Supplemental Material

SI. FIRST-ORDER BORN APPROXIMATION

In this section, we provide some more details on the calculation of the self-energy with the FB approximation. Essentially, a summation of $\mathbf{k}$ over the unperturbed retarded Green function [corresponding to the Hamiltonian in Eq. (1)] needs to be calculated and plugged into Eq. (7). It can be written as follows:

$$\begin{aligned}\mathcal{G}^{(0)}(\mathbf{k},\omega) = \frac{1}{2}\sum_s \left(\sigma_0 + s\sum_\alpha \sigma_\alpha \mathbf{v}_\alpha \cdot (\mathbf{k} - Q\,\mathbf{e}_{k_R})/\sqrt{\textstyle\sum_\beta [\mathbf{v}_\beta \cdot (\mathbf{k} - Q\,\mathbf{e}_{k_R})]^2}\right) \\ \times \left(\frac{\omega - E_s(\mathbf{k})/\hbar}{[\omega - E_s(\mathbf{k})/\hbar]^2 + \eta^2} - i\,\pi\,\delta[\omega - E_s(\mathbf{k}')/\hbar]\right) \equiv \sum_\nu [\mathrm{Re}\{\mathcal{G}_\nu^{(0)}(\mathbf{k},\omega)\} + i\,\mathrm{Im}\{\mathcal{G}_\nu^{(0)}(\mathbf{k},\omega)\}]\,\sigma_\nu.\end{aligned} \tag{S1}$$

Making use of the notation introduced on the last line of Eq. (S1), we obtain for the summation over $\mathrm{Im}\{\mathcal{G}_0^{(0)}(\mathbf{k},\omega)\}$ (assuming $v_R = v_z = v$ and $w_R = 0$):

$$\begin{aligned}\sum_{\mathbf{k}} \mathrm{Im}\{\mathcal{G}_0^{(0)}(\mathbf{k},\omega)\} &= -\frac{1}{2}\sum_s \pi \frac{\mathcal{V}}{(2\pi)^2} \int_0^{+\infty} \mathrm{d}k_R\, k_R \int_{-\infty}^{+\infty} \mathrm{d}k_z\; \delta(\omega - w_z k_z - s|v|\sqrt{(k_R - Q)^2 + k_z^2}) \\ &= -\frac{1}{2}\sum_s \pi \frac{\mathcal{V}}{(2\pi)^2} Q \int_{-Q}^{+\infty} \mathrm{d}q_R \int_{-\infty}^{+\infty} \mathrm{d}k_z\; \delta(\omega - w_z k_z - s|v|\sqrt{q_R^2 + k_z^2}),\end{aligned} \tag{S2}$$

with $q_R \equiv k_R - Q$.

In the weak tilt limit ($|w_z/v| \ll 1$) close to the nodal-ring energy ($|\omega/v| \ll Q$) and introducing polar coordinates with radius $k = \sqrt{q_R^2 + k_z^2}$ and polar angle θ ($k_z = k\cos\theta$), the solution k^* of the Dirac-delta function is given by:

$$k^* = \omega/(s|v| + w_z\cos\theta) \approx s\omega/|v| - w_z\omega\cos\theta/v^2. \tag{S3}$$

Plugging in this solution, we obtain:

$$\sum_{\mathbf{k}} \mathrm{Im}\{\mathcal{G}_0^{(0)}(\mathbf{k},\omega)\} = -\pi \frac{\mathcal{V}}{(2\pi)^2} \frac{Q}{2} \int_0^{2\pi} \mathrm{d}\theta\; \frac{|\omega|}{(|v| + s w_z\cos\theta)^2} \approx -\mathcal{V}\frac{Q|\omega|}{4v^2}. \tag{S4}$$

Similarly, we retrieve for $\mathrm{Im}\{\mathcal{G}_3^{(0)}(\mathbf{k},\omega)\}$:

$$\begin{aligned}\sum_{\mathbf{k}} \mathrm{Im}\{\mathcal{G}_3^{(0)}(\mathbf{k},\omega)\} &= -\frac{1}{2}\sum_s s\pi \frac{\mathcal{V}}{(2\pi)^2} \int_{-Q}^{+\infty} \mathrm{d}q_R\; (q_R + Q) \int_{-\infty}^{+\infty} \mathrm{d}k_z\; \frac{v k_z}{|v|\sqrt{q_R^2 + k_z^2}}\, \delta(\omega - w_z k_z - s|v|\sqrt{q_R^2 + k_z^2}) \\ &= -\frac{1}{2}\sum_s s\pi\vartheta(s\omega) \frac{\mathcal{V}}{(2\pi)^2} \frac{vQ|\omega|}{|v|} \int_0^{2\pi} \mathrm{d}\theta\; \frac{\cos\theta}{(|v| + s w_z\cos\theta)^2} \approx \mathcal{V}\frac{Q w_z|\omega|}{4v^3} \\ &= -\frac{w_z}{v}\sum_{\mathbf{k}} \mathrm{Im}\{\mathcal{G}_0^{(0)}(\mathbf{k},\omega)\}.\end{aligned} \tag{S5}$$

