Generalizations of single-peakedness

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Part I: Single-peaked orderings

Single-peaked orderings

Motivating example (Romero, 1978)

Suppose you are asked to order the following six objects in decreasing preference:

a_1 : 0 sandwich
a_2 : 1 sandwich
a_3 : 2 sandwiches
a_4 : 3 sandwiches
a_5 : 4 sandwiches
a_6 : more than 4 sandwiches

We write $a_i \prec a_j$ if a_i is preferred to a_j

Single-peaked orderings

a_1 : 0 sandwich
a_2 : 1 sandwich
a_3 : 2 sandwiches
a_4 : 3 sandwiches
a_5 : 4 sandwiches
a_6 : more than 4 sandwiches

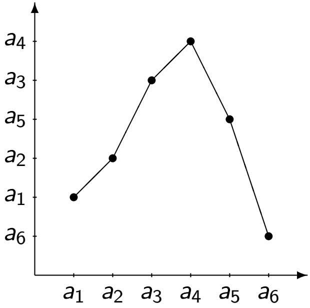
- after a good lunch: $a_1 \prec a_2 \prec a_3 \prec a_4 \prec a_5 \prec a_6$
- if you are starving: $a_6 \prec a_5 \prec a_4 \prec a_3 \prec a_2 \prec a_1$
- a possible intermediate situation: $a_4 \prec a_3 \prec a_5 \prec a_2 \prec a_1 \prec a_6$
- a quite unlikely preference: $a_6 \prec a_5 \prec a_2 \prec a_1 \prec a_3 \prec a_4$

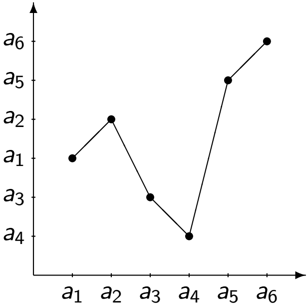
Single-peaked orderings

Let us represent graphically the latter two preferences with respect to the reference ordering $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$

$a_4 \prec a_3 \prec a_5 \prec a_2 \prec a_1 \prec a_6$

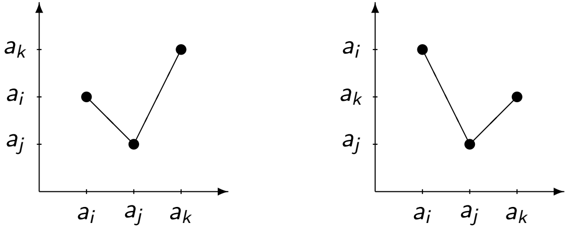
$a_6 \prec a_5 \prec a_2 \prec a_1 \prec a_3 \prec a_4$





Single-peaked orderings

Definition. (Black, 1948)
Let $\leq$ and $\preceq$ be total orderings on $X_n = \{a_1, \ldots, a_n\}$.
Then $\preceq$ is said to be *single-peaked for* $\leq$ if the following patterns are forbidden



Mathematically:

$$a_i < a_j < a_k \quad \Longrightarrow \quad a_j \prec a_i \quad \text{or} \quad a_j \prec a_k$$

Single-peaked orderings

$$a_i < a_j < a_k \quad \Longrightarrow \quad a_j \prec a_i \quad \text{or} \quad a_j \prec a_k$$

Let us assume that $X_n = \{a_1, \ldots, a_n\}$ is endowed with the ordering $a_1 < \cdots < a_n$

For $n = 4$

$$\begin{array}{ll}
a_1 \prec a_2 \prec a_3 \prec a_4 & a_4 \prec a_3 \prec a_2 \prec a_1 \\
a_2 \prec a_1 \prec a_3 \prec a_4 & a_3 \prec a_2 \prec a_1 \prec a_4 \\
a_2 \prec a_3 \prec a_1 \prec a_4 & a_3 \prec a_2 \prec a_4 \prec a_1 \\
a_2 \prec a_3 \prec a_4 \prec a_1 & a_3 \prec a_4 \prec a_2 \prec a_1
\end{array}$$

There are 2^{n-1} total orderings $\preceq$ on X_n that are single-peaked for $\leq$

Single-peaked orderings

Recall that a *weak ordering* (or *total preordering*) on X_n is a binary relation $\precsim$ on X_n that is total and transitive.

