RANK n SWAPPING ALGEBRA FOR GRASSMANNIAN

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Abstract. The *rank n swapping algebra* is the Poisson algebra defined on the ordered pairs of points on a circle using the linking numbers, where a subspace of $(\mathbb{K}^n \times \mathbb{K}^{n*})^r / \mathrm{GL}(n, \mathbb{K})$ is its geometric mode. In this paper, we find an injective Poisson homomorphism from the Poisson algebra on Grassmannian $G_{n,r}$ arising from boundary measurement map to the rank n swapping fraction algebra.

1. Introduction

We provide a geometric interpretation of bi-Poisson structure on Grassmannian using a circle.

In [L18], Labourie introduced the swapping algebra on ordered pairs of points on a circle using the linking numbers. It was used to characterize the Goldman Poisson structure [G84] on the character variety and the second Adler–Gel'fand–Dickey Poisson structure via Drinfel'd–Sokolov reduction [DS81] on the space $Opers_n$ of $\mathrm{SL}(n, \mathbb{R})$-opers with trivial holonomy for any $n > 1$. The ordered pairs of points on a circle should be understood as a pairing between a vector and a covector in a vector space $\mathbb{K}^n$ where $\mathbb{K}$ is a field of characteristic zero. When the dimension n is fixed, by [W39][CP76], all these pairings generate the polynomial ring of a subspace of $(\mathbb{K}^n \times \mathbb{K}^{n*})^r / \mathrm{GL}(n, \mathbb{K})$. The relations among these pairings are generated by $(n+1) \times (n+1)$ determinant relations. In [Su17], we show that these polynomial relations are Poisson ideal with respect to the swapping bracket and define the quotient algebra, called *rank n swapping algebra*. Actually, the swapping bracket depends on two parameters. We call (α, β)-*swapping bracket*, denoted by $\{\cdot,\cdot\}_{\alpha,\beta}$.

On the other hand, in [GSV09], Gekhtman et al. found two dimensional family of Poisson brackets on the open Schubert cell of Grassmannian induced from the boundary measurement map [Po06], denoted by $\{\cdot,\cdot\}_{B_{\alpha,\beta}}$. The parameters (α, β) are used to describe the R-matrix in [GSV09, Section 4]. Moreover, they show that these Poisson brackets are compatible with the natural cluster algebra structure [FZ02][GSV03], and the Grassmannian equipped with such Poisson bracket is a Poisson homogeneous space with respect to the natural action of GL_n equipped with an R-matrix Poisson–Lie structure [Se83]. In [GSSV12], Gekhtman et al. show that every cluster algebra compatible Poisson structure on the open Schubert cell of Grassmannian can be obtained as above.

2010 *Mathematics Subject Classification.* Primary 17B63; Secondary 14M15.

Key words and phrases. Circle, Poisson algebra, Grassmannian, rank n swapping algebra.

The author was partially supported by the Luxembourg National Research Fund(FNR) AFR bilateral grant COALAS 11802479-2. The author also acknowledge support from U.S. National Science Foundation grants DMS-1107452, 1107263, 1107367 RNMS:GEometric structures And Representation varieties (the GEAR Network).

Let $G^{\mathbf{I}}_{n,r}$ be the open Schubert cell of Grassmannian with respect to a subset $\mathbf{I}$ of $\{1,\cdots,r\}$. Let $\mathcal{Q}_n(\mathcal{P})$ be the fraction ring of the swapping ring where $\#\mathcal{P}=r$. We use the ratio of two $(n\times n)$ determinants as a coordinate.

Theorem 1.1 (Main theorem)**.** *We give an injective Poisson algebra homomorphism from a coordinate fraction ring of $G^{\mathbf{I}}_{n,r}$ to $\mathcal{Q}_n(\mathcal{P})$ with respect to the Poisson bracket $\{\cdot,\cdot\}_{B_{\alpha,\beta}}$ and the $(\beta-\alpha,\alpha+\beta)$-swapping bracket.*

In section 2, we recall the swapping algebra and the rank n swapping algebra. In section 3, we recall the Poisson structure on the Grassmannian induced from boundary measurement map and then prove the main theorem.

2. Rank n swapping algebra

In this section, we recall the swapping algebra [L18] and the rank n swapping algebra [Su17]. We prove some basic facts using the linking numbers on a circle. Lemma 2.10 ([Su17, Lemma 3.5, Remark 3.6]) is the key technical formula for computation.

2.1. Swapping algebra.

For $\mathcal{P}$ a cyclic subset of S^1, we represent an ordered pair (r,x) of $\mathcal{P}$ by the expression rx. Then we consider the associative commutative ring

$$\mathcal{Z}(\mathcal{P}):=\mathbb{K}[\{xy\}_{x,y\in\mathcal{P}}]/\{xx\}_{x\in\mathcal{P}} \tag{1}$$

over a field $\mathbb{K}$ of characteristic zero, where $\{xy\}_{x,y\in\mathcal{P}}$ are the set of variables. Then we equip $\mathcal{Z}(\mathcal{P})$ with a swapping bracket depending on the linking numbers, defined as follows.

Definition 2.1. [linking number] *Let (r,x,s,y) be a quadruple of points in $\mathcal{P}\subset S^1$. Let o be any point different from $r,x,s,y\in S^1$. Let σ be a homeomorphism from $S^1\backslash o$ to $\mathbb{R}$ with respect to the* ***anticlockwise*** *orientation of S^1. Let $\triangle(a)=-1;0;1$ whenever $a<0$; $a=0$; $a>0$ respectively.*

The linking number *between rx and sy is*

$$\begin{aligned}\mathcal{J}(rx,sy)&=\frac{1}{2}\cdot\triangle(\sigma(r)-\sigma(x))\cdot\triangle(\sigma(r)-\sigma(y))\cdot\triangle(\sigma(y)-\sigma(x))\\&-\frac{1}{2}\cdot\triangle(\sigma(r)-\sigma(x))\cdot\triangle(\sigma(r)-\sigma(s))\cdot\triangle(\sigma(s)-\sigma(x)).\end{aligned} \tag{2}$$

In fact, the value of $\mathcal{J}(rx,sy)$ belongs to $\{0,\pm1,\pm\frac{1}{2}\}$, and does not depend on the choice of the point o and depends only on the relative positions of r,x,s,y. In Figure 1, we describe five possible values of $\mathcal{J}(rx,sy)$.

