The low-dimensional algebraic cohomology of the Witt and the Virasoro algebra

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Abstract. In this contribution, we provide the results on the low-dimensional algebraic cohomology with values in the trivial and the adjoint module of the Witt and the Virasoro algebra over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$. A sketch of the various proofs is given, the details can be found in previous articles of the authors. More precisely, we give a summary of results known concerning the zeroth, first and second algebraic cohomology of the Witt and the Virasoro algebra, and we present some details about the more recent results related to the third algebraic cohomology. It has to be noted that we are dealing entirely with algebraic cohomology, meaning that our results are valid for any concrete realization of the Witt and the Virasoro algebra. Moreover, our results are independent of any underlying topology chosen.

1. Introduction

The Witt algebra and its universal central extension given by the Virasoro algebra are infinite-dimensional Lie algebras of outermost importance both in mathematics and in physics. The Virasoro algebra thus appears in two-dimensional conformal field theory and in String Theory. Contrary to higher dimensional cohomology, the low-dimensional cohomology comes with an easy interpretation in terms of known objects, such as invariants, central extensions, deformations, obstructions or crossed modules, see e.g. Gerstenhaber [1–5]. The analysis of these objects is also of interest in mathematics as they allow a better understanding of the Lie algebra itself. Also in physics, the low-dimensional cohomology is used, for example in the study of anomalies, see e.g. Roger [6].

The Witt and the Virasoro algebra have several concrete realizations. A geometrical realization of the Witt algebra is for example given by the algebra of meromorphic vector fields on the Riemann sphere $\mathbb{CP}^1$ that are holomorphic outside of zero and infinity. Another popular geometrical realization corresponds to the complexified Lie algebra of polynomial vector fields on the circle, $Vect_{Poly}(S^1)$. This Lie algebra is a dense subalgebra of the complexified Lie algebra of smooth vector fields on the circle, $Vect(S^1)$. The cohomology of the Lie algebra $Vect(S^1)$ is already known, as it has been computed by Gelfand and Fuks [7, 8] in the case of the trivial module and by Fialowski and Schlichenmaier [9] in the case of generic modules, including the adjoint module. For this representation of the Witt and the Virasoro algebra, it is sensible to consider continuous cohomology, i.e. to consider only continuous cochains. By restriction to the continuous sub-complex and by using density arguments, one can transfer the results for $Vect(S^1)$ to $Vect_{Poly}(S^1)$, i.e. the Witt algebra. Therefore, the continuous cohomology of the

Witt algebra is known.
However, the definition of the Witt and the Virasoro algebra is given by the Lie structure equation and is purely algebraic. In fact, the Witt algebra also has algebraic realizations, such as the algebra of derivations of the Laurent polynomials.
In the present article, we consider the low-dimensional algebraic cohomology of the Witt and the Virasoro independent of any concrete realization of these Lie algebras. We only use the primary definition via the Lie structure equation, making the results valid for any realization. In general, algebraic cohomology is much harder to compute than continuous cohomology, as the entire toolkit of topology and geometry cannot be used. Results on algebraic cohomology are thus somewhat scarce in the literature. By using elementary algebraic methods, Schlichenmaier showed in [10, 11] the vanishing of the second algebraic cohomology with values in the adjoint module of the Witt and the Virasoro algebra, see also Fialowski [12]. This result was already announced in the case of the Witt algebra in [13] by Fialowski, though no proof was given. The same result was obtained via elementary algebra in the supersymmetric case by Van den Hijligenberg and Kotchetkov in [14, 15]. The first algebraic cohomology is easy to compute, it was done explicitly for the Witt and the Virasoro algebra in [16]. The third algebraic cohomology with values in the trivial and the adjoint module was computed for the Witt and the Virasoro algebra by the authors in [16] and [17]. The aim of this article is to summarize the results obtained and to provide a sketch of the proofs involved.
In Section 2, we briefly introduce the Witt and the Virasoro algebra. The topic covered in Section 3 consists of the cohomology of Lie algebras. More precisely, after recalling the Chevalley-Eilenberg cohomology, we introduce the notion of the degree of a homogeneous cochain. Subsequently, we continue by recalling results known in the case of the zeroth, first and second algebraic cohomology of the Witt and the Virasoro algebra, with values in the adjoint and the trivial module. We also briefly mention the interpretation of each of these cohomologies. Finally, we end Section 3 by recalling the Theorem of Hochschild-Serre. In Section 4, we present our results obtained for the third algebraic cohomology of the Witt and the Virasoro algebra, with values in the adjoint and the trivial module. The results are presented along with a sketch of the proofs involved.
So far, we restricted our work to the low-dimensional algebraic cohomology. The reason is two-fold. On the one hand, higher dimensional cohomology such as the fourth cohomology and above, do not come with an easy interpretation in terms of known objects. Still, it would be interesting to see whether algebraic and continuous cohomology agree also in the case of higher dimensional cohomology. On the other hand though, their computation becomes increasingly difficult in the algebraic case. Comparison of the proofs provided for the first, second and third algebraic cohomology shows that there is no straightforward generalization when increasing the cohomological dimension.

