LOCAL CURVATURE ESTIMATES FOR THE RICCI-HARMONIC FLOW

YI LI

ABSTRACT. In this paper we give an explicit bound of $\Delta_{g(t)}u(t)$ and the local curvature estimates for the Ricci-harmonic flow

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)} + 4\nabla_{g(t)}u(t) \otimes \nabla_{g(t)}u(t), \quad \partial_t u(t) = \Delta_{g(t)}u(t)$$

under the condition that the Ricci curvature is bounded along the flow. In the second part these local curvature estimates are extended to a class of generalized Ricci flow, introduced by the author [42], whose stable points give Ricci-flat metrics on a complete manifold, and which is very close to the (K, N)-super Ricci flow recently defined by Xiangdong Li and Songzi Li [35]. Next we propose a conjecture for Einstein's scalar field equations motivated by a result in the first part and the bounded L^2-curvature conjecture recently solved by Klainerman, Rodnianski and Szeftel [26]. In the last two parts of this paper, we discuss two notions of "Riemann curvature tensor" in the sense of Wylie-Yeroshkin [23, 24, 67, 68], respectively, and Li [44], whose "Ricci curvature" both give the standard Bakey-Émery Ricci curvature [1], and the forward and backward uniqueness for the Ricci-harmonic flow.

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1. INTRODUCTION

In this paper we continue the study (see [42, 43]) of Ricci-harmonic flow

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)} + 4\nabla_{g(t)}u(t)\otimes\nabla_{g(t)}u(t), \quad \partial_t u(t) = \Delta_{g(t)}u(t) \tag{1.1}$$

on a complete n-dimensional manifold M, where $g(t)$ is a family of smooth Riemannian metrics on M and $u(t)$ is a family of smooth functions on M. The short time existence established by List [46, 47] and later extended by Müller [49, 51], says that, given an initial data (g_0, u_0) with g_0 and u_0 being, respectively, a smooth Riemannian metric and a smooth function on M, the system (1.1) exists over a maximal time interval $[0, T_{\max})$, where $T_{\max}$ is a finite number or infinity.

In the following we consider the Ricci-harmonic flow on $M\times[0,T]$ or on $M\times[0,T)$, depending on different situations, with $T\in(0,T_{\max})$. We brief our main results below.

Convention: From now on we always omit the time variable t and write $\Box, \Delta, \nabla, u, \mathrm{Rm}, \mathrm{Ric}, R, dV_t, |\cdot|$ for $\Box_{g(t)} := \partial_t - \Delta_{g(t)}, \Delta_{g(t)}, \nabla_{g(t)}, u(t), \mathrm{Rm}_{g(t)}, \mathrm{Ric}_{g(t)}, R_{g(t)}, dV_{g(t)}, |\cdot|_{g(t)}$ in the concrete computations, respectively. We write $\mathcal{P}\lesssim\mathcal{Q}$ or $\mathcal{Q}\gtrsim\mathcal{P}$ for two quantities $\mathcal{P}$ and $\mathcal{Q}$, if $\mathcal{P}\le C\,\mathcal{Q}$ for some uniform constant[1] C depending only on g_0, u_0, and the dimension n. The uniform constants C may vary from lines to lines.

(O) Some known results. It is known that the Ricci-harmonic flow shares the many properties with the Ricci flow. Here we list some results both for the Ricci-harmonic flow and the Ricci flow, see Table 1. Besides these results, there are other works on the Ricci-harmonic flow including gradient estimates, eigenvalues, entropies, functionals, and solitons, etc., see [8, 13, 18, 19, 20, 21, 43, 44, 50].

In particular, we mention a result on the gradient estimates of $u(t)$. Under the curvature condition

$$\sup_{M\times[0,T]}|\mathrm{Ric}_{g(t)}|_{g(t)}\le K, \tag{1.2}$$

Theorem B.2 shows that

$$|\nabla_{g(t)}u(t)|^2_{g(t)}\lesssim K \tag{1.3}$$

on $M\times[0,T]$, where $\lesssim$ depends only on n. Moreover, under a stronger condition

$$\sup_{M\times[0,T]}|\mathrm{Rm}_{g(t)}|_{g(t)}\le K, \tag{1.4}$$

[1] Given a flow on $M\times[0,T]$ or $M\times[0,T)$, a uniform constant C in this paper means a positive constant depending only on the initial data of the flow, M, and T. Of course, when T varies, C varies too.

TABLE 1. Ricci-harmonic flow (RHF) via Ricci flow (RH)

Known properties for $T_{\max} < \infty$	RF	RHF
$\lim_{t\to T_{\max}} \max_M \lvert \mathrm{Rm}_{g(t)}\rvert^2_{g(t)} = \infty$	[22]	[46][1]
$\lim_{t\to T_{\max}} \max_M \lvert \mathrm{Ric}_{g(t)}\rvert^2_{g(t)} = \infty$	[54][2]	[13]
either $\lim_{t\to T_{\max}} \max_{R_{g(t)}} = \infty$ *or* $\max_M R_{g(t)} \lesssim 1$ and $\lim_{t\to T_{\max}} \frac{\lvert W_{g(t)}\rvert_{g(t)} + \lvert \nabla^2_{g(t)} u(t)\rvert_{g(t)}}{R_{g(t)}} = \infty$	[7]	[43][3]
$T < \infty$, $n = 4$, $\lvert R_{g(t)}\rvert_{g(t)} \lesssim 1 \Rightarrow \int_M \lvert \mathrm{Ric}_{g(t)}\rvert^2_{g(t)} dV_{g(t)} \lesssim 1$ and $\int_M \lvert \mathrm{Rm}_{g(t)}\rvert^2_{g(t)} dV_{g(t)} \lesssim 1$	[4, 55]	[43]
Conjecture : $\lim_{t\to T_{\max}} \max_M R_{g(t)} = \infty$	see [7][4]	[44]
Pseudo-locality theorem	[52]	[18]

[1] For the general Ricci-harmonic flow, this result was proved by Muller [49].
[2] Recently, a new and elementary proof was given by Kotschwer, Munteaun, and Wang [32].
[3] Here $W_{g(t)}$ is the Weyl tensor of $g(t)$, and for RF we let $u(t) \equiv 0$. According to the evolution equation for $R_{g(t)}$ (see (A.8) – (A.9)) and the maximum principle, we can assume that $R_{g(t)} > 0$ for all t.
[4] This conjecture is due to Hamilton and was verified for the Kähler-Ricci flow by Zhang [72] and the Type-I Ricci flow by Enders, Müller and Topping [17].

Theorem B.2 also gives

$$|\nabla^2_{g(t)} u(t)|^2_{g(t)} \lesssim K \tag{1.5}$$

on $M \times [0, T]$, where $\lesssim$ depends only on n.

However, Cheng and Zhu [13] proved that under the condition (1.2) the Riemannian curvature remains bounded[2] and hence the Hessian $\nabla^2_{g(t)} u(t)$ is bounded along the flow.

(A) Gradient estimates under the condition (1.2). In this section, we assume that M is *closed*. It is clear from (A.8) that the gradient of $u(t)$ along the flow (1.1) is

[2] Their bound is implicit, however, following the argument of [32], we can give an explicit bound.

bounded[3], without any curvature condition, in terms of an initial data. According to Lemma A.1, integrating (A.8) over $M \times [0, T]$, one can prove that the total space-time L^2-norm of $\nabla^2_{g(t)} u(t)$ is bounded [4], i.e.,

$$\int_0^T \int_M \left| \nabla^2_{g(t)} u(t) \right|^2_{g(t)} dV_t dt \lesssim e^T$$

for any $T < T_{\max}$, without any curvature conditions such like (1.2) or (1.4). For fixed $T \in (0, T_{\max})$, one can get an upper bound, without any curvature conditions, for $\nabla^2_{g(t)} u(t)$ and then $\Delta_{g(t)} u(t)$ on $M \times [0, T]$, however this bound may depend on time T and even tends towards to infinity when $T \to T_{\max}$.

The first result in this paper is to obtain the time-independent (i.e., depends on $T_{\max}$) bound of $\Delta_{g(t)} u(t)$ under the curvature condition (1.2). We have mentioned that this time-independent bound has been implicitly obtained in [13], and our contribution is to obtain an *explicit* bound of $\Delta_{g(t)} u(t)$. Under the condition (1.2), one can prove (see Proposition 2.2)

$$\int_M |\Delta_{g(t)} u(t)|^2 dV_{g(t)} \leq C(1+K) e^{C(1+K)T} \leq C(1+K) e^{C(1+K)T_{\max}}, \ t \in [0, T], \tag{1.6}$$

for some uniform constant C. This L^2-norm bound together with the non-collapsing result (see Corollary 2.5) implies that

$$\begin{aligned} \left| \Delta_{g(t)} u(t) \right| &\leq \frac{C(1+K)}{(1+T)^{n/2}} \exp \left[C \left(1 + T + KT + e^{C\sqrt{K}} \right) \right] \\ &\leq C(1+K) \exp \left[C \left(1 + T_{\max} + KT_{\max} + e^{C\sqrt{K}} \right) \right] \end{aligned} \tag{1.7}$$

over $M \times [0, T]$, where C is a uniform constant.

Theorem 1.1. (See also Theorem 2.6) *If the Ricci-harmonic flow (1.1) satisfies the curvature condition (1.2), then $\Delta_{g(t)} u(t)$ is bounded and an explicit bound is given by (1.7).*

(B) Results for a generalized Ricci flow. The author [43] introduced a class of generalized Ricci flow (For motivation see **Section** 3), called $(\alpha_1, 0, \beta_1, \beta_2)$-*Ricci flow*:

$$\partial_t g(t) = -2 \mathrm{Ric}_{g(t)} + 2\alpha_1 \nabla_{g(t)} u(t) \otimes \nabla_{g(t)} u(t), \tag{1.8}$$

$$\partial_t u(t) = \Delta_{g(t)} u(t) + \beta_1 |\nabla_{g(t)} u(t)|^2_{g(t)} + \beta_2 u(t). \tag{1.9}$$

Here $\alpha_1, \beta_1, \beta_2$ are given constants. Under a technical condition "*regular*" (for definition, see subsection 3.1) the system (1.8) – (1.9) has the following estimate:

$$|\nabla_{g(t)} u(t)|_{g(t)} \leq C \tag{1.10}$$

for some uniform constant C depending only on $\alpha_1, \beta_1, \beta_2, n$. Thus, roughly speaking, the regular condition on constants $\alpha_1, \beta_1, \beta_2$ guarantees the boundedness of $\nabla_{g(t)} u(t)$.

[3] When M is complete and noncompact, the same result is still true for List's solution of the Ricci-harmonic flow, see Theorem A.3 and Theorem B.1.

[4] A general estimate is obtained in (3.28), where a generalized Ricci flow is considered.

The flow (1.8) – (1.9) is connected with the (K, N)-super Ricci flow introduced in [35], along which the W-entropy is constant. For more detail, see **Section** 3.

Theorem 1.2. (See also Theorem 3.2) *Let $(g(t), u(t))_{t\in[0,T)}$ be a solution to the regular $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flow (1.8) – (1.9) on a closed n-dimensional manifold M with $T \le \infty$ and the initial data (g_0, u_0). Assume that $S_{g(t)} + C \ge C_0 > 0$ along the flow for some uniform constants $C, C_0 > 0$. Then*

$$\frac{|\mathrm{Sin}_{g(t)}|_{g(t)}}{S_{g(t)} + C} \le C_1 + C_2 \max_{M\times[0,t]} \sqrt{\frac{|W_{g(s)}|_{g(s)} + |\nabla^2_{g(s)} u(s)|^2_{g(s)}}{S_{g(s)} + C}} \tag{1.11}$$

where

$$\begin{aligned} \mathrm{Sic}_{g(t)} &= \mathrm{Ric}_{g(t)} - \alpha_1 \nabla_{g(t)} u(t) \otimes \nabla_{g(t)} u(t), \\ S_{g(t)} &= \mathrm{tr}_{g(t)} \mathrm{Sic}_{g(t)} \ = \ R_{g(t)} - \alpha_1 |\nabla_{g(t)} u(t)|^2_{g(t)}, \\ \mathrm{Sin}_{g(t)} &:= \mathrm{Sic}_{g(t)} - \frac{S_{g(t)}}{n} g(t) \end{aligned}$$

and $W_{g(t)}$ is the Weyl tensor field of $g(t)$.

Here the technical assumption $S_{g(t)} + C \ge C_0 > 0$ is necessary in the theorem, since, due to the undermined signs of $\alpha_1, \beta_1, \beta_2$, we can not in general deduce any bounds for $S_{g(t)}$ from the evolution equations of (1.8) – (1.9) (see, for example, the evolution equation (3.11)). In the simplest case, $\alpha_1 \ge 0$ and $\beta_1 = \beta_2 = 0$ (i.e., Ricci-harmonic flow), we have a lower bound from Lemma A.1.

This result is obtained by applying Hamilton-Cao's method (see for example [7, 43]) on (1.8) – (1.9). More precisely, we consider the quantity

$$f_1 := \frac{|\mathrm{Sic}_{g(t)} + \frac{C}{n} g(t)|^2_{g(t)}}{(S_{g(t)} + C)^2}, \tag{1.12}$$

and deduce the following evolution inequality

$$\begin{aligned} \Box f &\le 2\langle \nabla f, \nabla \ln(S + C)\rangle \\ &\quad + 4(S + C) f \left(-f + \frac{2}{n-2} f^{1/2} + C + C\frac{|W|^2 + |\nabla^2 u|^2}{S + C} \right). \end{aligned} \tag{1.13}$$

A direct consequence of the maximum principle applied on (1.13) yields (1.11). In conclusion, we can derive the long-time existence of (1.8) – (1.9).

Corollary 1.3. (See also Corollary 3.3) *Let $(g(t), u(t))_{t\in[0,T)}$ be a solution to the regular $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flow (1.8) – (1.9) on a closed n-dimensional manifold M with $T \le \infty$ and the initial data (g_0, u_0). Then only one of the followings cases occurs:*

(a) *$T = \infty$;*
(b) *$T < \infty$ and $|\mathrm{Ric}_{g(t)}|_{g(t)} \le C$ for some uniform constant C;*
(c) *$T < \infty$ and $|\mathrm{Ric}_{g(t)}|_{g(t)} \to \infty$ as $t \to T$. In this case, there are only two subcases:*
   (c1) *$|R_{g(t)}|_{g(t)} \to \infty$,*
   (c2) *$|R_{g(t)}|_{g(t)} \le C$ for some uniform constant C and there exist some uniform constants $C_1, C_2 > 0$ such that $S_{g(t)} + C_1 \ge C_2 > 0$ and*

$$\frac{|W_{g(t)}|_{g(t)} + |\nabla^2_{g(t)} u(t)|^2_{g(t)}}{S_{g(t)} + C_1} \to \infty$$

as $t \to T$.

This is a general picture of the long time existence for regular $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flows, containing results in [7, 43]. Since the signs of $\alpha_1, \beta_1, \beta_2$ are not determined, we can not discard the case (b) which is of course true for Ricci-harmonic flow and Ricci flow.

In the case of dimension $n = 4$, we can consider another quantity

$$f_2 := \frac{|\mathrm{Sic}_{g(t)}|^2_{g(t)}}{S_{g(t)} + C}. \tag{1.14}$$

Since the sign of α_1 is undermined, some term in the evolution inequality (see (3.37)) for f_2 will contain the L^4-norm of $\nabla^2_{g(t)} u(t)$. Though we can prove

$$\left|\left|\nabla^2_{g(t)} u(t)\right|\right|_{L^1_{[0,T]} L^2(M)} \le C e^{CT} \le C e^{CT_{\max}} \tag{1.15}$$

for any $T \in (0, T_{\max})$, it is not clear whether we can get its (spatial) L^4-norm[5].

Corollary 1.4. (See also Corollary 3.6) *Let $(g(t), u(t))_{t\in[0,T)}$ be a solution to the regular $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flow (1.8) – (1.9) on a closed 4-manifold M with $T \le \infty$ and the initial data (g_0, u_0). Assume that $S_{g(t)} + C \ge C_0 > 0$ and $S^2_{g(t)} \le C_1 < \infty$ along the flow for some uniform constants $C, C_0, C_1 > 0$. Then*

$$\begin{aligned}\int_0^s \int_M |\mathrm{Sic}_{g(t)}|^2_{g(t)} dV_{g(t)} dt &\le C'(1+s)e^{C's} + \\ &\quad C'[1 - \mathrm{sgn}(\alpha_1, 0)]e^{C's} \int_0^s \int_M |\nabla^2_{g(t)} u(t)|^4 dV_{g(t)} dt,\end{aligned} \tag{1.16}$$

$$\begin{aligned}\int_0^s \int_M |\mathrm{Rm}_{g(t)}|^2_{g(t)} dV_{g(t)} dt &\le C'(1+s)e^{C's} + \\ &\quad C'[1 - \mathrm{sgn}(\alpha_1, 0)]e^{C's} \int_0^s \int_M |\nabla^2_{g(t)} u(t)|^4 dV_{g(t)} dt,\end{aligned} \tag{1.17}$$

for all $s \in [0, T)$, where $C' = C'(g_0, u_0, \alpha_1, \beta_1, \beta_2, C, C_0, C_1, A_1, \chi(M))$ is a uniform constant. Here $|\nabla_{g(t)} u(t)|^2_{g(t)} \le A_1$ holds along the flow (by the regularity) for some uniform constant $A_1 > 0$ (which depends only on g_0, u_0 and $\alpha_0, \beta_1, \beta_2$).

In the corollary, the shorthand notion $\mathrm{sgn}(\alpha_1, 0)$ is defined to be 1 if $\alpha_1 \ge 0$, and 0 otherwise. When α_1 is nonnegative, the inequalities (1.16) and (1.17) give us uniform $L^1_{[0,T)} L^2(M)$-norms to $\mathrm{Sic}_{g(t)}$ and $\mathrm{Rm}_{g(t)}$. Moreover, using an equality (above (3.43)) for Vol_t and $|\nabla_{g(t)} u(t)|^2_{g(t)} \le A_1$, the integral of $\mathrm{Sic}_{g(t)}$ can be replaced by the integral of $\mathrm{Ric}_{g(t)}$. Thus, in the case that $\alpha_1 \ge 0$, Corollary 1.4 gives uniform $L^1_{[0,T)} L^2(M)$-norms to $\mathrm{Ric}_{g(t)}$ and $\mathrm{Rm}_{g(t)}$.

To derive $L^1_{[0,T)} L^p(M)$-norms for $\mathrm{Sic}_{g(t)}$ and $\mathrm{Rm}_{g(t)}$, for simplicity, introduce the basic assumption **BA**, original defined in [4, 55] (see also [43]), for a solution $(g(t), u(t))_{t\in[0,T)}$ to the regular $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flow:

[5]Under the curvature condition (1.4), we can always get a bound for the L^4-norm of $\nabla^2_{g(t)} u(t)$ by Theorem B.2.

(a) M is a closed 4-manifold;
(b) $T < \infty$;
(c) $-1 \le S_{g(t)} \le 1$ along the flow;
(d) $|\nabla_{g(t)} u(t)|^2_{g(t)} \le A_1$ along the flow.

The last condition is obtained from the regularity of the flow and the third condition implies $S_{g(t)} + C \ge C_0 > 0$, where $C = 2$ and $C_0 = 1$.

Theorem 1.5. (See also Theorem 3.7) *Suppose that* $(g(t), u(t))_{t\in[0,T)}$ *satisfies* **BA**. *Then*

$$\begin{aligned}\int_0^s \int_M |\mathrm{Sic}_{g(t)}|^4_{g(t)} dV_{g(t)} dt &\le \widetilde{C}(1+s)e^{\widetilde{C}s}\\ &+ \widetilde{C}[1-\mathrm{sgn}(\alpha_1,0)]e^{\widetilde{C}s} \int_0^s \int_M |\nabla^2_{g(t)} u(t)|^4 dV_{g(t)} dt, \end{aligned} \tag{1.18}$$

$$\begin{aligned}\int_s^T \int_M |\mathrm{Sic}_{g(t)}|^2_{g(t)} dV_{g(t)} dt &\le \left[(T-s)e^T \mathrm{Vol}_0\right]^{\frac{4-p}{4}} e^{\widetilde{C}T} \Big[\widetilde{C}(1+T)\\ &+ \widetilde{C}[1-\mathrm{sgn}(\alpha_1,0)] \int_0^s \int_M |\nabla^2_{g(t)} u(t)|^4 dV_{g(t)} dt\Big]^{\frac{p}{4}}, \end{aligned} \tag{1.19}$$

for any $s \in [0,T)$. *Here* $\widetilde{C}$ *is a uniform constant which depends only on* $g_0, u_0, \alpha_1, \beta_1, \beta_2, A_1, \chi(M)$.