Furthermore, we obtain in a similar manner that $\mathrm{Im}\{\mathcal{G}_1^{(0)}(\mathbf{k},\omega)\} \propto \omega^2$ is negligible with respect to the other contributions, while $\mathcal{G}_2^{(0)} = 0$ by definition.

For the real part of the Green function, we get the following contributions:

$$\begin{aligned}\sum_{\mathbf{k}} \mathrm{Re}\{\mathcal{G}_0^{(0)}(\mathbf{k},\omega)\} &= \frac{1}{2}\sum_s \frac{\mathcal{V}}{(2\pi)^2}\mathcal{P} \int_{-q_R}^{+\infty} \mathrm{d}q_R\; (q_R + Q) \int_{-\infty}^{+\infty} \mathrm{d}k_z\; \frac{1}{\omega - w_z k_z - s|v|\sqrt{q_R^2 + k_z^2}} \\ &\to \frac{1}{2}\sum_s \frac{\mathcal{V}}{(2\pi)^2}\mathcal{P}\, Q \int_0^{2\pi} \mathrm{d}\theta \int_0^{k_{\mathrm{C}}^R} \mathrm{d}k\; \frac{k}{\omega - (s|v| + w_z\cos\theta)k} \approx -\mathcal{V}\frac{Q\omega}{2\pi v^2}\ln\left|\frac{v k_{\mathrm{C}}}{\omega}\right|,\end{aligned} \tag{S6}$$

where the arrow indicates the introduction of a cutoff wavevector k_{C} and we only keep the terms up to leading order in k_{C}. The cutoff wave vector represents the maximal distance to the nodal ring for the integration over $\mathbf{k}$ in both the radial and the axial directions. The energy scale of ω and the cutoff wave vector are assumed to obey the following constraints: $|\omega/v| \ll k_{\mathrm{C}} < Q$. Similarly, we obtain:

$$\begin{aligned}\sum_{\mathbf{k}} \mathrm{Re}\{\mathcal{G}_1^{(0)}(\mathbf{k},\omega)\} &= \frac{1}{2}\sum_s s\frac{\mathcal{V}}{(2\pi)^2}\mathcal{P}\int_{-Q}^{+\infty}\mathrm{d}q_R\,(q_R+Q)\int_{-\infty}^{+\infty}\mathrm{d}k_z\,\frac{vq_R}{|v|\sqrt{q_R^2+k_z^2}}\,\frac{1}{\omega-w_zk_z-s|v|\sqrt{q_R^2+k_z^2}} \\ &\to \frac{1}{2}\sum_s s\frac{\mathcal{V}}{(2\pi)^2}\frac{2v}{|v|}\mathcal{P}\int_0^{\pi}\mathrm{d}\theta\,\sin^2\theta\int_0^{k_{\mathrm{C}}}\mathrm{d}k\,\frac{k^2}{\omega-(s|v|+w_z\cos\theta)k} \approx -\mathcal{V}\frac{k_{\mathrm{C}}^2}{8\pi v},\end{aligned} \tag{S7}$$

having considered 4D spherical coordinates on the last line, and

$$\begin{aligned}\sum_{\mathbf{k}} \mathrm{Re}\{\mathcal{G}_3^{(0)}(\mathbf{k},\omega)\} &\to \frac{1}{2}\sum_s s\frac{\mathcal{V}}{(2\pi)^2}\mathcal{P}\,\frac{vQ}{|v|}\int_0^{2\pi}\mathrm{d}\theta\,\cos\theta\int_0^{k_{\mathrm{C}}}\mathrm{d}k\,\frac{k}{\omega-(s|v|+w_z\cos\theta)k} \\ &\approx \mathcal{V}\frac{w_zQ\omega}{2\pi v^3}\ln\left|\frac{vk_{\mathrm{C}}}{\omega}\right| = -\frac{w_z}{v}\sum_{\mathbf{k}}\mathrm{Re}\{\Sigma_0^{(\mathrm{FB})}(\omega)\}.\end{aligned} \tag{S8}$$

Combining all these results yields Eq. (9). Note that the angular dependence of the integrands in Eqs. (S6)-(S8) is different. The σ_0-term diverges in all directions, whereas the $\sigma_{3(1)}$ term only diverges away from the radial plane (axial direction). If an anisotropic cutoff is considered, the connection between Eq. (S6) and Eq. (S8) (last equality) would not hold for example. The crucial aspects for the phenomenology of exceptional lines and bulk Fermi ribbons, however, is the proportionality of self-energy terms to the Pauli matrices and their dependency on ω and they do not depend on the details of the cutoff implementation. Similar remarks can be made for the self-energy terms in the case of strong tilt.

