^n)\right)$.

(1) Suppose that $1 \leq s \leq l'$. If $i \leq l'$ and $i \neq s$, then y_s appears twice in the left side n-tuple $\left(x^{m'}, y^{l'} \backslash y_i \cup y_s, z^{p'}\right)$. We have

$$Q_{y_i} = \begin{cases} -\frac{1}{2} \cdot y_s v_t \cdot \Delta\left(\left(x^{m'}, y^{l'}, z^{p'}\right), (u^n)\right) & \text{if } i = s, \\ 0 & \text{if } s < i \leq l'. \end{cases}$$

Hence for $1 \leq s \leq \min\{l, l'\}$ the sum over t for y_s equals $-\frac{1}{2}$.

(2) Suppose that $l' < s \leq l$, the summation is null in this case. Thus we obtain

$$\left\{y_s v_t, \Delta\left(\left(x^{m'}, y^{l'}, z^{p'}\right), (u^n)\right)\right\} = 0.$$

Hence in this case the sum over t for y_s equals 0.

3. For the sum over t for z_s where $1 \leq s \leq p$:

Here $v \in S^1$ is strictly on the right side of $\overrightarrow{z_s v_t}$ for any possible s and t. By Lemma 2.6 Equation (9), and using $\Delta^L(z_s v_t)$ with respect to the left side of $\overrightarrow{z_s v_t}$, we have

$$\begin{aligned} &\left\{z_s v_t, \Delta\left(\left(x^{m'}, y^{l'}, z^{p'}\right), (u^n)\right)\right\} \\ &= \sum_{i=s}^{p'} \mathcal{J}(z_s v_t, z_i v) \cdot z_i v_t \cdot \Delta\left(\left(x^{m'}, y^{l'}, z^{p'} \backslash z_i \cup z_s\right), (u^n)\right) \\ &+ \sum_{i=1}^{l'} \mathcal{J}(z_s v_t, y_i v) \cdot y_i v_t \cdot \Delta\left(\left(x^{m'}, y^{l'} \backslash y_i \cup z_s, z^{p'}\right), (u^n)\right). \end{aligned} \tag{14}$$

(1) Suppose that $1 \leq s \leq p'$. If $i \leq p'$ and $i \neq s$, we have z_s appears twice in the left side n-tuple $\left(x^{m'}, y^{l'}, z^{p'} \backslash z_i \cup z_s\right)$. Thus we get

$$\begin{aligned} &\mathcal{J}(z_s v_t, z_i v) \cdot z_i v_t \cdot \Delta\left(\left(x^{m'}, y^{l'}, z^{p'} \backslash z_i \cup z_s\right), (u^n)\right) \\ = &\begin{cases} -\frac{1}{2} \cdot z_s v_t \cdot \Delta\left(\left(x^{m'}, y^{l'}, z^{p'}\right), (u^n)\right) & \text{if } i = s, \\ 0 & \text{if } s < i \leq p'. \end{cases} \end{aligned}$$

For the third line of Equation (14), since z_s appears twice in the left side n-tuple $\left(x^{m'}, y^{l'}\backslash y_i \cup z_s, z^{p'}\right)$, we obtain

$$\Delta\left(\left(x^{m'}, y^{l'}\backslash y_i \cup z_s, z^{p'}\right), (u^n)\right) = 0.$$

Thus we get

$$\left\{z_s v_t, \Delta\left(\left(x^{m'}, y^{l'}, z^{p'}\right), (u^n)\right)\right\} = -\frac{1}{2}\cdot z_s v_t \cdot \Delta\left(\left(x^{m'}, y^{l'}, z^{p'}\right), (u^n)\right).$$

Hence for $1 \leq s \leq \min\{p, p'\}$ the sum over t for z_s equals $-\frac{1}{2}$.

(2) Suppose that $p' < s \leq p$, **by our condition** (*), we have $l \geq l'$. We get

$$\begin{aligned} & \left\{z_s v_t, \Delta\left(\left(x^{m'}, y^{l'}, z^{p'}\right), (u^n)\right)\right\} \\ = & \sum_{i=1}^{l'} \mathcal{J}(z_s v_t, y_i v)\cdot y_i v_t \cdot \Delta\left(\left(x^{m'}, y^{l'}\backslash y_i \cup z_s, z^{p'}\right), (u^n)\right) \\ = & -\sum_{i=1}^{l'} y_i v_t \cdot \Delta\left(\left(x^{m'}, y^{l'}\backslash y_i \cup z_s, z^{p'}\right), (u^n)\right). \end{aligned}$$

Thus the sum over t for z_s equals

$$\frac{-1}{\det M \cdot B}\sum_{i=1}^{l'} \Delta\left(\left(x^m, y^l, z^p\backslash z_s \cup y_i\right), (v^n)\right)\cdot \Delta\left(\left(x^{m'}, y^{l'}\backslash y_i \cup z_s, z^{p'}\right), (u^n)\right).$$

Observe that $i \leq l' \leq l$, thus y_i appears twice in the left side n-tuple $\left(x^m, y^l, z^p\backslash z_s \cup y_i\right)$, hence $\Delta\left(\left(x^m, y^l, z^p\backslash z_s \cup y_i\right), (v^n)\right) = 0$. Thus the above summation is zero. Hence in this case the sum over t for z_s equals 0.

Sum over all the above cases, we conclude that

$$C = \frac{1}{2}\cdot \min\{m, m'\} - \frac{1}{2}\cdot\min\{l, l'\} - \frac{1}{2}\cdot \min\{p, p'\}.$$

□

Definition 5.9. [ORIENTED EDGE RATIO] *Given an ideal triangulation $\mathcal{T}$ and its n-triangulation $\mathcal{T}_n$ of D_k. For the vertex $V \in \mathcal{I}_n \cup \mathcal{J}_n$ associating to the marked triangle $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ and a triple of non-negative integers (m, l, p) with $m + l + p = n$, we express the vertex V by $v_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}^{m,l,p}$. To the* oriented edges

$$e_1 = \overrightarrow{v_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}^{m+1,l,p-1} v_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}^{m+1,l-1,p}},\ e_2 = \overrightarrow{v_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}^{m-1,l+1,p} v_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}^{m,l+1,p-1}},\ e_3 = \overrightarrow{v_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}^{m,l-1,p+1} v_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}^{m-1,l,p+1}}$$

of $\mathcal{T}_n$ without touching the vertices of $\mathcal{T}$, we associate

$$\begin{aligned} E_{e_1} &:= E\left(x^{m+1}, y^{l-1}, z^{p-1} | y_l, z_p\right), \\ E_{e_2} &:= E\left(x^{m-1}, y^{l+1}, z^{p-1} | z_p, x_m\right), \\ E_{e_3} &:= E\left(x^{m-1}, y^{l-1}, z^{p+1} | x_m, y_l\right). \end{aligned}$$