(C) A conjecture for the Einstein scalar field equations. This conjecture is based on the following, where M is a complete manifold,

Theorem 1.6. (See also Theorem 2.7) *Let* $(g(t), u(t))_{t\in[0,T]}$ *be a solution to the Ricci-haronic flow (1.1). Suppose there exist constants* $\rho, K, L, P > 0$ *and* $x_0 \in M$ *such that* $B_{g(0)}(x_0, \rho/\sqrt{K})$ *is compactly contained on M and the following conditions*[6]

$$|\mathrm{Ric}_{g(t)}|_{g(t)} \le K, \quad |\nabla_{g(t)} u(t)|_{g(t)} \le L, \quad |\nabla^2_{g(t)} u(t)|_{g(t)} \le P \tag{1.20}$$

hold on $B_{g(0)}(x_0, \rho/\sqrt{K}) \times [0,T]$. *For any* $p \ge 3$, *there is a constant C, depending only on n and p, such that*

$$\begin{aligned}\int_{B_{g(0)}(x_0,\rho/2\sqrt{K})} |\mathrm{Rm}_{g(t)}|^p_{g(t)} dV_{g(t)} &\le C\Lambda_1 e^{C\Lambda_2 T} \int_{B_{g(0)}(x_0,\rho/\sqrt{K})} |\mathrm{Rm}_{g(0)}|^p_{g(0)} dV_{g(0)}\\ &+ CK^p\left(1+\rho^{-2p}\right) e^{C\Lambda_2 T} \mathrm{Vol}_{g(t)}\left(B_{g(0)}\left(x_0, \frac{\rho}{\sqrt{K}}\right)\right). \end{aligned} \tag{1.21}$$

Here $\Lambda_1 := 1 + K$ *and* $\Lambda_2 := K + L + L^2 + P^2(1 + K^{-1})$.

This theorem extends [32] to the Ricci-harmonic flow. Along the argument in [32] we consider the quantity

$$\frac{d}{dt}\int_M |\mathrm{Rm}|^p \phi^{2p} dV_t$$

[6]The second assumption $|\nabla_{g(t)} u(t)|_{g(t)} \le L$ can not be derived by Theorem B.1. Actually, by Theorem B.1, one has the estimate

$$|\nabla_{g(t)} u(t)|_{g(t)} \le \rho^2 C_n \left(\frac{K}{\rho^2} + \frac{1}{t}\right) = C_n\left(K + \frac{\rho^2}{t}\right)$$

which depends on time t.

where ϕ is a Lipschitz function with support in $B_{g(0)}(x_0, \rho/\sqrt{K})$. For the Ricci-harmonic flow, extra integrals involving derivatives of $u(t)$ appear in the computation, however, these integrals can be treated with the help of the last assumptions in (1.20).

By Hölder's inequality we can get a similar upper bound for $p = 2$. From Theorem 1.6, we can get an upper bound for the L^2-norm of $\mathrm{Rm}_{g(t)}$. Motivated by the inequality (1.21), we in this section impose a conjecture for the Einstein scalar field equations, which is analogous to the corresponding conjecture for the Einstein vacuum equations proved by Klainerman, Rodnianski, and Szeftel [26, 56, 57, 58, 59, 60].

Consider Einstein's scalar field equation or Einstein Klein-Gordon system

$$\boldsymbol{R}_{\alpha\beta} - \frac{1}{2}\boldsymbol{R}\mathbf{g}_{\alpha\beta} = \boldsymbol{T}_{\alpha\beta}, \quad \boldsymbol{T}_{\alpha\beta} = 2\partial_\alpha u \partial_\beta u - \frac{1}{2}|\mathbf{D}u|^2\mathbf{g}, \tag{1.22}$$

where u is a smooth function on a four dimensional Lorentzian space-time $(\mathbf{M}, \mathbf{g})$, $\boldsymbol{R}_{\alpha\beta}$, $\boldsymbol{R}$, and $\mathbf{D}$ denote, respectively, the Ricci curvature tensor, scalar curvature, and the Levi-Civita connection of $\mathbf{g}$. In this case, the Einstein equation (1.15) can be written as

$$\boldsymbol{R}_{\alpha\beta} - 2\partial_\alpha u \partial_\beta u = 0.$$

As discussed in [53], we should impose a matter equation

$$\boldsymbol{\Delta} u = 0$$

for u, where $\boldsymbol{\Delta} := \mathbf{D}^\alpha \mathbf{D}_\alpha$. Hence we should consider a system of PDEs

$$\boldsymbol{R}_{\alpha\beta} - 2\partial_\alpha u \partial_\beta u = 0, \quad \boldsymbol{\Delta} u = 0. \tag{1.23}$$

An initial data set (Σ, g, k, u_0, u_1) for (1.23) consists of a three dimensional manifold Σ, a Riemannian metric g, a symmetric 2-tensor k, together with two functions u_0 and u_1 on Σ, all assumed to be smooth, verifying the constraint equations,

$$\nabla^j k_{ij} - \nabla_i \mathrm{tr} k = u_1 \nabla_i u_0, \tag{1.24}$$

$$R - |k|^2 + (\mathrm{tr} k)^2 = u_1^2 + |\nabla u_0|^2, \tag{1.25}$$

where ∇ is the Levi-Civita connection of g.

Given an initial data set (Σ, g, k, u_0, u_1), the Cauchy problem consists in finding a four-dimensional Lorentzian manifold $(\mathbf{M}, \mathbf{g})$ and a smooth function u on $\mathbf{M}$ satisfying (1.22), and also an embedding $\iota : \Sigma \to \mathbf{M}$ such that

$$\iota^* \mathbf{g} = g, \quad \iota^* u = u_0, \quad \iota^* K = k, \quad \iota^*(\boldsymbol{N} u) = u_1, \tag{1.26}$$

where $\boldsymbol{N}$ is the future-directed unit normal to $\iota(\Sigma)$ and K is the second fundamental form of $\iota(\Sigma)$. The local existence and uniqueness result for globally hyperbolic developments can be found in, for example, [53], Theorem 14.2. For stability and instability for Einstein's scalar field equation, we refer to [14, 15, 33, 34, 62, 63, 64, 65].

For Einstein's equations (i.e., $u = 0$ in (1.22), and the corresponding initial data set is denoted by (Σ, g, k)), Klainerman [25] proposed the following conjecture:

> *The Einstein vacuum equations admit local Cauchy developments for initial data sets (Σ, g, k) with locally finite L^2-curvature and locally finite L^2-norm of the first covariant derivatives of k.*

This conjecture was recently solved by Klainerman, Rodnianski and Szeftel [26]. Motivated by Theorem 1.6 and Klainerman's conjecture, we propose the following

Conjecture 1.7. (See also Conjecture 4.2) *The Einstein scalar field equations admit local Cauchy developments for initial data sets (Σ, g, k, u_0, u_1) with locally finite L^2-curvature, locally finite L^2-norm of the first covariant derivatives of k, locally finite L^2-norm of the covariant derivatives (up to second order) of u_0, and locally finite L^2-norm of the covariant derivatives (up to first order) of u_1.*

(D) Some notions on Riemann curvatures of Bakry-Émery Ricci curvature. We now compare our curvature Sm with $\alpha_1 = 2$ (see (3.10)) with a notion of curvature introduced recently by Wylie and Yeroshkin [68]. Let (M, g) be a Riemannian manifold with a smooth function u. Wylie and Yeroshkin introduced the following weighted connection

$$\nabla^u_X Y := \nabla_X Y - (Yu)X - (Xu)Y. \tag{1.27}$$

By Proposition 3.3 in [68], we have

$$R^u_{ijk\ell} = R_{ijk\ell} + \nabla_j\nabla_k u g_{i\ell} - \nabla_i\nabla_k u g_{j\ell} + \nabla_j u\nabla_k u g_{i\ell} - \nabla_i u\nabla_k u g_{j\ell}, \tag{1.28}$$

where $R^u_{ijk\ell} := \langle \mathrm{Rm}^\alpha(\partial_i, \partial_j)\partial_k, \partial_\ell\rangle$ and Rm^α is the induced Riemann curvature tensor associated to the connection ∇^u. The Ricci curvature associated to ∇^u is defined by

$$R^u_{jk} := g^{i\ell}R^u_{ijk\ell} = R_{ik} + (n-1)\nabla_j\nabla_k u + (n-1)\nabla_j u\nabla_k u. \tag{1.29}$$

Here the last formula also follows from Proposition 3.3 in [68]. Recall from (3.10) that (with $\alpha_1 = 2$)

$$S_{ijk\ell} = R_{ijk\ell} - \nabla_i u\nabla_k u g_{j\ell} - \nabla_i u\nabla_j u g_{k\ell}. \tag{1.30}$$

From now on, we are given a smooth function u on M and write

$$\begin{array}{rclrcl} R^{\mathbf{L}}_{ijk\ell} & := & S_{ijk\ell}, & R^{\mathbf{WY}}_{ijk\ell} & := & R^u_{ijk\ell}, \\ R^{\mathbf{L}}_{jk} & := & g^{i\ell}R^{\mathbf{L}}_{ijk\ell}, & R^{\mathbf{WY}}_{jk} & := & R^u_{jk} \;=\; g^{i\ell}R^{\mathbf{WY}}_{ijk\ell}, \\ R^{\mathbf{L}} & := & g^{jk}R^{\mathbf{L}}_{jk}, & R^{\mathbf{WY}} & := & g^{jk}R^{\mathbf{WY}}_{jk}. \end{array} \tag{1.31}$$

From (1.29) and (1.30), we have

$$\mathrm{Ric}^{\mathbf{L}} = \mathrm{Ric} - 2du\otimes du, \quad \mathrm{Ric}^{\mathbf{WY}} = \mathrm{Ric} + (n-1)du\otimes du + (n-1)\nabla^2 u.$$

Consider another Ricci curvature of $R^{\mathbf{WY}}_{ijk\ell}$:

$$\widehat{R}^{\mathbf{WY}}_{jk} := g^{i\ell}R^{\mathbf{WY}}_{ji\ell k} = R_{jk} + \left(\Delta u + |\nabla u|^2\right)g_{jk} - \nabla_j\nabla_k u - \nabla_j u\nabla_k u.$$

There are some relations between those two notions on "Riemann curvature tensors, e.g.,

$$R^{\mathbf{WY}}_{ijk\ell} - R^{\mathbf{L}}_{ijk\ell} = \nabla_i u\nabla_j u g_{k\ell} + \nabla_k u\nabla_j u g_{i\ell} + \nabla_j\nabla_k u g_{i\ell} - \nabla_i\nabla_k u g_{j\ell}, \tag{1.32}$$

and

$$\widehat{\mathrm{Ric}}^{\mathbf{WY}} - \mathrm{Ric}^{\mathbf{WY}} = \left(\Delta u + |\nabla u|^2\right)g - n\left(\nabla^2 u + du\otimes du\right). \tag{1.33}$$

We note that $\mathrm{Ric}^{\mathbf{L}}$ and $\mathrm{Ric}^{\mathbf{WY}}$ are actually the Ricci curvatures in the sense of Bakey-Émery [1]. We here use our notions to keep the paper smoothly.

We now have four different types of Ricci curvatures, Ric, $\mathrm{Ric}^{\mathbf{L}}$, $\mathrm{Ric}^{\mathbf{WY}}$, and $\widehat{\mathrm{Ric}}^{\mathbf{WY}}$, and three different types of scalar curvatures, R, $R^{\mathbf{L}}$, and $R^{\mathbf{WY}}$. In order to compare those quantities, we introduce a notation $\mathcal{P} \leq_{\mathbf{I},\mu} \mathcal{Q}$, which is an integral inequality with respect to the measure μ.

Given two scalar quantities $\mathcal{P}$, $\mathcal{Q}$ on (M, g), and a measure μ, we write $\mathcal{P} \leq_{\mathbf{I},\mu} \mathcal{Q}$ if the following inequality

$$\int_M \mathcal{P}\, d\mu \leq \int_{\mathcal{M}} \mathcal{Q}\, d\mu \tag{1.34}$$

holds. When $d\mu$ is the volume form dV, we simply write (1.34) as $\mathcal{P} \leq_{\mathbf{I}} \mathcal{Q}$. When $d\mu$ is the measured volume form $e^f dV$, we write (1.34) as $\mathcal{P}_{\leq \mathbf{I},f}\mathcal{Q}$. Similarly, we can define $\mathcal{P}_{\mathbf{I},\mu}\mathcal{Q}$.

Proposition 1.8. (See also Proposition 5.4) *For any measure μ on M and smooth function u on M, we have*

$$R^{\mathbf{L}} \leq_{\mathbf{I},\mu} R, \quad R \leq_{\mathbf{I}} R^{\mathbf{WY}}, \quad R =_{\mathbf{I},u} R^{\mathbf{WY}}. \tag{1.35}$$

This proposition shows that $R^{\mathbf{L}} \leq_{\mathbf{I}} R \leq_{\mathbf{I}} R^{\mathbf{WY}}$ and $R^{\mathbf{L}} \leq_{\mathbf{I},u} R =_{\mathbf{I},u} R^{\mathbf{WY}}$. Thus, in the sense of integrals, $R_{\mathbf{L}}$ is weaker and $R^{\mathbf{WY}}$ is stronger than R, respectively.

Next we consider the similar question on Ricci curvatures. Let (M, g) be a closed Riemannian manifold with a smooth function u, and μ be a given measure on M. Given two Ricci curvatures $\mathrm{Ric}^{\clubsuit}, \mathrm{Ric}^{\diamondsuit} \in \mathfrak{Ric}_4 := \{\mathrm{Ric}, \mathrm{Ric}^{\mathbf{L}}, \mathrm{Ric}^{\mathbf{WY}}, \widehat{\mathrm{Ric}}^{\mathbf{WY}}\}$, we say

$$\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{I},\mu} \mathrm{Ric}^{\diamondsuit} \tag{1.36}$$

if $\mathrm{Ric}^{\clubsuit}(X, X) \leq_{\mathbf{I},\mu} \mathrm{Ric}^{\diamondsuit}(X, X)$ holds for all vector fields $X \in \mathfrak{X}(M)$. Similarly we can define $\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{I}} \mathrm{Ric}^{\diamondsuit}$ and $\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{I},f} \mathrm{Ric}^{\diamondsuit}$. We say

$$\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{IK},\mu} \mathrm{Ric}^{\diamondsuit} \tag{1.37}$$

if $\mathrm{Ric}^{\clubsuit}(X, X) \leq_{\mathbf{I},\mu} \mathrm{Ric}^{\diamondsuit}(X, X)$ holds for all Killing vector fields $X \in \mathfrak{X}_{\mathbf{K}}(M)$, where $\mathfrak{X}_{\mathbf{K}}(M)$ is the space of all Killing vector fields on M. Similarly we can define $\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{IK}} \mathrm{Ric}^{\diamondsuit}$ and $\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{IK},f} \mathrm{Ric}^{\diamondsuit}$.

Consider the subset $\mathfrak{X}_{\mathbf{KC}}(M)$ of $\mathfrak{X}_{\mathbf{K}}(M)$, which consists of Killing vector fields on M with constant norm. we say

$$\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{IKC},\mu} \mathrm{Ric}^{\diamondsuit} \tag{1.38}$$

if $\mathrm{Ric}^{\clubsuit}(X, X) \leq_{\mathbf{I},\mu} \mathrm{Ric}^{\diamondsuit}(X, X)$ holds for all $X \in \mathfrak{X}_{\mathbf{KC}}(M)$. Similarly we can define $\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{IKC}} \mathrm{Ric}^{\diamondsuit}$ and $\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{IKC},f} \mathrm{Ric}^{\diamondsuit}$. We then obtain the following two results.

Theorem 1.9. (see also Theorem 5.6) *Let (M, g) be a closed Riemannian manifold with a smooth function u and μ be a given measure on M. Then we have*

(i) $\mathrm{Ric}^{\mathbf{L}} \leq_{\mathbf{I},\mu} \mathrm{Ric}$.
(ii) $\mathrm{Ric} \leq_{\mathbf{IKC}} \mathrm{Ric}^{\mathbf{WY}}$.
(iii) $\mathrm{Ric} \leq_{\mathbf{IKC}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}$.

A consequence of Theorem 1.9 indicates

$$\mathrm{Ric}^{\mathbf{L}} \le_{\mathbf{IKC}} \mathrm{Ric} \le_{\mathbf{IKC}} \mathrm{Ric}^{\mathbf{WY}} \quad \text{and} \quad \mathrm{Ric}^{\mathbf{L}} \le_{\mathbf{IKC}} \mathrm{Ric} \le_{\mathbf{IKC}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}. \tag{1.39}$$

Theorem 1.10. (See also Theorem 5.8) *Let (M, g) be a closed Riemannian manifold with a smooth function u and μ be a given measure on M. Then we have*

(i) $\mathrm{Ric} \le_{\mathbf{IKC},\widetilde{f}} \mathrm{Ric}^{\mathbf{WY}}$, *and*

(ii) $\mathrm{Ric} \le_{\mathbf{IKC},\widetilde{f}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}$.

where $\widetilde{f} := u - u_{\min} + c_0$ and $c_0 \ge 1/e$.

For a given odd-dimensional sphere, we can always find a Riemannian metric g and a Killing vector field X of constant length with respect to g.

Proposition 1.11. (See also Proposition 5.9) *On each of 28 homotopical seven-dimensional spheres M, there exist a Riemannian metric g and a nonzero vector field X, such that*

- $\mathrm{Ric}^{\mathbf{L}}(X, X) \le_{\mathbf{I}} \mathrm{Ric}(X, X) \le_{\mathbf{I}} \mathrm{Ric}^{\mathbf{WY}}(X, X)$ *and* $\mathrm{Ric}^{\mathbf{L}}(X, X) \le_{\mathbf{I}} \mathrm{Ric}(X, X) \le_{\mathbf{I}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}(X, X)$ *hold.*
- *for any smooth function u on M,* $\mathrm{Ric}^{\mathbf{L}}(X, X) \le_{\mathbf{I},\widetilde{f}} \mathrm{Ric}(X, X) \le_{\mathbf{I},\widetilde{f}} \mathrm{Ric}^{\mathbf{WY}}(X, X)$ *and* $\mathrm{Ric}^{\mathbf{L}}(X, X) \le \mathrm{Ric}(X, X) \le_{\mathbf{I},\widetilde{f}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}(X, X)$ *hold, where $\tilde{f} := u - u_{\min} + c_0$ with $c_0 \ge 1/e$.*

We say that a Riemannian metric g on M is of *cohomogeneity 1* if some compact Lie group G acts smoothly and isometrically on M and the space of orbits M/G with respect to this action is one-dimensional.

Proposition 1.12. (See also Proposition 5.10) *Let $n \ge 2$ and $\epsilon > 0$. On the sphere $\mathbb{S}^{2n-1}$, there are a (real-analytic) Riemannian metric g_ϵ, of cohomogeneity 1, with the property that all section curvatures of g_ϵ differ from 1 at most by ϵ, and a (real-analytic) nonzero vector field X_ϵ, such that*

- $\mathrm{Ric}^{\mathbf{L}}_{g_\epsilon}(X_\epsilon, X_\epsilon) \le_{\mathbf{I}} \mathrm{Ric}_{g_\epsilon}(X_\epsilon, X_\epsilon) \le_{\mathbf{I}} \mathrm{Ric}^{\mathbf{WY}}_{g_\epsilon}(X_\epsilon, X_\epsilon)$ *and* $\mathrm{Ric}^{\mathbf{L}}_{g_\epsilon}(X_\epsilon, X_\epsilon) \le_{\mathbf{I}} \mathrm{Ric}_{g_\epsilon}(X_\epsilon, X_\epsilon) \le_{\mathbf{I}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}_{g_\epsilon}(X_\epsilon, X_\epsilon)$ *hold.*
- *for any smooth function u on M,* $\mathrm{Ric}^{\mathbf{L}}_{g_\epsilon}(X_\epsilon, X_\epsilon) \le_{\mathbf{I},\widetilde{f}} \mathrm{Ric}_{g_\epsilon}(X_\epsilon, X_\epsilon) \le_{\mathbf{I},\widetilde{f}} \mathrm{Ric}^{\mathbf{WY}}_{g_\epsilon}(X_\epsilon, X_\epsilon)$ *and* $\mathrm{Ric}^{\mathbf{L}}_{g_\epsilon}(X_\epsilon, X_\epsilon) \le \mathrm{Ric}_{g_\epsilon}(X_\epsilon, X_\epsilon) \le_{\mathbf{I},\widetilde{f}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}_{g_\epsilon}(X_\epsilon, X_\epsilon)$ *hold, where $\tilde{f} := u - u_{\min} + c_0$ with $c_0 \ge 1/e$.*

Berestovskii and Nikonorov [2] observed that (see Remark 5.11) that if the $\mathbf{S}^1$-action obtain by Tuschmann [61] is free, then we can find for $\epsilon > 0$ a Killing vector field X_ϵ of unit length with respect to g_ϵ.

The nonnegativity of $R^{\mathbf{L}}_{ijk\ell}$ was used in [66] to prove the compactness for gradient shrinking Ricci harmonic solitons. There is no useful relation between $\mathrm{Rm}^{\mathbf{L}}$ and $\mathrm{Rm}^{\mathbf{WY}}$. More precisely, we can find (see Example 5.12) a Riemannian manifold (M, g) so that $\mathrm{Rm}^{\mathbf{L}}(X, Y, Y, X) < \mathrm{Rm}^{\mathbf{WY}}(X, Y, Y, X)$ for some triple (X, Y, u) of smooth vector fields X, Y and smooth function u, and $\mathrm{Rm}^{\mathbf{L}}(X, Y, Y, X) > \mathrm{Rm}^{\mathbf{WY}}(X, Y, Y, X)$ for another such triple (X', Y', u').

(E) Uniqueness for the Ricci-harmonic flow. The uniqueness problem for the Ricci flow was proved by Hamilton [22] in the compact setting, Chen-Zhu [12] for forward uniqueness of complete solutions, and Kotschwar [28, 29] for forward and backward uniqueness of complete solutions.

