Derived $\mathcal{D}$-Geometry

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Homotopy algebras, deformation theory and quantization

Derived $\mathcal{D}$-Geometry *

Combine

- Derived algebraic geometry (Toën-Vezzosi - derived stacks), and
- $\mathcal{D}$-Geometry (Beilinson-Drinfeld - jet bundle formalism)

and find a proper framework for a coordinate-free approach to BV

Costello, Gwilliam, Schreiber, ...

*Joint with Damjan Pištalo

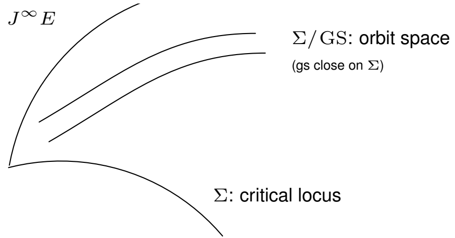
Koszul-Tate resolutions as cofibrant replacements of algebras over differential operators
(JHRS, 2018)
Homotopical algebraic context over differential operators
(JHRS, 2018)

Historical note

- **1967**: **Faddeev-Popov ghosts** in GQFT
  (consistency path integral formulation)
- $\pm$**1975**: **BRST** formalism
  (rationalization of ghosts)
- $\pm$**1982**: **BV** formalism
  (generalization of BRST for YMT to any Lagrangian GFT)
- $\pm$ **1995**: **Abstract BV** (e.g., AKSZ)
  $(S_0, \mathrm{sym}, S = S_0 + \ldots, \{S, S\} = 0, \{S, \bullet\}$ resolves quotient by $\mathrm{sym})$
- Interest: **Gauge anomalies, renormalization**
  $\mathcal{L} \rightsquigarrow \mathcal{L}(\xi)$, quantization independent of ξ, BV handles problem)

Classical Batalin-Vilkovisky complex

$$\mathrm{d}\,S = 0 \Leftrightarrow \mathrm{E}(t, q(t), \dot q(t), \ddot q(t)) = 0 \Leftrightarrow \mathrm{E}(t, q, \dot q, \ddot q)|_{j^\infty q(t)} = 0$$



LE algebra $(\mathcal{O}(J^\infty E)[C^\alpha_\rho], d)$

KT (Koszul-Tate) complex $(\mathcal{O}(J^\infty E)[\phi^*_{i\rho}, C^*_{\alpha\rho}], \delta_{\mathrm{KT}})$ resolves $\mathcal{O}(\Sigma)$

BV complex $(\mathcal{O}(J^\infty E)[\phi^*_{i\rho}, C^*_{\alpha\rho}, C^\alpha_\rho], \delta_{\mathrm{KT}} + d + \ldots)$ 'resolves' $\mathcal{O}(\Sigma/\mathrm{GS})$

Derived geometric Batalin-Vilkovisky complex

$\mathcal{O}(\Sigma)$, $\mathrm{d}\, S = 0$ (1) ; $\mathcal{O}(\Sigma/\mathrm{GS})$ (2)

$J^\infty E/\mathrm{GS}$ (1) (H: gs close off Σ) ; $\mathrm{d}\, S = 0$ (2)

$$J^\infty E/\,\mathrm{GS} \;\rightsquigarrow\; C = J^\infty E/\!/\,\mathrm{GS} \;\rightsquigarrow\; X_1 \rightrightarrows X_0 \;\rightsquigarrow\; \cdots X_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_1 \rightrightarrows X_0$$

$C = J^\infty E/\!/\,\mathrm{GS}$: ∞-groupoid

Spaces viewed as sheaves

$$\underline{\bullet} : \texttt{Aff} \ni c \mapsto \underline{c} := \mathrm{Hom}_{\texttt{Aff}}(-, c) \in \texttt{Fun}(\texttt{Aff}^{\mathrm{op}}, \texttt{Set})$$

$\mathrm{Lim}\, \underline{c} \simeq \underline{\mathrm{Lim}\, c}$

$\mathrm{Colim}\, \underline{c} \to \underline{\mathrm{Colim}\, c}$

trivial space $\leftrightsquigarrow$ representable sheaf

space $\leftrightsquigarrow$ sheaf

scheme $\leftrightsquigarrow$ locally representable sheaf

Spaces viewed as sheaves

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space ⇝ sheaf

scheme ⇝ locally representable sheaf

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trivial space $\leftsquigarrow\!\rightsquigarrow$ representable sheaf

space $\leftsquigarrow\!\rightsquigarrow$ sheaf

scheme $\leftsquigarrow\!\rightsquigarrow$ locally representable sheaf

∞-groupoids viewed as sheaves

$\bullet/\bullet$ of $\mathtt{Aff}$ $\rightsquigarrow$ ∞-groupoid $\rightsquigarrow$ $\mathtt{SSet}$ (simplicial set s.th. any horn has a filler)

$\cap$ of $\mathtt{Aff}$ $\rightsquigarrow$ $\otimes$ of $\mathtt{CA}$ $\rightsquigarrow$ $\overset{\mathbb{L}}{\otimes}$ of $\mathtt{DG_+CA}$