Each one of them is called oriented edge ratio *of the corresponding arrow.*

Lemma 5.10. *The image of the Fock–Goncharov coordinates by $\theta_{\mathcal{T}_n}$ are the products of direct edge ratios as in Figure 7.*

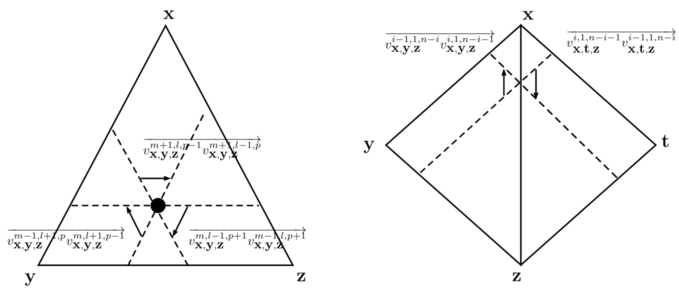

FIGURE 7. $\theta_{\mathcal{T}_n}(T_{m,l,p}(X,Y,Z))$ and $\theta_{\mathcal{T}_n}(\mathbb{B}_i(Y,T,Z,X))$

Proof. By Proposition 5.4, when $V \in \mathcal{J}_n$ is specified by $(\mathbf{x},\mathbf{y},\mathbf{z})$ and a triple of positive integers (m,l,p) with $m+l+p=n$, we have

$$\theta_{\mathcal{T}_n}(X_V) = E_{\overrightarrow{v^{m+1,l,p-1}_{\mathbf{x},\mathbf{y},\mathbf{z}} v^{m+1,l-1,p}_{\mathbf{x},\mathbf{y},\mathbf{z}}}} \cdot E_{\overrightarrow{v^{m-1,l+1,p}_{\mathbf{x},\mathbf{y},\mathbf{z}} v^{m,l+1,p-1}_{\mathbf{x},\mathbf{y},\mathbf{z}}}} \cdot E_{\overrightarrow{v^{m,l-1,p+1}_{\mathbf{x},\mathbf{y},\mathbf{z}} v^{m-1,l,p+1}_{\mathbf{x},\mathbf{y},\mathbf{z}}}}.$$

For $V \in \mathcal{I}'_n$ associating to $\overrightarrow{\mathbf{xz}}$ and $(i, n-i)$, suppose that two adjacent anticlockwise oriented ideal triangles $\overrightarrow{\mathbf{xyz}}$ and $\overrightarrow{\mathbf{xzt}}$ have a common edge $\overrightarrow{\mathbf{xz}}$, we have

$$\theta_{\mathcal{T}_n}(X_V) = -E_{\overrightarrow{v^{i-1,1,n-i}_{\mathbf{x},\mathbf{y},\mathbf{z}} v^{i,1,n-i-1}_{\mathbf{x},\mathbf{y},\mathbf{z}}}} \cdot E_{\overrightarrow{v^{i,1,n-i-1}_{\mathbf{x},\mathbf{t},\mathbf{z}} v^{i-1,1,n-i}_{\mathbf{x},\mathbf{t},\mathbf{z}}}}.$$

□

Definition 5.11. [LEVEL] *Given a triangulation $\mathcal{T}$ and its n-triangulation $\mathcal{T}_n$. For any vertex $\boldsymbol{x}$ of $\mathcal{T}$, we define the k-th* level *of $\boldsymbol{x}$ to be the union of the edges in*

$$\{v^{n-k,l,k-l}_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} v^{n-k,l-1,k-l+1}_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \mid \overline{\boldsymbol{xyz}} \textit{ is a triangle of } \mathcal{T} \textit{ for some } \boldsymbol{y},\boldsymbol{z},\ l=1,\cdots,k\}.$$

For any oriented edge e lying between i-th level and $(i+1)$-th level of $\boldsymbol{x}$, the sign $\epsilon_{\mathbf{x}}(e)$ of e with respect to $\boldsymbol{x}$ *is* $+1$ *(-1 resp.) if the arrow of e goes from the $(i+1)$-th level of $\boldsymbol{x}$ to the i-th level of $\boldsymbol{x}$ (otherwise resp.). For example $\epsilon_{\boldsymbol{x}}(e_1) = \epsilon_{\boldsymbol{x}}(e_2) = 1$ in Figure 8(1)(2).*

When the triangulation $\mathcal{T}$ is an ideal triangulation of D_k, the k-th level of $\boldsymbol{x}$ is topologically an interval, not a circle. In this case, for any two oriented edges e_1 and e_2 in $\mathcal{T}_n$, we say e_2 is after (before resp.) e_1 for $\boldsymbol{x}$, *if e_1 and e_2 both lie between i-th level and $(i+1)$-th level of $\boldsymbol{x}$ and e_2 is strictly after (before resp.) e_1 with respect to anticlockwise orientation centered at $\boldsymbol{x}$ as in Figure 8(1) ((2) resp.).*

Proposition 5.12. *Given an ideal triangulation $\mathcal{T}$ of D_k and its n-triangulation $\mathcal{T}_n$. Let e_1 (e_2 resp.) be the oriented edge of $\mathcal{T}_n$ lying inside the ideal triangle $\overline{\boldsymbol{xyz}}$ ($\overline{\boldsymbol{x'y'z'}}$ resp.) of $\mathcal{T}$. Then as in Figure 8 and 9, we have*

$$[E_{e_1}, E_{e_2}] = \begin{cases} \frac{1}{2}\cdot\epsilon_{\boldsymbol{u}}(e_1)\cdot\epsilon_{\boldsymbol{u}}(e_2), & e_2 \textit{ is after } e_1 \textit{ for } \boldsymbol{u}\in\{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\}\cap\{\boldsymbol{x'},\boldsymbol{y'},\boldsymbol{z'}\}; \\ -\frac{1}{2}\cdot\epsilon_{\boldsymbol{u}}(e_1)\cdot\epsilon_{\boldsymbol{u}}(e_2), & e_2 \textit{ is before } e_1 \textit{ for } \boldsymbol{u}\in\{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\}\cap\{\boldsymbol{x'},\boldsymbol{y'},\boldsymbol{z'}\}; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. The number of elements in $\#(\{\mathbf{x},\mathbf{y},\mathbf{z}\}\cap\{\mathbf{x'},\mathbf{y'},\mathbf{z'}\})$, denoted by $\mathcal{N}$, is zero or one or two or three. When $\mathcal{N}=0$, we have $[E_{e_1},E_{e_2}]=0$ since all the linking numbers are zero by setting the right side n-tuples properly.

