

Bowditch's Q-conditions and Minsky's primitive stability

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Abstract

For the action of the outer automorphism group of the rank two free group on the corresponding variety of $\mathrm{PSL}(2, \mathbb{C})$ characters, two domains of discontinuity have been known to exist that are strictly larger than the set of Schottky characters. One is introduced by Bowditch in 1998 (followed by Tan, Wong and Zhang in 2008) and the other by Minsky in 2013. We prove that these two domains are equal. We then show that they are contained in the set of characters having what we call the bounded intersection property.

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1 Introduction

Let F_n denote the free group on $n \geq 2$ generators, and set $G = \text{PSL}(2, \mathbb{C})$. The space of representations $\text{Hom}(F_n, G)$ admits a natural action of $\text{Aut}(F_n) \times \text{Aut}(G)$ given by

$$\begin{aligned} \text{Aut}(F_n) \times \text{Aut}(G) : \text{Hom}(F_n, G) &\longrightarrow \text{Hom}(F_n, G) \\ (f, g) &: \rho \longmapsto g \circ \rho \circ f^{-1} \end{aligned}$$

The G -character variety $\mathcal{X}(F_n)$ of F_n is then defined as the GIT quotient

$$\mathcal{X}(F_n) = \text{Hom}(F_n, G) // (\{Id\} \times \text{Inn}(G))$$

by the action of inner automorphisms of G . It then admits an action by the outer automorphism group

$$\text{Out}(F_n) = \text{Aut}(F_n)/\text{Inn}(F_n),$$

since the inner automorphism group $\text{Inn}(F_n)$ acts trivially.

Regarding the dynamical properties of the action of $\text{Out}(F_n)$ on $\mathcal{X}(F_n)$, one of the classical results is that this action is properly discontinuous on the open subset $\mathcal{S} \subset \mathcal{X}(F_n)$ consisting of Schottky characters. One may then ask if \mathcal{S} is a maximal domain of discontinuity for the action. Recently, Minsky [Min13] answered the question in the negative: an open subset \mathcal{PS} , consisting of *primitive stable* characters, is strictly larger than \mathcal{S}

$$\mathcal{S} \subsetneq \mathcal{PS} \subset \mathcal{X}(F_n)$$

and admits a properly discontinuous action of $\text{Out}(F_n)$. In particular, since \mathcal{S} is known to be the interior of the closed subset consisting of characters coming from injective representations with discrete image, we see that \mathcal{PS} contains characters coming from representations with non-discrete image. Thus Minsky's result shows that the geometric decomposition (discrete vs. dense) of the characters is different from the dynamical decomposition (properly discontinuous vs. ergodic) in this case. For more results about dynamics on representation spaces we refer the reader to the survey papers of Goldman [Gol06] (for surfaces and locally homogeneous geometric structures on them), Lubotzky [Lub11] (for free groups and various target groups), and Canary [Can15] (for word hyperbolic groups into semi-simple Lie groups with no compact factors).

We will focus on the special case of $n = 2$ in the present paper, where another domain of discontinuity for the action of $\text{Out}(F_2)$ has been known to exist. Bowditch [Bow98] introduced a notion called the *Q-conditions* and investigated the type-preserving characters in $\mathcal{X}(F_2)$ in an attempt to characterize the quasi-Fuchsian once-punctured torus groups. Generalizing Bowditch's work, Tan, Wong and Zhang [TWZ08a] tested the Q-conditions for *all* characters in $\mathcal{X}(F_2)$ and showed that the characters satisfying the Q-conditions form an open subset $\mathcal{BQ} \subset \mathcal{X}(F_2)$ and that $\text{Out}(F_2)$ acts on it properly discontinuously. As an interesting property of characters in \mathcal{BQ} , Bowditch and Tan-Wong-Zhang showed that a variation of McShane's identity holds for them.

In the first part of the paper we study the relation between these two domains \mathcal{BQ} and \mathcal{PS} . Very roughly speaking, the primitive stability of $\rho \in \text{Hom}(F_2, \text{PSL}(2, \mathbb{C}))$ means that under the orbit map of the ρ -action on \mathbb{H}^3 , the axes in F_2 of primitive words are sent to uniform quasi-geodesics (Definitions 4.3 and 4.4). On the other hand, the Q-conditions on ρ mean that for every primitive word, its cyclically reduced length and the translation distance of its ρ -action on \mathbb{H}^3 are comparable (Definition 4.5 and Theorem 4.6). It is not hard to see from the definitions that we have the inclusion

$$\mathcal{PS} \subset \mathcal{BQ} \subset \mathcal{X}(F_2).$$

A proof is given by Lupi in his thesis [Lup15, Proposition 2.9], but we also provide a different proof in Proposition 4.9 to make the paper self-contained.

A number of authors questioned if these two conditions are actually equivalent. See [Min13, p.64], [Can15, §.7.1] and [Lup15, §.4.2]. Also compare [Gol06, Problem 3.2]. Our first theorem answers the question in the affirmative:

Theorem I. *If a representation $\rho : F_2 \rightarrow \text{PSL}(2, \mathbb{C})$ satisfies the Q-conditions, then it is primitive stable. Consequently, we have the equality*

$$\mathcal{PS} = \mathcal{BQ}.$$

We prove this in Section 5.

1.1 Remark. (a) Lupi showed the same equality in his thesis [Lup15, Proposition 3.4 and Theorem 4.3] when the target group G is restricted to $\text{PSL}(2, \mathbb{R})$. His idea of proof, however,

does not seem to directly generalize to the case of $\mathrm{PSL}(2, \mathbb{C})$. We discuss this issue in Section 5.4.

(b) One may view Theorem I in analogy with a theorem of Delzant et al [DGLM11] that, for a word hyperbolic group acting on a metric space by isometries, an orbit map is a *quasi-isometric embedding* if and only if the action is *displacing*. In a sense, primitive stability (resp. the Q-conditions) is a weakening of the former (resp. the latter) property obtained by looking only at actions of primitive elements. We discuss this analogy in more detail in Section 5.5.

Let us briefly present our ideas of proof. The definitions of primitive stability and the Q-conditions involve the conjugacy classes of primitive elements of F_2 . The so-called *Christoffel words* are representatives of such classes and have many desirable properties for our purposes. Without giving the precise definition (Definition 2.8) let us exhibit a simple and motivational example in order to illustrate how to generate the Christoffel words and how the relevant geometry behaves asymptotically under the representations of Theorem I. See Figure 1.0.1.

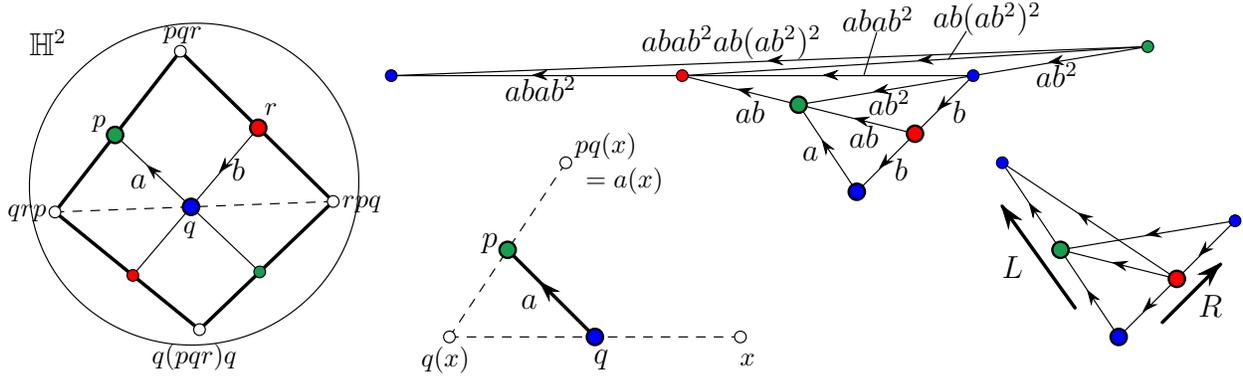


Figure 1.0.1: Idea of proof in the case of $G = \mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$.

The first picture shows a subgroup $W(p, q, r)$ of $\mathrm{Isom}^+(\mathbb{H}^2)$ generated by three orientation-preserving involutions p, q and r , whose fixed points are specified in the Klein projective model of \mathbb{H}^2 . If the product pqr is of infinite order, the group $W(p, q, r)$ is isomorphic to the free Coxeter group $\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ of rank three and contains the free group $F(a, b)$, where $a = pq$ and $b = qr$, as an index two subgroup. Note that $ab^{-1}a^{-1}b = pqr(qq)pqr = (pqr)^2$. The element pqr is an elliptic transformation in the picture; its fixed point and the fixed points of its three other conjugates form a quadrilateral. If pqr is parabolic or hyperbolic, then the quadrilateral is ideal or hyper-ideal, respectively. In all possible cases, however, the elements $a = pq$ and $b = qr$ are always hyperbolic translations and they identify the opposite edges of the quadrilateral. The quotient surface is a hyperbolic torus either with a cone-type singularity or with a cusp or with a funnel, depending on the nature of pqr .

The second picture represents a hyperbolic translation $a = pq$ as a directed edge. The fourth picture shows two operations L and R (left and right slides) which inductively generate other basis triples of $W(p, q, r)$ as well as other directed edges starting with the initial ones. The Christoffel words are the hyperbolic translations represented by the directed

edges so generated. They are primitive elements of $F(a, b)$. (For a more detailed account, see Section 2.3 as well as Appendix A.)

The third picture exhibits an example of the iteration $LRLR$ (read from right to left). Notice how quickly and uniformly the associated triangles degenerate with respect to the number of steps of LR -slides. In fact, if the initial triangle is acute then other triangles generated by the LR -slides are all obtuse. We remark that this process may be viewed as being reverse to the classical *trace minimizing algorithm* for checking discreteness of two-generator subgroups of $\mathrm{PSL}(2, \mathbb{R})$. See, for example, [Gil95, Chapter 2] and the references therein. This asymptotic behavior of triangles immediately imply that the representation $F(a, b)$ in $\mathrm{Isom}^+(\mathbb{H}^2)$ satisfies the Q-conditions and, after a little more work, primitive stability as well.

In the general case of representations in $\mathrm{PSL}(2, \mathbb{C})$, we exploit the fact (see Theorem 3.7) that an irreducible representation of $F(a, b)$ extends uniquely to the free Coxeter group $W(p, q, r)$, thereby giving us a triple of complete geodesics in \mathbb{H}^3 which are axes of the involutions p , q and r . For representations satisfying the Q-conditions (and hence irreducibility as well), we show that the triple of axes behaves analogously as in the third picture of Figure 1.0.1 under the LR -slides asymptotically. See Lemma 5.1 for a precise statement. Combined with the properties of Christoffel words, this enables us to show primitive stability.

In the second part of the paper (Section 6) we investigate a condition which we call the *bounded intersection property*. It is motivated by the work of Gilman and Keen [GK09], where they investigated palindromic elements in order to test the discreteness of irreducible representations $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$. Slightly strengthening their idea we consider the primitive elements that are palindromic in one of the three chosen bases (a, b) , (b, c) , (a, c) of F_2 , where $abc = 1$. The bounded intersection property is a certain geometric condition on the ρ -images of these palindromic primitive elements. See Definition 6.10 for details. This property also defines a subset $\mathcal{BI} \subset \mathcal{X}(F_2)$, which we show in Proposition 6.11 to be invariant under the action of $\mathrm{Out}(F_2)$.

We do not know if \mathcal{BI} is an open subset nor if $\mathrm{Out}(F_2)$ acts properly discontinuously on it. Nevertheless, we prove in Section 6.5 the following.

Theorem II. *If $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ satisfies the Q-conditions, then it has the bounded intersection property. That is, we have the inclusion $\mathcal{BQ} \subset \mathcal{BI}$. On the other hand, if ρ has discrete image and has the bounded intersection property, then it satisfies the Q-conditions.*

We were kindly informed by Caroline Series [Ser] that she also proved Theorem I as well as a result similar to Theorem II.

The rest of the paper is organized as follows. In Section 2 we recall some classical results about automorphisms of F_2 . We introduce the Christoffel words in a basis of F_2 and discuss their properties. In Section 3 we mostly follow Fenchel [Fen89] and discuss some basic geometry of \mathbb{H}^3 with an emphasis on half-turns and Coxeter extensions. We also introduce the amplitude of a right-angled hexagon. In Section 4 we recall the definition of primitive

stability and that of the Q-conditions, and prove that a primitive stable representation satisfies the Q-conditions. In Section 5 we give a proof of Theorem I. In Section 6 we discuss palindromic elements of F_2 , introduce the bounded intersection property and prove Theorem II. In Appendix A we discuss automorphisms of F_2 in view of its Coxeter extension.

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2 The free group of rank two and Christoffel words

We recall classical facts about the (outer) automorphism group of the free group F_2 of rank two. One often studies the group $\text{Out}(F_2)$ via its identification with the mapping class group of the one-holed torus. Since the representations of our interest are not (yet) known to be coming from a geometric structure on the surface, we do not discuss this identification here but rather proceed purely algebraically.

The eventual goal is to introduce Christoffel bases, which are explicit representatives of conjugacy classes of bases of F_2 . Their nice properties, especially Lemma 2.13, will be essential in our proof of Theorem I.

2.1 The automorphism group of F_2

The following results of Nielsen are all from his seminal paper [Nie17].

Let F_2 denote the free group on two generators. A *basis* of F_2 is an ordered pair of free generators. An element of F_2 is called *primitive* if it is a member of a basis. The abelianization $F_2/[F_2, F_2]$ of F_2 is isomorphic to the free abelian group \mathbb{Z}^2 of rank two. Bases and primitive elements of \mathbb{Z}^2 are defined analogously.

To be specific, let us fix one basis (a, b) of F_2 once and for all, and denote it by

$$\mathbf{e} = (a, b).$$

In order to emphasize this we shall often write

$$F_2 = F(\mathbf{e}) = F(a, b)$$

interchangeably. The abelianization homomorphism

$$(2.1) \quad \pi_{\mathbf{e}} : F_2 \rightarrow \mathbb{Z}^2$$

is then given by $\pi_e(a) = (1, 0)$ and $\pi_e(b) = (0, 1)$.

Nielsen showed the following.

2.2 Theorem (Nielsen). *Let x and y be elements of $F(a, b)$.*

- (a) *If the pair (x, y) is a basis of $F(a, b)$ then the 2×2 matrix with columns $\pi_e(x)$ and $\pi_e(y)$ belongs to $\mathrm{GL}(2, \mathbb{Z})$, that is, has determinant ± 1 . Conversely, any matrix in $\mathrm{GL}(2, \mathbb{Z})$ determines exactly one conjugacy class of a basis of $F(a, b)$.*
- (b) *The pair (x, y) is a basis of $F(a, b)$ if and only if the commutator $xyx^{-1}y^{-1}$ is conjugate either to the commutator $aba^{-1}b^{-1}$ or to its inverse $bab^{-1}a^{-1}$.*

Let us briefly discuss only the item (a) of the theorem. Note that a basis (x, y) of F_2 corresponds to a unique automorphism $f \in \mathrm{Aut}(F_2)$ defined by $f(a) = x$ and $f(b) = y$. Nielsen found a set of generators of $\mathrm{Aut}(F_2)$: they are the so-called elementary Nielsen transformations and correspond to the bases

$$(2.3) \quad (b, a), (a^{-1}, b), (a, b^{-1}), (a, ab), (ab, b),$$

respectively. Since an analogous result for $\mathrm{Aut}(\mathbb{Z}^2) \cong \mathrm{GL}(2, \mathbb{Z})$ is well-known, we see that the abelianization homomorphism π_e induces a surjective homomorphism

$$\mathrm{Aut}(F_2) \twoheadrightarrow \mathrm{Aut}(\mathbb{Z}^2).$$

Nielsen also showed that the kernel of this map is exactly the group $\mathrm{Inn}(F_2)$ of all inner automorphisms of F_2 . See also [LS77, Proposition I.4.5]. Hence we obtain an isomorphism

$$\mathrm{Out}(F_2) = \mathrm{Aut}(F_2)/\mathrm{Inn}(F_2) \cong \mathrm{Aut}(\mathbb{Z}^2) \cong \mathrm{GL}(2, \mathbb{Z}).$$

Therefore, the conjugacy classes of bases of F_2 are in one-to-one correspondence with the bases of \mathbb{Z}^2 . This implies, in turn, that the conjugacy classes of primitive elements of F_2 are in one-to-one correspondence with the primitive elements of \mathbb{Z}^2 . See also [OZ81, Corollary 3.2].

2.2 Unoriented primitive classes and the Farey triangulation

Both Bowditch's Q-conditions and Minsky's primitive stability are concerned with conjugacy classes of primitive elements of F_2 . As we shall see later in Section 4, these conditions also make sense for a slightly bigger classes. To be more precise, we consider the equivalence relation \sim on F_2 , where

$$a' \sim a \quad \text{iff} \quad a' \text{ is conjugate either to } a \text{ or to } a^{-1}.$$

Let us denote by Prim the set of all primitive elements of F_2 . For a primitive element $x \in \mathrm{Prim}$ we call its \sim -equivalence class $[x] \in \mathrm{Prim}/\sim \subset F_2/\sim$ the *unoriented primitive class* of x .

In view of the above one-to-one correspondence between primitive conjugacy classes of F_2 and primitive elements of \mathbb{Z}^2 , we see that the unoriented primitive classes of F_2 correspond to the projectivized primitive elements of \mathbb{Z}^2

$$(2.4) \quad \mathrm{Prim}/\sim \longleftrightarrow \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$$

and the action of $\text{Aut}(\mathbb{F}_2)$ on the unoriented primitive classes factors through $\text{PGL}(2, \mathbb{Z}) = \text{GL}(2, \mathbb{Z})/\{\pm\text{Id}\}$:

$$\text{Aut}(\mathbb{F}_2) \twoheadrightarrow \text{PGL}(2, \mathbb{Z}) \curvearrowright \mathbb{P}^1(\mathbb{Q}).$$

Note that the kernel of this homomorphism is $\text{Inn}^\epsilon(\mathbb{F}_2) := \text{Inn}(\mathbb{F}_2) \rtimes \langle \epsilon \rangle$, where ϵ is an involutory automorphism of $\mathbb{F}_2 = \mathbb{F}(a, b)$ defined by $\epsilon(a, b) = (a^{-1}, b^{-1})$.

2.5 Convention. When we write an element p/q of \mathbb{Q} for coprime integers p and q , we shall always assume $q > 0$. We formally write either $1/0$ or $(-1)/0$ for the infinity ∞ .

