Chapter 1

Krichever–Novikov type algebras. Definitions and Results

1.1. Introduction

The study of infinite dimensional (associative or Lie) algebras and their representations is a huge and rather involved field. Additional structures like a grading or additional informations about e.g. their origin will be indispensable to obtain insights and results.

Krichever-Novikov (KN) type algebras are an important class of infinite dimensional algebras. Roughly speaking, they are defined as algebras of meromorphic objects on compact Riemann surfaces, or equivalently on projective curves. The non-holomorphicity is controlled by a fixed finite set of points where poles are allowed. A splitting of this set of possible points of poles into two disjoint subsets will induce an "almost-grading" (see Definition 1.5.1 below). It is a weaker concept as a grading, but still powerful enough to act as a basic tool in representation theory. For example, highest weight representations still can be defined. Of course, central extensions of these algebras are also needed. They are forced, e.g. by representation theory and by quantization.

Examples of KN type algebras are the well-known algebras of Conformal Field Theory (CFT) the Witt algebra, the Virasoro algebra, the affine Lie algebras (affine Kac-Moody algebras), etc. They appear when the geometric setting consists of the

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Riemann Sphere, i.e. the genus zero Riemann surface, and the points of possible poles are $\{0\}$ and $\{\infty\}$. The almost-grading is now a honest grading.

Historically, starting from these well-known genus zero algebras, in 1986 Krichever and Novikov [KRI 87a], [KRI 87b], [KRI 89] suggested a global operator approach via KN objects. Still they only considered two possible points where poles are allowed and were dealing with the vector field and the function algebra. For work on affine algebras Sheinman [SHE 90] should be mentioned.

From the applications in CFT (e.g. string theory) but also from purely mathematical reasons, a multi-point theory is evident. In 1990 the author of the current review developed a systematic theory valid for all genus (including zero) and any fixed finite set of points where poles are allowed [SCH 90b], [SCH 90c], [SCH 90a], [SCH 90d]. These extensions were not at all straight-forward. The main point was to introduce a replacement of the graded algebra structure present in the “classical” case. Krichever and Novikov found that the already mentioned almost-grading (Definition 1.5.1) will be enough to allow for the standard constructions in representation theory. In [SCH 90a], [SCH 90d] it was realized that a splitting of the set A of points where poles are allowed, into two disjoint non-empty subsets $A = I \cup O$ is crucial for introducing an almost-grading. The corresponding almost-grading was explicitly given. In contrast to the classical situation, where there is only one grading, we will have a finite set of non-equivalent gradings and new interesting phenomena show up. This is already true for the genus zero case (i.e. the Riemann sphere case) with more than two points where poles are allowed. These algebras will be only almost-graded, see e.g. [SCH 93], [FIA 03], [FIA 05], [SCH 17].

Also other (Lie) algebras were introduced. In fact most of them come from a *Mother Poisson Algebra*, the algebra of meromorphic form, see Section 1.4.2. This algebra carries a (weak) almost-grading which gives the almost-grading for the other algebras. For the relevant algebras almost-graded central extensions are constructed and classified. In the case of genus zero in this way universal central extensions are obtained.

KN type algebras have a lot of interesting applications. They show up in the context of deformations of algebras, moduli spaces of marked curves, Wess-Zumino-Novikov-Witten (WZNW) models, Knizhnik-Zamolodchikov (KZ) equations, integrable systems, quantum field theories, symmetry algebras, and in many more domains of mathematics and theoretical physics. The KN type algebras carry a very rich representation theory. We have Verma modules, highest weight representations, Fermionic and Bosonic Fock representations, semi-infinite wedge forms, $b - c$ systems, Sugawara representations and vertex algebras.

In 2014 the author published the book *Krichever–Novikov type algebras. Theory and applications* [SCH 14b] which collects all the results, proofs and some applications of the multi-point KN algebras. There also a quite extensive list of references can be found, including articles published by physicists on applications in the field-theoretical context. For some applications in the context of integrable systems see also Sheinman, *Current algebras on Riemann surfaces* [SHE 12].

Recently a revived interest in the theory of KN type algebras appeared again in mathematics. The goal of this review is to give a gentle introduction to the KN type algebras in the multi-point setting, and to collect the basic definitions and results so that they are accessible for an interested audience not familiar with them yet. For the proof and more material we have to refer to the original articles and the corresponding parts of [SCH 14b]. There is a certain overlap with a previous survey of mine [SCH 16].

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1.2. The Virasoro Algebra and its Relatives

These algebras supply examples of non-trivial infinite dimensional Lie algebras. They are widely used in Conformal Field Theory. For the convenience of the reader we will start by recalling their conventional algebraic definitions.

The *Witt algebra* $\mathcal{W}$, sometimes also called Virasoro algebra without central term [1], is the Lie algebra generated as vector space over $\mathbb{C}$ by the basis elements $\{e_n \mid n \in \mathbb{Z}\}$ with Lie structure

$$[e_n, e_m] = (m - n)e_{n+m}, \quad n, m \in \mathbb{Z}. \tag{1.2.1}$$

The algebra $\mathcal{W}$ is more than just a Lie algebra. It is a graded Lie algebra. If we set for the degree $\deg(e_n) := n$ then

$$\mathcal{W} = \bigoplus_{n\in\mathbb{Z}} \mathcal{W}_n, \qquad \mathcal{W}_n = \langle e_n \rangle_{\mathbb{C}}. \tag{1.2.2}$$

Obviously, $\deg([e_n, e_m]) = \deg(e_n) + \deg(e_m)$.

Algebraically $\mathcal{W}$ can also be given as Lie algebra of derivations of the algebra of Laurent polynomials $\mathbb{C}[z, z^{-1}]$.

1. For some remarks what would be a correct naming, see the book [GUI 07].

Remark 1.2.1. In the purely algebraic context our field of definition $\mathbb{C}$ can be replaced by an arbitrary field $\mathbb{K}$ of characteristics 0.

For the Witt algebra the universal one-dimensional central extension is the *Virasoro algebra* $\mathcal{V}$. As vector space it is the direct sum $\mathcal{V} = \mathbb{C} \oplus \mathcal{W}$. If we set for $x \in \mathcal{W}$, $\hat{x} := (0, x)$, and $t := (1, 0)$ then its basis elements are $\hat{e}_n,\ n \in \mathbb{Z}$ and t with the Lie product [2].

$$[\hat{e}_n, \hat{e}_m] = (m-n)\hat{e}_{n+m} + \frac{1}{12}(n^3 - n)\delta_n^{-m}\, t, \qquad [\hat{e}_n, t] = [t, t] = 0, \tag{1.2.3}$$

for all $n, m \in \mathbb{Z}$. By setting $\deg(\hat{e}_n) := \deg(e_n) = n$ and $\deg(t) := 0$ the Lie algebra $\mathcal{V}$ becomes a graded algebra. The algebra $\mathcal{W}$ will only be a subspace, not a subalgebra of $\mathcal{V}$. But it will be a quotient. Up to equivalence of central extensions and rescaling the central element t, this is beside the trivial (splitting) central extension, the only central extension of $\mathcal{W}$.

Given $\mathfrak{g}$ a finite-dimensional Lie algebra (e.g. a finite-dimensional simple Lie algebra) then the tensor product of $\mathfrak{g}$ with the associative algebra of Laurent polynomials $\mathbb{C}[z, z^{-1}]$ carries a Lie algebra structure via

$$[x \otimes z^n, y \otimes z^m] := [x, y] \otimes z^{n+m}. \tag{1.2.4}$$

This algebra is called *current algebra* or *loop algebra* and denoted by $\overline{\mathfrak{g}}$. Again we consider central extensions. For this let β be a symmetric, bilinear form for $\mathfrak{g}$ which is invariant (i.e. $\beta([x, y], z) = \beta(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$). Then a central extension is given by

$$[\widehat{x \otimes z^n}, \widehat{y \otimes z^m}] := \widehat{[x, y] \otimes z^{n+m}} - \beta(x, y) \cdot m\, \delta_n^{-m} \cdot t. \tag{1.2.5}$$

This algebra is denoted by $\widehat{\mathfrak{g}}$ and called *affine Lie algebra*. With respect to the classification of Kac-Moody Lie algebras, in the case of a simple $\mathfrak{g}$ they are exactly the Kac-Moody algebras of untwisted affine type, [KAC 68], [KAC 90], [MOO 69].

To complete the description let me introduce the Lie superalgebra of Neveu-Schwarz type. The centrally extended superalgebra has as basis (we drop the $\hat{}$)

$$e_n,\ n \in \mathbb{Z}, \quad \varphi_m,\ m \in \mathbb{Z} + \frac{1}{2}, \quad t \tag{1.2.6}$$

2. Here δ_k^l is the Kronecker delta which is equal to 1 if $k = l$, otherwise zero.

with structure equations

$$
\begin{aligned}
[e_n, e_m] &= (m-n)e_{m+n} + \frac{1}{12}(n^3 - n)\,\delta_n^{-m}\, t, \\
[e_n, \varphi_m] &= (m - \frac{n}{2})\;\varphi_{m+n}, \\
[\varphi_n, \varphi_m] &= e_{n+m} - \frac{1}{6}\,(n^2 - \frac{1}{4})\,\delta_n^{-m}\, t.
\end{aligned}
\tag{1.2.7}
$$

By "setting $t = 0$" we obtain the non-extended superalgebra. The elements e_n (and t) are a basis of the subspace of even elements, the elements φ_m are a basis of the subspace of odd elements.

These algebras are Lie superalgebras. For completeness I recall their definition here.

Remark 1.2.2. (Definition of a Lie superalgebra). Let $\mathcal{S}$ be a vector space which is decomposed into even and odd elements $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$, i.e. $\mathcal{S}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space. Furthermore, let $[.,.]$ be a $\mathbb{Z}/2\mathbb{Z}$-graded bilinear map $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ such that for elements x, y of pure parity

$$
[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]. \tag{1.2.8}
$$

Here $\bar{x}$ is the parity of x, etc. These conditions say that

$$
[\mathcal{S}_{\bar{0}}, \mathcal{S}_{\bar{0}}] \subseteq \mathcal{S}_{\bar{0}}, \qquad [\mathcal{S}_{\bar{0}}, \mathcal{S}_{\bar{1}}] \subseteq \mathcal{S}_{\bar{1}}, \qquad [\mathcal{S}_{\bar{1}}, \mathcal{S}_{\bar{1}}] \subseteq \mathcal{S}_{\bar{0}}, \tag{1.2.9}
$$

and that $[x, y]$ is symmetric for x and y odd, otherwise anti-symmetric. Now $\mathcal{S}$ is a *Lie superalgebra* if in addition the *super-Jacobi identity* (for x, y, z of pure parity)

$$
(-1)^{\bar{x}\bar{z}}[x, [y, z]] + (-1)^{\bar{y}\bar{x}}[y, [z, x]] + (-1)^{\bar{z}\bar{y}}[z, [x, y]] = 0 \tag{1.2.10}
$$

is valid. As long as the type of the arguments is different from *(even, odd, odd)* all signs can be put to $+1$ and we obtain the form of the usual Jacobi identity. In the remaining case we get

$$
[x, [y, z]] + [y, [z, x]] - [z, [x, y]] = 0. \tag{1.2.11}
$$

By the definitions $\mathcal{S}_0$ is a Lie algebra.

1.3. The Geometric Picture

In the previous section I gave the Virasoro algebra and its relatives by purely algebraic means, i.e. by basis elements and structure equations. The full importance and strength will become more visible in a geometric context. Also from this geometric realization the need for a generalization as obtained via the Krichever–Novikov type algebras will become evident.

1.3.1. *The geometric realizations of the Witt algebra*

A geometric description of the Witt algebra over $\mathbb{C}$ can be given as follows.

Let W be the algebra of those meromorphic vector fields on the Riemann sphere $S^2 = \mathbb{P}^1(\mathbb{C})$ which are holomorphic outside $\{0\}$ and $\{\infty\}$. Its elements can be given as

$$v(z) = \tilde{v}(z)\frac{d}{dz} \qquad (1.3.1)$$

where $\tilde{v}$ is a meromorphic function on $\mathbb{P}^1(\mathbb{C})$, which is holomorphic outside $\{0, \infty\}$. Those are exactly the Laurent polynomials $\mathbb{C}[z, z^{-1}]$. Consequently, this subalgebra has the set $\{e_n, n \in \mathbb{Z}\}$ with $e_n = z^{n+1}\frac{d}{dz}$ as vector space basis. The Lie bracket of vector fields calculates as

$$[v, u] = \left(\tilde{v}\frac{d}{dz}\tilde{u} - \tilde{u}\frac{d}{dz}\tilde{v}\right)\frac{d}{dz}. \qquad (1.3.2)$$

Evaluated for the basis elements e_n this gives (1.2.1) and the algebra can be identified with the Witt algebra defined purely algebraically.

The subalgebra of global holomorphic vector fields is the 3-dimensional subspace $\langle e_{-1}, e_0, e_1\rangle_{\mathbb{C}}$. It is isomorphic to the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.