$\mathtt{Sh}(\mathtt{Aff}^{\mathrm{op}}, \mathtt{Set}) = \mathtt{Sh}(\mathtt{CA}, \mathtt{Set})$: spaces

$\mathtt{Sh}(\mathtt{CA}, \mathtt{SSet})$: stacks

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$\mathbf{DS} = \mathtt{Sh}(\mathtt{DG_+CA}, \mathtt{SSet})$: derived stacks

∞-groupoids viewed as sheaves

$\bullet/\bullet$ of $\mathtt{Aff}$ $\rightsquigarrow$ ∞-groupoid $\rightsquigarrow$ $\mathtt{SSet}$ (simplicial set s.th. any horn has a filler)

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$C = J^\infty E // \mathrm{GS}$: ∞-groupoid

$C = J^\infty E // \mathrm{GS} \in \mathbf{DS} = \mathtt{Sh}(\mathtt{DG_+CA}, \mathtt{SSet})$

Sheaf condition for $F \in \mathtt{Fun}(\mathtt{CA}, \mathtt{Set})$

$$\begin{array}{lccc} & \underline{U}_i \times_{\underline{U}} \underline{U}_j & \to & \underline{U}_j \\ & \downarrow & & \downarrow \;\; \underline{f}_j \;, \qquad U \in \mathtt{CA}^{\mathrm{op}} = \mathtt{Aff} \\ \underline{f}_i : & \underline{U}_i & \to & \underline{U} \end{array}$$

$$\begin{array}{ccccc} \coprod_{ij} \underline{U}_i \times_{\underline{U}} \underline{U}_j \rightrightarrows \coprod_i \underline{U}_i & & \longrightarrow & & \underline{U} \\ & \searrow & & \nearrow & \\ & & \mathrm{Colim}(\ldots) & & \end{array}$$

$$\begin{array}{ccccc} \prod_{ij} \underline{F}(\underline{U}_i \times_{\underline{U}} \underline{U}_j) \leftleftarrows \prod_i \underline{F}(\underline{U}_i) & & \longleftarrow & & \underline{F}(\underline{U}) \\ & \nwarrow & & \simeq & \\ & & \underline{F}(\mathrm{Colim}(\ldots)) & & \end{array}$$

$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{ij} \underline{F}(\underline{U}_i \times_{\underline{U}} \underline{U}_j)$: equalizer diag

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{ij} F(U_i \times_U U_j)$$

Homotopy theory in model categories

model category $\mathtt{M}$

weak equivalences W (weak homotopy equivalences)

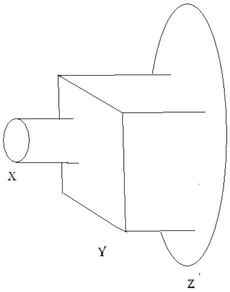
fibrations (‘good’ surjections)

cofibrations (‘good’ injections)

$$\boxed{\mathtt{Ho}(\mathtt{M}) = \mathtt{M}[W^{-1}]}$$

Homotopy colimits

$X \xrightarrow{f} Y \xrightarrow{g} Z \in \mathtt{Diag}(\mathtt{Top})$



Homotopy colimits

Model category $\mathtt{M}$

$\mathrm{Colim} : \mathtt{Diag}(\mathtt{M}) \to \mathtt{M}$

$\mathrm{HoColim}$

Homotopy colimits

Model category $\mathtt{M}$

$\mathrm{Colim} : \mathtt{Diag}(\mathtt{M}) \to \mathtt{M}$

$$\mathrm{HoColim} := \mathbb{L}\,\mathrm{Colim} : \mathtt{Ho}(\mathtt{Diag}(\mathtt{M})) \ni D \mapsto \mathrm{Colim}(\mathcal{C}_{\mathtt{Diag}(\mathtt{M})} D) \in \mathtt{Ho}(\mathtt{M})$$

Homotopy hypercovers

$\mathtt{Fun}(\mathtt{DG}_+\mathtt{CA}, \mathtt{SSet})$, $U \in \mathtt{DG}_+\mathtt{CA}^{\mathrm{op}} = \mathtt{D}_{-}\mathtt{Aff}$, $\underline{U} \in \mathtt{Fun}(\mathtt{DG}_+\mathtt{CA}, \mathtt{Set})$

$$
\begin{array}{ccccccc}
\cdots & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} & h_2 & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & h_1 & \rightrightarrows & h_0 \\
 & & p_2 \downarrow & & p_1 \downarrow & & p_0 \downarrow \\
\cdots & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} & \underline{U} & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & \underline{U} & \rightrightarrows & \underline{U}
\end{array}
$$

$p_\bullet : h_\bullet \to \underline{U}$ in $\mathtt{Fun}(\mathtt{DG}_+\mathtt{CA}, \mathtt{SSet})$, $\pi_0(h_\bullet) \to \pi_0(\underline{U})$ $\rightsquigarrow$ τ on $\mathtt{DG}_+\mathtt{CA}^{\mathrm{op}}$