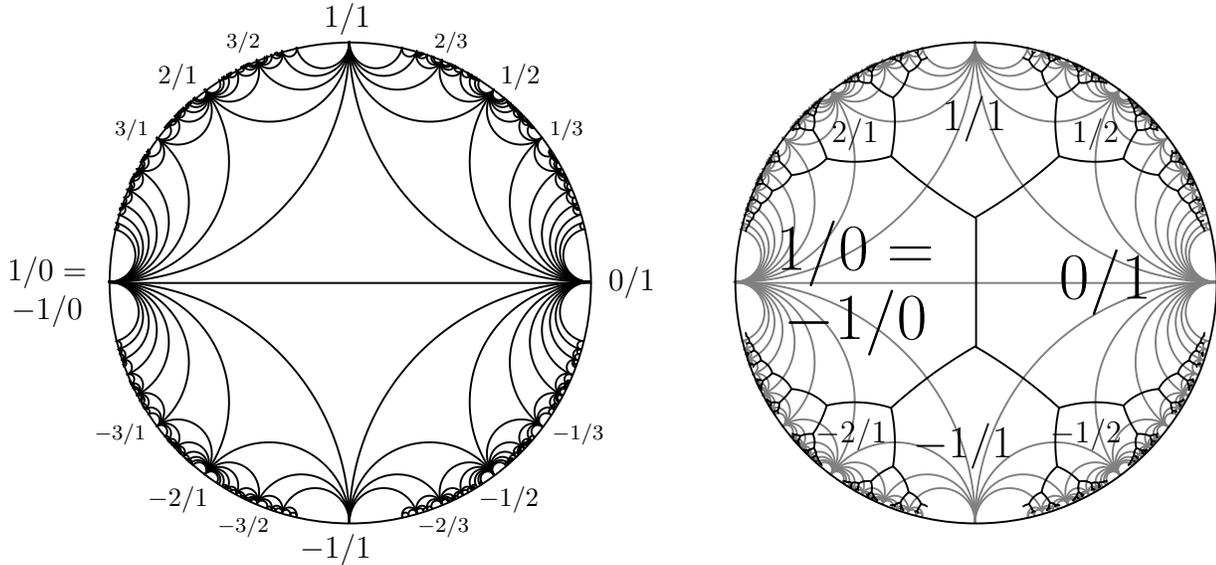


Figure 2.2.1: (Left) The Farey triangulation \mathcal{F} . (Right) Its dual tree with rational numbers written inside the complementary regions.

As the action of $\text{Aut}(\mathbb{F}_2)$ on Prim/\sim factors through $\text{PGL}(2, \mathbb{Z})$, we may visualize this action via the classical Farey triangulation. Let us take the upper half-plane of the complement $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ as a conformal model of the hyperbolic plane \mathbb{H}^2 . We denote $\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$, where the ideal boundary $\partial_\infty \mathbb{H}^2$ is identified with $\mathbb{P}^1(\mathbb{R})$. The *Farey triangulation* \mathcal{F} is a 2-dimensional simplicial complex realized as a subset

$$\mathcal{F} = \mathbb{H}^2 \cup \mathbb{P}^1(\mathbb{Q}) \subset \overline{\mathbb{H}^2}.$$

By definition, the set of 0-simplices are realized as $\mathbb{P}^1(\mathbb{Q})$. We denote

$$\mathbb{V} = \mathbb{P}^1(\mathbb{Q})$$

and call its elements the Farey vertices. The 1-simplices, called the Farey edges, are realized as complete geodesics connecting two Farey vertices p/q and r/s such that

$$\det \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1.$$

The 2-simplices, called the Farey triangles, are realized as ideal triangles bounded by three Farey edges. See Figure 2.2.1. For more about the Farey triangulation see, for example, [Hat, Chapter 1].

Abusing notations we shall often identify the vertex set $V = P^1(\mathbb{Q})$ with the set Prim/\sim of unoriented primitive classes via (2.4)

$$V = \text{Prim}/\sim.$$

and say a Farey vertex $[x] \in V$. The conjugacy class of a basis (x, y) of F_2 determines a *directed* Farey edge connecting vertices from $[x]$ to $[y]$, which we simply denote by

$$[x, y]$$

instead of $([x], [y])$. Note that three other conjugacy classes of bases (x, y^{-1}) , (x^{-1}, y) and (x^{-1}, y^{-1}) also determine the same directed edge $[x, y]$. Compare Theorem 2.2(a). By convention, however, whenever we write $[x, y]$ it will be assumed that the specific representative basis (x, y) is chosen. The directed edge corresponding to the distinguished basis $\mathbf{e} = (a, b)$ will often be denoted by $[\mathbf{e}] = [a, b]$; it runs from $\infty = 1/0$ to $0 = 0/1$.

We identified the Farey triangulation \mathcal{F} as a subset of the disk $\overline{\mathbb{H}^2}$. In most instances, however, we are interested only in its combinatorial structure and draw the pictures accordingly: for example, see Figure 2.3.1.

2.6 Remark. The dual object of the Farey triangulation is the complete binary tree. If it is properly embedded in the plane, it is called the *topograph* (see [Con97] and also [Hat]). The combinatorial structure of topograph is more convenient in Bowditch's theory, where one investigates flows on the topograph. See Bowditch [Bow98] as well as [TWZ08a] and [GMST]. Although we do not use the topograph directly, its tree structure underlies the inductive construction that we explain below using galleries and levels.

2.3 Levels and Christoffel words

Recall that we fixed a basis $\mathbf{e} = (a, b)$ of F_2 and that the distinguished directed Farey edge is denoted by $[\mathbf{e}] = [a, b]$. This naturally leads us to consider the 2-fold reflectional symmetry of the Farey triangulation with respect to $[\mathbf{e}]$ among other symmetries in $\text{PGL}(2, \mathbb{Z})$.

From the embedding $\mathcal{F} \subset \overline{\mathbb{H}^2}$ of the Farey triangulation, we can talk about the sides of directed Farey edges. The set of Farey vertices lying on the left-hand (resp. right-hand) side of $[x, y]$ will be denoted by $V^+[x, y]$ (resp. $V^-[x, y]$).

A single directed Farey edge $[x, y]$ *induces* directions on other edges as follows. The circle $\partial\mathbb{H}^2$ is the union of two semi-circles connecting $[x]$ and $[y]$, both of which are endowed with the linear order running from $[x]$ to $[y]$. We then direct other Farey edges according to this linear order; we say they are *co-directed* with $[x, y]$. For example, Figure 2.3.1(Left) shows the directions induced by $[\mathbf{e}]$. This will be used later for the inductive procedure in Definitions 2.8 and 6.3.

We now introduce the notion of levels relative to the distinguished edge $[\mathbf{e}]$. A *gallery* is the union $T_1 \cup T_2 \cup \dots \cup T_k$ of Farey triangles such that the triangles T_i and T_{i+1} are adjacent along an edge for $1 \leq i \leq k-1$. The *length* of a gallery is the number of triangles it contains.

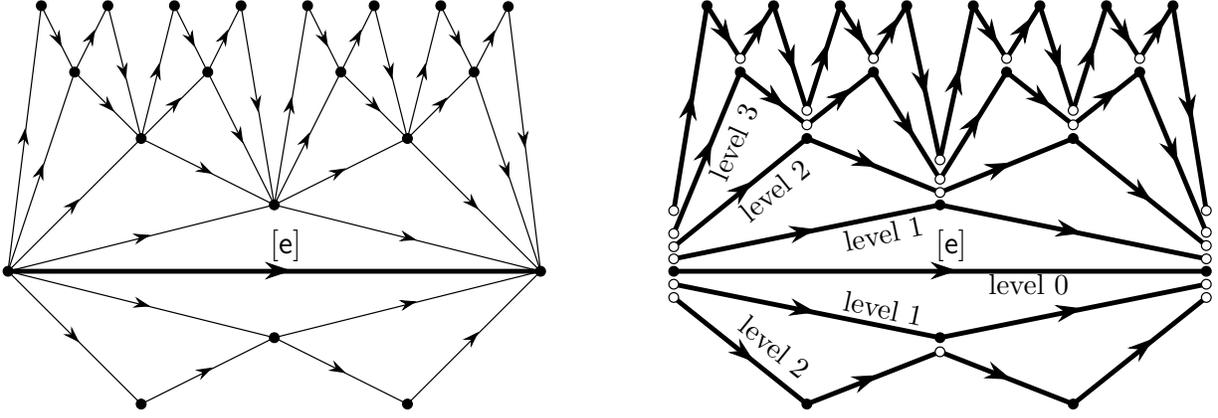


Figure 2.3.1: (Left) Farey edges with directions induced from the edge $[e]$. (Right) Levels with respect to the edge $[e]$.

2.7 Definition. (See Figure 2.3.1(Right).) Let X denote either a vertex distinct from $[a]$ or $[b]$, or an edge distinct from $[e]$, or a triangle. Then there exists a unique gallery of shortest length containing both X and $[e]$. The e -level $\text{Lv}_e(X)$ of X is defined to be the length of this gallery. The edge $[e]$ itself as well as its end points $[a]$ and $[b]$ are defined to be of e -level 0. There is no triangle of e -level 0.

Most of the time the reference edge $[e]$ is understood and we shall simply say *levels* instead of e -levels. While we defined the level combinatorially, it can also be defined using the geometry of \mathbb{H}^2 as in [GK11, Definition 3].

A vertex of level $k > 0$ is contained in a unique triangle of level k . A triangle of level $k > 0$ contains two edges of level k and one edge of level $k - 1$; if $k > 1$ its vertices are of level k , $k - 1$ and j (with $j \leq k - 2$), respectively.

A directed edge $[x, y]$ of level $k > 0$ contains vertices of level k and j (with $j \leq k - 1$), and is contained in two triangles of level k and of level $k + 1$, which will be denoted, respectively, by

$$[w; x, y] \quad \text{and} \quad [x, y; z],$$

where $\text{Lv}_e[w] \leq k - 1$ and $\text{Lv}_e[z] = k + 1$. The union of these triangles will be denoted by

$$[w; x, y; z]$$

and called the *quadrilateral* determined by $[x, y]$.

Following Bowditch [Bow98, p.704, p.706] we use the notion of levels in order to define functions on $V^\pm[a, b]$ inductively. Let S be a set and $B : S \times S \rightarrow S$ a binary operation. Given an S -valued function f defined on the vertices $[a]$ and $[b]$ of level 0, we can inductively extend it to a map $f : V^+[a, b] \rightarrow S$ as follows: a vertex $[z] \in V^+[a, b]$ of level $k + 1$ ($k \geq 0$) is contained in a unique triangle $[x, y; z]$ of level $k + 1$ such that $[x, y]$ is co-directed with $[a, b]$. Then the levels of $[x]$ and $[y]$ are at most k . Assuming, inductively, the values $f[x]$ and $f[y]$

have already been defined, we define $f[z] = B(f[x], f[y])$. Similarly, we extend f to $V^-[a, b]$ starting with (possibly new) initial values of f on $[a]$ and $[b]$.

Note that the level function Lv_e itself can be defined in this way by setting $S = \mathbb{Z}$ and $B(p, q) = \max\{p, q\} + 1$ starting with $Lv_e[a] = Lv_e[b] = 0$. More interesting examples are the Fibonacci function and the Christoffel function to be defined as follows. See Figure 2.3.2.

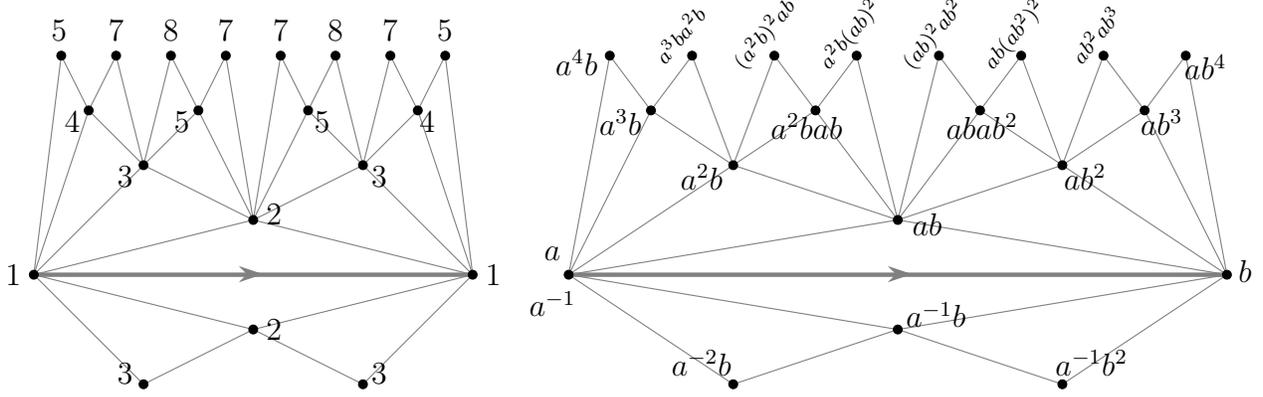


Figure 2.3.2: (Left) The Fibonacci function. (Right) The Christoffel function.

2.8 Definition. Let S and B be as discussed above.

- Let $S = \mathbb{N}$ and $B(p, q) = p + q$. The *e-Fibonacci function* Fib_e is defined inductively on V with initial values $\text{Fib}_e[a] = \text{Fib}_e[b] = 1$.
- Let $S = F(a, b)$ and $B(w_1, w_2) = w_1w_2$, the concatenation of w_1 and w_2 . The *e-Christoffel function* Ch_e is defined inductively on $V^+[a, b]$ with initial values $\text{Ch}_e[a] = a$ and $\text{Ch}_e[b] = b$, and on $V^-[a, b]$ with initial values $\text{Ch}_e[a] = a^{-1}$ and $\text{Ch}_e[b] = b$. The values of Ch_e are called *e-Christoffel words* (or *Christoffel words* in $\{a, b\}$).

Strictly speaking, the value $\text{Ch}_e[a]$ is indeterminate and can be either a or a^{-1} depending on the context. This will not cause any confusion later.

Before we delve into a few immediate properties of the Fibonacci and Christoffel functions, let us first adopt the following terminology: elements of the free semi-group generated by two symbols x and y are said to be *positive* in $\{x, y\}$.

Let $[w] \in V$ with $Lv_e[w] > 0$. Since Christoffel words are obtained by concatenation, it is clear that $\text{Ch}_e[w]$ is either positive in $\{a, b\}$ (if $[w] \in V^+[a, b]$) or positive in $\{a^{-1}, b\}$ (if $[w] \in V^-[a, b]$). This implies, in particular, that every e-Christoffel word is *cyclically reduced*, that is, its initial and terminal symbols are not inverse to each other. Observe also that if $[w] \in V$ corresponds to $p/q \in \mathbb{P}^1(\mathbb{Q})$ then $\pi_e(\text{Ch}_e[w]) = (p, q)$ (see (2.1), (2.4) and Convention 2.5). One can use the continued fraction expansion of p/q to find the corresponding e-Christoffel word. (See Appendix A.3 for an illustration of this reverse procedure.) Therefore, $\text{Ch}_e[w]$ is a cyclically reduced representative of its unoriented primitive class $[w]$.

It is also apparent from the definitions that, for each vertex $[w] \in V$, we have

$$(2.9) \quad \text{Fib}_e[w] = |\text{Ch}_e[w]|_e,$$

where $|\cdot|_e$ denotes the word length with respect to the basis $e = (a, b)$. See [Bow98, Lemma 2.2.1]. (As we shall see later in (4.1), the value $\text{Fib}_e[w]$ is also equal to the translation length $\ell_e(w)$.) We also observe that

$$(2.10) \quad \text{Fib}_e[w] \geq \text{Lv}_e[w] + 1$$

for all $[w] \in \mathbf{V}$.

Let us unravel the definition of e -Christoffel words a little further. Let $[x, y; z]$ be a Farey triangle of level $k + 1$ ($k \geq 0$) appearing in the inductive procedure and assume the edge $[x, y]$ of level k is co-directed with $[e]$. We then have $\text{Ch}_e[z] = \text{Ch}_e[x]\text{Ch}_e[y]$ by definition. If the pair $(\text{Ch}_e[x], \text{Ch}_e[y])$ is a basis of \mathbf{F}_2 , then the pairs $(\text{Ch}_e[x], \text{Ch}_e[z])$ and $(\text{Ch}_e[z], \text{Ch}_e[y])$ are obtained by elementary Nielsen transformations (the last two in (2.3)) and thus are bases as well. Therefore, for any Farey edge $[x, y]$ co-directed with $[e]$, we see inductively that the pair $(\text{Ch}_e[x], \text{Ch}_e[y])$ is a basis. We call such a pair an *e-Christoffel basis*.

2.11 Remark. The above inductive procedure basically amounts to considering the action of the free submonoid of $\text{Aut}(\mathbf{F}_2)$ generated by two automorphisms corresponding to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ in $\text{PGL}(2, \mathbb{Z})$. For more details, see Appendix A.3.

Suppose now $k > 0$ and let $[w; x, y]$ be the other Farey triangle adjacent to $[x, y]$. It is of level k . Then there are two cases depending on whether $\text{Lv}[x] < \text{Lv}[y]$ or $\text{Lv}[x] > \text{Lv}[y]$. In the former case we have $\text{Ch}_e[y] = \text{Ch}_e[x]\text{Ch}_e[w]$, and in the latter $\text{Ch}_e[x] = \text{Ch}_e[w]\text{Ch}_e[y]$.

To summarize, if (x, y) is an e -Christoffel basis with $\text{Lv}_e[x, y] \neq 0$ then it determines the quadrilateral $[x^{-1}y; x, y; xy]$. In particular, $\text{Lv}_e[x^{-1}y] < \text{Lv}_e[xy]$. We extract this property and apply it to arbitrary bases.

2.12 Definition. A basis $f = (x, y)$ of \mathbf{F}_2 is said to be *acute* relative to the basis $e = (a, b)$ if $\text{Lv}_e[x^{-1}y] < \text{Lv}_e[xy]$.

For any basis (x, y) with $\text{Lv}_e[x, y] \neq 0$, note that either (x, y) and (x^{-1}, y^{-1}) are acute relative to the basis e , or else (x^{-1}, y) and (x, y^{-1}) are acute relative to the basis e .

Lastly, suppose $[x, y]$ is a Farey edge co-directed with $[e]$ and $\text{Lv}_e[x, y] > 0$. In order to simplify the notation let us define $\mathbb{I}[x, y]$ to be $\mathbf{V}^+[x, y]$ if $[x, y]$ lies on the left of $[e]$, or to be $\mathbf{V}^-[x, y]$ if $[x, y]$ lies on the right of $[e]$. Then the following lemma is also immediate from the inductive construction of the Christoffel words.

2.13 Lemma. *Suppose $[x, y]$ is a Farey edge co-directed with $[e]$ and $\text{Lv}_e[x, y] > 0$. Then the e -Christoffel words in $\mathbb{I}[x, y]$ are positive words in $\{\text{Ch}_e[x], \text{Ch}_e[y]\}$.*

For example, the unoriented primitive class of the Christoffel word $(a^2b)^2ab$, which corresponds to $5/3$ in $\{a, b\}$, lies in $\mathbf{V}^+[a, ab]$. Since $(a^2b)^2ab = a(ab)a(ab)^2$, it corresponds to the positive slope $2/3$ in $\{a, ab\}$. See Figure 2.3.2(Right).

2.14 Remark. Although the Christoffel words are classical mathematical objects, the terminology was introduced rather recently in the 1990s. See [BLRS09] and [Aig13] for more detailed accounts on the theory of Christoffel words. See also [Coh71], [Coh72] and [KR07]. In the literature of Kleinian groups and low-dimensional topology, the Christoffel words also appeared in [Jør03], [JM79] and [OZ81] (with correction [GAR99]) as well as in [CMZ81] (in a somewhat weaker form). Compare also [GK11].