Defining a weak ordering on X_n amounts to defining an ordered partition of X_n

$$C_1 \prec \cdots \prec C_k$$

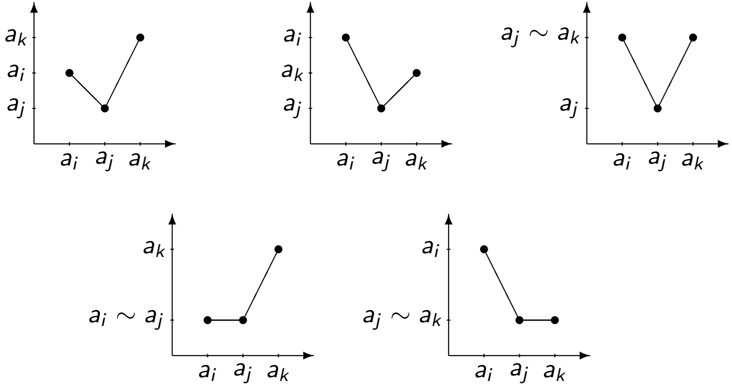
where $C_1, \ldots, C_k$ are the equivalence classes defined by $\sim$

For $n = 3$, we have 13 weak orderings

$a_1 \prec a_2 \prec a_3$	$a_1 \sim a_2 \prec a_3$	$a_1 \sim a_2 \sim a_3$
$a_1 \prec a_3 \prec a_2$	$a_1 \prec a_2 \sim a_3$	
$a_2 \prec a_1 \prec a_3$	$a_2 \prec a_1 \sim a_3$	
$a_2 \prec a_3 \prec a_1$	$a_3 \prec a_1 \sim a_2$	
$a_3 \prec a_1 \prec a_2$	$a_1 \sim a_3 \prec a_2$	
$a_3 \prec a_2 \prec a_1$	$a_2 \sim a_3 \prec a_1$	

Single-peaked orderings

Definition. (Black, 1948)
Let $\leq$ be a total ordering on X_n and let $\precsim$ be a weak ordering on X_n. Then $\precsim$ is said to be *single-plateaued for* $\leq$ if the following patterns are forbidden



Single-peaked orderings

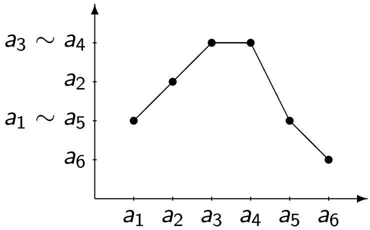
Mathematically:

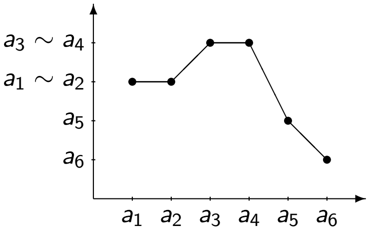
$$a_i < a_j < a_k \implies a_j \prec a_i \quad \text{or} \quad a_j \prec a_k \quad \text{or} \quad a_i \sim a_j \sim a_k$$

Examples

$a_3 \sim a_4 \prec a_2 \prec a_1 \sim a_5 \prec a_6$

$a_3 \sim a_4 \prec a_2 \sim a_1 \prec a_5 \prec a_6$





Part II: Quasitrivial and idempotent semigroups

Quasitriviality

Definition

$F\colon X_n^2 \to X_n$ is said to be

- *quasitrivial* (or *conservative*) if

$$F(x,y) \in \{x,y\} \qquad (x,y \in X_n)$$

- *idempotent* if

$$F(x,x) = x \qquad (x \in X_n)$$

Fact. If F is quasitrivial, then it is idempotent

Associative and quasitrivial operations

Definition.

The *projection operations* $\pi_1\colon X_n^2 \to X_n$ and $\pi_2\colon X_n^2 \to X_n$ are respectively defined by

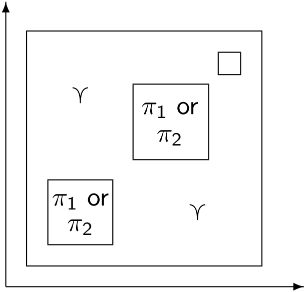
$$\begin{aligned} \pi_1(x,y) &= x, \qquad x,y \in X_n \\ \pi_2(x,y) &= y, \qquad x,y \in X_n \end{aligned}$$

Associative and quasitrivial operations

Assume that $X_n = \{a_1, \ldots, a_n\}$ is endowed with a weak ordering $\precsim$

Ordinal sum of projections

$$\mathrm{osp}_{\precsim} : X_n^2 \to X_n$$



If $\precsim$ is a total ordering, then $\mathrm{osp}_{\precsim} = \curlyvee$

Associative and quasitrivial operations

Theorem (Länger 1980, Kepka 1981)