In case of very strong tilt $|w_z/v| \gg 1$, the Dirac-delta function in Eq. (S2) has solutions within the integration boundaries when

$$-1 \le \frac{\omega - s|v|k}{w_zk} \le 1 \quad \Rightarrow \quad k \ge \frac{-s|v|\omega + |w_z\omega|}{w_z^2 - v^2} \equiv k_{\mathrm{min}}. \tag{S9}$$

This implies that we have to introduce a cutoff wavevector for the imaginary part as well, unlike in the case of weak tilt. Integrating over θ first and assuming an isotropic cutoff, we obtain:

$$\begin{aligned}\sum_{\mathbf{k}} \mathrm{Im}\{\mathcal{G}_0^{(0)}(\mathbf{k},\omega)\} &\to -\frac{1}{2}\sum_s \pi\frac{\mathcal{V}}{(2\pi)^2}2Q\int_{k_{\mathrm{min}}}^{k_{\mathrm{C}}}\mathrm{d}k\,\frac{k}{\sqrt{(w_zk)^2-(\omega-s|v|k)^2}} \approx -\mathcal{V}\frac{Qk_{\mathrm{C}}}{2\pi|w_z|}, \\ \sum_{\mathbf{k}} \mathrm{Im}\{\mathcal{G}_1^{(0)}(\mathbf{k},\omega)\} &\to -\frac{1}{2}\sum_s s\pi\frac{\mathcal{V}}{(2\pi)^2}\frac{2v}{|v|}\int_0^{\pi}\mathrm{d}\theta\,\sin^2\theta\int_0^{k_{\mathrm{C}}}\mathrm{d}k\,k^2\,\delta(\omega-w_zk\cos\theta-s|v|k) \\ &\approx -\frac{1}{2}\sum_s \pi\frac{\mathcal{V}}{(2\pi)^2}\frac{2v\omega}{w_z^2}\int_{k_{\mathrm{min}}}^{k_{\mathrm{C}}}\mathrm{d}k\,\frac{k}{\sqrt{(w_zk)^2-\omega^2}} \approx -\mathcal{V}\frac{vk_{\mathrm{C}}\omega}{2\pi|w_z|^3}, \\ \sum_{\mathbf{k}} \mathrm{Im}\{\mathcal{G}_3^{(0)}(\mathbf{k},\omega)\} &\to -\frac{1}{2}\sum_s s\pi\frac{\mathcal{V}}{(2\pi)^2}\frac{vQ}{|v|}\int_0^{2\pi}\mathrm{d}\theta\,\cos\theta\int_0^{k_{\mathrm{C}}}\mathrm{d}k\,k\,\delta(\omega-w_zk\cos\theta-s|v|k) \\ &= -\frac{1}{2}\sum_s s\pi\frac{\mathcal{V}}{(2\pi)^2}\frac{2vQ}{|v|w_z}\int_{k_{\mathrm{min}}}^{k_{\mathrm{C}}}\mathrm{d}k\,\frac{\omega-s|v|k}{\sqrt{(w_zk)^2-(\omega-s|v|k)^2}} \approx \mathcal{V}\frac{vQk_{\mathrm{C}}}{2\pi w_z|w_z|} \\ &= -\frac{v}{w_z}\sum_{\mathbf{k}}\mathrm{Im}\{\mathcal{G}_0^{(0)}(\mathbf{k},\omega)\}.\end{aligned} \tag{S10}$$

The summation over the real part of the Green function yields the following components:

$$\begin{aligned}
\sum_{\mathbf{k}} \mathrm{Re}\{\mathcal{G}_0^{(0)}(\mathbf{k},\omega)\} &\to \frac{1}{2}\sum_s \frac{\mathcal{V}}{(2\pi)^2}\, 4Q\,\mathcal{P}\int_0^{k_\mathrm{C}} \mathrm{d}k \int_0^1 \mathrm{d}x\, \frac{1}{\sqrt{1-x^2}} \frac{k(\omega - s|v|k)}{(\omega - s|v|k)^2 - (w_z k)^2 x^2} \\
&= \frac{1}{2}\sum_s \frac{\mathcal{V}}{(2\pi)^2}\, 2\pi Q \int_0^{k_\mathrm{max}} \mathrm{d}k\, \frac{k(\omega - s|v|k)}{|\omega - s|v|k|\sqrt{(\omega - s|v|k)^2 - (w_z k)^2}} \\
&\approx \frac{\mathcal{V}}{(2\pi)^2} \frac{2\pi Q\omega}{|\omega|} \int_0^{|\omega/w_z|} \mathrm{d}k\, \frac{k}{\sqrt{\omega^2 - w_z^2 k^2}} = \mathcal{V}\frac{Q\omega}{2\pi w_z^2},
\end{aligned} \tag{S11}$$

