On Rational Entailment for Propositional Typicality Logic*

Richard Booth[1], Giovanni Casini[2], Thomas Meyer[3], and Ivan Varzinczak[4]

[1]Cardiff University, United Kingdom. Email: *BoothR2@cardiff.ac.uk*
[2]CSC, University of Luxembourg, Luxembourg. Email: *giovanni.casini@uni.lu*
[3]University of Cape Town and CAIR, South Africa. Email: *tmeyer@cs.uct.ac.za*
[4]CRIL, Univ. Artois & CNRS, France. Email: *varzinczak@cril.fr*

Abstract

Propositional Typicality Logic (PTL) is a recently proposed logic, obtained by enriching classical propositional logic with a typicality operator capturing the most typical (alias normal or conventional) situations in which a given sentence holds. The semantics of PTL is in terms of ranked models as studied in the well-known KLM approach to preferential reasoning and therefore KLM-style rational consequence relations can be embedded in PTL. In spite of the non-monotonic features introduced by the semantics adopted for the typicality operator, the obvious Tarskian definition of entailment for PTL remains monotonic and is therefore not appropriate in many contexts. Our first important result is an impossibility theorem showing that a set of proposed postulates that at first all seem appropriate for a notion of entailment with regard to typicality cannot be satisfied simultaneously. Closer inspection reveals that this result is best interpreted as an argument for advocating the development of more than one type of PTL entailment. In the spirit of this interpretation, we investigate three different (semantic) versions of entailment for PTL, each one based on the definition of rational closure as introduced by Lehmann and Magidor for KLM-style conditionals, and constructed using different notions of minimality.

1 Introduction

Propositional Typicality Logic (PTL) [1, 2] is a recently proposed logic allowing for the representation of and reasoning with an explicit notion of *typicality*. It is obtained by enriching classical propositional logic with a *typicality operator* $\bullet$,

*This technical report presents an extended and elaborated version of a paper presented at the 24th International Joint Conference on Artificial Intelligence (IJCAI 2015).

the intuition of which is to refer to those most typical (or normal or conventional) situations in which a given sentence holds. PTL is characterised using a preferential semantics similar to that originally proposed by Shoham [25] and extensively developed by Kraus et al. [20] and Lehmann and Magidor [22] in the propositional case, and by others [4, 7, 8, 18, 24, 14, 15] in more expressive languages.

In spite of the non-monotonic features introduced by the adoption of a preferential semantics for $\bullet$, the obvious definition of entailment for PTL, i.e., the one based on a Tarskian notion of logical consequence, remains monotonic. Of course, such a notion of entailment is inappropriate in non-monotonic contexts, in particular when reasoning about typicality, as is already clear from an enriched version of the classical Tweety example: If birds typically fly, and penguins are birds (and that is all we know), we would expect to be able to conclude that typical penguins are typical birds, and therefore that typical penguins fly. Learning that penguins typically do not fly should lead us to conclude that penguins are not typical birds, and to retract the conclusions about typical penguins being typical birds, and about typical penguins flying.

In this paper, we investigate three semantic versions of entailment for PTL, constructed using three different forms of minimality. All these are based on the notion of rational closure as defined by Lehmann and Magidor [22] for KLM-style conditionals in a propositional setting. We show that they can be viewed as distinct extensions of rational closure, equivalent with respect to the conditional language originally proposed by Kraus et al., but different in the PTL framework.

We shall study the aforementioned forms of entailment in an abstract formal setting, obtained by proposing a set of postulates that, at first glance, seem appropriate for any notion of entailment with regard to typicality. Our first important result is a negative one, though. It is an impossibility result proving that the set of postulates cannot all be satisfied simultaneously. A more detailed analysis of the result shows that, instead of being viewed as negative, this result should rather be interpreted as an indication that PTL allows for different types of entailment, corresponding to different subsets of the full set of postulates we provide. In line with this argument, we define three types of entailment for PTL corresponding to distinct subsets of the postulates, referred to as *LM-entailment*, *PT-entailment*, and *PT'-entailment*, a modification of the latter. Our argument for more than one type of entailment for the same logic is in line with the proposal put forward by Lehmann in the context of entailment for conditional knowledge bases, where he proposes both *prototypical reasoning* and *presumptive reasoning* as acceptable forms of entailment [21]. The details of the distinct forms of entailment need not concern us here. Rather, what is important is the acknowledgement of the existence of more than one form of entailment for the same representational formalism.

The remainder of the present paper is structured as follows. Section 2 provides the background and notation for the rest of the work. In Section 3 we discuss the complexities surrounding a notion of entailment for PTL. In Section 4 we put forward our postulates and show the impossibility result. In Section 5 we define LM-entailment while Section 6 is devoted to the definition of PT-entailment, and Section 7 to the definition of PT'-entailment. Section 8 addresses the implications of the impossibility result, making the case for three forms of PTL entailment. Section 9 concludes and discusses future work.

2 Logical preliminaries

Let $\mathcal{P}$ be a finite set of propositional *atoms* with at least two elements.[1] We use $p, q, \ldots$ as meta-variables for atoms. Propositional sentences (and, in later sections, sentences of the richer language we shall introduce in Section 2.3 below) are denoted by $\alpha, \beta, \ldots$, and are recursively defined in the usual way: $\alpha ::= p \mid \neg\alpha \mid \alpha \land \alpha \mid \top \mid \bot$. All the other Boolean connectives ($\lor$, $\rightarrow$, $\leftrightarrow$, …) are defined in terms of $\neg$ and $\land$ in the standard way. With $\mathcal{L}$ we denote the set of all propositional sentences.

We denote by $\mathcal{U}$ the set of all propositional *valuations* $v : \mathcal{P} \longrightarrow \{0, 1\}$, i.e., $\mathcal{U} := \{0, 1\}^{\mathcal{P}}$. Whenever it eases the presentation, we shall represent valuations as sets of literals (i.e., atoms or negated atoms), with each literal indicating the truth-value of the respective atom. Thus, for the logic generated from $\mathcal{P} = \{p, q\}$, the valuation in which p is true and q is false will be represented as $\{p, \neg q\}$. Satisfaction of a sentence $\alpha \in \mathcal{L}$ by $v \in \mathcal{U}$ is defined in the usual truth-functional way and is denoted by $v \Vdash \alpha$.

2.1 KLM-style rational conditionals

In the conditional logic investigated by Kraus et al. [20], often referred to as the *KLM approach*, one is interested in (defeasible) conditionals of the form $\alpha \mid\!\sim \beta$, read as "typically, if α, then β" (or, depending on the example at hand, as "αs are typically βs" and variants thereof). For instance, if $\mathcal{P} = \{\mathsf{b}, \mathsf{f}, \mathsf{p}\}$, where b, f and p stand for, respectively, "being a bird", "being able to fly", and "being a penguin", the following are examples of defeasible conditionals: $\mathsf{b} \mid\!\sim \mathsf{f}$ (birds typically fly), $\mathsf{p} \land \mathsf{b} \mid\!\sim \neg\mathsf{f}$ (penguins that are birds typically do not fly).

Kraus et al. put forward the following list of properties that the conditional $\mid\!\sim$ ought to satisfy in order to be considered as appropriate in a nonmonotonic setting (these properties have been discussed at length in the nonmonotonic reasoning community and we shall not do so here):

$$(\text{Ref})\quad \alpha \mid\!\sim \alpha \qquad (\text{LLE})\quad \frac{\models \alpha \leftrightarrow \beta,\ \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma} \qquad (\text{And})\quad \frac{\alpha \mid\!\sim \beta,\ \alpha \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \land \gamma}$$

$$(\text{Or})\quad \frac{\alpha \mid\!\sim \gamma,\ \beta \mid\!\sim \gamma}{\alpha \lor \beta \mid\!\sim \gamma} \qquad (\text{RW})\quad \frac{\alpha \mid\!\sim \beta,\ \models \beta \rightarrow \gamma}{\alpha \mid\!\sim \gamma} \qquad (\text{CM})\quad \frac{\alpha \mid\!\sim \beta,\ \alpha \mid\!\sim \gamma}{\alpha \land \beta \mid\!\sim \gamma}$$

A conditional satisfying such properties is called a *preferential conditional*. We can require $\mid\!\sim$ to satisfy other properties as well, one of which is rational

[1] This (reasonable) assumption is needed for technical reasons.

monotonicity:

$$(\text{RM})\quad \frac{\alpha \mid\sim \gamma,\ \alpha \not\mid\sim \neg\beta}{\alpha \wedge \beta \mid\sim \gamma}$$

A preferential conditional also satisfying (RM) is called a *rational conditional*.

The semantics of KLM-style rational conditionals is given by structures called *ranked interpretations* [22]:

Definition 2.1 (Ranked interpretation) *A* ***ranked interpretation*** $\mathscr{R}$ *is a function from* $\mathcal{U}$ *to* $\mathbb{N} \cup \{\infty\}$ *satisfying the following* convexity *property: for every* $i \in \mathbb{N}$*, if* $\mathscr{R}(v) = i$*, then, for every* j *such that* $0 \leq j < i$*, there is a* $v' \in \mathcal{U}$ *for which* $\mathscr{R}(v') = j$*.*

Observe that $\mathscr{R}$ generates a modular order $\prec_{\mathscr{R}}$ on $\mathcal{U}$ as follows: $u \prec_{\mathscr{R}} v$ if and only if $\mathscr{R}(u) < \mathscr{R}(v)$ (where $i < \infty$ for every $i \in \mathbb{N}$). If there is no ambiguity, we will omit the subscript and refer to the modular order as $\prec$.[2]

In a ranked interpretation $\mathscr{R}$ the intuition is that valuations lower down in the ordering are deemed more normal (or typical) than those higher up, with those with an infinite rank (a rank of ∞) being regarded as so atypical as to be impossible.

The *possible* valuations in $\mathscr{R}$ are defined as follows: $\mathcal{U}^{\mathscr{R}} := \{u \in \mathcal{U} \mid \mathscr{R}(u) < \infty\}$. Given $\alpha \in \mathcal{L}$, we let $[\![\alpha]\!]^{\mathscr{R}} := \{v \in \mathcal{U}^{\mathscr{R}} \mid v \Vdash \alpha\}$. Given $\alpha, \beta \in \mathcal{L}$, we say $\mathscr{R}$ satisfies (is a ranked model of) the conditional $\alpha \mid\sim \beta$ (denoted $\mathscr{R} \Vdash \alpha \mid\sim \beta$) if all the $\prec$-minimal α-valuations also satisfy β, i.e., if $\min_{\prec}[\![\alpha]\!]^{\mathscr{R}} \subseteq [\![\beta]\!]^{\mathscr{R}}$. We say $\mathscr{R}$ is a ranked model of a set of conditionals $\mathcal{C}$ if $\mathscr{R} \Vdash \alpha \mid\sim \beta$ for every $\alpha \mid\sim \beta \in \mathcal{C}$.

Sometimes it is convenient to represent a ranked interpretation $\mathscr{R}$ as a partition $(L_0, \ldots, L_{n-1}, L_\infty)$ of $\mathcal{U}$ where, for $i \in \mathbb{N} \cup \{\infty\}$, $L_i = \{u \in \mathcal{U} \mid \mathscr{R}(u) = i\}$ and where n is some $i \in \mathbb{N}$ for which $L_i = \emptyset$. That is, for each $i \in \{0, \ldots, n-1, \infty\}$, L_i is the set of all valuations of rank i. We refer to such a ranked interpretation as an *n-rank* interpretation.

Observe that the partition above has a finite number of cells, but includes the possibility for some of the L_is to be empty. This is necessary for two reasons. First, the cell L_∞ (the set of all impossible valuations) may be empty. Second, as we shall see below, this representation will often be used to *compare* ranked interpretations. In cases where such ranked interpretations do not have the same number of non-empty cells, this representation allows us to represent them as having the same (finite) number of cells, say $(L_0, \ldots, L_{n-1}, L_\infty)$ and $(M_0, \ldots, M_{n-1}, M_\infty)$, where n is the smallest integer such that $L_i = M_i = \emptyset$.

Figure 1 depicts an example of a ranked interpretation for $\mathcal{P} = \{\mathsf{b}, \mathsf{f}, \mathsf{p}\}$ satisfying both $\mathsf{b} \mid\sim \mathsf{f}$ and $\mathsf{p} \wedge \mathsf{b} \mid\sim \neg\mathsf{f}$. (In our graphical representations of the ranked interpretations we frequently omit the rank ∞.)

[2] Recall that, given a set X, $\prec \subseteq X \times X$ is modular if and only if there is a ranking function $rk : X \longrightarrow \mathbb{N}$ s.t. for every $x, y \in X$, $x \prec y$ if and only if $rk(x) < rk(y)$. Note also that modular orders can be obtained from total preorders by imposing anti-symmetry.