Definition 2.2. [(α,β)-swapping bracket] *Let $\alpha,\beta\in\mathbb{K}$, the (α,β)-*swapping bracket *over $\mathcal{Z}(\mathcal{P})$ is defined by extending the following formula on arbitrary generators rx, sy to $\mathcal{Z}(\mathcal{P})$ using the* Leibniz's rule

$$\{rx,sy\}_{\alpha,\beta}=\mathcal{J}(rx,sy)\cdot(\alpha\cdot ry\cdot sx+\beta\cdot rx\cdot sy). \tag{3}$$

We denote the $(1,0)$-swapping bracket by $\{\cdot,\cdot\}$, called the swapping bracket.

It is easy to see that

$$\{\cdot,\cdot\}_{\alpha,\beta}=\alpha\{\cdot,\cdot\}_{1,0}+\beta\{\cdot,\cdot\}_{0,1}. \tag{4}$$

By direct computations, Labourie proved the following theorem.

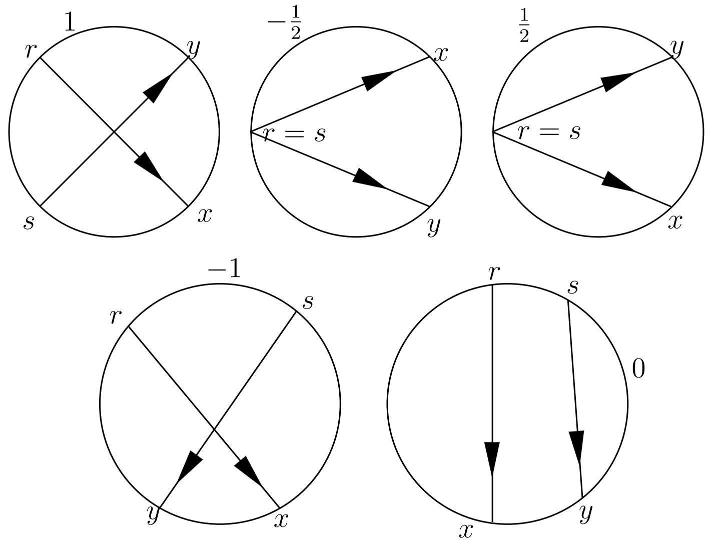

FIGURE 1. Linking number $\mathcal{J}(rx, sy)$ between rx and sy

Theorem 2.3. [LABOURIE [L18]] *The (α,β)-swapping bracket is Poisson.*

Let $\mathcal{Q}(\mathcal{P})$ be the field of fractions of $\mathcal{Z}(\mathcal{P})$. We extend the (α,β)-swapping bracket to $\mathcal{Q}(\mathcal{P})$ by

$$\{rx, \frac{1}{sy}\}_{\alpha,\beta} = -\frac{\{rx, sy\}_{\alpha,\beta}}{sy^2}. \tag{5}$$

Definition 2.4. [CROSS FRACTION] *Let x, y, z, t belong to $\mathcal{P}$ so that $x \neq t$ and $y \neq z$. The* cross fraction *of (x, y, z, t) is an element of $\mathcal{Q}(\mathcal{P})$:*

$$\frac{xz}{xt} \cdot \frac{yt}{yz}. \tag{6}$$

Let $\mathcal{B}(\mathcal{P})$ be the subring of $\mathcal{Q}(\mathcal{P})$ generated by cross fractions.

Definition 2.5. The (α,β)-swapping fraction (multifraction resp.) algebra of $\mathcal{P}$ *is the ring $\mathcal{Q}(\mathcal{P})$ ($\mathcal{B}(\mathcal{P})$ resp.) equipped with the (α,β)-swapping bracket, denoted by $(\mathcal{Q}(\mathcal{P}), \{\cdot,\cdot\}_{\alpha,\beta})$ ($(\mathcal{B}(\mathcal{P}), \{\cdot,\cdot\}_{\alpha,\beta})$ resp.).*

We define $(\mathcal{Q}(\mathcal{P}), \{\cdot,\cdot\})$ ($(\mathcal{B}(\mathcal{P}), \{\cdot,\cdot\})$ resp.) to be swapping fraction (multifraction resp.) algebra of $\mathcal{P}$.

We will prove that the ring $\mathcal{B}(\mathcal{P})$ is closed under $\{\cdot,\cdot\}_{\alpha,\beta}$ as follows.

Lemma 2.6. *For any $ab, \frac{xz}{xt} \cdot \frac{yt}{yz} \in \mathcal{Q}(\mathcal{P})$, we have*

$$\big\{ab, \frac{xz}{xt} \cdot \frac{yt}{yz}\big\}_{0,1} = 0. \tag{7}$$

As a consequence, for any $\mu, \nu \in \mathcal{B}(\mathcal{P})$, we obtain

$$\{\mu, \nu\}_{0,1} = 0. \tag{8}$$

Proof. By the Leibniz's rule, we have

$$\frac{\big\{ab, \frac{xz}{xt} \cdot \frac{yt}{yz}\big\}_{0,1}}{ab \cdot \frac{xz}{xt} \cdot \frac{yt}{yz}} = \frac{\big\{ab, xz\big\}_{0,1}}{ab \cdot xz} - \frac{\big\{ab, xt\big\}_{0,1}}{ab \cdot xt} + \frac{\big\{ab, yt\big\}_{0,1}}{ab \cdot yt} - \frac{\big\{ab, yz\big\}_{0,1}}{ab \cdot yz}. \tag{9}$$

By the cocycle identity of the linking number [L18] Definition 2.1.1(8):

$$\mathcal{J}(ab,cd)+\mathcal{J}(ab,de)+\mathcal{J}(ab,ec)=0, \tag{10}$$

the right hand side of Equation (9) equals

$$\begin{aligned}&\mathcal{J}(ab,xz)-\mathcal{J}(ab,xt)+\mathcal{J}(ab,yt)-\mathcal{J}(ab,yz)\\ &=\mathcal{J}(ab,tz)+\mathcal{J}(ab,zt)\\ &=0.\end{aligned} \tag{11}$$

□

As a consequence, we obtain

Proposition 2.7. *The ring $\mathcal{B}(\mathcal{P})$ is closed under $\{\cdot,\cdot\}_{\alpha,\beta}$.*

Proof. For any $\mu,\nu\in\mathcal{B}(\mathcal{P})$, by Lemma 2.6, we have

$$\{\mu,\nu\}_{\alpha,\beta}=\alpha\cdot\{\mu,\nu\}_{1,0}+\beta\cdot\{\mu,\nu\}_{0,1}=\alpha\cdot\{\mu,\nu\}_{1,0}. \tag{12}$$

By [Su17] Proposition 2.9, we obtain $\alpha\cdot\{\mu,\nu\}_{1,0}$ belongs to $\mathcal{B}(\mathcal{P})$. Hence we conclude that $\mathcal{B}(\mathcal{P})$ is closed under $\{\cdot,\cdot\}_{\alpha,\beta}$. □