Homotopy hypercovers

$\mathtt{Fun}(\mathtt{DG_+CA}, \mathtt{SSet})$, $U \in \mathtt{DG_+CA}^{\mathrm{op}} = \mathtt{D\_Aff}$, $\underline{U} \in \mathtt{Fun}(\mathtt{DG_+CA}, \mathtt{Set})$

$$
\begin{array}{ccccccc}
\cdots & \begin{smallmatrix}\to\\ \to\\ \to\\ \to\end{smallmatrix} & h_2 & \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix} & h_1 & \rightrightarrows & h_0 \\
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$$

$p_\bullet : h_\bullet \to \underline{U}$ in $\mathtt{Fun}(\mathtt{DG_+CA}, \mathtt{SSet})$, $\pi_0(h_\bullet) \to \pi_0(\underline{U})$ $\rightsquigarrow$ τ on $\mathtt{DG_+CA}^{\mathrm{op}}$

Homotopy sheaf condition for $F \in \mathtt{Fun}(\mathtt{DG_+CA}, \mathtt{SSet})$

$F \in \mathtt{Fun}(\mathtt{CA}, \mathtt{Set})$

$U \in \mathtt{CA}^{\mathrm{op}}$, $\coprod_{ij} \underline{U}_i \times_{\underline{U}} \underline{U}_j \rightrightarrows \coprod_i \underline{U}_i \to \underline{U}$, $\mathrm{Colim}(\ldots) \to \underline{U}$, $\underline{F}$: isom

$F \in \mathtt{Fun}(\mathtt{DG_+CA}, \mathtt{SSet})$

$U \in \mathtt{DG_+CA}^{\mathrm{op}}$, $p_\bullet : h_\bullet \to \underline{U}$, $\mathrm{HoColim}(h_\bullet) \to \underline{U}$, $\underline{F}$: isom

$$F(U) \xrightarrow{\sim} \underline{F}\,(\mathrm{HoColim}(h_\bullet)) = \mathrm{HoLim}\,\underline{F}(h_\bullet) = \mathrm{HoLim}_{[n]\in\Delta}\,\underline{F}(h_n)$$

Homotopical algebraic geometric context

In functors: $C = J^\infty E // \mathrm{GS} \;\in\; \mathbf{DS} = \mathtt{Sh}(\mathtt{DG_+CA}, \mathtt{SSet})$

Data: $\mathtt{DG_+CA}$, τ on $\mathtt{DG_+CA}^{\mathrm{op}}$

Additional data: $\mathbf{P} \subset \mathrm{Mor}(\mathtt{DG_+CA}^{\mathrm{op}})$

In spaces: $C = J^\infty E // \mathrm{GS}$: groupoid $s, t : X_1 \rightrightarrows X_0$

Groupoid in spaces / manifolds: $t \in \mathrm{Mor}(\mathcal{C}^\infty\,\mathtt{Man})$

$s : X_1 \ni (g, x) \mapsto x \in X_0 \quad t : X_1 \ni (g, x) \mapsto g \cdot x \in X_0$

Groupoid internal to $\mathbf{DS}$: $t \in \mathrm{Mor}(\mathbf{DS}) \quad \rightsquigarrow \quad \tilde{t} \in \mathrm{Mor}(\mathtt{DG_+CA}^{\mathrm{op}}) \quad \rightsquigarrow \quad \tilde{t} \in \mathbf{P}$

Condition: $(\mathtt{DG_+CA}, \tau, \mathbf{P})$ is a homotopical algebraic geometric context

Derived stacks over differential operators $\mathcal{D}$

$J^\infty E \to X\ ,\ \mathcal{O}(J^\infty E)$

$$\mathrm{d}\,S = 0 \Leftrightarrow d_t^k\ \mathrm{E}(t, q(t), \dot{q}(t), \ddot{q}(t)) = 0 \Leftrightarrow D_t^k\ \mathrm{E}(t, q, \dot{q}, \ddot{q})|_{j^\infty q(t)} = 0$$

$$d_t \ \rightsquigarrow\ D_t = \partial_t + \dot{q}\partial_q + \ddot{q}\partial_{\dot{q}} + \dots$$

$$\mathcal{D}(X)\cdot\mathcal{O}(J^\infty E)\ :\ \mathcal{O}(J^\infty E) \in \texttt{Mod}(\mathcal{D}) \cap \mathcal{D}\texttt{A}$$

$(\texttt{DG}_+\texttt{CK}\mathcal{D}\texttt{A}, \tau, \mathbf{P})$ is a homotopical algebraic geometric context

$$\mathbf{DK}\mathcal{D}\mathbf{S} = \texttt{Sh}(\texttt{DG}_+\texttt{CK}\mathcal{D}\texttt{A}, \texttt{SSet})$$

Derived stacks over differential operators $\mathcal{D}$

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$(\mathtt{DG_+CK\mathcal{D}A}, \tau, \mathbf{P})$ is a homotopical algebraic geometric context

$\mathbf{DK\mathcal{D}S} = \mathtt{Sh}(\mathtt{DG_+CK\mathcal{D}A}, \mathtt{SSet})$

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$(\mathtt{DG}_+\mathtt{C}\mathbb{K}\mathcal{D}\mathtt{A}, \tau, \mathbf{P})$ is a homotopical algebraic geometric context