2	$\{\mathsf{b}, \mathsf{f}, \mathsf{p}\}$
1	$\{\mathsf{b}, \neg\mathsf{f}, \neg\mathsf{p}\}$, $\{\mathsf{b}, \neg\mathsf{f}, \mathsf{p}\}$
0	$\{\neg\mathsf{b}, \neg\mathsf{f}, \neg\mathsf{p}\}$, $\{\neg\mathsf{b}, \mathsf{f}, \neg\mathsf{p}\}$, $\{\mathsf{b}, \mathsf{f}, \neg\mathsf{p}\}$

Figure 1: A ranked interpretation for $\mathcal{P} = \{\mathsf{b}, \mathsf{f}, \mathsf{p}\}$.

For a better understanding of the reasons behind the aforementioned properties and the semantic constructions, the reader is referred to the work of Kraus et al. [20, 22].

2.2 Rational closure

Given a set of conditionals $\mathcal{C}$, reasoning in the KLM framework amounts to the derivation of new conditionals from $\mathcal{C}$. Towards this end, Lehmann and Magidor [22] proposed what they refer to as *rational closure*. Here we focus on the semantic version of rational closure they present.

Their idea was to define a preference relation $\trianglelefteq_{\mathrm{LM}}$ over the set of possible ranked interpretations and then to base entailment on choosing only the most preferred, i.e., minimal w.r.t. $\trianglelefteq_{\mathrm{LM}}$, ranked models of $\mathcal{C}$.

The relation $\trianglelefteq_{\mathrm{LM}}$ can be described as follows. Consider any pair of ranked interpretations $\mathscr{R}_1 = (L_0, \ldots, L_{n-1}, L_\infty)$ and $\mathscr{R}_2 = (M_0, \ldots, M_{n-1}, M_\infty)$. Then,

$$\mathscr{R}_1 \trianglelefteq_{\mathrm{LM}} \mathscr{R}_2 \quad \text{if} \quad \begin{array}{ll} \textit{either} & L_i = M_i \text{ for all } i \in \{0, \ldots, n-1, \infty\}, \\ \textit{or} & L_j \supseteq M_j \text{ for the smallest } j \geq 0 \text{ s.t. } L_j \neq M_j. \end{array}$$

This is not exactly the semantic representation defined by Lehmann and Magidor, but this representation can easily be derived from other work on rational closure, such as that of Booth and Paris [3] and Giordano et al. [19]. The idea is that those ranked interpretations should be preferred in which as many valuations as possible are judged to be as plausible as the background knowledge $\mathcal{C}$ allows. Observe also that one of the consequences of this ordering is that, all other things being equal, a ranked interpretation in which a valuation is deemed to be possible will be preferred over one in which the same valuation is seen as impossible.

Clearly $\trianglelefteq_{\mathrm{LM}}$ forms a partial order over ranked interpretations. Lehmann and Magidor showed that for every set of conditionals $\mathcal{C}$, there exists a unique $\trianglelefteq_{\mathrm{LM}}$-minimum element $\mathscr{R}^{\mathrm{rc}}(\mathcal{C})$ among all the ranked models of $\mathcal{C}$. We will refer to this element as the *LM-minimum*. Then the rational closure of $\mathcal{C}$ is the set $\mid\!\sim^{\mathrm{rc}}_{\mathcal{C}} := \{(\alpha, \beta) \mid \mathscr{R}^{\mathrm{rc}}(\mathcal{C}) \Vdash \alpha \mid\!\sim \beta\}$. Rational closure is commonly viewed as the *basic* (although certainly not the only acceptable) form of entailment over propositional conditional knowledge bases, on which other, more venturous, forms of entailment can be constructed. It is therefore an appropriate choice on which to base our investigations into versions of entailment for PTL.

2.3 Propositional Typicality Logic

PTL [1] is a logical formalism explicitly allowing for the representation of and reasoning about a notion of *typicality*. Syntactically, it extends classical propositional logic with a *typicality operator* $\bullet$, the intuition of which is to capture the most typical (alias normal or conventional) situations or worlds. Here we shall briefly present the main results about PTL relevant for our purposes.

The language of PTL, denoted by $\mathcal{L}^\bullet$, is recursively defined by:

$$\alpha \ ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \top \mid \bot \mid \bullet\alpha$$

As before, p denotes an atom and all the other Boolean connectives are defined in terms of $\neg$ and $\wedge$.

Let $\mathcal{P} = \{\mathsf{b}, \mathsf{f}, \mathsf{o}, \mathsf{p}\}$, where b, f and p are as before and o represents "being an ostrich". The following are examples of $\mathcal{L}^\bullet$-sentences: $\bullet\mathsf{b}$ (being a typical bird), $\mathsf{o} \rightarrow \neg\bullet\mathsf{b}$ (ostriches are not typical birds), $(\mathsf{p} \vee \mathsf{o}) \leftrightarrow (\mathsf{b} \wedge \bullet\neg\mathsf{f})$ (being a penguin or an ostrich is equivalent to being a bird and being a typical non-flying creature).

Intuitively, a sentence of the form $\bullet\alpha$ is understood to refer to the typical situations in which α holds. Note that α can itself be a $\bullet$-sentence. The semantics of PTL is also in terms of ranked interpretations (see Definition 2.1). Satisfaction is defined inductively in the classical way, adding the following condition: $v \Vdash \bullet\alpha$ if $v \Vdash \alpha$ and there is no v' such that $v' \prec v$ and $v' \Vdash \alpha$. That is, given $\mathscr{R}$, $[\![\bullet\alpha]\!]^{\mathscr{R}} := \min_{\prec}[\![\alpha]\!]^{\mathscr{R}}$. In the ranked interpretation $\mathscr{R}$ of Figure 1, we have $[\![\bullet\mathsf{b}]\!]^{\mathscr{R}} = \{\{\mathsf{b}, \mathsf{f}, \neg\mathsf{p}\}\}$, $[\![\bullet\mathsf{p}]\!]^{\mathscr{R}} = \{\{\mathsf{b}, \neg\mathsf{f}, \mathsf{p}\}\}$ and $[\![\bullet(\mathsf{b} \wedge \neg\mathsf{f})]\!]^{\mathscr{R}} = \{\{\mathsf{b}, \neg\mathsf{f}, \neg\mathsf{p}\}, \{\mathsf{b}, \neg\mathsf{f}, \mathsf{p}\}\}$.

We say that $\alpha \in \mathcal{L}^\bullet$ is *satisfiable* in a ranked interpretation $\mathscr{R}$ if $[\![\alpha]\!]^{\mathscr{R}} \neq \emptyset$, otherwise α is *unsatisfiable* in $\mathscr{R}$. We say that $\mathscr{R}$ is a *ranked model* of α (denoted $\mathscr{R} \Vdash \alpha$) if $[\![\alpha]\!]^{\mathscr{R}} = \mathcal{U}^{\mathscr{R}}$.

A PTL *knowledge base* is a finite set of sentences $\mathcal{KB} \subseteq \mathcal{L}^\bullet$. We define $Mod(\mathcal{KB}) := \{\mathscr{R} \mid \mathscr{R} \Vdash \bigwedge \mathcal{KB}\}$.

A useful property of the typicality operator $\bullet$ is that it allows us to express KLM-style conditionals. That is, for every ranked interpretation $\mathscr{R}$ and every $\alpha, \beta \in \mathcal{L}$, $\mathscr{R} \Vdash \alpha \mid\!\sim \beta$ if and only if $\mathscr{R} \Vdash \bullet\alpha \rightarrow \beta$. The converse does not hold since it can be shown that there are $\mathcal{L}^\bullet$-sentences that cannot be expressed as a set of KLM-style $\mid\!\sim$-statements on $\mathcal{L}$ [2].

The representation result below, extending Theorem 3.12 of Lehmann and Magidor [22] to $\mathcal{L}^\bullet$, shows that the formalisation of the KLM rational conditional $\mid\!\sim$ inside PTL is appropriate.

Observation 1 (Booth et al. [2], Corollary 22) *Let $\mathscr{R}$ be a ranked interpretation and let $\mid\!\sim_{\mathscr{R}} := \{(\alpha, \beta) \mid \alpha, \beta \in \mathcal{L}^\bullet$ and $\mathscr{R} \Vdash \bullet\alpha \rightarrow \beta\}$. Then $\mid\!\sim_{\mathscr{R}}$ is a rational conditional. Conversely, for every rational conditional $\mid\!\sim$, there exists a ranked interpretation $\mathscr{R}$ such that, for every $\alpha, \beta \in \mathcal{L}^\bullet$, $\alpha \mid\!\sim \beta$ if and only if $\mathscr{R} \Vdash \bullet\alpha \rightarrow \beta$.*

For more details on PTL and the aforementioned properties, the reader is referred to the work by Booth et al. [2].

3 The entailment problem for PTL

The purpose of this section is to provide a more formal motivation for the remainder of the paper. From the perspective of knowledge representation and reasoning (KR&R), a central issue is that of what it means for a PTL sentence to *follow* from a (finite) PTL knowledge base $\mathcal{KB}$. An obvious approach to the matter is to embrace the notion of entailment advocated by Tarski [26] and largely adopted in the logic-based KR&R community.

Definition 3.1 (Ranked entailment and consequence) *Let $\mathcal{KB} \subseteq \mathcal{L}^\bullet$ and $\alpha \in \mathcal{L}^\bullet$. We say $\mathcal{KB}$* ***ranked-entails*** *α (noted $\mathcal{KB} \approx_0 \alpha$) if $Mod(\mathcal{KB}) \subseteq Mod(\alpha)$. Its associated* ***ranked consequence*** *operator is defined by setting, for $\mathcal{KB} \subseteq \mathcal{L}^\bullet$, $Cn_0(\mathcal{KB}) := \{\alpha \in \mathcal{L}^\bullet \mid \mathcal{KB} \approx_0 \alpha\}$.*

As we shall see below, this version of entailment is *not* appropriate in the context of PTL for a number of reasons. For one, consider the following definition of a conditional induced from a set of PTL sentences.

Definition 3.2 (Induced conditional relation) *Let $\mathcal{X} \subseteq \mathcal{L}^\bullet$. We define $\mid\!\sim_{\mathcal{X}} := \{(\alpha, \beta) \mid \alpha, \beta \in \mathcal{L}$ and $\bullet\alpha \rightarrow \beta \in \mathcal{X}\}$.*

It is worth investigating whether $\mid\!\sim_{Cn_0(\mathcal{KB})}$ is rational, i.e., whether it satisfies all the KLM properties for rationality from Section 2.1. The following proposition, which mimics a similar result by Lehmann and Magidor in the propositional case, shows that this is not the case:

Observation 2 (Booth et al. [2], Proposition 25) *$\mid\!\sim_{Cn_0(\mathcal{KB})}$ is a preferential conditional, but is not necessarily a rational conditional.*

Hence, ranked consequence as defined above delivers an induced defeasible conditional that is preferential but that need not be rational. This forms an argument against ranked entailment being an appropriate notion of entailment for PTL.

One of the principles to give serious consideration when investigating PTL entailment is the *presumption of typicality* [21, p. 63]. Informally, this means that one should assume that every situation is assumed to be as typical as possible. Sections 4 and 6 contain a formalisation of this principle. For now, we illustrate it with an example.

Example 3.1 Let $\mathcal{KB}_1 = \{\mathsf{p} \rightarrow \mathsf{b}, \bullet\mathsf{b} \rightarrow \mathsf{f}\}$ (penguins are birds, and typical birds fly). Given just this information about birds and penguins, it is reasonable to expect both $\bullet\mathsf{p} \rightarrow \bullet\mathsf{b}$ (typical penguins are typical birds) and therefore $\bullet\mathsf{p} \rightarrow \mathsf{f}$ (typical penguins fly) to follow from $\mathcal{KB}_1$. It is easy to see that with ranked entailment these requirements are not met, as ranked entailment is not ampliative, i.e., it does not allow for venturing beyond what is sanctioned by the knowledge base. □

Besides requiring PTL entailment to be ampliative, we also want it to be *defeasible*, that is, the conclusions derived under the presumption of typicality in an ampliative way can be retracted in case of new conflicting information. This is illustrated by the following example.

