Functional inequalities for Feynman-Kac semigroups

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Abstract

Using the tools of stochastic analysis, we prove various gradient estimates and Harnack inequalities for Feynman-Kac semigroups with possibly unbounded potentials. One of the main results is a derivative formula which can be used to characterize a lower bound on Ricci curvature using a potential.

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1 Introduction

Suppose M is a geodesically and stochastically complete and connected Riemannian manifold of dimension n, with Levi-Civita connection ∇ and Laplace-Beltrami operator Δ. Suppose $V \in L^2_{\text{loc}}(M)$ is bounded below. Then the operator $H := -\frac{1}{2}\Delta + V$ is essentially self-adjoint [3] and bounded below. Essential self-adjointness means that H can be uniquely extended from the domain $C_0^\infty(M)$ of smooth functions with compact support to a self-adjoint operator H with maximal domain $D(H) = \{f : f, Hf \in L^2(M)\}$. The corresponding semigroup generated by $-H$ consists of bounded self-adjoint operators on $L^2(M)$. Under certain conditions on V the semigroup is continuous and given by the Feynman-Kac formula

$$P_t^V f(x) = \mathbb{E}\left[\mathbb{V}_t^x f(X_t(x))\right].$$

Here $X(x)$ is a Brownian motion on M starting at x and

$$\mathbb{V}_t^x := e^{-\int_0^t V(X_s(x))ds}$$

for $x \in M$ and $t \geq 0$. A class of potentials that is of particular interest is those V for which

$$\lim_{t\downarrow 0} \sup_{x\in M} \mathbb{E}\left[\int_0^t |V(X_s(x))|ds\right] = 0.$$

Such V is said to belong to the Kato class, first introduced by Kato in [8] using an equivalent definition. The local Kato class consists of those potentials V for which $\mathbf{1}_K V$ belongs to the Kato class for any compact set $K \subset M$. This is a very large class of potentials, encompassing nearly all physically relevant phenomena, for which one can still expect the Feynman-Kac formula to make sense pointwise for all $x \in M$. We refer to Aizenman and Simon [1] and Simon [12] for a comprehensive account of all this, and to the more recent work of Güneysu [7] for generalizations to Schrödinger type operators on vector bundles.

Our focus will be on the derivative, or gradient, of the Feynman-Kac semigroup $P_t^V f$. For the heat semigroup $P_t f$, a probabilistic formula for the derivative $dP_t f$, not involving the derivative of f, was proved by Bismut in [2] using techniques from Malliavin calculus. A transparent generalization of this, using an argument based on martingales, was later given by Elworthy and Li in [6] and developed further by Thalmaier in [14]. Thalmaier's formulation is based on local martingales and allows to localize the contribution of Ricci curvature. A formula for $dP_t^V f$ was proved by Elworthy and Li in [6], some applications of which were recently explored in [10].

The present work is a continuation of the author's recent article [16], in which localized versions of the derivative formula were proved, together with an extension to the Hessian. In all cases, these derivative formulae involve the derivative of V. Therefore, as in [6] and [16], we will assume throughout that V is continuously differentiable.

In Section 2, we start by using the derivative formula from [16] to obtain explicit gradient estimates for $P_t^V f$. These estimates are then used to derive Theorem 3.2, which states that if

$$\sup_{x\in M} \mathbb{E}\left[\int_0^t |dV|(X_s(x))ds\right] < \infty \tag{1}$$

with the Ricci curvature of M bounded below, then

$$(dP_t^V f)(v) = \mathbb{E}\left[\mathbb{V}_t^x((df)_{X_t(x)}(//_t Q_t v) - f(X_t(x))\int_0^t (dV)_{X_s(x)}(//_s Q_s v)ds)\right]$$

for all $f \in C_b^1(M)$, $x \in M$, $v \in T_x M$ and $t \geq 0$. The composition $Q//^{-1}$ is the damped parallel transport along the paths of $X(x)$, determined by the solution to equation (3) below, and $C_b^1(M)$ denotes the space of all bounded continuously differentiable functions on M with bounded derivative. Note that condition (1) is satisfied if $|dV|$ belongs to the Kato class.

One application of this derivative formula is that it allows to characterize a lower bound on Ricci curvature using the potential V, as explained in Section 3. For example, suppose $V \in C^1(M)$ is bounded below with ∇V bounded and $K \in \mathbb{R}$. Then, according to Theorem 3.3, the following are equivalent:

(1) $\mathrm{Ric} \geq 2K$;

(2) if $f \in C_b^1(M)$ then

$$|\nabla P_t^V f| \leq e^{-Kt} P_t^V |\nabla f| + \|\nabla V\|_\infty \left(\frac{1-e^{-Kt}}{K}\right) P_t^V |f|$$

for all $t \geq 0$.

In particular, if the above gradient estimate is verified for some suitable V then it must hold for all such V.

The derivative formula stated above is also applied in Section 4 to prove a Harnack inequality, given by Theorem 4.1. Afterwards, in Section 5, we prove the counterpart to the Bismut formula for differentiation inside the semigroup, namely Theorem 5.3, and use it to derive two further derivative estimates, Propositions 5.4 and 5.5, with corresponding shift-Harnack inequalities, Theorems 5.6 and 5.7. Shift-Harnack inequalities were introduced by Wang in [18]. Exploring such inequalities in the present setting was motivated by a question posed by Feng-Yu Wang during a workshop at the University of Swansea in 2017.

2 Uniform gradient estimates

In [11], Priola and Wang derived uniform gradient estimates for semigroups generated by second order elliptic operators with irregular and unbounded coefficients and bounded, measurable potentials V. They did not require local Hölder continuity of the coefficients, instead replacing the gradient of the semigroup with the Lipschitz constant. Although we will only consider the particular operator $-\frac{1}{2}\Delta + V$, we do allow for the potential to be unbounded.