2.4 Level- n partition of Farey vertices

For each integer $n \geq 0$ we associate a partition of the vertex set V as follows. Let $V_{\leq n}$ be the (finite) subset of V consisting of all vertices of level at most n . Under the embedding $V = \mathbb{P}^1(\mathbb{Q}) \subset \mathbb{P}^1(\mathbb{R})$, consider the complement of $V_{\leq n}$ in $\mathbb{P}^1(\mathbb{R})$. It consists of 2^{n+1} open intervals I_j ($1 \leq j \leq 2^{n+1}$).

2.15 Definition. The *level- n partition* of V is defined to be the partition

$$V = V_{\leq n} \cup I_1 \cup I_2 \cup \cdots \cup I_{2^{n+1}},$$

where $I_j = I_j \cap \mathbb{P}^1(\mathbb{Q})$ for $1 \leq j \leq 2^{n+1}$. We still call each I_j an *interval* and cyclically order the indices as in Figure 2.4.1.

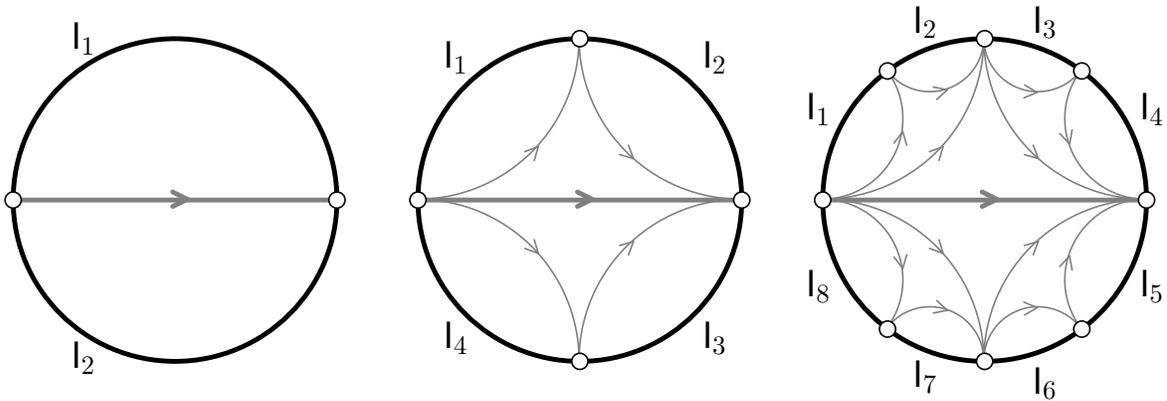


Figure 2.4.1: The level- n partitions of V for $n = 0, 1, 2$.

Note that each interval I_j is either of the form $V^+[x_j, y_j]$ if $1 \leq j \leq 2^n$ or of the form $V^-[x_j, y_j]$ if $2^n + 1 \leq j \leq 2^{n+1}$. As we did previously, we can simply write $I_j = I[x_j, y_j]$ for all j . By Lemma 2.13 the vertices in I_j have representatives that are positive words in $\{\text{Ch}_\epsilon[x_j], \text{Ch}_\epsilon[y_j]\}$. This property will be essential in our proof of Theorem I.

3 Representations of F_2 into $\text{PSL}(2, \mathbb{C})$

We discuss some classical facts on irreducible actions of F_2 on the hyperbolic space \mathbb{H}^3 . Mostly following the book [Fen89] of Fenchel, we emphasize the use of half-turns and the Coxeter extension. We also introduce the notion of amplitude of a right-angled hexagon.

3.1 Isometries of \mathbb{H}^3 and half-truns

We recall some of the basic definitions and classical results on the orientation-preserving isometries of the 3-dimensional hyperbolic space $(\mathbb{H}^3, d_{\mathbb{H}})$. We mainly refer to the book by Fenchel [Fen89].

As a model of $(\mathbb{H}^3, d_{\mathbb{H}})$ we use the upper half-space model $\mathbb{C} \times \mathbb{R}^+$. The group $\mathbf{Isom}^+(\mathbb{H}^3)$ of orientation preserving isometries of \mathbb{H}^3 can be identified, via the Poincaré extension, with the group $\mathbf{PSL}(2, \mathbb{C}) = \mathbf{SL}(2, \mathbb{C}) / \{\pm \text{Id}\}$ of projective automorphisms of $\mathbb{P}^1(\mathbb{C})$. See [Fen89, p.26] for more details. For a projective class $X \in \mathbf{PSL}(2, \mathbb{C})$ we write \tilde{X} to denote one of its lifts in $\mathbf{SL}(2, \mathbb{C})$.

Under the above identification, non-trivial isometries $X \in \mathbf{PSL}(2, \mathbb{C})$ of \mathbb{H}^3 are classified by the trace of their matrix representatives $\tilde{X} \in \mathbf{SL}(2, \mathbb{C})$: X is *elliptic* if $\text{tr} \tilde{X} \in (-2, 2)$, *parabolic* if $\text{tr} \tilde{X} \in \{-2, 2\}$, and *loxodromic* if $\text{tr} \tilde{X} \notin [-2, 2]$. See [Fen89, p.46]. If X is elliptic or loxodromic, then it preserves a geodesic line Axis_X in \mathbb{H}^3 , called the *axis* of X .

An elliptic isometry $P \in \mathbf{PSL}(2, \mathbb{C})$ is involutory, that is, of order two if and only if $\text{tr} \tilde{P} = 0$ if and only if $\tilde{P}^2 = -\text{Id}$; such an isometry will be called a *half-turn*. It is not hard to see that every orientation-preserving isometry X can be expressed

$$X = PQ$$

as a composition of two half-turns P and Q , although such a decomposition is never unique. See again [Fen89, pp.46-7].

Below we shall investigate this decomposition using matrices in $\mathbf{SL}(2, \mathbb{C})$. For our purpose we focus only on the case when the isometry X is elliptic or loxodromic. The *complex translation length* of X

$$\lambda_{\mathbb{H}}(X) = \ell_{\mathbb{H}}(X) + i\theta_{\mathbb{H}}(X) \in \mathbb{A} = \mathbb{C}/2\pi i\mathbb{Z}$$

is the complex number such that its real part $\ell_{\mathbb{H}}(X) \geq 0$ is the (real) *translation length* along the axis and its imaginary part $\theta_{\mathbb{H}}(X) \in \mathbb{R}/2\pi\mathbb{Z} \cong (-\pi, \pi]$ is the *rotation angle* measured according to the right-hand rule.

A half-turn $P \in \mathbf{PSL}(2, \mathbb{C})$ determines a geodesic Axis_P in \mathbb{H}^3 . The two lifts $\pm \tilde{P} \in \mathbf{SL}(2, \mathbb{C})$ of P have zero trace. An important fact is that we can consistently associate each of these matrices with one of the orientations of the geodesic Axis_P . See [Fen89, p.64, (1)-(3)].

3.1 Convention. In order to emphasize this fact we shall deliberately abuse notation: a traceless matrix $\tilde{P} \in \mathbf{SL}(2, \mathbb{C})$ will also denote the corresponding *oriented* geodesic as well.

Now let $X = PQ$ be a decomposition of an elliptic or loxodromic isometry X into two half-turns P and Q . Then the axes of P and Q intersect the axis of X orthogonally. Take arbitrary lifts $\tilde{P}, \tilde{Q} \in \mathbf{SL}(2, \mathbb{C})$ and orient Axis_X arbitrarily. Given such a triple $(\tilde{P}, \tilde{Q}; \text{Axis}_X)$ of oriented geodesics, Fenchel [Fen89, p.67] defines the *width* from \tilde{Q} to \tilde{P} along Axis_X

$$\eta(\tilde{Q}, \tilde{P}) = \eta(\tilde{Q}, \tilde{P}; \text{Axis}_X) := \pm \lambda_{\mathbb{H}}(H) \in \mathbb{C}/2\pi\mathbb{Z}$$

as the *signed* complex translation length of H , where H is the isometry that has axis Axis_X and maps \tilde{Q} to \tilde{P} , and the sign is provided in accordance with the orientation of Axis_X . (We shall always use the simpler notation $\eta(\tilde{Q}, \tilde{P})$ assuming implicitly that the orientation of Axis_X is understood.) See Figure 3.1.1. Let $\tilde{H} \in \mathbf{SL}(2, \mathbb{C})$ be an arbitrary lift of H . Then

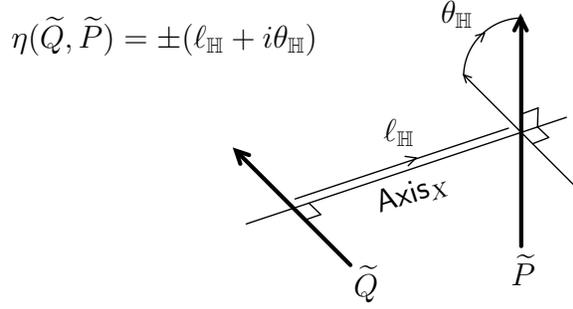


Figure 3.1.1: The width $\eta(\tilde{Q}, \tilde{P})$.

Fenchel [Fen89, p.68] shows that

$$X = H^2 = PQ, \quad \tilde{H}^2 = -\tilde{P}\tilde{Q}, \quad \frac{1}{2}\text{tr}\tilde{H}^2 = \cosh \eta(\tilde{Q}, \tilde{P}).$$

Let us set $\tilde{X} = \tilde{H}^2$ and assume that Axis_X is oriented naturally in the translation direction of X . Then $\eta(\tilde{Q}, \tilde{P})$ is one of the two *half* translation lengths of X , that is, $\lambda_{\mathbb{H}}(X)/2 = \eta(\tilde{Q}, \tilde{P})$ or $\eta(\tilde{Q}, \tilde{P}) + \pi i$, and we have

$$(3.2) \quad \frac{1}{2}\text{tr}\tilde{X} = \cosh \eta(\tilde{Q}, \tilde{P}) = \pm \cosh \frac{\lambda_{\mathbb{H}}(X)}{2}.$$

For a complex number $\eta = \ell + i\theta$ ($\ell \geq 0$) we have $\cosh \eta = \cosh \ell \cos \theta + i \sinh \ell \sin \theta$. So we see that $|\cosh \eta|^2 = \cosh^2 \ell - \sin^2 \theta = \sinh^2 \ell + \cos^2 \theta$, in particular, $\sinh \ell \leq |\cosh \eta| \leq \cosh \ell$. Thus from the above formula for $\text{tr}\tilde{X}$ we obtain

$$(3.3) \quad e^{\frac{1}{2}\ell_{\mathbb{H}}(X)} - 1 \leq |\text{tr}\tilde{X}| \leq e^{\frac{1}{2}\ell_{\mathbb{H}}(X)} + 1 \leq e^{\frac{1}{2}\ell_{\mathbb{H}}(X)+1}$$

For a number $h > 0$ its *parallel angle* $\alpha(h) \in (0, \pi/2)$ is defined by the formula

$$(3.4) \quad \sin \alpha(h) \cosh h = 1.$$

See Fenchel [Fen89, p.92]. A geometric meaning of $\alpha(\ell)$ is given in Figure 3.1.2(Left).

Suppose a (totally geodesic) plane \mathcal{H} intersects the axis of a loxodromic isometry Y transversely. The following lemma says that if the (real) translation length of Y is large enough then, regardless of the rotation angle of Y , the plane \mathcal{H} and its image $Y(\mathcal{H})$ are *ultra-parallel*, that is, their closures in $\overline{\mathbb{H}^3}$ are disjoint.

3.5 Lemma. *Let Y be a loxodromic isometry and $\mathcal{H} \subset \mathbb{H}^3$ a hyperplane intersecting Axis_Y with angle $\alpha \in (0, \pi/2]$. If the angle α is greater than the parallel angle $\alpha(\ell_{\mathbb{H}}(Y)/2)$ of the half (real) translation length of Y , then the hyperplane \mathcal{H} and the image $Y(\mathcal{H})$ are ultra-parallel.*

Proof. See Figure 3.1.2(Right). There we draw the (worst possible) case where the rotation angle $\theta_{\mathbb{H}}(Y)$ of Y is π . Since the proof follows immediately from the planar case, we omit the details. \square

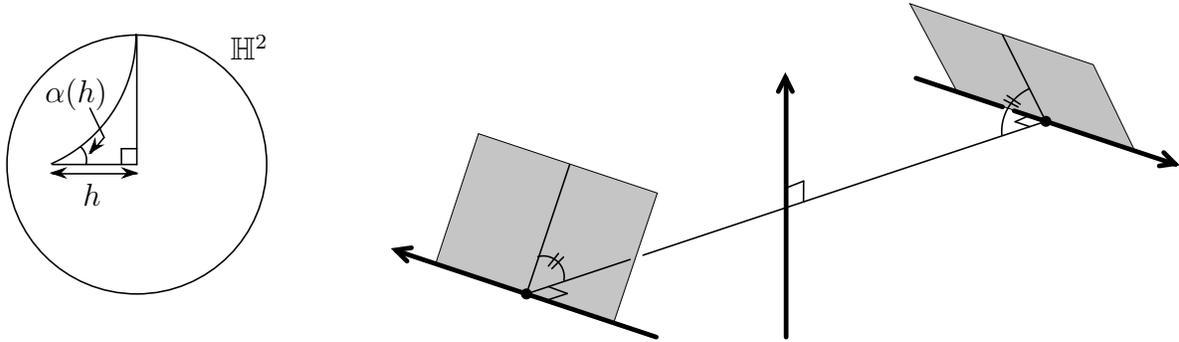


Figure 3.1.2: (Left) The parallel angle $\alpha(h)$ of h . (Right) Proof of Lemma 3.5

3.2 Irreducible representations and their Coxeter extensions

A homomorphism $\rho : F(a, b) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ represents an isometric action of $F(a, b)$ on \mathbb{H}^3 . Let (x, y) be another basis of F_2 . For simplicity we shall write

$$X = \rho(x), \quad Y = \rho(y), \quad Z = \rho(xy)^{-1}.$$

The trace of the commutator of X and Y is well-defined without ambiguity of sign for arbitrary lifts \tilde{X} and \tilde{Y} . It is also independent of the choice of basis (x, y) by Theorem 2.2(b), since $\mathrm{tr}A = \mathrm{tr}A^{-1}$ for $A \in \mathrm{SL}(2, \mathbb{C})$. We denote this number by

$$\kappa_\rho := \mathrm{tr}(XYX^{-1}Y^{-1}).$$

By definition, the representation ρ is *irreducible* if its action on $\mathbb{P}^1(\mathbb{C})$ has no fixed point. It is known that this is the case if and only if $\kappa_\rho \neq 2$. See, for example, [Gol09, Proposition 2.3.1].

Let $W(p, q, r) = \langle p, q, r \mid p^2 = q^2 = r^2 = 1 \rangle = \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ be the free (or universal) Coxeter group of rank three. The elements pq and qr freely generate a subgroup $\langle pq, qr \rangle$ of index two. (The reader may recall the discussion in the introduction and Figure 1.0.1.)

Given a basis $\mathbf{f} = (x, y)$ of F_2 , let

$$\psi_{\mathbf{f}} : F(x, y) \rightarrow W(p, q, r)$$

denote the embedding defined by $\psi_{\mathbf{f}}(x) = pq$ and $\psi_{\mathbf{f}}(y) = qr$. We say the group $W(p, q, r)$, together with $\psi_{\mathbf{f}}$, is the *Coxeter extension* of F_2 associated with the basis $\mathbf{f} = (x, y)$.

3.6 Remark. The lack of notational 3-fold symmetry in the definition of $\psi_{\mathbf{f}}$ is intentional and this way we emphasize the 2-fold symmetry of the Farey triangulation. In Section 6.3, however, we will define $\psi_{\mathbf{f}}$ differently in order to respect a 3-fold symmetry.

For irreducible representations of F_2 into $\mathrm{PSL}(2, \mathbb{C})$ we have the following classical theorem. For a proof see, for example, [Jør03, p.185] or [Fen89, II, p.94]. See also [Gol09, Theorem B and Theorem 3.2.2] for a more comprehensive account.

3.7 Theorem (Coxeter extension). *Let $\mathbf{f} = (x, y)$ be a basis of F_2 . If $\rho : F(x, y) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is an irreducible representation, then there exists a unique representation $\rho_{\mathbf{f}} : \mathbf{W}(p, q, r) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ such that $\rho = \rho_{\mathbf{f}} \circ \psi_{\mathbf{f}}$.*

We also call $\rho_{\mathbf{f}}$ the *Coxeter extension* of ρ associated with the basis $\mathbf{f} = (x, y)$. We simplify the notations by setting

$$(3.8) \quad P = \rho_{\mathbf{f}}(p), \quad Q = \rho_{\mathbf{f}}(q), \quad R = \rho_{\mathbf{f}}(r).$$

We then have $X = PQ$, $Y = QR$ and $Z = RP$.

Henceforth, we focus on the case of our most interest, namely, when X , Y and Z are all loxodromic. Then irreducibility of ρ implies that the half-turns P , Q and R are uniquely defined by the following properties. The axis Axis_Q is the common perpendicular of Axis_X and Axis_Y . Likewise, the axis Axis_P (resp. Axis_R) is the common perpendicular of Axis_Z and Axis_X (resp. Axis_Y). We denote by

$$\mathrm{Hex}(\rho, \mathbf{f}) := (\mathrm{Axis}_R, \mathrm{Axis}_Y, \mathrm{Axis}_Q, \mathrm{Axis}_X, \mathrm{Axis}_P, \mathrm{Axis}_Z)$$

the cyclically ordered sextuple of the geodesics, and call it the *right-angled hexagon* associated with ρ and $\mathbf{f} = (x, y)$.

3.3 Right-angled hexagons and their amplitudes

We continue with the previous discussion, so assume $\rho : F(x, y) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is irreducible and X , Y and Z are all loxodromic. Henceforth, we further assume that the geodesics Axis_X , Axis_Y , and Axis_Z are oriented naturally in accordance with the transformation directions of X , Y and Z .

Given the right-angled hexagon $\mathrm{Hex}(\rho, \mathbf{f})$ we will be mostly interested in the angular part of the width $\eta(\mathrm{Axis}_Y, \mathrm{Axis}_X)$ between Axis_X and Axis_Y . For the purpose of computation it will be convenient if we lift ρ to $\mathrm{SL}(2, \mathbb{C})$ and use the trace identities for matrices in $\mathrm{SL}(2, \mathbb{C})$.

More precisely, we take arbitrary lifts $\tilde{P}, \tilde{Q}, \tilde{R} \in \mathrm{SL}(2, \mathbb{C})$ of P, Q, R , respectively. This gives rise to a lift $\tilde{\rho} : F(x, y) \rightarrow \mathrm{SL}(2, \mathbb{C})$ of ρ defined by

$$\tilde{\rho}(x) = \tilde{X} = -\tilde{P}\tilde{Q}, \quad \tilde{\rho}(y) = \tilde{Y} = -\tilde{Q}\tilde{R}$$

In this case we have $\tilde{Z} = \tilde{Y}^{-1}\tilde{X}^{-1} = (-\tilde{R}\tilde{Q})(-\tilde{Q}\tilde{P}) = -\tilde{R}\tilde{P}$. Conversely, a lift $\tilde{\rho} : F(x, y) \rightarrow \mathrm{SL}(2, \mathbb{C})$ of ρ determines the lifts $\tilde{P}, \tilde{Q}, \tilde{R} \in \mathrm{SL}(2, \mathbb{C})$ uniquely (up to reverting their signs simultaneously).