Example 3.2 Assume $\bullet\mathsf{p} \rightarrow \bullet\mathsf{b}$ and $\bullet\mathsf{p} \rightarrow \mathsf{f}$ (somehow) could follow from $\mathcal{KB}_1$ in Example 3.1, but then we are informed that typical penguins do not fly. That is, let $\mathcal{KB}_2 = \mathcal{KB}_1 \cup \{\bullet\mathsf{p} \rightarrow \neg\mathsf{f}\}$. While we want $\mathsf{p} \rightarrow \neg\bullet\mathsf{b}$ (penguins are not typical birds) to follow from $\mathcal{KB}_2$, we do *not* want $\bullet\mathsf{p} \rightarrow \mathsf{f}$ to follow from $\mathcal{KB}_2$, which is not possible with ranked entailment. □

4 Towards a notion of entailment for PTL

We have seen that ranked entailment has some serious drawbacks in a non-monotonic context. Therefore, the question as to what logical consequence in PTL should mean remains mostly unanswered so far. In this section, we first specify and discuss a list of postulates formalising the requirements motivated in the last section and that, at first glance, seem reasonable for an appropriate notion of entailment in PTL. In the subsequent section, we consider specific alternatives to ranked entailment and check them against our postulates.

We start by introducing some notation. With $\approx_{?} \subseteq \mathscr{P}(\mathcal{L}^\bullet) \times \mathcal{L}^\bullet$, we denote any entailment relation on the language of PTL. Given an entailment relation $\approx_{?}$, its associated *consequence* operator is defined in the usual way by setting, for each $\mathcal{KB} \subseteq \mathcal{L}^\bullet$, $Cn_{?}(\mathcal{KB}) := \{\alpha \in \mathcal{L}^\bullet \mid \mathcal{KB} \approx_{?} \alpha\}$.

Following the tradition in the non-monotonic reasoning literature, the obvious starting point is to consider some of the basic properties of classical consequence operators.

P1 $\mathcal{KB} \subseteq Cn_{?}(\mathcal{KB})$ (Inclusion)

P2 $Cn_{?}(\mathcal{KB}) = Cn_{?}(Cn_{?}(\mathcal{KB}))$ (Idempotence)

Idempotence specifies that a consequence operator behaves as a 'once-off' operation, in the same spirit as that of a closure operator. It implies also its finitary version, the *Cumulativity Property*: If $\alpha \in Cn_{?}(\mathcal{KB})$, then $Cn_{?}(\mathcal{KB} \cup \{\alpha\}) = Cn_{?}(\mathcal{KB})$. There is agreement in the literature that both Inclusion and Cumulativity are desirable properties to have.

Ranked entailment, as defined in Section 3, satisfies Properties P1 and P2. Nevertheless, $Cn_0(\cdot)$, the associated consequence relation of ranked entailment, also satisfies the classical property of Monotonicity: If $\mathcal{KB} \subseteq \mathcal{KB}'$, then $Cn_0(\mathcal{KB}) \subseteq Cn_0(\mathcal{KB}')$. As seen in Example 3.1, this is a property that we do not want $Cn_{?}(\cdot)$ to satisfy (certainly not in general).

So, we require $Cn_{?}(\cdot)$ to be a *non-monotonic* consequence operator. This amounts to requiring $Cn_{?}(\cdot)$ to satisfy the following two postulates:

P3 For every $\mathcal{KB} \subseteq \mathcal{L}^\bullet$, $Cn_0(\mathcal{KB}) \subseteq Cn_{?}(\mathcal{KB})$ (Ampliativeness)

P4 For some $\mathcal{KB}, \mathcal{KB}' \subseteq \mathcal{L}^\bullet$, $\mathcal{KB} \subseteq \mathcal{KB}'$ but $Cn_?(\mathcal{KB}) \not\subseteq Cn_?(\mathcal{KB}')$ (Defeasibility)

Ampliativeness, a property generalising supra-classicality [23] (where the basic underlying entailment relation is classical), says that $Cn_?(\cdot)$ should be at least as venturous as its underlying ranked entailment. Defeasibility specifies that $Cn_?(\cdot)$ should be flexible enough to disallow previously derived conclusions in the light of new (possibly conflicting) information. In Example 3.1, assuming $\bullet \mathsf{p} \to \mathsf{f} \in Cn_?(\mathcal{KB}_1)$ is the case, then $\bullet \mathsf{p} \to \mathsf{f}$ should no longer be concluded if $\bullet \mathsf{p} \to \neg \mathsf{f}$ is added to $\mathcal{KB}_1$. Note that Defeasibility actually implies a *strict* version of Ampliativeness which says $Cn_?(\cdot)$ should in some cases be *more* venturous than its underlying ranked entailment. (Since, if $Cn_?(\mathcal{KB}) = Cn_0(\mathcal{KB})$ for all $\mathcal{KB}$, then $Cn_?(\cdot)$ is just ranked entailment, which is monotonic.)

P2 and P3 together imply that the closure operation $Cn_?(\cdot)$ gives as output a theory that is closed under $Cn_0(\cdot)$.

Lemma 4.1 *If $Cn_?(\cdot)$ satisfies P2 and P3, then, for every $\mathcal{KB}$,*

$$Cn_?(\mathcal{KB}) = Cn_0(Cn_?(\mathcal{KB}))$$

Proof:
Since $Cn_0(\cdot)$ clearly satisfies inclusion, $Cn_?(\mathcal{KB}) \subseteq Cn_0(Cn_?(\mathcal{KB}))$ is immediate. By P3 we have $Cn_0(Cn_?(\mathcal{KB})) \subseteq Cn_?(Cn_?(\mathcal{KB}))$, that, by P2, implies $Cn_0(Cn_?(\mathcal{KB})) \subseteq Cn_?(\mathcal{KB})$. □

Similarly to KLM in the propositional case, we would ideally like the defeasible conditional induced by $Cn_?(\mathcal{KB})$ (see Definition 3.2) to satisfy all the rationality properties:

P5 For every $\mathcal{KB} \subseteq \mathcal{L}^\bullet$, $\mathrel{|\!\sim}_{Cn_?(\mathcal{KB})}$ is a rational conditional relation on $\mathcal{L}$ (Conditional Rationality)

As observed above, P5 requires the defeasible conditional induced by $Cn_?(\mathcal{KB})$ to be rational—that is, to satisfy all the rationality properties. But from Theorem 3.12 of Lehmann and Magidor [22] it follows that every rational defeasible conditional can be obtained from a single ranked interpretation. So, from this it follows that requiring the defeasible conditional induced by $Cn_?(\mathcal{KB})$ to be rational amounts to requiring that the defeasible conditional be generated by a single ranked interpretation. That is, by courtesy of this result, P5 can also be rephrased as follows:

P5' For every $\mathcal{KB} \subseteq \mathcal{L}^\bullet$, there is a ranked interpretation $\mathscr{R}$ s.t., for every $\alpha, \beta \in \mathcal{L}$, $\alpha \mathrel{|\!\sim}_{Cn_?(\mathcal{KB})} \beta$ if and only if $\mathscr{R} \Vdash \bullet\alpha \to \beta$. ($\mathrel{|\!\sim}$ Single Model)

The next postulate we consider, which is easily shown to be a strengthening of P5, simply applies this same requirement, not just to defeasible statements, but to all statements expressible in PTL:

P6 For every $\mathcal{KB} \subseteq \mathcal{L}^\bullet$, there is a ranked interpretation $\mathscr{R}$ s.t., for all $\alpha \in \mathcal{L}^\bullet$, $\alpha \in Cn_?(\mathcal{KB})$ if and only if $\mathscr{R} \Vdash \alpha$ (Single Model)

An important special case of a PTL knowledge base is when the individual elements of $\mathcal{KB}$ correspond to KLM-style conditionals.

Definition 4.1 ((Propositional) conditional knowledge base) *A PTL knowledge base $\mathcal{KB}$ will be called a (propositional)* ***conditional knowledge base*** *if each element of $\mathcal{KB}$ is of the form $\bullet\alpha \to \beta$, for $\alpha, \beta \in \mathcal{L}$.*

The next postulate says that if $\mathcal{KB}$ is a propositional conditional knowledge base, then the result should coincide with Lehmann and Magidor's definition of rational closure:

P7 If $\mathcal{KB}$ is a conditional knowledge base, then $\mid\!\sim_{Cn_?(\mathcal{KB})} = \mid\!\sim^{\mathrm{rc}}_{\mathcal{KB}}$ (Extends Rational Closure)

Clearly, P7 implies P4.

The following property was shown by Lehmann and Magidor to be satisfied by the rational closure for conditional knowledge bases.

P8 Let $\alpha \in \mathcal{L}$. Then $\alpha \in Cn_?(\mathcal{KB})$ if and only if $\alpha \in Cn_0(\mathcal{KB})$ (Strict Entailment)

P8 states that $Cn_?(\cdot)$ should coincide with ranked entailment for those sentences not involving typicality. The motivation for Strict Entailment is twofold. First, it is a proposal for ranked entailment to be the lower bound for entailment w.r.t. classical sentences (those not involving typicality), a proposal that is not controversial. But secondly, it also requires entailment of classical sentences to correspond to exactly those sanctioned by ranked entailment. This can be viewed as adhering to the principle of *minimal change*. Being Tarskian, ranked entailment is monotonic, and the argument is therefore that, while non-monotonicity may be applicable for sentences involving typicality, it should not be applicable to classical statements.

We are also interested in a couple of progressively weaker versions of Strict Entailment (and the reasons will become clear later on). The first restricts it to hold only when $\mathcal{KB}$ is a conditional knowledge base.

P9 Let $\mathcal{KB}$ be a conditional knowledge base and $\alpha \in \mathcal{L}$. Then $\alpha \in Cn_?(\mathcal{KB})$ if and only if $\alpha \in Cn_0(\mathcal{KB})$ (Conditional Strict Entailment)

Note that P7 also implies P9. The latter implies that entailment for PTL coincides with classical propositional entailment in the case of propositional knowledge bases, as formalised by the next property.

P9' Let $\mathcal{KB} \subseteq \mathcal{L}$ and $\alpha \in \mathcal{L}$. Then $\alpha \in Cn_?(\mathcal{KB})$ if and only if $\mathcal{KB}$ entails α in classical propositional logic. (Classical Entailment)

Since for every $\mathcal{KB} \cup \{\alpha\} \subseteq \mathcal{L}$, $\mathcal{KB}$ entails α in classical propositional logic if and only if $\alpha \in Cn_0(\mathcal{KB})$, and any $\alpha \in \mathcal{L}$ is equivalent $\bullet\neg\alpha \to \bot$, P9' is indeed a weakening of P9 (provided that P8 also holds).

Finally, we consider another property shown by Lehmann and Magidor to be satisfied by the rational closure for conditional knowledge bases.

P10 Let $\alpha \in \mathcal{L}$. Then $\bullet\top \rightarrow \alpha \in Cn_?(\mathcal{KB})$ if and only if $\bullet\top \rightarrow \alpha \in Cn_0(\mathcal{KB})$ (Typical Entailment)

The motivation for P10 is similar to that for P8 above. Consequences of the form $\bullet\top \rightarrow \alpha$ are those for which α holds in the most typical situations. So, on the one hand, P10 is a proposal for ranked entailment to provide a lower bound for those statements holding in the most typical situations. But as with P8 above, it also provides an upper bound, thereby requiring that of those statements holding in typical situation exactly those sanctioned by ranked entailment ought to be regarded as being entailed by the knowledge base. The argument for this is that ranked entailment is monotonic and, applying the principle of minimal change, it is only when dealing with atypical situations that ranked entailment is not always sufficient.

Although these postulates all seem reasonable on their own, it turns out that they cannot all be satisfied simultaneously. In fact, this impossibility result already holds for a strict subset of the postulates.

Theorem 4.1 *There is no PTL consequence operator $Cn_?(\cdot)$ satisfying all of P1, P2, P3, P5, P8 and P10.*

Proof:
About (P5), requiring $\mid\!\sim_{Cn_?(\cdot)}$ to satisfy (RM) is equivalent to requiring that, for every knowledge base $\mathcal{KB}$ and whatever formulas α, β, γ, if $\bullet\alpha \rightarrow \gamma \in Cn_?(\cdot)$ and $\bullet\alpha \rightarrow \beta \notin Cn_?(\cdot)$, then we have $\bullet(\alpha \wedge \neg\beta) \rightarrow \gamma \in Cn_?(\cdot)$.

Assume $Cn_?(\cdot)$ satisfies the given properties, and let $\mathcal{KB} = \{\bullet\top \rightarrow p, \bullet\neg p \rightarrow \bullet q\}$. By Strict Entailment (P8), $p \notin Cn_?(\mathcal{KB})$ (because of e.g. the 2-rank model $(\{\{p, \neg q\}\}, \{\{\neg p, q\}\})$ of $\mathcal{KB}$). By Typical Entailment (P10), $\bullet\top \rightarrow \neg q \notin Cn_?(\mathcal{KB})$ (because of e.g. the 1-rank model $(\{\{p, q\}, \{p, \neg q\}\})$ of $\mathcal{KB}$). By Inclusion (P1) $\bullet\top \rightarrow p \in Cn_?(\mathcal{KB})$, and then by (RM) we must conclude that $\bullet(\top \wedge q) \rightarrow p \in Cn_?(\mathcal{KB})$, that is, $(\top \wedge q, p) \in \mid\!\sim_{Cn_?(\mathcal{KB})}$; since $\mid\!\sim_{Cn_?(\cdot)}$ must satisfy LLE, the latter implies $(q, p) \in \mid\!\sim_{Cn_?(\mathcal{KB})}$, that is, $\bullet q \rightarrow p \in Cn_?(\mathcal{KB})$.