It is a natural problem to prove the forward and backward uniqueness for the Ricci-harmonic flow. Actually, the forward uniqueness of the Ricci-harmonic flow when the underlying manifold is compact, was proved by List [46]. In this paper, we use the strategy of Kotschwar to prove the uniqueness for complete solutions of the Ricci-harmonic flow. List [46] has proved that (see Theorem A.3) if (M, g) is a complete and non-compact Riemannian manifold satisfying

$$|\mathrm{Rm}_g|_g + |u| + |\nabla_g u|_g + |\nabla^2_g u|_g \lesssim 1$$

then a local time existence holds for the Ricci-harmonic flow with the initial data (g, u), and moreover

$$|\mathrm{Rm}_{g(t)}|_{g(t)} + |u(t)| + |\nabla_{g(t)} u(t)|_{g(t)} + |\nabla^2_{g(t)} u(t)|_{g(t)} \lesssim 1.$$

Hence one may expect that the uniqueness of the Ricci-harmonic flow holds in the class $\{(g, u) : \mathrm{Rm}_g, \nabla_g u, \nabla^2_g u \text{ bounded}\}$. Surprisingly we prove that the uniqueness of the Ricci-harmonic flow holds in a larger class $\{(g, u) : \mathrm{Rm}_g \text{ bounded}\}$. The reason is that if the Riemann curvature is bounded along the flow (1.1) then all derivatives of $u(t)$ is still bounded by Theorem B.2. To state the results, consider the following curvature condition

$$\sup_{M\times[0,T]} \left(|\mathrm{Rm}_{g(t)}|_{g(t)} + |\mathrm{Rm}_{\tilde{g}(t)}|_{\tilde{g}(t)}\right) \leq K, \tag{1.40}$$

where K is some uniform constant.

Theorem 1.13. (See also Theorem 6.1) *Suppose that $(g(t), u(t))$ and $(\tilde{g}(t), \tilde{u}(t))$ are two smooth complete solutions of (1.1) satisfying (1.40). If $(g(0), u(0)) = (\tilde{g}(0), \tilde{u}(0))$, then $(g(t), u(t)) \equiv (\tilde{g}(t), \tilde{u}(t))$ for each $t \in [0, T]$.*

The Basic idea on proving Theorem 1.13 follows from the approach of Kotschwar [29] who considered the quantity

$$\begin{aligned} \mathcal{E}(t) \quad := \quad & \int_M \left[t^{-1}|g(t) - \tilde{g}(t)|^2_{g(t)} + t^{-\beta}\left|\Gamma_{g(t)} - \Gamma_{\tilde{g}(t)}\right|^2_{g(t)}\right. \\ & \left. + \left|\mathrm{Rm}_{g(t)} - \mathrm{Rm}_{\tilde{g}(t)}\right|^2_{g(t)}\right] e^{-\eta} dV_{g(t)} \end{aligned} \tag{1.41}$$

for the Ricci flow, where $\beta \in (0, 1)$ and η is a cutoff function (so that the integral is well-defined as t tends to zero). In our setting, the corresponding quantity for the Ricc-harmonic flow takes the form

$$\begin{aligned} \mathcal{E}(t) \quad := \quad & \int_M \left[t^{-1}|g(t) - \tilde{g}(t)|^2_{g(t)} + t^{-\beta}\left|\Gamma_{g(t)} - \Gamma_{\tilde{g}(t)}\right|^2_{g(t)} + \left|\mathrm{Rm}_{g(t)} - \mathrm{Rm}_{\tilde{g}(t)}\right|^2_{g(t)}\right. \\ & \left. + |u(t) - \tilde{u}(t)|^2_{g(t)} + \left|\nabla_{g(t)} u(t) - \nabla_{\tilde{g}(t)}\tilde{u}(t)\right|^2_{g(t)}\right] e^{-\eta} dV_{g(t)}. \end{aligned} \tag{1.42}$$

It can be showed

$$\mathcal{E}'(t) \leq N\mathcal{E}(t) \tag{1.43}$$

on $[0, T_0]$, for some $T_0 \ll 1$ and $N > 0$. From (1.43) together with the initial data $\mathcal{E}(0) = 0$, we get $\mathcal{E}(t) \equiv 0$ on $[0, T_0]$ and then on $[0, T]$.

Theorem 1.14. (See also Theorem 6.6) *Suppose that $(g(t), u(t))$ and $(\tilde{g}(t), \tilde{u}(t))$ are two smooth complete solutions of (1.1) satisfying (1.40). If $(g(T), u(T)) = (\tilde{g}(T), \tilde{u}(T))$, then $(g(t), u(t)) \equiv (\tilde{g}(t), \tilde{u}(t))$ for each $t \in [0, T]$.*

To prove Theorem 1.14, we use an idea of Kotschwar [28] and set

$$\begin{aligned} T &:= \mathrm{Rm} - \widetilde{\mathrm{Rm}}, \quad U := \nabla \mathrm{Rm} - \widetilde{\nabla}\widetilde{\mathrm{Rm}}, \\ h &:= g - \tilde{g}, \quad A := \nabla - \widetilde{\nabla}, \quad B := \nabla A \\ y &:= \nabla^2 u - \widetilde{\nabla}^2 \tilde{u}, \quad z := \nabla^3 u - \widetilde{\nabla}^3 \tilde{u}, \\ v &:= u - \tilde{u}, \quad w := \nabla u - \widetilde{\nabla}\tilde{u}, \quad x := \nabla w, \\ \mathbf{X} &:= T \oplus U \oplus y \oplus z, \quad \mathbf{Y} := h \oplus A \oplus B \oplus v \oplus w \oplus x. \end{aligned}$$

As the same argument of [28], we can prove

$$\begin{aligned} |\Box_{g(t)} \mathbf{X}|^2_{g(t)} &\lesssim |\mathbf{X}|^2_{g(t)} + |\mathbf{Y}|^2_{g(t)}, \\ |\partial_t \mathbf{Y}|^2_{g(t)} &\lesssim |\mathbf{X}|^2_{g(t)} + |\mathbf{Y}|^2_{g(t)} + |\nabla \mathbf{X}|^2_{g(t)} \end{aligned} \tag{1.44}$$

on any $[\delta, T]$, where $\delta \in (0, T)$. A result of Kotschwar (see Theorem 6.5 below) implies $\mathbf{X} = \mathbf{Y} \equiv 0$ on $[\delta, T]$, and then on $[0, T]$.

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2. Gradient and local curvature estimates

In this section we assume that $(g(t), u(t))_{[0,T]}$ is a solution of (1.1) on a closed n-dimensional manifold M and use the convention in **Section** 1. From the equation (A.8) in Lemma A.1, we see that $|\nabla u|^2$ is uniformly bounded, i.e.,

$$|\nabla u| \le L \tag{2.1}$$

on $M \times [0, T]$ for some uniform constant L depending only on the initial data $(g_0, u_0) = (g(0), u(0))$. We also notice from (A.9) that

$$R - 2|\nabla u|^2 \gtrsim -1 \Longrightarrow R \gtrsim -1. \tag{2.2}$$

Moreover, integrating over the space-time $M \times [0, T]$, we get

$$\begin{aligned} \frac{d}{dt} \int_M |\nabla u|^2 dV_t &= \int_M \partial_t |\nabla u|^2 dV_t + \int_M |\nabla u|^2 \partial_t dV_t \\ &= \int_M \left[-2|\nabla^2 u|^2 - 4|\nabla u|^4 - \left(R - 2|\nabla u|^2 \right) \right] dV_t \end{aligned} \tag{2.3}$$

and then

$$\frac{d}{dt} \int_M |\nabla u|^2 dV_t + 2 \int_M |\nabla^2 u|^2 dV_t = -4 \int_M |\nabla u|^4 dV_t - \int_M \left(R - 2|\nabla u|^2 \right) dV_t.$$

Denoting Vol_t the volume of $(M, g(t))$, we have

$$\int_0^t \int_M |\nabla^2 u|^2 dV_t dt \le L^2 \mathrm{Vol}_0 + C \int_0^t \mathrm{Vol}_s ds \tag{2.4}$$

from (2.1) and (2.2), where C is a uniform constant. Using (A.10) and (2.2), we have

$$\partial_t \mathrm{Vol}_t = \int_M \partial_t dV_t = \int_M \left(-R + 2|\nabla u|^2\right) dV_t \le C\mathrm{Vol}_t;$$

consequently,

$$\mathrm{Vol}_t \le e^{Ct}\mathrm{Vol}_0. \tag{2.5}$$

Plugging (2.5) into (2.4) we conclude that

$$\int_0^t \int_M |\nabla^2 u|^2 dV_t dt \le \left(L^2 + e^{Ct}\right) \mathrm{Vol}_0 \le C(1 + L^2)e^{Ct} \lesssim e^{Ct}. \tag{2.6}$$

According to (A.5) we obtain

$$\Box \Delta u = -4|\nabla u|^2 \Delta u + 2R_{ij}\nabla^i \nabla^j u - 4\nabla_i u \nabla_j u \nabla^i \nabla^j u. \tag{2.7}$$

In particular, the square of Δu satisfies

$$\begin{aligned}
\partial_t |\Delta u|^2 &= 2\Delta u \partial_t \Delta u = \Delta |\Delta u|^2 - 2|\nabla \Delta u|^2 - 8|\nabla u|^2 |\Delta u|^2 + 4\left(R_{ij}\nabla^i\nabla^j u\right)\Delta u \\
&\quad - 8\left(\nabla_i u \nabla_j u \nabla^i \nabla^j u\right)\Delta u \\
&\le \Delta |\Delta u|^2 - 2|\nabla \Delta u|^2 - 8|\nabla u|^2|\Delta u|^2 + 4\left(|\mathrm{Ric}| + 2|\nabla u|^2\right)|\nabla^2 u||\Delta u| \\
&\le \Delta|\Delta u|^2 + 2\left(|\mathrm{Ric}|^2 + 2|\nabla u|^2\right)|\Delta u|^2 + 2\left(|\mathrm{Ric}| + 2|\nabla u|^2\right)|\nabla^2 u|^2.
\end{aligned}$$

Taking integrations on both sides yields

$$\begin{aligned}
\frac{d}{dt}\int_M |\Delta u|^2 dV_t &= \int_M \partial_t |\Delta u|^2 dV_t + \int_M |\Delta u|^2 \left(-R + 2|\nabla u|^2\right) dV_t \\
&\le \int_M \left(2|\mathrm{Ric}| - R + 6|\nabla u|^2\right)|\Delta u|^2 dV_t + \int_M \left(2|\mathrm{Ric}| + 4|\nabla u|^2\right)|\nabla^2 u|^2 dV_t.
\end{aligned} \tag{2.8}$$

When the Ricci curvature is uniformly bounded, together with (2.6), we can prove that the L^2-norm of Δu is finite.

Proposition 2.1. *If the curvature condition (1.2) holds, then*

$$\int_M |\Delta u|^2 dV_t \le C(1 + K)e^{C(1+K)T} \tag{2.9}$$

for some uniform constant $C > 0$.

Proof. Compute, using (2.8) and (2.2),

$$\frac{d}{dt}\int_M |\Delta u|^2 dV_t \le \left(2K + C + 6L^2\right)\int_M |\Delta u|^2 dV_t + \left(2K + 4L^2\right)\int_M |\nabla^2 u|^2 dV_t.$$

Therefore

$$\frac{d}{dt}\left[e^{-(2K+C+6K^2)t}\int_M |\Delta u|^2 dV_t\right] \le \left(2K + 4L^2\right) e^{-(2K+C+6L^2)t}\int_M |\nabla^2 u|^2 dV_t.$$

From (2.6), we obtain

$$\begin{aligned}\int_M |\Delta u|^2 dV_t &\leq Ce^{(2K+C+6L^2)t} + \left(2K+4L^2\right)\int_0^t\int_M |\nabla^2 u|^2 dt \\ &\leq Ce^{(2K+C+6L^2)t} + C\left(2K+4L^2\right)\left(1+L^2\right)e^{Ct}\end{aligned}$$

which implies (2.9). □

2.1. **The boundedness of $\Delta_{g(t)}u(t)$.** According to [13], Chen and Zhu proved an analog of Sesum's theorem for the Ricci-harmonic flow. As a consequence, we see that $\Delta_{g(t)}u(t)$ is uniformly bounded. Our contribution in this paper is to give an *explicit* bound for $\Delta_{g(t)}u(t)$.

We first review a non-collapsing theorem for the Ricci-harmonic flow. Suppose that M is a closed manifold. For any Riemannian metric g, any smooth functions u, f, and any positive number τ, define (see [46])

$$\mathcal{W}(g,u,f,\tau) := \int_M \left[\tau\left(S_g + |\nabla_g f|_g^2\right) + f - n\right]\frac{e^{-f}}{(4\pi\tau)^{n/2}}dV_g \tag{2.10}$$

with $S_g := R_g - 2|\nabla_g u|_g^2$, and

$$\mu(g,u,\tau) := \inf\left\{\mathcal{W}(g,u,f,\tau) : f \in C^\infty(M) \text{ and } \int_M \frac{e^{-f}}{(4\pi\tau)^{n/2}}dV_g = 1\right\}. \tag{2.11}$$

Observe that

$$\mu(\tau g, u, \tau) = \mu(g,u,1), \quad \tau > 0. \tag{2.12}$$

Proposition 2.2. *If $(g(t), u(t), \tau(t))_{t\in[0,T]}$ solves*

$$\begin{aligned}\partial_t g(t) &= -2\mathrm{Ric}_{g(t)} + 4\nabla_{g(t)}u(t)\otimes\nabla_{g(t)}u(t), \\ \partial_g u(t) &= \Delta_{g(t)}u(t), \\ \frac{d}{dt}\tau(t) &= -1,\end{aligned} \tag{2.13}$$

then $\mu(g(t), u(t), \tau(t))$ is monotone nondecreasing in time t.

Proof. See [46, 47, 49, 51]. □

In the definition (2.10), introduce the function

$$w := \left[\frac{e^{-f}}{(4\pi\tau)^{n/2}}\right]^{1/2} \tag{2.14}$$

so that we can rewrite the functional $\mathcal{W}$ as

$$\begin{aligned}\mathcal{W}(g,u,f,\tau) &= \int_M \left[\tau\left(w^2 S_g + 4|\nabla_g w|_g^2\right) - \left(2\ln w + \frac{n}{2}\ln(4\pi\tau) + n\right)w^2\right]dV_g \\ &= \tau\int_M S_g w^2 dV_g - \left[\frac{n}{2}\ln(4\pi\tau) + n\right]\int_M w^2 dV_g \\ &\quad - 2\left[\int_M w^2 \ln w\, dV_g - 2\tau|\nabla_g w|_g^2 dV_g\right].\end{aligned} \tag{2.15}$$

The last term in (2.15) can be handed by the logarithmic Sobolev inequality (see [10], Lemma 17.1): for any $a > 0$ there exists a constant $C(a, g)$ such that if $\varphi > 0$ satisfies $\int_M \varphi^2 dV_g = 1$, then

$$\int_M \varphi^2 \ln \varphi \, dV_g - a \int_M |\nabla_g \varphi|_g^2 dV_g \leq C(a, g), \tag{2.16}$$

where

$$C(a, g) = a\mathrm{Vol}(M, g)^{-2/n} + \frac{n^2}{4ae^2 C_s(M, g)} \tag{2.17}$$

and $C_s(M, g)$ denotes the L^2-Sobolev constant.

Lemma 2.3. *Let M be a closed n-dimensional manifold. For any Riemannian metric g, any smooth function u, and any $\tau > 0$, we have*

$$\mu(g, u, \tau) \geq \tau S_{g,\min} - 2C(2\tau, g) - \frac{n}{2}\ln(4\pi\tau) - n, \tag{2.18}$$

$$\mu(g, u, \tau) \leq \tau S_{g,\mathrm{avg}} + \ln \mathrm{Vol}(M, g) - \frac{n}{2}\ln(4\pi\tau) - n. \tag{2.19}$$

Here $S_{g,\min} := \min_M S_g$ and $S_{g,\mathrm{avg}}$ denotes the average of S_g over (M, g).

Proof. Taking $f = \ln \mathrm{Vol}(M, g) - \frac{n}{2}\ln(4\pi\tau)$ in (2.11) gives the first inequality. The second estimate follows from (2.15) and (2.16). □

Lemma 2.4. *For every $n \geq 2$, $\rho \in (0, \infty)$, and $D > 0$, there exists $c = c(n, \rho, D) > 0$ such that if (M, g) is a closed n-dimensional Riemannian manifold, u is a smooth function on M, and if for some $r \in (0, \rho]$ and $A < \infty$ we have $\mu(g, u, r^2) > -A$, then for any $p \in M$ with $\mathrm{Ric}_g \geq -Dr^{-2}$ on $B_g(p, r)$ and $R_g \leq Dr^{-2}$ on $B_g(p, r)$, we have*

$$\mathrm{Vol}_g(B_g(p, r)) \geq \kappa r^n \tag{2.20}$$

where $\kappa := ce^{-A}$.

Proof. Since $S_g = R_g - 2|\nabla_g u|_g^2 \leq R_g$, the proof is almost the same as the proof of Proposition 5.37 in [11]. □

Actually, the constant c in Lemma 2.4 can be explicitly determined. We can take the constant c in such a way that it depends only on n and C. From the proof of Proposition 5.37 in [11], we have

$$\mu(g, u, r^2) \leq \ln \frac{\mathrm{Vol}_g(B_g(p, r))}{r^n} + C'(n, r) + \frac{1}{e}(4\pi)^{-n/2} e^{C_1(n,r)}$$

where

$$C'(n, r) := 36(4\pi)^{-n/2} e^{C_1(n,r)} + D, \quad C_1(n, r) := \frac{n}{2}\ln(4\pi) + \ln C(n, r)$$

and

$$C(n, r) = \int_0^r \frac{\sinh(\sqrt{K'}t)}{\sqrt{K'}} dt \Big/ \int_0^{r/2} \frac{\sinh(\sqrt{K'}t)}{\sqrt{K'}} dt, \quad K' := \frac{D}{(n-1)r^2}.$$

The last quantity $C(n,r)$ can be bounded as

$$\begin{aligned} C(n,r) &= \int_0^r \left(e^{\sqrt{K'}t} - e^{-\sqrt{K'}t}\right) dt \Big/ \int_0^{r/2} \left(e^{\sqrt{K'}t} - e^{-\sqrt{K'}t}\right) dt \\ &= \frac{e^{\sqrt{k'}t} + e^{-\sqrt{K'}t}|_0^r}{e^{\sqrt{K'}t} + e^{-\sqrt{K'}t}|_0^{r/2}} = \frac{e^{\sqrt{K'}r} + e^{-\sqrt{K'}r} - 2}{e^{\sqrt{K'}r/2} + e^{-\sqrt{K'}r/2} - 2} \\ &= \frac{(e^{\sqrt{K'}r/2} - e^{-\sqrt{K'}r/2})^2}{(e^{\sqrt{K'}r/4} - e^{-\sqrt{K'}r/4})^2} = \frac{(e^{\sqrt{K'}r} - 1)^2}{e^{\sqrt{K'}r/2}(e^{\sqrt{K'}r/2} - 1)^2} \\ &= \frac{(e^{\sqrt{K'}r/2} + 1)^2}{e^{\sqrt{K'}r/2}} = e^{\sqrt{K'}r/2} + 2 + e^{-\sqrt{K'}r/2} \leq 3 + e^{\sqrt{K'}r/2}. \end{aligned}$$

Hence

$$C(n,r) \leq 3 + e^{\sqrt{D/4(n-1)}} \tag{2.21}$$

and the constant c in Lemma 2.4 can be taken to be

$$c = C(n) \exp\left[-C(n) \exp\left(C(n)\sqrt{D}\right)\right], \tag{2.22}$$

where $C(n)$ is a uniform constant depending only on n.

From the rescaling

$$\left|\mathrm{Ric}_{r^2 g}\right|_{r^2 g} = r^{-2}|\mathrm{Ric}_g|_g, \quad B_{r^2 g}(p,r) = B_g(p,1), \tag{2.23}$$

we can conclude from Lemma 2.4 that

Corollary 2.5. *For every $n \geq 2$ and $D > 0$, there exists $c = c(n,D) > 0$ such that if (M,g) is a closed n-dimensional Riemannian manifold, u is a smooth function on M, and if for some $A < \infty$ we have $\mu(g,u,1) > -A$, then for any $p \in M$ with $|\mathrm{Ric}_g|_g \leq D$ on $B_g(p,1)$, we have*

$$\mathrm{Vol}_g(B_g(p,1)) \geq \kappa \tag{2.24}$$

where $\kappa = ce^{-A}$. Moreover, $c = c(n,D)$ can be taken to be given in (2.22) for some constant $C(n)$ depending only on n.