Let us first briefly consider the case in which the potential V is bounded. In this case, we have the following gradient estimate, which depends only on the supremum norm of V:

Proposition 2.1. *Suppose $V \in C^1(M)$ is bounded with $\mathrm{Ric} \geq 2K$. Then for all bounded measurable functions f we have*

$$\|dP_t^V f\|_\infty \leq \sqrt{\frac{2}{\pi t}} e^{K^- t} \left(1 + 2t\|V\|_\infty e^{-t \inf V}\right) \|f\|_\infty$$

for all $t > 0$.

Proof. According to the variation of constants formula we have

$$P_t^V f = P_t f - \int_0^t P_{t-s}(V P_s^V f) ds$$

and therefore

$$|dP_t^V f| \leq |dP_t f| + \int_0^t |dP_{t-s}(V P_s^V f)| ds. \tag{2}$$

It is well known that

$$|dP_t f| \leq \sqrt{\frac{2}{\pi t}} e^{K^- t} \|f\|_\infty$$

and similarly for the integrand. The result therefore follows by substituting these estimates into (2), integrating the latter. □

We now turn our attention to the case of unbounded V. Let us recall the Bismut formula for $P_t^V f$ that was proved in [16]. For this, denote by $//$ the stochastic parallel transport along the paths of $X(x)$ and by B the anti-development of $//$ to $T_x M$. Then B is a Brownian motion on $T_x M$ starting at the origin. Denote by Q the $\mathrm{End}(T_x M)$-valued solution to the ordinary differential equation

$$\frac{d}{dt} Q_t = -\frac{1}{2} //_t^{-1} \mathrm{Ric}^\sharp //_t Q_t \tag{3}$$

with $Q_0 = \mathrm{id}_{T_x M}$. The composition $Q//^{-1}$ is called the damped parallel transport. Now suppose D is a regular domain in M and denote by τ the first exit time of $X(x)$ from D. In [16] it was proved, for a bounded measurable function f and $x \in D$, that there is the following local derivative formula:

$$(dP_t^V f)_x = -\mathbb{E}\left[\mathbb{V}_t^x f(X_t(x)) \left(\int_0^t \langle Q_s \dot{h}_s, dB_s\rangle + dV(//_s Q_s h_s) ds\right)\right]$$

for all $t > 0$, where h is any bounded adapted process with paths belonging to the Cameron-Martin space $L^{1,2}([0,t]; \mathrm{Aut}(T_x M))$ such that $h_0 = 1$, $h_s = 0$ for $s \geq \tau \wedge t$ with $\mathbb{E}[\int_0^{\tau\wedge t} |\dot{h}_s|^2 ds] < \infty$. Such h can always be constructed, as shown for example in [4]. It follows that if the Ricci curvature is bounded below, in the sense that

$$\inf\{\mathrm{Ric}(v,v) : v \in TM, |v| = 1\} > -\infty,$$

and if

$$\kappa_V(t) := \sup_{x\in M} \mathbb{E}\left[\int_0^t |dV|(X_s(x)) ds\right] < \infty$$

then $dP^V f$ is uniformly bounded on $[\epsilon, t] \times M$ for all $\epsilon > 0$ and $t > 0$. Arguing as in the proof of [16, Theorem 7], it follows that in this case there is the following non-local, but explicit, version of the derivative formula:

$$(dP_t^V f)_x = \frac{1}{t}\mathbb{E}\left[\mathbb{V}_t^x f(X_t(x))\left(\int_0^t \langle Q_s, dB_s\rangle - (t-s)dV(//_s Q_s)ds\right)\right] \tag{4}$$

for all $t > 0$. Using formula (4), we can quickly derive some simple gradient estimates. For convenience, and without loss of generality, we will assume that V is non-negative. Note also that in the sequel, K will always denote a real-valued constant.

Theorem 2.2. *Suppose $V \in C^1(M)$ is non-negative with $\kappa_V(t) < \infty$ and $\mathrm{Ric} \geq 2K$. Then for all $f \in L^\infty(M)$ we have*

$$\|dP_t^V f\|_\infty \leq e^{K^- t}\left(\sqrt{\frac{2}{\pi t}} + \kappa_V(t)\right)\|f\|_\infty$$

for all $t > 0$.

Proof. Since for each $t > 0$ the stochastic integral $\int_0^t \langle Q_s, dB_s\rangle$ is a centred Gaussian random variable with variance $\int_0^t \|Q_s\|^2 ds \leq te^{2K^-}$ and since for centred Gaussian random variable with variance 1 we have $\mathbb{E}[|X|] = \sqrt{\frac{2}{\pi}}$, we have by formula (4), that

$$\begin{aligned} |dP_t^V f|(x) \leq{}& \|f\|_\infty \frac{1}{t}\left(\mathbb{E}\left[\left|\int_0^t \langle Q_s, dB_s\rangle\right|\right] + e^{K^- t}\mathbb{E}\left[\int_0^t (t-s)|dV|(X_s(x))ds\right]\right) \\ \leq{}& e^{K^- t}\|f\|_\infty\left(\sqrt{\frac{2}{\pi t}} + \mathbb{E}\left[\int_0^t |dV|(X_s(x))ds\right]\right) \end{aligned}$$

from which the result follows, by the definition of $\kappa_V(t)$. □

For $p > 1$ we can similarly obtain an estimate using the L^p norm (as opposed to the L^∞ norm), for which we should assume that

$$\kappa_{V,q}(t) := \sup_{x\in M} \mathbb{E}\left[\left(\int_0^t |dV|(X_s(x))ds\right)^q\right]^{\frac{1}{q}} < \infty$$

where q is the conjugate of p.