$\mathbf{D}\mathbb{K}\mathcal{D}\mathbf{S} = \mathtt{Sh}(\mathtt{DG}_+\mathtt{C}\mathbb{K}\mathcal{D}\mathtt{A}, \mathtt{SSet})$

Derived stacks over differential operators $\mathcal{D}$

$J^\infty E \to X\ ,\ \mathcal{O}(J^\infty E)$

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$(\mathtt{DG_+C}\mathbb{K}\mathcal{D}\mathtt{A}, \tau, \mathbf{P})$ is a homotopical algebraic geometric context

$\mathbf{D}\mathbb{K}\mathcal{D}\mathbf{S} = \mathtt{Sh}(\mathtt{DG_+C}\mathbb{K}\mathcal{D}\mathtt{A}, \mathtt{SSet})$

Derived stacks over differential operators $\mathcal{D}$

$J^\infty E \to X$, $\mathcal{O}(J^\infty E)$

$\mathrm{d}\, S = 0 \Leftrightarrow d_t^k\ \mathrm{E}(t, q(t), \dot{q}(t), \ddot{q}(t)) = 0 \Leftrightarrow D_t^k\ \mathrm{E}(t, q, \dot{q}, \ddot{q})|_{j^\infty q(t)} = 0$

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$\mathcal{D}(X) \cdot \mathcal{O}(J^\infty E)\ :\ \mathcal{O}(J^\infty E) \in \mathtt{Mod}(\mathcal{D}) \cap \mathcal{D}\mathtt{A}$

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Derived stacks over differential operators $\mathcal{D}$

$J^\infty E \to X \ , \ \mathcal{O}(J^\infty E)$

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$(\mathtt{DG}_+\mathtt{C}\mathbb{K}\mathcal{D}\mathtt{A}, \tau, \mathbf{P})$ is a homotopical algebraic geometric context

$\mathbf{D}\mathbb{K}\mathcal{D}\mathbf{S} = \mathtt{Sh}(\mathtt{DG}_+\mathtt{C}\mathbb{K}\mathcal{D}\mathtt{A}, \mathtt{SSet})$

Results I

- Cofibrantly generated model structure on
  - $\mathtt{DG}_+\mathcal{D}\mathtt{A}$
  - $\mathtt{DG}_+\,\mathcal{O}_X\text{-qc}\,\mathtt{CA}(\mathcal{D}_X)$, X smooth affine
- Cofibrations = retracts of relative Sullivan $\mathcal{D}$-algebras
- Specific explicit functorial TrivCof-Fib and Cof-TrivFib factorizations
- Homotopy sheaf condition for $F \in \mathtt{Fun}(\mathtt{DG}_+\mathcal{D}\mathtt{A}, \mathtt{SSet})$ is a fibrant object condition
  - Object-wise model structure on $\mathtt{Fun}(\mathtt{DG}_+\mathcal{D}\mathtt{A}, \mathtt{SSet})$
  - Bousfield localization w.r.t weq in $\mathtt{DG}_+\mathcal{D}\mathtt{A}^{\mathrm{op}}$
  - Bousfield localization w.r.t homotopy hypercovers

Results II

- $\mathtt{Mod}(\mathcal{A}) := \mathtt{Mod}_{\mathtt{DG}_+\mathcal{D}\mathtt{M}}(\mathcal{A})$, $\mathcal{A} \in \mathtt{DG}_+\mathcal{D}\mathtt{A}$, is a

  cofibrantly generated combinatorial proper symmetric monoidal model category which satisfies the monoid axiom, and

  $- \otimes_{\mathcal{A}} M : \mathtt{Mod}(\mathcal{A}) \to \mathtt{Mod}(\mathcal{A})$, $M \in \mathtt{Mod}(\mathcal{A})$ cofibrant, preserves weq

- $\mathtt{Alg}(\mathcal{A}) := \mathtt{CMon}(\mathtt{Mod}(\mathcal{A}))$, $\mathcal{A} \in \mathtt{DG}_+\mathcal{D}\mathtt{A}$, is a

  combinatorial proper model category, and

  $\mathcal{B} \otimes_{\mathcal{A}} - : \mathtt{Mod}(\mathcal{A}) \to \mathtt{Mod}(\mathcal{B})$, $\mathcal{B} \in \mathtt{Alg}(\mathcal{A})$ cofibrant, preserves weq

Koszul-Tate resolutions and Sullivan models (Dissertationes Math., 2018)

Koszul-Tate resolutions in 5 different fields

1. KTR of a quotient of a Noetherian commutative unital ring (homological algebra, Tate)
2. KTR of a higher-order reducible gauge theory (field theory, Henneaux-Teitelboim)
3. KTR of a compatibility complex (cohomological analysis of PDEs, Verbovetsky)
4. KTR as cofibrant replacement in a coslice category of $\mathtt{DG}_+\mathcal{D}\mathtt{A}$ (homotopy theory, PP)
5. KTR in $\mathcal{D}$-geometry (algebraic geometry, PP)