2. The Witt and the Virasoro Algebra

The Witt algebra $\mathcal{W}$ is an infinite-dimensional $\mathbb{Z}$-graded Lie algebra generated as a vector space over a base field $\mathbb{K}$ with characteristic zero by the basis elements $\{e_n \mid n \in \mathbb{Z}\}$, which satisfy the following Lie algebra structure equation:

$$[e_n, e_m] = (m-n)e_{n+m}, \qquad n, m \in \mathbb{Z}\,.$$

The Witt algebra becomes a $\mathbb{Z}$-graded Lie algebra by defining the degree of an element e_n by $deg(e_n) := n$. More precisely, the grading being given by one of its own elements, namely e_0: $[e_0, e_m] = me_m = deg(e_m)e_m$, the Witt algebra is an internally $\mathbb{Z}$-graded Lie algebra.
Let us briefly comment on the most popular realizations of the Witt algebra. An algebraic description of the Witt algebra is given in terms of the Lie algebra of derivations of the infinite-dimensional associative $\mathbb{K}$-algebra of Laurent polynomials $\mathbb{K}[Z^{-1}, Z]$.

By considering $\mathbb{K} = \mathbb{C}$, we obtain a geometrical realization of the Witt algebra given by the algebra of meromorphic vector fields on the Riemann sphere $\mathbb{CP}^1$ that are holomorphic outside of 0 and ∞. The generators of the Witt algebra for this realization are of the form $e_n = z^{n+1}\frac{d}{dz}$, where z corresponds to the quasi-global complex coordinate.
Finally, another geometrical realization of the Witt algebra is the complexified Lie algebra of polynomial vector fields on the circle S^1, in which case the generators are given by $e_n = e^{in\varphi}\frac{d}{d\varphi}$, where φ is the coordinate along S^1.

It is known that the Witt algebra has a unique non-trivial central extension $\mathcal{V}$, up to equivalence and rescaling, which in fact is a universal central extension:

$$0 \longrightarrow \mathbb{K} \xrightarrow{i} \mathcal{V} \xrightarrow{\pi} \mathcal{W} \longrightarrow 0\,, \tag{2.1}$$

where $\mathbb{K}$ is in the center of $\mathcal{V}$. This extension $\mathcal{V}$ corresponds to the Virasoro algebra.
On the level of vector spaces, the Virasoro algebra $\mathcal{V}$ is given as a direct sum $\mathcal{V} = \mathbb{K} \oplus \mathcal{W}$, with generators $\hat{e}_n := (0, e_n)$ and the one-dimensional central element $t := (1, 0)$. The generators fulfill the following Lie structure equation:

$$\begin{aligned} &[\hat{e}_n, \hat{e}_m] = (m-n)\hat{e}_{n+m} + \alpha(e_n, e_m)\cdot t \qquad n, m \in \mathbb{Z}\,, \\ &[\hat{e}_n, t] = [t, t] = 0\,, \end{aligned} \tag{2.2}$$