Theorem 2.3. *Suppose $V \in C^1(M)$ is non-negative with $\mathrm{Ric} \geq 2K$. Suppose $p > 1$, set $q = p/(p-1)$ and assume $\kappa_{V,q}(t) < \infty$. Then for all $f \in L^p(M)$ we have*

$$\|dP_t^V f\|_p \leq e^{K^- t}\left(\frac{C_q^{1/q}}{\sqrt{t}} + \kappa_{V,q}(t)\right)\|f\|_p$$

for all $t > 0$, where C_q is the constant from the Burkholder-Davis-Gundy inequality.

Proof. By the Burkholder-Davis-Gundy inequality, we see that

$$\begin{aligned} \mathbb{E}\left[\left|\int_0^t \langle Q_s, dB_s\rangle\right|^q\right] &\leq \mathbb{E}\left[\left(\sup_{0\leq s\leq t}\left|\int_0^s \langle Q_r, dB_r\rangle\right|\right)^q\right] \\ &\leq C_q \mathbb{E}\left[\left(\int_0^t \|Q_s\|^2 ds\right)^{\frac{q}{2}}\right] \end{aligned}$$

and so, by formula (4) and Hölder's inequality, we have

$$\begin{aligned}|dP_t^V f|(x) &\leq \mathbb{E}\left[|f|^p(X_t(x))\right]^{1/p} \frac{1}{t} \left(\mathbb{E}\left[\left| \int_0^t \langle Q_s, dB_s \rangle \right|^q \right]^{1/q} \right.\\ &\qquad \left. + e^{K^- t}\mathbb{E}\left[\left(\int_0^t (t-s)|dV|(X_s(x))ds \right)^q \right]^{\frac{1}{q}} \right)\\ &\leq e^{K^- t}\mathbb{E}\left[|f|^p(X_t(x))\right]^{1/p} \left(C_q^{1/q} \frac{1}{\sqrt{t}} + \kappa_{V,q}(t) \right)\end{aligned}$$

for all $t > 0$ and $x \in M$. Thus

$$\|dP_t^V f\|_p \leq e^{K^- t} \left(\frac{C_q^{1/q}}{\sqrt{t}} + \kappa_{V,q}(t) \right) \left(\int_M \mathbb{E}\left[|f|^p(X_t(x))\right] dx \right)^{1/p}$$

and the result follows by Fubini's theorem. □

Theorem 2.2 implies that if the Ricci curvature is bounded below then $dP^V f$ is bounded on $[\epsilon, \infty) \times M$ for each $\epsilon > 0$. A similar observation applies to Theorem 2.3. Our next step will be to use Theorems 2.2 and 2.3 to obtain quantitative estimates which are uniform across all space and time. The price we pay is the inclusion of a term involving Hf. These are generalizations of the estimate proved by Cheng, Thalmaier and the author in [4] and [5].

Theorem 2.4. *Suppose $V \in C^1(M)$ is non-negative with $\kappa_V(\delta) < \infty$ for some $\delta > 0$ and $\mathrm{Ric} \geq 2K$. Suppose $f \in C^2(M)$ with f, Hf bounded. Then the derivative $dP^V f$ is uniformly bounded on $[0, \infty) \times M$ and moreover*

$$\|dP_t^V f\|_\infty \leq e^{K^- \delta} \left(\left(\sqrt{\frac{2}{\pi\delta}} + \kappa_V(\delta) \right) \|f\|_\infty + \delta \left(\sqrt{\frac{8}{\pi\delta}} + \kappa_V(\delta) \right) \|Hf\|_\infty \right)$$

for all $t \geq 0$.

Proof. According the forward Kolmogorov equation we have

$$P_{\delta+t}^V f = P_t^V f - \int_0^\delta P_s^V (HP_t^V f) ds$$

and therefore

$$|dP_t^V f|(x) \leq |dP_{\delta+t}^V f|(x) + \int_0^\delta |dP_s^V (HP_t^V f)|(x) ds \tag{5}$$

for all $x \in M$. By Theorem 2.2 we have

$$|dP_{\delta+t}^V f|(x) \leq e^{K^- \delta} \|P_t^V f\|_\infty \left(\sqrt{\frac{2}{\pi\delta}} + \kappa_V(\delta) \right) \leq e^{K^- \delta} \|f\|_\infty \left(\sqrt{\frac{2}{\pi\delta}} + \kappa_V(\delta) \right)$$

and similarly

$$|dP_s^V (HP_t^V f)|(x) \leq e^{K^- s} \|Hf\|_\infty \left(\sqrt{\frac{2}{\pi s}} + \kappa_V(s) \right).$$

Consequently

$$|dP_t^V f|(x) \leq e^{K^- \delta} \left(\left(\sqrt{\frac{2}{\pi\delta}} + \kappa_V(\delta) \right) \|f\|_\infty + \left(\sqrt{\frac{8\delta}{\pi}} + \int_0^\delta \kappa_V(s) ds \right) \|Hf\|_\infty \right)$$

from which the result follows, since κ_V is non-decreasing. □

Theorem 2.5. *Suppose $V \in C^1(M)$ is non-negative with $\mathrm{Ric} \geq 2K$. Suppose $p > 1$, set $q = p/(p-1)$ and assume $\kappa_{V,q}(\delta) < \infty$ for some $\delta > 0$. Suppose $f \in C^2(M)$ with $f, Hf \in L^p(M)$. Then the derivative $dP^V f$ is uniformly L^p-bounded on $[0,\infty) \times M$ and moreover*

$$\|dP_t^V f\|_p \leq e^{K^-\delta} \left(\left(\frac{C_q^{1/q}}{\sqrt{\delta}} + \kappa_{V,q}(\delta) \right) \|f\|_p + \delta \left(\frac{2C_q^{1/q}}{\sqrt{\delta}} + \kappa_{V,q}(\delta) \right) \|Hf\|_p \right)$$

where C_q is the constant from the Burkholder-Davis-Gundy inequality.