Similarly, the algebra $\mathbb{C}[z, z^{-1}]$ can be given as the algebra of meromorphic functions on $S^2 = \mathbb{P}^1(\mathbb{C})$ holomorphic outside of $\{0, \infty\}$.

1.3.2. *Arbitrary genus generalizations*

In the geometric setup for the Virasoro algebra the objects are defined on the Riemann sphere and might have poles at most at two fixed points. For a global operator approach to conformal field theory and its quantization this is not sufficient. One needs Riemann surfaces of arbitrary genus. Moreover, one needs more than two points were singularities are allowed [3]. Such a generalizations were initiated by Krichever and Novikov [KRI 87a], [KRI 87b], [KRI 89], who considered arbitrary genus and the two-point case. As far as the current algebras are concerned see also Sheinman [SHE 90], [SHE 92], [SHE 93], [SHE 95]. The multi-point case was systematically examined by the current author [SCH 90b], [SCH 90c], [SCH 90a], [SCH 90d], [SCH 93] [SCH 96], [SCH 03b], [SCH 03a]. For some related approach see also Sadov [SAD 91].

3. The singularities correspond to points where free fields are entering the region of interaction or leaving it. In particular from the very beginning there is a natural decomposition of the set of points into two disjoint subsets.

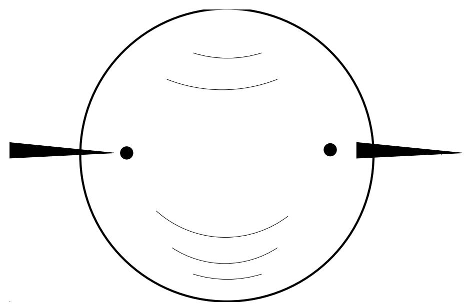

Figure 1.1. *Riemann surface of genus zero with one incoming and one outgoing point.*

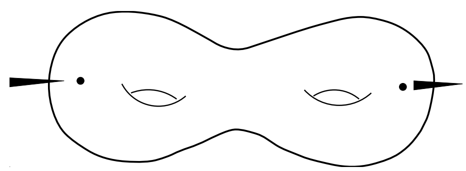

Figure 1.2. *Riemann surface of genus two with one incoming and one outgoing point.*

For the whole contribution let Σ be a compact Riemann surface without any restriction for the genus $g = g(\Sigma)$. Furthermore, let A be a finite subset of Σ. Later we will need a splitting of A into two non-empty disjoint subsets I and O, i.e. $A = I \cup O$. Set $N := \#A$, $K := \#I$, $M := \#O$, with $N = K + M$. More precisely, let

$$I = (P_1, \ldots, P_K), \quad \text{and} \quad O = (Q_1, \ldots, Q_M) \tag{1.3.3}$$

be disjoint ordered tuples of distinct points ("marked points", "punctures") on the Riemann surface. In particular, we assume $P_i \neq Q_j$ for every pair (i, j). The points in I are called the *in-points*, the points in O the *out-points*. Occasionally, we consider I and O simply as sets.

Sometimes we refer to the classical situation. By this we understand

$$\Sigma = \mathbb{P}^1(\mathbb{C}) = S^2, \quad I = \{z = 0\}, \quad O = \{z = \infty\}, \tag{1.3.4}$$

and the situation considered in Section 1.3.1.

The figures should indicate the geometric picture. Figure 1.1 shows the classical situation. Figure 1.2 is genus 2, but still two-point situation. Finally, in Figure 1.3 the case of a Riemann surface of genus 2 with two incoming points and one outgoing point is visualized.

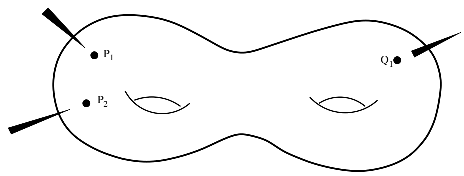

Figure 1.3. *Riemann surface of genus two with two incoming points and one outgoing point.*

Remark 1.3.1. We stress the fact, that these generalizations are needed also in the case of genus zero if one considers more than two points. Even in the case of genus zero and three points interesting algebras show up. See also [SCH 17].

1.3.3. *Meromorphic forms*

To introduce the elements of the generalized algebras (later called Krichever-Novikov type algebras) we first have to discuss forms of certain (conformal) weights. Recall that Σ is a compact Riemann surface of genus $g \geq 0$. Let A be a fixed finite subset of Σ. In fact we could allow for this and the following sections (as long as we do not talk about almost-grading) that A is an arbitrary subset. This includes the extremal cases $A = \emptyset$ or $A = \Sigma$.

Let $\mathcal{K} = \mathcal{K}_\Sigma$ be the canonical line bundle of Σ. Its local sections are the local holomorphic differentials. If $P \in \Sigma$ is a point and z a local holomorphic coordinate at P then a local holomorphic differential can be written as $f(z)dz$ with a local holomorphic function f defined in a neighborhood of P. A global holomorphic section can be described locally in coordinates $(U_i, z_i)_{i\in J}$ by a system of local holomorphic functions $(f_i)_{i\in J}$, which are related by the transformation rule induced by the coordinate change map $z_j = z_j(z_i)$ and the condition $f_i dz_i = f_j dz_j$. This yields

$$f_j = f_i \cdot \left(\frac{dz_j}{dz_i}\right)^{-1}. \qquad (1.3.5)$$

A meromorphic section of $\mathcal{K}$, i.e. a *meromorphic differential* is given as a collection of local meromorphic functions $(h_i)_{i\in J}$ with respect to a coordinate covering for which the transformation law (1.3.5) remains true. We will not make any distinction between the canonical bundle and its sheaf of sections, which is a locally free sheaf of rank 1.

In the following λ is either an integer or a half-integer. If λ is an integer then
(1) $\mathcal{K}^\lambda := \mathcal{K}^{\otimes\lambda}$ for $\lambda > 0$,
(2) $\mathcal{K}^0 := \mathcal{O}$, the trivial line bundle, and
(3) $\mathcal{K}^\lambda := (\mathcal{K}^*)^{\otimes(-\lambda)}$ for $\lambda < 0$.

Here $\mathcal{K}^*$ denotes the dual line bundle of the canonical line bundle. This is the holomorphic tangent line bundle, whose local sections are the holomorphic tangent vector fields $f(z)(d/dz)$.

If λ is a half-integer, then we first have to fix a "square root" of the canonical line bundle, sometimes called a *theta characteristics*. This means we fix a line bundle L for which $L^{\otimes 2} = \mathcal{K}$. After such a choice of L is done we set $\mathcal{K}^\lambda := \mathcal{K}_L^\lambda := L^{\otimes 2\lambda}$. In most cases we will drop the mentioning of L, but we have to keep the choice in mind. The fine-structure of the algebras we are about to define will depend on the choice. But the main properties will remain the same.

Remark 1.3.2. A Riemann surface of genus g has exactly 2^{2g} non-isomorphic square roots of $\mathcal{K}$. For $g = 0$ we have $\mathcal{K} = \mathcal{O}(-2)$, and $L = \mathcal{O}(-1)$, the tautological bundle, is the unique square root. Already for $g = 1$ we have four non-isomorphic ones. As in this case $\mathcal{K} = \mathcal{O}$ one solution is $L_0 = \mathcal{O}$. But we have also other bundles L_i, $i = 1, 2, 3$. Note that L_0 has a nonvanishing global holomorphic section, whereas this is not the case for L_1, L_2 and L_3. In general, depending on the parity of the dimension of the space of globally holomorphic sections, i.e. of $\dim \mathrm{H}^0(\Sigma, L)$, one distinguishes even and odd theta characteristics L. For $g = 1$ the bundle $\mathcal{O}$ is an odd, the others are even theta characteristics. The choice of a theta characteristic is also called a spin structure on Σ [ATI 71].

We set

$$\mathcal{F}^\lambda(A) := \{f \text{ is a global meromorphic section of } \mathcal{K}^\lambda \mid f \text{ is holomorphic on } \Sigma \setminus A\}. \quad (1.3.6)$$

Obviously this is a $\mathbb{C}$-vector space. To avoid cumbersome notation, we will often drop the set A in the notation if A is fixed and clear from the context. Recall that in the case of half-integer λ everything depends on the theta characteristic L.

Definition 1.3.3. The elements of the space $\mathcal{F}^\lambda(A)$ are called *meromorphic forms of weight* λ (with respect to the theta characteristic L).

Remark 1.3.4. In the two extremal cases for the set A we obtain $\mathcal{F}^\lambda(\emptyset)$ the global holomorphic forms, and $\mathcal{F}^\lambda(\Sigma)$ all meromorphic forms. By compactness each $f \in \mathcal{F}^\lambda(\Sigma)$ will have only finitely many poles. In the case that $f \not\equiv 0$ it will also have only finitely many zeros.

If f is a meromorphic λ-form it can be represented locally by meromorphic functions f_i via $f = f_i(dz_i)^{\otimes\lambda}$. If $f \not\equiv 0$ the local representing functions have only finitely many zeros and poles. Whether a point P is a zero or a pole of f does not depend on the coordinate z_i chosen. We can define for $P \in \Sigma$ the *order*

$$\mathrm{ord}_P(f) := \mathrm{ord}_P(f_i), \tag{1.3.7}$$

where $\mathrm{ord}_P(f_i)$ is the lowest nonvanishing order in the Laurent series expansion of f_i in the variable z_i around P. It will not depend on the coordinate z_i chosen.

The order $\mathrm{ord}_P(f)$ is (strictly) positive if and only if P is a zero of f. It is negative if and only if P is a pole of f. Moreover, its value gives the order of the zero and pole respectively. By compactness our Riemann surface Σ can be covered by finitely many coordinate patches. Hence, f can only have finitely many zeros and poles. We define the *(sectional) degree* of f to be

$$\mathrm{sdeg}(f) := \sum_{P\in\Sigma} \mathrm{ord}_P(f). \tag{1.3.8}$$

Proposition 1.3.5. *Let $f \in \mathcal{F}^\lambda$, $f \not\equiv 0$ then*

$$\mathrm{sdeg}(f) = 2\lambda(g-1). \tag{1.3.9}$$

For this and related results see [SCH 07b].

1.4. Algebraic Structures

Next we introduce algebraic operations on the vector space of meromorphic forms of arbitrary weights. This space is obtained by summing over all weights

$$\mathcal{F} := \bigoplus_{\lambda\in\frac{1}{2}\mathbb{Z}} \mathcal{F}^\lambda. \tag{1.4.1}$$

The basic operations will allow us to introduce finally the algebras we are heading for.

1.4.1. *Associative structure*

In this section A is still allowed to be an arbitrary subset of points in Σ. We will drop the subset A in the notation. The natural map of the locally free sheaves of rang one

$$\mathcal{K}^\lambda \times \mathcal{K}^\nu \to \mathcal{K}^\lambda \otimes \mathcal{K}^\nu \cong \mathcal{K}^{\lambda+\nu}, \quad (s,t) \mapsto s \otimes t, \tag{1.4.2}$$

defines a bilinear map

$$\cdot : \mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu}. \tag{1.4.3}$$

With respect to local trivialisations this corresponds to the multiplication of the local representing meromorphic functions

$$(s\,dz^\lambda, t\,dz^\nu) \mapsto s\,dz^\lambda \cdot t\,dz^\nu = s \cdot t\;dz^{\lambda+\nu}. \tag{1.4.4}$$

If there is no danger of confusion then we will mostly use the same symbol for the section and for the local representing function.

The following is obvious

Proposition 1.4.1. *The space $\mathcal{F}$ is an associative and commutative graded (over $\frac{1}{2}\mathbb{Z}$) algebra. Moreover, $\mathcal{A} = \mathcal{F}^0$ is a subalgebra and the $\mathcal{F}^\lambda$ are modules over $\mathcal{A}$.*

Of course, $\mathcal{A}$ is the algebra of those meromorphic functions on Σ which are holomorphic outside of A. In case that $A = \emptyset$, it is the algebra of global holomorphic functions. By compactness, these are only the constants, hence $\mathcal{A}(\emptyset) = \mathbb{C}$. In case that $A = \Sigma$ it is the field of all meromorphic functions $\mathcal{M}(\Sigma)$.

1.4.2. *Lie and Poisson algebra structure*

Next we define a Lie algebra structure on the space $\mathcal{F}$. The structure is induced by the map

$$\mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu+1}, \qquad (e,f) \mapsto [e,f], \tag{1.4.5}$$

which is defined in local representatives of the sections by

$$(e\,dz^\lambda, f\,dz^\nu) \mapsto [e\,dz^\lambda, f\,dz^\nu] := \left((-\lambda)e\frac{df}{dz} + \nu f\frac{de}{dz}\right) dz^{\lambda+\nu+1}, \tag{1.4.6}$$

and bilinearly extended to $\mathcal{F}$. Of course, we have to show the following

Proposition 1.4.2. *[SCH 14b, Prop. 2.6 and 2.7] The prescription $[.,.]$ given by (1.4.6) is well-defined and defines a Lie algebra structure on the vector space $\mathcal{F}$.*

Proposition 1.4.3. *[SCH 14b, Prop. 2.8] The subspace $\mathcal{L} = \mathcal{F}^{-1}$ is a Lie subalgebra, and the $\mathcal{F}^\lambda$'s are Lie modules over $\mathcal{L}$.*

Definition 1.4.4. An algebra $(\mathcal{B}, \cdot, [.,.])$ such that $\cdot$ defines the structure of an associative algebra on $\mathcal{B}$ and $[.,.]$ defines the structure of a Lie algebra on $\mathcal{B}$ is called a *Poisson algebra* if and only if the Leibniz rule is true, i.e.

$$\forall e,f,g \in \mathcal{B} : [e, f\cdot g] = [e,f]\cdot g + f \cdot [e,g]. \tag{1.4.7}$$

In other words, via the Lie product $[.,.]$ the elements of the algebra act as derivations on the associative structure.