2.2. Rank n swapping algebra.

Definition 2.8. *For any $d>1$ and any $x_1,\cdots,x_d,y_1,\cdots,y_d\in\mathcal{P}$, recall the notation*

$$\Delta\left((x_1,\cdots,x_d),(y_1,\cdots,y_d)\right):=\det\begin{pmatrix}x_1y_1 & \cdots & x_1y_d\\ \cdots & \cdots & \cdots\\ x_dy_1 & \cdots & x_dy_d\end{pmatrix}\in\mathcal{Z}_n(\mathcal{P}). \tag{13}$$

We call $(x_1,\cdots,x_d)$ ($(y_1,\cdots,y_d)$ resp) the left (right resp.) side n-tuple of *the determinant $\Delta\left((x_1,\cdots,x_d),(y_1,\cdots,y_d)\right)$.*

Definition 2.9. [THE RANK n SWAPPING RING $\mathcal{Z}_n(\mathcal{P})$] *For $n\geq 2$, let $R_n(\mathcal{P})$ be the ideal of $\mathcal{Z}(\mathcal{P})$ generated by*
$\{D\in\mathcal{Z}(\mathcal{P})\mid D=\Delta\left((x_1,\cdots,x_{n+1}),(y_1,\cdots,y_{n+1})\right),\forall x_1,,x_{n+1},y_1,\cdots,y_{n+1}\in\mathcal{P}\}$.
The rank n swapping ring *$\mathcal{Z}_n(\mathcal{P})$ is the quotient ring $\mathcal{Z}(\mathcal{P})/R_n(\mathcal{P})$.*

Lemma 2.10. [[Su17, Lemma 3.5, Remark 3.6]] *For any integer $m\geq 2$, suppose $x_1,\cdots,x_m$ ($y_1,\cdots,y_m$ resp.) in $\mathcal{P}$ are mutually distinct and anticlockwise ordered. Assume that a,b belong to $\mathcal{P}$ and $x_1,\cdots,x_l,y_1,\cdots,y_k$ are on the* ***right*** *side of the oriented edge $\overrightarrow{ab}$ (include coinciding with a or b) as illustrated in Figure 2. Let u (v resp.) be strictly on the left (right resp.) side of $\overrightarrow{ab}$. Let*

$$\begin{aligned}\Delta^R(ab)=&\sum_{d=1}^{l}\mathcal{J}(ab,x_du)\cdot x_db\cdot\Delta((x_1,\cdots,x_{d-1},a,x_{d+1},\cdots,x_m),(y_1,\cdots,y_m))\\ &+\sum_{d=1}^{k}\mathcal{J}(ab,uy_d)\cdot ay_d\cdot\Delta((x_1,\cdots,x_m),(y_1,\cdots,y_{d-1},b,y_{d+1},\cdots,y_m)),\end{aligned} \tag{14}$$

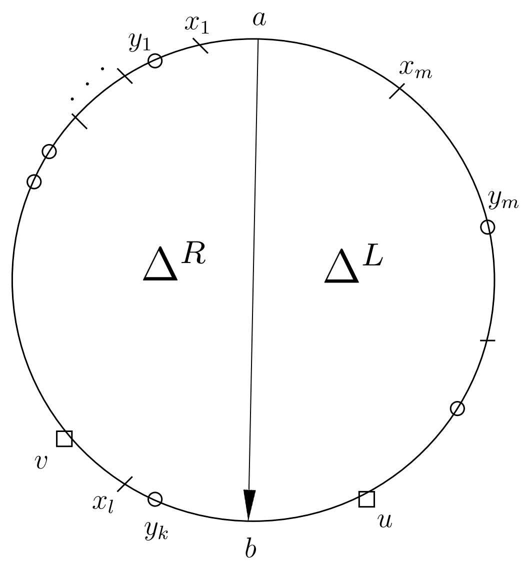

FIGURE 2. $\{ab, \Delta((x_1, \cdots, x_m), (y_1, \cdots, y_m))\}$

(15)
$$\Delta^L(ab) = \sum_{d=k+1}^{m} \mathcal{J}(ab, x_d v) \cdot x_d b \cdot \Delta((x_1, \cdots, x_{d-1}, a, x_{d+1}, \cdots, x_m), (y_1, \cdots, y_m))$$
$$+ \sum_{d=l+1}^{m} \mathcal{J}(ab, v y_d) \cdot a y_d \cdot \Delta((x_1, \cdots, x_m), (y_1, \cdots, y_{d-1}, b, y_{d+1}, \cdots, y_m)),$$

then we have

$$\{ab, \Delta((x_1, \cdots, x_m), (y_1, \cdots, y_m))\} = \Delta^R(ab) = \Delta^L(ab). \tag{16}$$

Lemma 2.11. *With the notations same as the lemma above, we have*

$$\{ab, \Delta((x_1, \cdots, x_m), (y_1, \cdots, y_m))\}_{0,1} = K \cdot ab \cdot \Delta((x_1, \cdots, x_m), (y_1, \cdots, y_m)),$$

for some constant K.

Proof. Since

$$\Delta((x_1, \cdots, x_m), (y_1, \cdots, y_m)) = \sum_{\sigma \in S_m} \epsilon_\sigma \prod_i x_i y_{\sigma(i)} \tag{17}$$

where ϵ_σ is the sign of σ in the permutation group S_m,

$$\begin{aligned} &\{ab, \Delta((x_1, \cdots, x_m), (y_1, \cdots, y_m))\}_{0,1} \\ &= ab \cdot \sum_{\sigma \in S_m} \epsilon_\sigma \Big(\sum_{i=1}^{m} \mathcal{J}(ab, x_i y_{\sigma(i)}) \Big) \prod_i x_i y_{\sigma(i)} \\ &= ab \cdot \sum_{\sigma \in S_m} \epsilon_\sigma K(\sigma) \prod_i x_i y_{\sigma(i)}, \end{aligned} \tag{18}$$

where $K(\sigma)=\sum_{i=1}^{m}\mathcal{J}(ab,x_iy_{\sigma(i)})$. For any transposition (kl) in the permutation group S_m, we get

$$\begin{aligned}&K(\sigma(kl))-K(\sigma)\\&=\mathcal{J}(ab,x_ky_{\sigma(l)})+\mathcal{J}(ab,x_ly_{\sigma(k)})-\mathcal{J}(ab,x_ky_{\sigma(k)})-\mathcal{J}(ab,x_ly_{\sigma(l)})\\&=\mathcal{J}(ab,y_{\sigma(k)}y_{\sigma(l)})+\mathcal{J}(ab,y_{\sigma(l)}y_{\sigma(k)})\\&=0.\end{aligned}\tag{19}$$

For any $\sigma',\sigma\in S_m$, σ' is related to σ by a sequence of transpositions. Thus $K(\sigma)=K(\sigma')$. Then we denote $K(\sigma)$ by K. Finally, we conclude that

$$\{ab,\Delta((x_1,\cdots,x_m),(y_1,\cdots,y_m))\}_{0,1}=K\cdot ab\cdot\Delta((x_1,\cdots,x_m),(y_1,\cdots,y_m)).\tag{20}$$

□

The following proposition is a consequence of the above two lemmas.