where $\alpha \in Z^2(\mathcal{W}, \mathbb{K})$ is the so-called Virasoro 2-cocycle, a representing element of which is given by:

$$\alpha(e_n, e_m) = -\frac{1}{12}(n^3 - n)\delta_{n+m,0}\,. \tag{2.3}$$

The cubic term n^3 is the most relevant term, whereas the linear term n corresponds to a coboundary[1].
By defining $deg(\hat{e}_n) := deg(e_n) = n$ and $deg(t) := 0$, the Virasoro algebra becomes also an internally $\mathbb{Z}$-graded Lie algebra.

3. The cohomology of Lie algebras

3.1. The Chevalley-Eilenberg cohomology

We will briefly recall the Chevalley-Eilenberg cohomology, for the convenience of the reader.
Let $\mathcal{L}$ be a Lie algebra, M a $\mathcal{L}$-module, and $C^q(\mathcal{L}, M)$ the space of q-multilinear alternating maps on $\mathcal{L}$ with values in M,

$$C^q(\mathcal{L}, M) := \mathrm{Hom}_{\mathbb{K}}(\wedge^q \mathcal{L}, M)\,.$$

We call q-cochains the elements of $C^q(\mathcal{L}, M)$, and we set by convention $C^0(\mathcal{L}, M) := M$. The coboundary operators δ_q are defined by:

$$\begin{aligned} \forall q \in \mathbb{N}, \qquad & \delta_q : C^q(\mathcal{L}, M) \to C^{q+1}(\mathcal{L}, M) : \psi \mapsto \delta_q\psi\,, \\ (\delta_q\psi)(x_1, \ldots x_{q+1}) : \quad & = \textstyle\sum_{1\le i<j\le q+1} (-1)^{i+j+1}\ \psi([x_i, x_j]\,, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1}) \\ & + \textstyle\sum_{i=1}^{q+1} (-1)^i\ x_i \cdot \psi(x_1, \ldots, \hat{x}_i, \ldots, x_{q+1})\,, \end{aligned} \tag{3.1}$$

with $x_1, \ldots, x_{q+1} \in \mathcal{L}$, $\hat{x}_i$ means that the entry x_i is omitted and the dot $\cdot$ stands for the module structure. In case of the adjoint module $M = \mathcal{L}$, we have $x \cdot y = [x, y]$ for $x \in \mathcal{L}$ and $y \in M$, while in the case of the trivial module $M = \mathbb{K}$, we have $x \cdot y = 0$. The coboundary operators fulfill

[1] The symbol $\delta_{i,j}$ is the Kronecker Delta, equaling one if $i = j$ and zero otherwise.

$\delta_{q+1} \circ \delta_q = 0 \; \forall \; q \in \mathbb{N}$, giving us a cochain complex $(C^*(\mathcal{L}, M), \delta)$ called the Chevalley-Eilenberg complex. The corresponding cohomology is the Chevalley-Eilenberg cohomology:

$$\mathrm{H}^q(\mathcal{L}, M) := Z^q(\mathcal{L}, M)/B^q(\mathcal{L}, M)\,,$$

where elements in $Z^q(\mathcal{L}, M) := \ker \delta_q$ are called q-cocycles and elements in $B^q(\mathcal{L}, M) := \mathrm{im}\ \delta_{q-1}$ are called q-coboundaries. For more details, we refer the reader to the original literature by Chevalley and Eilenberg [18].

