Single-peakedness in aggregation function theory

Jimmy Devillet

University of Luxembourg
Luxembourg

in collaboraton with Miguel Couceiro and Jean-Luc Marichal

Part I: Single-peaked orderings

Single-peaked orderings

Motivating example (Romero, 1978)

Suppose you are asked to order the following six objects in decreasing preference:

$$\begin{aligned} a_1 &: \quad \text{0 sandwich} \\ a_2 &: \quad \text{1 sandwich} \\ a_3 &: \quad \text{2 sandwiches} \\ a_4 &: \quad \text{3 sandwiches} \\ a_5 &: \quad \text{4 sandwiches} \\ a_6 &: \quad \text{more than 4 sandwiches} \end{aligned}$$

We write $a_i \prec a_j$ if a_i is preferred to a_j

Single-peaked orderings

a_1 : 0 sandwich
a_2 : 1 sandwich
a_3 : 2 sandwiches
a_4 : 3 sandwiches
a_5 : 4 sandwiches
a_6 : more than 4 sandwiches

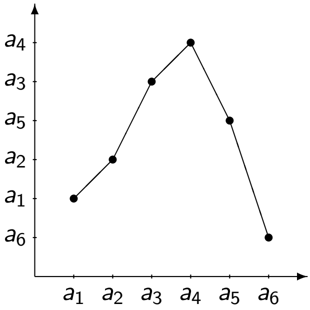
- after a good lunch: $a_1 \prec a_2 \prec a_3 \prec a_4 \prec a_5 \prec a_6$
- if you are starving: $a_6 \prec a_5 \prec a_4 \prec a_3 \prec a_2 \prec a_1$
- a possible intermediate situation: $a_4 \prec a_3 \prec a_5 \prec a_2 \prec a_1 \prec a_6$
- a quite unlikely preference: $a_6 \prec a_5 \prec a_2 \prec a_1 \prec a_3 \prec a_4$

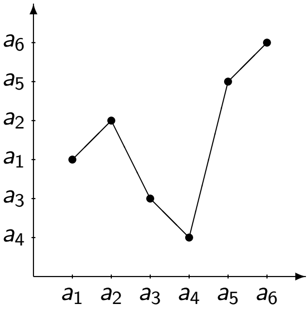
Single-peaked orderings

Let us represent graphically the latter two preferences with respect to the reference ordering $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$

$a_4 \prec a_3 \prec a_5 \prec a_2 \prec a_1 \prec a_6$

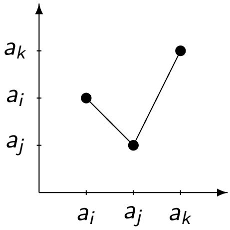
$a_6 \prec a_5 \prec a_2 \prec a_1 \prec a_3 \prec a_4$

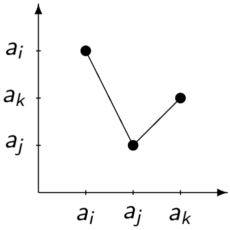




Single-peaked orderings

Definition. (Black, 1948)
Let $\leq$ and $\preceq$ be total orderings on $X_n = \{a_1, \ldots, a_n\}$.
Then $\preceq$ is said to be *single-peaked for* $\leq$ if the following patterns are forbidden





Mathematically:

$$a_i < a_j < a_k \quad \Longrightarrow \quad a_j \prec a_i \quad \text{or} \quad a_j \prec a_k$$

Single-peaked orderings

$$a_i < a_j < a_k \quad \Longrightarrow \quad a_j \prec a_i \quad \text{or} \quad a_j \prec a_k$$

Let us assume that $X_n = \{a_1, \ldots, a_n\}$ is endowed with the ordering $a_1 < \cdots < a_n$

For $n = 4$

$a_1 \prec a_2 \prec a_3 \prec a_4$	$a_4 \prec a_3 \prec a_2 \prec a_1$
$a_2 \prec a_1 \prec a_3 \prec a_4$	$a_3 \prec a_2 \prec a_1 \prec a_4$
$a_2 \prec a_3 \prec a_1 \prec a_4$	$a_3 \prec a_2 \prec a_4 \prec a_1$
$a_2 \prec a_3 \prec a_4 \prec a_1$	$a_3 \prec a_4 \prec a_2 \prec a_1$

There are 2^{n-1} total orderings $\preceq$ on X_n that are single-peaked for $\leq$

Single-peaked orderings

Recall that a *weak ordering* (or *total preordering*) on X_n is a binary relation $\precsim$ on X_n that is total and transitive.