Proposition 2.12. *$R_n(\mathcal{P})$ is a Poisson ideal of $\mathcal{Z}(\mathcal{P})$ with respect to the (α,β)-swapping bracket.*

Definition 2.13. [(α,β)-RANK n SWAPPING ALGEBRA OF $\mathcal{P}$] *The* (α,β)-rank n swapping algebra of $\mathcal{P}$ *is the ring $\mathcal{Z}_n(\mathcal{P})=\mathcal{Z}(\mathcal{P})/R_n(\mathcal{P})$ equipped with the (α,β)-swapping bracket, denoted by $(\mathcal{Z}_n(\mathcal{P}),\{\cdot,\cdot\}_{\alpha,\beta})$.*

We define $(\mathcal{Z}_n(\mathcal{P}),\{\cdot,\cdot\})$ to be rank n swapping algebra of $\mathcal{P}$.

By Theorem 4.7 in [Su17], $\mathcal{Z}_n(\mathcal{P})$ is an integral domain. Generators of $\mathcal{Z}_n(\mathcal{P})$ are non-zero divisors, so the cross fractions are well defined in the field of fractions of $\mathcal{Z}_n(\mathcal{P})$.

Let $\mathcal{Q}_n(\mathcal{P})$ be the field of fractions of $\mathcal{Z}_n(\mathcal{P})$. Let $\mathcal{B}_n(\mathcal{P})$ be the sub-fraction ring of $\mathcal{Z}_n(\mathcal{P})$ generated by cross fractions.

Definition 2.14. *Then, the* (α,β)-rank n swapping fraction (multifraction resp.) algebra of $\mathcal{P}$ *is $\mathcal{Q}_n(\mathcal{P})$ ($\mathcal{B}_n(\mathcal{P})$ resp.) equipped with the (α,β)-swapping bracket, denoted by $(\mathcal{Q}_n(\mathcal{P}),\{\cdot,\cdot\}_{\alpha,\beta})$ ($(\mathcal{B}_n(\mathcal{P}),\{\cdot,\cdot\}_{\alpha,\beta})$ resp.).*

We define $(\mathcal{Q}_n(\mathcal{P}),\{\cdot,\cdot\})$ ($(\mathcal{B}_n(\mathcal{P}),\{\cdot,\cdot\})$ resp.) to be the rank n swapping fraction (multifraction resp.) algebra of $\mathcal{P}$.

2.3. **$(n\times n)$-determinant ratio.** Let us recall the $(n\times n)$-determinant ratio in [Su15, Section 4].

Definition 2.15. [$(n\times n)$-DETERMINANT RATIO] *Let $x_1,\cdots,x_{n-1},y\in\mathcal{P}$ be different from each other. The* $(n\times n)$-determinant ratio *of $x_1,\cdots,x_{n-1},t,y$:*

$$E(x_1,\cdots,x_{n-1}|t,y):=\frac{\Delta\left((x_1,\cdots,x_{n-1},t),(v_1,\cdots,v_n)\right)}{\Delta\left((x_1,\cdots,x_{n-1},y),(v_1,\cdots,v_n)\right)}\tag{21}$$

for any $v_1,\cdots,v_n\in\mathcal{P}$ different from each other. By [Su15, Corollary 4.5]*, the $(n\times n)$-determinant ratio does not depend on $v_1,\cdots,v_n$ that we choose.*

The fraction ring $\mathcal{D}_n(\mathcal{P})$ generated by all the $(n\times n)$-determinant ratios is called $(n\times n)$-determinant ratio fraction ring.

Remark 2.16. *As a consequence of* [Su15, Corollary 4.5]*, the swapping bracket between any element in $\mathcal{Q}_n(\mathcal{P})$ and $E(x_1,\cdots,x_{n-1}|t,y)$ does not depend on the choice of right side n-tuple $(v_1,\cdots,v_n)$ of $E(x_1,\cdots,x_{n-1}|t,y)$. We can calculate the swapping bracket between two $(n\times n)$-determinant ratios with the right side n-tuples in any preferred position.*

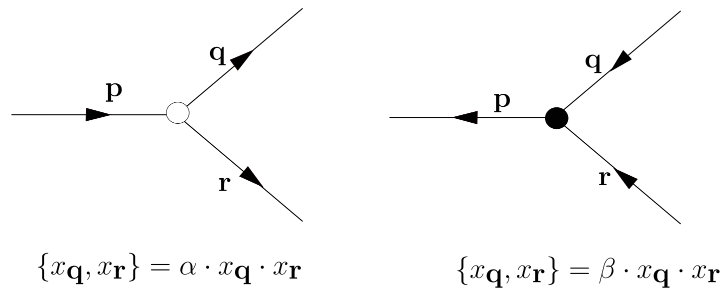

FIGURE 3.

3. Rank n swapping algebra on Grassmannian

3.1. Poisson structure on Grassmannian.

Definition 3.1. [GRASSMANNIAN] *Let n and r be two integers such that $0 < n \leq r$. Let $\mathbb{K}$ be a field of characteristic zero (example $\mathbb{R}$ or $\mathbb{C}$). The* Grassmannian *$\boldsymbol{Gr}(n,r)$ is the manifold of n-dimensional linear subspaces in $\mathbb{K}^r$.*

Let $\mathrm{Mat}^*(n,r)$ be the set of $(n \times r)$-matrices over $\mathbb{K}$ with rank n. Then

$$\mathbf{Gr}(n,r) = \mathrm{GL}_k \setminus \mathrm{Mat}^*(n,r) \tag{22}$$

where GL_k acts on $\mathrm{Mat}^*(n,r)$ by left multiplication.