3.2. Degree of a homogeneous cochain

We consider $\mathcal{L}$ to be a $\mathbb{Z}$-graded Lie algebra, $\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n$, and M a $\mathbb{Z}$-graded $\mathcal{L}$-module, i.e. $M = \bigoplus_{n\in\mathbb{Z}} M_n$. A q-cochain ψ is *homogeneous of degree* d if there exists a $d \in \mathbb{Z}$ such that for all q-tuple $x_1, \ldots, x_q$ of homogeneous elements $x_i \in \mathcal{L}_{deg(x_i)}$, we have:

$$\psi(x_1, \ldots, x_q) \in M_n \text{ with } n = \sum_{i=1}^{q} deg(x_i) + d\,.$$

Everything is compatible with the coboundary operator and hence the cohomology can be decomposed for all q:

$$\mathrm{H}^q(\mathcal{L}, M) = \bigoplus_{d\in\mathbb{Z}} \mathrm{H}^q_{(d)}(\mathcal{L}, M)\,.$$

In the case of internally graded Lie algebras and modules, the cohomology reduces to the degree-zero cohomology according to the result of Fuks [7]:

$$\begin{aligned} &\mathrm{H}^q_{(d)}(\mathcal{L}, M) = \{0\} \qquad \text{for } d \neq 0\,, \\ &\mathrm{H}^q(\mathcal{L}, M) = \mathrm{H}^q_{(0)}(\mathcal{L}, M)\,. \end{aligned} \tag{3.2}$$

In our case, the result can be applied as both the Witt algebra and the Virasoro algebra are internally graded Lie algebras. Besides, the modules considered here are also graded. This is obvious in the case of the adjoint module. In case of the trivial module $\mathbb{K}$, we have that $\mathbb{K}$ has a trivial grading given by $\mathbb{K} = \bigoplus_{n\in\mathbb{Z}} \mathbb{K}_n$ with $\mathbb{K}_0 = \mathbb{K}$ and $\mathbb{K}_n = \{0\}$ for $n \neq 0$.

3.3. Results on the algebraic cohomology of the Witt and the Virasoro algebra

For future reference, we briefly summarize in this section known results on the low-dimensional algebraic cohomology of the Witt and the Virasoro algebra, including the results highlighted in this contribution. Moreover, we shortly recall the interpretations of the low-dimensional cohomology in order to highlight its importance.

Let $\mathcal{L}$ be a Lie algebra and M a $\mathcal{L}$-module. The zeroth cohomology $\mathrm{H}^0(\mathcal{L}, M)$ corresponds to the space of elements of M invariant under $\mathcal{L}$, i.e. the $\mathcal{L}$-invariants of M. In the case of the Witt and the Virasoro algebra, we immediately obtain:

$$\begin{aligned} \mathrm{H}^0(\mathcal{W}, \mathbb{K}) &= \mathbb{K} \qquad \text{and} \qquad \mathrm{H}^0(\mathcal{W}, \mathcal{W}) = \{0\}\,, \\ \mathrm{H}^0(\mathcal{V}, \mathbb{K}) &= \mathbb{K} \qquad \text{and} \qquad \mathrm{H}^0(\mathcal{V}, \mathcal{V}) = \mathbb{K}\, t\,, \end{aligned}$$

where t is the central element of the Virasoro algebra.

The first cohomology with values in the trivial module corresponds to $\mathrm{H}^1(\mathcal{L}, \mathbb{K}) = (\mathcal{L}/[\mathcal{L}, \mathcal{L}])^*$ where $*$ stands for the dual space. The first cohomology with values in the adjoint module corresponds to outer derivations, $\mathrm{H}^1(\mathcal{L}, \mathcal{L}) = \mathrm{Out}(\mathcal{L})$. In [16], the first algebraic cohomology of the Witt and the Virasoro algebra was computed:

$$\begin{aligned} \mathrm{H}^1(\mathcal{W}, \mathbb{K}) &= \{0\} \qquad \text{and} \qquad \mathrm{H}^1(\mathcal{W}, \mathcal{W}) = \{0\}\,, \\ \mathrm{H}^1(\mathcal{V}, \mathbb{K}) &= \{0\} \qquad \text{and} \qquad \mathrm{H}^1(\mathcal{V}, \mathcal{V}) = \{0\}\,. \end{aligned}$$

