On quasitrivial semigroups

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Part I: Single-plateauedness

Weak orderings

Recall that a *weak ordering* (or *total preordering*) on a set X is a binary relation $\precsim$ on X that is total and transitive.

Defining a weak ordering on X amounts to defining an ordered partition of X

For $X = \{a_1, a_2, a_3\}$, we have 13 weak orderings

$a_1 \prec a_2 \prec a_3$	$a_1 \sim a_2 \prec a_3$	$a_1 \sim a_2 \sim a_3$
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Single-plateaued weak orderings

Definition. (Black, 1948)
Let $\leq$ be a total ordering on X and let $\precsim$ be a weak ordering on X. Then $\precsim$ is said to be *single-plateaued for* $\leq$ if

$$a_i < a_j < a_k \quad \Longrightarrow \quad a_j \prec a_i \quad \text{or} \quad a_j \prec a_k \quad \text{or} \quad a_i \sim a_j \sim a_k$$

Examples. On $X = \{a_1 < a_2 < a_3 < a_4 < a_5 < a_6\}$

$$a_3 \sim a_4 \prec a_2 \prec a_1 \sim a_5 \prec a_6 \qquad\qquad a_3 \sim a_4 \prec a_2 \sim a_1 \prec a_5 \prec a_6$$

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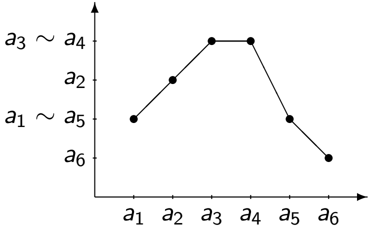

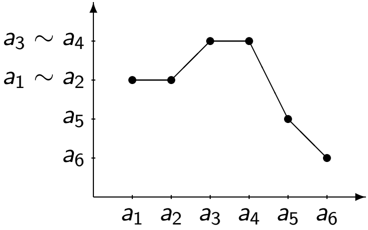

Part II: Quasitrivial semigroups

Quasitriviality

Definition

$F\colon X^2 \to X$ is said to be *quasitrivial* (or *conservative*) if

$$F(x,y) \ \in \ \{x,y\} \qquad x,y \in X$$

Example. $F = \max_{\leq}$ on $X = \{1,2,3\}$ endowed with the usual $\leq$

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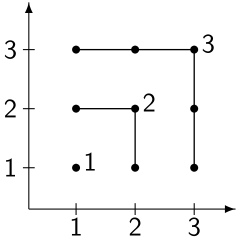

Projections

Definition.

The *projection operations* $\pi_1 \colon X^2 \to X$ and $\pi_2 \colon X^2 \to X$ are respectively defined by

$$\begin{aligned} \pi_1(x,y) &= x, \qquad x,y \in X \\ \pi_2(x,y) &= y, \qquad x,y \in X \end{aligned}$$

Quasitrivial semigroups

Theorem (Länger, 1980)

F is associative and quasitrivial

$$\Updownarrow$$

$$\exists \precsim \; : F|_{A\times B} = \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \quad \forall A, B \in X/\sim$$

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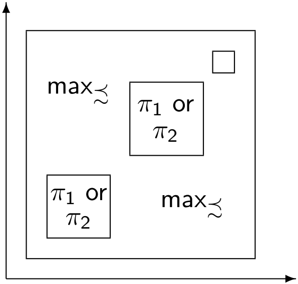

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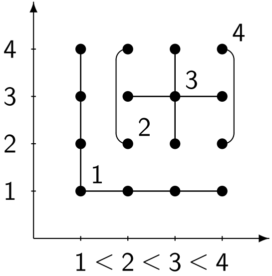
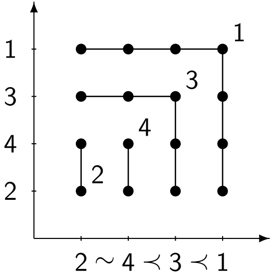




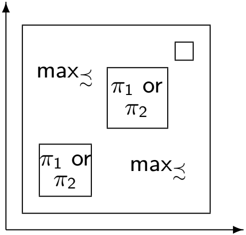
Order-preserving operations

Definition.