The right-angled hexagon $\mathrm{Hex}(\rho, \mathbf{f})$ is now oriented (in the sense of Fenchel [Fen89, p.79]). The cyclically ordered sextuple of the oriented geodesics will be denoted by

$$\mathrm{Hex}(\tilde{\rho}, \mathbf{f}) := (\tilde{R}, \mathrm{Axis}_Y, \tilde{Q}, \mathrm{Axis}_X, \tilde{P}, \mathrm{Axis}_Z).$$

See Figure 3.3.1.

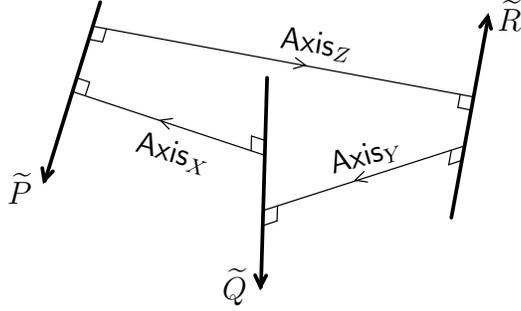


Figure 3.3.1: The right-angled hexagon $\text{Hex}(\tilde{\rho}, \mathbf{f})$.

For brevity let us set

$$\eta_X = \eta(\tilde{Q}, \tilde{P}), \quad \eta_Y = \eta(\tilde{R}, \tilde{Q}), \quad \eta_Z = \eta(\tilde{P}, \tilde{R}), \quad \eta_Q = \eta(\text{Axis}_Y, \text{Axis}_X).$$

In fact, the width η_Q is determined by $\tilde{\rho}$ only up to sign, since we may change the signs of $\tilde{P}, \tilde{Q}, \tilde{R}$ simultaneously and obtain the same lift $\tilde{\rho}$. However, if we set

$$(3.9) \quad \theta(\rho, \mathbf{f}) = |\text{Im } \eta_Q| \in [0, \pi]$$

this number is independent of the choice of the lift $\tilde{\rho}$. We call it the *angle* associated to ρ and \mathbf{f} .

We shall use the following form of the law of Cosines for $\text{Hex}(\tilde{\rho}, \mathbf{f})$:

$$\cosh \eta_Z = \cosh \eta_X \cosh \eta_Y + \sinh \eta_X \sinh \eta_Y \cosh \eta_Q.$$

See [Fen89, p.83]. Since we have from (3.2)

$$\begin{aligned} \text{tr } \tilde{X} &= -\text{tr } \tilde{P}\tilde{Q} = 2 \cosh \eta_X \\ \text{tr } \tilde{Y} &= -\text{tr } \tilde{Q}\tilde{R} = 2 \cosh \eta_Y \\ \text{tr } \tilde{Z} &= -\text{tr } \tilde{R}\tilde{P} = 2 \cosh \eta_Z \end{aligned}$$

the law of Cosines can be written as

$$(3.10) \quad \frac{2 \text{tr } \tilde{Z}}{\text{tr } \tilde{X} \text{tr } \tilde{Y}} = \frac{\cosh \eta_Z}{\cosh \eta_X \cosh \eta_Y} = 1 + \tanh \eta_X \tanh \eta_Y \cosh \eta_Q,$$

which we shall need later. Note that

$$(3.11) \quad \text{Re}(\tanh \eta_X) > 0 \text{ if and only if } \text{Re}(\eta_X) > 0,$$

the latter of which is the case as we assumed X is loxodromic and Axis_X is oriented naturally.

Lastly, to the oriented right-angled hexagon $\text{Hex}(\tilde{\rho}, \mathbf{f})$, Fenchel [Fen89, p.102] assigns a complex number, called its *amplitude*, defined by

$$\text{am}(\tilde{\rho}, \mathbf{f}) = -\frac{1}{2} \text{tr}(\tilde{P}\tilde{Q}\tilde{R}).$$

He then draws many formulas for the amplitude in geometric terms of the right-angled hexagon $\text{Hex}(\tilde{\rho}, \mathbf{f})$. One of the formulas we shall need later is the following [Fen89, p.103, (3)]:

$$(3.12) \quad \text{am}(\tilde{\rho}, \mathbf{f}) = -i \sinh \eta_X \sinh \eta_Y \sinh \eta_Q.$$

The amplitude is closely related to the commutator trace κ_ρ as follows:

$$(3.13) \quad \begin{aligned} \tilde{X}\tilde{Y}^{-1}\tilde{X}^{-1}\tilde{Y} &= \tilde{P}\tilde{Q}\tilde{R}(\tilde{Q}\tilde{Q})\tilde{P}\tilde{Q}\tilde{R} = -(\tilde{P}\tilde{Q}\tilde{R})^2, \\ \kappa_\rho = \text{tr}\tilde{X}\tilde{Y}^{-1}\tilde{X}^{-1}\tilde{Y} &= -\text{tr}(\tilde{P}\tilde{Q}\tilde{R})^2 = 2 - \text{tr}^2(\tilde{P}\tilde{Q}\tilde{R}) = 2 - 4\text{am}^2(\tilde{\rho}, \mathbf{f}). \end{aligned}$$

In particular, since κ_ρ is a constant independent of the basis $\mathbf{f} = (x, y)$, so is the modulus $|\text{am}(\tilde{\rho}, \mathbf{f})|$.

4 Primitive stability and the Q-conditions

We define two open conditions on isometric actions $\rho : \mathbb{F}_2 \rightarrow \text{PSL}(2, \mathbb{C}) \cong \text{Isom}^+(\mathbb{H}^3)$ of \mathbb{F}_2 on $(\mathbb{H}^3, d_{\mathbb{H}})$: Minsky's primitive stability (Section 4.2) and Bowditch's Q-conditions (Section 4.3). In Section 4.4 we show that the former implies the latter.

4.1 The isometric action of \mathbb{F}_2 on itself

Recall that we fixed a basis $\mathbf{e} = (a, b)$ of \mathbb{F}_2 . As before, we write $|w|_{\mathbf{e}} = |w|_{(a,b)}$ to denote the *word length* of $w \in \mathbb{F}_2$ with respect to the basis $\mathbf{e} = (a, b)$.

The *word metric* $d_{\mathbf{e}}$ on \mathbb{F}_2 is defined by $d_{\mathbf{e}}(u, v) = |u^{-1}v|_{\mathbf{e}}$ for $u, v \in \mathbb{F}_2$. Then \mathbb{F}_2 acts freely and isometrically on the metric space $(\mathbb{F}_2, d_{\mathbf{e}})$ by left multiplication. For an element $w \in \mathbb{F}_2$ its *translation length* $\ell_{\mathbf{e}}(w)$ is defined by

$$(4.1) \quad \ell_{\mathbf{e}}(w) := \inf_{u \in \mathbb{F}_2} d_{\mathbf{e}}(u, wu) \left(= \inf_{u \in \mathbb{F}_2} |u^{-1}wu|_{\mathbf{e}} = |r|_{\mathbf{e}} =: \|w\|_{\mathbf{e}} \right).$$

Algebraically, this can be interpreted as the *cyclically reduced length* $\|w\|_{\mathbf{e}}$ of w , that is, the word length $|r|_{\mathbf{e}}$ of a cyclically reduced representative r of the conjugacy class of w . Thus the translation length $\ell_{\mathbf{e}}(w)$ depends only on the \sim -equivalence class of w .

Note that if $w \in \mathbb{F}_2$ is not the identity then the action of w on $(\mathbb{F}_2, d_{\mathbf{e}})$ is *axial*: there exists a (discrete) geodesic $\mathbb{Z} \rightarrow (\mathbb{F}_2, d_{\mathbf{e}})$ which is invariant under the action of w and is translated non-trivially through the amount $\ell_{\mathbf{e}}(w)$. We denote the image of this geodesic by $\text{Axis}^{\mathbf{e}}(w)$ (or simply $\text{Axis}(w)$ when \mathbf{e} is understood) and call it the *axis* of w . Note that if $w = uru^{-1}$ then $\text{Axis}(w) = u\text{Axis}(r)$. If r is cyclically reduced, then $\text{Axis}(r)$ contains the cyclic orbit $\{r^i \mid i \in \mathbb{Z}\}$ of the identity; in particular, it passes through the identity.

Similarly, for any basis $\mathbf{f} = (x, y)$ different from \mathbf{e} , we denote by $\text{Axis}^{\mathbf{f}}(w) \subset (\mathbb{F}_2, d_{\mathbf{f}})$ the axis of a non-trivial element w with respect to \mathbf{f} .

4.2 Lemma. *Let $\mathbf{f} = (x, y)$ be an \mathbf{e} -Christoffel basis of \mathbf{F}_2 with $\text{Lv}_{\mathbf{e}}[x, y] > 0$. If w is a positive word in $\{x, y\}$, then the axis $\text{Axis}^{\mathbf{f}}(w)$ is of the form $\{g_i \mid i \in \mathbb{Z}\}$, where $g_0 = 1$ and $g_i^{-1}g_{i+1} \in \{x, y\}$ for all $i \in \mathbb{Z}$, and is contained in the axis $\text{Axis}(w) \subset (\mathbf{F}_2, d_{\mathbf{e}})$.*

Proof. If w is positive in $\{x, y\}$ then it is of the form $w = s_1s_2 \dots s_n$ for some $n > 0$ where each $s_i \in \{x, y\}$. Since $\mathbf{f} = (x, y)$ is an \mathbf{e} -Christoffel basis, x and y are \mathbf{e} -Christoffel words and they are both positive in $\{a, b\}$ or both positive in $\{a^{-1}, b\}$. (Recall the discussion after Definition 2.8.) Therefore, in either case, $w = s_1s_2 \dots s_n$ is linearly and cyclically reduced as a word in $\{a, b\}$. The lemma now follows. \square

4.2 Minsky's primitive stability

Minsky [Min13] introduced the notation of primitive stability and studied its basic properties. Although the definition makes sense for the free group \mathbf{F}_n of arbitrary rank $n \geq 2$, we restrict to the case $n = 2$ for simplicity.

Recall that we endowed \mathbf{F}_2 with a metric $d_{\mathbf{e}}$, where $\mathbf{e} = (a, b)$ is the distinguished basis. Suppose a representation $\rho : \mathbf{F}_2 \rightarrow \text{PSL}(2, \mathbb{C})$ is given. Upon choosing a base point $o \in \mathbb{H}^3$ the orbit map $\tau_{\rho, o} : \mathbf{F}_2 \rightarrow \mathbb{H}^3$ is defined by $\tau_{\rho, o}(w) = \rho(w)(o)$. Obviously, it is ρ -equivariant: $\tau_{\rho, o}(uw) = \rho(uw)(o) = \rho(u)(\tau_{\rho, o}(w))$ for $u, w \in \mathbf{F}_2$.

4.3 Definition. A representation $\rho : \mathbf{F}_2 \rightarrow \text{PSL}(2, \mathbb{C})$ is said to be *primitive stable* if there exist a point $o \in \mathbb{H}^3$ and constants $M, c > 0$ such that the $\tau_{\rho, o}$ -images of the axes $\text{Axis}(x) \subset (\mathbf{F}_2, d_{\mathbf{e}})$ of primitive elements x of \mathbf{F}_2 are (M, c) -quasi-geodesics in \mathbb{H}^3 , that is,

$$\frac{1}{M}d_{\mathbf{e}}(u, v) - c \leq d_{\mathbb{H}}(\rho(u)(o), \rho(v)(o)) \leq Md_{\mathbf{e}}(u, v) + c$$

for all $u, v \in \text{Axis}(x)$.

See [Min13, Definition 3.1]. Note that the property is preserved under the $\text{PSL}(2, \mathbb{C})$ -conjugacy action, so we can consider this definition in the character variety $\mathcal{X}(\mathbf{F}_2)$. As in the introduction we denote by $\mathcal{PS} \subset \mathcal{X}(\mathbf{F}_2)$ the subset consisting of all $\text{PSL}(2, \mathbb{C})$ -conjugacy classes of primitive stable representations.

The definition of primitive stability can be simplified a little. First of all, from the equivariance of $\tau_{\rho, o}$ and the triangle inequality in \mathbb{H}^3 , one can deduce that the constants (M, c) satisfying the upper inequality above exist for *any* isometric action ρ and for any choice of base point $o \in \mathbb{H}^3$. See [BH99, Lemma I.8.18] for example. Thus the lower inequality is the only essential requirement for primitive stability.

Furthermore, the requirement for each primitive element is rather for each (unoriented) primitive conjugacy class. For if $x = uru^{-1}$ then $\text{Axis}(x) = u\text{Axis}(r)$ and hence $\tau_{\rho, o}(\text{Axis}(x)) = \rho(u)\tau_{\rho, o}(\text{Axis}(r))$ by the equivariance of $\tau_{\rho, o}$. So it suffices to test the inequality only for cyclically reduced representatives.

To summarize, we can simplify the above definition as follows. Henceforth we shall always use this alternative definition.

4.4 Definition (Alternative). A representation $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is said to be *primitive stable* if there exist a point $o \in \mathbb{H}^3$ and constants $m, c > 0$ satisfying the following condition: for each unoriented primitive class $[x] \in \mathbf{V}$, there is a cyclically reduced representative x such that

$$m \cdot d_{\mathbb{e}}(u, v) - c \leq d_{\mathbb{H}}(\rho(u)(o), \rho(v)(o))$$

for all $u, v \in \mathrm{Axis}(x)$.

4.3 Bowditch's Q-conditions

Note that, for an element $X \in \mathrm{PSL}(2, \mathbb{C})$ and its two lifts $\pm \tilde{X} \in \mathrm{SL}(2, \mathbb{C})$, we have $\mathrm{tr}(-\tilde{X}) = -\mathrm{tr}\tilde{X}$ and $\mathrm{tr}\tilde{X} = \mathrm{tr}\tilde{X}^{-1}$. So, given a representation $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$, the complex modulus $|\mathrm{tr}(\rho(x))|$ is a well-defined function on the set $\mathbf{V} = \mathrm{Prim}/\sim$ of unoriented primitive classes of F_2 .

4.5 Definition. A representation $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is said to satisfy the *Q-conditions* if

- (i) $\rho(x)$ is loxodromic for all $[x] \in \mathbf{V}$, and
- (ii) $|\mathrm{tr}\rho(x)| \leq 2$ for only finitely many $[x] \in \mathbf{V}$.

See [Bow98, p.702] and [TWZ08a, p.765]. The conditions (i) and (ii) will be referred to as the first and second Q-condition, respectively. Since the trace function is invariant under conjugacy, this definition also makes sense in the character variety $\mathcal{X}(F_2)$. We denote by $\mathcal{BQ} \subset \mathcal{X}(F_2)$ the subset consisting of all $\mathrm{PSL}(2, \mathbb{C})$ -conjugacy classes of representations satisfying the Q-conditions.

One of the main results of Bowditch [Bow98, Theorem 2 and Proposition 4.9] and Tan, Wong and Zhang [TWZ08a, Theorem 3.3] is the following characterization of the second Q-condition. We rephrased their theorem using the equality (2.9).

4.6 Theorem. *Suppose a representation $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ satisfies the first Q-condition. Then ρ satisfies the Q-conditions if and only if there exist uniform positive constants $m = m(\rho)$ and $c = c(\rho)$ such that*

$$\log |\mathrm{tr}\rho(x)| \geq m \cdot \ell_{\mathbb{e}}(x) - c$$

for every $[x] \in \mathbf{V}$.

As a consequence we see that, for a representation ρ satisfying the Q-conditions, the set $\{\pm \mathrm{tr}\rho(x) \mid [x] \in \mathbf{V}\}$ is a discrete subset of \mathbb{C} .

In the course of proving the above theorem the authors also established the following:

4.7 Proposition. *If a representation $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ satisfies the Q-conditions, then there exists a uniform constant $N_{\rho} > 0$ with the following property: if a Farey edge $[x, y]$ has $\mathrm{Lv}[x, y] \geq N_{\rho}$ and determines the quadrilateral $[w; x, y; z]$ then*

$$|\mathrm{tr}\rho(w)| \leq |\mathrm{tr}\rho(z)|.$$

Consider the level- N_ρ partition (see Definition 2.15). Very roughly speaking, then the proposition implies that, for a representation ρ satisfying the Q-conditions, if we start with a Farey triangle and run the process of minimizing the complex modulus of vertex traces then the process always ends up with only a finitely many Farey triangles that are of level $< N_\rho$. Compare with the $\mathrm{PSL}(2, \mathbb{R})$ case in the introduction: if the axes of two hyperbolic translations intersect, then the trace minimizing process always stops at an acute (or right-angled) triangle.

Let (x, y) be an acute basis of F_2 with respect to $\mathbf{e} = (a, b)$ so that the Farey edge $[x, y]$ determines the quadrilateral $[x^{-1}y; x, y; xy]$. (See Definition 2.12.) For an arbitrary lift $\tilde{\rho}$ to $\mathrm{SL}(2, \mathbb{C})$ one can deduce the trace identity

$$\mathrm{tr}\tilde{\rho}(x^{-1}y) + \mathrm{tr}\tilde{\rho}(xy) = \mathrm{tr}\tilde{\rho}(x)\mathrm{tr}\tilde{\rho}(y)$$

from the Cayley-Hamilton theorem. (See, for example, [Gol09, Theorem A and Section 2.2] for a proof.) Then the inequality in Proposition 4.7 is easily seen to be equivalent to

$$(4.8) \quad \mathrm{Re} \left(\frac{\mathrm{tr}\tilde{\rho}(xy)}{\mathrm{tr}\tilde{\rho}(x)\mathrm{tr}\tilde{\rho}(y)} \right) \geq \frac{1}{2}.$$

4.4 Primitive stability implies the Q-conditions

Although the converse to our main theorem is well-known, we provide a proof for the sake of completeness. Our proof is essentially the same as the proof in [DGLM11, Proposition 4.0.5].

4.9 Proposition. *If $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is primitive stable, then it satisfies the Q-conditions.*

Proof. Suppose ρ is primitive stable. By Definition 4.4 (with $v = 1 \in F_2$) there exist a point $o \in \mathbb{H}^3$ and positive constants (m, c) such that, for each unoriented primitive class $[x] \in \mathbf{V}$, there is a cyclically reduced representative x such that

$$m|u|_{\mathbf{e}} - c \leq d_{\mathbb{H}}(o, \rho(u)(o))$$

for all $u \in \mathrm{Axis}(x)$. In particular, the ρ -orbit $\{\rho(x^n)(o) \mid n \in \mathbb{Z}\}$ of the point o under the cyclic subgroup $\langle x \rangle$ generated by x is a quasi-geodesic. This is the case only if $\rho(x)$ is loxodromic. Therefore, ρ satisfies the first Q-condition.