Proof. Write $\hat{g} := r^2 g$ for any given $r > 0$. Then the conditions $\mu(g,u,1) > -A$ and $|\mathrm{Ric}_g|_g \leq C$ on $B_h(p,1)$ become

$$\mu(\hat{g},u,r^2) = \mu(g,u,1) > -A, \quad |\mathrm{Ric}_{\hat{g}}|_{\hat{g}} = \frac{1}{r^2}|\mathrm{Ric}_g|_g \leq \frac{C}{r^2} \text{ on } B_{\hat{g}}(p,r).$$

We obtain from Lemma 2.4 that $\kappa r^n \leq \mathrm{Vol}_{\hat{g}}(B_{\hat{g}}(p,r)) = r^n \mathrm{Vol}_g(B_g(p,1))$. □

To prove the boundedness of $\Delta_{g(t)}u(t)$, we first verify that $\mu(g(t),u(t),1)$ is always bounded from below by some uniform constant. Set, for each real number τ,

$$C^\infty_\tau(M) := \left\{ f \in C^\infty(M) : \int_M \frac{e^{-f}}{(4\pi\tau)^{n/2}} dV_g = 1 \right\}.$$

The mapping

$$C^\infty_{\tau_2}(M) \ni f \longmapsto \tilde{f} := f + \frac{n}{2} \ln \frac{\tau_2}{\tau_1} \in C^\infty_{\tau_1}(M)$$

is one-to-one and onto. Choose $\tilde{f} \in C^\infty_{\tau_1}(M)$ so that $\mu(g,u,\tau_1) = \mathcal{W}(g,u,\tilde{f},\tau_1)$ and define $f := \tilde{f} - \frac{n}{2}\ln\frac{\tau_2}{\tau_1} \in C^\infty_{\tau_2}(M)$. Hence, for $\tau_1, \tau_2 > 0$,

$$\begin{aligned}
\mu(g,u,\tau_2) &\leq \mathcal{W}(g,u,f,\tau_2) \\
&= \int_M \left[\tau_2\left(S_g + |\nabla_g f|^2_g\right) + f - n\right] \frac{e^{-f}}{(4\pi\tau_2)^{n/2}} dV_g \\
&= \int_M \left[\tau_2\left(S_g + |\nabla \tilde{f}|^2_g\right) + \tilde{f} - n - \frac{n}{2}\ln\frac{\tau_2}{\tau_1}\right] \frac{e^{-\tilde{f}}}{(4\pi\tau_1)^{n/2}} dV_g \\
&= \int_M \left[\tau_1\left(S_g|\nabla_g \tilde{f}|^2_g\right) + \tilde{f} - n\right] \frac{e^{-\tilde{f}}}{(4\pi\tau_1)^{n/2}} dV_g \\
&\quad - \frac{n}{2}\ln\frac{\tau_2}{\tau_1} + (\tau_2 - \tau_1)\int_M \left(S_g + |\nabla_g \tilde{f}|^2_g\right) \frac{e^{-\tilde{f}}}{(4\pi\tau_1)^{n/2}} dV_g \\
&= \mu(g,u,\tau_1) - \frac{n}{2}\ln\frac{\tau_2}{\tau_1} + (\tau_2 - \tau_1)\int_M \left(S_g + |\nabla_g \tilde{f}|^2_g\right) \frac{e^{-\tilde{f}}}{(4\pi\tau_1)^{n/2}} dV_g
\end{aligned}$$

When $\tau_1 \geq \tau_2 > 0$, one has

$$\mu(g,u,\tau_2) \leq \mu(g,u,\tau_1) - \frac{n}{2}\ln\frac{\tau_2}{\tau_1} + (\tau_2 - \tau_1) S_{g,\min}. \tag{2.25}$$

In particular, if $0 < \tau(t) \leq 1$, then the inequality (2.25) implies

$$\mu(g(t),u(t),\tau(t)) \leq \mu(g(t),u(t),1) - \frac{n}{2}\ln\tau(t) + [\tau(t) - 1] S_{g(t),\min}. \tag{2.26}$$

By the monotonicity of the Ricci-harmonic flow, Proposition 2.2, we obtain from (2.26) that

$$\mu(g(t),u(t),1) \geq \mu(g(0),u(0),\tau(0)) + \frac{n}{2}\ln[\tau(0) - t] + [1 + t - \tau(0)] S_{g(t),\min}$$

when $1 + t - \tau(0) \geq 0$ and $\tau(0) - t > 0$. In particular, together with Lemma 2.4 and (2.4),

$$\mu(g(t),\mu(t),1) \geq \frac{n}{2}\ln\frac{\tau(0) - t}{4\pi\tau(0)} + (1+t) S_{g(0),\min} - 2C(2\tau(0), g(0)) - n \tag{2.27}$$

whenever $\tau(0) - 1 \leq t < \tau(0)$.

Theorem 2.6. *There exists a uniform constant C depending only on $n, g(0)$, and $u(0)$ such that the following statement is true: If $|\mathrm{Ric}_{g(t)}|_{g(t)} \leq K$ on $M \times [0,T]$, then*

$$|\Delta_{g(t)} u(t)|_{g(t)} \leq \frac{C(1+K)}{(1+T)^{n/2}} \exp\left[C\left(1 + T + 1 + KT + e^{C\sqrt{K}}\right)\right] \tag{2.28}$$

over any geodesic ball $B_{g(t)}(p, \sqrt{1+T})$. In particular, the estimate (2.28) holds on $M \times [0,T]$.

Proof. Let

$$\tilde{t} := \frac{t}{T+1}, \quad \widetilde{T} := \frac{T}{T+1}, \quad \tilde{g}(\tilde{t}) := \frac{1}{T+1} g\left((T+1)\tilde{t}\right), \quad \tilde{u}(\tilde{t}) := u\left((T+1)\tilde{t}\right).$$

Then $(\tilde{g}(\tilde{t}), \tilde{u}(\tilde{t}))_{\tilde{t}\in[0,\widetilde{T}]}$ is a solution of the Ricci flow with $\widetilde{T} \in (0,1)$. In this case we choose $\tilde{\tau}(0) := (1 + \widetilde{T})/2$ so that $\tilde{\tau}(0) - \tilde{t} \geq (1 + \widetilde{T})/2 - \widetilde{T} = (1 - \widetilde{T})/2 > 0$

and $1+\tilde{t}-\tilde{\tau}(0)\geq 1-(1+\widetilde{T})/2=(1-\widetilde{T})/2>0$. Therefore the estimate (2.27) applied to the rescaling Ricci-harmonic flow holds for all $\tilde{t}\in[0,\widetilde{T}]$, i.e.,

$$\mu(\tilde{g}(\tilde{t}),\tilde{u}(\tilde{t}),1)\geq\frac{n}{2}\ln\frac{1-\widetilde{T}}{1+\widetilde{T}}-(1+\widetilde{T})|\tilde{S}_{\tilde{g}(0),\min}|-2C(1+\widetilde{T},\tilde{g}(0))-\ln 4\pi-n.$$

Since the L^2-Sobolev constant $C_s(M,g)$ is invariant under scaling the metric, it follows from (2.22) and (A.9) that

$$\begin{aligned}\mu(\tilde{g}(\tilde{t}),\tilde{u}(\tilde{t}),1) &\geq \frac{n}{2}\ln\frac{1-\frac{T}{1+T}}{1+\frac{T}{1+T}}-\left(1+\frac{T}{1+T}\right)\left|(T+1)S_{g(0),\min}\right|-\ln 4\pi-n\\ &- 2\left[\frac{1+2T}{1+T}\left[(1+T)^{-n/2}\mathrm{Vol}(M,g(0))\right]^{-2/n}+\frac{n^2}{4e^{2\frac{1+2T}{1+T}}C_s(M,g(0))}\right]\\ &= -\frac{n}{2}\ln(1+2T)-(1+2T)\left[|S_{g(0),\min}|+\frac{2}{\mathrm{Vol}(M,g(0))^{2/n}}\right]\\ &- \ln 4\pi-n-\frac{n^2(1+T)}{2e^2(1+2T)C_s(M,g(0))}\\ &\geq -(1+2T)\left[\frac{n}{2}+|S_{g(0),\min}|+\frac{2}{\mathrm{Vol}(M,g(0))^{n/2}}\right]-\ln 4\pi-n-\frac{n^2}{2e^2C_s(M,g(0))}.\end{aligned}$$

Consequently,

$$\mu(\tilde{g}(\tilde{t}),\tilde{u}(\tilde{t}),1)\geq -C(1+2T) \tag{2.29}$$

for some uniform constant C depending only on $g(0)$ and $u(0)$. Because $|\widetilde{\mathrm{Ric}}_{\tilde{g}(\tilde{t})}|_{\tilde{g}(\tilde{t})}$ $=|\mathrm{Ric}_{g(t)}|_{g(t)}/(1+T)\leq K/(1+T)$ on $\tilde{B}_{\tilde{g}(\tilde{t})}(p,1)=B_{g(t)}(p,\sqrt{1+T})$, we have

$$\mathrm{Vol}_{\frac{1}{1+T}g(t)}\left(B_{\frac{1}{1+T}g(t)}(p,1)\right)\geq\kappa$$

where $\kappa=C(n)\exp[-C(n)\exp(C(n)\sqrt{K/(1+T)})]e^{-C(1+2T)}$. Thus

$$\begin{aligned}\mathrm{Vol}_{g(t)}\left(B_{g(t)}(p,\sqrt{1+T})\right) &\geq C(1+T)^{n/2}\exp\left[-C\left(1+2T+e^{C\sqrt{K/(1+T)}}\right)\right]\\ &\geq C(1+T)^{n/2}\exp\left[-C\left(1+2T+e^{C\sqrt{K}}\right)\right]. \end{aligned}\tag{2.30}$$

We now can prove the estimate (2.28). Suppose otherwise that

$$|\Delta_{g(t)}u(t)|>\frac{C(1+K)}{(1+T)^{n/2}}\exp\left\{C\left[(1+K)T+1+2T+e^{C\sqrt{K}}\right]\right\}$$

over some geodesic ball $B_{g(t)}(p,\sqrt{1+T})$ and for some time $t\in[0,T]$. On the other hand, from Proposition 2.2 and (2.30), we get

$$\begin{aligned}C(1+K)e^{C(1+K)T} &\geq \int_{\mathrm{Vol}_{g(t)}(B_{g(t)}(p,\sqrt{1+T}))}|\Delta_{g(t)}u(t)|^2dV_{g(t)}\\ &\geq \frac{2C(1+K)}{(1+T)^{n/2}}\exp\left\{2C\left[(1+K)T+1+2T+e^{C\sqrt{K}}\right]\right\}\cdot\mathrm{Vol}_{g(t)}\left(B_{g(t)}(p,\sqrt{1+T})\right)\\ &\geq 2C(1+K)e^{C(1+K)T}\exp\left[C\left(1+2T+e^{C\sqrt{K}}\right)\right]\geq 2C(1+K)e^{C(1+K)T}.\end{aligned}$$

This contradiction shows that we must have (2.28). □

2.2. **Local curvature estimates.** In this subsection we assume that

$$|\text{Ric}| \le K, \quad |\nabla u| \le L, \quad |\nabla^2 u| \le P$$

over an open subset Ω in M and ϕ is a Lipschitz function with support in Ω.

From Lemma A.1, we can deduce that

$$\begin{aligned}\Box|\text{Ric}|^2 &= -2|\nabla\text{Ric}|^2 + 4R_{pijq}R^{pq}R^{ij} - 8R_{pijq}R^{ij}\nabla^p u\nabla^q u \\ &\quad + 8\Delta u R^{ij}\nabla_i\nabla_j u - 8R^{ij}\nabla_i\nabla_k u\nabla^k\nabla_j u - 8R_{ij}R_k{}^j\nabla^i u\nabla u. \end{aligned} \tag{2.31}$$

In particular

$$\begin{aligned}|\nabla\text{Ric}|^2 &\le -\frac{1}{2}\Box|\text{Ric}|^2 + CK^2|\text{Rm}| + CKL^2|\text{Rm}| \\ &\quad + CK|\nabla^2 u||\Delta u| + CK|\nabla^2 u|^2 + CK^2L^2 \\ &\le -\frac{1}{2}\Box|\text{Ric}|^2 + CK(L^2+K)|\text{Rm}| + CKP^2 + CK^2L^2, \end{aligned} \tag{2.32}$$

by the fact at $|\Delta u| \le \sqrt{n}|\nabla^2 u|$. Similarly, from (A.6), we have

$$|\nabla\text{Rm}|^2 \le -\frac{1}{2}\Box|\text{Rm}|^2 + C|\text{Rm}|^3 + C|\text{Rm}|P^2 + CL^2|\text{Rm}|^2. \tag{2.33}$$

Moreover, we can prove that, see Lemma A.2,

$$\begin{aligned}\partial_t|\text{Rm}|^2 &= \nabla^2\text{Ric} * \text{Rm} + \text{Ric} * \text{Rm} * \text{Rm} \\ &\quad + \text{Rm} * \nabla^2 u * \nabla^2 u + \text{Rm} * \text{Rm} * \nabla u * \nabla u. \end{aligned} \tag{2.34}$$

As in [32], we consider the quantity

$$\frac{d}{dt}\int_M |\text{Rm}|^p\phi^{2p}dV_t$$

which can be rewritten as, using (2.34),

$$\begin{aligned}\frac{d}{dt}\int_M |\text{Rm}|^p\phi^{2p}dV_t &= \int_M (\partial_t|\text{Rm}|^p)\,\phi^{2p}dV + \int_M |\text{Rm}|^p\phi^{2p}\left(-R + 2|\nabla u|^2\right)dV_t \\ &= \frac{p}{2}\int_M |\text{Rm}|^{p-2}\Big[\nabla^2\text{Ric} * \text{Rm} + \text{Ric} * \text{Rm} * \text{Rm} + \text{Rm} * \nabla^2 u * \nabla^2 u \\ &\quad + \text{Rm} * \text{Rm} * \nabla u * \nabla u\Big]\phi^{2p}dV_t - \int_M R|\text{Rm}|^p\phi^{2p}dV_t + 2\int_M |\text{Rm}|^p|\nabla u|^2\phi^{2p}dV_t \\ &\le C\int_M |\text{Rm}|^{p-2}\left(\nabla^2\text{Ric} * \text{Rm}\right)\phi^{2p}dV_t + CK\int_M |\text{Rm}|^p\phi^{2p}dV_t \\ &\quad + CP^2\int_M |\text{Rm}|^{p-1}\phi^{2p}dV + CL^2\int_M |\text{Rm}|^p\phi^{2p}dV_t \\ &\quad + C\int_M |\text{Rm}|^{p-2}\left(\text{Rm} * \nabla u * \nabla^3 u\right)\phi^{2p}dV_t. \end{aligned}$$

From (2.5), (2.6), and (2.7) in [32], we know that

$$\begin{aligned}C\int_M |\text{Rm}|^{p-2}\left(\nabla^2\text{Ric} * \text{Rm}\right)\phi^{2p}dV_t &\le \frac{1}{K}\int_M |\nabla\text{Ric}|^2|\text{Rm}|^{p-1}\phi^{2p}dV_t \\ &\quad + CK\int_M |\nabla\text{Rm}|^2|\text{Rm}|^{p-3}\phi^{2p}dV_t + CK\int_M |\text{Rm}|^{p-1}|\nabla\phi|^2\phi^{2p-2}dV_t. \end{aligned}$$

Combining all terms yields

$$\frac{d}{dt}\int_M |\mathrm{Rm}|^p\phi^{2p}dV_t \le \frac{1}{K}\int_M |\nabla\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}\phi^{2p}dV_t$$

$$+\ CK\int_M |\nabla\mathrm{Rm}|^2|\mathrm{Rm}|^{p-3}\phi^{2p}dV_t + CK\int_M |\mathrm{Rm}|^{p-1}|\nabla\phi|^2\phi^{2p-2}dV_t \tag{2.35}$$

$$+\ C(K+L^2)\int_M |\mathrm{Rm}|^p\phi^{2p}dV + CP^2\int_M |\mathrm{Rm}|^{p-1}\phi^{2p}dV_t$$

In (2.35) the first two terms are "bad terms", since these contain derivatives of curvature. As in [32] we set

$$B_1 := \frac{1}{K}\int_M |\nabla\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}\phi^{2p}dV_t, \quad B_2 := \int_M |\nabla\mathrm{Rm}|^2|\mathrm{Rm}|^{p-3}\phi^{2p}dV_t.$$

We also introduce

$$\begin{aligned} A_1 &:= \int_M |\mathrm{Rm}|^p|\phi^{2p}dV_t, \quad A_2 := \int_M |\mathrm{Rm}|^{p-1}\phi^{2p}dV_t, \\ A_3 &:= \int_M |\mathrm{Rm}|^{p-1}|\nabla\phi|^2\phi^{2p-1}dV_t, \quad A_4 := \int_M |\mathrm{Rm}|^{p-1}|\nabla\phi|^2\phi^{2p-2}dV_t. \end{aligned}$$

Then the estimate (2.35) can be rewritten as

$$\frac{d}{dt}\int_M |\mathrm{Rm}|^p\phi^{2p}dV_t \le B_1 + CKB_2 + CKA_4 + C(K+L^2)A_1 + CP^2A_2. \tag{2.36}$$

Using (2.32) yields

$$\begin{aligned}
B_1 &\le \int_M \left[\frac{1}{2K}(\Delta-\partial_t)|\mathrm{Ric}|^2 + C(L^2+K)|\mathrm{Rm}| + C(P^2+KL^2)\right]|\mathrm{Rm}|^{p-1}\phi^{2p}dV_t \\
&= \frac{1}{2K}\int_M \left[(\Delta-\partial_t)|\mathrm{Ric}|^2\right]|\mathrm{Rm}|^{p-1}\phi^{2p}dV_t + C(L^2+K)A_1 + C(P^2+KL^2)A_2 \\
&= \frac{1}{2K}\int_M \left(\Delta|\mathrm{Ric}|^2\right)|\mathrm{Rm}|^{p-1}\phi^{2p}dV_t + C(L^2+K)A_1 + C(P^2+KL^2)A_2 \\
&\quad - \frac{1}{2K}\int_M \Big[\partial_t\left(|\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}\phi^{2p}dV_t\right) - |\mathrm{Ric}|^2\left(\partial_t|\mathrm{Rm}|^{p-1}\right)\phi^{2p}dV_t \\
&\qquad - |\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}\phi^{2p}\left(-R+2|\nabla u|^2\right)dV_t\Big] \\
&= -\frac{1}{2K}\left[\int_M \left\langle\nabla|\mathrm{Ric}|^2,\nabla|\mathrm{Rm}|^{p-1}\right\rangle\phi^{2p}dV_t + \int_M\left\langle\nabla|\mathrm{Ric}|^2,\nabla\phi^{2p}\right\rangle|\mathrm{Rm}|^{p-1}dV_t\right] \\
&\quad - \frac{1}{2K}\frac{d}{dt}\int_M |\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}\phi^{2p}dV_t + C(L^2+K)A_1 + C(P^2+KL^2)A_2 \\
&\quad + \frac{1}{2K}\int_M |\mathrm{Ric}|^2\left(\partial_t|\mathrm{Rm}|^{p-1}\right)\phi^{2p}dV_t + CKL^2A_2 + CKA_1 \\
&\le -\frac{1}{2K}\left[\int_M \left\langle\nabla|\mathrm{Ric}|^2,\nabla|\mathrm{Rm}|^{p-1}\right\rangle\phi^{2p}dV_t + \int_M\left\langle\nabla|\mathrm{Ric}|^2,\nabla\phi^{2p}\right\rangle|\mathrm{Rm}|^{p-1}dV_t\right] \\
&\quad - \frac{1}{2K}\frac{d}{dt}\int_M |\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}\phi^{2p}dV_t + \frac{1}{2K}\int_M |\mathrm{Ric}|^2\left(\partial_t|\mathrm{Rm}|^{p-1}\right)\phi^{2p}dV_t \\
&\quad + C(L^2+K)A_1 + C(P^2+KL^2)A_2.
\end{aligned}$$

From (2.10) and (2.11) in [32], one has

$$-\frac{1}{2K}\int_M \left\langle \nabla|\mathrm{Ric}|^2, \nabla|\mathrm{Rm}|^{p-1}\right\rangle \phi^{2p} dV_t \leq \frac{1}{10}B_1 + CKB_2, \tag{2.37}$$

$$-\frac{1}{2K}\int_M \left\langle \nabla|\mathrm{Ric}|^2, \nabla\phi^{2p}\right\rangle |\mathrm{Rm}|^{p-1} dV_t \leq \frac{1}{10}B_1 + CKA_4. \tag{2.38}$$

According to (2.34), we have

$$\begin{aligned}\frac{1}{2K}\int_M |\mathrm{Ric}|^2\left(\partial_t|\mathrm{Rm}|^{p-1}\right)\phi^{2p}dV_t &= \frac{p-1}{4K}\int_M |\mathrm{Ric}|^2\left(|\mathrm{Rm}|^{p-3}\partial_t|\mathrm{Rm}|^2\right)\phi^{2p}dV_t\\ &= \frac{C}{K}\int_M |\mathrm{Ric}|^2|\mathrm{Rm}|^{p-3}\phi^{2p}\Big[\nabla^2\mathrm{Ric}*\mathrm{Rm}+\mathrm{Ric}*\mathrm{Rm}*\mathrm{Rm}\\ &\quad + \mathrm{Rm}*\nabla^2 u*\nabla^2 u + \mathrm{Rm}*\mathrm{Rm}*\nabla u*\nabla u\Big]dV_t\\ &\leq \frac{C}{K}\int_M |\mathrm{Ric}|^2|\mathrm{Rm}|^{p-3}\phi^{2p}\left(\nabla^2\mathrm{Ric}*\mathrm{Rm}\right)dV_t + CKA_1 + C(P^2+KL^2)A_2\end{aligned}$$