$F\colon X^2 \to X$ is said to be *$\leq$-preserving* for some total ordering $\leq$ on X if for any $x, y, x', y' \in X$ such that $x \leq x'$ and $y \leq y'$, we have $F(x, y) \leq F(x', y')$

Order-preserving operations

$(*)$



single-plateauedness: $a < b < c \implies a \prec b$ or $a \prec c$ or $a \sim b \sim c$

Theorem (Couceiro et al., 2018)

F is associative, quasitrivial, and $\leq$-preserving

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$\exists \precsim$: F is of the form $(*)$ and $\precsim$ is single-plateaued for $\leq$

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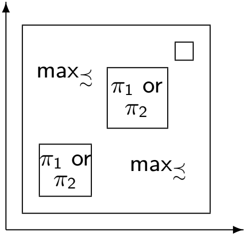

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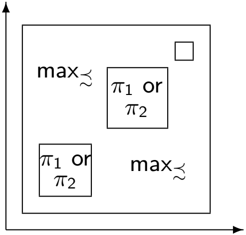

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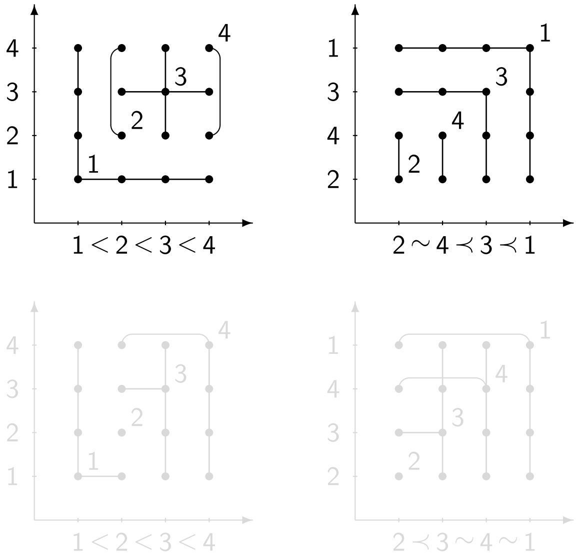

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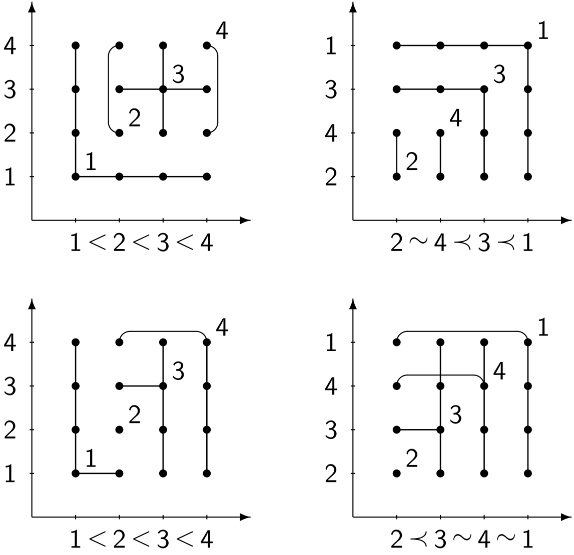

Quasitrivial n-ary semigroups

Definition

$F\colon X^n \to X$ is said to be

- *quasitrivial* if

$$F(x_1,\ldots,x_n) \in \{x_1,\ldots,x_n\} \qquad x_1,\ldots,x_n \in X$$

- *associative* if

$$\begin{aligned} & F(x_1,\ldots,x_{i-1},F(x_i,\ldots,x_{i+n-1}),x_{i+n},\ldots,x_{2n-1}) \\ = \quad & F(x_1,\ldots,x_i,F(x_{i+1},\ldots,x_{i+n}),x_{i+n+1},\ldots,x_{2n-1}) \end{aligned}$$

for all $x_1,\ldots,x_{2n-1} \in X$ and all $1 \leq i \leq n-1$.

Characterization?

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Characterization?

Reducibility

Definition

$F\colon X^n \to X$ and $G\colon X^2 \to X$ associative operations.

F is said to be *reducible to G* if

$$F(x_1, \ldots, x_n) = G(x_1, G(x_2, G(\ldots, G(x_{n-1}, x_n)\ldots)))$$

Example. On $X = \mathbb{R}$

$$F(x_1, x_2, x_3, x_4, x_5) = x_1 + x_2 + x_3 + x_4 + x_5,$$

and

$$G(x, y) = x + y$$

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Neutral elements

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$e \in X$ is said to be a *neutral element for F* if

$$F(x, e, \ldots, e) = F(e, x, e, \ldots, e) = \ldots = F(e, \ldots, e, x) = x,$$

for all $x \in X$

Example. $F(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \pmod{n-1}$

Proposition (Couceiro and D., 2019)

Any quasitrivial n-ary semigroup has at most two neutral elements.

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Theorem (Couceiro and D., 2019)

Any quasitrivial n-ary semigroup is reducible to a *semigroup.*

But the binary reduction is not necessarily quasitrivial nor unique.

Example.

$$F(x,y,z) = x + y + z \pmod{2}$$

$$G(x,y) = x + y \pmod{2} \qquad G'(x,y) = x + y + 1 \pmod{2}$$

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$F\colon X^n \to X$ associative and quasitrivial. TFAE

(i) F has a unique binary reduction

(ii) F has a quasitrivial binary reduction

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