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

(i) F is associative and quasitrivial

(ii) $F = \mathrm{osp}_{\precsim}$ for some weak ordering $\precsim$ on X_n

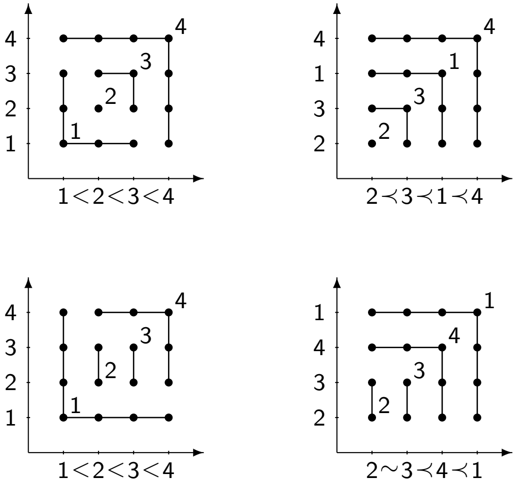
Corollary

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

(i) F is associative, quasitrivial, and commutative

(ii) $F = \curlyvee$ for some total ordering $\preceq$ on X_n

Associative and quasitrivial operations



Associative, quasitrivial, and order-preserving operations

Definition.

$F\colon X_n^2 \to X_n$ is said to be $\leq$-preserving for some total ordering $\leq$ on X_n if for any $x, y, x', y' \in X_n$ such that $x \leq x'$ and $y \leq y'$, we have $F(x,y) \leq F(x',y')$

Definition.

A *uninorm on* $(X_n, \leq)$ is an operation $F\colon X_n^2 \to X_n$ that

- has a neutral element $e \in X_n$ $\qquad (\Leftrightarrow \; F(x,e) = F(e,x) = x \quad \forall x \in X_n)$

and is

- associative
- commutative
- $\leq$-preserving

Associative, quasitrivial, and order-preserving operations

$\leq$: total ordering on X_n

Theorem (Couceiro et al., 2018)

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

(i) F is associative, quasitrivial, and $\leq$-preserving

(ii) $F = \mathrm{osp}_{\precsim}$ for some weak ordering $\precsim$ on X_n that is single-plateaued for $\leq$

Theorem (Couceiro et al., 2018)

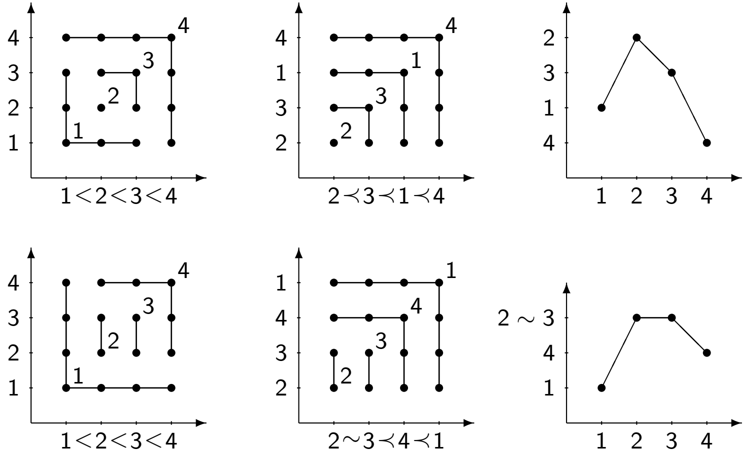
Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

(i) F is associative, quasitrivial, commutative, and $\leq$-preserving

(ii) $F = \curlyvee$ for some total ordering $\preceq$ on X_n that is single-peaked for $\leq$

(iii) F is an idempotent uninorm on X_n

Associative and quasitrivial operations



Associative, idempotent, and commutative operations

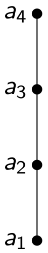
Lemma

Let $F\colon X_n^2 \to X_n$. The following assertions are equivalent.