Moreover, setting $u = x^n \in \mathrm{Axis}(x)$ in the above inequality, we have

$$m|x^n|_{\mathbf{e}} - c \leq d_{\mathbb{H}}(o, \rho(x^n)(o))$$

for any $n \in \mathbb{N}$. Therefore, we see that

$$\begin{aligned} \ell_{\mathbb{H}}(\rho(x)) &\geq \lim_{n \rightarrow \infty} \frac{1}{n} d_{\mathbb{H}}(o, \rho(x^n)(o)) \\ &\geq \lim_{n \rightarrow \infty} \frac{1}{n} (m|x^n|_{\mathbf{e}} - c) \\ &= m|x|_{\mathbf{e}} \\ &= m \cdot \ell_{\mathbf{e}}(x), \end{aligned}$$

where the first inequality (which is actually equality in the present case) follows from the triangle inequality and the fact that the limit on the right-hand side is independent of the point o . See, for example, [CDP90, Section 10.6] or [BH99, Exercise II.6.6]. Since $|\mathrm{tr}\rho(x)| \geq e^{\frac{1}{2}\ell_{\mathbb{H}}(\rho(x))} - 1$ by (3.3), we conclude that

$$|\mathrm{tr}\rho(x)| \geq e^{\frac{m}{2}\ell_e(x)} - 1 > 2$$

for almost all classes $[x] \in \mathbb{V}$, and thus ρ satisfies the second Q-condition. \square

4.10 Remark. Since \mathbb{H}^3 is Gromov δ -hyperbolic, we may use the Morse lemma (stability of quasi-geodesics) to give an alternative proof of the above proposition as in [Lup15, Proposition 2.9]. See also [Can15, Proposition 2.1]. The above proof is stronger in that it requires no assumption on the metric space on which F_2 acts isometrically.

5 The Q-conditions imply primitive stability

In this section we prove Theorem I. We first establish Lemma 5.1 as a consequence of Theorem 4.6 and Proposition 4.7. This lemma is crucial in that it enables us to consider the level- N partition of \mathbb{V} (Definition 2.15) for some $N > 0$ and to focus only one interval I_j thereof. This is done in Theorem 5.4, which is in fact slightly more general than what is needed for the proof of Theorem I, in the sense that, for such an interval $I_j = I(x, y)$, it deals not only with the primitive classes $[w] \in I(x, y)$ but with *all* positive words in x and y .

The proof of Theorem I starts in Section 5.3.

5.1 Lemma on the angle $\theta(\rho, \mathbf{f})$

Suppose $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ satisfies the Q-conditions. It is known [TWZ08b, Theorem 1.4] that ρ is irreducible. Since ρ is irreducible and satisfies the first Q-condition, we can follow the discussion in Section 3.3. In particular, the angle $\theta(\rho, \mathbf{f})$ is defined for any basis $\mathbf{f} = (x, y)$ of F_2 . See (3.9).

The following lemma will be crucial in our proof of Theorem I. The same statement also appears in [TX18, Proposition 4.12] with a different proof. Recall Definition 2.12 for acute bases.

5.1 Lemma. *Suppose $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ satisfies the Q-conditions. Let $\mathbf{f} = (x, y)$ be a basis of F_2 that is acute relative to the basis $\mathbf{e} = (a, b)$. Then the angle $\theta(\rho, \mathbf{f})$ converges to $0 \in [0, \pi]$ as $\mathrm{Lv}_{\mathbf{e}}[x, y] \rightarrow \infty$.*

Proof. Without loss of generality, we may assume that $\mathrm{Lv}_{\mathbf{e}}[x] < \mathrm{Lv}_{\mathbf{e}}[y]$. (The other case $\mathrm{Lv}_{\mathbf{e}}[x] > \mathrm{Lv}_{\mathbf{e}}[y]$ can be dealt with in a similar fashion.) Then we have

$$\mathrm{Lv}_{\mathbf{e}}[x, y] = \mathrm{Lv}_{\mathbf{e}}[y].$$

See Figure 2.3.1(Right). Furthermore, by (2.9), (2.10) and (4.1), we have the inequality $\ell_e(y) \geq \mathrm{Lv}_{\mathbf{e}}[y] + 1$ as well.

We now proceed as in Section 3.3 and employ the same notations therein. We take a lift $\tilde{\rho}$ to $\mathrm{SL}(2, \mathbb{C})$ and obtain the oriented right-angled hexagon $\mathrm{Hex}(\tilde{\rho}, \mathbf{f})$ by choosing a set of lifts of P, Q, R accordingly. From now on all the geometric quantities associated with the hexagon $\mathrm{Hex}(\tilde{\rho}, \mathbf{f})$ will be regarded as functions of the level $\mathrm{Lv}_e[x, y]$. Then, as we observed in (3.12) and (3.13), the number

$$|\mathrm{am}(\tilde{\rho}, \mathbf{f})| = |\sinh \eta_X \sinh \eta_Y \sinh \eta_Q|$$

is constant independent of $\mathrm{Lv}_e[x, y]$.

We first note that

$$\sinh \eta_X = 0 \quad \text{iff} \quad \eta_X = 0 \text{ or } i\pi \quad \text{iff} \quad \mathrm{tr} \tilde{X} = 2 \cosh \eta_X = \pm 2.$$

Since ρ satisfies the Q-conditions, we know from Theorem 4.6 that the set $\{\mathrm{tr} \tilde{X} \mid [x] \in \mathbf{V}\}$ does not accumulate at ± 2 . Thus there exists a uniform bound $\epsilon_\rho > 0$ such that $|\sinh \eta_X| \geq \epsilon_\rho$ for all $[x] \in \mathbf{V}$. On the other hand, by Theorem 4.6 again, we have

$$(5.2) \quad \log |2 \cosh \eta_Y| = \log |\mathrm{tr} \tilde{Y}| \geq m \cdot \ell_e(y) - c \geq m(\mathrm{Lv}_e[y] + 1) - c$$

for some uniform constants $m, c > 0$. So $|\sinh \eta_Y|$, as well as $|\cosh \eta_Y|$, diverges to infinity as $\mathrm{Lv}_e[x, y] = \mathrm{Lv}_e[y]$ tends to infinity. Therefore, since $|\mathrm{am}(\tilde{\rho}, \mathbf{f})|$ is constant, $|\sinh \eta_Q|$ must converge to 0 as $\mathrm{Lv}_e[x, y] \rightarrow \infty$. This means that the possible accumulation points of the set

$$\{\cosh \eta_Q \mid \mathbf{f} \text{ is a basis of } \mathbf{F}_2\}$$

are 1 and -1 .

However, if we consider only the bases that are acute relative to \mathbf{e} then -1 cannot be an accumulation point of the corresponding subset. To see this, suppose on the contrary that there is a sequence of acute bases $\{\mathbf{f}_i = (x_i, y_i)\}_i$ such that $\cosh \eta_{Q_i}$ converges to -1 as $i \rightarrow \infty$. Now recall the equation (3.10):

$$\frac{2 \mathrm{tr} \tilde{Z}_i}{\mathrm{tr} \tilde{X}_i \mathrm{tr} \tilde{Y}_i} = 1 + \tanh \eta_{X_i} \tanh \eta_{Y_i} \cosh \eta_{Q_i},$$

where $\tilde{Z}_i = (\tilde{X}_i \tilde{Y}_i)^{-1}$. Note from (3.11) that $\mathrm{Re}(\tanh \eta_{X_i}) > 0$, and from (5.2) that $\tanh \eta_{Y_i}$ converges to 1. Thus we have $\mathrm{Re}\left(\frac{\mathrm{tr} \tilde{Z}_i}{\mathrm{tr} \tilde{X}_i \mathrm{tr} \tilde{Y}_i}\right) < \frac{1}{2}$ for all sufficiently large i . But this contradicts Proposition 4.7 and the inequality (4.8), since we would eventually have $\mathrm{Lv}[x_i, y_i] \geq N_\rho$ for large i .

Therefore, we conclude that the set

$$\{\cosh \eta_Q \mid \mathbf{f} \text{ is an acute basis of } \mathbf{F}_2 \text{ relative to } \mathbf{e}\}$$

has a unique accumulation point 1. This means that η_Q , as well as $\theta(\rho, \mathbf{f})$, converges to 0 as $\mathrm{Lv}_e[x, y] \rightarrow \infty$. \square

5.3 *Remark.* By a similar argument one can show that the widths

$$\eta(\text{Axis}_X, \text{Axis}_Z) \quad \text{and} \quad \eta(\text{Axis}_Z, \text{Axis}_Y)$$

both converge to $\pi i \in \mathbb{A} = \mathbb{C}/2\pi i\mathbb{Z}$ as $\text{Lv}_e[x, y] \rightarrow \infty$. See also [TX18, Proposition 4.12]. In this sense we may say that, as $\text{Lv}_e[\mathbf{f}] \rightarrow \infty$, the hexagon $\text{Hex}(\rho, \mathbf{f})$ asymptotically becomes a “degenerate obtuse triangle” as in the case of $\text{PSL}(2, \mathbb{R}) \cong \text{Isom}^+(\mathbb{H}^2)$ illustrated in the introduction.

5.2 Positive words in a high level Christoffel basis

The following theorem *is* our main theorem and contains the most general statement, from which Theorem I will follow almost immediately.

5.4 Theorem. *Suppose $\rho : F_2 \rightarrow \text{PSL}(2, \mathbb{C})$ satisfies the Q-conditions. Then there exists a number $N > 0$ with the following property: if $\mathbf{f} = (x, y)$ is an \mathbf{e} -Christoffel basis of F_2 with $\text{Lv}_e[x, y] \geq N$ then there exist a point $o_{\mathbf{f}} \in \mathbb{H}^3$ and constants $m_{\mathbf{f}}, c_{\mathbf{f}} > 0$ such that, for every positive word w in $\{x, y\}$, the inequality*

$$m_{\mathbf{f}} \cdot d_e(u, v) - c_{\mathbf{f}} \leq d_{\mathbb{H}}(\rho(u)(o_{\mathbf{f}}), \rho(v)(o_{\mathbf{f}}))$$

holds for all $u, v \in \text{Axis}(w)$.

Proof. Suppose ρ satisfies the Q-conditions. By Lemma 5.1 we can take $N_1 > 0$ large enough that we have $-\pi/4 < \theta(\rho, \mathbf{f}) < \pi/4$ for every acute basis \mathbf{f} with $\text{Lv}_e[\mathbf{f}] \geq N_1$.

By Theorem 4.6, there are constants $m, c > 0$ such that $\log |\text{tr}\rho(w)| \geq m \cdot \ell_e(w) - c$ for all $[w] \in \mathbf{V}$. We have from (2.9) and (2.10) that $\ell_e(w) = |\text{Ch}_e[w]|_e = \text{Fib}_e[w] \geq \text{Lv}_e[w] + 1$. Note also from (3.3) that $\log |\text{tr}\rho(w)| < \frac{1}{2}\ell_{\mathbb{H}}(\rho(w)) + 1$. Thus we see that

$$\frac{1}{2}\ell_{\mathbb{H}}(\rho(w)) > \log |\text{tr}\rho(w)| - 1 \geq m \cdot \ell_e(w) - (c + 1) \geq m(\text{Lv}_e[w] + 1) - (c + 1)$$

for all $[w] \in \mathbf{V}$. Take $N_2 > 0$ big enough so that if $\text{Lv}_e[w] \geq N_2$ then the parallel angle (3.4) of the half translation length $\frac{1}{2}\ell_{\mathbb{H}}(\rho(w))$ is less than $\pi/4$.

Set $N = \max\{N_1, N_2\}$.

Suppose $\mathbf{f} = (x, y)$ is an \mathbf{e} -Christoffel basis and $\text{Lv}_e[x, y] \geq N$. In particular, \mathbf{f} is acute relative to \mathbf{e} (see the discussion preceding Definition 2.12), thus Lemma 5.1 applies to \mathbf{f} . Without loss of generality, we may further assume $\text{Lv}_e[x] < \text{Lv}_e[y]$ so that $\text{Lv}_e[x, y] = \text{Lv}_e[y]$.

Proceeding as in Section 3.3 we take a lift $\tilde{\rho}$ to $\text{SL}(2, \mathbb{C})$ and obtain the oriented right-angled hexagon $\text{Hex}(\tilde{\rho}, \mathbf{f})$ by choosing a set of lifts of P, Q, R accordingly. Take the (totally geodesic) plane $\mathcal{H} \subset \mathbb{H}^3$ which contains the geodesic \tilde{Q} and is perpendicular to Axis_X . See Figure 5.2.1.

Since the planes \mathcal{H} and $\rho(x)(\mathcal{H})$ are both orthogonal to Axis_X , they are ultra-parallel. Since $\text{Lv}_e[y] = \text{Lv}_e[x, y] \geq N_2$, the parallel angle of $\frac{1}{2}\ell_{\mathbb{H}}(\rho(y))$ is less than $\pi/4$. Since $\text{Lv}_e[x, y] \geq N_1$, we have $-\pi/4 < \theta(\rho, \mathbf{f}) < \pi/4$. This means that the angle θ' between \mathcal{H} and

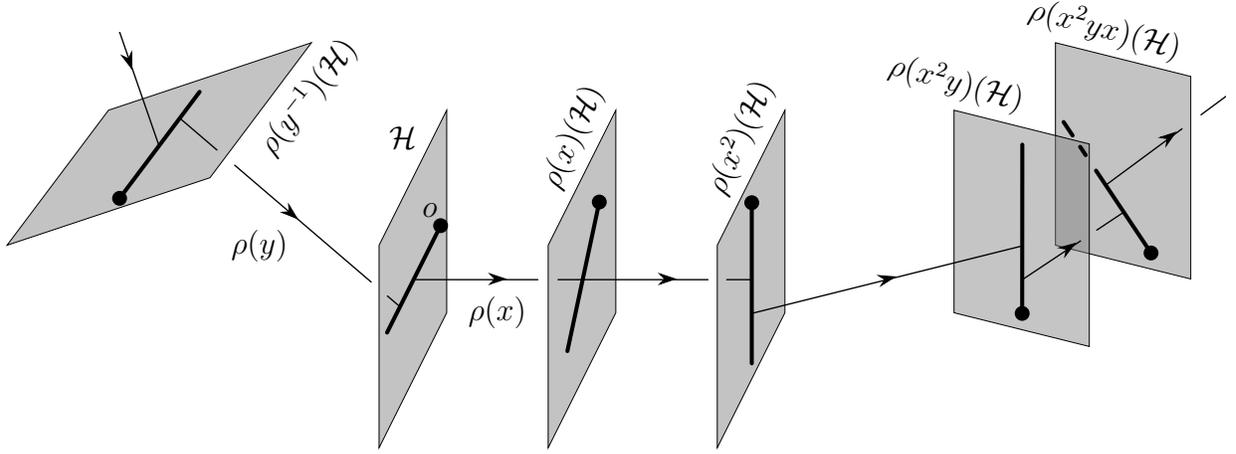


Figure 5.2.2: An example for the positive word $w = x^2 y x y$

These constants do not depend on w , but they depend only on the representation ρ , the bases $\mathbf{e} = (a, b)$ and $\mathbf{f} = (x, y)$, and the chosen point $o \in \mathcal{H}$. Let $u, v \in \text{Axis}(w)$. Then there are indices $j, k \in \mathbb{Z}$ such that u and v lie in the subintervals $[g_{j-1}, g_j]$ and $[g_k, g_{k+1}]$ of $\text{Axis}(w)$, respectively. Without loss of generality we may assume $j - 1 \leq k$. Then $u = g_{j-1}s$ and $v = g_k t$ where s and t are certain subwords either of x or of y . See Figure 5.2.3.

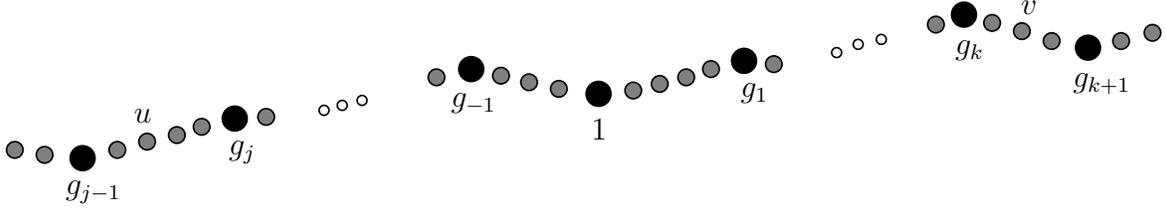


Figure 5.2.3: The discrete geodesic $\text{Axis}(w)$ in $(\mathbb{F}_2, d_{\mathbf{e}})$

From the triangle inequality, we have

$$\begin{aligned}
 & d_{\mathbb{H}}(\rho(u)(o), \rho(v)(o)) + 2dL \\
 & \geq d_{\mathbb{H}}(\rho(g_j)(o), \rho(u)(o)) + d_{\mathbb{H}}(\rho(u)(o), \rho(v)(o)) + d_{\mathbb{H}}(\rho(v)(o), \rho(g_k)(o)) \\
 & \geq d_{\mathbb{H}}(\rho(g_j)(o), \rho(g_k)(o)),
 \end{aligned}$$

and thus

$$\begin{aligned}
 d_{\mathbb{H}}(\rho(u)(o), \rho(v)(o)) & \geq d_{\mathbb{H}}(\rho(g_j)(o), \rho(g_k)(o)) - 2dL \\
 & \geq D(k - j) - 2dL \\
 & \geq \frac{D}{L}d_{\mathbf{e}}(u, v) - (2D + 2dL),
 \end{aligned}$$

since $L(k - j + 2) = d_{\mathbf{e}}(g_{j-1}, g_{k+1}) \geq d_{\mathbf{e}}(u, v)$. The proof is complete if we set $o_{\mathbf{f}} = o$, $m_{\mathbf{f}} = D/L$ and $c_{\mathbf{f}} = 2D + 2dL$. \square

5.3 Proof of Theorem I

Proof of Theorem I. Suppose ρ satisfies the Q-conditions. In order to show ρ is primitive stable, choose any base point $o \in \mathbb{H}^3$ and consider the associated orbit map $\tau_{\rho,o}$.

Take the number $N > 0$ guaranteed by Theorem 5.4 and consider the level- N partition of \mathbb{V} (Definition 2.15):

$$\mathbb{V} = \mathbb{V}_{\leq N} \cup \mathbb{I}_1 \cup \mathbb{I}_2 \cup \cdots \cup \mathbb{I}_{2^{N+1}}.$$

Recall that each interval \mathbb{I}_i is of the form $\mathbb{I}_i = \mathbb{I}[x_i, y_i]$ and $L_{\mathbb{V}_e}[x_i, y_i] = N$ for $1 \leq i \leq 2^{N+1}$.