Since by Inclusion (P1) $\bullet\neg p \rightarrow \bullet q \in Cn_?(\mathcal{KB})$, we have $\{\bullet q \rightarrow p, \bullet\neg p \rightarrow \bullet q\} \subset Cn_?(\mathcal{KB})$. Since $\bullet\neg p \rightarrow p \in Cn_0(\{\bullet q \rightarrow p, \bullet\neg p \rightarrow \bullet q\})$ and $Cn_0(\cdot)$ is monotonic, we have $\bullet\neg p \rightarrow p \in Cn_0(Cn_?(\mathcal{KB}))$. Then, by Lemma 4.1, that assumes P2 and P3, we have that $\bullet\neg p \rightarrow p \in Cn_?(\mathcal{KB})$.

Since $((\bullet\neg p \rightarrow p) \leftrightarrow p) \in Cn_0(\emptyset)$, we have that $p \in Cn_0(\{\bullet\neg p \rightarrow p\})$, that is, $p \in Cn_0(Cn_?(\mathcal{KB}))$, that is, by Lemma 4.1, $p \in Cn_?(\mathcal{KB})$, against (P8). □

While, at first glance, this seems to be a negative result, our contention is that it should be interpreted as an indication that a logic as expressive as PTL admits more than one form of entailment. We elaborate directly on this point in Section 8, and indirectly in Sections 5 and 6, where we define and discuss two instances of entailment for PTL.

5 LM-entailment

We now come to our first construction of an entailment relation in PTL. The idea is to try to lift the rational closure construction from conditional knowledge bases to arbitrary knowledge bases in $\mathcal{L}^\bullet$. We first observe that there is nothing to stop us from using the preference relation $\trianglelefteq_{\mathrm{LM}}$ (see Section 2.2) to compare ranked interpretations of *any* PTL knowledge base $\mathcal{KB}$. The question then is, does there always exist a *unique* LM-minimum element of the ranked models of $\mathcal{KB}$, as there does in the restricted conditional case? And if so, how can we construct it? We now answer these questions.

We assume as input a PTL knowledge base $\mathcal{KB} = \{\alpha_1, \ldots, \alpha_n\}$, where each sentence α_j is in *normal form*:

Definition 5.1 (Normal form) *$\alpha \in \mathcal{L}^\bullet$ is* ***in normal form*** *if it is of the form $\bigwedge_{i\leq t} \bullet\theta_i \to (\phi \vee \bigvee_{i\leq s} \bullet\psi_i)$, where $t, s \geq 0$ and the θ_i, ϕ and ψ_i are all purely propositional sentences.*

Theorem 5.1 *The normal form is complete for $\mathcal{L}^\bullet$, i.e., for every sentence $\alpha \in \mathcal{L}^\bullet$ there is a (finite) set of sentences $X \subseteq \mathcal{L}^\bullet$, each one in normal form, such that $Mod(\alpha) = Mod(\bigwedge X)$.*

Proof:
From the results by Booth et al. [1, Section 4], it follows that we need only consider sentences with non-nested instances of the typicality operator. So we let α be such a sentence. We let the set of *typicality* atoms be the propositional atoms occurring in $\mathcal{L}^\bullet$ together with every sentence of the form $\bullet\beta$ where β is a propositional sentence (we refer to the latter as *pure typicality atoms*). And we define the set of *typicality literals* in the obvious way: the set of typicality atoms and their negations. The set of *pure typicality literals* consists of the pure typicality atoms and their negations.

Now we define *typicality conjunctive normal form* as a conjunctive normal form defined on typicality atoms. It follows immediately that α can be rewritten as a sentence, say α', in typicality conjunctive normal form. Let X' be the set of conjuncts occurring in α'. We show below how to rewrite each conjunct in X' into a sentence in normal form. The resulting set X of sentences in normal form is the set referred to above.

By construction, each sentence $\gamma \in X'$ is a disjunction of typicality literals. We separate them into three disjoint sets, the set of propositional literals, the set of positive pure typicality literals (with cardinality of, say t, where $t \geq 0$) and the set of negative pure typicality literals (with cardinality of, say s, where $s \geq 0$). Let ϕ be the disjunction of propositional literals, denote the s positive pure typicality literals by $\psi_1, \ldots, \psi_s$, and the t negative pure typicality literals by $\theta_1, \ldots \theta_t$. It follows immediately that γ can be rewritten as the sentence $\bigwedge_{i\leq t} \theta_i \to (\phi \vee \bigvee_{i\leq s} \psi_i)$. □

For any ranked interpretation $\mathscr{R}$, and $S \subseteq \mathcal{U}^{\mathscr{R}}$, let $\mathscr{R}^\infty_S$ be the ranked interpretation such that $\mathscr{R}^\infty_S(v) = \mathscr{R}(v)$ for every $v \in S$, and $\mathscr{R}^\infty_S(v) = \infty$ for every $v \in \mathcal{U} \setminus S$. That is, $\mathscr{R}_S^\infty$ is the ranked interpretation obtained from $\mathscr{R}$ by turning all valuations not in S into impossible valuations. Similarly, let $\mathscr{R}_S^1$ be the ranked interpretation such that $\mathscr{R}_S^1(v) = \mathscr{R}(v)$ for every $v \in S$, and $\mathscr{R}_S^1(v) = \mathscr{R}(v) + 1$ for every $v \in \mathcal{U} \setminus S$. That is, $\mathscr{R}_S^1$ is the ranked interpretation obtained from $\mathscr{R}$ by increasing the rank of all valuations not in S by 1.

We now construct a sequence $(\mathscr{R}_0, \mathscr{R}_1, \ldots)$ of ranked interpretations as follows:

Step 1 Set $\mathscr{R}_0(v) := 0$ for all $v \in \mathcal{U}$, $S_0 := \emptyset$, and $i := 1$;

Step 2 $S_1 := \llbracket \mathcal{KB} \rrbracket^{\mathscr{R}_0}$ (*separate the valuations which satisfy $\mathcal{KB}$ w.r.t. the current ranked interpretation $\mathscr{R}_0$ from those that do not*);

Step 3 If $S_i = S_{i-1}$, then return $(\mathscr{R}_i)_{S_i}^\infty$ (*if there is no change in the new S_i then set the rank of those valuations that do not satisfy $\mathcal{KB}$ w.r.t. $\mathscr{R}_i$ to ∞ and return the interpretation that remains*);

Step 4 Otherwise $\mathscr{R}_i := (\mathscr{R}_{i-1})_{S_i}^1$ (*otherwise create a new ranked interpretation $\mathscr{R}_i$ by increasing the rank of every valuation not in S_i by 1*);

Step 5 $S_{i+1} := \llbracket \mathcal{KB} \rrbracket^{\mathscr{R}_i}$ and $i := i + 1$ (*separate the valuations which satisfy $\mathcal{KB}$ w.r.t. the current ranked interpretation $\mathscr{R}_i$ from those that do not, and increment i*);

Step 6 Go to Step 3.

Algorithm 1 below gives a compact description of the above steps.

Algorithm 1: LM-minimal

Input: $\mathcal{KB}$
Output: $\mathscr{R}^*_{\mathcal{KB}}$

$\mathcal{P}_{\mathcal{KB}} := \{p \mid p$ is a propositional letter occurring in $\mathcal{KB}\}$;
Let $\mathcal{U}$ be the universe of valuations for the vocabulary $\mathcal{P}_{\mathcal{KB}}$;
$\mathscr{R}_0(v) := 0$ for every $v \in \mathcal{U}$;
$S_0 := \emptyset$;
$S_1 := \llbracket \mathcal{KB} \rrbracket^{\mathscr{R}_0}$;
$i := 1$;
while $S_i \neq S_{i-1}$ **do**
    $\mathscr{R}_i := (\mathscr{R}_{i-1})_{S_i}^1$;
    $S_{i+1} := \llbracket \mathcal{KB} \rrbracket^{\mathscr{R}_i}$;
    $i := i + 1$;
$\mathscr{R}^*_{\mathcal{KB}} := (\mathscr{R}_i)_{S_i}^\infty$;
return $\mathscr{R}^*_{\mathcal{KB}}$

Example 5.1 Let us assume, for the sake of the example, that we are only talking about birds. Let $\mathcal{KB} := \{\bullet\top \to (\neg \mathsf{p} \wedge \neg \mathsf{r}), \bullet\mathsf{p} \to \bullet\neg\mathsf{f}, \bullet\mathsf{r} \to \bullet\mathsf{f}, \mathsf{p} \to \neg\mathsf{r}\}$ (the most typical things are neither penguins nor robins, typical penguins are typical non-flying birds, and typical robins are typical flying birds, penguins are not robins). The procedure initialises with all valuations being assigned the rank of 0. The only valuations that satisfy all three sentences w.r.t. $\mathscr{R}_0$ are those satisfying both $\neg\mathsf{p}$ and $\neg\mathsf{r}$. Thus $S_1 := [\![\mathcal{KB}]\!]^{\mathscr{R}_0} = \{\{\neg\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}, \{\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}\}$ and so we obtain $\mathscr{R}_1$ by changing the rank of all valuations not in S_1 to 1. Note that $[\![\bullet\neg\mathsf{f}]\!]^{\mathscr{R}_1} = \{\{\neg\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}\}$ and $[\![\bullet\mathsf{f}]\!]^{\mathscr{R}_1} = \{\{\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}\}$, so we can see that none of the valuations in $\mathcal{U} \setminus S_1$ is able to satisfy either $\bullet\mathsf{p} \rightarrow \bullet\neg\mathsf{f}$ or $\bullet\mathsf{r} \rightarrow \bullet\mathsf{f}$ w.r.t. $\mathscr{R}_1$. As a consequence, $S_2 := [\![\mathcal{KB}]\!]^{\mathscr{R}_1} = S_1$ and so the procedure terminates here with $\mathscr{R}^*_{\mathcal{KB}}$ as the ranked interpretation in which all valuations in S_1 ($\{\neg\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$ and $\{\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$) have rank 0 and all other valuations have rank ∞. See Figure 2 for the ranked interpretations generated by this example. □

$\mathscr{R}_0$

0	$\{\neg\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$, $\{\neg\mathsf{f}, \neg\mathsf{p}, \mathsf{r}\}$, $\{\neg\mathsf{f}, \mathsf{p}, \neg\mathsf{r}\}$, $\{\neg\mathsf{f}, \mathsf{p}, \mathsf{r}\}$, $\{\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$, $\{\mathsf{f}, \neg\mathsf{p}, \mathsf{r}\}$, $\{\mathsf{f}, \mathsf{p}, \neg\mathsf{r}\}$, $\{\mathsf{f}, \mathsf{p}, \mathsf{r}\}$

$\mathscr{R}_1$

1	$\{\neg\mathsf{f}, \neg\mathsf{p}, \mathsf{r}\}$, $\{\neg\mathsf{f}, \mathsf{p}, \mathsf{r}\}$, $\{\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$, $\{\mathsf{f}, \neg\mathsf{p}, \mathsf{r}\}$, $\{\mathsf{f}, \mathsf{p}, \neg\mathsf{r}\}$, $\{\mathsf{f}, \mathsf{p}, \mathsf{r}\}$
0	$\{\neg\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$, $\{\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$

$\mathscr{R}^*_{\mathcal{KB}}$

∞	$\{\neg\mathsf{f}, \neg\mathsf{p}, \mathsf{r}\}$, $\{\neg\mathsf{f}, \mathsf{p}, \mathsf{r}\}$, $\{\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$, $\{\mathsf{f}, \neg\mathsf{p}, \mathsf{r}\}$, $\{\mathsf{f}, \mathsf{p}, \neg\mathsf{r}\}$, $\{\mathsf{f}, \mathsf{p}, \mathsf{r}\}$
0	$\{\neg\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$, $\{\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$

$\mathscr{R}^*_{\mathcal{KB}}$ with the valuations of rank ∞ omitted:

0	$\{\neg\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$, $\{\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$

Figure 2: The ranked interpretations generated in Example 5.1.