Results III

- Field theoretic KTR resolves on-shell functions $\mathcal{O}(\Sigma)$

- $\mathcal{J} \in \mathcal{D}\mathtt{A}\,,\ \ \mathcal{I} \subset \mathcal{J}\ \ \mathcal{D}$-ideal, $\mathcal{J} \to \mathcal{J}/\mathcal{I}$ in $\mathtt{DG}_+\mathcal{D}\mathtt{A}$

  Homotopy theoretic KTR of $\mathcal{J}/\mathcal{I}$

  $\mathcal{J} \rightarrowtail \mathcal{J} \otimes \mathcal{S}V \stackrel{\sim}{\twoheadrightarrow} \mathcal{J}/\mathcal{I}$ in $\mathtt{DG}_+\mathcal{D}\mathtt{A}$, $\mathcal{J} \otimes \mathcal{S}V$ resolves $\mathcal{J}/\mathcal{I}$,

  $\mathcal{J} \otimes \mathcal{S}V$ cofibrant replacement of $\mathcal{J}/\mathcal{I}$ in $\mathcal{J} \downarrow \mathtt{DG}_+\mathcal{D}\mathtt{A}$

- X smooth scheme, $\mathcal{A}_X \in \mathtt{qcCA}(\mathcal{D}_X)$, $\varphi : \mathcal{A}_X \to \mathcal{B}_X$ in $\mathtt{DG}_+\mathtt{qcCA}(\mathcal{D}_X)$

  Algebraic geometric KTR of φ

  $\psi : \mathcal{C}_X \to \mathcal{B}_X$ in $\mathtt{DG}_+\mathtt{qcCA}(\mathcal{A}_X[\mathcal{D}_X])$, qis in $\mathtt{DG}_+\mathtt{qcM}(\mathcal{A}_X[\mathcal{D}_X])$,

  $\mathcal{C}_X$ of $\mathcal{D}_X$-Sullivan type

Results IV

- Homotopy theoretic KTR: same structure as field theoretic KTR
- Algebraic geometric KTR: extension of homotopy theoretic KTR to an arbitrary smooth scheme
- Existence: both KTRs do always exist
- Comparison theorems: all 5 KTRs are of the algebraic geometric type
- 5 different fields: dictionaries between their languages

Homotopical algebraic geometry and BV complex

Function algebras of derived $\mathbb{K}$-stacks

$$
\begin{array}{ccc}
\mathtt{D\_Aff} = \mathtt{DG_+}\mathbb{K}\mathtt{A}^{\mathrm{op}} & \stackrel{\iota}{\rightarrowtail} & \mathtt{DG}\mathbb{K}\mathtt{A}^{\mathrm{op}} \\
\Big\downarrow {\scriptstyle \underline{\bullet}_S} & & \\
\mathbf{DS} = \mathtt{Sh}(\mathtt{DG_+}\mathbb{K}\mathtt{A}, \mathtt{SSet}) & &
\end{array}
$$

Function algebras of derived $\mathbb{K}$-stacks

$$
\begin{array}{ccc}
\mathtt{D\_Aff} = \mathtt{DG}_+\mathbb{K}\mathtt{A}^{\mathrm{op}} & \overset{\iota}{\rightarrowtail} & \mathtt{DG}\mathbb{K}\mathtt{A}^{\mathrm{op}} \\
\Big\downarrow {\scriptstyle \underline{\bullet}_S} & \nearrow {\scriptstyle \mathrm{Kan}_{\underline{\bullet}_S}\,\iota} & \\
\mathbf{DS} = \mathtt{Sh}(\mathtt{DG}_+\mathbb{K}\mathtt{A}, \mathtt{SSet}) & &
\end{array}
$$

Function algebras of derived $\mathbb{K}$-stacks

$$\begin{array}{ccc}
\mathtt{D_Aff} = \mathtt{DG_+\mathbb{K}A}^{\mathrm{op}} & \xrightarrow{\ \iota\ } & \mathtt{DG\mathbb{K}A}^{\mathrm{op}} \\
\Big\downarrow \underline{\bullet}_S & \nearrow_{\mathcal{O}=\mathrm{Kan}_{\underline{\bullet}_S}\,\iota} & \\
\mathbf{DS} = \mathtt{Sh}(\mathtt{DG_+\mathbb{K}A}, \mathtt{SSet}) & &
\end{array}$$

Explanation

$C = J^\infty E // \mathrm{GS}$:

$(C, [\bullet, \bullet], R) \rightleftarrows d \in \mathrm{Der}_{-1}(\Gamma(\wedge C^*))\ , d^2 = 0$

$\Gamma(\wedge C^*) = \mathcal{O}(J^\infty E)[C^\alpha] = \{\sum F\, C^{\alpha_1} \ldots C^{\alpha_p}\}$

$d = R^i_\alpha(y) C^\alpha \partial_{y^i} + \frac{1}{2} C^\gamma_{\alpha\beta}(y) C^\alpha \wedge C^\beta \partial_{C^\gamma}$

$d = D^\rho_x(R^i_\alpha(C^\alpha))\partial_{\phi^i_\rho} + \frac{1}{2} D^\rho_x(C^\gamma_{\alpha\beta}(C^\alpha, C^\beta))\partial_{C^\gamma_\rho}$

Noether operators, Faddeev-Popov ghosts, fields

$\mathcal{O}(C) = \Gamma(\wedge C^*) = \mathcal{O}(J^\infty E)[C^\alpha] \in \mathtt{DG}\mathbb{K}\mathtt{A}$!