From the proof of (2.13) – (2.15) in [32], we can deduce that

$$\frac{C}{K}\int_M |\mathrm{Ric}|^2|\mathrm{Rm}|^{p-3}\phi^{2p}\left(\nabla^2\mathrm{Ric}*\mathrm{Rm}\right)dV \leq \frac{1}{5}B_1 + CKB_2 + CKA_4.$$

In summary, we arrive at

$$\begin{aligned}\frac{1}{2K}\int_M |\mathrm{Ric}|^2\left(\partial_t|\mathrm{Rm}|^{p-1}\right)\phi^{2p}dV_t &\leq \frac{1}{5}B_1 + CKB_2 + CKA_1\\ &\quad + C(P^2+KL^2)A_2 + CKA_4.\end{aligned} \tag{2.39}$$

Plugging (2.37), (2.38), and (2.39) into the inequality for B_1, we get

$$\begin{aligned}B_1 &\leq CKB_2 + C(K+L^2)A_1 + CKA_4\\ &\quad + C(P^2+KL^2)A_2 - \frac{1}{2K}\frac{d}{dt}\int_M |\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}\phi^{2p}dV_t.\end{aligned} \tag{2.40}$$

To deal with the term B_2, we use the evolution equation (2.33) and then obtain

$$\begin{aligned}B_2 &\leq \int_M \left[\frac{1}{2}\left(\Delta-\partial_t\right)|\mathrm{Rm}|^2 + C|\mathrm{Rm}|^3 + C(L^2+P^2)|\mathrm{Rm}|^2\right]|\mathrm{Rm}|^{p-3}\phi^{2p}dV_t\\ &= \frac{1}{2}\int_M\left(\Delta|\mathrm{Rm}|^2\right)|\mathrm{Rm}|^{p-3}\phi^{2p}dV_t + CA_1 + C(L^2+P^2)A_2\\ &\quad - \frac{1}{2}\int_M\left(\partial_t|\mathrm{Rm}|^2\right)|\mathrm{Rm}|^{p-3}\phi^{2p}dV_t\\ &\leq C\int_M |\nabla\mathrm{Rm}||\nabla\phi||\mathrm{Rm}|^{p-2}\phi^{2p-1}dV_t + CA_1 + C(L^2+P^2)A_2\\ &\quad - \frac{1}{2}\int_M\left(\partial_t|\mathrm{Rm}|^2\right)|\mathrm{Rm}|^{p-3}\phi^{2p}dV_t\\ &\leq \frac{1}{2}B_2 + CA_4 + CA_1 + C(L^2+P^2)A_2 - \frac{1}{2}\int_M\left(\partial_t|\mathrm{Rm}|^2\right)|\mathrm{Rm}|^{p-3}\phi^{2p}dV_t.\end{aligned}$$

As the proof of (2.18) – (2.19) in [32], we have

$$-\frac{1}{2}\int_M\left(\partial_t|\mathrm{Rm}|^2\right)|\mathrm{Rm}|^{p-3}\phi^{2p}dV_t = -\frac{1}{2}\int_M\Big[\partial_t\left(|\mathrm{Rm}|^2|\mathrm{Rm}|^{p-3}\phi^{2p}dV_t\right)$$

$$-|\mathrm{Rm}|^2\left(\partial_t|\mathrm{Rm}|^{p-3}\right)\phi^{2p}dV_t - |\mathrm{Rm}|^{p-1}\phi^{2p}\partial_t dV\Big]$$

$$= -\frac{1}{2}\frac{d}{dt}\int_M |\mathrm{Rm}|^{p-1}\phi^{2p}dV_t + \frac{p-3}{4}\int_M |\mathrm{Rm}|^{p-3}\left(\partial_t|\mathrm{Rm}|^2\right)\phi^{2p}dV_t$$
$$-\frac{1}{2}\int_M R|\mathrm{Rm}|^{p-1}\phi^{2p}dV_t + \int_M |\mathrm{Rm}|^{p-1}|\nabla u|^2\phi^{2p}dV_t$$

and therefore

$$-\frac{1}{2}\int_M\left(\partial_t|\mathrm{Rm}|^2\right)|\mathrm{Rm}|^{p-3}\phi^{2p}dV_t \le -\frac{1}{p-1}\frac{d}{dt}\int_M |\mathrm{Rm}|^{p-1}\phi^{2p}dV_t + CA_1 + CL^2A_2.$$

In summary,

$$B_2 \le -\frac{1}{p-1}\frac{d}{dt}\int_M |\mathrm{Rm}|^{p-1}\phi^{2p}dV_t + CA_1 + CA_4 + C(L^2+P^2)A_2. \tag{2.41}$$

From (2.36), (2.40), and (2.41), we finally obtain

$$\frac{d}{dt}\left[A_1 + CKA_2 + \frac{1}{2K}\int_M |\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}\phi^{2p}dV_t\right] \le C(K+L^2)A_1$$
$$+ CKA_4 + C(KL^2+KP^2+P^2+KL)A_2.$$

Theorem 2.7. *Let $(g(t),u(t))_{t\in[0,T]}$ be a solution to the Ricci-haronic flow. Suppose there exist constants $\rho, K, L, P > 0$ and $x_0 \in M$ such that $B_{g(0)}(x_0,\rho/\sqrt{K})$ is compactly contained on M and*

$$|\mathrm{Ric}_{g(t)}|_{g(t)} \le K, \quad |\nabla_{g(t)}u(t)|_{g(t)} \le L, \quad |\nabla^2_{g(t)}u(t)|_{g(t)} \le P$$

on $B_{g(0)}(x_0,\rho/\sqrt{K})\times[0,T]$. For any $p \ge 3$, there is a constant C, depending only on n and p, such that

$$\int_{B_{g(0)}(x_0,\rho/2\sqrt{K})} |\mathrm{Rm}_{g(t)}|^p_{g(t)}dV_{g(t)} \le C\Lambda_1 e^{C\Lambda_2 T}\int_{B_{g(0)}(x_0,\rho/\sqrt{K})} |\mathrm{Rm}_{g(0)}|^p_{g(0)}dV_{g(0)}$$
$$+ CK^p\left(1+\rho^{-2p}\right)e^{C\Lambda_2 T}\mathrm{Vol}_{g(t)}\left(B_{g(0)}\left(x_0,\frac{\rho}{\sqrt{K}}\right)\right).$$

Here $\Lambda_1 := 1+K$ and $\Lambda_2 := K + L + L^2 + P^2(1+K^{-1})$.

Proof. Choose $\Omega := B_{g(0)}(x_0,\rho/\sqrt{K})$ and

$$\phi := \left(\frac{\rho/\sqrt{K} - d_{g(0)}(x_0,\cdot)}{\rho/\sqrt{K}}\right)_+.$$

Then $e^{-2Kt}g(0) \le g(t) \le e^{2Kt}g(0)$ and $|\nabla_{g(t)}\phi|_{g(t)} \le e^{KT}|\nabla_{g(0)}\phi|_{g(0)} \le \sqrt{K}e^{KT}/\rho$ for any $t\in[0,T]$. Let

$$U := \int_M |\mathrm{Rm}|^p\phi^{2p}dV_t + CK\int_M |\mathrm{Rm}|^{p-1}\phi^{2p}dV_t + \frac{1}{2K}\int_M |\mathrm{Ric}|^2|\mathrm{Rm}|^{p-1}\phi^{2p}dV_t.$$

Then

$$U' \le C(K+L^2)U + CKA_4 + C(KL^2+KP^2+P^2+KL)\frac{U}{K}.$$

For A_4, we can estimate it as follows:

$$
\begin{aligned}
A_4 &= \int_M |\mathrm{Rm}|^{p-1}|\nabla\phi|^2\phi^{2p-2}dV_t \ \leq\ \int_{B_{g(0)}(x_0,\rho/\sqrt{K})}|\mathrm{Rm}|^{p-1}\phi^{2p-2}K\rho^{-2}e^{2KT}dV_t\\
&\leq \int_{B_{g(0)}(x_0,\rho/\sqrt{K})}\left[\frac{(|\mathrm{Rm}|^{p-1}\phi^{2p-2})^{\frac{p}{p-1}}}{\frac{p}{p-1}}+\frac{(K\rho^{-2}e^{2KT})^p}{p}\right]dV_t\\
&\leq A_1+K^pe^{2KpT}p\rho^{-2p}\mathrm{Vol}_{g(t)}\left(B_{g(0)}\left(x_0,\frac{\rho}{\sqrt{K}}\right)\right)\\
&\leq U+CK^p\rho^{-2p}e^{2KpT}\mathrm{Vol}_{g(t)}\left(B_{g(0)}\left(x_0,\frac{\rho}{\sqrt{K}}\right)\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
U' &\leq C\left[K+L^2+L+P^2\left(1+\frac{1}{K}\right)\right]U\\
&+CK^{p+1}\rho^{-2p}e^{2KpT}\mathrm{Vol}_{g(t)}\left(B_{g(0)}\left(x_0,\frac{\rho}{\sqrt{K}}\right)\right).
\end{aligned}
$$

Since, for each $\tau\in[0,T]$,

$$
\mathrm{Vol}_{g(t)}\left(B_{g(0)}\left(x_0,\frac{\rho}{\sqrt{K}}\right)\right)\leq e^{CKT}\mathrm{Vol}_{g(\tau)}\left(B_{g(0)}\left(x_0,\frac{\rho}{\sqrt{K}}\right)\right),
$$

as argued in (2.27) of [32], we deduce that

$$
U(\tau)\leq e^{C\Lambda_2T}\left[U(0)+C\rho^{-2p}K^p\mathrm{Vol}_{g(t)}\left(B_{g(0)}\left(x_0,\frac{\rho}{\sqrt{K}}\right)\right)\right], \tag{2.42}
$$

According to the Young inequality

$$
\begin{aligned}
\int_M|\mathrm{Rm}_{g(0)}|^{p-1}\phi^{2p}dV_{g(0)} &= \int_M\left(|\mathrm{Rm}|^{p-1}\phi^{2p-2}\right)\phi^2dV_{g(0)}\\
&\leq \frac{p-1}{p}\int_M|\mathrm{Rm}_{g(0)}|^p\phi^{2p}dV_{g(t)}+\frac{1}{p}\int_M\phi^{2p}dV_{g(0)},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
U(0) &\leq C(1+K)\int_M|\mathrm{Rm}_{g(0)}|^p\phi^{2p}dV_{g(0)}+CK\mathrm{Vol}_{g(0)}\left(B_{g(0)}\left(x_0,\frac{\rho}{\sqrt{K}}\right)\right)\\
&\leq C(1+K)\int_M|\mathrm{Rm}_{g(0)}|^{2p}_{g(0)}\phi^{2p}dV_{g(0)}\\
&\quad+CKe^{CKT}\mathrm{Vol}_{g(\tau)}\left(B_{g(0)}\left(x_0,\frac{\rho}{\sqrt{K}}\right)\right)\\
&\leq C(1+K)\int_{B_{g(0)}(x_0,\rho/\sqrt{K})}|\mathrm{Rm}_{g(0)}|^{2p}_{g(0)}dV_{g(0)}\\
&\quad+CKe^{CKT}\mathrm{Vol}_{g(\tau)}\left(B_{g(0)}\left(x_0,\frac{\rho}{\sqrt{K}}\right)\right),
\end{aligned}
$$

and $\phi\geq 1/2$ on $B_{g(0)}(x_0,\rho/2\sqrt{K})$, we complete the proof. □

The same method can be applied to the regular Ricci flow (see **Section** 3), since all computations only involve the evolution equations for the metrics, which take the same forms in the Ricci-harmonic flow.

3. RESULTS FOR A GENERALIZED RICCI FLOW

In this section we extend the main estimates in [43] to a generalized Ricci flow introduced in [42], where I proved that for any complete n-manifold M, the following two conditions are equivalent:

(i) there exists a Ricci-flat Riemannian metric on M;
(ii) there exists real numbers α, β, a smooth function u on M, and a Riemannian metric g on M such that

$$0 = -R_{ij} + \alpha\nabla_i\nabla_j u, \quad 0 = \Delta_g u + \beta|\nabla_g u|^2_g. \tag{3.1}$$

The main ingredient in the proof is Chen's result [9] which says that any complete noncompact steady gradient Ricci soliton has nonnegative scalar curvature.

Observe that the first equation in (3.1) is actual the vanishing of the ∞-Bakry-Émery Ricci tensor. Indeed, the N-Bakry-Émery Ricci tensor is defined by

$$\text{Ric}_{g,N,f} := \text{Ric}_g + \nabla^2 f - \frac{df \otimes df}{N-n} \tag{3.2}$$

for N finite, and

$$\text{Ric}_{g,\infty,f} := \text{Ric}_g + \nabla^2 f \tag{3.3}$$

for N infinite. Thus the first equation in (3.1) is equivalent to $\text{Ric}_{g,\infty,-\alpha u} = 0$.

Motivated by the above equivalence conditions I introduced the following generalized Ricci flow [42]:

$$\partial_t g(t) = -2\text{Ric}_{g(t)} + 2\alpha_1\nabla_{g(t)}u(t) \otimes \nabla_{g(t)}u(t) + 2\alpha_2\nabla^2_{g(t)}u(t), \tag{3.4}$$

$$\partial_t u(t) = \Delta_{g(t)}u(t) + \beta_1|\nabla_{g(t)}u(t)|^2_{g(t)} + \beta_2 u(t). \tag{3.5}$$

Here $\alpha_1, \alpha_2, \beta_1, \beta_2$ are given constants. This system is called $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-*Ricci flow*. In particular, when $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (2, 0, 0, 0)$, we get the Ricci-harmonic flow (1.1). In view of (3.2), we see that (3.4) can be written as

$$\partial_t g(t) = -2\text{Ric}_{g(t),N,-\alpha_2 u(t)},$$

where $N = n + \alpha_2^2/\alpha_1$ if $\alpha_1 \neq 0$, and $N = \infty$ if $\alpha_1 = 0$.

According to Proposition 2.12 in [42], we know that an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-Ricci flow is equivalent to the $(\alpha_1, 0, \beta_1 - \alpha_2, \beta_2)$-flow. By this reduction, in the following we main study the $(\alpha_1, 0, \beta_1, \beta_2)$-*Ricci flow*:

$$\partial_t g(t) = -2\text{Ric}_{g(t)} + 2\alpha_1\nabla_{g(t)}u(t) \otimes \nabla_{g(t)}u(t), \tag{3.6}$$

$$\partial_t u(t) = \Delta_{g(t)}u(t) + \beta_1|\nabla_{g(t)}u(t)|^2_{g(t)} + \beta_2 u(t). \tag{3.7}$$

Here $\alpha_1, \beta_1, \beta_2$ are given constants. Recall the notion $\Box_{g(t)} = \partial_t - \Delta_{g(t)}$ and introduce as in [43],

$$\text{Sic}_{g(t)} := \text{Ric}_{g(t)} - \alpha_1\nabla_{g(t)}u(t) \otimes \nabla_{g(t)}u(t), \tag{3.8}$$

$$S_{g(t)} := \text{tr}_{g(t)}\text{Sic}_{g(t)} = R_{g(t)} - \alpha_1|\nabla_{g(t)}u(t)|^2_{g(t)}. \tag{3.9}$$

Another interesting flow involving $(g(t), u(t))$ is the so-called *super-Ricci flow* introduced by X. D. Li and S. Z. Li [35] and in terms of our notions (3.2) and (3.3) can be written as, according to whether or not N is infinity

$$\frac{1}{2}\partial_t g(t) + \mathrm{Ric}_{g(t),N,u(t)} \geq K g(t)$$

which is called a (K, N)-*super Ricci flow*, where $N \in \mathbb{R}$, and, respectively,

$$\frac{1}{2}\partial_t g(t) + \mathrm{Ric}_{g(t),\infty,u(t)} \geq K g(t)$$

which is called a K-*super Perelman Ricci flow*. Under the super-Ricci flow, the authors studied Harnack inequalities [35, 37, 39], W-entropy formulas [35, 45, 38, 39, 40], (K, N)-Ricci solitons [38], etc. For example, they proved that if $(g(t), u(t))_{t\in[0,T]}$ satisfies

$$\frac{1}{2}\partial_t g(t) + \mathrm{Ric}_{g(t),N,u(t)} = 0, \quad \partial_t u(t) = \frac{1}{2}\mathrm{tr}_{g(t)}\left(\partial_t g(t)\right),$$

then the W-entropy $\mathcal{W}_N(f(t)) := \frac{d}{dt}\left[t\mathcal{H}_N(f(t))\right]$ with

$$\mathcal{H}_N(f(t)) := -\int_M f(t)\ln f(t)\, dV_{g(t)} - \frac{N}{2}\left[1 + \ln(4\pi t)\right]$$

is constant along the flow, where $f(t)$ is the fundamental solution to the heat equation $\partial_t f(t) = \Delta_{g(t)} f(t) - \langle \nabla_{g(t)} u(t), \nabla_{g(t)} f(t)\rangle_{g(t)}$. However, we can *not* apply this result to our flow (3.4) or

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t),N,-\alpha_2 u(t)},$$

since the second equation (3.5) may not satisfy the constraint equation $\partial_t u(t) = \frac{1}{2}\mathrm{tr}_{g(t)}\partial_t g(t)$.

In [43] we also introduced a "Riemann curvature" type for RHF

$$S_{ijk\ell} := R_{ijk\ell} - \frac{\alpha_1}{2}\left(g_{j\ell}\nabla_i u\nabla_k u + g_{k\ell}\nabla_i u\nabla_j u\right) \tag{3.10}$$

so that $S_{ij} = g^{k\ell}S_{ik\ell j} = g^{k\ell}S_{kij\ell} = R_{ij} - \alpha_1\nabla_i u\nabla_j u$. A related construction of "Riemann curvature" type for (3.2) is given in [68]. A detailed discussion of these two notions of "Riemann curvature" will be given in **Section** 5.

According to Lemma C.1, we have

Lemma 3.1. *Under the flow (3.6) – (3.7),*

$$\begin{aligned}
\Box S &= 2|\mathrm{Sic}|^2 + 2\alpha_1|\Delta u|^2 - 2\alpha_1\beta_2|\nabla u|^2 - 4\alpha_1\beta_1\nabla^i u\nabla^j u\nabla_i\nabla_j u \\
&= 2|\mathrm{Sic}|^2 + 2\alpha_1|\Delta u|^2 - 2\alpha_1\beta_2|\nabla u|^2 - 4\alpha_1\beta_1\langle\nabla^2 u, \nabla u\otimes\nabla u\rangle, \qquad (3.11)\\
\Box S_{ij} &= 2S_{kij\ell}S^{k\ell} - 2S_{ik}S^k{}_j + 2\alpha_1\Delta u\nabla_i\nabla_j u \\
&\quad - 2\alpha_1\beta_2\nabla_i u\nabla_j u - 2\alpha_1\beta_1\nabla^k u(\nabla_i u\nabla_j\nabla_k u + \nabla_j u\nabla_i\nabla_k u). \qquad (3.12)
\end{aligned}$$

3.1. **Long time existence.** Given an initial data (g_0, u_0), define $c_0 := |\nabla_{g_0}u_0|^2_{g_0}$. We always assume that c_0 is a positive number. According to Corollary 2.10 and Definition 2.11 in [42], we say the $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flow is *regular*, if $\alpha_1, \beta_1, \beta_2$ satisfy one of the following conditions:

(i) $\beta_2 \leq 0$ and $\alpha_1 \geq \beta_1^2$;
(ii) $\beta_2 > 0$ and $c_0^{-1}\beta_2 + \beta_1^2 \geq \alpha_1 > \beta_1^2$.

Then Corollary 2.10 in [42] tells us that

$$|\nabla u|^2 \lesssim 1 \tag{3.13}$$

along the $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flow equations (3.6) – (3.7), where $\lesssim$ depends only on $\alpha_1, \beta_1, \beta_2$ and c_0.

Theorem 3.2. *Let $(g(t), u(t))_{t\in[0,T)}$ be a solution to the regular $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flow on a closed n-dimensional manifold M with $T \leq \infty$ and the initial data (g_0, u_0). Assume that $S_{g(t)} + C \geq C_0 > 0$ along the flow for some uniform constants $C, C_0 > 0$. Then*

$$\frac{|\mathrm{Sin}_{g(t)}|_{g(t)}}{S_{g(t)} + C} \leq C_1 + C_2 \max_{M\times[0,t]} \sqrt{\frac{|W_{g(s)}|_{g(s)} + |\nabla^2_{g(s)} u(s)|^2_{g(s)}}{S_{g(s)} + C}} \tag{3.14}$$

where $\mathrm{Sin}_{g(t)} := \mathrm{Sic}_{g(t)} - \frac{S_{g(t)}}{n} g(t)$ is the trace-free part of $\mathrm{Sic}_{g(t)}$ and $W_{g(t)}$ is the Weyl tensor field of $g(t)$.

Here the assumption $S_{g(t)} + C > 0$ is necessary in the theorem, since, due to the undermined sign of $\alpha_1, \beta_1, \beta_2$, we can not in general deduce any bounds for $S_{g(t)}$ from the evolution equation (3.11). In the simplest case, $\alpha_1 \geq 0$ and $\beta_1 = \beta_2 = 0$ (i.e., Ricci-harmonic flow), we have a lower bound from (3.11).