The second cohomology with values in the trivial module $\mathrm{H}^2(\mathcal{L},\mathbb{K})$ classifies central extensions of $\mathcal{L}$ up to equivalence. In the case of the adjoint module, the second cohomology $\mathrm{H}^2(\mathcal{L},\mathcal{L})$ classifies infinitesimal deformations of $\mathcal{L}$ modulo equivalent deformations. In fact, the vanishing of this cohomology $\mathrm{H}^2(\mathcal{L},\mathcal{L})=\{0\}$ implies that the Lie algebra $\mathcal{L}$ is infinitesimally and formally rigid, see Fialowski and Fuchs [19], Fialowski [13, 20], Gerstenhaber [2–4], and Nijenhuis and Richardson [21]. However, contrary to the finite-dimensional case [2–4, 22], $\mathrm{H}^2(\mathcal{L},\mathcal{L})=\{0\}$ does not imply other types of rigidity in the infinite-dimensional case, see [9, 23–26]. Concerning the second cohomology for the Witt and the Virasoro algebra, the following results are known:

$$\begin{aligned} dim(\mathrm{H}^2(\mathcal{W},\mathbb{K}))=1 \qquad &\text{and} \qquad \mathrm{H}^2(\mathcal{W},\mathcal{W})=\{0\}\,,\\ \mathrm{H}^2(\mathcal{V},\mathbb{K})=\{0\} \qquad &\text{and} \qquad \mathrm{H}^2(\mathcal{V},\mathcal{V})=\{0\}\,. \end{aligned}$$

The first result $dim(\mathrm{H}^2(\mathcal{W},\mathbb{K}))=1$ is a well-known result, which states that the Witt algebra admits, up to equivalence and rescaling, only one non-trivial central extension, namely the Virasoro algebra. For an algebraic proof of this result, see e.g. the book by Kac, Raina and Rozhkovskaya [27]. The second result $\mathrm{H}^2(\mathcal{W},\mathcal{W})=\{0\}$ was shown in [10, 11] and [12]. This result leads to the fact that the Witt algebra is infinitesimally and formally rigid. The third result $\mathrm{H}^2(\mathcal{V},\mathbb{K})=\{0\}$ and the fourth result $\mathrm{H}^2(\mathcal{V},\mathcal{V})=\{0\}$ were shown in [10]. They imply that the Virasoro algebra admits no non-trivial central extensions and that it is infinitesimally and formally rigid, respectively.
The third cohomology with values in the adjoint module $\mathrm{H}^3(\mathcal{L},\mathcal{L})$ contains obstructions to the lifting of an infinitesimal deformation to a formal deformation. However, the third cohomology with values in a module M, $\mathrm{H}^3(\mathcal{L},M)$, also comes with a more constructive point of view in terms of crossed modules. In fact, the third cohomology $\mathrm{H}^3(\mathcal{L},M)$ classifies equivalence classes of crossed modules associated to $\mathcal{L}$ and M, see Wagemann [28] and Gerstenhaber [3, 5].
In the case of the Witt and the Virasoro algebra, we obtained the following results for the third cohomology with values in the trivial and the adjoint module:

$$\begin{aligned} dim(\mathrm{H}^3(\mathcal{W},\mathbb{K}))=1 \qquad &\text{and} \qquad \mathrm{H}^3(\mathcal{W},\mathcal{W})=\{0\}\,,\\ dim(\mathrm{H}^3(\mathcal{V},\mathbb{K}))=1 \qquad &\text{and} \qquad dim(\mathrm{H}^3(\mathcal{V},\mathcal{V}))=1\,. \end{aligned}$$

We proved the second result $\mathrm{H}^3(\mathcal{W},\mathcal{W})=\{0\}$ algebraically in [16]. This result states that there are no crossed modules associated to the Lie algebra $\mathcal{W}$ and the module $\mathcal{W}$. The remaining three results were proven by the authors in [17]. The three results imply that there is an equivalence class of a crossed module associated to $\mathcal{W}$ and $\mathbb{K}$, $\mathcal{V}$ and $\mathbb{K}$, as well as $\mathcal{V}$ and $\mathcal{V}$, respectively. For $\mathrm{H}^3(\mathcal{W},\mathbb{K})$ and $\mathrm{H}^3(\mathcal{V},\mathbb{K})$, an explicit algebraic expression of the generating cocycle of these spaces was given.