Proof. By (5) and Minkowski's inequality we have

$$\|dP_t^V f\|_p \leq \|dP_{t+\delta}^V f\|_p + \int_0^\delta \|dP_s^V (HP_t^V f)\|_p ds.$$

By Theorem 2.3 we have

$$\|dP_{t+\delta}^V f\|_p \leq e^{K^-\delta} \left(\frac{C_q^{1/q}}{\sqrt{\delta}} + \kappa_{V,q}(\delta) \right) \|f\|_p$$

and similarly

$$\|dP_s^V (HP_t^V f)\|_p \leq e^{K^- s} \left(\frac{C_q^{1/q}}{\sqrt{s}} + \kappa_{V,q}(s) \right) \|Hf\|_p$$

from which the result follows, as is the case $p = \infty$. □

3 Characterizations of Ricci curvature

It is well known that Ricci curvature can be characterized in terms of the gradient of the heat semigroup; see for example [17, Theorem 2.2.4]. Next we show that this characterization extends to the Feynman-Kac semigroups considered above. This shows a way in which the derivative dV, appearing in the gradient estimates of the previous section, occurs naturally:

Theorem 3.1. *Suppose $V \in C^1(M)$ is bounded below. Let $x \in M$ and $X \in T_xM$ with $|X| = 1$. Suppose $f \in C_0^\infty(M)$ with $\nabla f = X$, $\mathrm{Hess}\, f(x) = 0$ and set $\alpha := f(x)$. Then for any $p > 0$ we have*

$$\lim_{t \downarrow 0} \frac{P_t^V |\nabla f|^p(x) - |\nabla P_t^V f|^p(x)}{pt} = \frac{1}{2} \mathrm{Ric}(X,X) + \left(1 - \frac{1}{p}\right) V + \alpha dV(X).$$

Proof. By Taylor expansion at the point x we have

$$P_t^V |\nabla f|^p = |\nabla f|^p + t(\tfrac{1}{2}\Delta - V)|\nabla f|^p + o(t)$$

and also

$$|\nabla P_t^V f|^p = |\nabla f|^p + pt|\nabla f|^{p-2} \langle \nabla(\tfrac{1}{2}\Delta - V)f, \nabla f \rangle + o(t)$$

for small $t > 0$. Furthermore at the point x we have

$$(\tfrac{1}{2}\Delta - V)|\nabla f|^p = \frac{p}{2}|\nabla f|^{p-2} \left(\frac{1}{2}\Delta - \frac{2}{p} V \right) |\nabla f|^2 + \frac{p}{2} \left(\frac{p}{2} - 1 \right) |\nabla f|^{p-2} |\nabla |\nabla f|^2|^2$$

with $|\nabla|\nabla f|^2|^2(x) = |2(\mathrm{Hess}\, f(x))^\sharp(\nabla f)|^2 = 0$, and by the Bochner formula

$$\frac{1}{2}\Delta|\nabla f|^2 = \langle \nabla \Delta f, \nabla f \rangle + \mathrm{Ric}(\nabla f, \nabla f)$$

so therefore

$$\lim_{t\downarrow 0}\frac{P_t^V|\nabla f|^p - |\nabla P_t^V f|^p}{pt} = \frac{1}{2}\operatorname{Ric}(X,X) - \frac{1}{p}V + \langle \nabla(Vf),\nabla f\rangle$$
$$= \frac{1}{2}\operatorname{Ric}(X,X) + \left(1-\frac{1}{p}\right)V + \alpha dV(X)$$

as required. □

In the Section 2 we considered uniform estimates on $dP^V f$ which involved either the supremum norm of f and a constant which diverges for small time, or the supremum norms of both f and Hf and a constant which does not. In light of the previous theorem, we would also like to consider derivative estimates for functions belonging to $C_b^1(M)$. To do so we use the following derivative formula:

Theorem 3.2. *Suppose $V \in C^1(M)$ is bounded below with $\kappa_V(t) < \infty$. Suppose the Ricci curvature of M is bounded below with $f \in C_b^1(M)$. Then*

$$(dP_t^V f)(v) = \mathbb{E}\left[\mathbb{V}_t^x((df)_{X_t(x)}(//_tQ_tv) - f(X_t(x))\int_0^t (dV)_{X_s(x)}(//_sQ_sv)ds)\right] \qquad (6)$$

for all $x \in M$ and $v \in T_xM$.

Proof. First suppose $f \in C_0^\infty(M)$. Setting $f_s := P_{t-s}^V f$ and $N_s(v) := dP_{t-s}^V f(//_sQ_sv)$, using the definition of Q as the solution to equation (3) and by Itô's formula and the Weitzenböck formula

$$d\Delta f = \operatorname{tr}\nabla^2 df - df(\operatorname{Ric}^\sharp)$$

we see

$$dN_s(v) \overset{m}{=} df_s(//_s\partial_sQ_sv)ds + (\partial_s df_s)(//_sQ_sv)dt + \frac{1}{2}\operatorname{tr}\nabla^2(df_s)(//_sQ_sv)ds$$
$$= V(X_s(x))N_s(v)ds + f_s(X_s(x))dV(//_sQ_sv)ds$$

where $\overset{m}{=}$ denotes equality modulo the differential of a local martingale. It follows that

$$d(\mathbb{V}_s^x N_s(v)) \overset{m}{=} \mathbb{V}_s^x f_s(X_s(x))dV(//_sQ_sv)ds$$

so that

$$\mathbb{V}_s^x(dP_{t-s}^V f)_{X_s(x)}(//_sQ_sv) - \int_0^s \mathbb{V}_r^x P_{t-r}^V f(X_r(x))(dV)_{X_r(x)}(//_rQ_rv)dr \qquad (7)$$

is a local martingale. To verify that it is a true martingale, we see that

$$\mathbb{E}\left[\sup_{s\in[0,t]}\left(\mathbb{V}_s^x(dP_{t-s}^V f)_{X_s(x)}(//_sQ_sv) - \int_0^s \mathbb{V}_r^x P_{t-r}^V f(X_r(x))(dV)_{X_r(x)}(//_rQ_rv)dr\right)\right]$$
$$\leq e^{K^- t}|v|\left(\|dP^V f\|_{L^\infty([0,t]\times M)} + \|f\|_\infty \kappa_V(t)\right)$$

which is finite, by Theorem 2.4. Formula (6) follows by evaluating the expected value of (7) at times 0 and t, extending to all $f \in C_b^1(M)$ by approximation. □