Defining a weak ordering on X_n amounts to defining an ordered partition of X_n

$$C_1 \prec \cdots \prec C_k$$

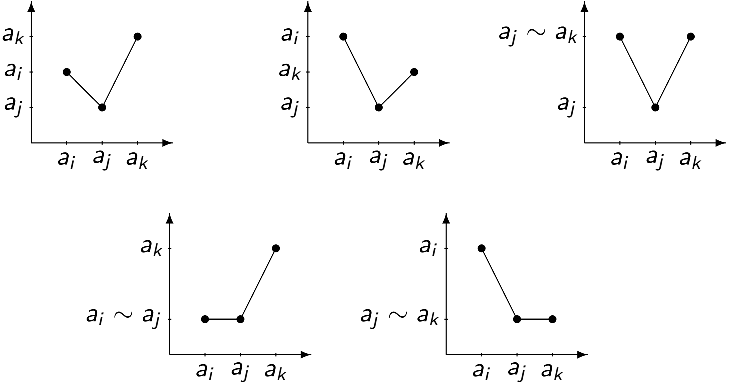
where $C_1, \ldots, C_k$ are the equivalence classes defined by $\sim$

For $n = 3$, we have 13 weak orderings

$$\begin{array}{lll}
a_1 \prec a_2 \prec a_3 & a_1 \sim a_2 \prec a_3 & a_1 \sim a_2 \sim a_3 \\
a_1 \prec a_3 \prec a_2 & a_1 \prec a_2 \sim a_3 & \\
a_2 \prec a_1 \prec a_3 & a_2 \prec a_1 \sim a_3 & \\
a_2 \prec a_3 \prec a_1 & a_3 \prec a_1 \sim a_2 & \\
a_3 \prec a_1 \prec a_2 & a_1 \sim a_3 \prec a_2 & \\
a_3 \prec a_2 \prec a_1 & a_2 \sim a_3 \prec a_1 &
\end{array}$$

Single-peaked orderings

Definition. (Black, 1948)
Let $\leq$ be a total ordering on X_n and let $\precsim$ be a weak ordering on X_n. Then $\precsim$ is said to be *single-plateaued for* $\leq$ if the following patterns are forbidden



Single-peaked orderings

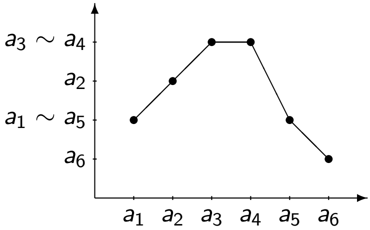
Mathematically:

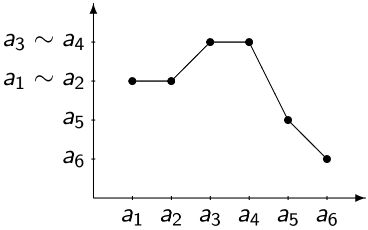
$$a_i < a_j < a_k \implies a_j \prec a_i \quad \text{or} \quad a_j \prec a_k \quad \text{or} \quad a_i \sim a_j \sim a_k$$

Examples

$a_3 \sim a_4 \prec a_2 \prec a_1 \sim a_5 \prec a_6$

$a_3 \sim a_4 \prec a_2 \sim a_1 \prec a_5 \prec a_6$





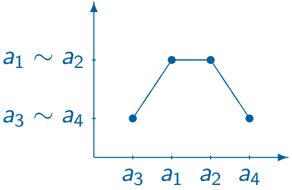
Single-peaked orderings

Q: Given $\precsim$ is it possible to find $\leq$ for which $\precsim$ is single-plateaued?

Example: On $X_4 = \{a_1, a_2, a_3, a_4\}$ consider $\precsim$ and $\precsim'$ defined by

$$a_1 \sim a_2 \prec a_3 \sim a_4 \qquad \text{and} \qquad a_1 \prec' a_2 \sim' a_3 \sim' a_4$$

Yes! Consider $\leq$ *defined by* $a_3 < a_1 < a_2 < a_4$



No!