When $\mathcal{N}=3$, suppose that the triangle $\overline{\mathbf{xyz}}$ of $\mathcal{T}$ has $\mathbf{x}\prec\mathbf{y}\prec\mathbf{z}\prec\mathbf{x}$ with respect to anticlockwise orientation of ∂D. Suppose that $x_{n-1}\prec\cdots\prec x_1\prec y_{n-1}\prec\cdots\prec$

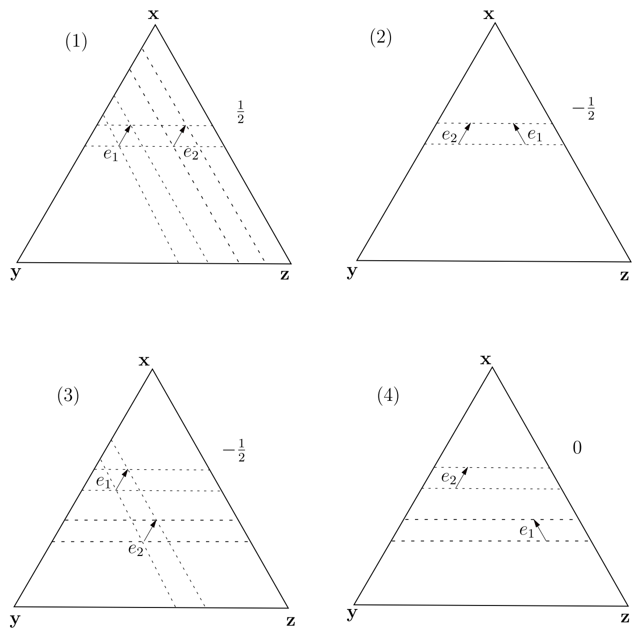

FIGURE 8. Computation of $[E_{e_1}, E_{e_2}]$. Consider the number of vertices that each vertex has e_1 and e_2 lying between two successive levels of that vertex. For (1)(3) the number is two, say $\mathbf{x}$ and $\mathbf{y}$; for (2)(4) the number is one, say $\mathbf{x}$.

$y_1 \prec z_{n-1} \prec \cdots \prec z_1 \prec x_{n-1}$ are anticlockwise ordered in $\mathcal{P}$ as shown in Figure 5. In the triangle $\overline{\mathbf{xyz}}$, for any given oriented edge, there are two different vertices in $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ such that for each one of them, say $\mathbf{u}$, this oriented edge lies between two successive levels of $\mathbf{u}$. Thus there is a common vertex, say $\mathbf{x}$, such that for the two oriented edges e_1, e_2 in $\mathcal{T}_n$, one lies between a-th level and $(a+1)$-th level of $\mathbf{x}$, and the other lies between a'-th level and $(a'+1)$-th level of $\mathbf{x}$. Since $[1/A, B] = -[A, B]$, we fix $\epsilon_{\mathbf{x}}(e_1) = \epsilon_{\mathbf{x}}(e_2) = 1$ without loss of generality. By symmetry, there are two cases as follows to be checked.

(1) Suppose that $e_1 = \overrightarrow{v^{a,b+1,c}_{\mathbf{x},\mathbf{y},\mathbf{z}} v^{a+1,b,c}_{\mathbf{x},\mathbf{y},\mathbf{z}}}$ and $e_2 = \overrightarrow{v^{a',b'+1,c'}_{\mathbf{x},\mathbf{y},\mathbf{z}} v^{a'+1,b',c'}_{\mathbf{x},\mathbf{y},\mathbf{z}}}$ where the non-negative integers a, b, c, a', b', c' satisfy $a+b+c = n-1$ and $a'+b'+c' = n-1$ as in Figure 8(1)(3).
   (a) If $b > b'$, we arrange $v_1, \cdots, v_n, u_1, \cdots, u_n$ so that $x_{n-1} \prec \cdots \prec x_1 \prec v_1 \prec \cdots \prec v_n \prec y_{n-1} \prec \cdots \prec y_1 \prec z_{n-1} \prec \cdots \prec z_1 \prec u_1 \prec \cdots \prec u_n \prec x_{n-1}$ as in Figure 5 for using Proposition 5.7. By Corollary 4.9, the swapping bracket between two oriented edge ratios does not depend on the right side n-tuples $(v^n), (u^n)$ that we choose. By Proposition 5.7 with the condition (*) $l \geq l'$ there, we have

$$\left[E\left(x^a, y^b, z^c | y_{b+1}, x_{a+1}\right), E\left(x^{a'}, y^{b'}, z^{c'} | y_{b'+1}, x_{a'+1}\right)\right]$$
$$= \left[(-1)^{2c+b} \frac{\Delta\left((x^a, y^{b+1}, z^c), (v^n)\right)}{\Delta\left((x^{a+1}, y^b, z^c), (v^n)\right)}, (-1)^{2c'+b'} \frac{\Delta\left((x^{a'}, y^{b'+1}, z^{c'}), (u^n)\right)}{\Delta\left((x^{a'+1}, y^{b'}, z^{c'}), (u^n)\right)}\right]$$

$$
\begin{aligned}
&= \left[\frac{\Delta\left((x^a,y^{b+1},z^c),(v^n)\right)}{\Delta\left((x^{a+1},y^b,z^c),(v^n)\right)},\frac{\Delta\left((x^{a'},y^{b'+1},z^{c'}),(u^n)\right)}{\Delta\left((x^{a'+1},y^{b'},z^{c'}),(u^n)\right)}\right]\\
&= \left[\Delta\left((x^a,y^{b+1},z^c),(v^n)\right),\Delta\left((x^{a'},y^{b'+1},z^{c'}),(u^n)\right)\right]\\
&- \left[\Delta\left((x^a,y^{b+1},z^c),(v^n)\right),\Delta\left((x^{a'+1},y^{b'},z^{c'}),(u^n)\right)\right]\\
&- \left[\Delta\left((x^{a+1},y^{b},z^c),(v^n)\right),\Delta\left((x^{a'},y^{b'+1},z^{c'}),(u^n)\right)\right]\\
&+ \left[\Delta\left((x^{a+1},y^{b},z^c),(v^n)\right),\Delta\left((x^{a'+1},y^{b'},z^{c'}),(u^n)\right)\right]\\
&= \left(\frac{1}{2}\cdot\min\{a,a'\}-\frac{1}{2}\cdot\min\{b+1,b'+1\}-\frac{1}{2}\cdot\min\{c,c'\}\right)\\
&- \left(\frac{1}{2}\cdot\min\{a,a'+1\}-\frac{1}{2}\cdot\min\{b+1,b'\}-\frac{1}{2}\cdot\min\{c,c'\}\right)\\
&- \left(\frac{1}{2}\cdot\min\{a+1,a'\}-\frac{1}{2}\cdot\min\{b,b'+1\}-\frac{1}{2}\cdot\min\{c,c'\}\right)\\
&+ \left(\frac{1}{2}\cdot\min\{a+1,a'+1\}-\frac{1}{2}\cdot\min\{b,b'\}-\frac{1}{2}\cdot\min\{c,c'\}\right)\\
&= \frac{1}{2}\cdot\left(\min\{a,a'\}-\min\{a,a'+1\}-\min\{a+1,a'\}+\min\{a+1,a'+1\}\right)\\
&- \frac{1}{2}\cdot\left(\min\{b,b'\}-\min\{b,b'+1\}-\min\{b+1,b'\}+\min\{b+1,b'+1\}\right)\\
&= \frac{1}{2}\cdot\left(\min\{a,a'\}-\min\{a,a'+1\}-\min\{a+1,a'\}+\min\{a+1,a'+1\}\right)\\
&= \begin{cases}\frac{1}{2} & \text{if } a'=a,\\ 0 & \text{if } a'\neq a.\end{cases}
\end{aligned}
$$

Note that in this case if $a'=a$, as in Figure 8(1) e_2 is after e_1 for $\mathbf{x}$.