Combining Theorems 3.1 and 3.2, we obtain the following equivalence:

Theorem 3.3. *Suppose $V \in C^1(M)$ is bounded below with ∇V bounded. Let $K \in \mathbb{R}$. Then the following are equivalent:*

(1) $\mathrm{Ric} \geq 2K$;

(2) *if* $f \in C_b^1(M)$ *then*

$$|\nabla P_t^V f| \leq e^{-Kt} P_t^V |\nabla f| + \|\nabla V\|_\infty \left(\frac{1-e^{-Kt}}{K}\right) P_t^V |f| \tag{8}$$

for all $t \geq 0$.

Proof. That (1) implies (2) follows immediately from Theorem 3.2. To prove that (2) implies (1), suppose $x \in M$ and $X \in T_x M$ with $|X| = 1$ and choose $f \in C_0^\infty(M)$ so that $f(x) = 0$, $\nabla f = X$ and $\mathrm{Hess}\, f(x) = 0$. Then, by applying Theorem 3.1 to the case $p = 1$, we obtain

$$\begin{aligned}
\frac{1}{2}\mathrm{Ric}(X,X) &= \lim_{t\downarrow 0} \frac{P_t^V |\nabla f|(x) - |\nabla P_t^V f|(x)}{t} \\
&\geq \lim_{t\downarrow 0} \left(\frac{1-e^{-Kt}}{Kt}\right) \left(K P_t^V |\nabla f|(x) - \|\nabla V\|_\infty P_t^V |f|(x)\right) \\
&= K|\nabla f|(x) - \|\nabla V\|_\infty |f(x)| \\
&= K
\end{aligned}$$

as required. □

For the case $V = 0$ it is well known that $\mathrm{Ric} \geq 2K$ if and only if (8) holds. So in fact $\mathrm{Ric} \geq 2K$ so long as (8) holds for *some* $V \in C^1(M)$ which is bounded below with ∇V bounded. Moreover, if (8) holds for *some* such V then it must hold for *all* such V.

4 Harnack inequalities

Denoting by ρ the Riemannian distance on M, we have also the following Harnack inequality for $P_t^V f$, the proof of which is based on that for the $V = 0$ case [17, Theorem 2.3.4]:

Theorem 4.1. *Suppose* $V \in C^1(M)$ *is non-negative with* ∇V *bounded and* $\mathrm{Ric} \geq 2K$. *Then for all bounded measurable functions* $f \geq 0$ *and* $p > 1$ *we have*

$$(P_t^V f)^p(x) \leq (P_t^V f^p)(y) \exp\left[\frac{p\rho^2(x,y)}{2(p-1)C_1(t,K)t} + \frac{t\rho(x,y)\|\nabla V\|_\infty}{2C_2(t,K)}\right]$$

for all $x, y \in M$ *and* $t > 0$, *where*

$$C_1(t,K) := \frac{e^{2Kt}-1}{2Kt}, \quad C_2(t,K) := \frac{Kt}{2}\coth\left(\frac{Kt}{2}\right).$$

Proof. Suppose first that $f \in C^2(M)$ with f bounded, $\inf f > 0$ and f constant outside a compact set. Given $x \neq y$ and $t > 0$, let $\gamma : [0,t] \to M$ be a geodesic from x to y of length $\rho(x,y)$. Let $v_s := \dot{\gamma}_s$, so that $|v_s| = \rho(x,y)/t$. Let

$$h(s) := t\left(\frac{e^{2Ks}-1}{e^{2Kt}-1}\right)$$

for $s \in [0,t]$, so that $h(0) = 0$ and $h(t) = t$. Let $y_s := \gamma_{h(s)}$ and define

$$\phi(s) := \log P_s^V((P_{t-s}^V f)^p)(y_s)$$

for $s \in [0,t]$. Then

$$\begin{aligned}
&\frac{d}{ds}\phi(s) \\
&= \frac{1}{P_s^V(P_{t-s}^V f)^p}\Bigg(\left(\frac{d}{ds}P_s^V\right)(P_{t-s}^V f)^p + P_s^V\left(\frac{d}{ds}(P_{t-s}^V f)^p\right) \\
&\qquad\qquad + \langle \nabla P_s^V(P_{t-s}^V f)^p, \dot{y}_s\rangle\Bigg)(y_s) \\
&= \frac{1}{P_s^V(P_{t-s}^V f)^p}\Bigg((\tfrac{1}{2}\Delta - V)P_s^V(P_{t-s}^V f)^p - pP_s^V((P_{t-s}^V f)^{p-1}(\tfrac{1}{2}\Delta - V)P_{t-s}^V f) \\
&\qquad\qquad + \langle \nabla P_s^V((P_{t-s}^V f)^p), \dot{y}_s\rangle\Bigg)(y_s) \\
&= \frac{1}{P_s^V(P_{t-s}^V f)^p}\Bigg(P_s^V(\tfrac{1}{2}\Delta(P_{t-s}^V f)^p) - pP_s^V((P_{t-s}^V f)^{p-1}\tfrac{1}{2}\Delta P_{t-s}^V f) \\
&\qquad\qquad + (p-1)P_s^V(V(P_{t-s}^V f)^p) + \langle \nabla P_s^V(P_{t-s}^V f)^p, \dot{y}_s\rangle\Bigg)(y_s).
\end{aligned}$$