Definition 3.2. [COORDINATE] *Let $\mathbf{I}$ be a n-element subset of $\{1, \cdots, r\}$. The* Plücker coordinate *$\Delta_{\mathbf{I}}$ of $\mathrm{Mat}^*(n,r)$ is the minor of the matrix formed by the columns of the matrix indexed by $\mathbf{I}$ with respect to lexicographical order.*

The Schubert cell

$$\boldsymbol{Gr}(n,r)^{\mathbf{I}} := \{A \in \boldsymbol{Gr}(n,r) \mid \Delta_{\mathbf{I}}(A) \neq 0,\ \mathbf{I} \text{ is lexicographically minimal}\}. \tag{23}$$

The Grassmannian $\boldsymbol{Gr}(n,r)$ is a disjoint union $\bigcup_{\mathbf{I}} \boldsymbol{Gr}(n,r)^{\mathbf{I}}$.

Observe that for any $g \in \mathrm{GL}_k$,

$$\Delta_{\mathbf{I}}(g \cdot A) = \det(g) \cdot \Delta_{\mathbf{I}}(A). \tag{24}$$

The subset $\mathbf{I}(i \to j)$ is obtained from $\mathbf{I}$ by replacing i by j. Then

$$\left\{ m_{ij} = \frac{\Delta_{\mathbf{I}(i \to j)}}{\Delta_{\mathbf{I}}} \right\}_{i \in \mathbf{I}, j \notin \mathbf{I}} \tag{25}$$

form a coordinate system *on the cell $\boldsymbol{Gr}(n,r)^{\mathbf{I}}$.*

Definition 3.3. [DIRECTED PLANAR GRAPH FOR $\mathbf{Gr}(n,r)$] *Let (V,E) be a* directed planar graph on a disk *with r vertices $v_1, \cdots, v_r$ ordered anticlockwise on the boundary, there are n ($r-n$ resp.) vertices on the boundary such that each one of these vertices has exactly one outgoing (incoming resp.) edge, each inner vertex has one incoming edge and two outgoing edges (called white vertex) or two incoming edges and one outgoing edge (called black vertex) as in Figure 3.*

To a vertex v and the edge e connecting to v, let $\mathbf{p}$ be (v,e), we assign a variable $x_{\mathbf{p}} \in \mathbb{R}\backslash\{0\}$. Let d be the cardinality of these pairs. We define a Poisson structure on $(\mathbb{R}\backslash\{0\})^d$ as follows. For a white (black resp.) vertex, labelling from incoming (outgoing resp.) edge clockwise $x_{\mathbf{p}}, x_{\mathbf{q}}, x_{\mathbf{r}}$, as in Figure 3, the Poisson bracket is $\{x_{\mathbf{q}}, x_{\mathbf{r}}\}_N = \alpha \cdot x_{\mathbf{q}} \cdot x_{\mathbf{r}}$ ($\{x_{\mathbf{q}}, x_{\mathbf{r}}\}_N = \beta \cdot x_{\mathbf{q}} \cdot x_{\mathbf{r}}$ resp.), otherwise zero.

Definition 3.4. [PERFECT PLANAR NETWORK] *Given* (V, E)*, let* w *be a map*

$$w : (\mathbb{R}\backslash\{0\})^d \to \mathbb{R}^{|E|} \tag{26}$$

such that the entry for the edge e *is* $w_e = x_{\boldsymbol{a}} x_{\boldsymbol{b}}$ *where* $\boldsymbol{a} = (v, e)$ *and* $\boldsymbol{b} = (u, e)$*. We define the* perfect planar network $N = (V, E, \{w_e\}_{e \in E})$*. Each perfect planar network defines a* space of edge weights E_N *to be* $w((\mathbb{R}\backslash\{0\})^d)$*.*

Definition 3.5. [BOUNDARY MEASUREMENT MAP] *The boundary measurement map* b *is a rational map from* E_N *to the cell* $\boldsymbol{Gr}(n, r)^{\mathbf{I}}$ *given as follows. Choose a unique representative in* $\mathrm{Mat}^*(n, r)$ *so that the sub* $n \times n$ *matrix formed by* $\mathbf{I}$ *columns* $I_{n\times n}$*, the other* (i, j)*-entry of the* $(n \times r)$*-matrix is defined to be the sum of the products of the weights of all paths starting at* v_i *and end at* v_j*.*

The map $b \circ w$ induces the $\{\cdot,\cdot\}_{B_{\alpha,\beta}}$ Poisson bracket on $\mathbf{Gr}(n, r)^{\mathbf{I}}$ from the Poisson bracket $\{\cdot,\cdot\}_N$ on $(\mathbb{R}\backslash\{0\})^d$.

We identity the set $\{1, 2, \cdots, r\}$ used to enumerate the columns with the anticlockwise ordered set $\mathcal{P} = \{a_1, a_2, \cdots, a_r\}$ on a circle.

Definition 3.6. [$\{\cdot,\cdot\}_{B_{\alpha,\beta}}$ POISSON BRACKET [GSV09, Theorem 3.3]] *Let* $\mathcal{P} = \{a_1, a_2, \cdots a_r\}$ *be ordered anticlockwise on a circle. The symbol* $\prec$ *is used to denote anticlockwise ordering on a circle. Let* parallel number *be*

$$s_{||}(a_i a_j, a_{i'} a_{j'}) = \begin{cases} 1 \quad if \quad a_i \prec a_{i'} \prec a_{j'} \prec a_j \prec a_i; \\ -1 \quad if \quad a_{i'} \prec a_i \prec a_j \prec a_{j'} \prec a_{i'}; \\ \dfrac{1}{2} \quad if \quad a_i = a_{i'} \prec a_{j'} \prec a_j \prec a_i \text{ or } a_i \prec a_{i'} \prec a_{j'} = a_j \prec a_i; \\ -\dfrac{1}{2} \quad if \quad a_{i'} = a_i \prec a_j \prec a_{j'} \prec a_{i'} \text{ or } a_{i'} \prec a_i \prec a_j = a_{j'} \prec a_{i'}; \\ 0 \quad otherwise. \end{cases} \tag{27}$$

As in Figure 4, the parallel number depends only on the corresponding position of four points. For any n*-element subset* $\mathbf{I}$*, the* $\{\cdot,\cdot\}_{B_{\alpha,\beta}}$ *Poisson bracket of* $\boldsymbol{Gr}(n, r)^{\mathbf{I}}$ *is given by*

$$\{m_{ij}, m_{i'j'}\}_{B_{\alpha,\beta}} = (\alpha-\beta)\cdot s_{||}(a_i a_j, a_{i'} a_{j'})\cdot m_{ij'}\cdot m_{i'j} + (\alpha+\beta)\cdot \mathcal{J}(a_i a_j, a_{i'} a_{j'})\cdot m_{ij}\cdot m_{i'j'}. \tag{28}$$

Remark 3.7. *When* $\boldsymbol{I} = \{1, 2, \cdots, n\}$*, the above Poisson bracket can be also found in* [BGY06]*. The parameters* (α, β) *are used to describe the R-matrix in* [GSV09, Section 4].

