VERIFICATION OF THE QUILLEN CONJECTURE IN THE RANK 2 IMAGINARY QUADRATIC CASE

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ABSTRACT. We confirm a conjecture of Quillen in the case of the mod 2 cohomology of arithmetic groups $\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-m})}[\frac{1}{2}])$, where $\mathcal{O}_{\mathbb{Q}(\sqrt{-m})}$ is an imaginary quadratic ring of integers. To make explicit the free module structure on the cohomology ring conjectured by Quillen, we compute the mod 2 cohomology of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\frac{1}{2}])$ via the amalgamated decomposition of the latter group.

INTRODUCTION

The Quillen conjecture on the cohomology of arithmetic groups has spurred a great deal of mathematics (according to [15]). On the cohomology of a linear arithmetic group, Quillen did find a module structure over the Chern classes of the containing group $\mathrm{GL}_n(\mathbb{C})$ (equivalently, of $\mathrm{SL}_n(\mathbb{C})$). Quillen did then conjecture that this module is free [18]. While the conjecture has been proven for large classes of groups of rank 2 matrices, as well as for some groups of rank 3 matrices, an obstruction against its validity has been found by Henn, Lannes and Schwartz [14], and this obstruction does occur at least for matrix ranks 14 and higher [13]. So the scope of the conjecture is not correct in Quillen's original statement, and the present paper contributes to efforts on determining the conjecture's correct range of validity. Considering the outcomes of previous research on this question, what seems likely, is that counterexamples may occur already at low matrix rank. In the present paper however, we confirm the Quillen conjecture over all of the Bianchi groups (the SL_2 groups over imaginary quadratic rings of integers), at the prime number 2:

Theorem 1. *Let $\mathcal{O}_{\mathbb{Q}(\sqrt{-m})}$ be a ring of imaginary quadratic integers. Then $\mathrm{H}^*(\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-m})}[\frac{1}{2}]); \mathbb{F}_2)$ is a free module over the ring generated by the Chern class c_2 of $\mathrm{SL}_2(\mathbb{C})$.*

Theorem 1 is a consequence of a theorem of Broto and Henn, as we shall explain in Section 2. To make explicit the free module structure on the cohomology ring in one example, we compute the mod 2 cohomology of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\frac{1}{2}])$ via the amalgamated decomposition

$$\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\frac{1}{2}]) \cong \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]) *_{\Gamma_0(\sqrt{-2})} \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$$

which is known from Serre's classical book [19], and which yields a Mayer–Vietoris long exact sequence on group cohomology that we evaluate.

For $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-1}][\frac{1}{2}])$, a computation of the mod 2 cohomology ring structure has already been achieved by Weiss [20], but the uniformizing element in the amalgamated decomposition is different for the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$ from the one for

the other imaginary quadratic rings. And after exclusion of the Gaussian integers, there is a general description of the mod 2 cohomology rings of the Bianchi groups and their subgroups [3]. So for our purposes, the Gaussian integers do not provide a typical example, while the ring $\mathbb{Z}[\sqrt{-2}]$ does. Moreover, having confirmed the Quillen conjecture for SL_2 over the Gaussian integers and over $\mathbb{Z}[\sqrt{-2}]$ removes one of the obstacles against $SL_4(\mathbb{Z}[\frac{1}{2}])$ to satisfy the Quillen conjecture (see diagram (1) in [20]), because in the centralizer spectral sequence for $SL_4(\mathbb{Z}[\frac{1}{2}])$ (compare with [12]), the stabilizers which have the highest complexity are of types $SL_2(\mathbb{Z}[\sqrt{-2}][\frac{1}{2}])$, $SL_2(\mathbb{Z}[\sqrt{-1}][\frac{1}{2}])$ and $SL_2(\mathbb{Z}[\sqrt{2}][\frac{1}{2}])$.

We follow Weiss's strategies for several aspects of our calculation, while we use a different cell complex and recent homological algebra techniques [3] to overcome specific technical difficulties entering with $SL_2(\mathbb{Z}[\sqrt{-2}][\frac{1}{2}])$. Then we arrive at the result stated in Theorem 3 below. To see how this illustrates the module structure predicted by the Quillen conjecture, let us state the latter over $SL_n(\mathbb{C})$).