We now need to show that: (*i*) the algorithm always terminates; (*ii*) it returns a ranked model of $\mathcal{KB}$, and (*iii*) for any other ranked model $\mathscr{R}$ of $\mathcal{KB}$, we have $\mathscr{R}^*_{\mathcal{KB}} \trianglelefteq_{\text{LM}} \mathscr{R}$. We know the following about (*i*) and (*ii*):

Lemma 5.1 *The following hold for each $i \geq 0$:*

1. *$S_i \subseteq S_{i+1}$, i.e., $[\![\mathcal{KB}]\!]^{\mathscr{R}_i} \subseteq [\![\mathcal{KB}]\!]^{\mathscr{R}_{i+1}}$;*
2. *For all $v_1, v_2 \in \mathcal{U}$, if $\mathscr{R}_i(v_1) < \mathscr{R}_i(v_2)$, then $v_1 \in [\![\mathcal{KB}]\!]^{\mathscr{R}_i}$;*
3. *$\mathscr{R}_i$ is a ranked interpretation.*

Proof:
See A.1. □

From Item 1 in Lemma 5.1 above, we know the algorithm terminates, since it generates a sequence of ranked interpretations (by Item 3) in which the set of valuations satisfying $\mathcal{KB}$ increases monotonically from one ranked interpretation to the next. Since each of these is finite, and since there is a finite number of valuations, the stopping criterion in Step 3 of the algorithm is guaranteed to occur eventually.

To show that the algorithm returns a ranked model of $\mathcal{KB}$ it suffices to show the following.

Lemma 5.2 *For every $\mathcal{KB}$ and every $i > 0$, $(\mathscr{R}_i)^{\infty}_{S_i}$ is a ranked model of $\mathcal{KB}$.*

Proof:
See A.2. □

So, at each stage of the algorithm, the current ranked interpretation, when those valuations not satisfying $\mathcal{KB}$ are excluded, forms a ranked model of $\mathcal{KB}$. Since the output $\mathscr{R}^*_{\mathcal{KB}}$ takes precisely this form we have the following result.

Proposition 5.1 *$\mathscr{R}^*_{\mathcal{KB}} \Vdash \bigwedge \mathcal{KB}$.*

Proof:
Follows from Lemma 5.2 and the construction of $\mathscr{R}^*_{\mathcal{KB}}$. □

Next we want to show that for any other ranked model $\mathscr{R}$ of $\mathcal{KB}$, we have $\mathscr{R}^*_{\mathcal{KB}} \trianglelefteq_{\mathrm{LM}} \mathscr{R}$.

Lemma 5.3 *Let $\mathscr{R}^*_{\mathcal{KB}} := (L_0, \ldots, L_{n-1}, L_\infty)$ and let $\mathscr{R} := (M_0, \ldots, M_{n-1}, M_\infty)$ be any other ranked model of $\mathcal{KB}$. Let $i \in \{0, \ldots, n-1\}$. If $L_j = M_j$ for all $j < i$, then $M_i \subseteq L_i$.*

Proof:
See A.3. □

From this lemma we can state:

Proposition 5.2 *Consider any $\mathcal{KB}$ and let $\mathscr{R}$ be a ranked model of $\mathcal{KB}$. Then $\mathscr{R}^*_{\mathcal{KB}} \trianglelefteq_{\mathrm{LM}} \mathscr{R}$.*

We are now in a position to define our first form of entailment for PTL.

Definition 5.2 (LM-entailment) *Let $\mathcal{KB} \subseteq \mathcal{L}^\bullet$ and $\alpha \in \mathcal{L}^\bullet$. We say $\mathcal{KB}$ **LM-entails** α, denoted $\mathcal{KB} \approx_{\mathrm{LM}} \alpha$, if $\mathscr{R}^*_{\mathcal{KB}} \Vdash \alpha$. Its corresponding consequence operator is defined as $Cn_{\mathrm{LM}}(\mathcal{KB}) := \{\alpha \in \mathcal{L}^\bullet \mid \mathscr{R}^*_{\mathcal{KB}} \Vdash \alpha\}$.*

The next result outlines which properties from the previous section are satisfied by $Cn_{\mathrm{LM}}(\cdot)$.

Theorem 5.2 *$Cn_{\mathrm{LM}}(\cdot)$ satisfies P1–P7, P9, and P10, but* **not** *P8.*

Proof:
For **P1**, Proposition 5.1 guarantees that $\mathscr{R}^*_{\mathcal{KB}}$ is a model of $\mathcal{KB}$. About **P2**, by Proposition 5.2, $\mathscr{R}^*_{\mathcal{KB}}$ is the LM-minimum model of $\mathcal{KB}$. If $\mathscr{R}^*_{\mathcal{KB}} \Vdash \alpha$, $\mathscr{R}^*_{\mathcal{KB}}$ must also be the LM-minimum model of $\mathcal{KB} \cup \{\alpha\}$. For **P3**, note that $\mathscr{R}^*_{\mathcal{KB}}$ is a ranked model of $\mathcal{KB}$ (Lemma 5.1, Item 3, plus Proposition 5.1), and so if $\alpha \in Cn_0(\mathcal{KB})$, then $\alpha \in \mathscr{R}^*_{\mathcal{KB}}$. **P4** is an immediate consequence of the satisfaction of **P7**.[3] **P5** is an immediate consequence of the satisfaction of **P6**.

[3]For this conclusion we need the requirement (specified in Section 2) that $\mathcal{P}$ contains at least two elements.

The latter holds by definition of $Cn_{\mathrm{LM}}(\mathcal{KB})$. For **P7**, see Section 2.2. **P9** is an immediate consequence of the satisfaction of **P7**.

Now consider **P10**. From right to left, it is an immediate consequence of **P3**. From left to right, assume there is a formula $\bullet\top \rightarrow \alpha$ that is in $Cn_{\mathrm{LM}}(\mathcal{KB})$, but not in $Cn_0(\mathcal{KB})$. It means that there is a ranked model $\mathscr{R}$ of $\mathcal{KB}$ that has in its lower layer a propositional valuation v s.t. $v \Vdash \neg\alpha$; but, given that the model $\mathscr{R}^*_{\mathcal{KB}}$ defining $Cn_{\mathrm{LM}}(\mathcal{KB})$ is the LM-minimum model of $\mathcal{KB}$, then also the lower layer of $\mathscr{R}^*_{\mathcal{KB}}$ must contain the valuation v, against the hypothesis.

Failure of **P8** can be seen in Example 5.1. There we have $\neg\mathsf{p} \in Cn_{\mathrm{LM}}(\mathcal{KB})$ (there is no penguin) because $\neg\mathsf{p}$ holds in both valuations occurring in $\mathscr{R}^*_{\mathcal{KB}}$. But $\neg\mathsf{p} \notin Cn_0(\mathcal{KB})$, because there does exist a ranked model $\mathscr{R}$ of $\mathcal{KB}$ for which $[\![\mathsf{p}]\!]^{\mathscr{R}} \neq \emptyset$, for instance the model $\mathscr{R}_2$ appearing in Example 6.1 below. Thus LM-entailment forces us to infer $\neg\mathsf{p}$ from $\mathcal{KB}$. □

In summary then, LM-entailment satisfies all our postulates, except for Strict Entailment (P8). Lest this be seen as a negative result, bear in mind that LM-entailment satisfies Conditional Strict Entailment (P9), the weakened version of Strict Entailment, and therefore also Classical Entailment.

In the next section we turn to a form of entailment satisfying Strict Entailment, but at the price of having to forego Conditional Rationality, and therefore the Single Model postulate as well.

6 PT-entailment

In this section we consider another option for entailment based on a version of minimality, and derived from the characterisation of rational closure by Giordano et al. [17, 19]. The general idea is to respect the principle of *presumption of typicality* (see Section 3), We shall refer to this form of entailment as *Presumption of Typicality* entailment, shortened to *PT-entailment.* Such a principle indicates the way in which the property (RM) should be satisfied. If we have $\alpha \mid\!\sim \gamma$ in our knowledge base $\mathcal{KB}$, then, in order to satisfy (RM), we have to add either $\alpha \mid\!\sim \neg\beta$ or $\alpha \wedge \beta \mid\!\sim \gamma$. The presumption of typicality requires that, whenever possible, we prefer the latter (that corresponds to a constrained application of monotonicity) over the former. Semantically, given the ranked models of a knowledge base $\mathcal{KB}$, this corresponds to considering only those models in which every valuation is taken as typical as possible, that is, it is 'pushed downward' in the model as much as possible, modulo the satisfaction of $\mathcal{KB}$.

In order to identify the interpretations that are necessary for the definition of a notion of entailment, we introduce a preference relation $\trianglelefteq_{\mathrm{PT}}$ on the set of ranked interpretations that follows directly from the presumption of typicality.

Definition 6.1 (Relation $\trianglelefteq_{\mathrm{PT}}$) *For two ranked interpretations $\mathscr{R}_1$ and $\mathscr{R}_2$, $\mathscr{R}_1 \trianglelefteq_{\mathrm{PT}} \mathscr{R}_2$ if and only if for every $w \in \mathcal{U}$, $\mathscr{R}_1(w) \leq \mathscr{R}_2(w)$. $\mathscr{R}_1 \triangleleft_{\mathrm{PT}} \mathscr{R}_2$ if and only if $\mathscr{R}_1 \trianglelefteq_{\mathrm{PT}} \mathscr{R}_2$ and not $\mathscr{R}_2 \trianglelefteq_{\mathrm{PT}} \mathscr{R}_1$.*

It is easy to check that $\trianglelefteq_{\mathrm{PT}}$ is a pre-order. Consistent with the principle of presumption of typicality, as a guideline in the choice of the relevant interpretations, the relation $\trianglelefteq_{\text{PT}}$ can be used to identify the relevant interpretations for the definition of a notion of entailment: we choose the models of $\mathcal{KB}$ in which the valuations are presumed to be as typical as possible, that is, the relevant models are those that are in $\min_{\trianglelefteq_{\text{PT}}} \mathit{Mod}(\mathcal{KB})$. Then, $\mathcal{KB}$ entails α if and only if α holds in all the (preferred) models in $\min_{\trianglelefteq_{\text{PT}}} \mathit{Mod}(\mathcal{KB})$.

If we consider knowledge bases composed only of classical non-monotonic conditionals $\alpha \mathrel{|\!\sim} \beta$, it corresponds exactly to LM-minimality as defined in the previous section. Nevertheless, given the extra expressive power of PTL, we obtain the surprising result that the two semantic constructions are not equivalent anymore. Moreover, in the present context, this notion of minimality can give back a number of minimal models, as the following example shows.

Example 6.1 Consider the knowledge base $\mathcal{KB}$ from Example 5.1. Then, one can see that $\min_{\trianglelefteq_{\text{PT}}} \mathit{Mod}(\mathcal{KB}) = \{\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3\}$, where:

$\mathscr{R}_1$:

0	$\{\neg\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$, $\{\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$,

$\mathscr{R}_2$:

2	$\{\mathsf{f}, \mathsf{p}, \neg\mathsf{r}\}$
1	$\{\neg\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$, $\{\neg\mathsf{f}, \mathsf{p}, \neg\mathsf{r}\}$
0	$\{\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$

$\mathscr{R}_3$:

2	$\{\neg\mathsf{f}, \neg\mathsf{p}, \mathsf{r}\}$
1	$\{\mathsf{f}, \neg\mathsf{p}, \mathsf{r}\}$, $\{\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$
0	$\{\neg\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$

In Example 6.1, note that $\mathscr{R}_1$ is the LM-minimum of $\mathcal{KB}$. In fact, it is easy to check from the characterisation of rational closure in Section 3 and Definition 6.1 that the LM-minimum of $\mathcal{KB}$ is always in $\min_{\trianglelefteq_{\text{PT}}} \mathit{Mod}(\mathcal{KB})$.

Proposition 6.1 *For every knowledge base $\mathcal{KB}$, the* LM*-minimum of $\mathcal{KB}$ is in $\min_{\trianglelefteq_{\text{PT}}} \mathit{Mod}(\mathcal{KB})$.*

Proof:
Consider the definition of the preference relation for LM-minimality.

$$\mathscr{R}_1 \trianglelefteq_{\text{LM}} \mathscr{R}_2 \text{ if and only if } \begin{array}{ll} \textit{either} & L_i = M_i \text{ for all } i \in \{0, \ldots, n-1, \infty\}, \\ \textit{or} & L_j \supseteq M_j \text{ for the smallest } j \geq 0 \text{ s.t. } L_j \neq M_j. \end{array}$$

Let $\mathscr{R} = (L_0, \ldots, L_{n-1}, L_\infty)$ be a model of a knowledge base $\mathcal{KB}$ that is minimal w.r.t. $\trianglelefteq_{\text{LM}}$, that is, there is no model $\mathscr{R}'$ s.t. $\mathscr{R}' \triangleright_{\text{LM}} \mathscr{R}$. We have to prove that it is minimal also w.r.t. $\trianglelefteq_{\text{PT}}$. If this were not the case, we would have a model $\mathscr{R}' = (M_0, \ldots, M_{n-1}, M_\infty)$ of $\mathcal{KB}$ s.t. there is a valuation w s.t. $\mathscr{R}'(w) < \mathscr{R}(w)$. But then we would have that for some $i \in \mathbb{N}$, $L_i \subseteq M_i$, and $L_j = M_j$ for all $0 \leq j < i$. Hence $\mathscr{R}'$ would be preferred also w.r.t. $\trianglelefteq_{\text{LM}}$. $\square$

We are now ready for the definition of our second type of entailment:

Definition 6.2 (PT-Entailment) *Let $\mathcal{KB} \subseteq \mathcal{L}^\bullet$ and $\alpha \in \mathcal{L}^\bullet$. We say $\mathcal{KB}$* PT-entails *α, denoted $\mathcal{KB} \mathrel{|\!\!\approx}_{\mathrm{PT}} \alpha$, if and only if $\min_{\trianglelefteq_{\mathrm{PT}}} \mathit{Mod}(\mathcal{KB})) \subseteq \mathit{Mod}(\alpha)$.*

Its corresponding consequence operator $\mathit{Cn}_{\mathrm{PT}}(\cdot)$ is inferentially *weaker* than $\mathit{Cn}_{\mathrm{LM}}(\cdot)$, since it is defined on a possibly larger set of models.