3.4. The Hochschild-Serre Spectral Sequence

For the convenience of the reader, we will recall the Hochschild-Serre spectral sequence, which will be used in the proofs later on.

Theorem [Hochschild-Serre [29, 30]] 3.4.1. *For every ideal $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$, there is a convergent first quadrant spectral sequence:*

$$E_2^{pq}=\mathrm{H}^p(\mathfrak{g}/\mathfrak{h},\mathrm{H}^q(\mathfrak{h},M)))\Rightarrow \mathrm{H}^{p+q}(\mathfrak{g},M)\,,$$

with M being a $\mathfrak{g}$-module and via $\mathfrak{h}\hookrightarrow\mathfrak{g}$ also a $\mathfrak{h}$-module.

A concise proof of this well-known result can be found for example in the textbook of Weibel [31]. The original literature is given by the articles [29, 30] by Hochschild and Serre.

4. Results for the third algebraic cohomology of the Witt and the Virasoro algebra

In [16], the authors proved the vanishing of the third algebraic cohomology of the Witt algebra with values in the adjoint module.

Theorem 4.1. *The third algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char$(\mathbb{K}) = 0$ and values in the adjoint module is zero, i.e.*

$$\mathrm{H}^3(\mathcal{W}, \mathcal{W}) = \{0\}\,.$$

Sketch of the proof. The proof is accomplished in three steps.

- In a first step, the entire analysis can be reduced to the degree zero cohomology. In fact, the Witt algebra is an internally graded algebra, and in that case the same obviously holds true for the adjoint module. Therefore, due to the result of Fuks (3.2), we only need to consider degree zero cochains, meaning we can write the 3-cochains as $\psi(e_i, e_j, e_k) = \psi_{i,j,k}\ e_{i+j+k}$ with suitable coefficients $\psi_{i,j,k} \in \mathbb{K}$. The coboundary and the cocycle condition can be rewritten in terms of these coefficients. The main aim of the proof thus reduces to proving that all the coefficients have to be zero up to coefficients from a coboundary.
- The second step consists in performing a cohomological change $\psi' = \psi - \delta_2\phi$, where $\phi(e_i, e_j) = \phi_{i,j}\ e_{i+j}$ is a degree zero 2-cochain. The aim is to put as many coefficients $\psi'_{i,j,k}$ as possible equal to zero by defining $\phi_{i,j}$ in an appropriate and consistent way. This is achieved by using recurrence relations obtained from the coboundary condition.
- In the last step, we use the cocycle condition evaluated on suitable combinations of the basis elements e_i in order to find non-trivial relations between the coefficients $\psi'_{i,j,k}$, which finally lead to the result $\psi'_{i,j,k} = 0\ \forall\ i, j, k \in \mathbb{Z}$.

□

In [17], the authors proved that the third algebraic cohomology of the Virasoro algebra with values in the adjoint module is one-dimensional.

Theorem 4.2. *The third algebraic cohomology of the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with char$(\mathbb{K}) = 0$ and values in the adjoint module is one-dimensional, i.e.*

$$dim(\mathrm{H}^3(\mathcal{V}, \mathcal{V})) = 1\,.$$

Proof. The short exact sequence (2.1) of Lie algebras is also a short exact sequence of $\mathcal{V}$-modules, leading to a long exact sequence in cohomology,

$$\cdots \to \mathrm{H}^2(\mathcal{V}, \mathcal{W}) \to \mathrm{H}^3(\mathcal{V}, \mathbb{K}) \to \mathrm{H}^3(\mathcal{V}, \mathcal{V}) \to \mathrm{H}^3(\mathcal{V}, \mathcal{W}) \to \dots\,. \tag{4.1}$$