Fix an index i ($1 \leq i \leq 2^{N+1}$) and let $[w] \in \mathbb{I}_i = \mathbb{I}[x_i, y_i]$. Then the vertex $[w]$ has a representative $\text{Ch}_e[w]$ which is an e-Christoffel word (Definition 2.8). In particular, $\text{Ch}_e[w]$ is cyclically reduced. Furthermore, the word $\text{Ch}_e[w]$ is a positive word in $\{\text{Ch}_e[x_i], \text{Ch}_e[y_i]\}$ by Lemma 2.13, and the pair $\mathbf{f}_i = (\text{Ch}_e[x_i], \text{Ch}_e[y_i])$ is an e-Christoffel basis. Therefore, by Theorem 5.4 again, there exist $o_i \in \mathbb{H}^3$ and $m_i, c_i > 0$ such that

$$m_i \cdot d_e(u, v) - c_i \leq d_{\mathbb{H}}(\rho(u)(o_i), \rho(v)(o_i)),$$

for all $u, v \in \text{Axis}(w)$. Let $r_i = d_{\mathbb{H}}(o, o_i)$. Then, by the triangle inequality, we have

$$\begin{aligned} d_{\mathbb{H}}(\rho(u)(o), \rho(v)(o)) &\geq d_{\mathbb{H}}(\rho(u)(o_i), \rho(v)(o_i)) - d_{\mathbb{H}}(\rho(u)(o_i), \rho(u)(o)) - d_{\mathbb{H}}(\rho(v)(o), \rho(v)(o_i)) \\ &\geq m_i \cdot d_e(u, v) - (c_i + 2r_i), \end{aligned}$$

for all $u, v \in \text{Axis}(w)$.

These constants m_i, c_i and r_i depend only on the interval \mathbb{I}_i . Let (m_0, c_0) be some positive constants for which the analogous inequalities of Definition 4.4 hold for the *finite* number of vertices in $\mathbb{V}_{\leq N}$ with respect to the base point o . By setting

$$\begin{aligned} m &= \min\{m_0, m_i \mid 1 \leq i \leq 2^{N+1}\} \\ c &= \max\{c_0, c_i + 2r_i \mid 1 \leq i \leq 2^{N+1}\}, \end{aligned}$$

we conclude that ρ satisfies Definition 4.4 with constants $m, c > 0$ and base point o . \square

5.4 On Lupi's proof

Lupi recently showed in his thesis [Lup15, Proposition 3.4 and Theorem 4.3] that, for representations $\rho : F(a, b) \rightarrow \text{PSL}(2, \mathbb{R})$, primitive stability is equivalent to the Q-conditions. In order to sketch his idea of proof, we first remark that, for $\text{PSL}(2, \mathbb{R})$ representations, the only non-trivial case to analyze is the non-Schottky representations where the axes of hyperbolic transformations a and b intersect, namely, the case we discussed in the introduction. See Figure 1.0.1.

There we have a quadrilateral Q whose opposite edges are paired by a and b . Thus as a quotient of Q we obtain a hyperbolic torus with one cone-type singularity. The image of such representation is not discrete in general, unless the cone-angle is a rational multiple of π . Nevertheless, we may still think of Q as a sort of fundamental domain and investigate its translates under the action. In fact, a crucial observation made by Lupi is that, for a

primitive element w with $\text{Axis}(w) = \{u_i\}_{i \in \mathbb{Z}}$ in $F(a, b)$, the translates $\rho(u_i)(Q)$ of Q do not overlap and form an infinite staircase-like strip. See [Lup15, Figure 3.3]. From this one can easily see that the image of $\text{Axis}(w)$ under an orbit map is a uniform quasi-geodesic.

For $\text{PSL}(2, \mathbb{C})$ -representations satisfying the Q-conditions, however, there seems to be no “pseudo-fundamental domain” in \mathbb{H}^3 as simple as the quadrilateral Q in \mathbb{H}^2 . For example, consider a slight perturbation of the above example out of $\text{PSL}(2, \mathbb{R})$ and its Coxeter extension $W(p, q, r)$. We obtain four complete geodesics which are axes of pqr and of its three other conjugates as in Figure 1.0.1. The four axes are in skew position so that, in general, no (totally geodesic) plane in \mathbb{H}^3 contains any two of them. So there is no direct analogue of the quadrilateral Q .

We may try to form a domain using other types of surfaces than planes. For example, the three axes of pqr , p and qrp (see Figure 1.0.1 again) can be used to define a ruled surface, that is, the union of all complete geodesics which intersect the three axes simultaneously. For a slight perturbation of the above example, we may then form a “twisted” quadrilateral prism bounded by such ruled surfaces. However, it seems rather hard to analyze such an object. For example, it is not immediate to know how long its combinatorial structure is preserved under perturbation and how far the distance is between two opposite surfaces of the quadrilateral.

5.5 On primitive displacing actions

Let us explain why we may view Theorem I in analogy with a theorem of Delzant et al [DGLM11].

Let (X, d_X) be a metric space and $\rho : \Gamma \rightarrow \text{Isom}(X)$ an isometric action of a group Γ . For an element $w \in \Gamma$ its translation length $\ell_X(w)$ for the ρ -action is defined as

$$\ell_X(w) = \inf_{x \in X} d_X(x, \rho(w)(x)).$$

Endow Γ with a word metric d_e for a generating set e , and consider the isometric action of Γ on itself by left multiplication. The translation length of $w \in \Gamma$ for this action will be denoted by $\ell_e(w)$.

The action $\rho : \Gamma \rightarrow \text{Isom}(X)$ is called *displacing* if there are positive constants m and c such that

$$\ell_X(w) \geq m \cdot \ell_e(w) - c$$

for all $w \in \Gamma$. Delzant et al showed that if Γ is word hyperbolic then the action ρ is displacing if and only if an orbit map of ρ is a quasi-isometric embedding. See [DGLM11, Lemma 2.0.1, Proposition 2.2.1, Corollary 4.0.6].

Since the free group F_2 is word hyperbolic, we may view our Theorem I in analogy with the above theorem. Namely, if we focus only on actions of primitive elements, the Q-conditions, by Theorem 4.6 and (3.3), can be thought of as a weakening of the displacing property, while primitive stability is obviously a weakening of quasi-isometric embedding.

Primitive stability is originally defined by Minsky [Min13] for free groups of arbitrary rank. The Q-conditions also make sense for such free groups, but Theorem 4.6 is only known

to hold for F_2 . One may then ask if it holds for any F_n . Or, regardless, we rather consider the following definition: a representation $\rho : F_n \rightarrow \mathbf{Isom}(X)$ ($n \geq 2$) is said to be *primitive displacing* if there are positive constants m and c such that

$$\ell_X(w) \geq m \cdot \ell_e(w) - c$$

for all primitive elements $w \in F_n(\mathbf{e})$. Then it is natural to ask:

5.6 Question. For which metric space X , is it true that $\rho : F_n \rightarrow \mathbf{Isom}(X)$ ($n \geq 2$) is primitive stable if and only if it is primitive displacing?

In fact, one direction, primitive stability implies primitive displacing, is true with the same proof as in Proposition 4.9. So the question is really about the other. Our Theorem I tells us the equivalence is true for $n = 2$ and $X = \mathbb{H}^3$. The next case one may want to explore seems to be when $n = 2$ and X is a rank-1 symmetric space of non-compact type, e.g., \mathbb{H}^4 . In the absence of trigonometric rules, one may employ certain large-scale geometric arguments in order to draw a conclusion that is analogous to Lemma 5.1, assuming the primitive displacing property and using Theorem 2.2(b).

6 Bounded intersection property

In this section we investigate representations $\rho : F_2 \rightarrow \mathbf{PSL}(2, \mathbb{C})$ satisfying a new condition which we call the bounded intersection property. It is described using primitive elements in F_2 that are palindromic in a fixed basis. Our definition is motivated by the work of Gilman and Keen [GK09, Theorem 6.6], where they use palindromic elements of F_2 to give a sufficient condition for a representation to be discrete.

In Section 6.5 we prove Theorem II that the bounded intersection property is implied by the Q-conditions.

6.1 Tri-coloring

For the forthcoming discussion we need the *tri-coloring* of the Farey triangulation.

We fix a basis

$$\mathbf{e} = (a, b)$$

with abelianization $\pi_e : F_2 \rightarrow \mathbb{Z}^2$ as in (2.1). Consider the homomorphism given by reduction modulo 2

$$\mathbf{PGL}(2, \mathbb{Z}) \twoheadrightarrow \mathbf{PGL}(2, \mathbb{Z}/2) \cong \mathbf{Sym}_3.$$

Its kernel, the *level two congruence subgroup* $\mathbf{PGL}(2, \mathbb{Z}/2)_{(2)}$, is isomorphic to the ideal triangle reflection group $T^*(\infty, \infty, \infty) \cong \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ with fundamental domain any Farey triangle.

Reduction modulo 2

$$\mathbf{V} = \mathbf{P}^1(\mathbb{Q}) \twoheadrightarrow \mathbf{P}^1(\mathbb{Z}/2) = \{0/1, 1/1, 1/0\}$$

partitions the vertex set V into three $\text{PGL}(2, \mathbb{Z}/2)_{(2)}$ -orbits “colored” by $0/1$, $1/1$ and $1/0$, respectively. Note that the three vertices of a Farey triangle are all colored differently. A Farey edge is then colored by the same color of its opposite vertex. See Figure 6.2.1. For example, the vertex $[a]$ is colored by $1/0$ (black), the vertex $[b]$ by $0/1$ (white), and the edge $[a, b]$ by $1/1$ (gray). If a vertex $[w] \in V$ is colored by $c \in \{0/1, 1/1, 1/0\}$ then we shall often say $[w]$ is a c -vertex. Similarly for edges.

For a more detailed discussion of tri-coloring, see [GMST, Section 2].

6.2 Primitive palindromes

We shall be interested in primitive elements that are palindromic.

6.1 Definition. A word $w = w(a, b) \in F_2$ is said to be *palindromic* in a basis $e = (a, b)$ (or, simply, an *e-palindrome*) if it reads the same forward and backward, that is,

$$(6.2) \quad w(a, b)^{-1} = w(a^{-1}, b^{-1}).$$

Gilman and Keen [GK11] introduced an algorithm of finding a palindromic representative in each primitive conjugacy class whose cyclically reduced length (4.1) is odd. This algorithm can be described by slightly modifying the inductive procedure for the Christoffel function (Definition 2.8(b)) as follows. Co-direct all Farey edges with the directed edge $[e] = [a, b]$ as in Section 2.3.

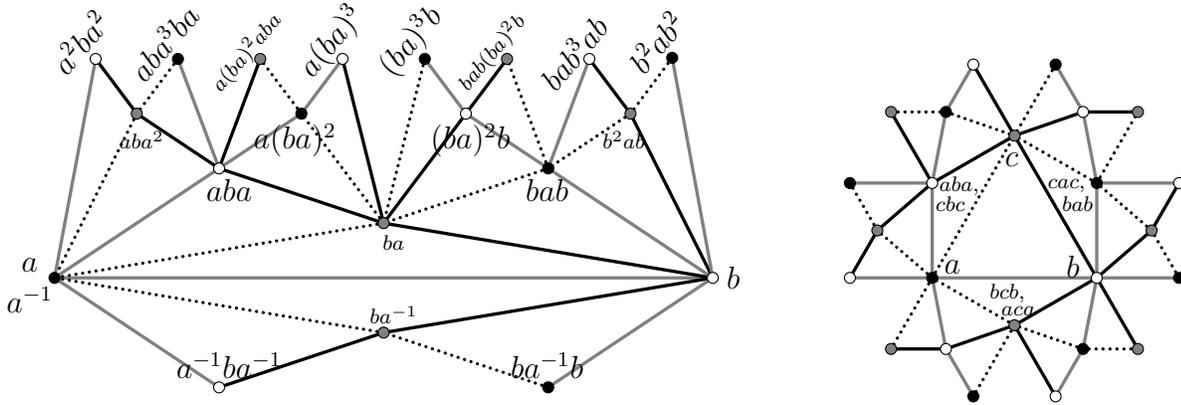


Figure 6.2.1: (Left) The e -palindrome function. (Right) The \tilde{e} -palindromes.

6.3 Definition. Let $S = F(a, b)$ and define $B : S \times S \rightarrow S$ by

$$B(w_1, w_2) = \begin{cases} w_2w_1 & \text{if } [w_1, w_2] \text{ and } [a, b] \text{ have the same color,} \\ w_1w_2 & \text{otherwise.} \end{cases}$$

The *e-palindrome function* Pal_e is defined inductively on $V^+[a, b]$ with initial values $\text{Pal}_e[a] = a$ and $\text{Pal}_e[b] = b$, and on $V^-[a, b]$ with initial values $\text{Pal}_e[a] = a^{-1}$ and $\text{Pal}_e[b] = b$.

See Figure 6.2.1(Left). Observe that $\text{Pal}_e[w]$ is an e -palindrome if and only if $[w] \in \mathbf{V}$ has color 1/0 or 0/1 (black or white). On the other hand, $\text{Pal}_e[w]$ is a product of two e -palindromes if and only if $[w]$ has color 1/1 (gray). These are re-statements of [GK11, Theorem 2.1], since $[w]$ has color 1/1 (gray) if and only if its cyclically reduced length $\|w\|_e = \text{Fib}_e[w]$ is even, that is, $1 + 1 \pmod{2}$. We also note that, according to [Fig06, Theorem 1.(1)], $\text{Pal}_e[w]$ and its inverse are the only e -palindromic representatives belonging to the unoriented conjugacy class $[w] \in \mathbf{V}$.

An *e -palindromic basis* (x, y) is a basis such that both x and y are e -palindromes. As with the e -Christoffel bases, it is apparent that if $[x, y]$ is a Farey edge then the pair $(\text{Pal}_e[x], \text{Pal}_e[y])$ is a basis. If, in addition, $[x, y]$ and $[a, b]$ have the same color, then $(\text{Pal}_e[x], \text{Pal}_e[y])$ is an e -palindromic basis. Compare [Fig06, Theorem 2] and [KR07, Theorem 5.5].

6.4 Lemma. *Let $\mathbf{f} = (x, y) = (\text{Pal}_e[x], \text{Pal}_e[y])$ be an e -palindromic basis (so that $[x, y]$ and $[a, b]$ have the same color). Then*

- (a) *the basis $\mathbf{e} = (a, b)$ is an \mathbf{f} -palindromic basis;*
- (b) *the set of e -palindromes is the same as the set of \mathbf{f} -palindromes.*

Proof. (a) Without loss of generality, we may assume that $[x]$ and $[y]$ are both in $\mathbf{V}^+[a, b]$ and the edge $[x, y]$ is co-directed with $[a, b]$. Let \mathcal{G} denote the shortest gallery containing both $[a, b]$ and $[x, y]$. Let $\{[g_k] \in \mathbf{V} \mid 1 \leq k \leq n\}$ be the set of all 1/1-vertices (gray vertices) in \mathcal{G} . Then we can decompose \mathcal{G} into a chain of *simple galleries* \mathcal{G}_k ($1 \leq k \leq n$)

$$\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_n,$$

where each \mathcal{G}_k is the star of $[g_k]$ in \mathcal{G} , namely, the union of all Farey triangles in \mathcal{G} containing $[g_k]$. The intersection $\mathcal{G}_k \cap \mathcal{G}_{k+1}$ of two adjacent simple galleries is a gray edge $[a_k, b_k]$, so that each \mathcal{G}_k is the shortest gallery connecting two gray edges $[a_{k-1}, b_{k-1}]$ and $[a_k, b_k]$, where $(a_0, b_0) = (a, b)$ and $(a_n, b_n) = (x, y)$. See Figure 6.2.2(Bottom).

Note that the property of being palindromic is transitive in the following sense: if a word $w = w(x', y')$ is an (x', y') -palindrome and if (x', y') is an (x, y) -palindromic basis, then $w = w(x, y)$ is an (x, y) -palindrome. Thus, by induction, it is enough to prove the lemma when $n = 1$ and the gallery \mathcal{G} is simple.

There are two possibilities of a simple gallery:

$$(x, y) = (a(ba)^i, a(ba)^{i+1}) \quad \text{or} \quad (x, y) = ((ba)^{i+1}b, (ba)^ib)$$

for some integer $i \geq 0$. See Figure 6.2.2(Top). As the proofs are similar we shall consider only the first case. Then we have $x^{-1}y = ba$ and

$$\begin{aligned} a &= x(ba)^{-i} = x(y^{-1}x)^i, \\ b &= x^{-1}ya^{-1} = x^{-1}y(x^{-1}y)^i x^{-1} = (x^{-1}y)^{i+1}x^{-1}. \end{aligned}$$

Therefore, both a and b are \mathbf{f} -palindromes. The proof is complete.

(b) This follows from (a) and the transitive property of palindromes we noted above. \square

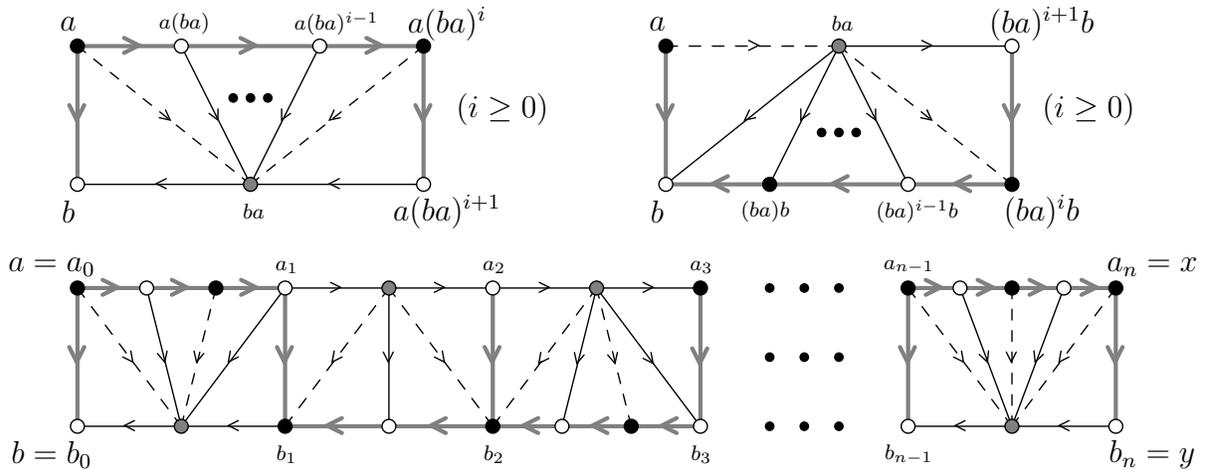


Figure 6.2.2: (Top) Two simple galleries. (Bottom) A generic gallery connecting $[a, b]$ and $[x, y]$.

Following [GMST, Section 2.2] we define a *basic triple* as an ordered triple $(x, y, z) \in \text{Prim}^3$ such that (x, y) is a basis and $xyz = 1$. A basis $\mathbf{f} = (x, y)$ extends uniquely to a basic triple (x, y, z) , which we denote by $\tilde{\mathbf{f}}$. It is clear that this correspondence is one-to-one between bases and basic triples.

6.5 Definition. Let $\tilde{\mathbf{f}} = (x, y, z)$ be a basic triple. An element $w \in F_2$ is said to be an $\tilde{\mathbf{f}}$ -*palindrome* if it is palindromic in either (x, y) or (y, z) or (z, x) (when written in these bases).