with $x \equiv \cos\theta$, $k_\mathrm{max} \equiv k_\mathrm{min}$ from Eq. (S9), and where we have made use of the following identity:

$$\int_0^1 \mathrm{d}x\, \frac{1}{\sqrt{1-x^2}} \frac{1}{a^2 - b^2x^2} = \begin{cases} -i\pi/(2|a|\sqrt{b^2-a^2}) & (|b| > |a| > 0) \\ \pi/(2|a|\sqrt{a^2-b^2}) & (|a| > |b| > 0) \end{cases}, \tag{S12}$$

for $a, b \in \mathbb{R}$, and

$$\begin{aligned}
\sum_{\mathbf{k}} \mathrm{Re}\{\mathcal{G}_1^{(0)}(\mathbf{k},\omega)\} &\to \frac{1}{2}\sum_s s\frac{\mathcal{V}}{(2\pi)^2} \frac{4vw_z}{|v|}\mathcal{P}\int_0^{k_\mathrm{C}} \mathrm{d}k \int_0^1 \mathrm{d}x\, \frac{k^3\sqrt{1-x^2}x}{(\omega - s|v|k)^2 - (w_z k)^2 x^2} \\
&\approx -\frac{1}{2}\sum_s s\frac{\mathcal{V}}{(2\pi)^2} \frac{4vw_z}{|vw_z^3|} \int_{k_\mathrm{min}}^{k_\mathrm{C}} \mathrm{d}k\, \sqrt{(w_z k)^2 - (\omega - s|v|k)^2}\, \mathrm{arccosh}(|w_z k|/|\omega - s|v|k|) \\
&\approx -\mathcal{V}\frac{vk_\mathrm{C}\omega}{\pi^2 w_z^3}\, \mathrm{arccosh}(|w_z/v|),
\end{aligned} \tag{S13}$$

where we have made use of the following identity:

$$\int_0^1 \mathrm{d}x\, \frac{\sqrt{1-x^2}x}{a^2 - b^2x^2} = \begin{cases} -i\pi\sqrt{b^2-a^2}/(2|b|^3) + 1/b^2 - \sqrt{b^2-a^2}\,\mathrm{arccosh}(|b/a|)/|b|^3 & (|b| > |a| > 0) \\ 1/b^2 - \sqrt{a^2-b^2}\,\arcsin(|b/a|)/|b|^3 & (|a| > |b| > 0) \end{cases}, \tag{S14}$$

and

$$\begin{aligned}
\sum_{\mathbf{k}} \mathrm{Re}\{\mathcal{G}_3^{(0)}(\mathbf{k},\omega)\} &\to \frac{1}{2}\sum_s s\frac{\mathcal{V}}{(2\pi)^2} \frac{4vw_zQ}{|v|}\mathcal{P}\int_0^{k_\mathrm{C}} \mathrm{d}k \int_0^1 \mathrm{d}x\, \frac{1}{\sqrt{1-x^2}} \frac{k^2x^2}{(\omega - s|v|k)^2 - (w_z k)^2 x^2} \\
&= \frac{1}{2}\sum_s s\frac{\mathcal{V}}{(2\pi)^2} \frac{4\pi vQ}{2|v|w_z}\mathcal{P}\int_0^{k_\mathrm{max}} \mathrm{d}k\, \frac{|\omega - s|v|k|}{\sqrt{(\omega - s|v|k)^2 - (w_z k)^2}} \\
&\approx -\frac{\mathcal{V}}{(2\pi)^2} \frac{2\pi vQ\omega}{w_z|\omega|} \int_0^{|\omega/w_z|} \mathrm{d}k\, \frac{k}{\sqrt{\omega^2 - w_z^2k^2}} = -\mathcal{V}\frac{vQ\omega}{2\pi w_z^3} = -\frac{v}{w_z}\sum_{\mathbf{k}} \mathrm{Re}\{\mathcal{G}_0^{(0)}(\mathbf{k},\omega)\},
\end{aligned} \tag{S15}$$

where we have made use of the following identity:

$$\int_0^1 \mathrm{d}x\, \frac{1}{\sqrt{1-x^2}} \frac{x^2}{a^2 - b^2x^2} = \begin{cases} -\pi[1 + i|a|/\sqrt{b^2-a^2}]/(2b^2) & (|b| > |a| > 0) \\ -\pi[1 - |a|/\sqrt{a^2-b^2}]/(2b^2) & (|a| > |b| > 0) \end{cases}. \tag{S16}$$

Combining all these results yields Eq. (10).