Concerning the second cohomology, it is known that $\mathrm{H}^2(\mathcal{V}, \mathcal{W}) \cong \mathrm{H}^2(\mathcal{W}, \mathcal{W})$ and also $\mathrm{H}^2(\mathcal{W}, \mathcal{W}) = \{0\}$, see Section 3.3 and the original literature [10].
Concerning the third cohomology, we have $\mathrm{H}^3(\mathcal{V}, \mathcal{W}) \cong \mathrm{H}^3(\mathcal{W}, \mathcal{W})$ due to Theorem 4.3 below and $\mathrm{H}^3(\mathcal{W}, \mathcal{W}) = \{0\}$ due to Theorem 4.1. Consequently, the long exact sequence (4.1) reduces to:

$$0 \to \mathrm{H}^3(\mathcal{V}, \mathbb{K}) \to \mathrm{H}^3(\mathcal{V}, \mathcal{V}) \to 0\,. \tag{4.2}$$

Since $dim(\mathrm{H}^3(\mathcal{V}, \mathbb{K})) = 1$ by Theorem 4.4 below, the two-term exact sequence (4.2) above yields the announced result $dim(\mathrm{H}^3(\mathcal{V}, \mathcal{V})) = 1$. □

Theorem 4.3.

$$\begin{aligned} &\textit{If} && \mathrm{H}^j(\mathcal{W}, \mathcal{W}) = 0 \quad \textit{for} \quad k-2 \le j \le k-1\,,\\ &\textit{then} && \mathrm{H}^k(\mathcal{V}, \mathcal{W}) \cong \mathrm{H}^k(\mathcal{W}, \mathcal{W})\,.\\ &\textit{In particular,} && \mathrm{H}^3(\mathcal{V}, \mathcal{W}) \cong \mathrm{H}^3(\mathcal{W}, \mathcal{W})\,. \end{aligned}$$

Sketch of the proof. The proof is obtained via the Hochschild-Serre spectral sequence. Taking $\mathfrak{g} = \mathcal{V}$ and $\mathfrak{h} = \mathbb{K}$ in Theorem 3.4.1, the second page of the Hochschild-Serre spectral sequence becomes $E_2^{p,q} = \mathrm{H}^p(\mathcal{W}, \mathrm{H}^q(\mathbb{K}, M))$. Using $\mathcal{W}$ as a $\mathcal{V}$-module and denoting by $\varphi_i : \mathrm{H}^i(\mathcal{W}, \mathcal{W}) \to \mathrm{H}^{i+2}(\mathcal{W}, \mathcal{W})$ the maps on the E_2 level, we obtain from the third page $E_3 = E_\infty$ the following result:

$$\mathrm{H}^k(\mathcal{V}, \mathcal{W}) \cong \frac{\mathrm{H}^k(\mathcal{W}, \mathcal{W})}{\mathrm{im}\ \varphi_{k-2}} \bigoplus \ker\left(\varphi_{k-1} : \mathrm{H}^{k-1}(\mathcal{W}, \mathcal{W}) \to \mathrm{H}^{k+1}(\mathcal{W}, \mathcal{W})\right) ,$$

and in particular, if $\mathrm{H}^j(\mathcal{W}, \mathcal{W}) = \{0\}$ for $k-1 \leq j \leq k$, then

$$\mathrm{H}^{k+1}(\mathcal{V}, \mathcal{W}) \cong \mathrm{H}^{k+1}(\mathcal{W}, \mathcal{W}) .$$

Applying this to $k = 2$ and using $\mathrm{H}^1(\mathcal{W}, \mathcal{W}) = \mathrm{H}^2(\mathcal{W}, \mathcal{W}) = \{0\}$, see Section 3.3 or [16] and [10], we obtain $\mathrm{H}^3(\mathcal{V}, \mathcal{W}) \cong \mathrm{H}^3(\mathcal{W}, \mathcal{W})$. □

Theorem 4.4. *The third algebraic cohomology of the Witt and the Virasoro algebra over a field* $\mathbb{K}$ *with* $char(\mathbb{K}) = 0$ *and values in the trivial module is one-dimensional, i.e.*

$$dim(\mathrm{H}^3(\mathcal{W}, \mathbb{K})) = dim(\mathrm{H}^3(\mathcal{V}, \mathbb{K})) = 1 .$$