Proof. As in [43], consider the quantity

$$f := \frac{|\mathrm{Sin}_{g(t)}|^2_{g(t)}}{(S_{g(t)} + C)^\gamma} = \frac{|\mathrm{Sic}_{g(t)} + \frac{C}{n} g(t)|^2_{g(t)}}{(S_{g(t)} + C)^\gamma} - \frac{1}{n}(S_{g(t)} + C)^{2-\gamma}, \quad \gamma > 0 \tag{3.15}$$

and set

$$\mathrm{Sic}'_{g(t)} := \mathrm{Sic}_{g(t)} + \frac{C}{n} g(t), \quad S'_{g(t)} := S_{g(t)} + C. \tag{3.16}$$

From the identity (3.21) in [11], we have

$$\begin{aligned} \Box \frac{|\mathrm{Sic}'|^2}{(S')^\gamma} &= \frac{1}{(S')^\gamma}\Box|\mathrm{Sic}'|^2 - \gamma\frac{|\mathrm{Sic}'|^2}{(S')^{\gamma+1}}\Box S - \gamma(\gamma+1)\frac{|\mathrm{Sic}'|^2}{(S')^{\gamma+2}}|\nabla S'|^2 \\ &+ \frac{2\gamma}{(S')^{\gamma+1}}\left\langle \nabla|\mathrm{Sic}'|^2, \nabla S'\right\rangle. \end{aligned}$$

Using (3.12), we get

$$\begin{aligned} \Box|\mathrm{Sic}|^2 &= -2|\nabla \mathrm{Sic}|^2 + 4\mathrm{Sm}(\mathrm{Sic}, \mathrm{Sic}) + 2\left\langle \mathrm{Sic}, 2\alpha_1 \Delta u \nabla^2 u - 2\alpha_1\beta_2 \nabla u \otimes \nabla u\right\rangle \\ &- 4\alpha_1\beta_1\left\langle \mathrm{Sic}, \nabla u \otimes \nabla |\nabla u|^2\right\rangle \end{aligned} \tag{3.17}$$

where $\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic}) = S_{kij\ell}S^{ij}S^{k\ell}$. As in [43], we can prove

$$\begin{aligned}
\Box|\mathrm{Sic}'|^2 &= \Box|\mathrm{Sic}|^2 + \frac{2C}{n}\Box S \\
&= -2|\nabla\mathrm{Sic}'|^2 + 4\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic}) + \frac{4C}{n}|\mathrm{Sic}|^2 \\
&\quad + 4\alpha_1\left\langle \mathrm{Sic}', \Delta u\nabla^2 u - \beta_2\nabla u\otimes\nabla u\right\rangle - 4\alpha_1\beta_1\left\langle \mathrm{Sic}', \nabla u\otimes\nabla|\nabla u|^2\right\rangle \\
&= -2|\nabla\mathrm{Sic}'|^2 + 4\mathrm{Sm}(\mathrm{Sic}',\mathrm{Sic}') - \frac{4C}{n}|\mathrm{Sic}'|^2 + \frac{4C^2}{n^2}S' \\
&\quad + 4\alpha_1\left\langle \mathrm{Sic}', \Delta u\nabla^2 u - \beta_2\nabla u\otimes\nabla u\right\rangle - 4\alpha_1\beta_1\left\langle \mathrm{Sic}', \nabla u\otimes\nabla|\nabla u|^2\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
\Box\frac{|\mathrm{Sic}'|^2}{(S')^\gamma} &= -\frac{2}{(S')^\gamma}|\nabla\mathrm{Sic}'|^2 - \frac{2\gamma|\mathrm{Sic}'|^4}{(S')^{\gamma+1}} + \frac{4}{(S')^\gamma}\mathrm{Sm}(\mathrm{Sic}',\mathrm{Sic}') \\
&\quad - \gamma(\gamma+1)\frac{|\mathrm{Sic}'|^2|\nabla S'|^2}{(S')^{\gamma+2}} + \frac{2\gamma}{(S')^{\gamma+1}}\langle\nabla|\mathrm{Sic}'|^2,\nabla S'\rangle + \frac{4C^2}{n^2}\frac{S'}{(S')^\gamma} \\
&\quad - \frac{2C}{n}\frac{2(1-\gamma)S'+\gamma C}{(S')^{\gamma+1}}|\mathrm{Sic}'|^2 + \frac{4\alpha_1}{(S')^\gamma}\langle\mathrm{Sic}',\Xi\rangle - \frac{2\alpha_1\gamma|\mathrm{Sic}'|^2}{(S')^{\gamma+1}}\mathrm{tr}\Xi
\end{aligned}$$

where $\mathrm{tr}\Xi$ is the trace of Ξ with respect to $g(t)$, and

$$\Xi := \Delta u\nabla^2 u - \beta_2\nabla u\otimes\nabla u - \beta_1\nabla u\otimes\nabla|\nabla u|^2. \tag{3.18}$$

From the identities

$$\begin{aligned}
\left\langle\nabla\frac{|\mathrm{Sic}'|^2}{(S')^\gamma},\nabla S'\right\rangle &= \frac{1}{(S')^\gamma}\langle\nabla|\mathrm{Sic}'|^2,\nabla S'\rangle - \frac{\gamma}{(S')^{\gamma+1}}|\nabla S'|^2|\mathrm{Sic}'|^2, \\
|S'\nabla\mathrm{Sic}'|^2 &= |Z'|^2 - |\mathrm{Sic}'|^2|\nabla S'|^2 + S'\langle\nabla|\mathrm{Sic}'|^2,\nabla S'\rangle,
\end{aligned}$$

where Z' is a 3-tensor with components $Z'_{ijk} = S'\nabla_i S'_{jk} - S'_{jk}\nabla_i S'$, we have

$$\begin{aligned}
\Box\frac{|\mathrm{Sic}'|^2}{(S')^\gamma} &= \frac{2(\gamma-1)}{S'}\left\langle\nabla\frac{|\mathrm{Sic}'|^2}{(S')^\gamma},\nabla S'\right\rangle - \frac{2}{(S')^{\gamma+2}}|Z'|^2 - \frac{2\gamma|\mathrm{Sic}'|^4}{(S')^{(\gamma+1)}} \\
&\quad - \frac{(2-\gamma)(\gamma-1)}{(S')^{(\gamma+1)}}|\mathrm{Sic}'|^2|\nabla S'|^2 + \frac{4}{(S')^\gamma}\mathrm{Sm}(\mathrm{Sic}',\mathrm{Sic}') + \frac{4C^2}{n^2}\frac{S'}{(S')^\gamma} \\
&\quad - \frac{2C}{n}\frac{2(1-\gamma)S'+\gamma C}{(S')^{\gamma+1}}|\mathrm{Sic}'|^2 + \frac{4\alpha_1}{(S')^\gamma}\langle\mathrm{Sic}',\Xi\rangle - \frac{2\alpha_1\gamma|\mathrm{Sic}'|^2}{(S')^{\gamma+1}}\mathrm{tr}\Xi.
\end{aligned}$$

The identity

$$\Box(S')^{2-\gamma} = (2-\gamma)(S')^{1-\gamma}\Box S' - (2-\gamma)(1-\gamma)(S')^{-\gamma}|\nabla S'|^2$$

implies

$$\begin{aligned}
\Box f &= 2(\gamma-1)\langle\nabla f,\nabla\ln S'\rangle - \frac{2}{(S')^{\gamma+2}}|Z'|^2 - (2-\gamma)(\gamma-1)|\nabla\ln S'|^2 f \\
&\quad + \mathscr{D}_1 + \mathscr{D}_2 + \mathscr{D}_3.
\end{aligned} \tag{3.19}$$

where

$$
\begin{aligned}
\mathscr{D}_1 &:= -\frac{2(2-\gamma)}{n}(S')^{1-\gamma}|\mathrm{Sic}'|^2 + \frac{4}{(S')^\gamma}\mathrm{Sm}(\mathrm{Sic}',\mathrm{Sic}') - \frac{2\gamma|\mathrm{Sic}'|^4}{(S')^{1+\gamma}},\\
&= \frac{2}{(S')^{\gamma+1}}\left[(2-\gamma)|\mathrm{Sic}'|^2|\mathrm{Sin}|^2 - 2\left(|\mathrm{Sic}'|^4 - S'\mathrm{Sm}(\mathrm{Sic}',\mathrm{Sic}')\right)\right],\\
&= \frac{2}{(S')^{\gamma+1}}\Big[-\gamma(S')^{2\gamma}f^2 + \left(\frac{2n-4}{n(n-1)} - \frac{\gamma}{n}\right)(S')^{\gamma+2}f - \frac{4(S')^4}{n-2}\frac{\mathrm{Sin}^3}{(S')^3}\\
&\quad + 2(S')^3 W\left(\frac{\mathrm{Sin}}{S'},\frac{\mathrm{Sin}}{S'}\right) + \frac{2\alpha_1}{n-2}\left\langle (S')^2\mathrm{Sic}' - \frac{n}{2}S'\mathrm{Sic}'^2, \nabla u\otimes\nabla u\right\rangle\\
&\quad - \frac{2}{n-1}\left(\frac{C}{n} + \frac{\alpha|\nabla u|^2}{n-1}\right)\left(\frac{n-1}{n}(S')^3 - (S')^{\gamma+1}f\right)\Big],\\
\mathscr{D}_2 &:= \frac{4C}{n}\left[\frac{CS'}{n(S')^2} - \frac{(1-\gamma)S' + \frac{1}{2}\gamma C}{(S')^{\gamma+1}}|\mathrm{Sic}'|^2 - \frac{2-\gamma}{2n}\frac{C-2S'}{(S')^{\gamma-1}}\right],\\
&= \frac{4C}{n^2}\left[\frac{C}{S'} + \frac{C}{(S')^{\gamma-1}} + \frac{1}{(S')^{\gamma-2}} + nf\left(\gamma - 1 - \frac{\gamma C}{2S'}\right)\right],\\
\mathscr{D}_3 &:= \frac{4\alpha_1}{(S')^\gamma}\langle \mathrm{Sic}',\Xi\rangle - \frac{2\alpha_1\mathrm{tr}\Xi}{(S')^{\gamma+1}}\left(\gamma|\mathrm{Sic}'|^2 + \frac{2-\gamma}{n}|S'|^2\right)
\end{aligned}
$$

and $\mathrm{Sin}^3 = \mathrm{Sin}_{ij}\mathrm{Sin}^j{}_k\mathrm{Sin}^{ki}$ and $(\mathrm{Sic}'^2)_{ij} = S'_{ik}S'^k_j$. Some detailed computations can be found in [43].

In particular, for $\gamma = 2$, we have from (3.19) that

$$
\Box f = 2\langle \nabla f, \nabla \ln S'\rangle - 2\left|\nabla\left(\frac{\mathrm{Sin}}{S'}\right)\right|^2 + \mathscr{D}_1 + \mathscr{D}_2 + \mathscr{D}_3, \tag{3.20}
$$

where

$$
\begin{aligned}
\mathscr{D}_1 &= 4S'\Big[-f^2 - \frac{f}{n(n-1)} - \frac{2}{n-2}\frac{\mathrm{Sin}^3}{(S')^3} + \frac{1}{S'}W\left(\frac{\mathrm{Sin}}{S'},\frac{\mathrm{Sin}}{S'}\right)\\
&\quad - \frac{1}{S'}\left(\frac{C}{n} + \frac{\alpha|\nabla u|^2}{n-2}\right)\left(\frac{1}{n} - \frac{f}{n-1}\right) + \frac{1}{S'}\frac{\alpha}{n-2}\left\langle\frac{\mathrm{Sin}^2}{(S')^2}, \nabla u\otimes\nabla u\right\rangle\\
&\quad + \frac{1}{S'}\frac{\alpha|\nabla u|^2}{2n(n-2)}\Big],\\
\mathscr{D}_2 &= \frac{4C}{n^2}\left[\frac{C}{S'} + \frac{C}{(S')^3} + nf\left(1 - \frac{C}{S'}\right)\right],\\
\mathscr{D}_3 &= \frac{4\alpha_1}{S'}\left[\left\langle\frac{\mathrm{Sic}'}{S'},\Xi\right\rangle - f\mathrm{tr}\Xi\right].
\end{aligned}
$$

Since the flow is regular, we have a uniform upper bound for $|\nabla u|$, together with $S' \geq C_0 > 0$, and then

$$
\begin{aligned}
\mathscr{D}_1 &\leq 4S'\left[-f^2 - \frac{f}{n(n-1)} + \frac{2}{n-2}f^{3/2} + \tilde{C}\frac{|W|}{S'}f + \tilde{C} + \tilde{C}f\right],\\
\mathscr{D}_2 &\leq 4S'\left(\tilde{C} + \tilde{C}f\right),\\
\mathscr{D}_3 &\leq 4\tilde{C}S'\left(f + f^{1/2}\right)\frac{|\nabla^2 u|^2}{S'}
\end{aligned}
$$

for some uniform constant $\tilde{C}$ depending only on on $n, C_0, C, \alpha_1, \beta_1, \beta_2, g_0$, and u_0. Without loss of generality, we may assume that $f \geq 1$. In this case we have

$$\Box f \leq 2\langle \nabla f, \nabla \ln S' \rangle + 4S' f \left(-f + \frac{2}{n-2} f^{1/2} + \tilde{C} + \tilde{C} \frac{|W| + |\nabla^2 u|^2}{S'} \right).$$

Now the maximum principle yields the desired estimate. $\Box$

As immediate consequence, we have the following

Corollary 3.3. *Let $(g(t), u(t))_{t\in[0,T)}$ be a solution to the regular $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flow on a closed n-dimensional manifold M with $T \leq 0$ and the initial data (g_0, u_0). Then only one of the followings cases occurs:*

(a) $T = \infty$;
(b) $T < \infty$ *and* $|\text{Ric}_{g(t)}|_{g(t)} \lesssim 1$;
(c) $T < \infty$ *and* $\text{Ric}_{g(t)}|_{g(t)} \to \infty$ *as* $t \to T$. *In this case, there are only two subcases:*
  (c1) $|R_{g(t)}|_{g(t)} \to \infty$,
  (c2) $|R_{g(t)}|_{g(t)} \lesssim 1$ *and there exist some uniform constants* $C_1, C_2 > 0$ *such that* $S_{g(t)} + C_1 \geq C_2 > 0$ *and*

$$\frac{|W_{g(t)}|_{g(t)} + |\nabla^2_{g(t)} u(t)|^2_{g(t)}}{S_{g(t)} + C_1} \to \infty$$

  as $t \to T$.

This is a general property of the long time existence for a regular Ricci flow, generalizing results in [7, 43]. Since the signs of $\alpha_1, \beta_1, \beta_2$ are not determined, we can not discard the case (b) which is true for Ricci-harmonic flow and Ricci flow.

3.2. **Bounded scalar curvature.** We assume that $(g(t), u(t))_{t\in[0,T)}$ is a solution to the regular $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flow on a closed 4-dimensional manifold M with $T \leq \infty$ and the initial data (g_0, u_0), and also assume that $S_{g(t)} + C \geq C_0 > 0$ along the flow for some uniform constants $C, C_0 > 0$. According to (3.13) one has

$$|\nabla_{g(t)} u(t)|^2_{g(t)} \leq A_1 \tag{3.21}$$

along the flow.

In the proof of Theorem 3.2 we actually proved the following identity

$$\begin{aligned} \Box \frac{|\text{Sic}|^2}{S+C} &= -2\frac{|Z|^2}{(S+C)^3} - 2\frac{|\text{Sic}|^4}{(S+C)^2} + 4\frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+C} \\ &\quad - \frac{2\alpha_1}{(S+C)^2}\left[\text{tr}\Xi|\text{Sic}|^2 - 2(S+C)\langle \text{Sic}, \Xi\rangle\right], \end{aligned} \tag{3.22}$$

where Z is the 3-tensor with components

$$Z_{ijk} = S' \nabla_i S_{jk} - S_{jk} \nabla_i S' = (S+C)\nabla_i S_{jk} - S_{jk}\nabla_i S, \tag{3.23}$$

and Ξ is given in (3.18). Observe that the last term in (3.21) can be written as

$$\begin{aligned} &\text{tr}\Xi|\text{Sic}|^2 - 2(S+C)\langle \text{Sic}, \Xi\rangle \\ = &\left[(S+C)\left|\Delta u \frac{\text{Sic}}{\sqrt{S+C}} - \sqrt{S+C}\nabla^2 u\right|^2 - (S+C)^2|\nabla^2 u|^2\right] \end{aligned}$$

$$- \beta_2 \left[(S+C) \left(|\nabla u| \frac{\mathrm{Sic}}{\sqrt{S+C}} - \frac{\sqrt{S+C}}{|\nabla u|} \nabla u \otimes \nabla u \right)^2 - (S+C)^2 |\nabla u|^2 \right]$$

$$- 2\beta_1 (S+C) \left[\langle \nabla^2 u, \nabla u \otimes \nabla u \rangle \frac{|\mathrm{Sic}|^2}{S+C} - 2 \left\langle \mathrm{Sic}, \frac{1}{2} \nabla u \otimes \nabla |\nabla u|^2 \right\rangle \right].$$

To further analysis, we need to know an estimate for $|\nabla^2 u|^2$. Recall that

$$\Box |\nabla u|^2 = 2\beta_2 |\nabla u|^2 - 2|\nabla^2 u|^2 - 2\alpha_1 |\nabla u|^4 + 4\beta_1 \langle \nabla u \otimes \nabla u, \nabla^2 u \rangle. \tag{3.24}$$

Given any $\epsilon > 0$, we have from (3.24) that

$$\begin{aligned} \Box |\nabla u|^2 &\leq 2\beta_2 |\nabla u|^2 - 2|\nabla^2 u|^2 - 2\alpha_1 |\nabla u|^4 + 4|\beta_1| \left(\epsilon |\nabla^2 u|^2 + \frac{1}{4\epsilon} |\nabla u|^4 \right) \\ &= 2\beta_2 |\nabla u|^2 - 2\left(1 - 2\epsilon|\beta_1|\right) |\nabla^2 u|^2 - 2\left(\alpha_1 - \frac{|\beta_1|}{2\epsilon} \right) |\nabla u|^4. \end{aligned}$$

From the evolution equation $\partial_t dV_t = -S dV_t$, we arrive at

$$\begin{aligned} \frac{d}{dt} \int_M |\nabla u|^2 dV_t &= \int_M \partial_t |\nabla u|^2 dV_t + \int_M |\nabla u|^2 \partial_t dV_t \\ &= \int_M \Box |\nabla u|^2 dV_t - \int_M S |\nabla u|^2 dV_t \\ &\leq 2\beta_2 \int_M |\nabla u|^2 dV_t - 2\left(1 - 2\epsilon|\beta_1|\right) \int_M |\nabla^2 u|^2 dV_t \\ &\quad - \int_M S |\nabla u|^2 dV_t - 2\left(\alpha_1 - \frac{|\beta_1|}{2\epsilon} \right) \int_M |\nabla u|^4 dV_t \end{aligned}$$

and then

$$\begin{aligned} \frac{d}{dt} \int_M |\nabla u|^2 dV_t &\leq -2\left(1 - 2\epsilon|\beta_1|\right) \int_M |\nabla^2 u|^2 dV_t \\ &\quad + (2|\beta_2| + C) \int_M |\nabla u|^2 - 2\left(\alpha_1 - \frac{|\beta_1|}{2\epsilon} \right) \int_M |\nabla u|^4 dV_t, \end{aligned} \tag{3.25}$$

because of $S + C \geq C_0 > 0$.