Consider the basic triple $\tilde{\mathbf{e}} = (a, b, c)$ associated to the distinguished basis $\mathbf{e} = (a, b)$. In addition to the (a, b) -palindrome function (Definition 6.3) we may consider the (b, c) -palindrome function and the (c, a) -palindrome function as well. See Figure 6.2.1(Right), where the vertices $[a]$, $[b]$ and $[c]$ are colored black, white and gray, respectively. We noted previously that $\text{Pal}_{(a,b)}[w]$ is an (a, b) -palindrome unless $[w]$ is colored gray. Analogous statements are true of the functions $\text{Pal}_{(b,c)}$ and $\text{Pal}_{(c,a)}$. Thus, for example, a black vertex contains an (a, b) -palindrome and a (c, a) -palindrome. Therefore, we can conclude as follows.

6.6 Lemma. *If a primitive element is not conjugate to $a^{\pm 1}$, $b^{\pm 1}$ or $c^{\pm 1}$, then its conjugacy class contains at least one and at most two $\tilde{\mathbf{e}}$ -palindromes.*

One can actually show that the two $\tilde{\mathbf{e}}$ -palindromic representatives are distinct elements of F_2 , but we shall not need this fact.

6.3 Geometry of palindromes

Let $\mathbf{e} = (a, b)$ be the distinguished basis of F_2 and $\tilde{\mathbf{e}} = (a, b, c)$ the associated basic triple. As in Section 3.2 (but recall Remark 3.6) we denote by $W(p, q, r)$ the free Coxeter group of

rank three, and by

$$\psi_{\mathbf{e}} : \mathbf{F}(a, b) \rightarrow \mathbf{W}(p, q, r)$$

the embedding defined by $\psi_{\mathbf{e}}(a) = qr$ and $\psi_{\mathbf{e}}(b) = rp$. We then have $\psi_{\mathbf{e}}(c) = pq$. Observe that the conjugation by the involutory element r inverts $\psi_{\mathbf{e}}(a)$ and $\psi_{\mathbf{e}}(b)$:

$$\begin{aligned} r \cdot \psi_{\mathbf{e}}(a) \cdot r &= rq = \psi_{\mathbf{e}}(a^{-1}) \\ r \cdot \psi_{\mathbf{e}}(b) \cdot r &= pr = \psi_{\mathbf{e}}(b^{-1}). \end{aligned}$$

Thus for any word $w = w(a, b) \in \mathbf{F}(a, b)$ we have

$$(6.7) \quad r \cdot \psi_{\mathbf{e}}(w(a, b)) \cdot r = \psi_{\mathbf{e}}(w(a^{-1}, b^{-1})).$$

See also the end of Appendix A.2.

Let $\rho : \mathbf{F}_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ be an irreducible representation and $\rho_{\mathbf{e}}$ be the Coxeter extension of ρ such that $\rho = \rho_{\mathbf{e}} \circ \psi_{\mathbf{e}}$ (see Theorem 3.7). We simplify the notations as in (3.8) so that, for example, \mathbf{Axis}_R denotes the axis of $\rho_{\mathbf{e}}(r)$.

The ρ -image of a palindrome has the following geometric property. See also [GK09, Lemma 5.1].

6.8 Lemma. *Suppose $\rho : \mathbf{F}_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is irreducible and $w = w(a, b)$ is palindromic in $\mathbf{e} = (a, b)$. If $\rho(w)$ is elliptic or loxodromic, then its axis intersects \mathbf{Axis}_R orthogonally. If $\rho(w)$ is parabolic, then its fixed point is an ideal end point of \mathbf{Axis}_R .*

Proof. Applying $\rho_{\mathbf{e}}$ to (6.7) we have $R \cdot \rho(w(a, b)) \cdot R = \rho(w(a^{-1}, b^{-1}))$. Since $w = w(a, b)$ is palindromic, we have from (6.2) that

$$R \cdot \rho(w) \cdot R = \rho(w)^{-1}.$$

Thus the involution R preserves the set of fixed points of $\rho(w)$ in the ideal boundary $\partial\mathbb{H}^3$. If $\rho(w)$ is elliptic or loxodromic, then R exchanges the end points of $\mathbf{Axis}_{\rho(w)}$, hence \mathbf{Axis}_R is orthogonal to $\mathbf{Axis}_{\rho(w)}$. If $\rho(w)$ is parabolic, then its fixed point is one of the end points of \mathbf{Axis}_R . \square

6.9 Remark. The converse of the lemma is not true in general. That is, for a non-palindromic element w' , $\rho(w')$ may satisfy the same conclusion. Such a counterexample arises, for example, when there is an element $u \in \mathbf{F}_2$ such that $\rho(u)$ is elliptic or loxodromic and $\mathbf{Axis}_{\rho(u)}$ coincides with \mathbf{Axis}_R . For if $w' = wuw^{-1}$ with w palindromic then both $\rho(w)$ and $\rho(w')$ have axes orthogonal to \mathbf{Axis}_R . But w' cannot be palindromic if there is no cancellation in wuw^{-1} as a word in (a, b) .

In order to find an explicit counterexample, let ρ be a $\mathrm{PSL}(2, \mathbb{R})$ -representation such that $\rho(a)$ and $\rho(b)$ are hyperbolic translations with the same translation length ℓ where $\cosh \ell = (1 + \sqrt{3})/2$ and their axes intersect orthogonally. Then one can verify that the element $\rho(a^2ba^{-1}b^{-2})$ is elliptic with axis \mathbf{Axis}_R . Let $w = (a^2ba^{-1}b^{-2})a(b^2ab^{-1}a^{-2})$. Then w is not palindromic but $\rho(w)$ is hyperbolic with axis intersecting \mathbf{Axis}_R orthogonally.

6.4 Bounded intersection property

Let $\mathbf{e} = (a, b)$ and $\tilde{\mathbf{e}} = (a, b, c)$ as before, and we continue to use the same notation as in the preceding section.

Of particular interest are irreducible representations satisfying the first Q-condition. Given such a representation $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$, the image $\rho(w)$ of any primitive element w admits an axis $\mathrm{Axis}_{\rho(w)}$ in \mathbb{H}^3 . If a primitive element w is palindromic in (a, b) (resp. (b, c) and (c, a)) then by Lemma 6.8 the axis $\mathrm{Axis}_{\rho(w)}$ is orthogonal to Axis_R (resp. Axis_P and Axis_Q).

Consider the following subsets of \mathbb{H}^3 :

$$\begin{aligned} J_P &= \{\mathrm{Axis}_P \cap \mathrm{Axis}_{\rho(w)} \mid w \text{ is a primitive } (b, c)\text{-palindrome}\}, \\ J_Q &= \{\mathrm{Axis}_Q \cap \mathrm{Axis}_{\rho(w)} \mid w \text{ is a primitive } (c, a)\text{-palindrome}\}, \\ J_R &= \{\mathrm{Axis}_R \cap \mathrm{Axis}_{\rho(w)} \mid w \text{ is a primitive } (a, b)\text{-palindrome}\}. \end{aligned}$$

6.10 Definition. A representation $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ has the *bounded intersection property* with respect to $\mathbf{e} = (a, b)$ if it satisfies the following conditions:

- (i) it is irreducible;
- (ii) all primitive elements are sent to loxodromic elements;
- (iii) the sets J_P , J_Q and J_R are all bounded.

Note that the property (ii) is the first Q-condition (Definition 4.5(i)). The bounded intersection property of ρ is clearly invariant under conjugation of $\mathrm{PSL}(2, \mathbb{C})$, so the definition also makes sense in the character variety $\mathcal{X}(F_2)$. We denote by $\mathcal{BI} \subset \mathcal{X}(F_2)$ the subset consisting of all $\mathrm{PSL}(2, \mathbb{C})$ -conjugacy classes of representations having the bounded intersection property.

The above definition was made after a choice of basis $\mathbf{e} = (a, b)$. As we shall see below, however, it is in fact independent of the choice. Thus the set \mathcal{BI} is invariant under the action of $\mathrm{Out}(F_2)$.

6.11 Proposition. *The bounded intersection property is independent of the choice of basis.*

Proof. Suppose a representation $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ has the bounded intersection property with respect to $\mathbf{e} = (a, b)$ and the associated basic triple (a, b, c) . Let (x, y) be another basis and (x, y, z) the associated basic triple. Since (i) and (ii) of Definition 6.10 are satisfied regardless of bases, it remains to check the third property (iii) for (x, y) . So we consider the Coxeter extension (S, T, U) associated to (x, y, z) , so that $\rho(x) = TU$, $\rho(y) = US$ and $\rho(z) = ST$. Then our goal is to show that the associated subsets of \mathbb{H}^3

$$\begin{aligned} J_S &= \{\mathrm{Axis}_S \cap \mathrm{Axis}_{\rho(w)} \mid w \text{ is a primitive } (y, z)\text{-palindrome}\}, \\ J_T &= \{\mathrm{Axis}_T \cap \mathrm{Axis}_{\rho(w)} \mid w \text{ is a primitive } (z, x)\text{-palindrome}\}, \\ J_U &= \{\mathrm{Axis}_U \cap \mathrm{Axis}_{\rho(w)} \mid w \text{ is a primitive } (x, y)\text{-palindrome}\} \end{aligned}$$

are all bounded.

We start with the set J_S associated with the basis (y, z) . The color of the Farey edge $[y, z]$ matches one of the colors of $[a, b]$, $[b, c]$ and $[c, a]$. Without loss of generality we may assume that it matches the color of $[e] = [a, b]$. Consider the e -palindromic basis

$$f := (\text{Pal}_e[y], \text{Pal}_e[z]).$$

Since $[f] = [y, z]$ as directed edges, f must be conjugate to one of the bases

$$g \in \{(y, z), (y^{-1}, z), (y, z^{-1}), (y^{-1}, z^{-1})\}.$$

So there is an element $g \in F_2$ such that

$$gfg^{-1} = g.$$

Then the following equalities hold:

$$\begin{aligned} J_S &= \{\text{Axis}_S \cap \text{Axis}_{\rho(w)} \mid w \text{ is a primitive } (y, z)\text{-palindrome}\} \\ &= \{\text{Axis}_S \cap \text{Axis}_{\rho(w)} \mid w \text{ is a primitive } g\text{-palindrome}\} \\ &= \{\text{Axis}_S \cap \text{Axis}_{\rho(w)} \mid w \text{ is a primitive } (gfg^{-1})\text{-palindrome}\} \\ &= \{\text{Axis}_S \cap \text{Axis}_{\rho(gwg^{-1})} \mid w \text{ is a primitive } f\text{-palindrome}\} \\ &= \rho(g) \cdot \{\text{Axis}_R \cap \text{Axis}_{\rho(w)} \mid w \text{ is a primitive } f\text{-palindrome}\} \\ &= \rho(g) \cdot \{\text{Axis}_R \cap \text{Axis}_{\rho(w)} \mid w \text{ is a primitive } e\text{-palindrome}\} \\ &= \rho(g) \cdot J_R. \end{aligned}$$

For the fourth equality, note that (gfg^{-1}) -palindromes are of the form gwg^{-1} for some f -palindrome w . The sixth equality holds since f is an e -palindromic basis: by Lemma 6.4(b) w is an e -palindrome if and only if it is an f -palindrome, and by Lemma 6.8 the Axis_R , which is the common perpendicular of $\text{Axis}_{\rho(a)}$ and $\text{Axis}_{\rho(b)}$, is also the common perpendicular of $\text{Axis}_{\rho(\text{Pal}_e[y])}$ and $\text{Axis}_{\rho(\text{Pal}_e[z])}$. For the fifth equality $\rho(g) \cdot \text{Axis}_R = \text{Axis}_S$, note from $gfg^{-1} = g$ that Axis_S is the common perpendicular of $\text{Axis}_{\rho(y)} = \text{Axis}_{\rho(g\text{Pal}_e[y]g^{-1})}$ and $\text{Axis}_{\rho(z)} = \text{Axis}_{\rho(g\text{Pal}_e[z]g^{-1})}$. All the remaining equalities are either immediate or by definition.

Since $\rho(g)$ is an isometry of \mathbb{H}^3 and J_R is a bounded subset, the set J_S is also bounded. In a similar fashion we can show that J_T and J_U are bounded as well. The proof is complete. \square

6.12 Remark. Note that the $\mathbb{Z}/3$ -symmetry in Definition 6.10(iii) is essential in the above proof. Under the conditions (i) and (ii) of Definition 6.10, we do not know if the boundedness of one set, say, J_R implies that of the other sets J_P and J_Q . Compare with the definition in [GK09], where only one set J_R is taken into consideration.

By Proposition 6.11 and Theorem II to be proved below, we know that $\mathcal{BI} \subset \mathcal{X}(F_2)$ is invariant under the action of $\text{Out}(F_2)$ and contains the open subset \mathcal{BQ} .

6.13 Question. Is the subset \mathcal{BI} open? Does $\text{Out}(F_2)$ act on \mathcal{BI} properly discontinuously? Do we have the equality $\mathcal{BQ} = \mathcal{BI}$?

6.5 Proof of Theorem II

Proof. In order to prove the first claim of the theorem, suppose $\rho : F_2 \rightarrow \mathrm{PSL}(2, \mathbb{C})$ satisfies the Q-conditions. Then it satisfies Definition 6.10(i) by [TWZ08b, Theorem 1.4]. It also satisfies Definition 6.10(ii) which is the first Q-condition.

To prove Definition 6.10(iii) fix a basis $\mathbf{f} = (x, y)$ and the associated basic triple (x, y, z) . As in the proof of Proposition 6.11 we consider the Coxeter extension (S, T, U) associated to (x, y, z) , so that $\rho(x) = TU$, $\rho(y) = US$ and $\rho(z) = ST$. Then we have to show that the sets J_S , J_T and J_U are all bounded. We first show that the set J_U is bounded. So consider the basis (x, y) and the corresponding word metric $d_{(x,y)}$ on F_2 .

Since ρ satisfies the Q-conditions, it is primitive stable by Theorem I. Let $o \in \mathbb{H}^3$ be a point and (M, c) a pair of positive constants given by the definition of primitive stability (Definition 4.3). Denote by $\tau_{\rho,o}$ the corresponding orbit map sending F_2 into \mathbb{H}^3 . Then the $\tau_{\rho,o}$ -image of all primitive axes in $(F_2, d_{(x,y)})$ are sent to (M, c) -quasi-geodesics in \mathbb{H}^3 . By the Morse lemma (see, for example, [BH99, Theorem III.H.1.7]), we know that there is a constant $D > 0$ depending on (M, c) such that for any primitive element w we have

$$d_{\mathrm{Haus}}(\tau_{\rho,o}(\mathrm{Axis}(w)), \mathrm{Axis}_{\rho(w)}) \leq D,$$

where d_{Haus} is the Hausdorff distance for subsets of \mathbb{H}^3 .

Assume that w is palindromic in (x, y) as well. Then w is cyclically reduced and its axis $\mathrm{Axis}(w)$ in $(F_2, d_{(x,y)})$ passes through the identity. Therefore, the image $\tau_{\rho,o}(\mathrm{Axis}(w))$ contains the point o . This implies that

$$d_{\mathbb{H}}(o, \mathrm{Axis}_{\rho(w)}) \leq D.$$

Let w and w' be primitive (x, y) -palindromes. By Proposition 6.8, we know that both $\mathrm{Axis}_{\rho(w)}$ and $\mathrm{Axis}_{\rho(w')}$ are orthogonal to Axis_U . Therefore the distance between them is realized by their intersection points with Axis_U . Let n (resp. n') be the nearest-point projection of o to the geodesic $\mathrm{Axis}_{\rho(w)}$ (resp. $\mathrm{Axis}_{\rho(w')}$). Then we have

$$d_{\mathbb{H}}(\mathrm{Axis}_{\rho(w)}, \mathrm{Axis}_{\rho(w')}) \leq d_{\mathbb{H}}(n, n') \leq d_{\mathbb{H}}(n, o) + d_{\mathbb{H}}(o, n') \leq 2D,$$

that is, the distance between the two intersection points in Axis_U is bounded above by $2D$. Therefore, we conclude that the diameter of the set J_U is bounded above by $4D$.

For the subsets J_S and J_T we consider the metrics $d_{(y,z)}$ and $d_{(z,x)}$ on F_2 , respectively. Then the same proof shows that they are bounded as well. Therefore, the representation ρ has the bounded intersection property.

We now prove the second claim of the theorem. So suppose ρ has a discrete image (which, in the present case of $G = \mathrm{PSL}(2, \mathbb{C})$, is equivalent to having a discrete orbit in \mathbb{H}^3) and satisfies the bounded intersection property with respect to a basic triple $\tilde{\mathbf{e}} = (a, b, c)$. Since ρ satisfies the first Q-condition by Definition 6.10(ii), it remains to check the second Q-condition.

So suppose, on the contrary, that ρ does not satisfy the second Q-condition. Then there are infinitely many unoriented primitive classes $[w] \in \mathbf{V}$ such that $|\mathrm{tr}\rho(w)| \leq 2$. Recall that

there are only three types of $\tilde{\epsilon}$ -palindromes (Definition 6.5) and each class $[w] \in \mathbf{V}$ contains at least one $\tilde{\epsilon}$ -palindromes (Lemma 6.6). Thus we may assume, without loss of generality, that there are infinitely many primitive elements w that are (x, y) -palindromes and satisfy $|\text{tr}\rho(w)| \leq 2$.

Then we claim that, for any point $o \in \text{Axis}_R$, its images under such $\rho(w)$'s will be contained in a compact subset of \mathbb{H}^3 and hence have an accumulation point. The claim follows from the bounded intersection property, since the axes of $\rho(w)$ are all orthogonal to Axis_R and the translation lengths of $\rho(w)$ are bounded above by a uniform constant (recall (3.3)). This contradicts the assumption that ρ has discrete image. Therefore, ρ must satisfy the second Q-condition as well. \square

A Appendix: The structure of $\text{Aut}(\mathbf{F}_2)$

We review the structure of $\text{Aut}(\mathbf{F}_2)$ via the isomorphism $\text{Aut}(\mathbf{F}_2) \cong \text{Aut}(\mathbf{W}_3)$, where \mathbf{W}_3 denotes the free Coxeter group of rank three. One may compare our discussion with those in [Gol03, Appendix] and [GMST, Section 2], for example.

As an application we explain the inductive algorithm of generating the Christoffel words in terms of $\text{Aut}(\mathbf{W}_3)$.

A.1 On $\text{Aut}(\mathbf{W}_3)$

Let $\mathbf{W}_3 = \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2 = \langle r_1, r_2, r_3 \mid r_1^2 = r_2^2 = r_3^2 = 1 \rangle$ be the free (or universal) Coxeter group of rank three. Here we collect a few facts scattered in the literature regarding the structure of $\text{Aut}(\mathbf{W}_3)$. We find it helpful to visualize the group $\text{Aut}(\mathbf{W}_3)$ from the representation of \mathbf{W}_3 in the group $\text{Isom}^+(\mathbb{E}^2)$ or $\text{Isom}^+(\mathbb{H}^2)$ rather than from the Cayley graph of \mathbf{W}_3 . See Figure A.1.1.