It is a straightforward calculation to verify that the energy-independent imaginary contribution to the summation over the Green function in case of very strong radial tilt ($|w_R/v| \gg 1$) is proportional to σ_1 rather than σ_3 in case of strong axial tilt. Following the same procedure as for axial tilt, we get:

$$
\begin{aligned}
\sum_{\mathbf{k}} \mathrm{Im}\{\mathcal{G}_0^{(0)}(\mathbf{k},\omega)\} &= \frac{1}{2}\sum_s \pi \frac{\mathcal{V}}{(2\pi)^2} \int\limits_{-Q}^{+\infty} \mathrm{d}q_R\ (q_R+Q) \int\limits_{-\infty}^{+\infty} \mathrm{d}k_z\ \delta(\omega - w_R q_R - s|v|\sqrt{q_R^2+k_z^2}) \\
&= \underbrace{-\frac{1}{2}\sum_s \pi \frac{\mathcal{V}}{(2\pi)^2} \int\limits_0^{\pi} \mathrm{d}\theta\ \sin\theta \int\limits_0^{+\infty} \mathrm{d}k\ k^2\, \delta(\omega - w_R k \sin\theta - s|v|k)}_{(\mathrm{A})} - \underbrace{(w_R \leftrightarrow -w_R)}_{(\mathrm{B})} \\
&\quad + \underbrace{-\frac{1}{2}\sum_s \pi \frac{\mathcal{V}}{(2\pi)^2} Q \int\limits_0^{2\pi} \mathrm{d}\theta \int\limits_0^{+\infty} \mathrm{d}k\ k\, \delta(\omega - w_R k \sin\theta - s|v|k)}_{(\mathrm{C})},
\end{aligned} \tag{S17}
$$

with the terms evaluating as follows:

$$
\begin{aligned}
(\mathrm{A}) &\to -\frac{1}{2}\sum_s \pi \frac{\mathcal{V}}{(2\pi)^2} \frac{2}{w_R} \int\limits_{k_{\min}}^{k_{\mathrm{C}}} \mathrm{d}k\ \frac{k(\omega - s|v|k)}{\sqrt{(w_R k)^2 - (\omega - s|v|k)}} \approx -\mathcal{V}\frac{k_{\mathrm{C}}\omega}{2\pi w_R |w_R|}, \\
(\mathrm{C}) &\to -\frac{1}{2}\sum_s \pi \frac{\mathcal{V}}{(2\pi)^2} 4Q \int\limits_{k_{\min}}^{k_{\mathrm{C}}} \mathrm{d}k\ \frac{k}{\sqrt{(w_R k)^2 - (\omega - s|v|k)}} \approx -\mathcal{V}\frac{Q k_{\mathrm{C}}}{\pi |w_R|},
\end{aligned} \tag{S18}
$$

resulting in

$$
\sum_{\mathbf{k}} \mathrm{Im}\{\mathcal{G}_0^{(0)}(\mathbf{k},\omega)\} \approx -\mathcal{V}\frac{Q k_{\mathrm{C}}}{\pi |w_R|}. \tag{S19}
$$

Similarly, we obtain:

$$
\begin{aligned}
\sum_{\mathbf{k}} \mathrm{Im}\{\mathcal{G}_1^{(0)}(\mathbf{k},\omega)\} &= -\frac{1}{2}\sum_s s\pi \frac{\mathcal{V}}{(2\pi)^2} \frac{v}{|v|} \int\limits_{-Q}^{+\infty} \mathrm{d}q_R\ (q_R+Q) \int\limits_{-\infty}^{+\infty} \mathrm{d}k_z\ \frac{q_R}{\sqrt{q_R^2+k_z^2}} \delta(\omega - w_R q_R - s|v|\sqrt{q_R^2+k_z^2}) \\
&= \underbrace{-\frac{1}{2}\sum_s s\pi \frac{\mathcal{V}}{(2\pi)^2} \frac{v}{|v|} \int\limits_0^{\pi} \mathrm{d}\theta\ \sin^2\theta \int\limits_0^{+\infty} \mathrm{d}k\ k^2\, \delta\left(\omega - w_R k \sin\theta - s|v|k\right)}_{(\mathrm{D})} + \underbrace{(w_R \leftrightarrow -w_R)}_{(\mathrm{E})} \\
&\quad + \underbrace{-\frac{1}{2}\sum_s s\pi \frac{\mathcal{V}}{(2\pi)^2} \frac{vQ}{|v|} \int\limits_0^{\pi} \mathrm{d}\theta\ \sin\theta \int\limits_0^{+\infty} \mathrm{d}k\ k\, \delta\left(\omega - w_R k \sin\theta - s|v|k\right)}_{(\mathrm{F})} - \underbrace{(w_R \leftrightarrow -w_R)}_{(\mathrm{G})},
\end{aligned} \tag{S20}
$$