(b) If $b<b'$, we have $b'>b$. Using the computation in Case (1a), we get

$$
\begin{aligned}
&\left[E\left(x^a,y^b,z^c|y_{b+1},x_{a+1}\right),E\left(x^{a'},y^{b'},z^{c'}|y_{b'+1},x_{a'+1}\right)\right]\\
=\;& -\left[E\left(x^{a'},y^{b'},z^{c'}|y_{b'+1},x_{a'+1}\right),E\left(x^a,y^b,z^c|y_{b+1},x_{a+1}\right)\right]\\
=\;& \begin{cases}-\frac{1}{2} & \text{if } a'=a,\\ 0 & \text{if } a'\neq a.\end{cases}
\end{aligned}
$$

Note that in this case if $b<b'$ and $a'=a$, the oriented edge e_2 is before e_1 for $\mathbf{x}$.

(c) If $b=b'$ and $c<c'$, as in Figure 8(3) e_2 is before e_1 for $\mathbf{y}$ and $a\neq a'$. By Proposition 5.7 with the condition (*) $p\leq p'$ there, similarly we obtain

$$
\begin{aligned}
&\left[E\left(x^a,y^b,z^c|y_{b+1},x_{a+1}\right),E\left(x^{a'},y^{b'},z^{c'}|y_{b'+1},x_{a'+1}\right)\right]\\
=\;& \frac{1}{2}\cdot\left(\min\{a,a'\}-\min\{a,a'+1\}-\min\{a+1,a'\}+\min\{a+1,a'+1\}\right)\\
-\;& \frac{1}{2}\cdot\left(\min\{b,b'\}-\min\{b,b'+1\}-\min\{b+1,b'\}+\min\{b+1,b'+1\}\right)
\end{aligned}
$$

$$= \quad -\frac{1}{2}.$$

If $b = b'$ and $c > c'$, we have e_2 is after e_1 for $\mathbf{y}$, $c' < c$ and $a \neq a'$. Using the above computation, we get

$$\begin{aligned}
& \left[E\left(x^a, y^b, z^c | y_{b+1}, x_{a+1}\right), E\left(x^{a'}, y^{b'}, z^{c'} | y_{b'+1}, x_{a'+1}\right)\right] \\
= \quad & -\left[E\left(x^{a'}, y^{b'}, z^{c'} | y_{b'+1}, x_{a'+1}\right), E\left(x^a, y^b, z^c | y_{b+1}, x_{a+1}\right)\right] = \frac{1}{2}.
\end{aligned}$$

If $b = b'$ and $c = c'$, then $a = a'$. Thus in this case we have

$$[E_{e_1}, E_{e_2}] = 0.$$

(2) Suppose that $e_1 = \overrightarrow{v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{a,b,c+1} v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{a+1,b,c}}$ and $e_2 = \overrightarrow{v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{a',b'+1,c'} v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{a'+1,b',c'}}$ where the non-negative integers a, b, c, a', b', c' satisfy $a+b+c = n-1$ and $a'+b'+c' = n-1$ as in Figure 8(2)(4).

(a) If $b > b'$, by Proposition 5.7 with the condition (*) $l \geq l'$ there, we have

$$\begin{aligned}
& \left[E\left(x^a, y^b, z^c | z_{c+1}, x_{a+1}\right), E\left(x^{a'}, y^{b'}, z^{c'} | y_{b'+1}, x_{a'+1}\right)\right] \\
= \quad & \left[(-1)^{b+c}\frac{\Delta\left((x^a, y^b, z^{c+1}), (v^n)\right)}{\Delta\left((x^{a+1}, y^b, z^c), (v^n)\right)}, (-1)^{b'+2c'}\frac{\Delta\left((x^{a'}, y^{b'+1}, z^{c'}), (u^n)\right)}{\Delta\left((x^{a'+1}, y^{b'}, z^{c'}), (u^n)\right)}\right] \\
= \quad & \left[\frac{\Delta\left((x^a, y^b, z^{c+1}), (v^n)\right)}{\Delta\left((x^{a+1}, y^b, z^c), (v^n)\right)}, \frac{\Delta\left((x^{a'}, y^{b'+1}, z^{c'}), (u^n)\right)}{\Delta\left((x^{a'+1}, y^{b'}, z^{c'}), (u^n)\right)}\right] \\
= \quad & \left(\frac{1}{2}\cdot\min\{a, a'\} - \frac{1}{2}\cdot\min\{b, b'+1\} - \frac{1}{2}\cdot\min\{c+1, c'\}\right) \\
- \quad & \left(\frac{1}{2}\cdot\min\{a, a'+1\} - \frac{1}{2}\cdot\min\{b, b'\} - \frac{1}{2}\cdot\min\{c+1, c'\}\right) \\
- \quad & \left(\frac{1}{2}\cdot\min\{a+1, a'\} - \frac{1}{2}\cdot\min\{b, b'+1\} - \frac{1}{2}\cdot\min\{c, c'\}\right) \\
+ \quad & \left(\frac{1}{2}\cdot\min\{a+1, a'+1\} - \frac{1}{2}\cdot\min\{b, b'\} - \frac{1}{2}\cdot\min\{c, c'\}\right) \\
= \quad & \frac{1}{2}\cdot\left(\min\{a, a'\} - \min\{a, a'+1\} - \min\{a+1, a'\} + \min\{a+1, a'+1\}\right) \\
= \quad & \begin{cases} \frac{1}{2} & \text{if } a' = a, \\ 0 & \text{if } a' \neq a. \end{cases}
\end{aligned}$$

Note that in this case if $a' = a$, then e_2 is after e_1 for $\mathbf{x}$.