2-quasilinear weak orderings

Definition.
We say that $\precsim$ is *2-quasilinear* if

$$a \prec b \sim c \sim d \quad \Longrightarrow \quad a, b, c, d \text{ are not pairwise distinct}$$

Proposition (D., Marichal, Teheux)

We have

$$\precsim \text{ is 2-quasilinear} \quad \Longleftrightarrow \quad \exists \leq \text{ for which } \precsim \text{ is single-plateaued}$$

Part II: Aggregation functions

Associativity and quasitrivial operations

Definition

$F\colon X_n^2 \to X_n$ is said to be

- *associative* if

$$F(F(x,y),z) \quad = \quad F(x,F(y,z)) \qquad x,y,z \in X_n$$

- *quasitrivial* (or *conservative*) if

$$F(x,y) \ \in \ \{x,y\} \qquad x,y \in X_n$$

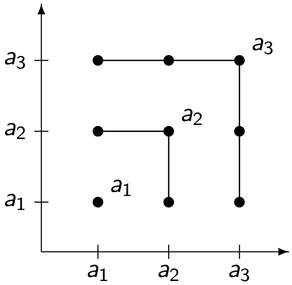
- *idempotent* if

$$F(x,x) \quad = \quad x \qquad x \in X_n$$

Fact. If F is quasitrivial, then it is idempotent.

Associativity and quasitrivial operations

Example. $F = \max_{\leq}$ on $X_3 = \{a_1, a_2, a_3\}$



Associative and quasitrivial operations

Definition

The *projection operations* $\pi_1 \colon X_n^2 \to X_n$ and $\pi_2 \colon X_n^2 \to X_n$ are respectively defined by

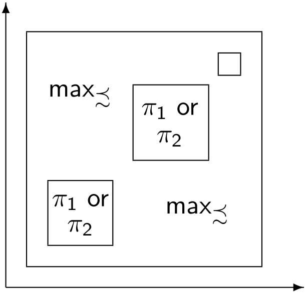
$$\begin{aligned} \pi_1(x,y) &= x, \qquad x,y \in X_n \\ \pi_2(x,y) &= y, \qquad x,y \in X_n \end{aligned}$$

Associative and quasitrivial operations

Assume that $X_n = \{a_1, \ldots, a_n\}$ is endowed with a weak ordering $\precsim$

Ordinal sum of projections

$$\mathrm{osp}_{\precsim} : X_n^2 \to X_n$$



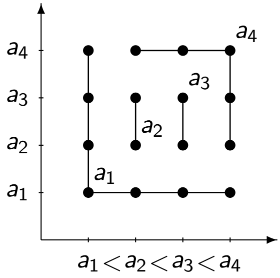
Associative and quasitrivial operations

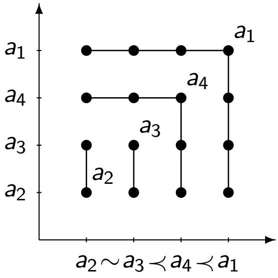
Theorem (Länger 1980)

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

(i) F is associative and quasitrivial

(ii) $F = \mathrm{osp}_{\precsim}$ for some weak ordering $\precsim$ on X_n





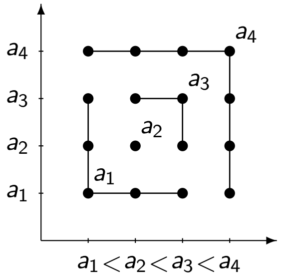
Associative, quasitrivial, and commutative operations

Corollary

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

(i) F is associative, quasitrivial, and commutative

(ii) $F = \max_{\preceq}$ for some total ordering $\preceq$ on X_n



Aggregation functions

Definition.

$F\colon X_n^2 \to X_n$ is said to be *$\leq$-preserving* for some total ordering $\leq$ on X_n if for any $x, y, x', y' \in X_n$ such that $x \leq x'$ and $y \leq y'$, we have $F(x,y) \leq F(x',y')$

Definition.

An *aggregation function on $(X_n, \leq)$* is an operation $F\colon X_n^2 \to X_n$ that

- is $\leq$-preserving

and satisfies

- $F(a_1, a_1) = a_1$ and $F(a_n, a_n) = a_n$

Example. $F = \max_{\leq}$ on $X_3 = \{a_1, a_2, a_3\}$

Uninorms

Definition.