Conjecture 2 (Quillen). *Let ℓ be a prime number. Let K be a number field with $\zeta_\ell \in K$, and S a finite set of places containing the infinite places and the places over ℓ. Then the natural inclusion $\mathcal{O}_{K,S} \hookrightarrow \mathbb{C}$ makes $H^*(SL_n(\mathcal{O}_{K,S}); \mathbb{F}_\ell)$ a free module over the cohomology ring $H^*_{cts}(SL_n(\mathbb{C}); \mathbb{F}_\ell)$.*

Theorem 3. *Denote by e_4 the image of the second Chern class of the natural representation of $SL_2(\mathbb{C})$, so $\mathbb{F}_2[e_4]$ is the image of $H^*_{cts}(SL_2(\mathbb{C}); \mathbb{F}_2)$. Then the cohomology ring $H^*(SL_2(\mathbb{Z}[\sqrt{-2}][\frac{1}{2}]); \mathbb{F}_2)$ is the free module of rank 9 over $\mathbb{F}_2[e_4]$ with basis $\{x_2, x_3, y_3, z_3, s_3, x_4, s_4, s_5, s_6\}$, where the subscript of the classes specifies their degree as a cohomology class.*

Organization of the paper. In Section 1, we recall the amalgamated decomposition $SL_2(\mathbb{Z}[\sqrt{-2}][\frac{1}{2}]) \cong SL_2(\mathbb{Z}[\sqrt{-2}]) *_{\Gamma_0(\sqrt{-2})} SL_2(\mathbb{Z}[\sqrt{-2}])$ on which our calculation is based, as well as its general form which we use in Section 2 in the proof of Theorem 1. In Section 3, we describe a cell complex for the involved congruence subgroup $\Gamma_0(\sqrt{-2})$. We use it to compute the cohomology of the latter in Section 3.1. In Section 4, we determine the maps induced on cohmology by the two injections of $\Gamma_0(\sqrt{-2})$ into $SL_2(\mathbb{Z}[\sqrt{-2}])$ that characterize the amalgamated product. Finally in Section 5, we conclude the proof of Theorem 3.

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1. THE AMALGAMATED DECOMPOSITION

For a quadratic ring $\mathcal{O}_K$, in this section we decompose $SL_2(\mathcal{O}_K[\frac{1}{2}])$ as a product of two copies of $SL_2(\mathcal{O}_K)$ amalgamated over a suitable congruence subgroup. It is described in Serre's book *Trees* [19] how to do such a decomposition in general, and example decompositions are given for $SL_2(\mathbb{Z}[\frac{1}{p}])$ in Serre's book as well as for $SL_2(\mathbb{Z}[\sqrt{-1}][\frac{1}{2}])$ in [20].

The following theorem is implicit in the section 1.4 of chapter II of [19].

Let K be a field equipped with a *discrete valuation* v. Recall that v is a homomorphism from $(K \setminus \{0\}, \cdot)$ to $(\mathbb{Q}, +)$ and that

$$v(x + y) \geq \min(v(x), v(y)) \text{ for } x, y \in K,$$

with the convention that $v(0) = +\infty$.

Let $\mathcal{O}$ be the subset of K on which the valuation is non-negative. Then $\mathcal{O}$ is a subring of K and called the Discrete Valuation Ring. A *uniformizer* is an arbitrary element $\pi \in \mathcal{O} \setminus \{0\}$ with $v(\pi) > 0$ such that for all $x \in \mathcal{O}$ with $v(x) > 0$, the inequality $v(\pi) \leq v(x)$ holds.

Theorem 4 (Serre). *Let A be a dense sub-ring of K. Then for any uniformizer π, we have the decomposition*

$$\mathrm{SL}_2(A) \cong \mathrm{SL}_2(\mathcal{O} \cap A) *_{\Gamma_0(\pi)} \mathrm{SL}_2(\mathcal{O} \cap A),$$

where $\Gamma_0(\pi)$ is the subgroup of $\mathrm{SL}_2(\mathcal{O} \cap A)$ of upper triangular matrices modulo the ideal (π) of $\mathcal{O} \cap A$, injecting

- *as the natural inclusion into the first factor of type $\mathrm{SL}_2(\mathcal{O} \cap A)$,*
- *via the formula*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \pi^{-1} d\pi & \pi^{-1} c \\ b\pi & a \end{pmatrix}$$

  into the second factor of type $\mathrm{SL}_2(\mathcal{O} \cap A)$.

Note that in the above formula, we have replaced Serre's original matrix by an obviously equivalent one suggested by Weiss [20]. This has been done with the purpose to have an alternative description of the injection in question as conjugation by the matrix $\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$.

We can now apply Serre's above theorem to quadratic number fields.

Definition 5. Let K be a quadratic number field, let p be a prime number, and let v_p be the p-adic valuation of $\mathbb{Q}$. We define a function v_p^N on K by

$$v_p^N(x) := v_p(N(x)),$$

where $N(x)$ is the number-theoretic norm of $x \in K$.

Note that as K is quadratic, $N(x) = x\overline{x}$ with $\overline{x}$ the Galois conjugate of x, and for K imaginary quadratic, the Galois conjugate is the complex conjugate.

Lemma 6 (recall of basic algebraic number theory). *The function v_p^N*

  (i) *is a valuation of K,*
  (ii) *extends the valuation $2v_p$*
      (the latter being the double of the p-adic valuation of $\mathbb{Q}$),
  (iii) *and takes its values exclusively in $\mathbb{Z}$.*

An elementary proof can be found in the first preprint version of the present paper.