with

$$
\begin{aligned}
(\mathrm{D}) &\to -\frac{1}{2}\sum_s s\pi \frac{\mathcal{V}}{(2\pi)^2} \frac{2v}{|v| w_R^2} \int\limits_{k_{\min}}^{k_{\mathrm{C}}} \mathrm{d}k\ \frac{(\omega - s|v|k)^2}{\sqrt{(w_R k)^2 - (\omega - s|v|k)^2}} \approx \mathcal{V}\frac{v k_{\mathrm{C}}\omega}{\pi |w_R|^3}, \\
(\mathrm{F}) &\to -\frac{1}{2}\sum_s s\pi \frac{\mathcal{V}}{(2\pi)^2} \frac{2vQ}{|v| w_R} \int\limits_{k_{\min}}^{k_{\mathrm{C}}} \mathrm{d}k\ \frac{\omega - s|v|k}{\sqrt{(w_R k)^2 - (\omega - s|v|k)^2}} \approx \mathcal{V}\frac{v Q k_{\mathrm{C}}}{2\pi w_R |w_R|},
\end{aligned} \tag{S21}
$$

leading to

$$\sum_{\mathbf{k}} \mathrm{Im}\{\mathcal{G}_1^{(0)}(\mathbf{k},\omega)\} \approx \mathcal{V}\frac{vQk_{\mathrm{C}}}{\pi w_R|w_R|} = -\frac{v}{w_z}\sum_{\mathbf{k}} \mathrm{Im}\{\mathcal{G}_0^{(0)}(\mathbf{k},\omega)\}, \tag{S22}$$

and finally

$$\sum_{\mathbf{k}} \mathrm{Im}\{\mathcal{G}_3^{(0)}(\mathbf{k},\omega)\} = -\frac{1}{2}s\pi\frac{\mathcal{V}}{(2\pi)^2}\frac{v}{|v|}\int_{-Q}^{+\infty}\mathrm{d}q_R\,(q_R+Q)\int_{-\infty}^{+\infty}\mathrm{d}k_z\,\frac{k_z}{\sqrt{q_R^2+k_z^2}}\delta(\omega - w_R q_R - s|v|\sqrt{q_R^2+k_z^2}) = 0. \tag{S23}$$

SII. SELF-CONSISTENT BORN APPROXIMATION

The SB approximation, considering a scalar disorder potential ($S_{1,2,3}=0$) and energies close to the nodal loop ($\omega \to 0$), leads to the following self-consistency equation for the retarded self-energy $\Sigma^{(\mathrm{SB})} \equiv \lim_{\omega\to 0}\Sigma^{(\mathrm{SB})}(\omega)$:

$$\Sigma^{(\mathrm{SB})} = \frac{\gamma}{(2\pi)^3}\int \mathrm{d}^3k\,\frac{1}{\omega - H(\mathbf{k})/\hbar - \Sigma^{(\mathrm{SB})}}. \tag{S24}$$

We consider the following Hamiltonian for a NLSM with nodal ring with radius Q in the $k_z=0$ plane, positive velocities $v_R=v_z=v>0$ and positive axial tilt ($w_R=0$, $w_z>0$):

$$H(\mathbf{k})/\hbar = w_z k_z\,\sigma_0 + (vQ/2)[(k_R/Q)^2-1]\,\sigma_1 + vk_z\,\sigma_3. \tag{S25}$$