3.2. Main theorem.

Theorem 3.8. [MAIN THEOREM] *Let* $\mathcal{P} = \{a_1 \prec \cdots \prec a_r \prec a_1\}$ *be ordered anticlockwise on a circle, let the field of fractions* $\mathcal{Q}_n(\mathcal{P})$ *of the rank* n *swapping ring* $\mathcal{Z}_n(\mathcal{P})$ *equipped with the* $(\beta - \alpha, \alpha + \beta)$*-swapping bracket* $\{\cdot,\cdot\}_{\beta-\alpha,\alpha+\beta}$*. For any* n*-element subset* $\mathbf{I} = \{k_1, \cdots, k_n\}$ *with* $k_1 < \cdots < k_n$*. If* $i \in \mathbf{I}$*, let* $\widehat{a_i}$ *be* $a_{k_1}, \cdots, a_{k_n}$ *with* a_i *removed. There is an injective algebra homomorphism*

$$\theta_{\alpha,\beta} : \mathbb{K}[\{m_{ij}\}_{i\in\mathbf{I}, j\notin\mathbf{I}}] \to \mathcal{Q}_n(\mathcal{P}) \tag{29}$$

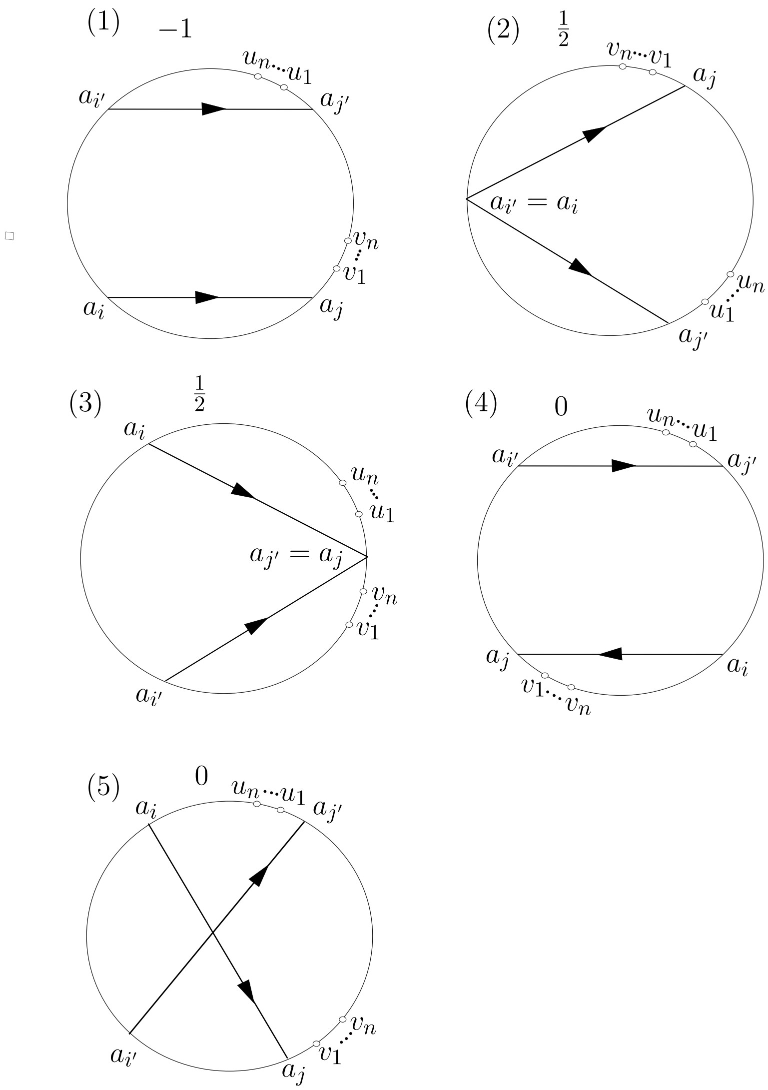

FIGURE 4. Parallel number $s_{||}(a_i a_j, a_{i'} a_{j'})$, arrangement of $(v_1, \cdots, v_n)$ and $(u_1, \cdots, u_n)$

defined by extending the following formula on arbitrary generator m_{ij} *to the polynomial ring*

$$\theta_{\alpha,\beta}\left(m_{ij}\right) = E\left(\widehat{a_i}|a_j, a_i\right). \tag{30}$$

Then the algebra homomorphism $\theta_{\alpha,\beta}$ *is Poisson with respect to* $\{\cdot,\cdot\}_{B_{\alpha,\beta}}$ *and* $\{\cdot,\cdot\}_{\beta-\alpha,\alpha+\beta}$.

We prove the main theorem by several steps. Firstly, we send the ratios of $(n\times n)$-determinant in $\mathbb{K}^n$ to the $(n \times n)$-determinant ratios for $\mathcal{Z}_n(\mathcal{P})$. The injectivity follows the same argument in [Su15, Proposition 5.3] using [Su17, Theorem 4.6].

Notation 3.9. *Let us denote* $\frac{\{A,B\}_{\alpha,\beta}}{A\cdot B}$ *by* $[A,B]_{\alpha,\beta}$.

Moreover, we denote $[A,B]_{1,0}$ *by* $[A,B]$.

Lemma 3.10.

(31)
$$\left\{E\left(\widehat{a_i}|a_j,a_i\right),E\left(\widehat{a_{i'}}|a_{j'},a_{i'}\right)\right\}_{0,1}=\mathcal{J}(a_ia_j,a_{i'}a_{j'})\cdot E\left(\widehat{a_i}|a_j,a_i\right)\cdot E\left(\widehat{a_{i'}}|a_{j'},a_{i'}\right),$$

Proof. Let $(u_1,\cdots,u_n)$ be the right side n-tuple of the two $(n\times n)$-determinant ratios. By the Leibniz's rule, we obtain

(32)
$$\begin{aligned}&\left[E\left(\widehat{a_i}|a_j,a_i\right),E\left(\widehat{a_{i'}}|a_{j'},a_{i'}\right)\right]_{0,1}\\&=\left[\Delta\left((\widehat{a_i},a_j),(u^n)\right),\Delta\left((\widehat{a_{i'}},a_{j'}),(u^n)\right)\right]-\left[\Delta\left((\widehat{a_i},a_j),(u^n)\right),\Delta\left((\widehat{a_{i'}},a_{i'}),(u^n)\right)\right]\\&-\left[\Delta\left((\widehat{a_i},a_i),(u^n)\right),\Delta\left((\widehat{a_{i'}},a_{j'}),(u^n)\right)\right]+\left[\Delta\left((\widehat{a_i},a_i),(u^n)\right),\Delta\left((\widehat{a_{i'}},a_{i'}),(u^n)\right)\right]\end{aligned}$$