A *uninorm on* $(X_n, \leq)$ is an operation $F\colon X_n^2 \to X_n$ that

- has a neutral element $e \in X_n$ $\qquad (\Leftrightarrow \; F(x,e) = F(e,x) = x \;\; \forall x \in X_n)$

and is

- associative
- commutative
- $\leq$-preserving

Associative, quasitrivial, and order-preserving operations

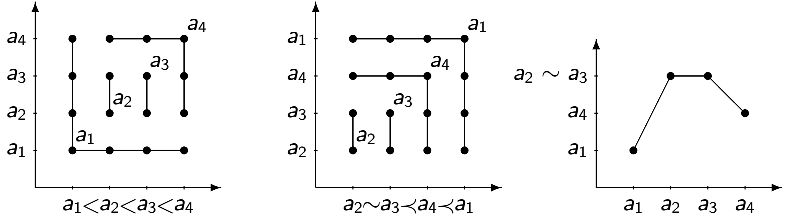
$\leq$: total ordering on X_n

Theorem

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

(i) F is associative, quasitrivial, and $\leq$-preserving

(ii) $F = \mathrm{osp}_{\precsim}$ for some weak ordering $\precsim$ on X_n that is single-plateaued for $\leq$



Associative, quasitrivial, and order-preserving operations

$\leq$: total ordering on X_n

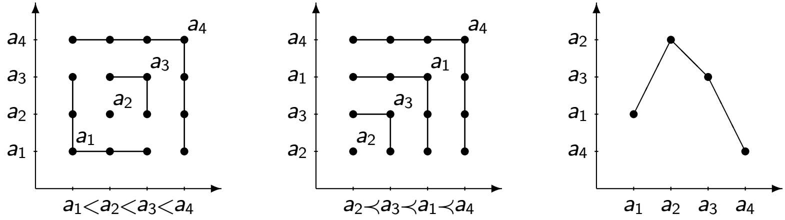
Theorem

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

(i) F is associative, quasitrivial, commutative, and $\leq$-preserving

(ii) $F = \max_{\preceq}$ for some total ordering $\preceq$ on X_n that is single-peaked for $\leq$

(iii) F is an idempotent uninorm on X_n



Order-preservable operations

Definition.

We say that $F\colon X_n^2 \to X_n$ is *order-preservable* if it is $\leq$-preserving for some $\leq$

Q: Given an associative and quasitrivial F, is it order-preservable?

$$\text{2-quasilinearity}: \quad a \prec b \sim c \sim d \quad \Longrightarrow \quad a, b, c, d \text{ are not pairwise distinct}$$

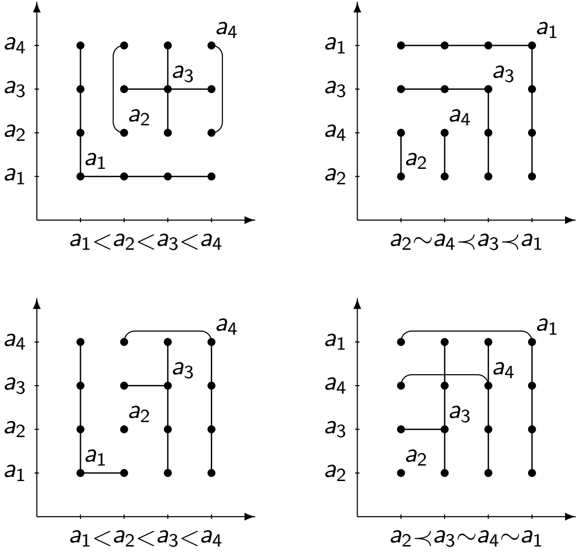
Theorem (D., Marichal, Teheux)

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

(i) F is associative, quasitrivial, and order-preservable

(ii) $F = \mathrm{osp}_{\precsim}$ for some weak ordering $\precsim$ on X_n that is 2-quasilinear

Order-preservable operations



Final remarks

1. We have introduced and identified a number of integer sequences in `http://oeis.org`
   - Number of associative and quasitrivial operations: A292932
   - Number of associative, quasitrivial, and $\leq$-preserving operations: A293005
   - Number of weak orderings on X_n that are single-plateaued for $\leq$: A048739
   - $\cdots$
2. Most of our characterizations still hold on arbitrary sets X (not necessarily finite)

Some references

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