(b) If $b \leq b'$, then $b' \geq b$. Using the computation in Case (2a), we get

$$\begin{aligned}
& \left[E\left(x^a, y^b, z^c | z_{c+1}, x_{a+1}\right), E\left(x^{a'}, y^{b'}, z^{c'} | y_{b'+1}, x_{a'+1}\right)\right] \\
= \quad & -\left[E\left(x^{a'}, y^{b'}, z^{c'} | y_{b'+1}, x_{a'+1}\right), E\left(x^a, y^b, z^c | z_{c+1}, x_{a+1}\right)\right] \\
= \quad & -\frac{1}{2}\cdot\left(\left(\min\{a, a'\} - \min\{a, a'+1\} - \min\{a+1, a'\} + \min\{a+1, a'+1\}\right)\right)
\end{aligned}$$

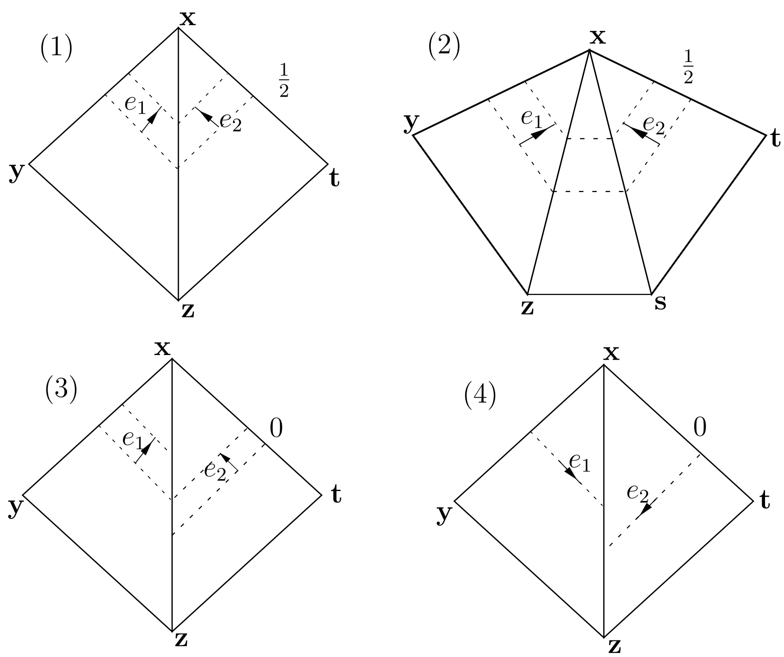

FIGURE 9. Computation of $[E_{e_1}, E_{e_2}]$ when the triangles for two oriented edge ratios have two common vertices or one common vertex.

$$= \begin{cases} -\frac{1}{2} & \text{if } a' = a, \\ 0 & \text{if } a' \neq a. \end{cases}$$

Note that in this case if $a' = a$, then e_2 is before e_1 for $\mathbf{x}$ as in Figure 8(2).

We conclude that for $\mathcal{N} = 3$

$$[E_{e_1}, E_{e_2}] = \begin{cases} \frac{1}{2} \cdot \epsilon_{\mathbf{u}}(e_1) \cdot \epsilon_{\mathbf{u}}(e_2), & e_2 \; is \; after \; e_1 \; for \; \mathbf{u} \in \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}; \\ -\frac{1}{2} \cdot \epsilon_{\mathbf{u}}(e_1) \cdot \epsilon_{\mathbf{u}}(e_2), & e_2 \; is \; before \; e_1 \; for \; \mathbf{u} \in \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}; \\ 0, & otherwise. \end{cases}$$

Suppose $\mathcal{N} = 2$ as in Figure 9(1)(3)(4). If $a \leq a'$ ((a, b, c) and (a', b', c') are defined similarly as the case $\mathcal{N} = 3$ with respect to $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and $(\mathbf{x}, \mathbf{z}, \mathbf{t})$), we can combine the points $t_1, \cdots, t_{c'} \in \mathcal{P}$ with the points $x_1, \cdots, x_{a'} \in \mathcal{P}$ and apply Proposition 5.7 with the condition $l \geq l' = 0$ there. If $a \geq a'$, we use $[E_{e_1}, E_{e_2}] = -[E_{e_2}, E_{e_1}]$ for arguing the same way as above. The case $\mathcal{N} = 1$ follows in a similar way. □

Proof of Theorem 5.5. To prove the theorem, we have to verify that

$$[\theta_{\mathcal{T}_n}(X_A), \theta_{\mathcal{T}_n}(X_\tau)] = \frac{\theta_{\mathcal{T}_n}(\{X_A, X_\tau\}_n)}{\theta_{\mathcal{T}_n}(X_A) \cdot \theta_{\mathcal{T}_n}(X_\tau)} = \varepsilon_{A\tau},$$

for any $A, \tau \in \mathcal{I}'_n \cup \mathcal{J}_n$.

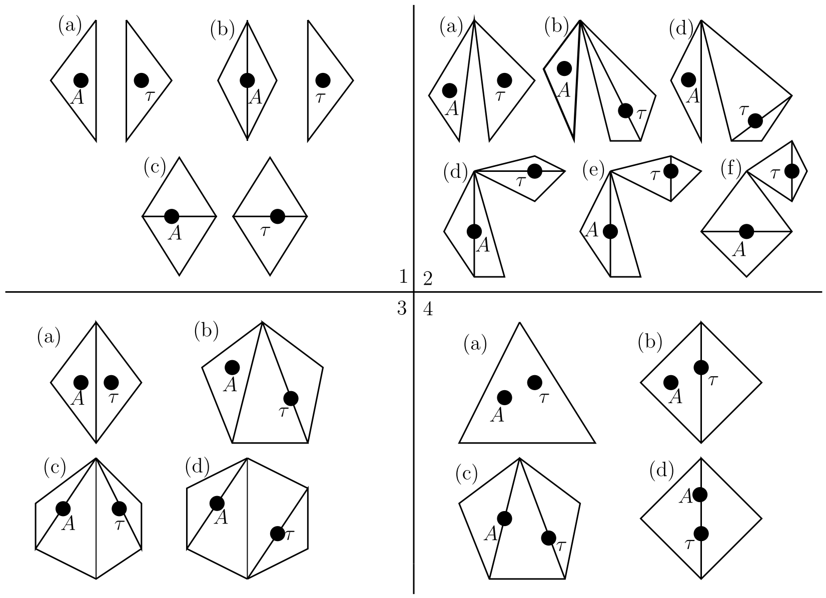

FIGURE 10. All the cases up to symmetry

Each $V \in \mathcal{I}'_n$ is related to a graph with 4-gon and an edge in the ideal triangulation $\mathcal{T}$, and each $V \in \mathcal{J}_n$ is related to a triangle in the ideal triangulation $\mathcal{T}$. By symmetry, we have the following possible cases as shown in Figure 10:

(1) Two graphs associated with A and τ are separated by a line;
(2) Two graphs associated with A and τ have one common point;
(3) Two graphs associated with A and τ have two common points;
(4) other cases.

Following Lemma 5.10, we write $\theta_{\mathcal{T}_n}(X_A)$ and $\theta_{\mathcal{T}_n}(X_\tau)$ as the products of directed edge ratios. Then we use Proposition 5.12 to compute their swapping bracket case by case. We prove the case in Figure 10 $4(a)$ explicitly and leave the details for the other cases to the reader.