(1) $\beta_1 = 0$. In this case, the inequality becomes

$$\frac{d}{dt} \int_M |\nabla u|^2 dV_t \leq -2 \int_M |\nabla^2 u|^2 dV_t + (2|\beta_2| + C) \int_M |\nabla u|^2 dV_t - 2\alpha_1 \int_M |\nabla u|^4 dV_t.$$

When $\alpha_1 \geq 0$, we furthermore have

$$\frac{d}{dt} \int_M |\nabla u|^2 dV_t \leq -2 \int_M |\nabla^2 u|^2 dV_t + (2|\beta_2| + C) \int_M |\nabla u|^2 dV_t$$

or in this form

$$\frac{d}{dt} \left[e^{-(2|\beta_2| + C)t} \int_M |\nabla u|^2 dV_t \right] \leq -2 e^{-(2|\beta_2| + C)t} \int_M |\nabla^2 u|^2 dV_t.$$

Integrating over the interval $[0, t]$ and using (3.21), we obtain

$$2 \int_0^t \int_M |\nabla^2 u|^2 dV_t dt + \int_M |\nabla u|^2 dV_t \leq e^{(2|\beta_2| + C)t} A_1 \mathrm{Vol}_0 \tag{3.26}$$

where Vol_0 is the volume of the initial metric g_0. When $\alpha_1 < 0$, we similar have

$$\frac{d}{dt}\int_M |\nabla u|^2 dV_t \le -2\int_M |\nabla^2 u|^2 dV_t + (2|\beta_2| + 2|\alpha_1|A_1 + C)\int_M |\nabla u|^2 dV_t.$$

Replacing $|\beta_2|$ by $|\beta_2| + |\alpha_1|A_1$ in (3.26), the case that α_1 is negative implies

$$2\int_0^t \int_M |\nabla^2 u|^2 dV_t dt + \int_M |\nabla u|^2 dV_t \le e^{(2|\beta_2|+2|\alpha_1|A_1+C)t} A_1 \mathrm{Vol}_0. \tag{3.27}$$

(2) $\beta_1 \neq 0$. In this case we choose $\epsilon = 1/4|\beta_1|$ in (3.25) and obtain

$$\begin{aligned}\frac{d}{dt}\int_M |\nabla u|^2 dV_t \quad &\le \quad -\int_M |\nabla^2 u|^2 dV_t + (2|\beta_2| + C)\int_M |\nabla u|^2 dV_t \\ &\quad - 2(\alpha_1 - 2\beta_1^2)\int_M |\nabla u|^4 dV_t.\end{aligned}$$

A similar argument used to obtain equations (3.26) and (3.27) we get

$$\int_0^t \int_M |\nabla^2 u|^2 dV_t dt + \int_M |\nabla u|^2 dV_t \le e^{(2|\beta_2|+2|\alpha_1 - 2\beta_1^2|A_1+C)t} A_1 \mathrm{Vol}_0. \tag{3.28}$$

Finally, from (3.27) and (3.28), we have

$$\int_0^t A_2(t)\,dt \le e^{(2|\beta_2|+2|\alpha_1 - 2\beta_1^2|A_1+C)t} A_1 \mathrm{Vol}_0 \tag{3.29}$$

where

$$A_2(t) := \int_M |\nabla^2 u|^2 dV_t. \tag{3.30}$$

Introduce

$$\Lambda := \frac{1}{(S+C)^2}\left[\mathrm{tr}\Xi|\mathrm{Sic}|^2 - 2(S+C)\langle \mathrm{Sic}, \Xi\rangle\right], \quad f := \frac{|\mathrm{Sic}|^2}{S+C} \tag{3.31}$$

and rewrite (3.22) as

$$\Box f = -2\frac{|Z|^2}{(S+C)^3} - 2f^2 + 4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+C} - 2\alpha_1\Lambda. \tag{3.32}$$

To determine a lower bound for $\alpha_1\Lambda$ we consider the following cases.

(i) $\alpha_1 \ge 0$. In this case we shall also find a lower bound for Λ.

(ia) When $\beta_2 \le 0$, we have

$$\begin{aligned}\Lambda \quad &\ge \quad -|\nabla^2 u|^2 - |\beta_2||\nabla u|^2 - 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{S+C} f - 2|\beta_1|\left(f + \frac{|\nabla u|^4|\nabla^2 u|^2}{S+C}\right) \\ &\ge \quad -|\nabla^2 u|^2 - |\beta_2||\nabla u|^2 - 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{C_0} f - 2|\beta_1|\left(f + \frac{|\nabla^2 u|^2|\nabla u|^4}{C_0}\right).\end{aligned}$$

(ib) When $\beta_2 > 0$, we get the same estimate in (ia), where, in the case, the term $-|\beta_2||\nabla u|^2$ is now replaced by

$$-\frac{\beta_2}{S+C}\left(|\nabla u|\frac{\mathrm{Sic}}{\sqrt{S+C}} - \frac{\sqrt{S+C}}{|\nabla u|}\nabla u \otimes \nabla u\right)^2$$

which is bounded below by

$$-\frac{2\beta_2}{S+C}\left(f|\nabla u|^2 + \frac{S+C}{|\nabla u|^2}|\nabla u|^4\right) \ge -2\beta_2|\nabla u|^2\left(1 + \frac{f}{C_0}\right).$$

From (ia) – (ib), we obtain for any β_2

$$\begin{aligned}\Lambda \;&\geq\; -|\nabla^2 u|^2 - 2|\beta_2||\nabla u|^2\left(1+\frac{f}{C_0}\right)\\ &\quad - 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{C_0}f - 2|\beta_1|\left(f+\frac{|\nabla^2 u|^2|\nabla u|^4}{C_0}\right).\end{aligned} \tag{3.33}$$

(ii) $\alpha_1 < 0$. In this case we shall find an upper bound for Λ.
(iia) When $\beta_2 > 0$, we have

$$\begin{aligned}\Lambda \;&\leq\; \frac{1}{(S+C)^2}\left[(S+C)\left|\Delta u\frac{\mathrm{Sic}}{\sqrt{S+C}} - \sqrt{S+C}\nabla^2 u\right|^2 + \beta_2(S+C)^2|\nabla u|^2\right]\\ &\quad + 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{S+C}f + 2|\beta_1|\left(f+\frac{|\nabla u|^4|\nabla^2 u|^2}{S+C}\right)\\ &= \frac{1}{S+C}\left|\Delta u\frac{\mathrm{Sic}}{\sqrt{S+C}} - \sqrt{S+C}\nabla^2 u\right|^2 + \beta_2|\nabla u|^2\\ &\quad + 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{S+C}f + 2|\beta_1|\left(f+\frac{|\nabla u|^4|\nabla^2 u|^2}{S+C}\right)\\ &\leq \frac{2}{S+C}\left[(\Delta u)^2 f + (S+C)|\nabla^2 u|^2\right] + \beta_2|\nabla u|^2\\ &\quad + 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{S+C}f + 2|\beta_1|\left(f+\frac{|\nabla u|^4|\nabla^2 u|^2}{S+C}\right)\\ &\leq 2|\nabla^2 u|^2\left(1+\frac{4f}{C_0}\right) + \beta_2|\nabla u|^2\\ &\quad + 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{S+C}f + 2|\beta_1|\left(f+\frac{|\nabla u|^4|\nabla^2 u|^2}{S+C}\right)\end{aligned}$$

(iib) When $\beta_2 \leq 0$, we also have

$$\begin{aligned}\Lambda \;&\leq\; 2|\nabla^2 u|^2\left(1+\frac{4f}{C_0}\right) - 2\beta_2|\nabla u|^2\left(1+\frac{f}{C_0}\right)\\ &\quad + 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{S+C}f + 2|\beta_1|\left(f+\frac{|\nabla u|^4|\nabla^2 u|^2}{S+C}\right).\end{aligned}$$

From (iia) – (iib), we obtain for any β_2

$$\begin{aligned}\Lambda \;&\leq\; 2|\nabla^2 u|^2\left(1+\frac{4f}{C_0}\right) + 2|\beta_2||\nabla u|^2\left(1+\frac{f}{C_0}\right)\\ &\quad + 2|\beta_1|\frac{|\nabla^2 u||\nabla u|^2}{C_0}f + 2|\beta_1|\left(f+\frac{|\nabla u|^4|\nabla^2 u|^2}{C_0}\right).\end{aligned} \tag{3.34}$$

Lemma 3.4. *If* $\alpha_1 \geq 0$, *one has*

$$\begin{aligned} \frac{d}{dt}\int_M f\,dV_t &\leq \int_M \left[-2f^2 + 4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+C} - fS\right] dV_t \\ &+ \int_M 4\alpha_1\left[|\beta_1| + \frac{A_1(|\beta_2| + |\beta_1||\nabla^2 u|)}{C_0}\right] f\,dV_t \\ &+ 2\alpha_1\left(1 + \frac{2A_1^2|\beta_1|}{C_0}\right) A_2 + 4\alpha_1 A_1|\beta_2|\mathrm{Vol}_t. \end{aligned} \tag{3.35}$$

If $\alpha_1 < 0$, *one has*

$$\begin{aligned} \frac{d}{dt}\int_M f\,dV_t &\leq \int_M \left[-2f^2 + 4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+C} - fS\right] dV_t \\ &- \int_M 4\alpha_1\left[|\beta_1| + \frac{A_1(|\beta_2| + |\beta_1||\nabla^2 u|) + 4|\nabla^2 u|^2}{C_0}\right] f\,dV_t \\ &- 2\alpha_1\left(2 + \frac{2A_1^2|\beta_1|}{C_0}\right) A_2 - 4\alpha_1 A_1|\beta_2|\mathrm{Vol}_t. \end{aligned} \tag{3.36}$$

Here Vol_t *denotes the volume of* $g(t)$.

This follows immediately from (3.32) – (3.34). According to (3.35) and (3.36) we have

$$\begin{aligned} \frac{d}{dt}\int_M f\,dV_t &\leq \int_M \left[-2f^2 + 4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+C} - fS\right] dV_t + \int_M 4|\alpha_1|\Bigg[|\beta_1| + \\ &\quad \frac{A_1(|\beta_2| + |\beta_1||\nabla^2 u|) + 4(1-\mathrm{sgn}(\alpha_1,0))|\nabla^2 u|^2}{C_0}\Bigg] f\,dV_t \\ &+ 4|\alpha_1|\left(1 + \frac{A_1^2|\beta_1|}{C_0}\right) A_2 + 4|\alpha_1|A_1|\beta_2|\mathrm{Vol}_t, \end{aligned} \tag{3.37}$$

where $\mathrm{sgn}(\alpha_1,0) = 1$ if $\alpha_1 \geq 0$, and otherwise 0. For $\alpha_1 \geq 0$, the above estimates shall imply integrals bounds for $|\mathrm{Sic}|$, $|\mathrm{Sm}|$ as in [43]. On the other hand, the case $\alpha_1 < 0$ will prevent us to obtain the previous-type of estimates, since we have no control on $|\nabla^2 u|^2$ even in the integral sense. Hence in the case that α_1 is negative, we will impose another condition[7]:

$$|\nabla^2_{g(t)} u(t)|^2_{g(t)} \leq \widetilde{A}_1 \tag{3.38}$$

along the flow for some uniform constant $\widetilde{A}_1$. Note that the condition (3.38) is stronger than (3.29). A weakened condition is

$$||\nabla^2_{g(t)} u(t)||_{L^4(M,g(t))} \leq \widehat{A}_1 \tag{3.39}$$

along the flow for some uniform constant $\widehat{A}_1$.

[7]For the Ricci-harmonic flow or the $(\alpha,0,0,0)$-Ricci flow, the estimate (3.38) is always true provided that the curvature condition (1.4) holds.

In the the case of dimension 4, we have the following Gauss-Bonnet-Chern formula

$$32\pi^2\chi(M) = \int_M \left[|\mathrm{Rm}|^2 - 4|\mathrm{Ric}|^2 + R^2\right] dV_g \tag{3.40}$$

for any Riemannian metric g on M, where $\chi(M)$ is the Euler characteristic number of M. Applying the formula (3.40) to $g(t)$ and noting that (see Lemma 3.1 in [43])

$$\begin{aligned} |\mathrm{Rm}|^2 - 4|\mathrm{Ric}|^2 + R^2 &= |\mathrm{Sm}|^2 - 4|\mathrm{Sic}|^2 + S^2 \\ &\quad - \frac{13}{2}\alpha_1^2|\nabla u|^4 - 9\alpha_1 \mathrm{Sic}(\nabla u, \nabla u) + 2\alpha_1 S|\nabla u|^2 \end{aligned}$$

we have (see (3.12) in [43])

$$\begin{aligned} \int_M \left[|\mathrm{Sm}|^2 - 4|\mathrm{Sic}|^2 + S^2\right] dV_t &= 32\pi^2\chi(M) + \frac{13}{2}\alpha_1^2 \int_M |\nabla u|^4 dV_t \\ &\quad + 9\alpha_1 \int_M \mathrm{Sic}(\nabla u, \nabla u)\, dV_t - 2\alpha_1 \int_M S|\nabla u|^2 dV_t. \end{aligned} \tag{3.41}$$

Moreover we also have (see (3.15) in [43])

$$\begin{aligned} \int_M \left[-2f^2 + 4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+C} - fS\right] dV_t &\le \int_M \left(-f^2 + 36Cf + 574S^2\right) dV_t \\ &\quad + 8\left[32\pi^2\chi(M) + 13\alpha_1^2 A_1^2 \mathrm{Vol}_0 e^{(2|\beta_2| + 2|\alpha_1 - 2\beta_1^2|A_1 + C)t}\right] \end{aligned} \tag{3.42}$$

where we used (3.28) to control the integral

$$\int_M |\nabla u|^4 dV_t.$$

Plugging (3.42) into (3.37) and using (2.5) yields

$$\begin{aligned} \frac{d}{dt}\int_M f\, dV_t &\le \int_M \left(-\frac{1}{2}f^2 + C_1 f + C_2 S^2\right) dV_t \\ &\quad + C_3 A_2 + C_4\left[1 - \mathrm{sgn}(\alpha_1, 0)\right] \int_M |\nabla^2 u|^4 dV_t + C_5 e^{C_6 t} + C_7. \end{aligned} \tag{3.43}$$

where

$$\begin{aligned} C_1 &= 36C + 4|\alpha_1||\beta_1| + \frac{4|\alpha_1|A_1|\beta_2|}{C_0} = C_1(C, C_0, \alpha_1, \beta_1, \beta_2, A_1), \\ C_2 &= 574, \\ C_3 &= \frac{16|\alpha_1|^2 A_1^2|\beta_1|^2}{C_0^2} + 4|\alpha_1|\left(1 + \frac{A_1^2|\beta_1|}{C_0}\right) = C_3(C_0, \alpha_1, \beta_1, A_1), \\ C_4 &= \frac{256|\alpha_1|^2}{C_0^2} = C_4(C_0, \alpha_1), \\ C_5 &= (104\alpha_1^2 A_1^2 + 4|\alpha_1|A_1|\beta_2|)\mathrm{Vol}_0 = C_5(\alpha_1, \beta_2, A_1, \mathrm{Vol}_0), \\ C_6 &= 2|\beta_2| + 2|\alpha_1 - 2\beta_1^2||A_1| + C = C_6(C, \alpha_1, \beta_1, \beta_2, A_1), \\ C_7 &= 256\pi^2\chi(M). \end{aligned} \tag{3.44}$$

Note that C_1, C_6 are linear functions of A_1 and C_3, C_5 are quadratic functions of A_1. Furthermore, C_2, C_7 are constants depending only on the topological quantities of M. Finally, C_4 depends only on α_1, and the term containing C_4 in (3.43) vanishes provided that $\alpha_1 \ge 0$.

Theorem 3.5. *Let $(g(t),u(t))_{t\in[0,T)}$ be a solution to the regular $(\alpha_1,0,\beta_1,\beta_2)$-Ricci flow on a closed 4-manifold M with $T\leq\infty$ and the initial data (g_0,u_0). Assume that $S_{g(t)}+C\geq C_0>0$ along the flow for some uniform constants $C,C_0>0$. Then*

$$\begin{aligned}&\int_M\frac{|\mathrm{Sic}_{g(s)}|^2_{g(s)}}{S_{g(s)}+C}dV_{g(s)}+\int_0^s\int_M\frac{|\mathrm{Sic}_{g(t)}|^4_{g(t)}}{(S_{g(t)}+C)^2}dV_{g(t)}dt\\ \leq\ & C'(1+s)e^{C's}+C'e^{C's}\int_0^s\int_M S^2dV_tdt\\ &+C'[1-\mathrm{sgn}(\alpha_1,0)]e^{C's}\int_0^s\int_M|\nabla^2u|^4dV_tdt,\end{aligned}\tag{3.45}$$

$$\begin{aligned}\int_M|\mathrm{Sic}_{g(s)}|_{g(s)}dV_{g(s)}\ \leq\ & C'(1+s)e^{C's}+C'e^{C's}\int_0^s\int_M S^2dV_tdt\\ &+C'[1-\mathrm{sgn}(\alpha_1,0)]e^{C's}\int_0^s\int_M|\nabla^2u|^4dV_tdt,\end{aligned}\tag{3.46}$$

$$\begin{aligned}\int_0^s\int_M|\mathrm{Sic}_{g(t)}|^2_{g(t)}dV_{g(t)}dt\ \leq\ & C'(1+s)e^{C's}+C'e^{C's}\int_0^s\int_M S^2dV_tdt\\ &+C'[1-\mathrm{sgn}(\alpha_1,0)]e^{C's}\int_0^s\int_M|\nabla^2u|^4dV_tdt,\end{aligned}\tag{3.47}$$

$$\begin{aligned}\int_0^s\int_M|\mathrm{Sm}_{g(t)}|^2_{g(t)}dV_{g(t)}dt\ \leq\ & C'(1+s)e^{C's}+C'e^{C's}\int_0^s\int_M S^2dV_tdt\\ &+C'[1-\mathrm{sgn}(\alpha_1,0)]e^{C's}\int_0^s\int_M|\nabla^2u|^4dV_tdt,\end{aligned}\tag{3.48}$$

for all $s\in[0,T)$, where $C'=C'(g_0,u_0,\alpha_1,\beta_1,\beta_2,C,C_0,A_1,\chi(M))$ is a uniform constant. Here $|\nabla_{g(t)}u(t)|^2_{g(t)}\leq A_1$ holds along the flow (by the regularity) for some uniform constant $A_1>0$ (which depends only on g_0,u_0 and α_0,β_1,β_2).

Proof. Integrating (3.43) over $[0,s]$ we obtain

$$\begin{aligned}&e^{-C_1t}\int_M f\,dV_t+\frac{1}{2}e^{-C_1s}\int_0^s\int_M f^2dV_tdt\\ \leq\ &\int_0^s\left[C_3A_2e^{-C_1t}+C_7e^{-C_1t}+C_5e^{(C_6-C_1)t}\right]dt+C_2\int_0^s e^{-C_1t}\int_M S^2dV_tdt\\ &+C_4\int_0^s e^{-C_1t}[1-\mathrm{sgn}(\alpha_1,0)]\int_M|\nabla^2u|^4dV_tdt+\int_M f\,dV_t\Big|_{t=0}.\end{aligned}$$

Using (3.29), the above inequality becomes

$$\begin{aligned}\int_M f\,dV_s+\int_0^s\int_M f^2dV_tdt\ \leq\ & 2e^{C_1s}\left(\int_M f\,dV_t\Big|_{t=0}\right)\\ +2C_3e^{(C_1+C_6)s}+\frac{2C_7}{C_1}e^{C_1s}+2C_5\cdot&\begin{cases}se^{C_1s}, & C_1=C_6,\\ e^{(C_1+C_6)s}/|C_1-C_6|, & C_1\neq C_6,\end{cases}\\ +2C_2e^{C_1s}\int_0^s\int_M S^2dV_tdt+2C_5&[1-\mathrm{sgn}(\alpha_1,0)]e^{C_1s}\int_0^s\int_M e^{-C_1t}|\nabla^2u|^4dV_tdt.\\ \leq\ & C_8(1+s)e^{(C_1+C_6)s}+2C_2e^{C_1s}\int_0^s\int_M S^2dV_tdt\end{aligned}$$

$$+ 2C_5[1-\operatorname{sgn}(\alpha_1,0)]e^{C_1 s}\int_0^s\int_M e^{-C_1 t}|\nabla^2 u|^4 dV_t dt$$

for some uniform constant C_8 which depends only on $g_0, u_0, \alpha_1, \beta_1, \beta_2, C, C_0$, and $A_1, \chi(M)$.

The second and third estimates follows from (see (3.22) and below in [43])

$$|\mathrm{Sic}| \le 2\frac{|\mathrm{Sic}|^2}{S+C} + \frac{C}{2}, \quad |\mathrm{Sic}|^2 \le 8\frac{|\mathrm{Sic}|^4}{(S+C)^2} + \frac{C^2}{4},$$

and $\mathrm{Vol}_t \le e^{Ct}\mathrm{Vol}_0$.

The last estimate follows from (3.41), as the argument in [43]. □

Corresponding to case $(c2)$ in Corollary 3.3, we obtain

Corollary 3.6. *Let $(g(t), u(t))_{t\in[0,T)}$ be a solution to the regular $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flow on a closed 4-manifold M with $T \le \infty$ and the initial data (g_0, u_0). Assume that $S_{g(t)} + C \ge C_0 > 0$ and $S^2_{g(t)} \le C_1 < \infty$ along the flow for some uniform constants $C, C_0, C_1 > 0$. Then*

$$\begin{aligned}\int_0^s\int_M |\mathrm{Sic}_{g(t)}|^2_{g(t)} dV_{g(t)} dt &\le C'(1+s)e^{C's} \\ &\quad + C'[1-\operatorname{sgn}(\alpha_1,0)]e^{C's}\int_0^s\int_M |\nabla^2 u|^4 dV_t dt,\end{aligned} \tag{3.49}$$

$$\begin{aligned}\int_0^s\int_M |\mathrm{Sm}_{g(t)}|^2_{g(t)} dV_{g(t)} dt &\le C'(1+s)e^{C's} \\ &\quad + C'[1-\operatorname{sgn}(\alpha_1,0)]e^{C's}\int_0^s\int_M |\nabla^2 u|^4 dV_t dt,\end{aligned} \tag{3.50}$$

for all $s \in [0,T)$, where $C' = C'(g_0, u_0, \alpha_1, \beta_1, \beta_2, C, C_0, C_1, A_1, \chi(M))$ is a uniform constant. Here $|\nabla_{g(t)} u(t)|^2_{g(t)} \le A_1$ holds along the flow (by the regularity) for some uniform constant $A_1 > 0$ (which depends only on g_0, u_0 and $\alpha_0, \beta_1, \beta_2$).

To get the $L^1_{[0,T)}L^p(M)$-estimate of $\mathrm{Sic}_{g(t)}$, introduce the basic assumption **BA** for a solution $(g(t), u(t))_{t\in[0,T)}$ to the regular $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flow:

(a) M is a closed 4-manifold;
(b) $T < \infty$;
(c) $-1 \le S_{g(t)} \le 1$ along the flow;
(d) $|\nabla_{g(t)} u(t)|^2_{g(t)} \le A_1$ along the flow.