Theorem 1.4.5. *[SCH 14b, Thm. 2.10] The space $\mathcal{F}$ with respect to $\cdot$ and $[.,.]$ is a Poisson algebra.*

Next we consider important substructures. We already encountered the subalgebras $\mathcal{A}$ and $\mathcal{L}$. But there are more structures around.

1.4.3. *The vector field algebra and the Lie derivative*

First we look again on the Lie subalgebra $\mathcal{L} = \mathcal{F}^{-1}$. Here the Lie action respect the homogeneous subspaces $\mathcal{F}^{\lambda}$. As forms of weight -1 are vector fields, it could also be defined as the Lie algebra of those meromorphic vector fields on the Riemann surface Σ which are holomorphic outside of A. For vector fields we have the usual Lie bracket and the usual Lie derivative for their actions on forms. For the vector fields we have (again naming the local functions with the same symbol as the section) for $e, f \in \mathcal{L}$

$$[e, f]_{|} = [e(z)\frac{d}{dz}, f(z)\frac{d}{dz}] = \left(e(z)\frac{df}{dz}(z) - f(z)\frac{de}{dz}(z)\right)\frac{d}{dz}\,. \tag{1.4.8}$$

For the Lie derivative we get

$$\nabla_e(f)_{|} = L_e(g)_{|} = e\,.\,g_{|} = \left(e(z)\frac{df}{dz}(z) + \lambda f(z)\frac{de}{dz}(z)\right)\frac{d}{dz}\,. \tag{1.4.9}$$

Obviously, these definitions coincide with the definitions already given in (1.4.6). But now we obtained a geometric interpretation.

1.4.4. *The algebra of differential operators*

If we look at $\mathcal{F}$, considered as Lie algebra, more closely, we see that $\mathcal{F}^0$ is an abelian Lie subalgebra and the vector space sum $\mathcal{F}^0 \oplus \mathcal{F}^{-1} = \mathcal{A} \oplus \mathcal{L}$ is also a Lie subalgebra. In an equivalent way it can also be constructed as semidirect sum of $\mathcal{A}$ considered as abelian Lie algebra and $\mathcal{L}$ operating on $\mathcal{A}$ by taking the derivative.

Definition 1.4.6. The *Lie algebra of differential operators of degree* ≤ 1 is defined as the semidirect sum of $\mathcal{A}$ with $\mathcal{L}$ and is denoted by $\mathcal{D}^1$.

In terms of elements the Lie product is

$$[(g, e), (h, f)] = (e\,.\,h - f\,.\,g\,,\ [e, f]). \tag{1.4.10}$$

Using the fact, that $\mathcal{A}$ is an abelian subalgebra in $\mathcal{F}$ this is exactly the definition for the Lie product given for this algebra. Hence, $\mathcal{D}^1$ is a Lie algebra.

The projection on the second factor $(g, e) \mapsto e$ is a Lie homomorphism and we obtain a short exact sequences of Lie algebras

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{D}^1 \longrightarrow \mathcal{L} \longrightarrow 0\,. \tag{1.4.11}$$

Hence, $\mathcal{A}$ is an (abelian) Lie ideal of $\mathcal{D}^1$ and $\mathcal{L}$ a quotient Lie algebra. Obviously, $\mathcal{L}$ is also a subalgebra of $\mathcal{D}^1$.

Proposition 1.4.7. *The vector space $\mathcal{F}^\lambda$ becomes a Lie module over $\mathcal{D}^1$ by the operation*

$$(g, e).f := g \cdot f + e.f, \quad (g, e) \in \mathcal{D}^1(A), f \in \mathcal{F}^\lambda(A). \tag{1.4.12}$$

1.4.5. *Differential operators of all degree*

We want to consider also differential operators of arbitrary degree acting on $\mathcal{F}^\lambda$. This is obtained via some universal constructions. First we consider the universal enveloping algebra $U(\mathcal{D}^1)$. We denote its multiplication by $\odot$ and its unit by $\mathbf{1}$.

The universal enveloping algebra contains many elements which act in the same manner on $\mathcal{F}^\lambda$. For example, if h_1 and h_2 are functions different from constants then $h_1 \cdot h_2$ and $h_1 \odot h_2$ are different elements of $U(\mathcal{D}^1)$. Nevertheless, they act in the same way on $\mathcal{F}^\lambda$.

Hence, we will divide out further relations

$$\mathcal{D} = U(\mathcal{D}^1)/J, \quad \text{respectively} \quad \mathcal{D}_\lambda = U(\mathcal{D}^1)/J_\lambda \tag{1.4.13}$$

with the two-sided ideals

$$J := (\ a \odot b - a \cdot b,\ \mathbf{1} - 1 \mid a, b \in \mathcal{A}\ ),$$
$$J_\lambda := (\ a \odot b - a \cdot b,\ \mathbf{1} - 1,\ a \odot e - a \cdot e + \lambda\, e\,.\,a \mid a, b \in \mathcal{A}, e \in L\ ).$$

We can show that for all λ the $\mathcal{F}^\lambda$ are modules over $\mathcal{D}$, and for a fixed λ the space $\mathcal{F}^\lambda$ is a module over $\mathcal{D}_\lambda$.

Denote by $\mathrm{Diff}(\mathcal{F}^\lambda)$ the associative algebra of algebraic differential operators as defined in ([GRO 71, IV,16.8,16.11] and [BER 75]). Let $D \in \mathcal{D}$ and assume that D is one of the generators

$$D = a_0 \odot e_1 \odot a_1 \odot e_2 \odot \cdots \odot a_{n-1} \odot e_n \odot a_n \tag{1.4.14}$$

with $e_i \in \mathcal{L}$ and $a_i \in \mathcal{A}$ (written as element in $U(\mathcal{D}^1)$) then

Proposition 1.4.8. *[SCH 14b, Prop. 2.14] Every element $D \in \mathcal{D}$ respectively of $\mathcal{D}_\lambda$ of the form (1.4.14) operates as (algebraic) differential operator of degree $\leq n$ on $\mathcal{F}^\lambda$.*

In fact, we get (associative) algebra homomorphisms

$$\mathcal{D} \to \mathrm{Diff}(\mathcal{F}^\lambda), \qquad \mathcal{D}_\lambda \to \mathrm{Diff}(\mathcal{F}^\lambda) \; . \tag{1.4.15}$$

In case that the set A of points where poles are allowed is finite and non-empty the complement $\Sigma \setminus A$ is affine [HAR 77, p.297]. Hence, as shown in [GRO 71] every differential operator can be obtained by successively applying first order operators, i.e. by applying elements from $U(\mathcal{D}^1)$. In other words the homomorphisms (1.4.15) are surjective.

1.4.6. *Lie superalgebras of half forms*

Recall from Remark 1.2.2 the definition of a Lie superalgebra.

With the help of our associative product (1.4.2) we will obtain examples of Lie superalgebras. First we consider

$$\cdot \; \mathcal{F}^{-1/2} \times \mathcal{F}^{-1/2} \to \mathcal{F}^{-1} = \mathcal{L} \; , \tag{1.4.16}$$

and introduce the vector space $\mathcal{S}$ with the product

$$\mathcal{S} := \mathcal{L} \oplus \mathcal{F}^{-1/2}, \quad [(e,\varphi),(f,\psi)] := ([e,f] + \varphi \cdot \psi, e \,.\, \varphi - f \,.\, \psi). \tag{1.4.17}$$

The elements of $\mathcal{L}$ are denoted by $e, f, \ldots$, and the elements of $\mathcal{F}^{-1/2}$ by $\varphi, \psi, \ldots$.

The definition (1.4.17) can be reformulated as an extension of $[.,.]$ on $\mathcal{L}$ to a super-bracket (denoted by the same symbol) on $\mathcal{S}$ by setting

$$[e,\varphi] := -[\varphi, e] := e \,.\, \varphi = (e\frac{d\varphi}{dz} - \frac{1}{2}\varphi\frac{de}{dz})(dz)^{-1/2} \tag{1.4.18}$$

and

$$[\varphi,\psi] := \varphi \cdot \psi \; . \tag{1.4.19}$$

We call the elements of $\mathcal{L}$ elements of even parity, and the elements of $\mathcal{F}^{-1/2}$ elements of odd parity. For such elements x we denote by $\bar{x} \in \{\bar{0}, \bar{1}\}$ their parity.

The sum (1.4.17) can also be described as $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$, where $\mathcal{S}_{\bar{i}}$ is the subspace of elements of parity $\bar{i}$.

Proposition 1.4.9. *[SCH 14b, Prop. 2.15] The space $\mathcal{S}$ with the above introduced parity and product is a Lie superalgebra.*

Remark 1.4.10. The choice of the theta characteristics corresponds to choosing a *spin structure* on Σ. For the relation of the Neveu-Schwarz superalgebra to the geometry of graded Riemann surfaces see Bryant [BRY 90].

1.4.7. *Jordan superalgebra*

Leidwanger and Morier-Genoux introduced in [LEI 12] a *Jordan superalgebra* in our geometric setting. They put

$$\mathcal{J} := \mathcal{F}^0 \oplus \mathcal{F}^{-1/2} = \mathcal{J}_{\bar{0}} \oplus \mathcal{J}_{\bar{1}}. \tag{1.4.20}$$

Recall that $\mathcal{A} = \mathcal{F}^0$ is the associative algebra of meromorphic functions. They define the (Jordan) product $\circ$ via the algebra structures for the spaces $\mathcal{F}^\lambda$ by

$$\begin{aligned} f \circ g &:= f \cdot g \quad \in \mathcal{F}^0, \\ f \circ \varphi &:= f \cdot \varphi \quad \in \mathcal{F}^{-1/2}, \\ \varphi \circ \psi &:= [\varphi, \psi] \quad \in \mathcal{F}^0. \end{aligned} \tag{1.4.21}$$

By rescaling the second definition with the factor 1/2 one obtains a *Lie anti-algebra* as introduced by Ovsienko [OVS 11]. See [LEI 12] for more details and additional results on representations.

1.4.8. *Higher genus current algebras*

We fix an arbitrary finite-dimensional complex Lie algebra $\mathfrak{g}$. Our goal is to generalize the classical current algebra to higher genus. For this let (Σ, A) be the geometrical data consisting of the Riemann surface Σ and the subset of points A used to define $\mathcal{A}$, the algebra of meromorphic functions which are holomorphic outside of the set $A \subseteq \Sigma$.

Definition 1.4.11. The *higher genus current algebra* associated to the Lie algebra $\mathfrak{g}$ and the geometric data (Σ, A) is the Lie algebra $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}(A) = \overline{\mathfrak{g}}(\Sigma, A)$ given as vector space by $\overline{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}$ with the Lie product

$$[x \otimes f, y \otimes g] = [x, y] \otimes f \cdot g, \qquad x, y \in \mathfrak{g}, \quad f, g \in \mathcal{A}. \tag{1.4.22}$$

Proposition 1.4.12. $\overline{\mathfrak{g}}$ *is a Lie algebra.*

As usual we will suppress the mentioning of (Σ, A) if not needed. The elements of $\overline{\mathfrak{g}}$ can be interpreted as meromorphic functions $\Sigma \to \mathfrak{g}$ from the Riemann surface Σ to the Lie algebra $\mathfrak{g}$, which are holomorphic outside of A.

Later we will introduce central extensions for these current algebras. They will generalize affine Lie algebras, respectively affine Kac-Moody algebras of untwisted type.

For some applications it is useful to extend the definition by considering differential operators (of degree ≤ 1) associated to $\overline{\mathfrak{g}}$. We define $\mathcal{D}^1_{\mathfrak{g}} := \overline{\mathfrak{g}} \oplus \mathcal{L}$ and take in the summands the Lie product defined there and put additionally

$$[e, x \otimes g] := -[x \otimes g, e] := x \otimes (e.g). \tag{1.4.23}$$

This operation can be described as semidirect sum of $\overline{\mathfrak{g}}$ with $\mathcal{L}$ and we get

Proposition 1.4.13. *[SCH 14b, Prop. 2.15] $\mathcal{D}^1_{\mathfrak{g}}$ is a Lie algebra.*

1.4.9. *Krichever–Novikov type algebras*

Above the set A of points where poles are allowed was arbitrary. In case that A is finite and moreover $\#A \geq 2$ the constructed algebras are called Krichever–Novikov (KN) type algebras. In this way we get the KN vector field algebra, the function algebra, the current algebra, the differential operator algebra, the Lie superalgebra, etc. The reader might ask what is so special about this situation so that these algebras deserve special names. In fact in this case we can endow the algebra with a (strong) almost-graded structure. This will be discussed in the next section. The almost-grading is a crucial tool for extending the classical result to higher genus. Recall that in the classical case we have genus zero and $\#A = 2$.