Suppose $A \in \mathcal{J}_n$ ($\tau \in \mathcal{J}_n$ resp.) is specified by $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and a triple of positive integers (m, l, p) where $m+l+p = n$ ((m', l', p') where $m'+l'+p' = n$ resp.). Let $e_1 = \overrightarrow{v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{m+1,l,p-1} v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{m+1,l-1,p}}$, $e_2 = \overrightarrow{v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{m-1,l+1,p} v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{m,l+1,p-1}}$, $e_3 = \overrightarrow{v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{m,l-1,p+1} v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{m-1,l,p+1}}$, $e'_1 = \overrightarrow{v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{m'+1,l',p'-1} v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{m'+1,l'-1,p'}}$, $e'_2 = \overrightarrow{v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{m'-1,l'+1,p'} v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{m',l'+1,p'-1}}$, $e'_3 = \overrightarrow{v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{m',l'-1,p'+1} v_{\mathbf{x},\mathbf{y},\mathbf{z}}^{m'-1,l',p'+1}}$. So

$$\theta_{\mathcal{T}_n}(X_A) = E_{e_1} \cdot E_{e_2} \cdot E_{e_3}, \quad \theta_{\mathcal{T}_n}(X_\tau) = E_{e'_1} \cdot E_{e'_2} \cdot E_{e'_3}.$$

By the Leibniz's rule, we have

$$\begin{aligned} & \left[\theta_{\mathcal{T}_n}(X_A), \theta_{\mathcal{T}_n}(X_\tau)\right] = \left[E_{e_1} E_{e_2} E_{e_3}, E_{e'_1} E_{e'_2} E_{e'_3}\right] \\ = & \sum_{j=1}^{3} \sum_{i=1}^{3} \left[E_{e_i}, E_{e'_j}\right]. \end{aligned}$$

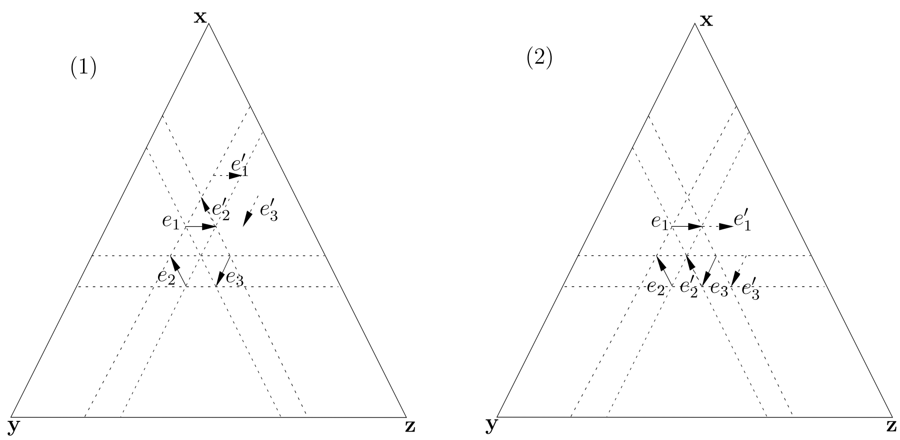

FIGURE 11. Checking $[\theta_{\mathcal{T}_n}(X_A), \theta_{\mathcal{T}_n}(X_\tau)] = \varepsilon_{A\tau}$.

Then we use Proposition 5.12 for computing $\left[E_{e_i}, E_{e'_j}\right]$ for any $i, j = 1, 2, 3$. Firstly, by observing Figure 11(1), we have

$$\sum_{i=1}^{3} \left[E_{e_i}, E_{e'_j}\right] = 0$$

if e'_j lies outside the three strips formed by the dashed lines. Then, we compute case by case for e'_j lying inside these stripes. We get the following property, denoted by $(**)$:

$$\sum_{i=1}^{3} \left[E_{e_i}, E_{e'_j}\right] \neq 0$$

if and only if e'_j lies strictly inside the hexagon formed by the vertices of e_1, e_2, e_3. The property (**) provides us the only six possible edges for e'_j that we have $\sum_{i=1}^{3} \left[E_{e_i}, E_{e'_j}\right] \neq 0$. Of course, when $(m, l, p) = (m', l', p')$, we have

$$[\theta_{\mathcal{T}_n}(X_A), \theta_{\mathcal{T}_n}(X_\tau)] = 0 = \varepsilon_{A\tau}.$$

When $(m, l, p) \neq (m', l', p')$, without loss of generality, suppose that $l > l'$.

(1) When $l - 1 > l'$, we have $l - 1 \geq l' + 1$. As shown in Figure 11(1), none of e'_1, e'_2, e'_3 lie strictly inside the hexagon formed by the vertices of e_1, e_2, e_3. By property (**), we obtain

$$[\theta_{\mathcal{T}_n}(X_A), \theta_{\mathcal{T}_n}(X_\tau)] = 0 = \varepsilon_{A\tau}.$$

(2) When $l - 1 = l'$:

   (a) When $m' \neq m$ and $m' \neq m + 1$, none of e'_1, e'_2, e'_3 lie strictly inside the hexagon formed by the vertices of e_1, e_2, e_3. By property (**), we obtain

   $$[\theta_{\mathcal{T}_n}(X_A), \theta_{\mathcal{T}_n}(X_\tau)] = 0 = \varepsilon_{A\tau}.$$

   (b) when $m' = m$, by Figure 11(2), we have

   $$[\theta_{\mathcal{T}_n}(X_A), \theta_{\mathcal{T}_n}(X_\tau)] = \left[E_{e_1}E_{e_2}E_{e_3}, E_{e'_1}E_{e'_2}E_{e'_3}\right]$$

$$
\begin{aligned}
&= \sum_{i=1}^{3}\left[E_{e_i}, E_{e'_1}\right] + \sum_{i=1}^{3}\left[E_{e_i}, E_{e'_2}\right] + \sum_{i=1}^{3}\left[E_{e_i}, E_{e'_3}\right] \\
&= 0 + \left(0 + \frac{1}{2} + \frac{1}{2}\right) + 0 \\
&= 1.
\end{aligned}
$$

(c) Similarly, when $m' = m+1$, we have

$$\left[\theta_{\mathcal{T}_n}\left(X_A\right), \theta_{\mathcal{T}_n}\left(X_\tau\right)\right] = -1.$$

In the cases of Figure 10 4(a), we conclude that

$$\left[\theta_{\mathcal{T}_n}\left(X_A\right), \theta_{\mathcal{T}_n}\left(X_\tau\right)\right] = \varepsilon_{A\tau}.$$

□

6. Compatibility between any $\theta_{\mathcal{T}_n}$ and $\theta_{\mathcal{T}'_n}$

For any two ideal triangulations $\mathcal{T}$ and $\mathcal{T}'$ of D_k, a finite sequence of flips can transform $\mathcal{T}$ to $\mathcal{T}'$. To prove that $\theta_{\mathcal{T}_n}$ and $\theta_{\mathcal{T}'_n}$ are compatible, it is enough to prove the compatibility when $\mathcal{T}'$ is obtained from $\mathcal{T}$ by one flip at the edge e. Let $\mathcal{T}_n$ and $\mathcal{T}'_n$ be the n-triangulations of $\mathcal{T}$ and $\mathcal{T}'$ respectively. As shown in [FG06, Section 10.3], the flip f_e is a composition of the mutations where each mutation corresponds to a Plücker relation for $\mathbb{K}^n$. Let us denote the transition map for the flip f_e by μ_e^X. It induces a rational map μ_e^{X*} from $\mathcal{FX}(\mathcal{T}'_n)$ to $\mathcal{FX}(\mathcal{T}_n)$.

