Characterizations and classifications of quasitrivial semigroups

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Part I: Single-plateauedness and 2-quasilinearity

Weak orderings

Recall that a *weak ordering* (or *total preordering*) on a set X is a binary relation $\precsim$ on X that is total and transitive.

Defining a weak ordering on X amounts to defining an ordered partition of X

For $X = \{a_1, a_2, a_3\}$, we have 13 weak orderings

$a_1 \prec a_2 \prec a_3$	$a_1 \sim a_2 \prec a_3$	$a_1 \sim a_2 \sim a_3$
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$a_2 \prec a_1 \prec a_3$	$a_2 \prec a_1 \sim a_3$	
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\end{array}$$

Single-plateaued weak orderings

Definition. (Black, 1948)
Let $\leq$ be a total ordering on X and let $\precsim$ be a weak ordering on X.
Then $\precsim$ is said to be *single-plateaued for* $\leq$ if

$$a_i < a_j < a_k \implies a_j \prec a_i \quad \text{or} \quad a_j \prec a_k \quad \text{or} \quad a_i \sim a_j \sim a_k$$

Examples. On $X = \{a_1 < a_2 < a_3 < a_4 < a_5 < a_6\}$

$a_3 \sim a_4 \prec a_2 \prec a_1 \sim a_5 \prec a_6$

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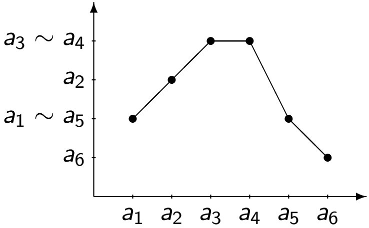

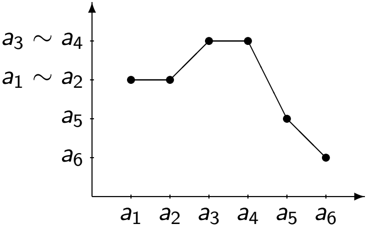

Single-plateaued weak orderings

Q: Given $\precsim$ is it possible to find $\leq$ for which $\precsim$ is single-plateaued?

Example: On $X = \{a_1, a_2, a_3, a_4\}$ consider $\precsim$ and $\precsim'$ defined by

$$a_1 \sim a_2 \prec a_3 \sim a_4 \qquad \text{and} \qquad a_1 \prec' a_2 \sim' a_3 \sim' a_4$$

Yes! Consider $\leq$ *defined by* $a_3 < a_1 < a_2 < a_4$

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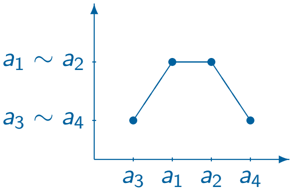

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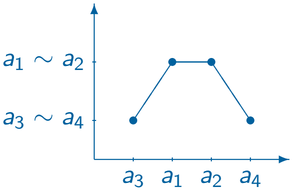

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2-quasilinear weak orderings

Definition.
We say that $\precsim$ is *2-quasilinear* if

$$a \prec b \sim c \sim d \quad \Longrightarrow \quad a, b, c, d \text{ are not pairwise distinct}$$

Proposition

Assume the axiom of choice.

$$\precsim \text{ is 2-quasilinear} \quad \Longleftrightarrow \quad \exists \leq \text{ for which } \precsim \text{ is single-plateaued}$$

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Part II: Quasitrivial semigroups

Quasitriviality

Definition

$F\colon X^2 \to X$ is said to be *quasitrivial* (or *conservative*) if

$$F(x,y) \in \{x,y\} \qquad x,y \in X$$

Example. $F = \max_{\leq}$ on $X = \{1,2,3\}$ endowed with the usual $\leq$

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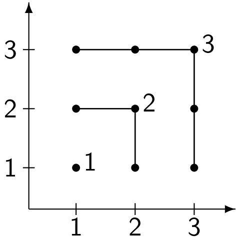

Projections

Definition.