Proposition 6.2 *$\mathit{Cn}_{\mathrm{PT}}(\cdot)$ satisfies P1–P4 and P7–P10.*

Proof:
P1. $\mathit{Cn}_{PT}(\mathcal{KB})$ is defined using only models of $\mathcal{KB}$.
P2. If every PT-minimal model of $\mathcal{KB}$ is also a model of α, they must also be the PT-minimal models of $\mathcal{KB} \cup \{\alpha\}$. Assume $\mathscr{R}$ is a PT-minimal model of $\mathcal{KB}$ and $\mathscr{R} \Vdash \alpha$, but $\mathscr{R} \notin \min_{\trianglelefteq_{\mathrm{PT}}} \mathit{Mod}(\mathcal{KB} \cup \{\alpha\})$, it means that there is a model $\mathscr{R}'$ of $\mathcal{KB} \cup \{\alpha\}$ s.t. $\mathscr{R}' \trianglelefteq_{\mathrm{PT}} \mathscr{R}$, but in such a case $\mathscr{R}$ would not be in $\min_{\trianglelefteq_{\mathrm{PT}}} \mathit{Mod}(\mathcal{KB})$.

Now assume $\mathscr{R}$ is a PT-minimal model of $\mathcal{KB} \cup \{\alpha\}$ but not of $\mathcal{KB}$, since there is a $\mathcal{KB}$-model $\mathscr{R}'$ s.t. $\mathscr{R}' \trianglelefteq_{\mathrm{PT}} \mathscr{R}$ and $\mathscr{R}' \in \min_{\trianglelefteq_{\mathrm{PT}}} \mathit{Mod}(\mathcal{KB})$; but in such a case, $\mathscr{R}'$ would not satisfy α, against the hypothesis.
P3. Every model in $\min_{\trianglelefteq_{\mathrm{PT}}} \mathit{Mod}(\mathcal{KB})$ is by definition a ranked model of $\mathcal{KB}$, hence it takes part to the definition of $\mathit{Cn}_0(\mathcal{KB})$. So, if $\alpha \in \mathit{Cn}_?(\mathcal{KB})$, then $\alpha \in \mathscr{R}^*$.
P4. It is an immediate consequence of the satisfaction of **P7**.[4]
P7. See the analagous result by Giordano et al. [19, Section 2.3.2]; in particular Theorem 2, that implies that in case of a conditional KB the use of PT-minimality leads to a single minimal model, characterising Rational Closure.
P8. Let α be a propositional formula s.t. $\alpha \notin \mathit{Cn}_0(\mathcal{KB})$: then there is a ranked model $\mathscr{R}$ of $\mathcal{KB}$ s.t. $\mathscr{R}(v) \leq \infty$ for some v s.t. $v \Vdash \neg\alpha$. Either $\mathscr{R}$ is a PT-minimal model of $\mathcal{KB}$ itself, or there is a PT-minimal model $\mathscr{R}'$ of $\mathcal{KB}$ s.t. $\mathscr{R}' \trianglelefteq_{\mathrm{PT}} \mathscr{R}$; that is, it must be the case that $\mathscr{R}'(v) \leq \infty$ for some model $\mathscr{R}' \in \min_{\trianglelefteq_{\mathrm{PT}}} \mathit{Mod}(\mathcal{KB})$, that in turn implies that $\alpha \notin \mathit{Cn}_{PT}(\mathcal{KB})$.
P9. It is an immediate consequence of the satisfaction of **P7**.
P10. It is a direct consequence of Proposition 6.1 and the satisfaction of **P10** for LM-entailment. □

Unfortunately, *Conditional Rationality* (P5) is not valid and therefore, neither is the Single Model postulate (P6).

Theorem 6.1 *There is some $\mathcal{KB}$ such that the conditional induced by $\mathit{Cn}_{\mathrm{PT}}(\mathcal{KB})$ is not a rational conditional.*

To see this, consider Example 6.1: we have $\bullet\neg\mathsf{p} \rightarrow \neg\mathsf{r} \in \mathit{Cn}_{\mathrm{PT}}(\mathcal{KB})$ (typical non-penguins are not robins—since we know the most typical things are not robins), but neither $\bullet\neg\mathsf{p} \rightarrow \neg\mathsf{f} \in \mathit{Cn}_{\mathrm{PT}}(\mathcal{KB})$, nor $\bullet(\neg\mathsf{p} \wedge \mathsf{f}) \rightarrow \neg\mathsf{r} \in \mathit{Cn}_{\mathrm{PT}}(\mathcal{KB})$, which means the rational monotonicity property (RM) is not satisfied.

[4] As in Theorem 5.2, for this conclusion we need the requirement (specified in Section 2) that $\mathcal{P}$ contains at least two elements.

On the other hand, observe that $\neg\mathsf{p} \notin Cn_{\mathrm{PT}}(\mathcal{KB})$. Recall from the proof of Theorem 5.2 that we used the fact that $\neg\mathsf{p} \in Cn_{\mathrm{LM}}(\mathcal{KB})$ to show that LM-entailment does not satisfy Strict Entailment (P8).

7 PT'-entailment

As we have shown above, relying on LM-minimality results in the loss of property P8 (Strict Entailment), while using PT-minimality results in the loss of the uniqueness of the minimal model (P6) and the rationality of our conditional reasoning (P5). In this section we consider a third possible form of entailment—one in which we aim to augment the inferential power w.r.t. ranked entailment while preserving P8. In doing so we still rely on PT-minimality, but among the PT-minimal models we consider only the ones with the *maximal* sets of possible valuations (w.r.t. $\subseteq$). That is, we let $\min^{\supseteq}_{\trianglelefteq_{\mathrm{PT}}} Mod(\mathcal{KB}) := \{\mathscr{R} \in \min_{\trianglelefteq_{\mathrm{PT}}} Mod(\mathcal{KB}) \mid$ there is no $\mathscr{R}' \in \min_{\trianglelefteq_{\mathrm{PT}}} Mod(\mathcal{KB})$ s.t. $\mathcal{U}^{\mathscr{R}'} \supset \mathcal{U}^{\mathscr{R}}\}$.

The corresponding entailment relation $\approx_{\mathrm{PT}'}$ can be defined as follows.

Definition 7.1 (PT'-Entailment) *Let $\mathcal{KB} \subseteq \mathcal{L}^{\bullet}$ and $\alpha \in \mathcal{L}^{\bullet}$. We say $\mathcal{KB}$* PT'-entails *α, denoted $\mathcal{KB} \models\!\!\approx_{\mathrm{PT}'} \alpha$, if and only if $\min^{\supseteq}_{\trianglelefteq_{\mathrm{PT}}} Mod(\mathcal{KB}) \subseteq Mod(\alpha)$.*

For example, in Example 6.1 we would consider only $\mathscr{R}_2$ and $\mathscr{R}_3$.

Our first result regarding PT'-entailment is that it is inferentially stronger than PT-entailment.

Proposition 7.1 *For every formula α, if $\mathcal{KB} \models\!\!\approx_{\mathrm{PT}} \alpha$ then $\mathcal{KB} \models\!\!\approx_{\mathrm{PT}'} \alpha$. Conversely, there is a fomula α s.t. $\mathcal{KB} \models\!\!\approx_{\mathrm{PT}'} \alpha$ and $\mathcal{KB} \not\models\!\!\approx_{\mathrm{PT}} \alpha$.*

Proof:
Note firstly that $\min^{\supseteq}_{\trianglelefteq_{\mathrm{PT}}} Mod(\mathcal{KB}) \subseteq \min_{\trianglelefteq_{\mathrm{PT}}} Mod(\mathcal{KB})$ for every $\mathcal{KB}$ implies that $\models\!\!\approx_{\mathrm{PT}} \subseteq \models\!\!\approx_{\mathrm{PT}'}$. Moreover, observe from Example 7.1 that $\mathcal{KB}' \models\!\!\approx_{\mathrm{PT}'} \bullet\top \rightarrow \neg\mathsf{f}$ but $\mathcal{KB}' \not\models\!\!\approx_{\mathrm{PT}} \bullet\top \rightarrow \neg\mathsf{f}$. □

Example 7.1 Consider the knowledge base $\mathcal{KB}' := \{\bullet\top \rightarrow (\neg\mathsf{p} \wedge \neg\mathsf{r}), \bullet\mathsf{p} \rightarrow \neg\mathsf{f}, \bullet\mathsf{r} \rightarrow \bullet\mathsf{f}, \mathsf{p} \rightarrow \neg\mathsf{r}\}$, which is a modified version of the knowledge $\mathcal{KB}$ from Example 5.1. The only difference is that now we state that typical penguins are non-flying birds, not that they are typical non-flying birds.

Then, one can check that $\min_{\trianglelefteq_{\mathrm{PT}}} Mod(\mathcal{KB}') = \{\mathscr{R}_1, \mathscr{R}_2\}$, where:

$\mathscr{R}_1$:

2	$\{\mathsf{f}, \mathsf{p}, \neg\mathsf{r}\}$
1	$\{\neg\mathsf{f}, \mathsf{p}, \neg\mathsf{r}\}$
0	$\{\neg\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$, $\{\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$,

$\mathscr{R}_2$:

2	$\{\neg\mathsf{f}, \neg\mathsf{p}, \mathsf{r}\}$, $\{\mathsf{f}, \mathsf{p}, \neg\mathsf{r}\}$
1	$\{\mathsf{f}, \neg\mathsf{p}, \mathsf{r}\}$, $\{\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$, $\{\neg\mathsf{f}, \mathsf{p}, \neg\mathsf{r}\}$
0	$\{\neg\mathsf{f}, \neg\mathsf{p}, \neg\mathsf{r}\}$

while $\min^{\supseteq}_{\trianglelefteq_{\mathrm{PT}}} \mathit{Mod}(\mathcal{KB}') = \{\mathscr{R}_2\}$.

Unfortunately, while PT'-entailment is an improvement over PT-entailment in terms of inferential strength, it is weaker than PT-entailment when it comes to the satisfaction of the list of desirable properties. That is, it satisfies, and fails to satisfy, the same properties as PT-entailment, except for Typical Entailment (P10), which PT-entailment satisfies, but PT'-entailment does not.

Proposition 7.2 *$Cn_{\mathrm{PT}'}(\cdot)$ satisfies P1–P4 and P7–P9, but does not satify P5, P6, and P10.*

Proof:
Regarding **P1, P2, P3, P4**, and **P9** the proof for $Cn_{PT'}(\cdot)$ is the same as for $Cn_{PT}(\cdot)$ (Proposition 6.2 above).
Regarding the failure of **P5**, consider Example 6.1. In this example, while $\min_{\trianglelefteq_{\mathrm{PT}}} \mathit{Mod}(\mathcal{KB}) = \{\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3\}$, we have that $\min^{\supseteq}_{\trianglelefteq_{\mathrm{PT}}} \mathit{Mod}(\mathcal{KB}) = \{\mathscr{R}_2, \mathscr{R}_3\}$. We can use the same case used in the proof of Theorem 6.1: we have $\mathcal{KB} \mathrel{|\!\!\approx}_{\mathrm{PT}'} \bullet(\neg \mathsf{p}) \rightarrow \neg \mathsf{r}$, but neither $\mathcal{KB} \mathrel{|\!\!\approx}_{\mathrm{PT}'} \bullet(\neg \mathsf{p}) \rightarrow \neg \mathsf{f}$, nor $\mathcal{KB} \mathrel{|\!\!\approx}_{\mathrm{PT}'} \bullet(\neg \mathsf{p} \wedge \mathsf{f}) \rightarrow \neg \mathsf{r}$ hold.
The failure of **P5** immediately implies the failure of **P6**.
P7. As pointed out in Proposition 6.2, in case we are dealing with a conditional KB, deciding PT-minimality over a consistent conditional KB gives back a single minimal model, characterising Rational Closure. It follows immediately that such a model is also the only PT'-minimal one.
P8. Again, it follows from the satisfaction of P8 for PT-entailment (see Proposition 6.2). Let $\mathcal{KB}$ be a knowledge base and α be a propositional formula. If there is a PT-minimal model $\mathscr{R}$ s.t. $\mathscr{R}(v) \leq \infty$ for some $v \not\Vdash \alpha$, then, by definition, there must be also in $\min^{\supseteq}_{\trianglelefteq_{\mathrm{PT}}} \mathit{Mod}(\mathcal{KB})$ a model $\mathscr{R}'$ of $\mathcal{KB}$ s.t. $\mathscr{R}'(v) \leq \infty$.
For the failure of **P10**, we consider Example 7.1 and to the case used in the proof of Proposition 7.1: $\mathcal{KB}' \mathrel{|\!\!\approx}_{\mathrm{PT}'} \bullet\top \rightarrow \neg \mathsf{f}$ but, since $\mathcal{KB}' \mathrel{|\!\!\not\approx}_{\mathrm{PT}} \bullet\top \rightarrow \neg \mathsf{f}$ and $\mathrel{|\!\!\approx}_{\mathrm{PT}}$ satisfies *Ampliativeness* (**P3**), $\bullet\top \rightarrow \neg \mathsf{f}$ is not in $Cn_0(\mathcal{KB}')$.