Proposition 6.1. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two ideal triangulations of D_k such that $\mathcal{T}'$ is obtained from $\mathcal{T}$ by a flip f_e at the edge e. For any $Q \in \mathcal{FX}(\mathcal{T}'_n)$, we have*

$$\theta_{\mathcal{T}_n} \circ \mu_e^{X*}(Q) = \theta_{\mathcal{T}'_n}(Q).$$

Proof. Since the homomorphism $\theta_{\mathcal{T}_n}$ is induced from the homomorphism χ_n, to prove the proposition, it is enough to prove the Plücker relation in $\mathcal{Z}_n(\mathcal{P})$. □

The proof of the following Plücker relation for the rank n swapping algebra $\mathcal{Z}_n(\mathcal{P})$ also works for the Plücker relation for $\mathbb{K}^n$ even for the degenerate case. We are not aware of an appropriate reference for this fact, and provide a proof here.

Lemma 6.2. [Plücker relation] *For any $x^{n-2}, a, b, c, d, u_1, \cdots, u_n \in \mathcal{P}$, we have the following equality in $\mathcal{Z}_n(\mathcal{P})$*

$$
\begin{aligned}
&\Delta\left(\left(x^{n-2}, a, d\right), \left(u_1, \cdots, u_n\right)\right) \cdot \Delta\left(\left(x^{n-2}, b, c\right), \left(u_1, \cdots, u_n\right)\right) \\
&+ \Delta\left(\left(x^{n-2}, a, b\right), \left(u_1, \cdots, u_n\right)\right) \cdot \Delta\left(\left(x^{n-2}, c, d\right), \left(u_1, \cdots, u_n\right)\right) \\
&= \Delta\left(\left(x^{n-2}, a, c\right), \left(u_1, \cdots, u_n\right)\right) \cdot \Delta\left(\left(x^{n-2}, b, d\right), \left(u_1, \cdots, u_n\right)\right).
\end{aligned}
\tag{15}
$$

Proof. Let $\widehat{u_j} := u_1, \cdots, u_{j-1}, u_{j+1}, \cdots, u_n$. We have

$$
\begin{aligned}
&\Delta\left(\left(x^{n-2}, a, d\right), \left(u^n\right)\right) \cdot \Delta\left(\left(x^{n-2}, b, c\right), \left(u^n\right)\right) \\
&+ \Delta\left(\left(x^{n-2}, a, b\right), \left(u^n\right)\right) \cdot \Delta\left(\left(x^{n-2}, c, d\right), \left(u^n\right)\right) \\
&- \Delta\left(\left(x^{n-2}, a, c\right), \left(u^n\right)\right) \cdot \Delta\left(\left(x^{n-2}, b, d\right), \left(u^n\right)\right) \\
&= \sum_{j=1}^{n} \Delta\left(\left(x^{n-2}, a\right), \left(\widehat{u_j}\right)\right) \cdot \left\{(-1)^{j+n} \cdot du_j \cdot \Delta\left(\left(x^{n-2}, b, c\right), \left(u^n\right)\right)\right. \\
&\left. +(-1)^{j+n} \cdot bu_j \cdot \Delta\left(\left(x^{n-2}, c, d\right), \left(u^n\right)\right) - (-1)^{j+n} \cdot cu_j \cdot \Delta\left(\left(x^{n-2}, b, d\right), \left(u^n\right)\right)\right\}
\end{aligned}
\tag{16}
$$

For $i = 1, \cdots, n-2$, we get

$$(-1)^{n+i} \cdot \Delta\left(\left(x^{n-2}, a, x_i\right), (u^n)\right) = \sum_{j=1}^{n} (-1)^{i+j} \cdot \Delta\left(\left(x^{n-2}, a\right), (\widehat{u_j})\right) \cdot x_i u_j = 0.$$

Using the above formula, we obtain that the right hand side of Equation (16) equals

$$\begin{aligned}
&-\sum_{j=1}^{n} \Delta\left(\left(x^{n-2}, a\right), (\widehat{u_j})\right) \cdot \\
&\left\{\sum_{i=1}^{n-2} (-1)^{i+j} x_i u_j \cdot \Delta\left((x_1, \cdots, x_{i-1}, x_{i+1} \cdots, x_{n-2}, b, c, d), (u^n)\right)\right. \\
&+(-1)^{n-1+j} \cdot b u_j \cdot \Delta\left(\left(x^{n-2}, c, d\right), (u^n)\right) + (-1)^{n+j} \cdot c u_j \cdot \Delta\left(\left(x^{n-2}, b, d\right), (u^n)\right) \\
&\left.+(-1)^{n+1+j} \cdot d u_j \cdot \Delta\left(\left(x^{n-2}, b, c\right), (u^n)\right)\right\} \\
&= -\sum_{j=1}^{n} \Delta\left(\left(x^{n-2}, a\right), (\widehat{u_j})\right) \cdot \Delta\left(\left(x^{n-2}, b, c, d\right), (u^n, u_j)\right) \\
&= 0.
\end{aligned}$$

□

Since χ_n does not depend on the ideal triangulation, as a consequence, we again prove the following result.

Corollary 6.3. *The rank n Fock–Goncharov Poisson bracket $\{\cdot,\cdot\}_n$ does not depend on the ideal triangulation.*

7. From cross fractions to $(n \times n)$-determinant ratios

We will relate a cross fraction to a product of two $(n \times n)$-determinant ratios through the following identification of points on circles.

Definition 7.1. *For $n \geq 2$, let $\mathcal{P} = \{s \prec w \prec \cdots \prec t \prec s\}$ and*

$$\mathcal{P}_{n-1} = \{s_{n-1} \prec \cdots \prec s_1 \prec w_{n-1} \prec \cdots \prec w_1 \prec \cdots \prec t_{n-1} \prec \cdots \prec t_1 \prec s_{n-1}\},$$

where each $r \in \mathcal{P}$ corresponds to $n-1$ anticlockwise ordered points $r_{n-1}, \ldots, r_1$ nearby in $\mathcal{P}_{n-1}$ as in Figure 12(1)(2). Let $\mathcal{RT}_n(\mathcal{P})$ be the sub fraction ring of $\mathcal{Q}_n(\mathcal{P})$ generated by all elements like $\frac{yx}{zx}$. Let $\mathcal{DR}_n(\mathcal{P}_{n-1})$ be the sub fraction ring of $\mathcal{Q}_n(\mathcal{P}_{n-1})$ generated by all elements like $E\left(x^{n-1}|y_1, z_1\right)$.