LE algebra

Explanation

$C = J^\infty E // \mathrm{GS}$: gauge Lie algebroid $C \to J^\infty E$

$(C, [\bullet, \bullet], R) \rightleftarrows d \in \mathrm{Der}_{-1}(\Gamma(\wedge C^*))$, $d^2 = 0$

$\Gamma(\wedge C^*) = \mathcal{O}(J^\infty E)[C^\alpha] = \{ \sum F \, C^{\alpha_1} \dots C^{\alpha_p} \}$

$d = R^i_\alpha(y) C^\alpha \partial_{y^i} + \frac{1}{2} C^\gamma_{\alpha\beta}(y) C^\alpha \wedge C^\beta \partial_{C^\gamma}$

$d = D^\rho_x(R^i_\alpha(C^\alpha))\partial_{\phi^i_\rho} + \frac{1}{2} D^\rho_x(C^\gamma_{\alpha\beta}(C^\alpha, C^\beta))\partial_{C^\gamma_\rho}$

Noether operators, Faddeev-Popov ghosts, fields

$\mathcal{O}(C) = \Gamma(\wedge C^*) = \mathcal{O}(J^\infty E)[C^\alpha] \in \mathtt{DG}\mathbb{K}\mathtt{A}$!

LE algebra

Explanation

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LE algebra

Derived critical locus: the problem

$C = J^\infty E // \mathrm{GS} \in \mathbf{DS} = \mathtt{Sh}(\mathtt{DG}_+\mathbb{K}\mathtt{A}, \mathtt{SSet})$ (1) , $C_{\{\mathrm{d}\, S=0\}}$? (2) , BV ?

$S \in \mathcal{O}(C) \in \mathtt{DG}\mathbb{K}\mathtt{A}$ (given), $\mathrm{d}\, S : C \ni m \mapsto \mathrm{d}_m\, S \in T^*C$?

$$\begin{array}{ccc} C_{\{\mathrm{d}\, S=0\}} & \longrightarrow & C \\ \downarrow & \circlearrowleft & \downarrow 0 \\ C & \xrightarrow{\mathrm{d}\, S} & T^*C \end{array}$$

Homotopy pullback ?

Derived critical locus: an almost meaningful problem

$$
\begin{array}{ccc}
\mathcal{O}(T^*C) & \xrightarrow{\;i_{\mathrm{d}\,S}=i_S\;} & \mathcal{O}(C) \\
\Big\downarrow i_0 & \circlearrowleft & \Big\downarrow \\
\mathcal{O}(C) & \longrightarrow & \underbrace{\mathcal{O}(C_{\{\mathrm{d}\,S=0\}})}_{\text{BV complex ?}}
\end{array}
$$

Homotopy pushout in $\mathtt{DG}\mathbb{K}\mathtt{A}_{\mathcal{O}(C)}$?

Derived stacks over the derived stack C: $\mathcal{O} : \mathbf{DS}/C \to \mathtt{DG}\mathbb{K}\mathtt{A}^{\mathrm{op}}/C$

H: $C \in \mathtt{DG}\mathbb{K}\mathtt{A}^{\mathrm{op}} = \mathtt{DAff}(\mathbb{K})$!

$$\mathtt{DG}\mathbb{K}\mathtt{A}^{\mathrm{op}}/C \simeq \mathtt{DAff}(\mathbb{K})/C \simeq (\mathcal{O}(C)/\mathtt{DG}\mathbb{K}\mathtt{A})^{\mathrm{op}} \simeq \mathtt{DG}\mathbb{K}\mathtt{A}^{\mathrm{op}}_{\mathcal{O}(C)}$$

Derived critical locus: an almost meaningful problem

$$\begin{array}{ccc}
\mathcal{O}(T^*C) & \xrightarrow{\ i_{\mathrm{d}\,S} = i_S\ } & \mathcal{O}(C) \\
\Big\downarrow {\scriptstyle i_0} & \circlearrowleft & \Big\downarrow \\
\mathcal{O}(C) & \longrightarrow & \underbrace{\mathcal{O}(C_{\{\mathrm{d}\,S=0\}})}_{\text{BV complex ?}}
\end{array}$$

Homotopy pushout in $\mathtt{DG\mathbb{K}A}_{\mathcal{O}(C)}$?