Now we can turn our attention to $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\frac{1}{2}])$. Choose $K := \mathbb{Q}(\sqrt{-2})$, equipped with the valuation v_2^N defined above. Then we can expect $\mathcal{O} = \mathbb{Z}_{(2)}[\sqrt{-2}]$. Furthermore, we expect $A := \mathbb{Z}[\sqrt{-2}][\frac{1}{2}]$ to be dense in $\mathbb{Q}(\sqrt{-2})$ with respect to the topology induced by v_2^N; and $\mathcal{O} \cap A = \mathbb{Z}[\sqrt{-2}]$.

As $v_2^N(\sqrt{-2}) = 1$ is minimal on the part of $\mathcal{O}$ on which v_2^N is positive, we can choose $\sqrt{-2}$ as a uniformizer. Then Theorem 4 yields that

$$\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\tfrac{1}{2}]) \cong \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]) *_{\Gamma_0(\sqrt{-2})} \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]),$$

where $\Gamma_0(\sqrt{-2})$ is the subgroup of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ of upper triangular matrices modulo the ideal $(\sqrt{-2})$ of $\mathbb{Z}[\sqrt{-2}]$, injecting

- as the natural inclusion into the first factor of type $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$,
- via the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & \sqrt{-2}^{-1}c \\ b\sqrt{-2} & a \end{pmatrix}$$

into the second factor of type $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$.

We note that $v_2^N(2) = 2$, so 2 is not a uniformizer.

2. The module over the Chern class ring

In this section, we shall explain how Theorem 1 can be seen as a consequence of a theorem of Broto and Henn.

For the first approach, we define the *depth* of an ideal I in a finitely generated module M over a Noetherian ring as in standard commutative algebra textbooks [16]: We assume $I \cdot M \neq M$. A sequence of elements $x_1, \ldots, x_n$ of I of positive degree is called *regular* on M if x_1 is not a zero divisor on M and x_{i+1} is not a zero divisor on the quotient $M/_{(x_1,\ldots,x_i)M}$. Under these conditions, any two maximal regular sequences have the same finite length, called the *depth*.

The following lemma is the special case of depth 1 of a classical commutative algebra result. The reader can easily work out a proof for it.

Lemma 7. *Let K be a field and A be a positively graduated, connected K-algebra. Let M be a graduated A-module with $M_n = 0$ for $n < 0$ and M_n a finite-dimensional K-vector space for all $n \in \mathbb{N} \cup \{0\}$. Let x be an element in the maximal ideal of A which operates on M injectively by multiplication. Then M is a free $K[x]$-module.*

Proof of Theorem 1. Theorem 1.1 of [6] states, in its variant provided by remark 2.3 of the same paper:
Let p be a prime, G be a discrete group of finite virtual cohomological dimension, X a G-space and C a central elementary Abelian p-subgroup of G acting trivially on X. If $\mathrm{H}^*(X, \mathbb{F}_p)$ is finite dimensional over $\mathbb{F}_p$, then the depth of $\mathrm{H}_G^*(X, \mathbb{F}_p)$ is at least as big as the rank of C.

We set $p = 2$, $G = \mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-m})}[\tfrac{1}{2}])$ and C its central subgroup generated by minus the identity matrix (note that the following arguments also work for G an arbitrary discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ with center C). Then as we know from Borel and Serre [5], G acts properly discontinuously on the space X constructed as the direct product of $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2$ and the Bruhat-Tits building associated to the 2-adic group $\mathrm{SL}_2(\mathbb{Q}(\sqrt{-m})_2)$. Moreover, C acts trivially on X. As X is finite-dimensional, the virtual cohomological dimension of the discrete group G is finite. We consider $\mathrm{H}_G^*(X, \mathbb{F}_2)$ as an $\mathrm{H}^*(G, \mathbb{F}_2)$-module via the algebra map $\mathrm{H}^*(G, \mathbb{F}_2) \to \mathrm{H}_G^*(X, \mathbb{F}_2)$ induced by projection from X to a point. Then as $\mathrm{H}^*(X, \mathbb{F}_2)$ is finite dimensional over $\mathbb{F}_2$, Broto and Henn's above theorem provides us at least depth 1 for the maximal ideal in $\mathrm{H}_G^*(X, \mathbb{F}_2)$ constituted by the elements of strictly positive degree, and hence a regular sequence $x_1, \ldots, x_n$ with $n \geq 1$, where $x_1 \in \mathrm{H}^*(G, \mathbb{F}_2)$ is of strictly

positive degree and operates on $\mathrm{H}_G^*(X, \mathbb{F}_2)$ injectively by multiplication. Now we can apply Lemma 7 in order to obtain the free module structure on $\mathrm{H}_G^*(X, \mathbb{F}_2)$ As X is contractible, $\mathrm{H}^*(G, \mathbb{F}_2) = \mathrm{H}_G^*(X, \mathbb{F}_2)$. $\qquad\square$