Strictly speaking, a KN type algebra should be considered to be one of the above algebras for $2 \leq \#A < \infty$ together with a fixed chosen almost-grading, induced by the splitting $A = I \cup O$ into two disjoint non-empty subset, see Definition 1.5.1.

1.5. Almost-Graded Structure

1.5.1. *Definition of almost-gradedness*

In the classical situation discussed in Section 1.2 the algebras introduced in the last section are graded algebras. In the higher genus case and even in the genus zero case with more than two points where poles are allowed there is no non-trivial grading anymore. As realized by Krichever and Novikov [KRI 87a] there is a weaker concept, an almost-grading, which to a large extend is a valuable replacement of a honest grading. Such an almost-grading is induced by a splitting of the set A into two non-empty and disjoint sets I and O. The (almost-)grading is fixed by exhibiting certain basis elements in the spaces $\mathcal{F}^\lambda$ as homogeneous.

Definition 1.5.1. Let $\mathcal{L}$ be a Lie or an associative algebra such that $\mathcal{L} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n$ is a vector space direct sum, then $\mathcal{L}$ is called an *almost-graded* (Lie-) algebra if

(i) $\dim \mathcal{L}_n < \infty$,

(ii) There exists constants $L_1, L_2 \in \mathbb{Z}$ such that

$$\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathcal{L}_h, \qquad \forall n, m \in \mathbb{Z}.$$

The elements in $\mathcal{L}_n$ are called *homogeneous* elements of degree n, and $\mathcal{L}_n$ is called *homogeneous subspace* of degree n.

If $\dim \mathcal{L}_n$ is bounded with a bound independent of n we call $\mathcal{L}$ *strongly almost-graded*. If we drop the condition that $\dim \mathcal{L}_n$ is finite dimensional we call $\mathcal{L}$ *weakly almost-graded*.

In a similar manner almost-graded modules over almost-graded algebras are defined. We can extend in an obvious way the definition to superalgebras, respectively even to more general algebraic structures. Note that this definition makes complete sense also for more general index sets $\mathbb{J}$. In fact we will consider the index set $\mathbb{J} = (1/2)\mathbb{Z}$ in the case of superalgebras. The even elements (with respect to the super-grading) will have integer degree, the odd elements half-integer degree.

1.5.2. *Separating cycle and Krichever-Novikov pairing*

Before we give the almost-grading we introduce an important structure. Let C_i be positively oriented (deformed) circles around the points P_i in I, $i = 1, \ldots, K$ and C_j^* positively oriented circles around the points Q_j in O, $j = 1, \ldots, M$.

A cycle C_S is called a separating cycle if it is smooth, positively oriented of multiplicity one and if it separates the in-points from the out-points. It might have more than one component. In the following we will integrate meromorphic differentials on Σ without poles in $\Sigma \setminus A$ over closed curves C. Hence, we might consider C and C' as equivalent if $[C] = [C']$ in $\mathrm{H}_1(\Sigma \setminus A, \mathbb{Z})$. In this sense we write for every separating cycle

$$[C_S] = \sum_{i=1}^{K} [C_i] = -\sum_{j=1}^{M} [C_j^*]. \tag{1.5.1}$$

The minus sign appears due to the opposite orientation. Another way for giving such a C_S is via level lines of a "proper time evolution", for which I refer to [SCH 14b, Section 3.9].

Given such a separating cycle C_S (respectively cycle class) we define a linear map

$$\mathcal{F}^1 \to \mathbb{C}, \qquad \omega \mapsto \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega. \tag{1.5.2}$$

The map will not depend on the separating line C_S chosen, as two of such will be homologous and the poles of ω are only located in I and O.

Consequently, the integration of ω over C_S can also be described over the special cycles C_i or equivalently over C_j^*. This integration corresponds to calculating residues

$$\omega \quad \mapsto \quad \frac{1}{2\pi \mathrm{i}}\int_{C_S} \omega \;=\; \sum_{i=1}^{K} \operatorname{res}_{P_i}(\omega) \;=\; -\sum_{l=1}^{M} \operatorname{res}_{Q_l}(\omega). \tag{1.5.3}$$

Definition 1.5.2. The pairing

$$\mathcal{F}^{\lambda} \times \mathcal{F}^{1-\lambda} \to \mathbb{C}, \quad (f,g) \mapsto \langle f, g\rangle := \frac{1}{2\pi \mathrm{i}}\int_{C_S} f \cdot g, \tag{1.5.4}$$

between λ and $1-\lambda$ forms is called *Krichever-Novikov (KN) pairing*.

Note that the pairing depends not only on A (as the $\mathcal{F}^{\lambda}$ depend on it) but also critically on the splitting of A into I and O as the integration path will depend on it. Once the splitting is fixed the pairing will be fixed too.

In fact there exit dual basis elements (see (1.5.9)) hence the pairing is non-degenerate.

1.5.3. *The homogeneous subspaces*

Given the vector spaces $\mathcal{F}^{\lambda}$ of forms of weight $\mathcal{L}$ we will now single out subspaces $\mathcal{F}^{\lambda}_m$ of degree m by giving a basis of these subspaces. This has been done in the 2-point case by Krichever and Novikov [KRI 87a] and in the multi-point case by the author [SCH 90b], [SCH 90c], [SCH 90a], [SCH 90d], see also Sadov [SAD 91]. See in particular [SCH 14b, Chapters 3,4,5] for a complete treatment. All proofs of the statements to come can be found there.

Depending on whether the weight λ is integer or half-integer we set $\mathbb{J}_{\lambda} = \mathbb{Z}$ or $\mathbb{J}_{\lambda} = \mathbb{Z} + 1/2$. For $\mathcal{F}^{\lambda}$ we introduce for $m \in \mathbb{J}_{\lambda}$ subspaces $\mathcal{F}^{\lambda}_m$ of dimension K, where $K = \#I$, by exhibiting certain elements $f^{\lambda}_{m,p} \in \mathcal{F}^{\lambda}$, $p = 1,\dots,K$ which constitute a basis of $\mathcal{F}^{\lambda}_m$. The elements are the elements of degree m. As explained in the following, the degree is in an essential way related to the zero orders of the elements at the points in I.

Let $I = (P_1, P_2, \dots, P_K)$ then we require for the zero-order at the point $P_i \in I$ of the element $f^{\lambda}_{n,p}$

$$\operatorname{ord}_{P_i}(f^{\lambda}_{n,p}) = (n+1-\lambda) - \delta_i^p, \quad i = 1,\dots,K\ . \tag{1.5.5}$$

The prescription at the points in O is made in such a way that the element $f^\lambda_{m,p}$ is essentially uniquely given. Essentially unique means up to multiplication with a constant[4]. After fixing as additional geometric data a system of coordinates z_l centered at P_l for $l = 1, \dots, K$ and requiring that

$$f^\lambda_{n,p}(z_p) = z_p^{n-\lambda}(1 + O(z_p))(dz_p)^\lambda \tag{1.5.6}$$

the element $f_{n,p}$ is uniquely fixed. In fact, the element $f^\lambda_{n,p}$ only depends on the first order jet of the coordinate z_p.

Example. Here we will not give the general recipe for the prescription at the points in O. Just to give an example which is also an important special case, assume $O = \{Q\}$ is a one-element set. If either the genus $g = 0$, or $g \geq 2$, $\lambda \neq 0,\ 1/2,\ 1$ and the points in A are in generic position then we require

$$\operatorname{ord}_Q(f^\lambda_{n,p}) = -K \cdot (n + 1 - \lambda) + (2\lambda - 1)(g - 1). \tag{1.5.7}$$

In the other cases (e.g. for $g = 1$) there are some modifications at the point in O necessary for finitely many n.

Theorem 1.5.3. *[SCH 14b, Thm. 3.6] Set*

$$\mathcal{B}^\lambda := \{\, f^\lambda_{n,p} \mid n \in \mathbb{J}_\lambda,\ p = 1, \dots, K \,\}. \tag{1.5.8}$$

Then (a) $\mathcal{B}^\lambda$ is a basis of the vector space $\mathcal{F}^\lambda$.

(b) The introduced basis $\mathcal{B}^\lambda$ of $\mathcal{F}^\lambda$ and $\mathcal{B}^{1-\lambda}$ of $\mathcal{F}^{1-\lambda}$ are dual to each other with respect to the Krichever-Novikov pairing (1.5.4), i.e.

$$\langle f^\lambda_{n,p}, f^{1-\lambda}_{-m,r} \rangle = \delta^r_p\, \delta^m_n, \quad \forall n, m \in \mathbb{J}_\lambda, \quad r, p = 1, \dots, K. \tag{1.5.9}$$

In particular, from part (b) of the theorem it follows that the Krichever-Novikov pairing is non-degenerate. Moreover, any element $v \in \mathcal{F}^{1-\lambda}$ acts as linear form on $\mathcal{F}^\lambda$ via

$$\Phi_v : \mathcal{F}^\lambda \mapsto \mathbb{C}, \quad w \mapsto \Phi_v(w) := \langle v, w \rangle. \tag{1.5.10}$$

Via this pairing $\mathcal{F}^{1-\lambda}$ can be considered as restricted dual of $\mathcal{F}^\lambda$. The identification depends on the splitting of A into I and O as the KN pairing depends on it. The full space $(\mathcal{F}^\lambda)^*$ can even be described with the help of the pairing in a "distributional interpretation" via the distribution $\Phi_{\hat{v}}$ associated to the formal series

$$\hat{v} := \sum_{m \in \mathbb{J}_\lambda} \sum_{p=1}^{K} a_{m,p} f^{1-\lambda}_{m,p}, \quad a_{m,p} \in \mathbb{C}\ . \tag{1.5.11}$$

4. Strictly speaking, there are some special cases where some constants have to be added such that the Krichever-Novikov duality (1.5.9) is valid.

The dual elements of $\mathcal{L}$ will be given by the formal series (1.5.11) with basis elements from $\mathcal{F}^2$, the quadratic differentials, the dual elements of $\mathcal{A}$ correspondingly from $\mathcal{F}^1$, the differentials, and the dual elements of $\mathcal{F}^{-1/2}$ correspondingly from $\mathcal{F}^{3/2}$.

It is quite convenient to use special notations for elements of some important weights:

$$\begin{aligned} e_{n,p} &:= f_{n,p}^{-1}, \quad \varphi_{n,p} := f_{n,p}^{-1/2}, \quad A_{n,p} := f_{n,p}^{0}, \\ \omega^{n,p} &:= f_{-n,p}^{1}, \quad \Omega^{n,p} := f_{-n,p}^{2}. \end{aligned} \tag{1.5.12}$$

In view of (1.5.9) for the forms of weight 1 and 2 we invert the index n and write it as a superscript.

Remark 1.5.4. It is also possible (and for certain applications necessary) to write explicitely down the basis elements $f_{n,p}^{\lambda}$ in terms of "usual" objects defined on the Riemann surface Σ. For genus zero they can be given with the help of rational functions in the quasi-global variable z. For genus one (i.e. the torus case) representations with the help of Weierstraß σ and Weierstraß $\wp$ functions exists. For genus ≥ 1 there exists expressions in terms of theta functions (with characteristics) and prime forms. Here the Riemann surface has first to be embedded into its Jacobian via the Jacobi map. See [SCH 14b, Chapter 5], [SCH 90c], [SCH 93] for more details.

1.5.4. *The algebras*

Theorem 1.5.5. *[SCH 14b, Thm. 3.8] There exists constants R_1 and R_2 (depending on the number and splitting of the points in A and on the genus g) independent of λ and ν and independent of $n, m \in \mathbb{J}$ such that for the basis elements*

$$\begin{aligned} f_{n,p}^{\lambda} \cdot f_{m,r}^{\nu} = \; & f_{n+m,r}^{\lambda+\nu} \delta_p^r \\ & + \sum_{h=n+m+1}^{n+m+R_1} \sum_{s=1}^{K} a_{(n,p)(m,r)}^{(h,s)} f_{h,s}^{\lambda+\nu}, \qquad a_{(n,p)(m,r)}^{(h,s)} \in \mathbb{C}, \\ [f_{n,p}^{\lambda}, f_{m,r}^{\nu}] = \; & (-\lambda m + \nu n)\, f_{n+m,r}^{\lambda+\nu+1} \delta_p^r \\ & + \sum_{h=n+m+1}^{n+m+R_2} \sum_{s=1}^{K} b_{(n,p)(m,r)}^{(h,s)} f_{h,s}^{\lambda+\nu+1}, \quad b_{(n,p)(m,r)}^{(h,s)} \in \mathbb{C}. \end{aligned} \tag{1.5.13}$$

In generic situations and for $N = 2$ points one obtains $R_1 = g$ and $R_2 = 3g$.