The homomorphism *μ from $\mathcal{RT}_n(\mathcal{P})$ to $\mathcal{DR}_n(\mathcal{P}_{n-1})$ is defined by extending the following formula on the generators to $\mathcal{RT}_n(\mathcal{P})$ using Leibniz's rule*

$$\mu\left(\frac{yx}{zx}\right) = E\left(x^{n-1}|y_1, z_1\right).$$

Proposition 7.2. *The homomorphism μ is well-defined and injective.*

Proof. By Theorem 4.2 $B_{n\mathbb{K}}/S_{n\mathbb{K}} \cong \mathcal{Z}_n(\mathcal{P})$ and $B'_{n\mathbb{K}}/S'_{n\mathbb{K}} \cong \mathcal{Z}_n(\mathcal{P}_{n-1})$. We consider the geometric model instead. We embed $B_{n\mathbb{K}}/S_{n\mathbb{K}}$ into $B'_{n\mathbb{K}}/S'_{n\mathbb{K}}$ with respect to the homomorphism μ in the following way. We associate a vector for $B_{n\mathbb{K}}/S_{n\mathbb{K}}$ corresponding to x on the left to a vector for $B'_{n\mathbb{K}}/S'_{n\mathbb{K}}$ corresponding to x_1 on the left, a covector in $B_{n\mathbb{K}}/S_{n\mathbb{K}}$ corresponding to x on the right to the wedge of $(n-1)$ vectors in $B'_{n\mathbb{K}}/S'_{n\mathbb{K}}$ corresponding to $x_1, \cdots, x_{n-1}$ on the left. Thus any relation in $\mathcal{RT}_n(\mathcal{P})$ generated by the $(n+1)\times(n+1)$ determinants in $R_n(\mathcal{P})$ is one-to-one

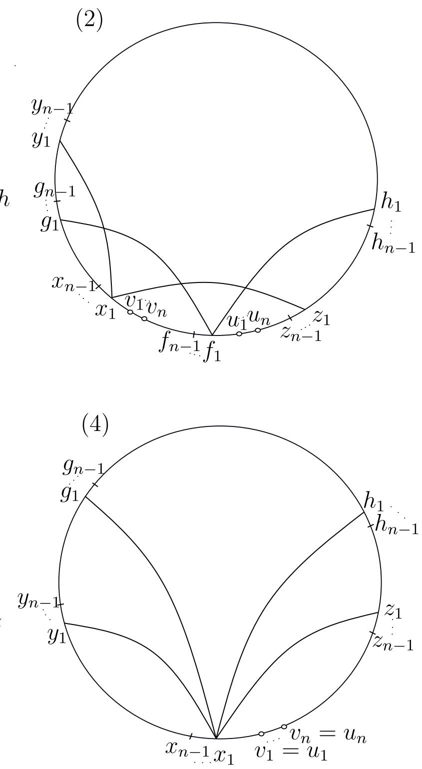

Figure 12.

correspondence with a relation in $\mu(\mathcal{RT}_n(\mathcal{P}))$ obtained from replacing each ordered pair by a $(n \times n)$-determinant with fixed right side n-tuple. We conclude that the homomorphism μ is well-defined and injective. $\square$

Theorem 7.3. *The injective homomorphism μ is Poisson with respect to the swapping bracket.*

Proof. It is enough to prove the theorem for generators in $\mathcal{RT}_n(\mathcal{P})$. For two arbitrary generators $\frac{yx}{zx}, \frac{gf}{hf} \in \mathcal{RT}_n(\mathcal{P})$, we want to show that

$$\mu\left(\left[\frac{yx}{zx}, \frac{gf}{hf}\right]\right) = \left[\mu\left(\frac{yx}{zx}\right), \mu\left(\frac{gf}{hf}\right)\right].$$

Firstly we have

$$\mu\left(\left[\frac{yx}{zx}, \frac{gf}{hf}\right]\right) = \mu([yx, gf]) - \mu([yx, hf]) - \mu([zx, gf]) + \mu([zx, hf]). \tag{17}$$

When $x \neq f$, we arrange v^n (u^n resp.) immediately after x_1 (f_1 resp.) with respect to the anticlockwise orientation as in Figure 12(2). Then we can use the fact that the linking number only depends on the corresponding position of the four points.

This crucial arrangement allows us to get

$$\begin{aligned}
&[\Delta((x^{n-1},y_1),(v^n)),\Delta((f^{n-1},g_1),(u^n))]\\
&=\frac{\mathcal{J}(y_1v_1,g_1u_1)\cdot\Delta((x^{n-1},g_1),(v^n))\cdot\Delta((f^{n-1},y_1),(u^n))}{\Delta((x^{n-1},y_1),(v^n))\cdot\Delta((f^{n-1},g_1),(u^n))}\\
&=\mathcal{J}(y_1x_1,g_1f_1)\cdot E(x^{n-1}|g_1,y_1)\cdot E(f^{n-1}|y_1,g_1)\\
&=\mathcal{J}(yx,gf)\cdot\mu\left(\frac{gx}{yx}\right)\cdot\mu\left(\frac{yf}{gf}\right)\\
&=\mu\left([yx,gf]\right).
\end{aligned}\tag{18}$$

We have similar formulas for the other three terms in the right hand side of Equation (17). Thus we obtain

$$\begin{aligned}
&\mu\left(\left[\frac{yx}{zx},\frac{gf}{hf}\right]\right)\\
&=[\Delta((x^{n-1},y_1),(v^n)),\Delta((f^{n-1},g_1),(u^n))]-[\Delta((x^{n-1},y_1),(v^n)),\Delta((f^{n-1},h_1),(u^n))]\\
&-[\Delta((x^{n-1},z_1),(v^n)),\Delta((f^{n-1},g_1),(u^n))]+[\Delta((x^{n-1},z_1),(v^n)),\Delta((f^{n-1},h_1),(u^n))]\\
&=\left[E(x^{n-1}|y_1,z_1),E(f^{n-1}|g_1,h_1)\right]=\left[\mu\left(\frac{yx}{zx}\right),\mu\left(\frac{gf}{hf}\right)\right].
\end{aligned}\tag{19}$$

When $x=f$, we arrange the same way as above and $v_i=u_i$ for $i=1,\cdots,n$ as in Figure 12(4). By explicit computation, we get the same results as Equation (18) and Equation (19) in this case. We conclude that the homomorphism μ is Poisson with respect to the swapping bracket. $\square$

As a consequence, for $n=3$, our main theorem generalizes [Su14, Chapter 3] if we replace the elements like $E\left(x^2|y_1,z_1\right)$ by $\frac{yx}{zx}$. The advantage of the expression $\frac{yx}{zx}$ is that it allows us to get rid of $x_2\in\mathcal{P}$, but it only works for $n\leq 3$. Using the Poisson homomorphism μ, we have the following result.

Corollary 7.4. *The Poisson homomorphism in* [L18, Theorem 10.7.2] *is still Poisson after replacing each cross fraction by a product of two $(n\times n)$-determinant ratios.*

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