Since

$$\Delta(P_{t-s}^V f)^p = p(P_{t-s}^V f)^{p-1}\Delta P_{t-s}^V f + p(p-1)(P_{t-s}^V f)^{p-2}|\nabla P_{t-s}^V f|^2,$$

it follows from Theorem 3.3 that

$$\begin{aligned}
&\frac{d}{ds}\phi(s) \\
&= \frac{1}{P_s^V(P_{t-s}^V f)^p}\Bigg(P_s^V\left(\frac{p(p-1)}{2}(P_{t-s}^V f)^{p-2}|\nabla P_{t-s}^V f|^2\right) \\
&\qquad\qquad + (p-1)P_s^V(V(P_{t-s}^V f)^p) + \langle \nabla P_s^V((P_{t-s}^V f)^p), \dot{y}_s\rangle\Bigg)(y_s) \\
&\geq \frac{1}{P_s^V(P_{t-s}^V f)^p}P_s^V\Bigg((P_{t-s}^V f)^p\Bigg(\frac{p(p-1)}{2}|\nabla \log P_{t-s}^V f|^2 + (p-1)V \\
&\qquad\qquad - \frac{p\rho(x,y)}{t}h'(s)e^{-Ks}|\nabla \log P_{t-s}^V f| \\
&\qquad\qquad - \frac{\rho(x,y)}{t}h'(s)\|\nabla V\|_\infty\left(\frac{1-e^{-Ks}}{K}\right)\Bigg)\Bigg)(y_s) \\
&\geq -\frac{p\rho^2(x,y)h'(s)^2e^{-2Ks}}{2(p-1)t^2} - \frac{\rho(x,y)}{t}h'(s)\|\nabla V\|_\infty\left(\frac{1-e^{-Ks}}{K}\right).
\end{aligned}$$

Since

$$h'(s) = 2Kt\left(\frac{e^{2Ks}}{e^{2Kt}-1}\right)$$

we have

$$\frac{d}{ds}\phi(s) \geq -\frac{2p\rho^2(x,y)K^2e^{2Ks}}{(p-1)(e^{2Kt}-1)^2} - 2\rho(x,y)\left(\frac{e^{2Ks}-e^{Ks}}{e^{2Kt}-1}\right)\|\nabla V\|_\infty$$

which integrated between 0 and t yields the inequality. By approximations and the monotone class theorem, as in [17, Theorem 2.3.3], this extends to all non-negative, bounded measurable functions. □

5 Shift-Harnack inequalities

In this section we prove two further differentiation formulae, which complement formula (4), and use them to deduce shift-Harnack inequalities, first introduced by Wang in [18] and similar to Theorem 4.1 except that the shift in space variable takes place *inside* the semigroup. The approach will be similar to that of [15]. In particular, we start by supposing that α is a bounded continuously differentiable 1-form and α_s a solution to the equation

$$\frac{\partial}{\partial s}\alpha_s = (\tfrac{1}{2}\Delta - V)\alpha_s \tag{9}$$

for $s \in (0,t]$ with $\alpha_0 = \alpha$. Here Δ denotes the Hodge Laplacian $-(d^\star + d)^2$ acting on 1-forms, were $d^\star$ denotes the codifferential operator.

Proposition 5.1. *Suppose $V \in C^1(M)$ is bounded below and that h_s is a bounded adapted process with paths belonging to the Cameron-Martin space $L^{1,2}([0,t];[0,\infty))$. Then*

$$\mathbb{V}_s^x d^\star \alpha_{t-s}(X_s(x))h_s + \mathbb{V}_s^x \alpha_{t-s}\left(//_s Q_s \left(\int_0^s \dot{h}_r Q_r^{-1} dB_r\right)\right) + \int_0^s \mathbb{V}_r^x \langle dV, \alpha_{t-r}\rangle (X_r(x))h_r dr \tag{10}$$

is a local martingale.

Proof. Since $d^\star$ commutes with Δ, with

$$-d^\star(V\alpha_s) = -Vd^\star\alpha_s + \langle dV, \alpha_s\rangle,$$

by equation (9) we have

$$\frac{\partial}{\partial s} d^\star\alpha_s = (\tfrac{1}{2}\Delta - V)d^\star\alpha_s + \langle dV, \alpha_s\rangle.$$

Consequently, by Itô's formula, we find that

$$n_s := \mathbb{V}_s^x (d^\star\alpha_{t-s})(X_s(x)) + \int_0^s \mathbb{V}_r^x \langle dV, \alpha_{t-r}\rangle (X_r(x))dr$$

is a local martingale and therefore so is

$$n_s h_s - \int_0^s n_r \dot{h}_r dr \tag{11}$$

with the latter starting at $d^\star\alpha_t h_0$. Moreover

$$\begin{aligned}\mathbb{V}_s^x (d^\star\alpha_{t-s})(X_s(x))\dot{h}_s ds &= -\mathbb{V}_s^x \sum_{i=1}^n (\nabla_{//_s e_i}\alpha_{t-s})(//_s e_i)\dot{h}_s ds \\ &= -\mathbb{V}_s^x \Big\langle \sum_{i=1}^n (\nabla_{//_s e_i}\alpha_{t-s})(//_s Q_s)dB_s^i, \dot{h}_s Q_s^{-1} dB_s \Big\rangle\end{aligned}$$

where $\{e_i\}_{i=1}^n$ is any orthonormal basis of $T_x M$. Since

$$d(\mathbb{V}_s^x \alpha_{t-s}(//_s Q_s)) = \mathbb{V}_s^x \sum_{i=1}^n (\nabla_{//_s e_i}\alpha_{t-s})(//_s Q_s)dB_s^i \tag{12}$$