The theorem says in particular that with respect to both the associative and Lie structure the algebra $\mathcal{F}$ is weakly almost-graded. The reason why we only have weakly almost-gradedness is that

$$\mathcal{F}^\lambda = \bigoplus_{m\in\mathbb{J}_\lambda} \mathcal{F}^\lambda_m, \qquad \text{with} \quad \dim \mathcal{F}^\lambda_m = K, \tag{1.5.14}$$

and if we add up for a fixed m all λ we get that our homogeneous spaces are infinite dimensional.

In the definition of our KN type algebra only finitely many λs are involved, hence the following is immediate

Theorem 1.5.6. *The Krichever-Novikov type vector field algebras $\mathcal{L}$, function algebras $\mathcal{A}$, differential operator algebras $\mathcal{D}^1$, Lie superalgebras $\mathcal{S}$, and Jordan superalgebras $\mathcal{J}$ are all (strongly) almost-graded algebras and the corresponding modules $\mathcal{F}^\lambda$ are almost-graded modules.*

We obtain with $n \in \mathbb{J}_\lambda$

$$\begin{gathered} \dim \mathcal{L}_n = \dim \mathcal{A}_n = \dim \mathcal{F}^\lambda_n = K, \\ \dim \mathcal{S}_n = \dim \mathcal{J}_n = 2K, \quad \dim \mathcal{D}^1_n = 3K\,. \end{gathered} \tag{1.5.15}$$

If $\mathcal{U}$ is any of these algebras, with product denoted by $[\ ,\ ]$ then

$$[\mathcal{U}_n, \mathcal{U}_m] \subseteq \bigoplus_{h=n+m}^{n+m+R_i} \mathcal{U}_h, \tag{1.5.16}$$

with $R_i = R_1$ for $\mathcal{U} = \mathcal{A}$ and $R_i = R_2$ otherwise.

For further reference let us specialize the lowest degree term component in (1.5.13) for certain special cases.

$$\begin{aligned} A_{n,p} \cdot A_{m,r} &= A_{n+m,r}\, \delta^p_r \ + \ \text{h.d.t.} \\ A_{n,p} \cdot f^\lambda_{m,r} &= f^\lambda_{n+m,r}\, \delta^p_r \ + \ \text{h.d.t.} \\ [e_{n,p}, e_{m,r}] &= (m-n) \cdot e_{n+m,r}\, \delta^p_r \ + \ \text{h.d.t.} \\ e_{n,p} \cdot f^\lambda_{m,r} &= (m+\lambda n) \ \cdot f^\lambda_{n+m,r}\, \delta^p_r \ + \ \text{h.d.t.} \end{aligned} \tag{1.5.17}$$

Here h.d.t. denote linear combinations of basis elements of degree between $n+m+1$ and $n+m+R_i$, Finally, the almost-grading of $\mathcal{A}$ induces an almost-grading of the current algebra $\overline{\mathfrak{g}}$ by setting $\overline{\mathfrak{g}}_n = \mathfrak{g} \otimes \mathcal{A}_n$. We obtain

$$\overline{\mathfrak{g}} = \bigoplus_{n \in \mathbb{Z}} \overline{\mathfrak{g}}_n, \quad \dim \overline{\mathfrak{g}}_n = K \cdot \dim \mathfrak{g}. \tag{1.5.18}$$

1.5.5. *Triangular decomposition and filtrations*

Let $\mathcal{U}$ be one of the above introduced algebras (including the current algebra). On the basis of the almost-grading we obtain a triangular decomposition of the algebras

$$\mathcal{U} = \mathcal{U}_{[+]} \oplus \mathcal{U}_{[0]} \oplus \mathcal{U}_{[-]}, \tag{1.5.19}$$

where

$$\mathcal{U}_{[+]} := \bigoplus_{m>0} \mathcal{U}_m, \quad \mathcal{U}_{[0]} = \bigoplus_{m=-R_i}^{m=0} \mathcal{U}_m, \quad \mathcal{U}_{[-]} := \bigoplus_{m<-R_i} \mathcal{U}_m. \tag{1.5.20}$$

By the almost-gradedness the $[+]$ and $[-]$ subspaces are (infinite dimensional) subalgebras. The $[0]$ spaces in general not. Sometimes we call them *critical strips*.

With respect to the almost-grading of $\mathcal{F}^\lambda$ we introduce a filtration

$$\begin{gathered} \mathcal{F}^\lambda_{(n)} := \bigoplus_{m \geq n} \mathcal{F}^\lambda_m, \\ \ldots. \quad \supseteq \quad \mathcal{F}^\lambda_{(n-1)} \quad \supseteq \quad \mathcal{F}^\lambda_{(n)} \quad \supseteq \quad \mathcal{F}^\lambda_{(n+1)} \quad \ldots. \end{gathered} \tag{1.5.21}$$

Proposition 1.5.7. *[SCH 14b, Prop. 3.15]*

$$\mathcal{F}^\lambda_{(n)} = \{ \, f \in \mathcal{F}^\lambda \mid \operatorname{ord}_{P_i}(f) \geq n - \lambda, \forall i = 1, \ldots, K \, \}. \tag{1.5.22}$$

1.6. Central Extensions

Central extension of our algebras appear naturally in the context of quantization and regularization of actions. Of course they are also of independent mathematical interest.

1.6.1. *Central extensions and cocycles*

For the convenience of the reader let us repeat the relation between central extensions and the second Lie algebra cohomology with values in the trivial module. A central extension of a Lie algebra W is a special Lie algebra structure on the vector space direct sum $\widehat{W} = \mathbb{C} \oplus W$. If we denote $\hat{x} := (0, x)$ and $t := (1, 0)$ then the Lie structure is given by

$$[\hat{x}, \hat{y}] = \widehat{[x, y]} + \psi(x, y) \cdot t, \quad [t, \widehat{W}] = 0, \quad x, y \in W\,, \tag{1.6.1}$$

with bilinear form ψ. The map $x \mapsto \hat{x} = (0, x)$ is a linear splitting map. $\widehat{W}$ will be a Lie algebra, e.g. will fulfill the Jacobi identity, if and only if ψ is an antisymmetric bilinear form and fulfills the Lie algebra 2-cocycle condition

$$0 = d_2\psi(x, y, z) := \psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y). \tag{1.6.2}$$

Two central extensions are equivalent if they essentially correspond only to the choice of different splitting maps. A 2-cochain ψ is a coboundary if there exists a linear form $\varphi : W \to \mathbb{C}$ such that

$$\psi(x, y) = \varphi([x, y]). \tag{1.6.3}$$

Every coboundary is a cocycle. The second Lie algebra cohomology $\mathrm{H}^2(W, \mathbb{C})$ of W with values in the trivial module $\mathbb{C}$ is defined as the quotient of the space of 2-cocycles modulo coboundaries. Moreover, two central extensions are equivalent if and only if the difference of their defining 2-cocycles ψ and ψ' is a coboundary. In this way the second Lie algebra cohomology $\mathrm{H}^2(W, \mathbb{C})$ classifies equivalence classes of central extensions. The class [0] corresponds to the trivial central extension. In this case the splitting map is a Lie homomorphism. We construct central extensions of our algebras by exhibiting such Lie algebra 2-cocycles.

Clearly, equivalent central extensions are isomorphic. The opposite is not true. In our case we can always rescale the central element by multiplying it with a nonzero scalar. This is an isomorphism but not an equivalence of central extensions. Nevertheless it is an irrelevant modification. Hence we will be mainly interested in central extensions modulo equivalence and rescaling. They are classified by $[0]$ and the elements of the projectivized cohomology space $\mathbb{P}(\mathrm{H}^2(W, \mathbb{C}))$.

In the classical case we have $\dim \mathrm{H}^2(\mathcal{W}, \mathbb{C}) = 1$, hence there are only two essentially different central extensions, the splitting one given by the direct sum $\mathbb{C} \oplus \mathcal{W}$ of Lie algebras and the up to equivalence and rescaling unique non-trivial one, the Virasoro algebra $\mathcal{V}$.

1.6.2. ***Geometric cocycles***

The cocycle of the Witt algebra

$$\frac{1}{12}(n^3 - n)\delta_n^{-m} \tag{1.6.4}$$

to define the Virasoro algebra is very special. Obviously it does not make any sense in the higher genus and/or multi-point case. We need to find a geometric description. For this we have first to introduce connections.

1.6.2.1. *Projective and affine connections*

Let $(U_\alpha, z_\alpha)_{\alpha\in J}$ be a covering of the Riemann surface by holomorphic coordinates with transition functions $z_\beta = f_{\beta\alpha}(z_\alpha)$.

Definition 1.6.1. (a) A system of local (holomorphic, meromorphic) functions $R = (R_\alpha(z_\alpha))$ is called a (holomorphic, meromorphic) *projective connection* if it transforms as

$$R_\beta(z_\beta)\cdot(f'_{\beta,\alpha})^2 = R_\alpha(z_\alpha)+S(f_{\beta,\alpha}), \qquad \text{with} \quad S(h) = \frac{h'''}{h'}-\frac{3}{2}\left(\frac{h''}{h'}\right)^2, \tag{1.6.5}$$

the Schwartzian derivative. Here $'$ denotes differentiation with respect to the coordinate z_α.

(b) A system of local (holomorphic, meromorphic) functions $T = (T_\alpha(z_\alpha))$ is called a (holomorphic, meromorphic) *affine connection* if it transforms as

$$T_\beta(z_\beta)\cdot(f'_{\beta,\alpha}) = T_\alpha(z_\alpha) + \frac{f''_{\beta,\alpha}}{f'_{\beta,\alpha}}. \tag{1.6.6}$$

Every Riemann surface admits a holomorphic projective connection [HAW 66],[GUN 66]. Given a point P then there exists always a meromorphic affine connection holomorphic outside of P and having maximally a pole of order one there [SCH 90d].

From their very definition it follows that the difference of two affine (projective) connections will be a (quadratic) differential. Hence, after fixing one affine (projective) connection all others are obtained by adding (quadratic) differentials.

Next we introduce in a geometric way by integration of certain differentials, associated to pairs of Lie algebra elements, over arbitrary smooth curves. For the proofs that the following expressions are indeed 2-cocycles we refer to [SCH 90d] (and [SCH 14b]).

1.6.2.2. *The function algebra $\mathcal{A}$*

We consider it as abelian Lie algebra. Let C be an arbitrary smooth but not necessarily connected curve. We set

$$\psi^1_C(g,h) := \frac{1}{2\pi\mathrm{i}}\int_C g\,dh, \quad g,h\in\mathcal{A}. \tag{1.6.7}$$

1.6.2.3. *The current algebra* $\overline{\mathfrak{g}}$

For $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{A}$ we fix a symmetric, invariant, bilinear form β on $\mathfrak{g}$ (not necessarily non-degenerate). Recall, that invariance means that we have $\beta([x,y],z) = \beta(x,[y,z])$ for all $x, y, z \in \mathfrak{g}$. Then a cocycle is given as

$$\psi^2_{C,\beta}(x \otimes g, y \otimes h) := \beta(x,y) \cdot \frac{1}{2\pi \mathrm{i}} \int_C g dh, \quad x, y \in \mathfrak{g}, \; g, h \in \mathcal{A}. \tag{1.6.8}$$

1.6.2.4. *The vector field algebra* $\mathcal{L}$

Here it is a little bit more delicate. First we have to choose a (holomorphic) projective connection R. We define

$$\psi^3_{C,R}(e,f) := \frac{1}{24\pi \mathrm{i}} \int_C \left(\frac{1}{2}(e'''f - ef''') - R \cdot (e'f - ef') \right) dz \,. \tag{1.6.9}$$

Only by the term coming with the projective connection it will be a well-defined differential, i.e. independent of the coordinate chosen. Another choice of a projective connection will result in a cohomologous one. Hence, the equivalence class of the central extension will be the same.

1.6.2.5. *The differential operator algebra* $\mathcal{D}^1$

For the differential operator algebra the cocycles of type (1.6.7) for $\mathcal{A}$ can be extended by zero on the subspace $\mathcal{L}$. The cocycles for $\mathcal{L}$ can be pulled back. In addition there is a third type of cocycles mixing $\mathcal{A}$ and $\mathcal{L}$:

$$\psi^4_{C,T}(e,g) := \frac{1}{24\pi \mathrm{i}} \int_C (eg'' + Teg')dz, \qquad e \in \mathcal{L}, g \in \mathcal{A}, \tag{1.6.10}$$

with an affine connection T, with at most a pole of order one at a fixed point in O. Again, a different choice of the connection will not change the cohomology class.