3. THE CELL COMPLEX FOR THE CONGRUENCE SUBGROUP

Due to the above amalgamated decomposition, we want to study the action of the congruence subgroup

$$\Gamma_0(\sqrt{-2}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]) \ \Big| \ c \in \langle \sqrt{-2} \rangle \right\}$$

in the Bianchi group $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ on a suitable cell complex (a 2-dimensional retract of hyperbolic 3-space). For this purpose, we are in the fortunate situation that the symmetric space $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2$ acted on by $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ is isometric to real hyperbolic 3-space $\mathcal{H}^3$. We can use the upper half-space model for $\mathcal{H}^3$, where as a set, $\mathcal{H}^3 = \{(z, \zeta) \in \mathbb{C} \times \mathbb{R} \mid \zeta > 0\}$. Then we can use Poincaré's explicit formulas for the action: For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$, the action of γ on $\mathcal{H}^3$ is given by $\gamma \cdot (z, \zeta) = (z', \zeta')$, where

$$z' = \frac{\left(\overline{cz + d} \right)(az + b) + \zeta^2 \bar{c}a}{|cz + d|^2 + \zeta^2 |c|^2}, \qquad \zeta' = \frac{|\det \gamma| \zeta}{|cz + d|^2 + \zeta^2 |c|^2}.$$

Luigi Bianchi [4] has constructed a fundamental polyhedron for the action on $\mathcal{H}^3$ of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$, and we use Lakeland's method [2, Section 6] to find a set translates of it under elements of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ outside $\Gamma_0(\sqrt{-2})$, which constitute a fundamental domain $\mathcal{F}$ for $\Gamma_0(\sqrt{-2})$, strict in its interior: No two points in its interior can be identified by the action of an element of $\Gamma_0(\sqrt{-2})$. A cumbersome aspect of Bianchi's fundamental polyhedron, and hence also of $\mathcal{F}$, is that it is not compact, but open at cusps which sit in the boundary of $\mathcal{H}^3$. We shall remedy this aspect with an $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$-equivariant retraction, namely a retraction of $\mathcal{H}^3$ onto a 2-dimensional cell complex, along geodesic arcs away from the cusps, which commutes with the $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$-action. Our choice of $\mathcal{F}$, and on it Mendoza's $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$-equivariant retraction [17] away from all cusps, are depicted in Figure 1(A). An alternative choice would have been to use Flöge's $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$-equivariant retraction [11] away only from cusps in the $\Gamma_0(\sqrt{-2})$-orbit of ∞, and then to use a Borel-Serre compactification on the remaining cusps; the outcome of that alternative is shown in [2, figure 1]. In Figure 1(B), we list the coordinates of the vertices of $\mathcal{F}$, and in Figure 1(C), we display the compact fundamental domain for $\Gamma_0(\sqrt{-2})$ obtained from $\mathcal{F}$ in the 2-dimensional retract of $\mathcal{H}^3$. By boundary identifications on the latter compact fundamental domain, we obtain the quotient space of the 2-dimensional retract modulo the $\Gamma_0(\sqrt{-2})$-action, as drawn in Figure 1(D). The fundamental domain in Figure 1(C) is subject to edge identifications $\gg$, $\ggg$ and $\ggg\!\!\!\!\ggg$ carried out by $\begin{pmatrix} 1 & 0 \\ -\sqrt{-2} & 1 \end{pmatrix}$, and $>$ carried out by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, both of which are in $\Gamma_0(\sqrt{-2})$. With the notation $\omega := \sqrt{-2}$, the non-trivial edge stabilizers are generated by the order-4-matrices

$$A = \begin{pmatrix} 1 & \omega \\ \omega & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 - \omega & -\omega \\ 2 & 1 + \omega \end{pmatrix}, \quad C = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix},$$

(A) Fundamental polyhedron for the action of $\Gamma_0(\sqrt{-2})$ on hyperbolic 3-space. We retract it $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$-equivariantly away from the cusps at 0 and ∞, along the dotted edges.
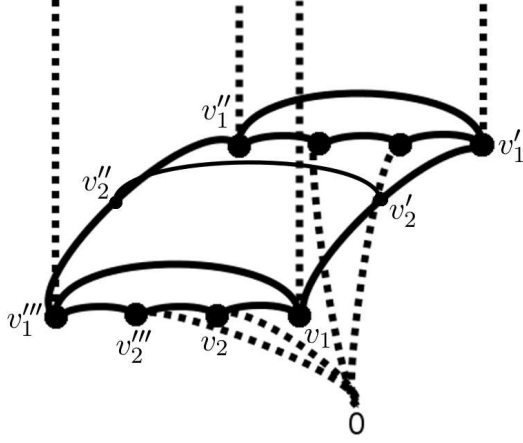
(B) Coordinates of the above vertices in upper half-space. Denote $\sqrt{-2}$ by ω, denote the height square by ζ^2 and project to the boundary plane at height $\zeta = 0$.