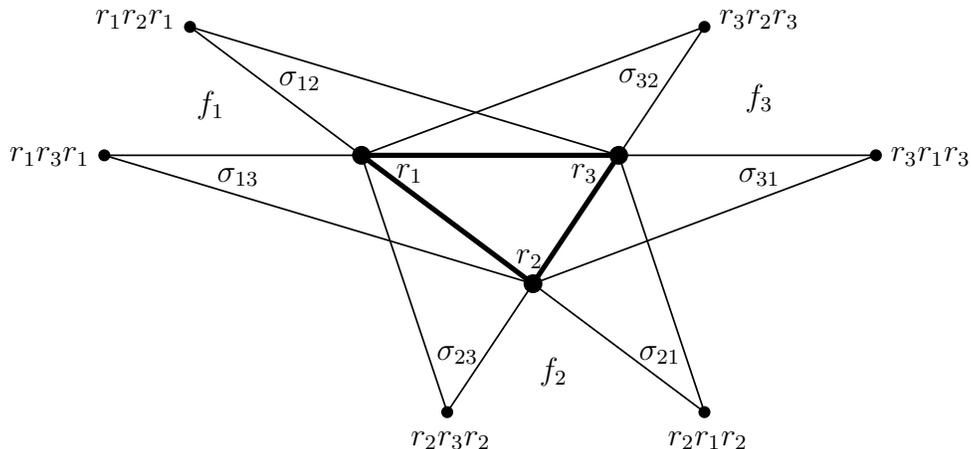


Figure A.1.1: Special involutory automorphisms σ_{ij} in $\text{Aut}(\mathbf{W}_3)$

Let $\text{Sym}_3 = \{1, s_{123}, s_{321}, t_{12}, t_{23}, t_{31}\}$ denote the group of automorphisms of W_3 permuting the free factors. Let $\text{Spe}(W_3)$ denote the group of automorphisms of W_3 that preserve the conjugacy classes of r_1 , r_2 and r_3 ; this group is known to be generated by the following *involutory* automorphisms of W_3 (see [Müh98, Theorem]):

$$\begin{aligned}\sigma_{12}(r_1, r_2, r_3) &= (r_1, r_1 r_2 r_1, r_3), & \sigma_{13}(r_1, r_2, r_3) &= (r_1, r_2, r_1 r_3 r_1), \\ \sigma_{23}(r_1, r_2, r_3) &= (r_1, r_2, r_2 r_3 r_2), & \sigma_{21}(r_1, r_2, r_3) &= (r_2 r_1 r_2, r_1, r_3), \\ \sigma_{31}(r_1, r_2, r_3) &= (r_3 r_1 r_3, r_2, r_3), & \sigma_{32}(r_1, r_2, r_3) &= (r_1, r_3 r_2 r_3, r_3).\end{aligned}$$

We set and choose

$$\begin{aligned}\iota_1 &= \sigma_{12}\sigma_{13} = \sigma_{13}\sigma_{12}, & \iota_2 &= \sigma_{21}\sigma_{23} = \sigma_{23}\sigma_{21}, & \iota_3 &= \sigma_{31}\sigma_{32} = \sigma_{32}\sigma_{31}, \\ f_1 &\in \{\sigma_{12}, \sigma_{13}\}, & f_2 &\in \{\sigma_{23}, \sigma_{21}\}, & f_3 &\in \{\sigma_{31}, \sigma_{32}\}.\end{aligned}$$

Note that ι_k ($k = 1, 2, 3$) is the conjugation by r_k and thus we have $\text{Inn}(W_3) = \langle \iota_1, \iota_2, \iota_3 \rangle \cong W_3$. Then it can be shown that

$$\begin{aligned}\text{Aut}(W_3) &= \text{Spe}(W_3) \rtimes \text{Sym}_3 \\ &= (\langle \iota_1, \iota_2, \iota_3 \rangle \rtimes \langle f_1, f_2, f_3 \rangle) \rtimes \text{Sym}_3 \\ &= (\text{Inn}(W_3) \rtimes W_3) \rtimes \text{Sym}_3.\end{aligned}$$

See [Fra02, Theorem 2.11] or [CG90, Lemma 3.5 and §7] for a proof. The action of $\langle f_1, f_2, f_3 \rangle$ on $\langle \iota_1, \iota_2, \iota_3 \rangle$ corresponds to the action of $\langle f_1, f_2, f_3 \rangle$ on $W_3 = \langle r_1, r_2, r_3 \rangle$. For example, if $f_1 = \sigma_{12}$ then we have

$$f_1 \iota_1 f_1 = \iota_1, \quad f_1 \iota_2 f_1 = \iota_1 \iota_2 \iota_1, \quad f_1 \iota_3 f_1 = \iota_3.$$

From the above we then have

$$\begin{aligned}\text{Out}(W_3) &= \text{Aut}(W_3)/\text{Inn}(W_3) \\ &\cong W_3 \rtimes \text{Sym}_3 \\ &= \langle [f_1], [f_2], [f_3] \rangle \rtimes \{1, s_{123}, s_{321}, t_{12}, t_{23}, t_{31}\} \\ &\cong \text{PGL}(2, \mathbb{Z}) \\ &\cong T^*(2, 3, \infty),\end{aligned}$$

where the last group denotes the $(2, 3, \infty)$ -triangle reflection group of \mathbb{H}^2 . See Figure A.1.2, where we draw \mathbb{H}^2 in the Klein projective model and shaded a fundamental domain for the action of $T^*(2, 3, \infty)$.

The short exact sequence

$$1 \rightarrow \text{Inn}(W_3) \rightarrow \text{Aut}(W_3) \rightarrow \text{Out}(W_3) \rightarrow 1$$

does not split, that is, the whole group $\text{Out}(W_3)$ does not lift to $\text{Aut}(W_3)$. This is because the f_i 's cannot be chosen to have the full symmetry of Sym_3 ; see Figure A.1.1. At best, they can be chosen so as to admit the symmetry of $\mathbb{Z}/3 = \{1, s_{123}, s_{321}\}$, say, $f_1 = \sigma_{12}$, $f_2 = \sigma_{23}$ and $f_3 = \sigma_{31}$, and we can lift only the part $W_3 \rtimes \mathbb{Z}/3$ of $\text{Out}(W_3)$.

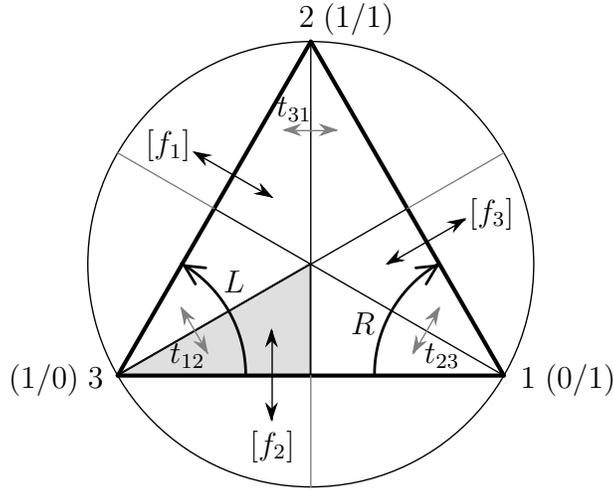


Figure A.1.2: $\text{Out}(W_3)$ is isomorphic to $\text{PGL}(2, \mathbb{Z}) \cong T^*(2, 3, \infty)$

A.2 The isomorphism $\text{Aut}(W_3) \cong \text{Aut}(F_2)$

Given a basis triple (r_1, r_2, r_3) of W_3 , the pair of elements $(a, b) := (r_1 r_2, r_2 r_3)$ freely generate a subgroup $F_2 = F(a, b)$ of index two. One can show that the correspondence

$$(A.1) \quad (r_1, r_2, r_3) \longleftrightarrow (r_1 r_2, r_2 r_3) = (a, b)$$

induces an isomorphism

$$\text{Aut}(W_3) \cong \text{Aut}(F_2).$$

See Figure A.2.1. Compare [CD02, p. 146] or [PRW10, Remark 2].

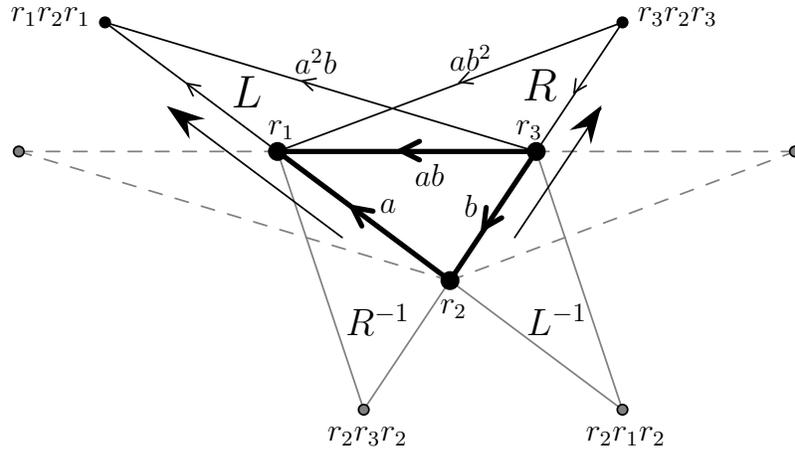


Figure A.2.1: The isomorphism $\text{Aut}(W_3) \cong \text{Aut}(F_2)$

Under the above isomorphism, the automorphisms σ_{12} , σ_{23} and σ_{31} correspond to the

automorphisms $\tilde{\mathfrak{J}}_2$, $\tilde{\mathfrak{J}}_3$ and $\tilde{\mathfrak{J}}_1$ of $\mathbf{Aut}(\mathbf{F}_2)$ defined by the following bases of $\mathbf{F}(a, b)$:

$$(A.2) \quad \begin{aligned} (r_1, r_1 r_2 r_1, r_3) &\longleftrightarrow (r_2 r_1, r_1 r_2 r_1 r_3) = (a^{-1}, a^2 b), \\ (r_1, r_2, r_2 r_3 r_2) &\longleftrightarrow (r_1 r_2, r_3 r_2) = (a, b^{-1}), \\ (r_3 r_1 r_3, r_2, r_3) &\longleftrightarrow (r_3 r_1 r_3 r_2, r_2 r_3) = (b^{-1} a^{-1} b^{-1}, b). \end{aligned}$$

These appear also in [GMST, §.2.1 (6)]. On the other hand, the (opposite) automorphisms σ_{13} , σ_{21} and σ_{32} correspond to

$$\begin{aligned} (r_1, r_2, r_1 r_3 r_1) &\longleftrightarrow (r_1 r_2, r_2 r_1 r_3 r_1) = (a, a^{-1} b^{-1} a^{-1}), \\ (r_2 r_1 r_2, r_2, r_3) &\longleftrightarrow (r_2 r_1, r_2 r_3) = (a^{-1}, b), \\ (r_1, r_3 r_2 r_3, r_3) &\longleftrightarrow (r_1 r_3 r_2 r_3, r_3 r_2) = (ab^2, b^{-1}). \end{aligned}$$

The inner automorphism $\iota_2 \in \mathbf{Aut}(\mathbf{W}_3)$ corresponds to the involutory automorphism $\epsilon \in \mathbf{Aut}(\mathbf{F}_2)$ (defined in Section 2.2):

$$\iota_2(r_1, r_2, r_3) = (r_2 r_1 r_2, r_2, r_2 r_3 r_2) \longleftrightarrow (r_2 r_1, r_3 r_2) = (a^{-1}, b^{-1}) = \epsilon(a, b).$$

A.3 Finding primitive representatives

We observed that $\mathbf{Out}(\mathbf{W}_3)$ does not lift to $\mathbf{Aut}(\mathbf{W}_3)$. Its subgroup $\mathbf{W}_3 \cong \langle [f_1], [f_2], [f_3] \rangle$, however, can be lifted to $\mathbf{Aut}(\mathbf{W}_3)$ and still acts transitively on the set of Farey triangles. Therefore, in order to find representatives of the \sim -equivalence classes in $\mathbf{V} = \mathbf{Prim}/\sim \subset \mathbf{F}_2/\sim$, we may take the orbit of a basis triple of \mathbf{W}_3 under the action of the (lifted) subgroup $\mathbf{W}_3 \cong \langle f_1, f_2, f_3 \rangle$ of $\mathbf{Aut}(\mathbf{W}_3)$, then convert its members to bases of \mathbf{F}_2 using the correspondence (A.1). For example, if we choose $f_1 = \sigma_{12}$, $f_2 = \sigma_{23}$ and $f_3 = \sigma_{31}$, then the action is given by $\tilde{\mathfrak{J}}_2$, $\tilde{\mathfrak{J}}_3$ and $\tilde{\mathfrak{J}}_1$ as in (A.2). But this example has a drawback that the formulas look somewhat complicated due to lack of symmetry.

Instead of lifting $\mathbf{W}_3 \subset \mathbf{Out}(\mathbf{W}_3)$ we rather lift the free submonoid of $\mathbf{Out}(\mathbf{W}_3) \cong \mathbf{PGL}(2, \mathbb{Z})$ generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. See Remark 2.11. In this way we can find “half” of the representatives. Compare [ADP99, Definition 2.6 and Lemma 5.2]. The lifted automorphisms are given by

$$\begin{aligned} L &:= \sigma_{12} t_{12} : (r_1, r_2, r_3) \mapsto (r_1 r_2 r_1, r_1, r_3) \longleftrightarrow (r_1 r_2, r_1 r_3) = (a, ab), \\ R &:= \sigma_{32} t_{23} : (r_1, r_2, r_3) \mapsto (r_1, r_3, r_3 r_2 r_3) \longleftrightarrow (r_1 r_3, r_2 r_3) = (ab, b). \end{aligned}$$

See Figures A.1.2 and A.2.1 again. Let \mathcal{L} and \mathcal{R} denote the corresponding automorphisms of \mathbf{F}_2 :

$$\mathcal{L}(a) = a, \mathcal{L}(b) = ab, \quad \text{and} \quad \mathcal{R}(a) = ab, \mathcal{R}(b) = b.$$

We also consider the functions $\underline{\mathcal{L}}, \underline{\mathcal{R}} : \mathbf{F}_2 \times \mathbf{F}_2 \rightarrow \mathbf{F}_2 \times \mathbf{F}_2$ defined by

$$\begin{aligned} \underline{\mathcal{L}}(x, y) &= (x, xy) \\ \underline{\mathcal{R}}(x, y) &= (xy, y). \end{aligned}$$

Using the pair \mathcal{L} and \mathcal{R} (or the pair $\underline{\mathcal{L}}$ and $\underline{\mathcal{R}}$) we can generate the e-Christoffel bases with $\mathbf{e} = (a, b)$. (See the paragraph prior to Remark 2.11.)

Suppose a positive rational number p/q has continued fraction expansion

$$p/q = [n_0; n_1, n_2, \dots, n_k] = [n_0; n_1, n_2, \dots, n_k - 1, 1].$$

Let $\text{Ch}_e(p/q)$ denote the e-Christoffel word associated to p/q . Then we have

$$\text{Ch}_e(p/q) = a'b',$$

where

$$\begin{aligned} (a', b') &= \mathcal{L}^{n_0} \mathcal{R}^{n_1} \mathcal{L}^{n_2} \dots \mathcal{L}^{n_k-1}(a, b) \\ &= \underline{\mathcal{L}}^{n_k-1} \dots \underline{\mathcal{L}}^{n_2} \underline{\mathcal{R}}^{n_1} \underline{\mathcal{L}}^{n_0}(a, b), \end{aligned}$$

or

$$\begin{aligned} (a', b') &= \mathcal{L}^{n_0} \mathcal{R}^{n_1} \mathcal{L}^{n_2} \dots \mathcal{R}^{n_k-1}(a, b) \\ &= \underline{\mathcal{R}}^{n_k-1} \dots \underline{\mathcal{L}}^{n_2} \underline{\mathcal{R}}^{n_1} \underline{\mathcal{L}}^{n_0}(a, b) \end{aligned}$$

depending on the parity of k .

Let us illustrate this using an example. Consider the continued fraction expansion

$$\frac{17}{10} = 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{3}}} = [1; 1, 2, 3] = [1; 1, 2, 2, 1].$$

Then we have

$$\begin{array}{ll} (a, b) \xrightarrow{\mathcal{R}^2} (ab^2, b) & (a, b) \xrightarrow{\underline{\mathcal{L}}^1} (a, ab) \\ \xrightarrow{\mathcal{L}^2} (a(a^2b)^2, a^2b) & \xrightarrow{\underline{\mathcal{R}}^1} (a^2b, ab) \\ \xrightarrow{\mathcal{R}^1} (ab((ab)^2b)^2, (ab)^2b) & \xrightarrow{\underline{\mathcal{L}}^2} (a^2b, (a^2b)^2ab) \\ \xrightarrow{\mathcal{L}^1} (a^2b((a^2b)^2ab)^2, (a^2b)^2ab), & \xrightarrow{\underline{\mathcal{R}}^2} (a^2b((a^2b)^2ab)^2, (a^2b)^2ab). \end{array}$$

That is,

$$(a', b') = \mathcal{L}^1 \mathcal{R}^1 \mathcal{L}^2 \mathcal{R}^2(a, b) = \underline{\mathcal{R}}^2 \underline{\mathcal{L}}^2 \underline{\mathcal{R}}^1 \underline{\mathcal{L}}^1(a, b) = (a^2b((a^2b)^2ab)^2, (a^2b)^2ab)$$

Thus we obtain

$$\text{Ch}_e(17/10) = a'b' = a^2b((a^2b)^2ab)^3.$$

We emphasize the difference between $\mathcal{L}^1 \mathcal{R}^1 \mathcal{L}^2 \mathcal{R}^2$ and $\underline{\mathcal{R}}^2 \underline{\mathcal{L}}^2 \underline{\mathcal{R}}^1 \underline{\mathcal{L}}^1$. The former is a sequence of “substitutions” as \mathcal{L} and \mathcal{R} are automorphisms of \mathbb{F}_2 defined by their actions on the

distinguished basis $\mathbf{e} = (a, b)$, while the latter is a sequence of left or right “concatenation” dictated by the functions $\underline{\mathcal{L}}$ and $\underline{\mathcal{R}}$. For practical purposes the latter method is preferable since it is compatible with the inductive generation of \mathbf{e} -Christoffel words as in Figure 1.0.1 and Definition 2.8.

Lastly, we remark that both L and R preserve the element $r_1r_2r_3$:

$$\begin{aligned} L(r_1r_2r_3) &= (r_1r_2r_1)(r_1)(r_3) = r_1r_2r_3, \\ R(r_1r_2r_3) &= (r_1)(r_3)(r_3r_2r_3) = r_1r_2r_3. \end{aligned}$$

Since we have

$$(r_1r_2r_3)^2 = (r_1r_2r_3)(r_2r_2)(r_1r_2r_3) = ab^{-1}a^{-1}b$$

(as in (3.13)), the corresponding automorphisms \mathcal{L} and \mathcal{R} in $\text{Aut}(\mathbb{F}_2)$ preserve the commutator $[a, b^{-1}] = ab^{-1}a^{-1}b$. Of course, one can check this directly from their definitions. Compare [CD02, p. 145 and p. 147], [ASWY07, Lemma 2.1.7] and [KR07, Remark 5.6(c)].

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