1.6.2.6. *The Lie superalgebra* $\mathcal{S}$

Here we have to take into account that it is not a Lie algebra. Hence, the Jacobi identity has to be replaced by the super-Jacobi identity. The conditions for being a cocycle for the superalgebra cohomology will change too. Recall the definition of the algebra from Section 1.4.6, in particular that the even elements (parity 0) are the vector fields and the odd elements (parity 1) are the half-forms. A bilinear form c is a cocycle if the following is true. The bilinear map c will be symmetric if both x and y are odd, otherwise it will be antisymmetric:

$$c(x,y) = -(-1)^{\bar{x}\bar{y}} c(x,y). \tag{1.6.11}$$

The super-cocycle condition reads as

$$(-1)^{\bar{x}\bar{z}} c(x,[y,z]) + (-1)^{\bar{y}\bar{x}} c(y,[z,x]) + (-1)^{\bar{z}\bar{y}} c(z,[x,y]) \;=\; 0. \tag{1.6.12}$$

With the help of c we can define central extensions in the Lie superalgebra sense. If we put the condition that the central element is even then the cocycle c has to be an even map and c vanishes for pairs of elements of different parity.

By convention we denote vector fields by $e, f, g, ...$ and -1/2-forms by $\varphi, \psi, \chi, ..$ and get

$$c(e, \varphi) = 0, \quad e \in \mathcal{L},\ \varphi \in \mathcal{F}^{-1/2}. \tag{1.6.13}$$

The super-cocycle conditions for the even elements is just the cocycle condition for the Lie subalgebra $\mathcal{L}$. The only other nonvanishing super-cocycle condition is for the *(even,odd,odd)* elements and reads as

$$c(e, [\varphi, \psi]) - c(\varphi, e\,.\,\psi) - c(\psi, e\,.\,\varphi) = 0. \tag{1.6.14}$$

Here the definition of the product $[e, \psi] := e\,.\,\psi$ was used.

If we have a cocycle c for the algebra $\mathcal{S}$ we obtain by restriction a cocycle for the algebra $\mathcal{L}$. For the mixing term we know that $c(e, \psi) = 0$. A naive try to put just anything for $c(\varphi, \psi)$ (for example 0) will not work as (1.6.14) relates the restriction of the cocycle on $\mathcal{L}$ with its values on $\mathcal{F}^{-1/2}$.

Proposition 1.6.2. *[SCH 13] Let C be any closed (differentiable) curve on Σ not meeting the points in A, and let R be any (holomorphic) projective connection then the bilinear extension of*

$$\begin{aligned}
\Phi_{C,R}(e, f) &:= \frac{1}{24\pi \mathrm{i}} \int_C \left(\frac{1}{2}(e'''f - ef''') - R \cdot (e'f - ef') \right) dz \\
\Phi_{C,R}(\varphi, \psi) &:= -\frac{1}{24\pi \mathrm{i}} \int_C (\varphi'' \cdot \psi + \varphi \cdot \psi'' - R \cdot \varphi \cdot \psi)\, dz \\
\Phi_{C,R}(e, \varphi) &:= 0
\end{aligned} \tag{1.6.15}$$

gives a Lie superalgebra cocycle for $\mathcal{S}$, hence defines a central extension of $\mathcal{S}$. A different projective connection will yield a cohomologous cocycle.

Note that the $\Phi_{C,R}$ restricted to $\mathcal{L}$ gives $\Psi^3_{C,R}$.

A similar formula was given by Bryant in [BRY 90]. By adding the projective connection in the second part of (1.6.15) he corrected some formula appearing in [BON 88]. He only considered the two-point case and only the integration over a separating cycle. See also [KRE 13] for the multi-point case, where still only the integration over a separating cycle is considered.

In contrast to the differential operator algebra case the two parts cannot be prescribed independently. Only with the same integration path (more precisely, homology class) and the given factors in front of the integral it will work. The reason for this is that (1.6.14) relates both.

1.6.3. *Uniqueness and classification of central extensions*

The above introduced cocycles depend on the choice of the connections R and T. Different choices will not change the cohomology class. Hence, this ambiguity does not disturb us. What really matters is that they depend on the integration curve C chosen.

In contrast to the classical situation, for the higher genus and/or multi-point situation there are many essentially different closed curves and hence many non-equivalent central extensions defined by the integration.

But we should take into account that we want to extend the almost-grading from our algebras to the centrally extended ones. This means we take $\deg \hat{x} := \deg x$ and assign a degree $deg(t)$ to the central element t, and still we want to obtain almost-gradedness.

This is possible if and only if our defining cocycle ψ is *local* in the following sense (the name was introduced in the two point case by Krichever and Novikov in [KRI 87a]). There exists $M_1, M_2 \in \mathbb{Z}$ such that

$$\forall n, m : \quad \psi(W_n, W_m) \neq 0 \implies M_1 \leq n + m \leq M_2. \tag{1.6.16}$$

Here W stands for any of our algebras (including the supercase). Very important, "local" is defined in terms of the almost-grading, and the almost-grading itself depends on the splitting $A = I \cup O$. Hence what is "local" depends on the splitting too.

We will call a cocycle *bounded* (from above) if there exists $M \in \mathbb{Z}$ such that

$$\forall n, m : \quad \psi(W_n, W_m) \neq 0 \implies n + m \leq M. \tag{1.6.17}$$

Similarly bounded from below can be defined. Locality means bounded from above and from below.

Given a cocycle class we call it *bounded* (respectively *local*) if and only if it contains a representing cocycle which is bounded (respectively local). Not all cocycles in a bounded class have to be bounded. If we choose as integration path a separating cocycle C_S, or one of the C_i then the above introduced geometric cocycles are local, respectively bounded. Recall that in this case integration can be done by calculating residues at the in-points or at the out-points. All these cocycles are cohomologically nontrivial. The theorems in the following concern the opposite direction. They were treated in my works [SCH 03b], [SCH 03a], [SCH 13]. See also [SCH 14b] for a complete and common treatment.

The following result for the vector field algebra $\mathcal{L}$ gives the principal structure of the classification results.

Theorem 1.6.3. *[SCH 03b], [SCH 14b, Thm. 6.41] Let $\mathcal{L}$ be the Krichever–Novikov vector field algebra with a given almost-grading induced by the splitting $A = I \cup O$.*
(a) The space of bounded cohomology classes is K-dimensional ($K = \#I$). A basis is given by setting the integration path in (1.6.9) to C_i, $i = 1, \ldots, K$ the little (deformed) circles around the points $P_i \in I$.
(b) The space of local cohomology classes is one-dimensional. A generator is given by integrating (1.6.9) over a separating cocycle C_S, i.e.

$$\psi^3_{C_S,R}(e,f) = \frac{1}{24\pi \mathrm{i}} \int_{C_S} \left(\frac{1}{2}(e'''f - ef''') - R \cdot (e'f - ef') \right) dz \; . \qquad (1.6.18)$$

(c) Up to equivalence and rescaling there is only one non-trivial one-dimensional central extension $\widehat{\mathcal{L}}$ of the vector field algebra $\mathcal{L}$ which allows an extension of the almost-grading.

Remark 1.6.4. In the classical situation, Part (c) shows also that the Virasoro algebra is the unique non-trivial central extension of the Witt algebra (up to equivalence and rescaling). This result extends to the more general situation under the condition that one fixes the almost-grading, hence the splitting $A = I \cup O$. Here I like to repeat the fact that for $\mathcal{L}$ depending on the set A and its possible splittings into two disjoint subsets there are different almost-gradings. Hence, the "unique" central extension finally obtained will also depend on the splitting. Only in the two point case there is only one splitting possible. In the case that the genus $g \geq 1$ there are even integration paths possible in the definition of (1.6.9) which are not homologous to a separating cycle of any splitting. Hence, there are other central extensions possible not corresponding to any almost-grading.

The above theorem is a model for all other classification results. We will always obtain a statement about the bounded (from above) cocycles and then for the local cocycles.

If we consider the function algebra $\mathcal{A}$ as an abelian Lie algebra then every skew-symmetric bilinear form will be a non-trivial cocycle. Hence, there is no hope of uniqueness. But if we add the condition of $\mathcal{L}$-invariance, which is given as

$$\psi(e.g, h) + \psi(g, e.h) = 0, \quad \forall e \in \mathcal{L},\ g, h \in \mathcal{A} \qquad (1.6.19)$$

things will change.

Let us denote the subspace of local cohomology classes by H^2_{loc}, and the subspace of local and $\mathcal{L}$-invariant cohomology classes by $\mathrm{H}^2_{\mathcal{L},loc}$. Note that the conditions are only required for at least one representative in the cohomology class. We collect a part of the results for the cocycle classes of the other algebras in the following theorem.

Theorem 1.6.5. *[SCH 14b, Cor. 6.48]*

(a) $\dim \mathrm{H}^2_{\mathcal{L},loc}(\mathcal{A}, \mathbb{C}) = 1$,

(b) $\dim \mathrm{H}^2_{loc}(\mathcal{L}, \mathbb{C}) = 1$,

(c) $\dim \mathrm{H}^2_{loc}(\mathcal{D}^1, \mathbb{C}) = 3$,

(d) $\dim \mathrm{H}^2_{loc}(\overline{\mathfrak{g}}, \mathbb{C}) = 1$ *for* $\mathfrak{g}$ *a simple finite-dimensional Lie algebra,*

(e) $\dim \mathrm{H}^2_{loc}(\mathcal{S}, \mathbb{C}) = 1$,

A basis of the cohomology spaces are given by taking the cohomology classes of the cocycles (1.6.7), (1.6.9), (1.6.10), (1.6.8), (1.6.15) obtained by integration over a separating cycle C_S.

Consequently, we obtain also for these algebras the corresponding result about uniqueness of almost-graded central extensions. For the differential operator algebra we get three independent cocycles. This generalizes results of [ARB 88] for the classical case.

For results on the bounded cocycle classes we have to multiply the dimensions above by $K = \#I$. For the supercase with odd central element the bounded cohomology vanishes.

For $\mathfrak{g}$ a reductive Lie algebra and if the cocycle is $\mathcal{L}$-invariant if restricted to the abelian part, a complete classification of local cocycle classes for both $\overline{\mathfrak{g}}$ and $\mathcal{D}^1_{\mathfrak{g}}$ can be found in [SCH 03a], [SCH 14b, Chapter 9].

I like to mention that in all the applications I know of, the cocycles coming from representations, regularizations, etc. are local. Hence, the uniqueness or classification result above can be used.

1.7. Examples and Generalizations

1.7.1. *The genus zero and three-point situation*

For illustration let us consider the three-point KN type algebras of genus zero. We consider the Riemann sphere $S^2 = \mathbb{P}^1$ and a set A consisting of 3 points. Given any triple of 3 points there exists always an analytic automorphism of $\mathbb{P}^1$ mapping this triple to $\{a, -a, \infty\}$, with $a \neq 0$. In fact $a = 1$ would suffice. Without restriction we can take

$$I := \{\infty\}, \quad O := \{a, -a\} \ .$$

Due to the symmetry of the situation it is more convenient to take a symmetrized basis of $\mathcal{A}$ (with $k \in \mathbb{Z}$)

$$A_{2k} := (z-a)^k(z+a)^k, \qquad A_{2k+1} := z(z-a)^k(z+a)^k \ , \tag{1.7.1}$$

for $\mathcal{L}$ (with $k \in \mathbb{Z}$)

$$V_{2k} := z(z-a)^k(z+a)^k \frac{d}{dz}, \qquad V_{2k+1} := (z-a)^{k+1}(z+a)^{k+1} \frac{d}{dz} \ , \tag{1.7.2}$$

and for the $-1/2$-forms

$$\varphi_{2k-1/2} := (z-a)^k(z+a)^k(\frac{d}{dz})^{-1/2}, \quad \varphi_{2k+1/2} := z(z-a)^k(z+a)^k(\frac{d}{dz})^{-1/2} . \tag{1.7.3}$$

Also we inverted the grading. By straight-forward calculations we obtain for the algebras the following structures.

The function algebra.

$$A_n \cdot A_m = \begin{cases} A_{n+m}, & n \text{ or } m \text{ even}, \\ A_{n+m} + a^2 \otimes A_{n+m-2}, & n \text{ and } m \text{ odd}. \end{cases} \tag{1.7.4}$$

The vector field algebra.

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} + a^2 V_{n+m-2}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)a^2 V_{n+m-2}, & n \text{ odd}, \ m \text{ even}. \end{cases} \tag{1.7.5}$$

The current algebra.

$$[x \otimes A_n, y \otimes A_m] = \begin{cases} [x,y] \otimes A_{n+m}, & n \text{ or } m \text{ even}, \\ [x,y] \otimes A_{n+m} + a^2 [x,y] \otimes A_{n+m-2}, & n \text{ and } m \text{ odd}. \end{cases} \tag{1.7.6}$$

The structure equations for the superalgebra look similar and can be easily calculated.