Vertex	Projection z	ζ^2
v_1	$-\frac{1}{2} - \frac{\omega}{2}$	$1/4$
v_1'	$\frac{1}{2} - \frac{\omega}{2}$	$1/4$
v_1''	$\frac{1}{2} + \frac{\omega}{2}$	$1/4$
v_1'''	$-\frac{1}{2} + \frac{\omega}{2}$	$1/4$
v_2	$-\frac{1}{2} - \frac{\omega}{4}$	$1/8$
v_2'	$-\frac{\omega}{2}$	$1/2$
v_2''	$\frac{\omega}{2}$	$1/2$
v_2'''	$-\frac{1}{2} + \frac{\omega}{4}$	$1/8$

(C) A fundamental domain (strict in its interior) for the action of $\Gamma_0(\sqrt{-2})$ on the 2-dimensional retract is given by the three quadrangles with marked vertices from Figure 1(A).
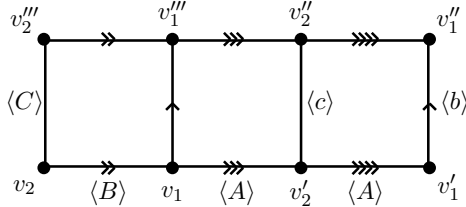
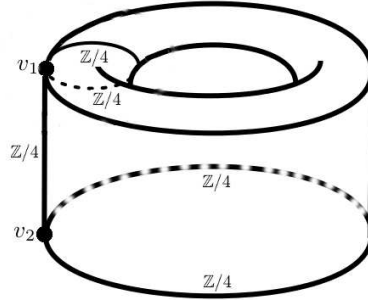(D) Quotient space of the latter fundamental domain by its edge identifications.

FIGURE 1. The cell complex for $\Gamma_0(\sqrt{-2})$

respectively their conjugates by the above mentioned edge identifications. A fundamental domain for $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ is given by the quadrangle $(v_2', v_1', v_1'', v_2'')$. In $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$, there are two additional edge stabilizer generators,

$$b = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \text{ of order 6, and } c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ of order 4,}$$

which are not in $\Gamma_0(\sqrt{-2})$.

3.1. **The cohomology of the congruence subgroup.** From Figure 1(D), we see that the orbit space of the 2-dimensional retract X of hyperbolic space under the action of $\Gamma_0(\sqrt{-2})$ has the homotopy type of a 2-torus, so

$$\dim_{\mathbb{F}_2} \mathrm{H}^p(_{\Gamma_0(\sqrt{-2})}\backslash X; \mathbb{F}_2) = \begin{cases} 0, & p > 2, \\ 1, & p = 2, \\ 2, & p = 1, \\ 1, & p = 0. \end{cases}$$

We also see in Figure 1(D) that the non-central 2-torsion subcomplex X_s, namely the union of the cells of X whose cell stabilizers in $\Gamma_0(\sqrt{-2})$ contain elements of order a power of 2, and which are not in the center of $\Gamma_0(\sqrt{-2})$, has an orbit space $_{\Gamma_0(\sqrt{-2})}\backslash X_s$ of shape $\mathsf{O\!\!-\!\!O}$. Following [3] and [2], we define the co-rank c to be the rank of the cokernel of

$$\mathrm{H}^1(_{\Gamma_0(\sqrt{-2})}\backslash X; \mathbb{F}_2) \to \mathrm{H}^1(_{\Gamma_0(\sqrt{-2})}\backslash X_s; \mathbb{F}_2)$$

induced by the inclusion $X_s \subset X$. Again inspecting Figure 1(D), we can see that the co-rank c vanishes. Then [2, lemma 20] yields that the $d_2^{p,2q}$-differentials of the associated equivariant spectral sequence (cf. Section 4 below) vanish. Computing a presentation for the Abelianization $(\Gamma_0(\sqrt{-2}))^{\mathrm{ab}} \cong \mathrm{H}_1(\Gamma_0(\sqrt{-2}); \mathbb{Z})$ from Figure 1, we can observe that the $d_2^{p,4q+1}$-differentials vanish as well. Then we apply [2, corollary 27], and obtain the following result.

Proposition 8. $\dim_{\mathbb{F}_2} \mathrm{H}^p(\Gamma_0(\sqrt{-2}); \mathbb{F}_2) = \begin{cases} 5, & p \equiv 4 \ or \ 5 \mod 4, \\ 6, & p \equiv 2 \ or \ 3 \mod 4 \\ 4, & p = 1, \\ 1, & p = 0. \end{cases}$