The *projection operations* $\pi_1 \colon X^2 \to X$ and $\pi_2 \colon X^2 \to X$ are respectively defined by

$$\begin{aligned} \pi_1(x, y) &= x, \qquad x, y \in X \\ \pi_2(x, y) &= y, \qquad x, y \in X \end{aligned}$$

Quasitrivial semigroups

Theorem (Länger, 1980)

$$F \text{ is associative and quasitrivial}$$

$$\Updownarrow$$

$$\exists \precsim \ : F|_{A\times B} = \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \quad \forall A, B \in X/\sim$$

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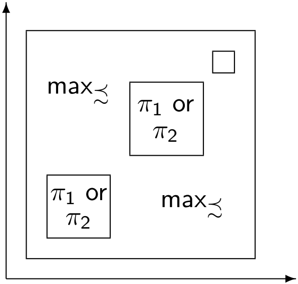

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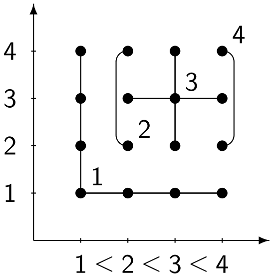

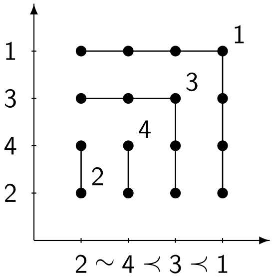

Order-preservable operations

Definition.

$F\colon X^2 \to X$ is said to be *$\leq$-preserving* for some total ordering $\leq$ on X if for any $x, y, x', y' \in X$ such that $x \leq x'$ and $y \leq y'$, we have $F(x,y) \leq F(x',y')$

Definition.

We say that $F\colon X^2 \to X$ is *order-preservable* if it is $\leq$-preserving for some $\leq$

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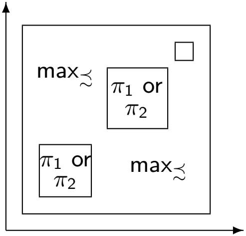

2-quasilinearity : $a \prec b \sim c \sim d \implies a, b, c, d$ are not pairwise distinct

Theorem

Assume the axiom of choice.

$$F \text{ is associative, quasitrivial, and order-preservable}$$

$$\Updownarrow$$

$$\exists \precsim \; : F \text{ is of the form } (*) \text{ and } \precsim \text{ is 2-quasilinear}$$

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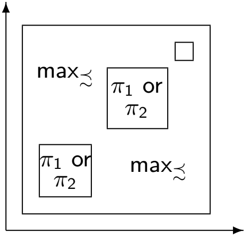

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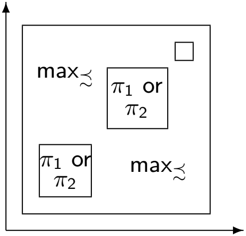

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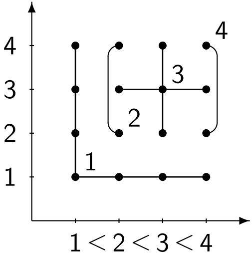

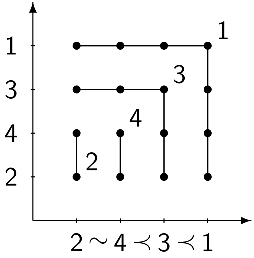

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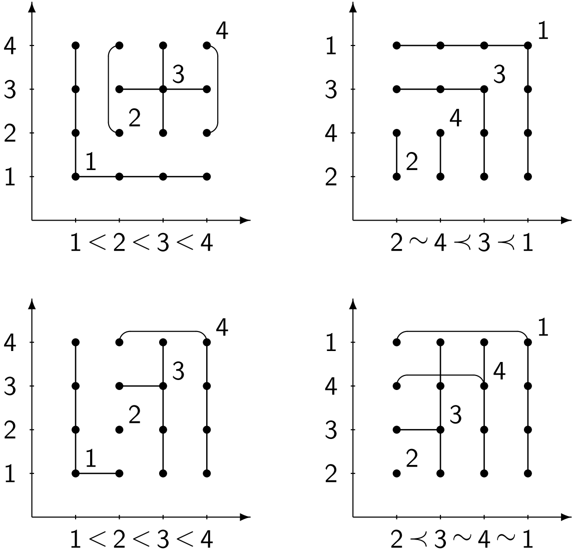

Final remarks

In arXiv: 1811.11113 and *Quasitrivial semigroups: characterizations and enumerations (Semigroup Forum, 2018)*

1. Characterizations and classifications of quasitrival semigroups by means of certain equivalence relations

2. Characterization of associative, quasitrivial, and order-preserving operations by means of single-plateauedness

3. New integer sequences (http://www.oeis.org)
   - Number of quasitrivial semigroups: A292932
   - Number of associative, quasitrivial, and order-preserving operations: A293005
   - Number of associative, quasitrivial, and order-preservable operations: Axxxxxx
   - ...

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