The central extensions can be given by determining the cocycle values by calculating the residues of the integrand at ∞. For example the local cocycle $\psi^1_{C_S}$ for the function algebra calculates as (see [FIA 05, A.13 and A.14])

$$\frac{1}{2\pi \mathrm{i}}\int_{C_S} A_n dA_m = \begin{cases} -n\delta_m^{-n}, & n, m \text{ even}, \\ 0, & n, m \text{ different parity}, \\ -n\delta_m^{-n} + a^2(-n+1)\delta_m^{-n+2}, & n, m \text{ odd}. \end{cases} \tag{1.7.7}$$

The **affine algebra** is now given as the almost-graded central extension $\widehat{\mathfrak{g}}_{\beta,S}$ of the current algebra given by the cocycle

$$\psi^2_{C_S,\beta}(x\otimes A_n, y\otimes A_m) = \beta(x,y)\cdot\frac{1}{2\pi \mathrm{i}}\int_{C_S} A_n dA_m = \beta(x,y)\cdot\psi^1_{C_S}(A_n, A_m). \tag{1.7.8}$$

Three-point $\mathfrak{sl}(2,\mathbb{C})$-current algebra for genus 0.
Given a simple Lie algebra $\mathfrak{g}$ with generators and structure equations the relations above can be written in these terms. An important example is $\mathfrak{sl}(2,\mathbb{C})$ with the standard generators

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We set $e_n := e\otimes A_n$, $n\in\mathbb{Z}$ and in the same way f_n and h_n. Recall that the invariant bilinear form $\beta(x,y) = \mathrm{tr}(x\cdot y)$. We calculate

$$[e_n, f_m] = \begin{cases} h_{n+m}, & n \text{ or } m \text{ even}, \\ h_{n+m} + a^2 h_{n+m-2}, & n \text{ and } m \text{ odd}, \end{cases} \tag{1.7.9}$$

$$[h_n, e_m] = \begin{cases} 2e_{n+m}, & n \text{ or } m \text{ even}, \\ 2e_{n+m} + 2a^2 e_{n+m-2}, & n \text{ and } m \text{ odd}, \end{cases} \tag{1.7.10}$$

$$[h_n, f_m] = \begin{cases} -2f_{n+m}, & n \text{ or } m \text{ even}, \\ -2f_{n+m} - 2a^2 f_{n+m-2}, & n \text{ and } m \text{ odd}. \end{cases} \tag{1.7.11}$$

For the central extension we obtain

$$[e_n, f_m] = \begin{cases} h_{n+m} - n\delta_m^{-n}, & n \text{ or } m \text{ even}, \\ h_{n+m} + a^2 h_{n+m-2} - n\delta_m^{-n} - a^2(n-1)\delta_m^{-n+2}, & n \text{ and } m \text{ odd}, \end{cases}$$

(1.7.12)

and

$$[h_n, h_m] = \begin{cases} -2n\delta_m^{-n}, & n, m \text{ even}, \\ 0, & n, m \text{ different parity}, \\ -2n\delta_m^{-n} + 2a^2(-n+1)\delta_m^{-n+2}, & n, m \text{ odd}. \end{cases} \quad (1.7.13)$$

For the other commutators we do not have contributions to the center.

1.7.2. *Genus zero multi-point algebras – integrable systems*

Already the Witt and Virasoro algebra in genus zero with two points where poles are allowed are mathematically highly interesting objects which have e.g. a non-trivial representation theory. If we remain on the Riemann sphere but now allow more than two poles we obtain an even more demanding mathematical theory. For the multi-point case the related systems are important. For example the classical Knizhnik-Zamolodchikov models of Conformal Field Theory (CFT) are of this type, see e.g. [KNI 84]. Integrable systems show up.

Due to the connection between CFT and statistical mechanics it is not a surprise that the genus zero multi-point Krichever–Novikov algebras turn out to be related to algebras appearing in statistical mechanics. For example the *Onsager algebra* appears as subalgebra of the three-point, $g = 0$, $\mathfrak{sl}(2, \mathbb{C})$-Krichever–Novikov algebra. In this context see e.g. the work of Terwilliger and collaborators [HAR 07], [BEN 07], [ITO 08].

For the genus zero multi-point situation quite a number of publications appeared. Some references are [SCH 90c], [FIA 03], [FIA 05], [FIA 07], [BRE 91], [BRE 95], [SCH 07a], [ANZ 92], [COX 08]. Recently, the author [SCH 17] gave a thorough and unified treatment of universal central extensions of the genus zero algebras.

From the point of view of symmetries of integrable systems the concept of *automorphic Lie algebras* shows up. It was e.g. developed by Lombardo, Mikailov, and Sanders in [LOM 05a], [LOM 05b], [LOM 10]. Invariant objects under finite subgroups of $PGL(2, \mathbb{C})$, the symmetry group of the Riemann sphere, are studied. Of course, there are relations to the $g = 0$, multi-point Krichever–Novikov type algebras. Chopp [CHO 11] obtained some results for the genus one multi-point setting.

1.7.3. *Deformations*

As the second Lie algebra cohomology of the Witt and Virasoro algebra in their adjoint module vanishes [SCH 11], [FIA 12], [FIA 90] both are formally and infinitesimally rigid. This means that all formal (and infinitesimal) families with special fiber these algebras are equivalent to the trivial one. If we consider the examples of Section 1.7.1 parameterized by a variable a, then they are non-trivial (even locally non-trivial) families which have themselves as special elements for $a = 0$ the classical algebras. The geometric context is clear: the two points a and $-a$ move together. By Fialowski and Schlichenmaier [FIA 03], [FIA 05], [FIA 07] the above algebras and similar families of algebras on tori, were used to exhibit the fact, that e.g. the Witt and Virasoro algebra despite their formal rigidity allow non-trivial algebraic-geometric deformations. This is an effect that cannot appear in the finite-dimensional algebra setting. For families on tori see the above quoted results, respectively [SCH 14b, Chapter 12]. See also [SCH 90c], [BRE 90], [BRE 94], [DEC 90], [RUF 92].

1.8. Lax Operator Algebras

Recently, a new class of current type algebras appeared, the Lax operator algebras. As the naming indicates, they are related to integrable systems [SHE 11]. The algebras were introduced by Krichever [KRI 02], and Krichever and Sheinman [KRI 07]. Here I will report on their definition. See the book [SHE 12] of Sheinman for more details.

Compared to the KN current type algebra we will allow additional singularities which will play a special role. The points where these singularities are allowed are called *weak singular points*. The set of such points is denoted by

$$W = \{\gamma_s \in \Sigma \setminus A \mid s = 1, \ldots, R\}. \tag{1.8.1}$$

Let $\mathfrak{g}$ be one of the classical matrix algebras $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$, $\mathfrak{so}(n)$, $\mathfrak{sp}(2n)$. We assign to every point γ_s a vector $\alpha_s \in \mathbb{C}^n$ (respectively $\in \mathbb{C}^{2n}$ for $\mathfrak{sp}(2n)$). The system

$$\mathcal{T} := \{(\gamma_s, \alpha_s) \in \Sigma \times \mathbb{C}^n \mid s = 1, \ldots, R\} \tag{1.8.2}$$

is called *Tyurin data*.

Remark 1.8.1. In case that $R = n \cdot g$ and for generic values of (γ_s, α_s) with $\alpha_s \neq 0$ the tuples of pairs $(\gamma_s, [\alpha_s])$ with $[\alpha_s] \in \mathbb{P}^{n-1}(\mathbb{C})$ parameterize semi-stable rank n and degree $n\,g$ framed holomorphic vector bundles as shown by Tyurin [TJU 65]. Hence, the name Tyurin data.

We consider $\mathfrak{g}$-valued meromorphic functions [5]

$$L : \Sigma \to \mathfrak{g}, \tag{1.8.3}$$

5. Strictly speaking, the interpretation as function is a little bit misleading, as they behave under differentiation like operators on trivialized sections of vector bundles.

which are holomorphic outside $W \cup A$, have at most poles of order one (respectively of order two for $\mathfrak{sp}(2n)$) at the points in W, and fulfill certain conditions at W depending on $\mathcal{T}$. To describe them let us fix local coordinates w_s centered at γ_s, $s = 1, \ldots, R$. For $\mathfrak{gl}(n)$ the conditions are as follows. For $s = 1, \ldots, R$ we require that there exist $\beta_s \in \mathbb{C}^n$ and $\kappa_s \in \mathbb{C}$ such that the matrix-valued function L has the following expansion at $\gamma_s \in W$

$$L(w_s) = \frac{L_{s,-1}}{w_s} + L_{s,0} + \sum_{k>0} L_{s,k} w_s^k, \tag{1.8.4}$$

with

$$L_{s,-1} = \alpha_s \beta_s^t, \quad \operatorname{tr}(L_{s,-1}) = \beta_s^t \alpha_s = 0, \quad L_{s,0}\, \alpha_s = \kappa_s \alpha_s. \tag{1.8.5}$$

In particular, if $L_{s,-1}$ is non-vanishing then it is a rank 1 matrix, and if $\alpha_s \neq 0$ then it is an eigenvector of $L_{s,0}$. The requirements (1.8.5) are independent of the chosen coordinates w_s.

It is not at all clear that the commutator of two such matrix functions fulfills again these conditions. But it is shown in [KRI 07] that they indeed close to a Lie algebra (in fact in the case of $\mathfrak{gl}(n)$ they constitute an associative algebra under the matrix product). If one of the $\alpha_s = 0$ then the conditions at the point γ_s correspond to the fact, that L has to be holomorphic there. If all α_s's are zero or $W = \emptyset$ we obtain back the current algebra of KN type. For the algebra under consideration here, in some sense the Lax operator algebras generalize them. In the bundle interpretation of the Tyurin data the KN case corresponds to the trivial rank n bundle.

For $\mathfrak{sl}(n)$ the only additional condition is that in (1.8.4) all matrices $L_{s,k}$ are trace-less. The conditions (1.8.5) remain unchanged.

In the case of $\mathfrak{so}(n)$ one requires that all $L_{s,k}$ in (1.8.4) are skew-symmetric. In particular, they are trace-less. Following [KRI 07] the set-up has to be slightly modified. First only those Tyurin parameters α_s are allowed which satisfy $\alpha_s^t \alpha_s = 0$. Then, (1.8.5) is changed in the following way:

$$L_{s,-1} = \alpha_s \beta_s^t - \beta_s \alpha_s^t, \quad \operatorname{tr}(L_{s,-1}) = \beta_s^t \alpha_s = 0, \quad L_{s,0}\, \alpha_s = \kappa_s \alpha_s. \tag{1.8.6}$$

For $\mathfrak{sp}(2n)$ we consider a symplectic form $\hat{\sigma}$ for $\mathbb{C}^{2n}$ given by a non-degenerate skew-symmetric matrix σ. The Lie algebra $\mathfrak{sp}(2n)$ is the Lie algebra of matrices X such that $X^t \sigma + \sigma X = 0$. The condition $\operatorname{tr}(X) = 0$ will be automatic. At the weak singularities we have the expansion

$$L(w_s) = \frac{L_{s,-2}}{w_s^2} + \frac{L_{s,-1}}{w_s} + L_{s,0} + L_{s,1} w_s + \sum_{k>1} L_{s,k} w_s^k. \tag{1.8.7}$$

The condition (1.8.5) is modified as follows (see [KRI 07]): there exist $\beta_s \in \mathbb{C}^{2n}$, $\nu_s, \kappa_s \in \mathbb{C}$ such that

$$L_{s,-2} = \nu_s \alpha_s \alpha_s^t \sigma, \quad L_{s,-1} = (\alpha_s \beta_s^t + \beta_s \alpha_s^t)\sigma, \quad {\beta_s}^t \sigma \alpha_s = 0, \quad L_{s,0}\, \alpha_s = \kappa_s \alpha_s. \tag{1.8.8}$$

Moreover, we require $\alpha_s^t \sigma L_{s,1} \alpha_s = 0$. Again under the point-wise matrix commutator the set of such maps constitute a Lie algebra.

It is possible to introduce an almost-graded structure for these Lax operator algebras induced by a splitting of the set $A = I \cup O$. This is done for the two-point case in [KRI 07] and for the multi-point case in [SCH 14a]. From the applications there is again a need to classify almost-graded central extensions.

The author obtained this jointly with O. Sheinman in [SCH 08] for the two-point case. For the multi-point case see [SCH 14a]. For the Lax operator algebras associated to the simple algebras $\mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(n)$ it will be unique (meaning: given a splitting of A there is an almost-grading and with respect to this there is up to equivalence and rescaling only one non-trivial almost-graded central extension). For $\mathfrak{gl}(n)$ we obtain two independent local cocycle classes if we assume $\mathcal{L}$-invariance on the reductive part. Both in the definition of the cocycle and in the definition of $\mathcal{L}$-invariance a connection shows up.

Remark 1.8.2. Recently, Sheinman extended the set-up to G_2 [SHE 14] and moreover gave a recipe for all semi-simple Lie algebras [SHE 15].

1.9. Fermionic Fock Space

1.9.1. *Semi-Infinite forms and fermionic Fock space representations*

Our Krichever-Novikov vector field algebras $\mathcal{L}$ have as Lie modules the spaces $\mathcal{F}^\lambda$. These representations are not of the type physicists are usually interested in. There are neither annihilation nor creation operators which can be used to construct the full representation out of a vacuum state.

To obtain representation with the required properties the almost-grading again comes into play. First, using the grading of $\mathcal{F}^\lambda$ it is possible to construct starting from $\mathcal{F}^\lambda$, the forms of weight $\lambda \in 1/2\mathbb{Z}$, the *semi-infinite wedge forms* $\mathcal{H}^\lambda$s.