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$$\mathtt{DG\mathbb{K}A}^{\mathrm{op}}/C \;\simeq\; \mathtt{DAff}(\mathbb{K})/C \;\simeq\; (\mathcal{O}(C)/\mathtt{DG\mathbb{K}A})^{\mathrm{op}} \;\simeq\; \mathtt{DG\mathbb{K}A}^{\mathrm{op}}_{\mathcal{O}(C)}$$

Derived critical locus: an almost meaningful problem

$$\begin{array}{ccc} \mathcal{O}(T^*C) & \xrightarrow{i_{\mathrm{d}\,S}=i_S} & \mathcal{O}(C) \\ \downarrow{\scriptstyle i_0} & \circlearrowleft & \downarrow \\ \mathcal{O}(C) & \longrightarrow & \underbrace{\mathcal{O}(C_{\{\mathrm{d}\,S=0\}})}_{\text{BV complex ?}} \end{array}$$

Homotopy pushout in $\mathtt{DG}\mathbb{K}\mathtt{A}_{\mathcal{O}(C)}$?

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Definition of the data

$\mathcal{O}(T^*C) = \Gamma(\wedge TC) = \mathcal{S}_{\mathcal{O}(C)} \operatorname{Der} \mathcal{O}(C) \in \mathtt{DG}\mathbb{K}\mathtt{A}_{\mathcal{O}(C)}$?

$\ldots \longrightarrow \operatorname{Der}_k \mathcal{O}(C) \stackrel{[d_{\mathcal{O}(C)}, \bullet]}{\longrightarrow} \operatorname{Der}_{k-1} \mathcal{O}(C) \longrightarrow \ldots \in \mathtt{DG}\mathbb{K}\mathtt{M}_{\mathcal{O}(C)}$!

$i_S : \mathcal{S}_{\mathcal{O}(C)} \operatorname{Der} \mathcal{O}(C) \to \mathcal{S}_{\mathcal{O}(C)} \mathcal{O}(C)$

$i_0 : \mathcal{S}_{\mathcal{O}(C)} \operatorname{Der} \mathcal{O}(C) \to \mathcal{S}_{\mathcal{O}(C)} 0$

Derived critical locus: a meaningful problem

$$\begin{array}{ccc} \mathcal{S}_{\mathcal{O}(C)}\operatorname{Der}\mathcal{O}(C) & \xrightarrow{\;i_S\;} & \mathcal{S}_{\mathcal{O}(C)}\mathcal{O}(C) \\ \Big\downarrow i_0 & \circlearrowleft & \Big\downarrow \\ \mathcal{S}_{\mathcal{O}(C)}0 & \longrightarrow & \underbrace{\mathcal{O}(C_{\{\mathrm{d}\,S=0\}})}_{\text{BV complex ?}} \end{array}$$

Homotopy pushout in $\mathtt{DG}\mathbb{K}\mathtt{A}_{\mathcal{O}(C)}$

$$\mathcal{S}_{\mathcal{O}(C)}\ \mathrm{HoColim}\ \begin{array}{ccc} \operatorname{Der}\mathcal{O}(C) & \xrightarrow{\;i_S\;} & \mathcal{O}(C) \\ \Big\downarrow i_0 & \circlearrowleft & \Big\downarrow \\ 0 & \longrightarrow & \star \end{array} \;=\; \mathcal{S}_{\mathcal{O}(C)}\ \mathrm{Colim}\ \begin{array}{ccc} \mathcal{C}\circ & \xrightarrow{\;i_S\;} & \mathcal{C}\diamond \\ \Big\downarrow \mathcal{C}i_0 & \circlearrowleft & \Big\downarrow \\ \mathcal{C}0 & \longrightarrow & \star \end{array}$$

$$\mathcal{C}i_0 : \circ = \operatorname{Der}\mathcal{O}(C) \to \mathcal{C}0 \;=\; \operatorname{Der}\mathcal{O}(C)[1] \oplus_{\mathrm{id}} \operatorname{Der}\mathcal{O}(C)$$

Derived critical locus: a meaningful problem

$$\begin{array}{ccc}
\mathcal{S}_{\mathcal{O}(C)}\operatorname{Der}\mathcal{O}(C) & \xrightarrow{\;i_S\;} & \mathcal{S}_{\mathcal{O}(C)}\mathcal{O}(C) \\
\downarrow{\scriptstyle i_0} & \circlearrowleft & \downarrow \\
\mathcal{S}_{\mathcal{O}(C)}0 & \longrightarrow & \underbrace{\mathcal{O}(C_{\{\mathrm{d}\,S=0\}})}_{\text{BV complex ?}}
\end{array}$$

Homotopy pushout in $\mathtt{DG}\mathbb{K}\mathtt{A}_{\mathcal{O}(C)}$

$$\mathcal{S}_{\mathcal{O}(C)}\ \mathrm{HoColim}\ \begin{array}{ccc}
\operatorname{Der}\mathcal{O}(C) & \xrightarrow{\;i_S\;} & \mathcal{O}(C) \\
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Derived critical locus: a meaningful problem

$$\begin{array}{ccc} \mathcal{S}_{\mathcal{O}(C)}\operatorname{Der}\mathcal{O}(C) & \xrightarrow{\;i_S\;} & \mathcal{S}_{\mathcal{O}(C)}\mathcal{O}(C) \\ \Big\downarrow i_0 & \circlearrowleft & \Big\downarrow \\ \mathcal{S}_{\mathcal{O}(C)}0 & \longrightarrow & \underbrace{\mathcal{O}(C_{\{\mathrm{d}\,S=0\}})}_{\text{BV complex ?}} \end{array}$$