Note that this Hamiltonian has a quadratic term, which vanishes close to the nodal loop and regularizes the nonphysical cone tops of the linearized model, present in Eq. (1). Solving for $\Sigma^{(\mathrm{SB})}$ in Eq. (S24), we get:

$$\begin{aligned}
\Sigma_0^{(\mathrm{SB})} &= \frac{\gamma}{2\pi}\left[\left(-\frac{\omega - \mathrm{Re}\{\Sigma_0^{(\mathrm{SB})}\} + c_z\mathrm{Re}\{\Sigma_3^{(\mathrm{SB})}\}}{c_z^2-1} - i\,\mathrm{Im}\{\Sigma_0^{(\mathrm{SB})}\}\right)\int \mathrm{d}k\;J - c_z\int \mathrm{d}k\;vkJ\right],\\
\Sigma_3^{(\mathrm{SB})} &= \frac{\gamma}{2\pi}\left[\left(c_z\frac{\omega - \mathrm{Re}\{\Sigma_0^{(\mathrm{SB})}\} + c_z\mathrm{Re}\{\Sigma_3^{(\mathrm{SB})}\}}{c_z^2-1} + i\,\mathrm{Im}\{\Sigma_3^{(\mathrm{SB})}\}\right)\int \mathrm{d}k\;J + \int \mathrm{d}k\;vkJ\right],
\end{aligned} \tag{S26}$$

with $c_z \equiv w_z/v$,

$$\begin{aligned}
J &\equiv \frac{1}{2\pi}\int \mathrm{d}k_R\;k_R\,\det(k_R^2/Q^2)^{-1},\\
\det(x) &\equiv (\omega - w_z k_z - \Sigma_0^{(\mathrm{SB})})^2 - (vQ/2)^2(x-1)^2 - (vk_z + \Sigma_3^{(\mathrm{SB})})^2,
\end{aligned} \tag{S27}$$

and $k \equiv k_z + q$, with:

$$q \equiv \frac{1}{v}\frac{c_z(\omega - \mathrm{Re}\{\Sigma_0^{(\mathrm{SB})}\}) + \mathrm{Re}\{\Sigma_3^{(\mathrm{SB})}\}}{c_z^2-1}. \tag{S28}$$

For $c_z > 1$, we get the following solutions for the integration over J and vkJ, respectively:

$$\begin{aligned}
-I_1 &\equiv \frac{\gamma}{2\pi}\int \mathrm{d}k\;J \approx -\frac{\gamma Q}{4v^2\sqrt{c_z^2-1}},\\
iI_2 &\equiv \frac{\gamma}{2\pi}\int \mathrm{d}k\;vkJ \approx i\frac{\gamma Q k_{\mathrm{C}}}{2\pi v\sqrt{c_z^2-1}}\mathrm{sgn}(-c_z\mathrm{Im}\{\Sigma_0^{(\mathrm{SB})}\} + \mathrm{Im}\{\Sigma_3^{(\mathrm{SB})}\}),
\end{aligned} \tag{S29}$$

with cutoff wave vector k_{C} for the integration over k along the axial direction (note that this cutoff implementation differs from that of the FB approximation, but that it does not affect the phenomenology of the quasiparticle spectrum). The newly introduced integration constants, I_1 and I_2, are approximately real, such that the self-energy becomes:

$$\begin{aligned}
\Sigma_0^{(\mathrm{SB})} &= \frac{I_1}{c_z^2(1+I_1)+I_1-1}\omega - i\frac{c_z I_2}{1-I_1},\\
\Sigma_3^{(\mathrm{SB})} &= -\frac{c_z I_1}{c_z^2(1+I_1)+I_1-1}\omega + i\frac{I_2}{1+I_1} = -c_z\mathrm{Re}\{\Sigma_0^{(\mathrm{SB})}\} - i\,\mathrm{Im}\{\Sigma_0^{(\mathrm{SB})}\}/c_z.
\end{aligned} \tag{S30}$$

In the limit $c_z \gg 1$, $I_{1,2} \ll 1$ we obtain:

$$\begin{aligned}\Sigma_0^{(\mathrm{SB})} &\approx i\,\gamma \frac{Qk_\mathrm{C}}{2\pi w_z},\\ \Sigma_3^{(\mathrm{SB})} &\approx -i\,\gamma \frac{vQk_\mathrm{C}}{2\pi w_z^2},\end{aligned} \tag{S31}$$

recovering the result of Eq. (10) that is responsible for the appearance of a bulk Fermi ribbon.

For $c_z < 1$, the integrations over J and vkJ yield:

$$\begin{aligned}\frac{\gamma}{2\pi}\int \mathrm{d}k\; J &\approx -\frac{\gamma Q}{2\pi v^2\sqrt{1-c_z^2}} \ln\left|\frac{k_\mathrm{C}}{q}\right| - i\frac{\gamma Q\,\mathrm{sgn}(\Gamma)}{4v^2\sqrt{1-c_z^2}} \equiv -H_1 - iH_2,\\ \frac{\gamma}{2\pi}\int \mathrm{d}k\; vkJ &\approx -\frac{\gamma Q[c_z(\omega - \mathrm{Re}\{\Sigma_0^{(\mathrm{SB})}\}) + \mathrm{Re}\{\Sigma_3^{(\mathrm{SB})}\}]}{2\pi v^2(1-c_z^2)^{3/2}} \equiv -H_3[c_z(\omega - \mathrm{Re}\{\Sigma_0^{(\mathrm{SB})}\}) + \mathrm{Re}\{\Sigma_3^{(\mathrm{SB})}\}],\end{aligned} \tag{S32}$$