it follows that

$$\int_0^s \mathbb{V}_r^x (d^\star\alpha_{t-r})(X_r(x))\dot{h}_r dr + \mathbb{V}_s^x \alpha_{t-s}(//_s Q_s)\left(\int_0^s \dot{h}_r Q_r^{-1} dB_r\right) \tag{13}$$

is also a local martingale. Using the fact that (11) and (13) are local martingales, and since by integration by parts

$$-\int_0^s \int_0^r \mathbb{V}_u^x \langle dV, \alpha_{t-u} \rangle (X_u(x)) du \dot{h}_r dr = -\int_0^s \mathbb{V}_r^x \langle dV, \alpha_{t-r} \rangle (X_r(x)) dr h_s + \int_0^s \mathbb{V}_r^x \langle dV, \alpha_{t-r} \rangle (X_r(x)) h_r dr,$$

we easily verify that (10) is also a local martingale. □

Theorem 5.2. *Suppose $V \in C^1(M)$ is bounded below with ∇V and Ric bounded. Suppose α is a bounded continuously differentiable 1-form with $d^\star \alpha$ bounded. Then*

$$P_t^V (d^\star \alpha)(x) = -\frac{1}{t} \mathbb{E} \left[\mathbb{V}_t^x \alpha \left(//_t Q_t \int_0^t Q_s^{-1} (dB_s + (t-s) //_s^{-1} \nabla V ds) \right) \right]$$

for all $x \in M$ and $t > 0$.

Proof. According to the assumptions of the theorem, with $h_s = (t-s)/t$, the local martingale (10) is a true martingale. Furthermore, by (12), it follows that

$$\alpha_t = \mathbb{E} \left[\mathbb{V}_t \alpha (//_t Q_t) \right]$$

so the result follows by evaluating the martingale (10) at the times 0 and t, and applying the Markov property to the term involving ∇V. □

Given a vector field Y, the Bismut formula (4) provides a probabilistic expression for the derivative $Y(P_t^V f)$ which does not involve derivatives of f. By Theorem 5.2, and using the facts that $\operatorname{div} Y = -d^\star Y^\flat$ and $\operatorname{div}(fY) = Yf + f \operatorname{div} Y$, we obtain the following theorem which provides a similar expression for the derivative $P_t^V (Y(f))$:

Theorem 5.3. *Suppose $V \in C^1(M)$ is bounded below with ∇V and Ric bounded. Suppose f is a bounded C^1 function and Y a bounded C^1 vector field for which $\operatorname{div} Y$ and $Y(f)$ are also bounded. Then*

$$\begin{aligned} & P_t^V (Y(f))(x) \\ = & -\mathbb{E} \left[\mathbb{V}_t^x f(X_t(x)) (\operatorname{div} Y)(X_t(x)) \right] \\ & + \frac{1}{t} \mathbb{E} \left[\mathbb{V}_t^x f(X_t(x)) \left\langle Y(X_t(x)), //_t Q_t \int_0^t Q_s^{-1} (dB_s + (t-s) //_s^{-1} \nabla V ds) \right\rangle \right] \end{aligned}$$

for all $x \in M$ and $t > 0$.

By Theorem 5.3, we have the following two propositions:

Proposition 5.4. *Suppose $V \in C^1(M)$ is non-negative with ∇V bounded and $2K \leq \mathrm{Ric} \leq 2L$ for constants K and L. Suppose f is a bounded C^1 function and Y a bounded C^1 vector field for which $\operatorname{div} Y$ and $Y(f)$ are also bounded. Then*

$$|P_t^V (Y(f))|(x) \leq \alpha_t^V (Y) (P_t^V f^2)^{\frac{1}{2}} (x)$$

for all $x \in M$ and $t > 0$, where

$$\begin{aligned} \alpha_t^V (Y) := & \ \| \operatorname{div} Y \|_\infty + \|Y\|_\infty \frac{e^{-Kt}}{\sqrt{t}} \left(\frac{e^{2Lt} - 1}{2Lt} \right)^{\frac{1}{2}} \\ & + \|Y\|_\infty \|\nabla V\|_\infty \frac{e^{-Kt}}{L} \left(\frac{1}{\frac{Lt}{2} \left(\coth \left(\frac{Lt}{2} \right) - 1 \right)} - 1 \right). \end{aligned}$$

Proof. By Theorem 5.3, we have

$$\begin{aligned}
&|P_t^V(Y(f))|(x)\\
\le\ & \|Y\|_\infty \frac{e^{-tK}}{t}\mathbb{E}\left[\left(\int_0^t Q_s^{-1}dB_s\right)^2\right]^{\frac{1}{2}} (P_t^V f^2)^{\frac{1}{2}}(x)\\
&+\left(\|\operatorname{div} Y\|_\infty + \|Y\|_\infty\|\nabla V\|_\infty \frac{e^{-tK}}{t}\int_0^t e^{Ls}(t-s)ds\right)(P_t^V f^2)^{\frac{1}{2}}(x)\\
=\ & \alpha_t^V(Y)(P_t^V f^2)^{\frac{1}{2}}(x)
\end{aligned}$$

as required. □

Proposition 5.5. *Suppose $V \in C^1(M)$ is non-negative with ∇V bounded and $2K \le \operatorname{Ric} \le 2L$ for constants K and L. Suppose $f > 0$ is a bounded C^1 function and Y a bounded C^1 vector field for which $\operatorname{div} Y$ and $Y(f)$ are also bounded. Then*

$$|P_t^V(Y(f))|(x) \le \delta(P_t^V(f\log f)(x) - P_t^V f \log P_t^V f(x)) + \beta_t^V(\delta, Y)P_t^V f(x)$$

for all $x \in M$, $t > 0$ and $\delta > 0$, where

$$\begin{aligned}
\beta_t^V(\delta, Y) :=\ & \|\operatorname{div} Y\|_\infty + \|Y\|_\infty^2 \frac{e^{-2Kt}}{2t\delta}\left(\frac{e^{2Lt}-1}{2Lt}\right)\\
&+\|Y\|_\infty\|\nabla V\|_\infty \frac{e^{-Kt}}{L}\left(\frac{1}{\frac{Lt}{2}\left(\coth\left(\frac{Lt}{2}\right)-1\right)}-1\right).
\end{aligned}$$