4. THE MAPS ON EQUIVARIANT SPECTRAL SEQUENCES

In order to evaluate the Mayer–Vietoris long exact sequence of the amalgamated decomposition $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\frac{1}{2}]) \cong \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]) *_{\Gamma_0(\sqrt{-2})} \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$, we need to find out what the two injections of $\Gamma_0(\sqrt{-2})$ into $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$, namely i (the natural inclusion) and j (the conjugation map defined in Theorem 4) induce on the mod 2 cohomology rings of these groups. For this purpose, we make use of the equivariant spectral sequences

$$E_1^{p,q} = \bigoplus_{\sigma \ p\text{-cell}} \mathrm{H}^q(\Gamma_\sigma; \mathbb{F}_2) \text{ converging to } \mathrm{H}^{p+q}(\Gamma; \mathbb{F}_2),$$

for Γ being either $\Gamma_0(\sqrt{-2})$ or $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$. The notation Γ_σ stands for the stabilizer of σ in Γ, and the p-cells indexing the above direct sum run through a set of orbit representatives. As we let the p-cells for $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ run through a fundamental domain contained in the one for $\Gamma_0(\sqrt{-2})$, we get two compatible equivariant spectral sequences, and can compute the maps on $\mathrm{H}^{p+q}(\Gamma; \mathbb{F}_2)$ from the maps between the two E_1-pages. In Section 3.1, we have seen that the d_2-differentials are all trivial for $\Gamma_0(\sqrt{-2})$, and having in mind the cylindrical shape of $_{\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])}\backslash X$ obtained from the edge identification on the quadrangle $(v_2', v_1', v_1'', v_2'')$, they are trivial as well for $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$; therefore, this computation splits into two parts: firstly, the map on bottom rows $E_2^{p,0} \cong \mathrm{H}^p(_\Gamma\backslash X; \mathbb{F}_2)$, where X is the 2-dimensional retract of hyperbolic space; and secondly the maps supported on cells with 2-torsion in their stabilizers. The result of the first part is the following.

Proposition 9. *The injections i and j induce a map*

$$\bigoplus_2 \mathrm{H}^*(_{\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])}\backslash X; \mathbb{F}_2) \xrightarrow{(i^*, j^*)} \mathrm{H}^*(_{\Gamma_0(\sqrt{-2})}\backslash X; \mathbb{F}_2)$$

which is injective, respectively surjective, except for $\ker(i^0, j^0) \cong \mathbb{F}_2$ and $\mathrm{coker}(i^2, j^2) \cong \mathbb{F}_2$.

Proof. The two generators of $\mathrm{H}^1({}_{\Gamma_0(\sqrt{-2})}\backslash X; \mathbb{F}_2)$ are supported on the loops obtained from the edges stabilized by to the matrices A, respectively C. Considering again the cylindrical shape of ${}_{\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])}\backslash X$ obtained from the edge identification on the quadrangle $(v_2', v_1', v_1'', v_2'')$, the generator of $\mathrm{H}^1({}_{\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])}\backslash X; \mathbb{F}_2)$ is supported on the loop obtained from the edge stabilized by the matrix c. The matrices c and $C = i(C)$ are conjugate in $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ via the matrix $h := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ which sends the quadrangle $(v_2, v_1, v_1''', v_2''')$ to the quadrangle $(v_2', v_1', v_1'', v_2'')$. Also, the matrix $j(A) = c^{-1}Cc$ is conjugate to c via h. Hence (i^1, j^1) identifies the two loops of $\bigoplus_2 \mathrm{H}^1({}_{\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])}\backslash X; \mathbb{F}_2)$ with those of $\mathrm{H}^1({}_{\Gamma_0(\sqrt{-2})}\backslash X; \mathbb{F}_2)$, and thus is an isomorphism. The ranks of (i^0, j^0) and (i^2, j^2) are obvious. $\square$

For the second part of the computation, we notice that apart from the ± 1 matrices fixing all of X, only vertices and edges admit non-trivial stabilizers, so in degrees $q > 0$, the $E_1^{p,q}$ terms are concentrated in the three columns $p \in \{0, 1, 2\}$. And for $E_2^{p,q}(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]), X; \mathbb{F}_2)$ (we shall use this notation with arguments to distinguish between the two spectral sequences), they are concentrated in the two columns $p \in \{0, 1\}$, because $\mathrm{H}^2({}_{\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])}\backslash X; \mathbb{F}_2) = 0$. As we have seen in Section 3.1 that $d_2^{p,q}(\Gamma_0(\sqrt{-2}), X; \mathbb{F}_2) = 0$, the $p = 2$ column has stationary terms $E_2^{2,q}(\Gamma_0(\sqrt{-2}), X; \mathbb{F}_2) \cong \mathrm{H}^2({}_{\Gamma_0(\sqrt{-2})}\backslash X; \mathbb{F}_2) \cong \mathbb{F}_2$ which remain as the $E_\infty^{2,q}$-term for all $q \geq 0$. So the following lemma gives us all the information that we need in order to achieve the computation.