with:

$$\Gamma \equiv (\omega - \mathrm{Re}\{\Sigma_0^{(\mathrm{SB})}\} + c_z\mathrm{Re}\{\Sigma_3^{(\mathrm{SB})}\})(-\mathrm{Im}\{\Sigma_0^{(\mathrm{SB})}\} + c_z\mathrm{Im}\{\Sigma_3^{(\mathrm{SB})}\}). \tag{S33}$$

In the limit $c_z \ll 1$, $H_{1,2,3} \ll 1$, we obtain:

$$\begin{aligned}\Sigma_0^{(\mathrm{SB})} &\approx -H_1\omega - i\,H_2\omega \approx -\frac{\gamma Q\omega}{2\pi v^2}\ln\left|\frac{k_\mathrm{C}}{q}\right| - i\,\frac{\gamma Q|\omega|}{4v^2},\\ \Sigma_3^{(\mathrm{SB})} &\approx c_z H_1\omega + i\,c_z H_2\omega = -c_z\Sigma_0^{(\mathrm{SB})}.\end{aligned} \tag{S34}$$

Up to a cutoff-independent constant in the real part, this is in exact agreement with the result based on the FB approximation in Eq. (9). Note that we have neglected the $\propto \sigma_1$ self-energy correction that leads to a renormalization of the nodal-ring radius in this section, as it is irrelevant for the appearance of a bulk Fermi ribbon.

SIII. HIGHER-ORDER CORRECTIONS

In Fig. 2, we have plotted the complex quasiparticle spectrum for strong axial tilt while neglecting the $\propto \omega$ terms in the self-energy, resulting in a flat bulk Fermi ribbon with a sharp square-root singularity at its edges. We can also solve for the spectrum while taking into account the higher-order corrections. From Eq. (10), we can write the self-energy as follows:

$$\Sigma(\omega) = (\Theta\omega - i/\tau)(\sigma_0 - \sigma_3/c_z) + (\Xi\omega + i\Phi\omega)\,\sigma_1, \tag{S35}$$

with real parameters $\Theta, 1/\tau, \Xi$ and Φ, all being proportional to γ. Plugging this into Eq. (7) and solving for $E_s(\mathbf{k})$ with Eq. (8), yields:

$$\begin{aligned}E_s(\mathbf{k})/\hbar &= \frac{g(k_R,k_z) + s\sqrt{g(k_R,k_z)^2 + fh(k_R,k_z)}}{f}, \qquad f \equiv (1-\Theta)^2 - \Theta^2/c_z^2 - (\Xi + i\,\Phi)^2,\\ g(k_R,k_z) &\equiv [1-(1-1/c_z^2)\Theta](w_zk_z - i/\tau) + v(k_R - Q)(\Xi + i\Phi),\\ h(k_R,k_z) &\equiv v^2(k_R-Q)^2 - (1-1/c_z^2)(w_zk_z - i/\tau)^2.\end{aligned} \tag{S36}$$

A flat bulk Fermi ribbon is realized for values of k_R and k_z that satisfy:

$$g(k_R,k_z) = C, \qquad 0 > g(k_R,k_z)^2 + fh(k_R,k_z) \in \mathbb{R}, \tag{S37}$$

with constant C. Only when neglecting Θ, Ξ and Φ is this realized by $k_z = 0$ and $|k_R - Q| < 1/|w_z\tau|$. In general, these conditions cannot be met. In Fig. S1, the exact spectrum, according to Eq. (S36) is presented for the same parameters as considered in Fig. S1, with the flat bulk Fermi ribbon and square-root singularities only approximately realized.

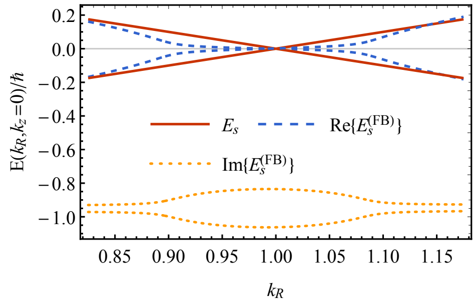

FIG. S1. The spectrum of a type-II nodal-line semimetal with strong axial tilt (along k_z) and disorder, according to the solution of Eq. (S36), evaluated close to the nodal ring with $k_z = 0$. The same parameter set as in Fig. 2 has been considered.