By Lemma 2.11, we have

$$\left\{E\left(\widehat{a_i}|a_j,a_i\right),E\left(\widehat{a_{i'}}|a_{j'},a_{i'}\right)\right\}_{0,1}=K\cdot E\left(\widehat{a_i}|a_j,a_i\right)\cdot E\left(\widehat{a_{i'}}|a_{j'},a_{i'}\right),\tag{33}$$

where K is some constant. By Equation (32), using the same argument as in Lemma 2.11, K is the summation of $4\cdot n^2$ linking numbers. We observe that all the terms associated to $\widehat{a_{i'}}$ or $\widehat{a_i}$ canceled except the following four terms:

$$\begin{aligned}K&=\left(\mathcal{J}(a_ju_n,a_{j'}u_n)-\mathcal{J}(a_ju_n,a_{i'}u_n)-\mathcal{J}(a_iu_n,a_{j'}u_n)+\mathcal{J}(a_iu_n,a_{i'}u_n)\right)\\&=\mathcal{J}(a_ia_j,a_{i'}a_{j'}).\end{aligned}\tag{34}$$

□

By Equation (4), to prove Theorem 3.8, it is enough to prove

(35)
$$\left\{E\left(\widehat{a_i}|a_j,a_i\right),E\left(\widehat{a_{i'}}|a_{j'},a_{i'}\right)\right\}_{1,0}=-s_{||}\left(a_ia_j,a_{i'}a_{j'}\right)\cdot E\left(\widehat{a_i}|a_{j'},a_i\right)\cdot E\left(\widehat{a_{i'}}|a_j,a_{i'}\right).$$

The main technique that we use to prove the theorem is the following lemma.

Lemma 3.11. [[Su15, Lemma 5.6]] *For* $n\geq 2$*, let* $M=(c_sd_t)_{s,t=1}^n$ *be a* $(n\times n)$*-matrix with* $c_s,d_t\in\mathcal{P}$*, let* M_{st} *be the determinant of the matrix obtained from* M *by deleting the* s*-th row and the* t*-th column. Let* $B\in\mathcal{Q}_n(\mathcal{P})$*, we have*

$$\{\det M,B\}=\sum_{s=1}^n\sum_{t=1}^n(-1)^{s+t}\cdot\det M_{st}\cdot\{c_sd_t,B\}\tag{36}$$

in $\mathcal{Q}_n(\mathcal{P})$.

Proof of Theorem 3.8. We prove the case (1) in Figure 4.

Suppose that $(v_1,\cdots,v_n)$ ($(u_1,\cdots,u_n)$ resp.) is the right side n-tuple for the term on the left (right resp.) hand side in each $[\cdot,\cdot]$ bracket below. By Remark 2.16, we arrange the points $(v_1,\cdots,v_n)$ ($(u_1,\cdots,u_n)$ resp.) in between two successive points of $\mathbf{I}$ where one of them is a_j ($a_{j'}$ resp.) as in Figure 4. For all the cases, such arrangements simplify the computation a lot, and allows us to compute in a similar way.

We compute $D=\left[\Delta\left((\widehat{a_i},a_j),(v^n)\right),\Delta\left((\widehat{a_{i'}},a_{j'}),(u^n)\right)\right]$. By Lemma 3.11, we have

$$D=\frac{1}{\det M\cdot B}\sum_{s=1}^n\sum_{t=1}^n(-1)^{s+t}M_{st}\left\{c_sd_t,B\right\},\tag{37}$$

for $\det M=\Delta\left((\widehat{a_i},a_j),(v^n)\right)$ and $B=\Delta\left((\widehat{a_{i'}},a_{j'}),(u^n)\right)$. We fix c_s and compute the sum $\frac{1}{\det M\cdot B}\sum_{t=1}^n(-1)^{s+t}M_{st}\left\{c_sd_t,B\right\}$ over t, where the summation is called *the sum over* t *for* c_s for short:

(1) When $c_s = a_{i'}$, recall $\mathbf{I} = \{k_1, \cdots, k_n\}$ with $k_1 < \cdots < k_n$ and suppose $i' = k_l$. By Lemma 2.10 Equation (14), we use $\Delta^R(a_{i'}v_t)$ with respect to the right side of $\overrightarrow{a_{i'}v_t}$, then we have

$$\begin{aligned} &\{a_{i'}v_t, \Delta\left(\left(\widehat{a_{i'}}, a_{j'}\right), (u^n)\right)\} \\ &= \sum_{d=l+1}^{n} a_{k_d}v_t \cdot (-1)^{d-l} \cdot \Delta\left(\left(\widehat{a_{k_d}}, a_{j'}\right), (u^n)\right). \end{aligned} \tag{38}$$

Let $\widehat{a_{i'}, a_i}$ be $a_{k_1}, \cdots, a_{k_n}$ with $a_{i'}, a_i$ removed. Then the sum over t for $a_{i'}$ equals

$$\begin{aligned} &\frac{1}{\det M \cdot B} \sum_{d=l+1}^{n} \sum_{t=1}^{n} (-1)^{l+t} M_{lt} \cdot a_{k_d}v_t \cdot (-1)^{d-l} \cdot \Delta\left(\left(\widehat{a_{k_d}}, a_{j'}\right), (u^n)\right) \\ &= \frac{1}{\det M \cdot B} \sum_{d=l+1}^{n} (-1)^{n-1+d} \cdot \Delta\left(\left(\widehat{a_{i'}, a_i}, a_{k_d}, a_j\right), (v^n)\right) \cdot \Delta\left(\left(\widehat{a_{k_d}}, a_{j'}\right), (u^n)\right). \end{aligned} \tag{39}$$

If $k_d \neq i$, we have $a_{k_d} \in \widehat{a_{i'}, a_i}$. Then $\Delta\left(\left(\widehat{a_{i'}, a_i}, a_{k_d}, a_j\right), (u^n)\right) = 0$. Thus the right hand side of Equation (39) equals

$$\frac{\Delta\left(\left(\widehat{a_{i'}}, a_j\right), (v^n)\right) \cdot \Delta\left(\left(\widehat{a_i}, a_{j'}\right), (u^n)\right)}{\Delta\left(\left(\widehat{a_i}, a_j\right), (v^n)\right) \cdot \Delta\left(\left(\widehat{a_{i'}}, a_{j'}\right), (u^n)\right)}. \tag{40}$$

(2) When $c_s = a_m \neq a_{i'}$, we have

$$\{a_m v_t, \Delta\left(\left(\widehat{a_{i'}}, a_{j'}\right), (u^n)\right)\} = \begin{cases} \frac{1}{2} \cdot a_m v_t \cdot \Delta\left(\left(\widehat{a_{i'}}, a_{j'}\right), (u^n)\right) & \text{if } a_m \in \widehat{a_{i'}, a_i}, \\ 0 & \text{if } a_m = a_j. \end{cases} \tag{41}$$

Thus the sum over t for a_m equals $\frac{1}{2}$ if $a_m \neq a_j$, equals 0 if $a_m = a_j$.