Lemma 10. *In degrees $q > 0$, $p \in \{0, 1\}$, the map*

$$\bigoplus_2 E^{p,q}(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]), X; \mathbb{F}_2) \xrightarrow{(i^*, j^*)} E_2^{p,q}(\Gamma_0(\sqrt{-2}), X; \mathbb{F}_2)$$

is surjective, with kernel

$q = 4k+4$	$\langle e_4^i + e_4^j \rangle$	0
$q = 4k+3$	$\langle b_3^i, b_3^j \rangle$	$\langle e_2^{\langle A \rangle, i} b_1^i + e_2^{\langle A \rangle, j} b_1^j \rangle$
$q = 4k+2$	0	$\langle e_2^{\langle A \rangle, i} + e_2^{\langle A \rangle, j} \rangle$
$q = 4k+1$	0	$\langle b_1^i + b_1^j \rangle$
$k \in \mathbb{N} \cup \{0\}$	$p = 0$	$p = 1,$

where the subscripts specify the degrees of the cohomology classes, and the superscripts can be ignored (they only serve for tracking back the origin of a class in the calculation). Throughout the column $p = 2$, there is a constant term $\mathbb{F}_2$ in the corresponding cokernel.

Proof. From Proposition 9, we already know the contribution of the orbit spaces: a term of type $\mathbb{F}_2$ in rows $q \equiv 0 \mod 4$ in the column $p = 0$, and a constant term $\mathbb{F}_2$ throughout the column $p = 2$ in the corresponding cokernel. Complementary to this, there is a contribution of the non-central 2-torsion subcomplexes $X_s(\Gamma)$, which we are now going to determine. The maps i and j from the stabilizers on

$X_s(\Gamma_0(\sqrt{-2}))$ to the stabilizers on $X_s(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]))$,
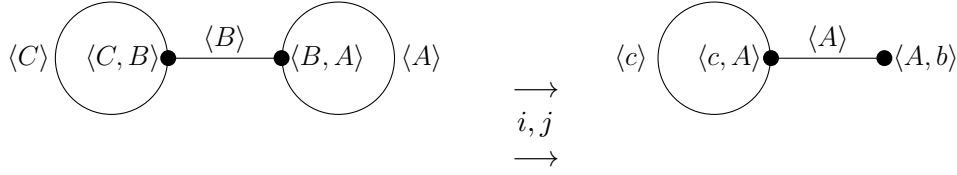
are determined by the following conjugacies (marked with the symbol "$\sim$") in $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$:

$$j(A) = c^{-1}Cc \sim c \sim C = i(C),$$
$$j(B) = -c^{-1}Bc \sim A = j(C) = i(A) \sim i(B).$$

The assignments $j(C) = A$, respectively $i(A) = A$, induce on the $E_2^{1,q}(\Gamma_0(\sqrt{-2}), X, \mathbb{F}_2)$ terms, which have their generators suppported on the loops obtained from the edges stabilized by C, respectively A, zero maps coming from the edge stabilized by A in $X_s(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]))$, because the latter supports the zero class in $E_2^{1,q}(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]), X, \mathbb{F}_2)$. Therefore, we can ignore those two assignments, and work only with the remaining ones. For the map i, we have remaining

On mod 2 cohomology rings, this induces

For the map j, we have remaining

On mod 2 cohomology rings, this induces

Assembling the maps i and j to (i,j), we can now see that (i^*, j^*) is surjective on the $E_1^{p,q}$ terms in the two columns $p \in \{0, 1\}$. To compute the kernel of (i^*, j^*), we compare the E_2 pages:

The E_2 page for the action of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ on X is concentrated in the two columns $p \in \{0, 1\}$, with the following generators, where the superscripts specify

the stabilizer of the supporting cell.

$q = 4k+4$	$\langle e_4^{\langle A,c\rangle}\rangle$	$\langle (e_2^{\langle c\rangle})^2\rangle$
$q = 4k+3$	$\langle x_3^{\langle A,c\rangle}, b_3^{\langle A\rangle}\rangle$	$\langle e_2^{\langle c\rangle} b_1^{\langle c\rangle}, e_2^{\langle A\rangle} b_1^{\langle A\rangle}\rangle$
$q = 4k+2$	$\langle x_2^{\langle A,c\rangle}, y_2^{\langle A,c\rangle}\rangle$	$\langle e_2^{\langle c\rangle}, e_2^{\langle A\rangle}\rangle$
$q = 4k+1$	$\langle x_1^{\langle A,c\rangle}\rangle$	$\langle b_1^{\langle c\rangle}\rangle$
$k \in \mathbb{N}\cup\{0\}$	$p=0$	$p=1$

The E_2 page for the action of $\Gamma_0(\sqrt{-2})$ on X is concentrated in the three columns $p \in \{0,1,2\}$, with the following generators.