□

8 Making sense of the impossibility result

Theorem 4.1 in Section 4 shows that there is no PTL consequence operator satisfying all of our postulates—more specifically, none satisfying P1, P2, P3, P5, P8, and P10. This raises the important question of which of these postulates ought to be foregone in the search for an appropriate form of PTL entailment. In trying to find an answer to this question, it is useful to consider the three forms of entailment we proposed in the previous sections. The answer seems to be that it makes sense to consider (at least) two forms of entailment for PTL, represented here by LM-entailment and PT-entailment. PT'-entailment is not viewed as a viable option, given that it satisfies fewer properties than PT-entailment. In essence then, it comes down to a choice between having a form of entailment that satisfies Strict Entailment (PT-entailment), and one that satisfies the Single Model postulate and Conditional Rationality, i.e., is based on a rational conditional (LM-entailment).

The advantage of LM-entailment is that it satisfies all postulates except for Strict Entailment, which includes not only Single Model and Conditional Rationality, but also Conditional Strict Entailment and Classical Entailment, the weakened versions of Strict Entailment. On the other hand, the argument for PT-entailment is that the Single Model property is too restrictive in the context of full PTL, and ought to be dropped. That is, in a logic as expressive as PTL in which there are not any restrictions on the typicality operator, any form of entailment based on minimality, and adhering to the presumption of typicality, as outlined in Section 6, is likely to violate the Single Model property.

The point of view that different forms of entailment can be appropriate in enriched versions of propositional logic, particularly enriched versions dealing with aspects of typicality, is not surprising, nor new. Lehmann [21] makes the case for two forms of entailment for the conditional logic discussed in Section 2.1 on which PTL is based. He draws a distinction between *prototypical reasoning*, corresponding to rational closure as discussed in Section 2.2, and *presumptive reasoning*. The details of the differences between prototypical and presumptive reasoning is not that important for our purposes here. The important point is that differences in context will determine which form of entailment is appropriate. It is our contention that the same principle applies to the differences between LM-entailment and PT-entailment.

As we have seen above, the difference between these two forms of entailment comes down to a choice between Strict Entailment on the one hand, Conditional Rationality (and Single Model) on the other hand. Employing LM-entailment ensures that we remain rational (i.e., satisfying all the KLM properties), but at the cost of going beyond Tarskian monotonicity for typicality-free sentences. Conversely, making use of PT-entailment allows us to remain Tarskian for typicality-free sentences, but forces us to forego rationality, and in particular, the rational monotonicity property RM. Intuitively then, LM-entailment is the more permissive form of entailment here. Not only do we remain rational, unlike PT-entailment, but we do so at the cost of allowing the entailment of more typicality-free sentences than permitted by PT-entailment. We conclude this section with an example illustrating this point.

Example 8.1 Consider again the knowledge base $\mathcal{KB} := \{\bullet\top \rightarrow (\neg\mathsf{p}\wedge\neg\mathsf{r}), \bullet\mathsf{p} \rightarrow \bullet\neg\mathsf{f}, \bullet\mathsf{r} \rightarrow \bullet\mathsf{f}, \mathsf{p} \rightarrow \neg\mathsf{r}\}$ introduced in Example 5.1. It is not hard to verify that both LM-entailment and PT-entailment sanction the conclusion that typical non-robins are not penguins ($\mathcal{KB} \approx_{\mathrm{LM}} \bullet(\neg\mathsf{r}) \rightarrow \neg\mathsf{p}$ and $\mathcal{KB} \approx_{\mathrm{PT}} \bullet(\neg\mathsf{r}) \rightarrow \neg\mathsf{p}$), and do *not* allow for the entailment that typical non-robins cannot fly ($\mathcal{KB} \not\approx_{\mathrm{LM}} \bullet(\neg\mathsf{r}) \rightarrow \neg\mathsf{f}$ and $\mathcal{KB} \not\approx_{\mathrm{PT}} \bullet(\neg\mathsf{r}) \rightarrow \neg\mathsf{f}$). This leaves us with a choice. On the one hand it is reasonable to conclude from this that typical *flying* non-robins are not penguins. In fact, rational monotonicity requires of us to be able to draw this conclusion. But in order to do so, we need to be able to conclude that there are no penguins, which violates Strict Entailment. This is the route followed by

LM-entailment. The other option would be to insist that we do not have enough information to conclude that there are no penguins, but in the process of doing so, also forego the conclusion that typical flying non-robins are not penguins. That is, we insist on Strict Entailment at the expense of rational monotonicity. This is the path followed by PT-entailment. □

9 Conclusion

The focus of this paper is an investigation into the entailment problem for the logic PTL. We approached the problem from two angles: an abstract formal perspective, in which a set of appropriate postulates were presented and discussed, and a constructive perspective, in which three specific entailment relations were defined and studied. The primary conclusion to be drawn from this investigation is that a logic as expressive as PTL supports more than one form of entailment. This conclusion is supported from the abstract perspective via an impossibility result, as well as through the constructive approach via the definition of two of the three distinct types of PTL entailment: LM-entailment and PT-entailment. While both forms of entailment are generalisations of *rational closure*, only one, LM-entailment, retains all the rationality properties associated with rational closure, formalised as the Conditional Rationality postulate (P5). However, it does not satisfy Strict Entailment (P8), a postulate which requires an entailment relation to remain Tarskian for conclusions not involving typicality, although it satisfies weakened versions of Strict Entailment (P9 and P9′). On the other hand, the other form of entailment we studied, PT-entailment, satisfies P8, but not Conditional Rationality (P5).

The framework of Booth et al. [1, 2] is, to the best of our knowledge, the first attempt to introduce a full-fledged typicality operator into propositional logic. In terms of other related work, the closest we are aware of is the restricted form of typicality for description logics by Giordano et al. [16]. However, a consequence of their restricted use of typicality is that a propositional version of their logic would correspond to a KLM-style conditional logic in which rational closure behaves well, and which is much less expressive than PTL.

Britz et al. [6] and Giordano et al. [16] have investigated the connection between the KLM approach and Gödel-Löb modal logic, which is closely related to PTL. Exploiting this connection should deliver an axiomatisation of an inference relation corresponding to ranked entailment, but it does not seem useful for modelling entailment relations based on minimisation as LM- and PT-entailment.

For future work, an obvious open question is whether our conjecture, that the subsets of postulates satisfied by LM-entailment and PT-entailment respectively provide appropriate abstract formalisations of two distinct forms of PTL entailment, can be formalised through representation theorems. From a computational perspective, it is worth investigating whether, as is the case for rational closure for conditional logics, the computation of (the different forms of) PTL entailment can be reduced to a series of classical entailment checks.

Our results in the propositional setting pave the way for an investigation of appropriate forms of entailment in other, more expressive, preferential approaches, such as preferential description logics [8, 18, 5, 10, 12] and modal logics of defeasibility [7, 9, 11, 13]. The move to logics with more structure is of a challenging nature, and a simple rephrasing of our approach to these logics may not deliver the expected results. We are currently investigating these issues.

Acknowledgements

The work of Giovanni Casini and of Thomas Meyer has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 690974 (MIREL project). The work of Thomas Meyer has been supported in part by the National Research Foundation of South Africa (grant No. UID 98019).

References

[1] R. Booth, T. Meyer, and I. Varzinczak. PTL: A propositional typicality logic. In L. Fariñas del Cerro, A. Herzig, and J. Mengin, editors, *Proceedings of the 13th European Conference on Logics in Artificial Intelligence (JELIA)*, number 7519 in LNCS, pages 107–119. Springer, 2012.

[2] R. Booth, T. Meyer, and I. Varzinczak. A propositional typicality logic for extending rational consequence. In E. Fermé, D. Gabbay, and G. Simari, editors, *Trends in Belief Revision and Argumentation Dynamics*, volume 48 of *Studies in Logic – Logic and Cognitive Systems*, pages 123–154. King's College Publications, 2013.

[3] R. Booth and J. Paris. A note on the rational closure of knowledge bases with both positive and negative knowledge. *Journal of Logic, Language and Information*, 7(2):165–190, 1998.

[4] C. Boutilier. Conditional logics of normality: A modal approach. *Artificial Intelligence*, 68(1):87–154, 1994.

[5] K. Britz, G. Casini, T. Meyer, K. Moodley, U. Sattler, and I. Varzinczak. Rational defeasible reasoning for description logics. Technical report, University of Cape Town, South Africa, 2017.

[6] K. Britz, J. Heidema, and T. Meyer. Modelling object typicality in description logics. In A. Nicholson and X. Li, editors, *Proceedings of the 22nd Australasian Joint Conference on Artificial Intelligence*, number 5866 in LNAI, pages 506–516. Springer, 2009.

[7] K. Britz, T. Meyer, and I. Varzinczak. Preferential reasoning for modal logics. *Electronic Notes in Theoretical Computer Science*, 278:55–69, 2011. Proceedings of the 7th Workshop on Methods for Modalities (M4M'2011).

[8] K. Britz, T. Meyer, and I. Varzinczak. Semantic foundation for preferential description logics. In D. Wang and M. Reynolds, editors, *Proceedings of the 24th Australasian Joint Conference on Artificial Intelligence*, number 7106 in LNAI, pages 491–500. Springer, 2011.

[9] K. Britz and I. Varzinczak. Defeasible modalities. In *Proceedings of the 14th Conference on Theoretical Aspects of Rationality and Knowledge (TARK)*, pages 49–60, 2013.

[10] K. Britz and I. Varzinczak. Introducing role defeasibility in description logics. In L. Michael and A. Kakas, editors, *Proceedings of the 15th European Conference on Logics in Artificial Intelligence (JELIA)*, number 10021 in LNCS, pages 174–189. Springer, 2016.

[11] K. Britz and I. Varzinczak. From KLM-style conditionals to defeasible modalities, and back. *Journal of Applied Non-Classical Logics (JANCL)*, 2017.

[12] K. Britz and I. Varzinczak. Toward defeasible $\mathcal{SROIQ}$. In *Proceedings of the 30th International Workshop on Description Logics*, 2017.

[13] K. Britz and I. Varzinczak. Preferential accessibility and preferred worlds. *Journal of Logic, Language and Information (JoLLI)*, 2018.

[14] G. Casini and U. Straccia. Rational closure for defeasible description logics. In T. Janhunen and I. Niemelä, editors, *Proceedings of the 12th European Conference on Logics in Artificial Intelligence (JELIA)*, number 6341 in LNCS, pages 77–90. Springer-Verlag, 2010.

[15] G. Casini and U. Straccia. Defeasible inheritance-based description logics. *Journal of Artificial Intelligence Research (JAIR)*, 48:415–473, 2013.

[16] L. Giordano, V. Gliozzi, N. Olivetti, and G. Pozzato. $\mathcal{ALC}+T$: a preferential extension of description logics. *Fundamenta Informaticae*, 96(3):341–372, 2009.

[17] L. Giordano, V. Gliozzi, N. Olivetti, and G. Pozzato. A minimal model semantics for nonmonotonic reasoning. In L. Fariñas del Cerro, A. Herzig, and J. Mengin, editors, *Proceedings of the 13th European Conference on Logics in Artificial Intelligence (JELIA)*, number 7519 in LNCS, pages 228–241. Springer, 2012.

[18] L. Giordano, V. Gliozzi, N. Olivetti, and G. Pozzato. A non-monotonic description logic for reasoning about typicality. *Artificial Intelligence*, 195:165–202, 2013.