Combing the above results, we obtain

$$\begin{aligned} &[\Delta\left(\left(\widehat{a_i} | a_j\right), (v^n)\right), \Delta\left(\left(\widehat{a_{i'}} | a_{j'}\right), (u^n)\right)] \\ &= \frac{\Delta\left(\left(\widehat{a_{i'}}, a_j\right), (v^n)\right) \cdot \Delta\left(\left(\widehat{a_i}, a_{j'}\right), (u^n)\right)}{\Delta\left(\left(\widehat{a_i}, a_j\right), (v^n)\right) \cdot \Delta\left(\left(\widehat{a_{i'}}, a_{j'}\right), (u^n)\right)} + \frac{n-2}{2} \\ &= \frac{\Delta\left(\left(\widehat{a_{i'}}, a_j\right), (u^n)\right) \cdot \Delta\left(\left(\widehat{a_i}, a_{j'}\right), (u^n)\right)}{\Delta\left(\left(\widehat{a_i}, a_j\right), (u^n)\right) \cdot \Delta\left(\left(\widehat{a_{i'}}, a_{j'}\right), (u^n)\right)} + \frac{n-2}{2} \\ &= \frac{E\left(\widehat{a_{i'}} | a_j, a_{i'}\right) \cdot E\left(\widehat{a_i} | a_{j'}, a_i\right)}{E\left(\widehat{a_i} | a_j, a_i\right) \cdot E\left(\widehat{a_{i'}} | a_{j'}, a_{i'}\right)} + \frac{n-2}{2}. \end{aligned} \tag{42}$$

By similar computation, we get

$$[\Delta\left(\left(\widehat{a_i}, a_j\right), (v^n)\right), \Delta\left(\left(\widehat{a_{i'}}, a_{i'}\right), (u^n)\right)] = \frac{n-1}{2}, \tag{43}$$

$$[\Delta\left(\left(\widehat{a_i}, a_i\right), (v^n)\right), \Delta\left(\left(\widehat{a_{i'}}, a_{j'}\right), (u^n)\right)]_{\beta-\alpha,0} = \frac{n-1}{2}, \tag{44}$$

and

$$[\Delta\left(\left(\widehat{a_i}, a_i\right), (v^n)\right), \Delta\left(\left(\widehat{a_{i'}}, a_{i'}\right), (u^n)\right)]_{\beta-\alpha,0} = \frac{n}{2}. \tag{45}$$

Thus we have

$$
\begin{aligned}
\left[E\left(\widehat{a_i}|a_j,a_i\right),E\left(\widehat{a_{i'}}|a_{j'},a_{i'}\right)\right] &= \frac{E\left(\widehat{a_i}|a_{j'},a_i\right)\cdot E\left(\widehat{a_{i'}}|a_j,a_{i'}\right)}{E\left(\widehat{a_i}|a_j,a_i\right)\cdot E\left(\widehat{a_{i'}}|a_{j'},a_{i'}\right)}+ \\
&+\frac{n-2}{2}-\frac{n-1}{2}-\frac{n-1}{2}+\frac{n}{2} \\
&=\frac{E\left(\widehat{a_i}|a_{j'},a_i\right)\cdot E\left(\widehat{a_{i'}}|a_j,a_{i'}\right)}{E\left(\widehat{a_i}|a_j,a_i\right)\cdot E\left(\widehat{a_{i'}}|a_{j'},a_{i'}\right)}.
\end{aligned} \tag{46}
$$

In this case, since $s_{||}\left(a_ia_j,a_{i'}a_{j'}\right)=-1$, we obtain

$$
\left\{E\left(\widehat{a_i}|a_j,a_i\right),E\left(\widehat{a_{i'}}|a_{j'},a_{i'}\right)\right\}=-s_{||}\left(a_i,a_j,a_{i'},a_{j'}\right)\cdot E\left(\widehat{a_i}|a_{j'},a_i\right)\cdot E\left(\widehat{a_{i'}}|a_j,a_{i'}\right). \tag{47}
$$

The proof for the other cases are similar with respect to the arrangements in Figure 4. All the terms cancel out except the term in Equation (40) is replaced by

$$
-s_{||}\left(a_i,a_j,a_{i'},a_{j'}\right)\cdot\frac{\Delta\left(\left(\widehat{a_{i'}},a_j\right),\left(v^n\right)\right)\cdot\Delta\left(\left(\widehat{a_i},a_{j'}\right),\left(u^n\right)\right)}{\Delta\left(\left(\widehat{a_i},a_j\right),\left(v^n\right)\right)\cdot\Delta\left(\left(\widehat{a_{i'}},a_{j'}\right),\left(u^n\right)\right)}. \tag{48}
$$

Note that for the case (3) in Figure 4, the contribution is the same as above, but coming from the fact

$$
\left[a_jv_t,\Delta\left(\left(\widehat{a_{i'}},a_{j'}\right),\left(u^n\right)\right)\right]=-\frac{1}{2}. \tag{49}
$$

Hence we obtain for all the cases, we have

$$
\left\{E\left(\widehat{a_i}|a_j,a_i\right),E\left(\widehat{a_{i'}}|a_{j'},a_{i'}\right)\right\}=-s_{||}\left(a_i,a_j,a_{i'},a_{j'}\right)\cdot E\left(\widehat{a_i}|a_{j'},a_i\right)\cdot E\left(\widehat{a_{i'}}|a_j,a_{i'}\right). \tag{50}
$$

Combing with Lemma 3.10 and Equation (28), we conclude that $\theta_{\alpha,\beta}$ is Poisson with respect to $\{\cdot,\cdot\}_{B_{\alpha,\beta}}$ and $\{\cdot,\cdot\}_{\beta-\alpha,\alpha+\beta}$. □

Acknowledgements

I thank Yau Mathematical Sciences Center at Tsinghua University and Luxembourg University for their hospitality.

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