Proof. By Theorem 5.3 and [13, Lemma 6.45], the latter being essentially Jensen's inequality, we have

$$\begin{aligned}
&|P_t^V(Y(f))|(x)\\
\le\ & \|\operatorname{div} Y\|_\infty P_t^V f(x)\\
&+\left|\frac{1}{t}\mathbb{E}\left[\nabla_t f(X_t(x))\left\langle Y(X_t(x)), //_t Q_t\int_0^t Q_s^{-1}(dB_s + (t-s)//_s^{-1}\nabla V ds)\right\rangle\right]\right|\\
\le\ & \delta(P_t^V(f\log f)(x) - P_t^V f(x)\log P_t^V f(x))\\
&+\delta P_t^V f(x)\log \mathbb{E}\left[\exp\left[\|Y\|_\infty \frac{e^{-Kt}}{t\delta}\left|\int_0^t Q_s^{-1}dB_s\right|\right]\right]\\
&+\left(\|\operatorname{div} Y\|_\infty + \|Y\|_\infty\|\nabla V\|_\infty \frac{e^{-Kt}}{t}\int_0^t e^{Ls}(t-s)ds\right)P_t^V f(x)\\
\le\ & \delta(P_t^V(f\log f)(x) - P_t^V f \log P_t^V f(x)) + \beta_t^V(\delta, Y)P_t^V f(x)
\end{aligned}$$

as required. □

These estimates can be used to derive shift-Harnack inequalities, as introduced by Wang for Markov operators on a Banach space; [18, Proposition 2.3]. In particular, suppose as in [15] that $\{F_s : s \in [0,1]\}$ is a C^1 family of diffeomorphisms of M with $F_0 = \mathrm{id}_M$. For each $s \in [0,1]$ define a vector field Y_s on M by

$$Y_s := (DF_s)^{-1}\frac{d}{ds}F_s$$

and assume that Y_s and $\operatorname{div} Y_s$ are uniformly bounded.

Theorem 5.6. *Suppose $V \in C^1(M)$ is non-negative with ∇V bounded and $2K \leq \mathrm{Ric} \leq 2L$ for constants K and L. Suppose $f \geq 0$ is a bounded measurable function and that Y_s and $\operatorname{div} Y_s$ are uniformly bounded. Then*

$$P_t^V f(x) \leq P_t^V (f \circ F_1)(x) + \left(\int_0^1 (\alpha_t^V)^2 (Y_s) ds \right)^{\frac{1}{2}} \left(P_t^V f^2 \right)^{\frac{1}{2}} (x)$$

for all $x \in M$ and $t \geq 0$, where α_t^V is defined as in Proposition 5.4.

Proof. It suffices to prove for f continuously differentiable. Since

$$\frac{d}{ds}(f \circ F_s) = \nabla_{V_s}(f \circ F_s),$$

and following the proof of [18, Proposition 2.3], we see for all $r > 0$ that

$$\begin{aligned} &\frac{d}{ds} P_t^V \left(\frac{f}{1+rsf}(F_{1-s}) \right) \\ &\quad = -r P_t^V \left(\frac{f^2}{(1+rsf)^2}(F_{1-s}) \right) - P_t^V \left(Y_{1-s} \left(\frac{f}{1+rsf}(F_{1-s}) \right) \right) \end{aligned}$$

for all $s \in [0,1]$. Applying Proposition 5.4 and proceeding as in the proof of [18, Proposition 2.3], integrating over $s \in [0,1]$ and minimizing over $r > 0$, we easily obtain the desired inequality. □

Theorem 5.7. *Suppose $V \in C^1(M)$ is non-negative with ∇V bounded and $2K \leq \mathrm{Ric} \leq 2L$ for constants K and L. Suppose $f \geq 0$ is a bounded measurable function and that Y_s and $\operatorname{div} Y_s$ are uniformly bounded. Then*

$$(P_t^V f)^p(x) \leq P_t^V (f^p \circ F_1)(x) \exp \left[\int_0^1 \frac{p}{1+(p-1)s} \beta_t^V \left(\frac{p-1}{1+(p-1)s}, Y_s \right) ds \right]$$

for all $x \in M$, $t \geq 0$ and $p > 1$, where β_t is defined as in Proposition 5.5.

Proof. As for the previous theorem, it suffices to prove for f continuously differentiable. The result extends to more general f by approximation. Applying Proposition 5.5, as in the proof of [18, Proposition 2.3] with $\beta(s) := 1 + (p-1)s$ for $s \in [0,1]$, we obtain

$$\frac{d}{ds} \log \left(P_t^V \left(f^{\beta(s)}(F_s) \right)(x) \right)^{\frac{p}{\beta(s)}} \geq -\frac{p}{\beta(s)} \beta_t^V \left(\frac{p-1}{\beta(s)}, Y_s \right)$$

for $s \in [0,1]$. Integrating over s yields the result. □

Looking ahead, it would now be desirable to weaken our assumptions on dV, such as to suppose simply that dV exists in some weak sense and is locally Kato, or indeed to eliminate terms involving dV altogether. To do the latter could however be difficult, since dV appears natually in Theorem 3.1. Assumptions involving dV do appear elsewhere in the literature, such as in the celebrated work of Li and Yau [9].

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