$q = 4k+4$	$\langle e_4^{\langle C,B\rangle} + e_4^{\langle B,A\rangle}\rangle$	$\langle (e_2^{\langle C\rangle})^2, (e_2^{\langle A\rangle})^2\rangle$	$\mathbb{F}_2$
$q = 4k+3$	$\langle x_3^{\langle C,B\rangle}, x_3^{\langle B,A\rangle}\rangle$	$\langle e_2^{\langle C\rangle} b_1^{\langle C\rangle}, e_2^{\langle B\rangle} b_1^{\langle B\rangle}, e_2^{\langle A\rangle} b_1^{\langle A\rangle}\rangle$	$\mathbb{F}_2$
$q = 4k+2$	$\langle x_2^{\langle C,B\rangle}, y_2^{\langle C,B\rangle}, x_2^{\langle B,A\rangle}, y_2^{\langle B,A\rangle}\rangle$	$\langle e_2^{\langle C\rangle}, e_2^{\langle B\rangle}, e_2^{\langle A\rangle}\rangle$	$\mathbb{F}_2$
$q = 4k+1$	$\langle x_1^{\langle C,B\rangle}, x_1^{\langle B,A\rangle}\rangle$	$\langle b_1^{\langle C\rangle} + b_1^{\langle A\rangle}\rangle$	$\mathbb{F}_2$
$k \in \mathbb{N}\cup\{0\}$	$p=0$	$p=1$	$p=2$

This yields the claimed kernel (adding i, respectively j to the superscripts to specify the relevant copy of $E^{p,q}(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]), X; \mathbb{F}_2)$ for the pre-image), and also yields the claimed cokernel. $\square$

5. Investigating the module structure of the cohomology ring

With the above preparation, we will in this section conclude the proof of Theorem 3. Combining Proposition 9 and Lemma 10, we can see that the injections i and j induce a map on cohomology of groups,

$$\bigoplus_2 \mathrm{H}^*(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]); \mathbb{F}_2) \overset{(i^*, j^*)}{\longrightarrow} \mathrm{H}^*(\Gamma_0(\sqrt{-2}); \mathbb{F}_2)$$

which has the following kernel and cokernel dimensions over $\mathbb{F}_2$.

$q = 4k+5$	0	1
$q = 4k+4$	2	1
$q = 4k+3$	3	1
$q = 4k+2$	1	1
$q = 1$	0	0
$k \in \mathbb{N}\cup\{0\}$	$\dim_{\mathbb{F}_2}\ker(i^*,j^*)$	$\dim_{\mathbb{F}_2}\mathrm{coker}(i^*,j^*)$

In the Mayer–Vietoris long exact sequence on group cohomology with $\mathbb{F}_2$-coefficients derived from the amalgamated decomposition

$$\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\tfrac{1}{2}]) \cong \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]) *_{\Gamma_0(\sqrt{-2})} \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$$

with respect to the maps i and j,

$$\cdots \longleftarrow \mathrm{H}^{n+1}(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\tfrac{1}{2}]))$$
$$\mathrm{H}^n(\Gamma_0(\sqrt{-2})) \underset{(i^n, j^n)}{\longleftarrow} \bigoplus_2 \mathrm{H}^n(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])) \longleftarrow \mathrm{H}^n(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\tfrac{1}{2}]))$$
$$\cdots$$

the above calculated dimensions yield

$$1 \qquad 0 \longleftarrow \dim_{\mathbb{F}_2} \mathrm{H}^5(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\tfrac{1}{2}]))$$

$$1 \qquad 2 \longleftarrow \dim_{\mathbb{F}_2} \mathrm{H}^4(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\tfrac{1}{2}]))$$

$$1 \qquad 3 \longleftarrow \dim_{\mathbb{F}_2} \mathrm{H}^3(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\tfrac{1}{2}]))$$

$$1 \qquad 1 \longleftarrow \dim_{\mathbb{F}_2} \mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\tfrac{1}{2}]))$$

$$0 \qquad 0 \longleftarrow \dim_{\mathbb{F}_2} \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\tfrac{1}{2}]))$$

We identify $e_4 := e_4^i + e_4^j$ as the class, multiplication by which yields the 4-periodicity exposed in Lemma 10. Let us set the following names for the remaining classes: $x_4 := e_2^{\langle A \rangle, i} b_1^i + e_2^{\langle A \rangle, j} b_1^j$, $x_3 := b_3^i$, $y_3 := b_3^j$, $z_3 := e_2^{\langle A \rangle, i} + e_2^{\langle A \rangle, j}$, $x_2 := b_1^i + b_1^j$ and s_{q+2} for the image of the generator of $\mathrm{H}^2({}_{\Gamma_0(\sqrt{-2})} \backslash X; \mathbb{F}_2)$ that is generating the $E_2^{2,q}$ term of the equivariant spectral sequence for $\Gamma_0(\sqrt{-2})$ in rows $q \in \{1, 2, 3, 4\}$. Theorem 1 tells us that $\mathrm{H}^*(\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}][\tfrac{1}{2}]); \mathbb{F}_2)$ is a free module over $\mathbb{F}_2[e_4]$, hence the above remaining classes constitute a basis $\{x_2, x_3, y_3, z_3, s_3, x_4, s_4, s_5, s_6\}$ for it. Thus we get the result stated in Theorem 3.
We can use our result to calculate dimension bounds for $\mathrm{H}^q(\mathrm{GL}_2(\mathbb{Z}[\sqrt{-2}][\tfrac{1}{2}]); \mathbb{F}_2)$, see [10].

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