Homotopy pushout in $\mathtt{DG}\mathbb{K}\mathtt{A}_{\mathcal{O}(C)}$

$$\mathcal{S}_{\mathcal{O}(C)}\ \mathrm{HoColim} \begin{array}{ccc} \operatorname{Der}\mathcal{O}(C) & \xrightarrow{\;i_S\;} & \mathcal{O}(C) \\ \Big\downarrow i_0 & \circlearrowleft & \Big\downarrow \\ 0 & \longrightarrow & \star \end{array} \;=\; \mathcal{S}_{\mathcal{O}(C)}\ \mathrm{Colim} \begin{array}{ccc} \mathcal{C}\circ & \xrightarrow{\;i_S\;} & \mathcal{C}\diamond \\ \Big\downarrow \mathcal{C}i_0 & \circlearrowleft & \Big\downarrow \\ \mathcal{C}0 & \longrightarrow & \star \end{array}$$

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Derived critical locus: a meaningful problem

$$
\begin{array}{ccc}
\mathcal{S}_{\mathcal{O}(C)} \operatorname{Der} \mathcal{O}(C) & \xrightarrow{i_S} & \mathcal{S}_{\mathcal{O}(C)} \mathcal{O}(C) \\
\downarrow{\scriptstyle i_0} & \circlearrowleft & \downarrow \\
\mathcal{S}_{\mathcal{O}(C)} 0 & \longrightarrow & \underbrace{\mathcal{O}(C_{\{\mathrm{d}\, S=0\}})}_{\text{BV complex ?}}
\end{array}
$$

Homotopy pushout in $\mathtt{DG}\mathbb{K}\mathtt{A}_{\mathcal{O}(C)}$

$$
\mathcal{S}_{\mathcal{O}(C)} \ \mathrm{HoColim} \begin{array}{ccc}
\operatorname{Der} \mathcal{O}(C) & \xrightarrow{i_S} & \mathcal{O}(C) \\
\downarrow{\scriptstyle i_0} & \circlearrowleft & \downarrow \\
0 & \longrightarrow & \star
\end{array}
= \mathcal{S}_{\mathcal{O}(C)} \ \mathrm{Colim} \begin{array}{ccc}
\mathcal{C}\circ & \xrightarrow{i_S} & \mathcal{C}\diamond \\
\downarrow{\scriptstyle \mathcal{C} i_0} & \circlearrowleft & \downarrow \\
\mathcal{C}0 & \longrightarrow & \star
\end{array}
$$

$$
\mathcal{C} i_0 : \circ = \operatorname{Der} \mathcal{O}(C) \to \mathcal{C}0 = \operatorname{Der} \mathcal{O}(C)[1] \oplus_{\mathrm{id}} \operatorname{Der} \mathcal{O}(C)
$$

Derived critical locus: the solution

$$\mathcal{S}_{\mathcal{O}(C)}\left(\begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{Der}_k\,\mathcal{O}(C) & \xrightarrow{[d_{\mathcal{O}(C)},\bullet]} & \mathrm{Der}_{k-1}\,\mathcal{O}(C) & \longrightarrow & \cdots \\ & & \bigoplus & \searrow^{i_S} & \bigoplus & & \\ \cdots & \longrightarrow & \mathcal{O}_{k-1}(C) & \xrightarrow{d_{\mathcal{O}(C)}} & \mathcal{O}_{k-2}(C) & \longrightarrow & \cdots \end{array}\right) = \underbrace{\mathcal{O}(C_{\{\mathrm{d}\,S=0\}})}_{\text{BV complex ?}}$$

LE algebra $(\mathcal{O}(C), d_{\mathcal{O}(C)}) = (\mathcal{O}(J^\infty E)[C^\alpha], d_{\mathcal{O}(C)})$

KT complex $(\mathcal{O}(J^\infty E)[\phi_i^*, C_\alpha^*], \delta_{\mathrm{KT}})$,

Derived critical locus: the solution

$$\mathcal{S}_{\mathcal{O}(C)}\left(\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{Der}_k\,\mathcal{O}(C) & \xrightarrow{[d_{\mathcal{O}(C)},\bullet]} & \mathrm{Der}_{k-1}\,\mathcal{O}(C) & \longrightarrow \cdots \\ & \bigoplus & \searrow^{i_S} & \bigoplus & \\ \cdots \longrightarrow & \mathcal{O}_{k-1}(C) & \xrightarrow{d_{\mathcal{O}(C)}} & \mathcal{O}_{k-2}(C) & \longrightarrow \cdots \end{array}\right) = \underbrace{\mathcal{O}(C_{\{\mathrm{d}\,S=0\}})}_{\text{BV complex ?}}$$

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KT complex $(\mathcal{O}(J^\infty E)[\phi_i^*, C_\alpha^*], \delta_{\mathrm{KT}})$, $\partial_{y^i}, \partial_{C^\alpha}$

Figure: Photo by Adam Bouse on Unsplash

THANK YOU FOR YOUR INTEREST