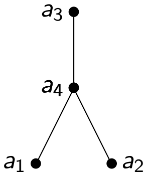
(i) F is associative, idempotent, and commutative

(ii) $F = \curlyvee$ for some join-semilattice ordering $\preceq$ on X_n

Example. On $X_4 = \{a_1, a_2, a_3, a_4\}$, consider the total ordering $\leq$ and the join-semilattice ordering $\preceq$



$\leq$

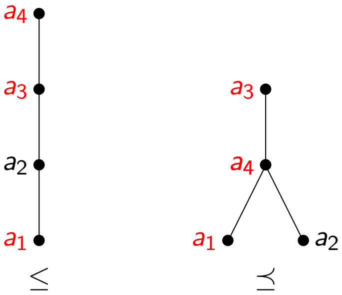


$\preceq$

Towards a generalization

$\leq$: total ordering on X_n

$\preceq$: join-semilattice ordering on X_n

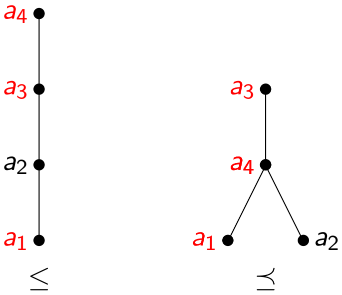


$$\curlyvee(a_1, a_4) = a_4 \;\text{ and }\; \curlyvee(a_3, a_4) = a_3 \quad \Rightarrow \quad \curlyvee \;\text{ is not } \leq\text{-preserving}$$

What are the $\preceq$ for which $\curlyvee$ are $\leq$-preserving?

Towards a generalization

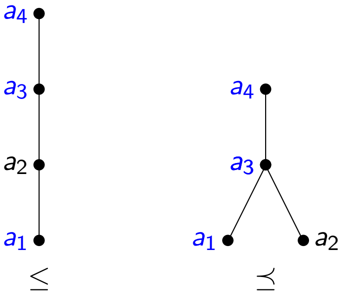
$$a \leq b \leq c \quad \Longrightarrow \quad b \preceq a \curlyvee c \qquad (*)$$



$\preceq$ does not satisfy $(*)$

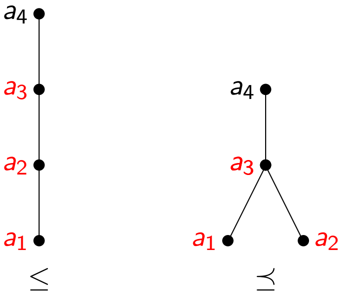
Towards a generalization

$$a \leq b \leq c \quad \Longrightarrow \quad b \preceq a \curlyvee c \qquad (*)$$



$\preceq$ satisfies $(*)$

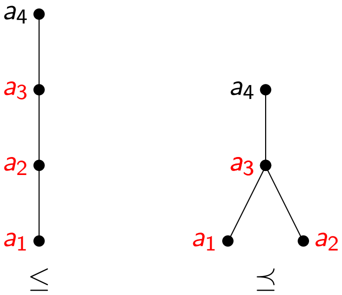
Towards a generalization



$$\Upsilon(a_1, a_2) = a_3 \quad \text{and} \quad \Upsilon(a_2, a_2) = a_2 \quad \Rightarrow \quad \Upsilon \ \text{ is not } \leq\text{-preserving}$$

Towards a generalization

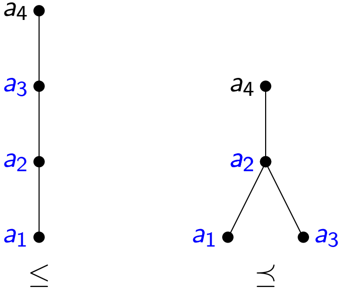
$$a < b < c \quad \Longrightarrow \quad (a \neq b \curlyvee c \quad \text{and} \quad c \neq a \curlyvee b) \qquad (**)$$



$\preceq$ satisfies $(*)$ but not $(**)$

Towards a generalization

$$a < b < c \quad \Longrightarrow \quad (a \neq b \curlyvee c \quad \text{and} \quad c \neq a \curlyvee b) \qquad (**)$$



$\preceq$ satisfies $(*)$ and $(**)$

Also, $\curlyvee$ is $\leq$-preserving

Nondecreasingness

Definition. We say that $\preceq$ is *nondecreasing for* $\leq$ if it satisfies $(*)$ and $(**)$

$$\boxed{F \text{ is associative, idempotent, and commutative iff } F = \curlyvee}$$

Theorem (Devillet et al., 2018)

For any $F\colon X^2 \to X$, the following are equivalent.

(i) F is associative, idempotent, commutative, and $\leq$-preserving

(ii) $F = \curlyvee$ for some $\preceq$ that is nondecreasing for $\leq$

Nondecreasingness

C_n : nth Catalan number

Proposition (Devillet et al., 2018)

C_n is

- the number of nondecreasing join-semilattice orders on X_n
- the number of associative, idempotent, commutative, and $\leq$-preserving binary operations on X_n

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