The vector space $\mathcal{H}^\lambda$ is generated by basis elements which are formal expressions of the type

$$\Phi = f^\lambda_{(i_1)} \wedge f^\lambda_{(i_2)} \wedge f^\lambda_{(i_3)} \wedge \cdots, \tag{1.9.1}$$

where $(i_1) = (m_1, p_1)$ is a double index indexing our basis elements. The indices are in strictly increasing lexicographical order. They are stabilizing in the sense that they will increase exactly by one starting from a certain index which depends on Φ. The action of $\mathcal{L}$ should be extended by Leibniz rule from $\mathcal{F}^\lambda$ to $\mathcal{H}^\lambda$. But a problem arises. For elements of the critical strip $\mathcal{L}_{[0]}$ of the algebra $\mathcal{L}$ it might happen that they produce infinitely many contributions. The action has to be regularized (as physicists like to call it), which is a well-defined mathematical procedure.

Here the almost-grading has another appearance. By the (strong) almost-graded module structure of $\mathcal{F}^\lambda$ the algebra $\mathcal{L}$ can be embedded into the Lie algebra of both-sided infinite matrices

$$\overline{gl}(\infty) := \{A = (a_{ij})_{i,j\in\mathbb{Z}} \mid \exists r = r(A), \text{ such that } a_{ij} = 0 \text{ if } |i-j| > r\ \}, \quad (1.9.2)$$

with "finitely many diagonals". The embedding will depend on the weight λ. For $\overline{gl}(\infty)$ there exists a procedure for the regularization of the action on the semi-infinite wedge product [DAT 82], [KAC 81], see also [KAC 87]. In particular, there is a unique non-trivial central extension $\widehat{gl}(\infty)$. If we pull-back the defining cocycle for the extension we obtain a central extension $\widehat{\mathcal{L}}_\lambda$ of $\mathcal{L}$ and the required regularization of the action of $\widehat{\mathcal{L}}_\lambda$ on $\mathcal{H}^\lambda$. As the embedding of $\mathcal{L}$ depends on the weight λ the cocycle will depend too. The pull-back cocycle will be local. Hence, by the classification results of Section 1.6.3 it is the unique central extension class defined by (1.6.9) integrated over C_S (up to a rescaling).

In $\mathcal{H}^\lambda$ there are invariant subspaces, which are generated by a certain "vacuum vectors". The subalgebra $\mathcal{L}_{[+]}$ annihilates the vacuum, the central element and the other elements of degree zero act by multiplication with a constant and the whole representation space is generated by $\mathcal{L}_{[-]} \oplus \mathcal{L}_{[0]}$ from the vacuum.

As the function algebra $\mathcal{A}$ operates as multiplication operators on $\mathcal{F}^\lambda$ the above representation can be extended to the algebra $\mathcal{D}^1$ (see details in [SCH 90d], [SCH 14b]) after one passes to central extensions. The cocycle again is local and hence, up to coboundary, it will be a certain linear combination of the 3 generating cocycles for the differential operator algebra. In fact its class will be

$$c_\lambda \cdot [\psi^3_{C_S}] + \frac{2\lambda - 1}{2}[\psi^4_{C_S}] - [\psi^1_{C_S}], \qquad c_\lambda := -2(6\lambda^2 - 6\lambda + 1). \qquad (1.9.3)$$

Recall that ψ^3 is the cocycle for the vector field algebra, ψ^1 the cocycle for the function algebra, and ψ^4 the mixing cocycle. Note that the expression for c_λ appears also in Mumford's formula [SCH 07b] relating divisors on the moduli space of curves.

For $\mathcal{L}$ we could rescale the central element. Hence essentially, the central extension $\widehat{\mathcal{L}}$ did not depend on the weight. Here this is different. The central extension $\widehat{\mathcal{D}^1}_\lambda$

depends on it. Furthermore, the representation on $\mathcal{H}^\lambda$ gives a projective representation of the algebra of $\mathcal{D}_\lambda$ of differential operators of all orders. It is exactly the combination (1.9.3) which lifts to a cocycle for $\mathcal{D}_\lambda$ and gives a central extension $\widehat{\mathcal{D}}_\lambda$.

For the centrally extended algebras $\widehat{\mathfrak{g}}$ in a similar way fermionic Fock space representations can be constructed, see [SHE 01], [SCH 99].

1.9.2. b – c *systems*

Related to the above there are other quantum algebra systems which can be realized on $\mathcal{H}^\lambda$. On the space $\mathcal{H}^\lambda$ the forms $\mathcal{F}^\lambda$ act by wedging elements $f^\lambda \in \mathcal{F}^\lambda$ in front of the semi-infinite wedge form, i.e.

$$\Phi \mapsto f^\lambda \wedge \Phi. \tag{1.9.4}$$

Using the Krichever-Novikov duality pairing (1.5.4) and by contracting the elements in the semi-infinite wedge forms, the forms $f^{1-\lambda} \in \mathcal{F}^{1-\lambda}$ will act on them too. For Φ a basis element (1.9.1) of $\mathcal{H}^\lambda$ the contraction is defined via

$$i(f^{1-\lambda})\Phi = \sum_{l=1}^{\infty}(-1)^{l-1}\langle f^{1-\lambda}, f^\lambda_{i_l}\rangle \cdot f^\lambda_{(i_1)} \wedge f^\lambda_{(i_2)} \wedge \cdots \check{f}^\lambda_{(i_l)}. \tag{1.9.5}$$

Here $\check{f}^\lambda_{(i_l)}$ indicates as usual that this element will not be there anymore.

Both operations create a Clifford algebra like structure, which is sometimes called a $b-c$ system, see [SCH 14b, Chapters 7 and 8].

1.10. Sugawara Representation

In the classical set-up the (two-dimensional) Sugawara construction relates to a representation of the classical affine Lie algebra $\widehat{\mathfrak{g}}$ a representation of the Virasoro algebra, see e.g. [KAC 90], [KAC 87]. In joint work with O. Sheinman the author succeeded in extending it to arbitrary genus and the multi-point setting [SCH 98]. For an updated treatment, incorporating also the uniqueness results of central extensions, see [SCH 14b, Chapter 10]. Here we will give a very rough sketch.

We start with an admissible representation V of a centrally extended current algebra $\widehat{\mathfrak{g}}$. Admissible means, that the central element operates as constant $\times$ identity, and that every element v in the representation space will be annihilated by the elements in $\widehat{\mathfrak{g}}$ of sufficiently high degree (which depends on the element v).

For simplicity let $\mathfrak{g}$ be either abelian or simple and β the non-degenerate symmetric invariant bilinear form used to construct $\widehat{\mathfrak{g}}$ (now we need that it is non-degenerate). Let

$\{u_i\},\ \{u^j\}$ be a system of dual basis elements for $\mathfrak{g}$ with respect to β, i.e. $\beta(u_i, u^j) = \delta_i^j$. Note that the Casimir element of $\mathfrak{g}$ can be given by $\sum_i u_i u^i$. For $x \in \mathfrak{g}$ we consider the family of operators $x(n,p)$ given by the operation of $x \otimes A_{n,p}$ on V. We group them together in a formal sum

$$\widehat{x}(Q) := \sum_{n\in\mathbb{Z}} \sum_{p=1}^{K} x(n,p)\omega^{n,p}(Q), \quad Q \in \Sigma. \tag{1.10.1}$$

Such a formal sum is called a field if applied to a vector $v \in V$ it gives again a formal sum (now of elements from V) which is bounded from above. By the condition of admissibility $\widehat{x}(Q)$ is a field. It is of conformal weight one, as the one-differentials $\omega^{n,p}$ show up.

The current operator fields are defined as [6]

$$J_i(Q) := \widehat{u_i}(Q) = \sum_{n,p} u_i(n,p)\omega^{n,p}(Q). \tag{1.10.2}$$

The Sugawara operator field $T(Q)$ is defined by

$$T(Q) := \frac{1}{2}\sum_i {:}J_i(Q)J^i(Q){:} \ . \tag{1.10.3}$$

Here :...: denotes some normal ordering, which is needed to make the product of two fields again a field. The *standard normal ordering* is defined as

$${:}x(n,p)y(m,r){:} \quad := \begin{cases} x(n,p)y(m,r), & (n,p) \le (m,r) \\ y(m,r)x(n,p), & (n,p) > (m,r) \end{cases} \tag{1.10.4}$$

where the indices (n,p) are lexicographically ordered. By this prescription the annihilation operator, i.e. the operators of positive degree, are brought as much as possible to the right so that they act first.

As the current operators are fields of conformal weights one the Sugawara operator field is a field of weight two. Hence we write it as

$$T(Q) = \sum_{k\in\mathbb{Z}} \sum_{p=1}^{K} L_{k,p} \cdot \Omega^{k,p}(Q) \tag{1.10.5}$$

with certain operators $L_{k,p}$. The $L_{k,p}$ are called *modes of the Sugawara field* T or just Sugawara operators.

6. For simplicity we drop mentioning the range of summation here and in the following when it is clear.

Let 2κ be the eigenvalue of the Casimir operator in the adjoint representation. For $\mathfrak{g}$ abelian $\kappa = 0$. For $\mathfrak{g}$ simple and β normalized such that the longest roots have square length 2 then κ is the dual Coxeter number. Recall that the central element t acts on the representation space V as $c \cdot id$ with a scalar c. This scalar is called the *level* of the representation. The key result is (where $x(g)$ denotes the operator corresponding to the element $x \otimes g$)

Proposition 1.10.1. *[SCH 14b, Prop. 10.8] Let $\mathfrak{g}$ be either an abelian or a simple Lie algebra. Then*

$$[L_{k,p}, x(g)] = -(c+\kappa) \cdot x(e_{k,p} \,.\, g)\,. \tag{1.10.6}$$

$$[L_{k,p}, \widehat{x}(Q)] = (c+\kappa) \cdot (e_{k,p} \,.\, \widehat{x}(Q))\,. \tag{1.10.7}$$

Recall that $e_{k,p}$ are the KN basis elements for the vector field algebra $\mathcal{L}$.

In the next step the commutators of the operators $L_{k,p}$ can be calculated. In the case the $c+\kappa = 0$, called the *critical level*, these operators generate a subalgebra of the center of $\mathfrak{gl}(V)$. If $c+\kappa \neq 0$ (i.e. at a non-critical level) the $L_{k,p}$ can be replaced by rescaled elements $L^*_{k,p} = \frac{-1}{c+\kappa} L_{k,p}$ and we denote by $T[..]$ the linear representation of $\mathcal{L}$ induced by

$$T[e_{k,p}] = \mathcal{L}^*_{k,p}. \tag{1.10.8}$$

The result is that T defines a projective representation of $\mathcal{L}$ with a local cocycle. This cocycle is up to rescaling our geometric cocycle $\psi^3_{C_S,R}$ with a suitable projective connection [7] R. In detail,

$$T[[e,f]] = [T[e], T[f]] + \frac{c \dim \mathfrak{g}}{c+\kappa}\, \psi^3_{C_S,R}(e,f) id. \tag{1.10.9}$$

Consequently, by setting

$$T[\hat{e}] := T[e], \quad T[t] := \frac{c \dim \mathfrak{g}}{c+\kappa}\, id\,. \tag{1.10.10}$$

we obtain a honest Lie representation of the centrally extended vector field algebra $\widehat{\mathcal{L}}$ given by this local cocycle. For the general reductive case, see [SCH 14b, Section 10.2.1].

7. The projective connection takes care of the "up to coboundary". It is induced by the normal ordering prescription

1.11. Application to Moduli Space

This application deals with *Wess-Zumino-Novikov-Witten models* and the *Knizhnik-Zamolodchikov Connection*. Despite the fact, that it is a very important application, the following description is very condensed. More can be found in [SCH 99], [SCH 04]. See also [SCH 14b], [SHE 12].

Wess-Zumino-Novikov-Witten (WZNW) models are defined on the basis of a fixed finite-dimensional simple (or semi-simple) Lie algebra $\mathfrak{g}$. One considers families of representations of the affine algebras $\widehat{\mathfrak{g}}$ (which is an almost-graded central extension of $\overline{\mathfrak{g}}$) defined over the moduli space of Riemann surfaces of genus g with $K+1$ marked points and splitting of type $(K,1)$. The single point in O will be a reference point. The data of the moduli of the Riemann surface and the marked points enter the definition of the algebra $\widehat{\mathfrak{g}}$ and the representation. The construction of certain co-invariants yields a special vector bundle of finite rank over moduli space, called the vector bundle of conformal blocks, or Verlinde bundle. With the help of the Krichever Novikov vector field algebra, and using the Sugawara construction, the *Knizhnik-Zamolodchikov (KZ) connection* is given. It is projectively flat. An essential fact is that certain elements in the critical strip $\mathcal{L}_{[0]}$ correspond to infinitesimal deformations of the moduli and to moving the marked points. This gives a global operator approach in contrast to the semi-local approach of Tsuchia, Ueno, and Yamada [TSU 89].

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