Sketch of the proof. The proof of $dim(\mathrm{H}^3(\mathcal{W}, \mathbb{K})) = 1$ can easily be extended to the proof of $dim(\mathrm{H}^3(\mathcal{V}, \mathbb{K})) = 1$, so that both results can be obtained simultaneously.
The proof consists basically of three steps. In a first step, we use the fact that both the Witt algebra and the Virasoro algebra are internally graded Lie algebras, and that the trivial module $\mathbb{K}$ comes with a trivial grading, which allows us to reduce the analysis to the degree zero cohomology due to (3.2).
In a second step, the aim is to find a degree zero 3-cocycle of $\mathrm{H}^3(\mathcal{W}, \mathbb{K})$ which is not a coboundary. The definition of this 3-cocycle can easily be extended to a non-trivial 3-cocycle of $\mathrm{H}^3(\mathcal{V}, \mathbb{K})$. Inspired by the result $dim(\mathrm{H}^3(Vect(S^1), \mathbb{R})) = 1$ in continuous cohomology, we consider the generator of this space given by the so-called Godbillon-Vey 3-cocycle, see e.g. the book by Guieu and Roger [32] and also the original literature [8]. Expressed in our basis and in our algebraic setting, the Godbillon-Vey cocycle becomes:

$$\Psi(e_n, e_m, e_k) = \begin{cases} 0 & \text{for } k \neq -(n+m) \\ A \begin{vmatrix} 1 & 1 & 1 \\ n & m & -(n+m) \\ n^2 & m^2 & (n+m)^2 \end{vmatrix} & \text{for } k = n+m \end{cases}$$

$$\Leftrightarrow \Psi(e_n, e_m, e_k) = \begin{cases} 0 & \text{for } k \neq -(n+m) \\ A(m-n)(2m+n)(m+2n) & \text{for } k = n+m \end{cases} ,$$

where A is a non-vanishing constant. Inserting this definition of Ψ into the cocycle condition, it is straightforward to verify that Ψ is a 3-cocycle for $\mathrm{H}^3(\mathcal{W}, \mathbb{K})$. Furthermore, evaluating Ψ on the combination of generators given by e_1, e_0 and e_{-1}, we obtain a value different from zero for Ψ. However, every coboundary evaluated on the same combination of generators e_1, e_0 and e_{-1}, gives zero. Hence, the 3-cocycle Ψ cannot be a coboundary. The map Ψ can be trivially extended to a cochain of $\mathrm{H}^3(\mathcal{V}, \mathbb{K})$ by defining $\Psi(e_n, e_m, t) = 0$. A similar direct verification to the one done in the case of the Witt algebra shows that the extended Ψ is a non-trivial cocycle of $\mathrm{H}^3(\mathcal{V}, \mathbb{K})$.
In the final step, we show that there are no other non-trivial 3-cocycles than Ψ in $\mathrm{H}^3(\mathcal{W}, \mathbb{K})$ or $\mathrm{H}^3(\mathcal{V}, \mathbb{K})$, up to equivalence. Let ψ be any arbitrary degree zero 3-cocycle of $\mathcal{W}$ or $\mathcal{V}$. We consider the linear combination given by:

$$\psi\prime = \psi - \frac{\psi(e_{-1}, e_1, e_0)}{2}\Psi . \tag{4.3}$$

Obviously, $\psi\prime$ satisfies $\psi\prime(e_{-1}, e_1, e_0) = 0$. In a separate proposition, we prove that any degree zero 3-cocycle $\widetilde{\psi}$ satisfying $\widetilde{\psi}(e_{-1}, e_1, e_0) = 0$ must be a coboundary. The proposition is proved by elementary but tedious algebraic computations, similar to the ones used in the proof of Theorem 4.1. The proposition and (4.3) together then imply that every arbitrary ψ must be a multiple of Ψ up to coboundaries, which allows to conclude. □

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