[19] L. Giordano, V. Gliozzi, N. Olivetti, and G. Pozzato. Semantic characterization of rational closure: From propositional logic to description logics. *Artificial Intelligence*, 226:1–33, 2015.

[20] S. Kraus, D. Lehmann, and M. Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44:167–207, 1990.

[21] D. Lehmann. Another perspective on default reasoning. *Annals of Mathematics and Artificial Intelligence*, 15(1):61–82, 1995.

[22] D. Lehmann and M. Magidor. What does a conditional knowledge base entail? *Artificial Intelligence*, 55:1–60, 1992.

[23] D. Makinson. *Bridges from Classical to Nonmonotonic Logic*, volume 5 of *Texts in Computing*. King's College Publications, 2005.

[24] J. Quantz and V. Royer. A preference semantics for defaults in terminological logics. In *Proceedings of the 3rd International Conference on Principles of Knowledge Representation and Reasoning (KR)*, pages 294–305, 1992.

[25] Y. Shoham. *Reasoning about Change: Time and Causation from the Standpoint of Artificial Intelligence*. MIT Press, 1988.

[26] A. Tarski. *On some fundamental concepts of metamathematics. [1930] Logic, Semantics, Metamathematics. Papers from 1923 to 1938, translated by J.H. Woodger. Pages 30–36.* Clarendon Press, 1956.

Appendix

A Proof of Lemmas 5.1, 5.2 and 5.3

A.1 Proof of Lemma 5.1

Lemma 5.1 *The following hold for each $i \geq 0$:*

1. *$S_i \subseteq S_{i+1}$, i.e., $[\![\mathcal{KB}]\!]^{\mathscr{R}_i} \subseteq [\![\mathcal{KB}]\!]^{\mathscr{R}_{i+1}}$;*
2. *For all $v_1, v_2 \in \mathcal{U}$, if $\mathscr{R}_i(v_1) < \mathscr{R}_i(v_2)$, then $v_1 \in [\![\mathcal{KB}]\!]^{\mathscr{R}_i}$;*
3. *$\mathscr{R}_i$ is a ranked interpretation.*

Proof:
We show all three simultaneously by complete induction on i. So, assume all of Items 1, 2 and 3 hold for all $m < i$. We will show this implies all three hold also for i. We assume each $\alpha \in \mathcal{KB}$ is in normal form.
1. $[\![\mathcal{KB}]\!]^{\mathscr{R}_i} \subseteq [\![\mathcal{KB}]\!]^{\mathscr{R}_{i+1}}$.
Let $v \in [\![\mathcal{KB}]\!]^{\mathscr{R}_i}$ and let $\alpha \in \mathcal{KB}$ with $\alpha = \bigwedge_{i \leq t} \bullet\theta_i \rightarrow (\phi \vee \bigvee_{i \leq s} \bullet\psi_i)$ (for some $s, t \geq 0$). We must show $v \in [\![\alpha]\!]^{\mathscr{R}_{i+1}}$. Since $v \in [\![\mathcal{KB}]\!]^{\mathscr{R}_i}$ we know $v \in [\![\alpha]\!]^{\mathscr{R}_i}$. Hence we know that one of the following must hold:

- $v \notin [\![\bullet\theta_k]\!]^{\mathscr{R}_i}$ for some k: This means (since θ_k is propositional) v is not $\prec^{\mathscr{R}_i}$-minimal in $[\![\theta_k]\!]^{\mathscr{R}_i} = [\![\theta_k]\!]^{\mathscr{R}_{i+1}}$. But then it is also not $\prec^{\mathscr{R}_{i+1}}$-minimal since, by construction, if $\mathscr{R}_i(v) \leq \mathscr{R}_i(w)$ then $\mathscr{R}_{i+1}(v) \leq \mathscr{R}_{i+1}(w)$. Hence in this case $v \notin [\![\bullet\theta_k]\!]^{\mathscr{R}_{i+1}}$.
- $v \in [\![\phi]\!]^{\mathscr{R}_i}$: In this case also $v \in [\![\phi]\!]^{\mathscr{R}_{i+1}}$, since $[\![\phi]\!]^{\mathscr{R}_i} = [\![\phi]\!]^{\mathscr{R}_{i+1}}$ (because ϕ is purely propositional).
- $v \in [\![\bullet\psi_k]\!]^{\mathscr{R}_i}$ for some k: This means v is $\prec^{\mathscr{R}_i}$-minimal in $[\![\psi_k]\!]^{\mathscr{R}_i}$. But then it is also $\prec^{\mathscr{R}_{i+1}}$-minimal, since we assumed $v \in [\![\mathcal{KB}]\!]^{\mathscr{R}_i} = S_{i+1}$, and so by construction of $\mathscr{R}_{i+1}$ we have that $\mathscr{R}_{i+1}(w) < \mathscr{R}_{i+1}(v)$ if and only if $\mathscr{R}_i(w) < \mathscr{R}_i(v)$ for all $w \in \mathcal{U}$. Since $[\![\psi_k]\!]^{\mathscr{R}_i} = [\![\psi_k]\!]^{\mathscr{R}_{i+1}}$ (since ψ_k is purely propositional) we obtain that v is $\prec^{\mathscr{R}_{i+1}}$-minimal in $[\![\psi_k]\!]^{\mathscr{R}_{i+1}}$, i.e., $v \in [\![\bullet\psi_k]\!]^{\mathscr{R}_{i+1}}$.

Thus in all three possible cases we obtain $v \in [\![\alpha]\!]^{\mathscr{R}_{i+1}}$ as required.
2. $\mathscr{R}_i(v_1) < \mathscr{R}_i(v_2)$ implies $v_1 \in [\![\mathcal{KB}]\!]^{\mathscr{R}_i}$.
Suppose $\mathscr{R}_i(v_1) < \mathscr{R}_i(v_2)$. Observe that, by construction, this can only be the case if $i > 0$. Then either $\mathscr{R}_{i-1}(v_1) < \mathscr{R}_{i-1}(v_2)$ or $v_2 \notin S_i$. If $\mathscr{R}_{i-1}(v_1) < \mathscr{R}_{i-1}(v_2)$ then, by the inductive hypothesis, $v_1 \in [\![\mathcal{KB}]\!]^{\mathscr{R}_{i-1}}$, while if $v_2 \notin S_i$, then $v_1 \in S_i = [\![\mathcal{KB}]\!]^{\mathscr{R}_{i-1}}$. So either way we get $v_1 \in [\![\mathcal{KB}]\!]^{\mathscr{R}_{i-1}}$ and so we get the desired conclusion by applying $[\![\mathcal{KB}]\!]^{\mathscr{R}_{i-1}} \subseteq [\![\mathcal{KB}]\!]^{\mathscr{R}_i}$ which was just proved in Item 1 above.
3. $\mathscr{R}_i$ is a ranked interpretation.
By construction it immediately follows that $\mathscr{R}_i$ is a function from $\mathcal{U}$ to $\mathbb{N} \cup \{\infty\}$.

We need to show the convexity property: if $\mathscr{R}_i(u) = j$ then, for every k such that $0 \leq k < j$, there is a $v \in \mathcal{U}$ for which $\mathscr{R}_i(v) = k$. If $i = 0$, this follows immediately (since $\mathscr{R}_0(u) = 0$ for all $u \in \mathcal{U}$). Otherwise we have by the inductive hypothesis that $\mathscr{R}_{i-1}$ is a ranked interpretation. We have two cases. (1) $S_i = S_{i-1}$: Then $\mathscr{R}_i = (\mathscr{R}_{i-1})^{\infty}_{S_i}$ from which convexity follows immediately. (2) $S_i \neq S_{i-1}$: Then $\mathscr{R}_i = (\mathscr{R}_{i-1})^{1}_{S_i}$ from which convexity also follows immediately.

□

A.2 Proof of Lemma 5.2

Lemma 5.2 *For every $\mathcal{KB}$ and every $i > 0$, $(\mathscr{R}_i)^{\infty}_{S_i}$ is a ranked model of $\mathcal{KB}$.*

Proof:
Let $\mathscr{R}$ denote $(\mathscr{R}_i)^{\infty}_{S_i}$. We need to show that for every valuation $v \in \mathcal{U}^{\mathscr{R}}$, i.e., every $v \in S_i = [\![\mathcal{KB}]\!]^{\mathscr{R}_{i-1}}$, and for every $\alpha \in \mathcal{KB}$, we have $v \in [\![\alpha]\!]^{\mathscr{R}}$. Since $v \in [\![\alpha]\!]^{\mathscr{R}_{i-1}}$ we know one of the following must hold (recalling that α is expressed in normal form $\bigwedge_{i \leq t} \bullet\theta_i \rightarrow (\phi \vee \bigvee_{i \leq s} \bullet\psi_i)$.):

- $v \notin [\![\bullet\theta_k]\!]^{\mathscr{R}_{i-1}}$ for some k: This means v is not $\prec^{\mathscr{R}_{i-1}}$-minimal in $[\![\theta_k]\!]^{\mathscr{R}_{i-1}}$. But then it is also not $\prec^{\mathscr{R}}$-minimal in $[\![\theta_k]\!]^{\mathscr{R}} = [\![\theta_k]\!]^{\mathscr{R}_{i-1}} \cap S_i$, since if $w \prec^{\mathscr{R}_{i-1}} v$ and $w \in [\![\theta_k]\!]^{\mathscr{R}_{i-1}}$, then from the former we know $w \in S_i$ by Item 2 of the previous lemma. Hence in this case $v \notin [\![\bullet\theta_k]\!]^{\mathscr{R}}$.
- $v \in [\![\phi]\!]^{\mathscr{R}_{i-1}}$: In this case also $v \in [\![\phi]\!]^{\mathscr{R}}$, since $[\![\phi]\!]^{\mathscr{R}} = [\![\phi]\!]^{\mathscr{R}_{i-1}} \cap S_i$ (because ϕ is purely propositional).
- $v \in [\![\bullet\psi_k]\!]^{\mathscr{R}_{i-1}}$ for some k: This means v is $\prec^{\mathscr{R}_{i-1}}$-minimal in $[\![\psi_k]\!]^{\mathscr{R}_{i-1}}$. But then it is also $\prec^{\mathscr{R}}$-minimal in $[\![\psi_k]\!]^{\mathscr{R}} = [\![\psi_k]\!]^{\mathscr{R}_{i-1}} \cap S_i$, since $\prec^{\mathscr{R}_{i-1}} \subseteq \prec^{\mathscr{R}}$. Hence $v \in [\![\bullet\psi_k]\!]^{\mathscr{R}}$.

Thus in all three possible cases we obtain $v \in [\![\alpha]\!]^{\mathscr{R}}$ as required. □

A.3 Proof of Lemma 5.3

Lemma 5.3 *Let $\mathscr{R}^*_{\mathcal{KB}} := (L_0, \ldots, L_{n-1}, L_\infty)$ and let $\mathscr{R} := (M_0, \ldots, M_{n-1}, M_\infty)$ be any other ranked model of $\mathcal{KB}$. Let $i \in \{0, \ldots, n-1\}$. If $L_j = M_j$ for all $j < i$, then $M_i \subseteq L_i$.*

Proof:
Let $v \in M_i$. By construction, $S_i = [\![\mathcal{KB}]\!]^{\mathscr{R}_{i-1}}$ where $\mathscr{R}_{i-1} = (L_0, \ldots, L_{i-1}, (\mathcal{U} \setminus \bigcup_{j<i} L_j, \emptyset)$. Let $\alpha \in \mathcal{KB}$, with $\alpha = \bigwedge_{i \leq t} \bullet\theta_i \rightarrow (\phi \vee \bigvee_{i \leq s} \bullet\psi_i)$ (for some $s, t \geq 0$). We must show v satisfies α in $\mathscr{R}_{i-1}$, so assume v satisfies $\neg\phi$ and is a minimal θ_k-state in $\mathscr{R}_{i-1}$ for all k. We must show v is a minimal ψ_k-state in $\mathscr{R}_{i-1}$ for at least one k. Since we assume $M_j = L_j$ for all $j < i$, we have $\mathscr{R}_{i-1} = (M_0, \ldots, M_{i-1}, (\mathcal{U} \setminus \bigcup_{j<i} M_j), \emptyset)$. Since $v \in M_i$, we can show that, for *any* propositional sentence λ, we have that v is a minimal λ-state in $(M_0, \ldots, M_{i-1}, (\mathcal{U} \setminus \bigcup_{j<i} M_j), \emptyset)$ if and only if it is a minimal λ-state in $(M_0, \ldots, M_i, \emptyset)$. Thus, from the fact that $(M_0, \ldots, M_i, \emptyset)$ is a ranked model of $\mathcal{KB}$, we obtain our conclusion. □