The last condition is obtained from the regularity of the flow and the third condition implies $S_{g(t)} + C \ge C_0 > 0$, where $C = 2$ and $C_0 = 1$.

Under **BA**, given $\epsilon > 0$, one has (see the proof of Theorem 3.4 in [43])

$$\begin{aligned}&\int_M\left[-2f^2 + 4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+2} - fS\right]dV_t \\ \le\ &\int_M\left[-\left(2-\frac{4}{\epsilon^2}\right)f^2 + (12\epsilon^2+1)f + \epsilon^2\left(|\mathrm{Sm}|^2 - 4|\mathrm{Sic}|^2 + S^2\right)\right]dV_t\end{aligned}$$

where $f = |\mathrm{Sic}|^2/(S+2)$. In this case, $C_0 = 1$, so that (3.37) becomes

$$\frac{d}{dt}\int_M f\,dV_t \le \int_M\left[-2f^2 + 4\frac{\mathrm{Sm}(\mathrm{Sic},\mathrm{Sic})}{S+2} - fS\right]dV_t$$

$$
\begin{aligned}
&+ \int_M 4|\alpha_1| \left[|\beta_1| + A_1(|\beta_2| + |\beta_1||\nabla^2 u|) + 4(1-\mathrm{sgn}(\alpha_1,0))|\nabla^2 u|^2 \right] f\, dV_t \\
&\qquad + 4|\alpha_1|(1+A_1^2|\beta_1|) + 4|\alpha_1|A_1|\beta_2|e^{2t}\mathrm{Vol}_0. \\
&\leq \int_M \left[-\left(2 - \frac{12}{\epsilon^2}\right) f^2 + \left(12\epsilon^2 + 1 + 4|\alpha_1\beta_1| + 4|\alpha_1\beta_2|A_1\right) f \right] dV_t \\
&+ \epsilon^2 \int_M \left(|\mathrm{Sm}|^2 - 4|\mathrm{Sic}|^2 + S^2 \right) dV_t + 4|\alpha_1|(1+A_1^2|\beta_1|)A_2 + 4|\alpha_1|A_1|\beta_2|e^{2t}\mathrm{Vol}_0 \\
&\quad + \epsilon^2 \int_M \left[(\alpha_1\beta_1)^2 A_1^2 |\nabla^2 u|^2 + 256|\alpha_1|^2(1-\mathrm{sgn}(\alpha_1,0))|\nabla^2 u|^4 \right] dV_t.
\end{aligned}
$$

On the other hand, the identity (3.41) implies (see (3.13) in [43])

$$
\int_M \left[|\mathrm{Sm}|^2 - 4|\mathrm{Sic}|^2 + S^2 \right] dV_t \leq 32\pi^5 \chi(M) + 13\alpha_1^2 \int_M |\nabla u|^4 dV_t + \frac{243}{26} \int_M f\, dV_t.
$$

Therefore, taking $\epsilon^2 = 12$ and using (3.28),

$$
\begin{aligned}
\frac{d}{dt}\int_M f\, dV_t &\leq \int_M \left(-f^2 + \widetilde{C}_1 f \right) dV_t + \widetilde{C}_2 A_2 + \widetilde{C}_3 + \widetilde{C}_4 e^{\widetilde{C}_5 t} \\
&\quad + \widetilde{C}_6 [1 - \mathrm{sgn}(\alpha_1, 0)] \int_M |\nabla^2 u|^4 dV_t, \qquad (3.51)
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{C}_1 &= 145 + 4|\alpha_1\beta_1| + 4|\alpha_1\beta_2|A_1 + \frac{1458}{13} = \widetilde{C}_1(\alpha_1, \beta_1, \beta_2, A_1), \\
\widetilde{C}_2 &= 4|\alpha_1|(1 + A_1^2|\beta_1|) + 12(\alpha_1\beta_1)^2 A_1^2 = \widetilde{C}_2(\alpha_1, \beta_1, A_1), \\
\widetilde{C}_3 &= 385\pi^5 \chi(M), \\
\widetilde{C}_4 &= \left(156\alpha_1^2 A_1 + 4|\alpha_1|A_1|\beta_2|\right) \mathrm{Vol}_0 = \widetilde{C}_4(\alpha_1, \beta_2, A_1, \mathrm{Vol}_0), \\
\widetilde{C}_5 &= 2|\beta_2| + 2|\alpha_1 - 2\beta_1^2|A_1 + 2 = \widetilde{C}_5(\alpha_1, \beta_1, \beta_2, A_1), \\
\widetilde{C}_6 &= 3072|\alpha_1|^2 = \widetilde{C}_6(\alpha_1).
\end{aligned}
$$

Theorem 3.7. *Suppose that $(g(t), u(t))_{t\in[0,T)}$ satisfies* **BA**. *Then*

$$
\begin{aligned}
\int_M |\mathrm{Sic}_{g(s)}|^2_{g(s)} dV_{g(s)} &\leq \widetilde{C}(1+s)e^{\widetilde{C}s} \\
&\quad + \widetilde{C}[1 - \mathrm{sgn}(\alpha_1,0)] e^{\widetilde{C}s} \int_0^s \int_M |\nabla^2_{g(t)} u(t)|^4 dV_{g(t)} dt, \qquad (3.52) \\
\int_M |\mathrm{Sm}_{g(s)}|^2_{g(s)} dV_{g(s)} &\leq \widetilde{C}(1+s)e^{\widetilde{C}s} \\
&\quad + \widetilde{C}[1 - \mathrm{sgn}(\alpha_1,0)] e^{\widetilde{C}s} \int_0^s \int_M |\nabla^2_{g(t)} u(t)|^4 dV_{g(t)} dt, \qquad (3.53) \\
\int_0^s \int_M |\mathrm{Sic}_{g(t)}|^4_{g(t)} dV_{g(t)} dt &\leq \widetilde{C}(1+s)e^{\widetilde{C}s} \\
&\quad + \widetilde{C}[1 - \mathrm{sgn}(\alpha_1,0)] e^{\widetilde{C}s} \int_0^s \int_M |\nabla^2_{g(t)} u(t)|^4 dV_{g(t)} dt, \qquad (3.54) \\
\int_s^T \int_M |\mathrm{Sic}_{g(t)}|^p_{g(t)} dV_{g(t)} dt &\leq \left[(T-s)e^T \mathrm{Vol}_0 \right]^{\frac{4-p}{4}} e^{\widetilde{C}T} \left[\widetilde{C}(1+T) \right. \\
&\quad \left. + \widetilde{C}[1 - \mathrm{sgn}(\alpha_1,0)] \int_0^T \int_M |\nabla^2_{g(t)} u(t)|^4 dV_{g(t)} dt \right]^{\frac{p}{4}} \qquad (3.55)
\end{aligned}
$$

for any $s \in [0, T)$ *and* $0 < p < 4$*. Here* $\widetilde{C}$ *is a uniform constant which depends only on* $g_0, u_0, \alpha_1, \beta_1, \beta_2, A_1, \chi(M)$.

Proof. From (3.51) and (3.29), one has

$$e^{-\widetilde{C}_1 s}\int_M f\,dV_s + e^{-\widetilde{C}_1 s}\int_0^s\int_M f^2 dV_t dt \;\leq\; \int_M f\,dV_t\Big|_{t=0}$$

$$+ \widetilde{C}_2 e^{\widetilde{C}_5 s} + \frac{\widetilde{C}_3}{\widetilde{C}_1}\left(1 - e^{-\widetilde{C}_1 s}\right) + \widetilde{C}_4\cdot\begin{cases} s, & \widetilde{C}_1 = \widetilde{C}_5, \\ e^{(\widetilde{C}_1+\widetilde{C}_5)s}/|\widetilde{C}_1+\widetilde{C}_5|, & \widetilde{C}_1 \neq \widetilde{C}_5, \end{cases}$$

$$+ \widetilde{C}_6[1 - \mathrm{sgn}(\alpha_1, 0)]\int_0^s\int_M e^{-\widetilde{C}_1 t}|\nabla^2 u|^4 dV_t dt$$

and hence

$$\int_M f\,dV_s + \int_0^s\int_M f^2 dV_t dt \;\leq\; \widetilde{C}_7(1+s)e^{(\widetilde{C}_1+\widetilde{C}_5)s}$$

$$+ \widetilde{C}_6[1 - \mathrm{sgn}(\alpha_1, 0)]e^{\widetilde{C}_1 s}\int_0^s\int_M e^{-\widetilde{C}_1 t}|\nabla^2 u|^4 dV_t dt.$$

for some uniform constant $\widetilde{C}_7$ which depends only on $g_0, u_0, \alpha_1, \beta_1, \beta_2, A_1, \chi(M)$.

Because $|S \leq 1$, we have $\frac{1}{3}|\mathrm{Sic}|^2 \leq f \leq |\mathrm{Sic}|^2$. The above estimate immediately implies (3.52) and (3.54). The second estimate follows from (3.41) as the argument in [43] (see the proof of Theorem 3.5). The last estimate follows from $\mathrm{Vol}_t \leq e^t \mathrm{Vol}_0$ and the Hölder inequality. □

According to our definition of Sic, Sm in (3.10), we see that the boundedness of Sic, Sm is equivalent to the boundedness of Sic, Rm for any regular Ricci flow.

Corollary 3.8. *Suppose that* $(g(t), u(t))_{t\in[0,T)}$ *satisfies* **BA**. *Then*

$$\int_M |\mathrm{Ric}_{g(s)}|^2_{g(s)} dV_{g(s)} \;\leq\; \widetilde{C}(1+s)e^{\widetilde{C}s} + \widetilde{C}[1 - \mathrm{sgn}(\alpha_1, 0)]e^{\widetilde{C}s}\int_0^s\int_M |\nabla^2_{g(t)} u(t)|^4 dV_{g(t)} dt, \tag{3.56}$$

$$\int_M |\mathrm{Rm}_{g(s)}|^2_{g(s)} dV_{g(s)} \;\leq\; \widetilde{C}(1+s)e^{\widetilde{C}s} + \widetilde{C}[1 - \mathrm{sgn}(\alpha_1, 0)]e^{\widetilde{C}s}\int_0^s\int_M |\nabla^2_{g(t)} u(t)|^4 dV_{g(t)} dt, \tag{3.57}$$

for any $s \in [0, T)$*. Here* $\widetilde{C}$ *is a uniform constant which depends only on* $g_0, u_0, \alpha_1, \beta_1, \beta_2,$ $A_1, \chi(M)$.

4. Bounded L^2-curvature conjecture for the Einstein scalar field equations

From Theorem 2.7, we can get an upper bound for the L^2-norm of $\mathrm{Rm}_{g(t)}$. Motivated by this estimate, we in this section impose a conjecture for the Einstein scalar field equations, which is analogous to the corresponding conjecture for the Einstein vacuum equations proved by Klainerman, Rodnianski, and Szeftel [26, 56, 57, 58, 59, 60].

4.1. **Initial value problem.** In this section we recall some basic results for Einstein scalar field equations from [53]. Consider Einstein's equation

$$\boldsymbol{R}_{\alpha\beta} - \frac{1}{2}\boldsymbol{R}\mathbf{g}_{\alpha\beta} = \boldsymbol{T}_{\alpha\beta} \tag{4.1}$$

where $\boldsymbol{R}_{\alpha\beta}$ and $\boldsymbol{R}$denote, respectively, the Ricci curvature tensor and scalar curvature of a four dimensional Lorentzian space-time $(\mathbf{M}, \mathbf{g})$. If the energy-momentum tensor $\boldsymbol{T}_{\alpha\beta}$ is chosen as

$$\boldsymbol{T}_{\alpha\beta} = 2\partial_\alpha u \partial_\beta u - \frac{1}{2}|\mathbf{D}u|^2\mathbf{g}, \tag{4.2}$$

where $\mathbf{D}$ is the Levi-Civita connection of $\mathbf{g}$ and u is a smooth function on $\mathbf{M}$. In this case, the Einstein equation (4.1) can be written as

$$\boldsymbol{R}_{\alpha\beta} - 2\partial_\alpha u \partial_\beta u = 0.$$

As discussed in [53], we should impose a matter equation

$$\boldsymbol{\Delta} u = 0$$

for u, where $\boldsymbol{\Delta} := \mathbf{D}^\alpha \mathbf{D}_\alpha$. Hence we should consider a system of PDEs

$$\boldsymbol{R}_{\alpha\beta} - 2\partial_\alpha u \partial_\beta u = 0, \quad \boldsymbol{\Delta} u = 0, \tag{4.3}$$

which is called the Einstein scalar field equation or the Einstein-Klein-Gordon equation.

An *initial data set* (Σ, g, k, u_0, u_1) for (4.3) consists of a three dimensional manifold Σ, a Riemannian metric g, a symmetric 2-tensor k, together with two functions u_0 and u_1 on Σ, all assumed to be smooth, verifying the constraint equations,

$$\nabla^j k_{ij} - \nabla_i \mathrm{tr} k = u_1 \nabla_i u_0, \tag{4.4}$$

$$R - |k|^2 + (\mathrm{tr} k)^2 = u_1^2 + |\nabla u_0|^2, \tag{4.5}$$

where ∇ is the Levi-Civita connection of g.

Given an initial data set (Σ, g, k, u_0, u_1), the *Cauchy problem* consists in finding a four-dimensional Lorentzian manifold $(\mathbf{M}, \mathbf{g})$ and a smooth function u on $\mathbf{M}$ satisfying (4.3), and also an embedding $\iota : \Sigma \to \mathbf{M}$ such that

$$\iota^* \mathbf{g} = g, \quad \iota^* u = u_0, \quad \iota^* K = k, \quad \iota^*(\boldsymbol{N} u) = u_1, \tag{4.6}$$

where $\boldsymbol{N}$ is the future-directed unit normal to $\iota(\Sigma)$ and K is the second fundamental form of $\iota(\Sigma)$.

The local existence and uniqueness result for globally hyperbolic developments can be found in [53], Theorem 14.2. For stability and instability for Einstein's scalar field equation, we refer to [14, 15, 33, 34, 62, 63, 64, 65].

4.2. **Bounded L^2-curvature conjecture for Einstein's equations.** For Einstein's equations (i.e., $u = 0$ in (4.3), and the initial data is denoted by (Σ, g, k)), Klainerman [25] proposed the following conjecture:

> *The Einstein vacuum equations admits local Cauchy developments for initial data sets (Σ, g, k) with locally finite L^2-curvature and locally finite L^2-norm of the first covariant derivatives of k.*

This conjecture was recently solved by Klainerman, Rodnianski and Szeftel [26]. To give a precise result, we assume that the space-time $(\mathbf{M}, \mathbf{g})$ to be foliated by the level surfaces $\Sigma_t = \mathfrak{t}^{-1}(t)$ of a time function $\mathfrak{t}$. Let $\boldsymbol{T}$ denote the unit normal to Σ_t, and let k the second fundamental form of Σ_t, i.e., $k_{ij} := -\mathbf{g}(\mathbf{D}_i \boldsymbol{T}, e_j)$, where $(e_i)_{1 \leq i \leq 3}$ denote an arbitrary frame on Σ_t. We also assume that the Σ_t-foliation is maximal, i.e., we have

$$\mathrm{tr}_g k = 0$$

where $g = g(t)$ is the induced metric on Σ_t.

Theorem 4.1. (Klainerman-Rodnianski-Szeftel, 2015) *Let $(\mathbf{M}, \mathbf{g})$ an asymptotically flat solution to the Einstein vacuum equations together with a maximal foliation by space-like hyper-surfaces Σ_t defined as level hyper-surfaces of a time function $\mathfrak{t}$. Assume that the initial slice (Σ, g, k) is such that the Ricci curvature $\mathrm{Ric} \in L^2(\Sigma)$, $\nabla k \in L^2(\Sigma)$, and Σ has a strictly positive volume radius on scales ≤ 1, i.e.,*

$$r_{\mathrm{vol}}(\Sigma, 1) := \inf_{p \in \Sigma} \inf_{r \in (0,1]} \frac{\mathrm{vol}_g(B_g(p, r))}{r^3} > 0. \tag{4.7}$$

Then there exists a time

$$T := T\left(||\mathrm{Ric}||_{L^2(\Sigma)}, ||\nabla k||_{L^2(\Sigma)}, r_{\mathrm{vol}}(\Sigma, 1)\right) > 0$$

and a constant

$$C := C\left(||\mathrm{Ric}||_{L^2(\Sigma)}, ||\nabla k||_{L^2(\Sigma)}, r_{\mathrm{vol}}(\Sigma, 1)\right) > 0$$

such that the following control

$$||\mathbf{Rm}||_{L^\infty_{[0,T]} L^2(\Sigma_t)} \leq C, \quad ||\nabla k||_{L^\infty_{[0,T]} L^2(\Sigma_t)} \leq C, \quad \inf_{t \in [0,T]} r_{\mathrm{vol}}(\Sigma_t, 1) \geq \frac{1}{C},$$

holds on $t \in [0, T]$.

4.3. **Bounded L^2-curvature conjecture for the Einstein scalar field equations.** Motivated by Theorem 2.7 and Theorem 4.1, we propose the following

Conjecture 4.2. *The Einstein scalar field equations admit local Cauchy developments for initial data sets (Σ, g, k, u_0, u_1) with locally finite L^2-curvature, locally finite L^2-norm of the first covariant derivatives of k, locally finite L^2-norm of the covariant derivatives (up to second order) of u_0, and locally finite L^2-norm of the covariant derivatives (up to first order) of u_1.*

An interesting question related to Theorem 4.1 is

Question 4.3. *Can we extend Theorem 4.1 to the Einstein scalar field equations?*

5. Sm AND WYLIE-YEROSHKIN RIEMANN CURVATURE

In this section we compare our curvature Sm with $\alpha_1 = 2$ (3.10) with a notion of curvature introduced recently by Wylie and Yeroshkin [68]. For other notions of sectional curvatures, see [23, 24, 67].

Let (M, g) be a Riemannian manifold of dimension n with a smooth function u. Wylie and Yeroshkin introduced the following weighted connection

$$\nabla^u_X Y := \nabla_X Y - (Yu)X - (Xu)Y. \tag{5.1}$$

By Proposition 3.3 in [68], we have

$$R^u_{ijk\ell} = R_{ijk\ell} + \nabla_j \nabla_k u g_{i\ell} - \nabla_i \nabla_k u g_{j\ell} + \nabla_j u \nabla_k u g_{i\ell} - \nabla_i u \nabla_k u g_{j\ell}, \tag{5.2}$$

where $R^u_{ijk\ell} := \langle \mathrm{Rm}^\alpha(\partial_i, \partial_j)\partial_k, \partial_\ell \rangle$ and Rm^α is the induced Riemann curvature tensor associated to the connection ∇^u. The Ricci curvature associated to ∇^u is defined by

$$R^u_{jk} := g^{i\ell} R^u_{ijk\ell} = R_{ik} + (n-1)\nabla_j \nabla_k u + (n-1)\nabla_j u \nabla_k u. \tag{5.3}$$

Here the last formula also follows from Proposition 3.3 in [68].

Recall from (3.10) that (with $\alpha_1 = 2$)

$$S_{ijk\ell} = R_{ijk\ell} - \nabla_i u \nabla_k u g_{j\ell} - \nabla_i u \nabla_j u g_{k\ell}. \tag{5.4}$$

From now on, we are given a smooth function u on M and write

$$\begin{aligned} R^{\mathbf{L}}_{ijk\ell} &:= S_{ijk\ell}, \quad R^{\mathbf{WY}}_{ijk\ell} := R^u_{ijk\ell}, \\ R^{\mathbf{L}}_{jk} &:= g^{i\ell} R^{\mathbf{L}}_{ijk\ell}, \quad R^{\mathbf{WY}}_{jk} := R^u_{jk} = g^{i\ell} R^{\mathbf{WY}}_{ijk\ell}, \\ R^{\mathbf{L}} &:= g^{jk} R^{\mathbf{L}}_{jk}, \quad R^{\mathbf{WY}} := g^{jk} R^{\mathbf{WY}}_{jk}. \end{aligned} \tag{5.5}$$

From (5.3) and (5.4), we have

$$\mathrm{Ric}^{\mathbf{L}} = \mathrm{Ric} - 2du \otimes du, \quad \mathrm{Ric}^{\mathbf{WY}} = \mathrm{Ric} + (n-1) du \otimes du + (n-1)\nabla^2 u.$$

Remark 5.1. *We note that* $\mathrm{Ric}^{\mathbf{L}}$ *and* $\mathrm{Ric}^{\mathbf{WY}}$ *are actually the Ricci curvatures in the sense of Bakey-Émery* [1]. *We here use our notions to keep the paper smoothly.*

There is another type of Ricci curvature given by

$$\widehat{R}^{\mathbf{WY}}_{jk} := g^{i\ell} R^{\mathbf{WY}}_{ji\ell k} = R_{jk} + \left(\Delta u + |\nabla u|^2\right) g_{jk} - \nabla_j \nabla_k u - \nabla_j u \nabla_k u. \tag{5.6}$$

Lemma 5.2. **(Basic identities for $R^{\mathbf{L}}_{ijk\ell}$ and $R^{\mathbf{WY}}_{ijk\ell}$)** *We have*

$$R^{\mathbf{WY}}_{ijk\ell} - R^{\mathbf{L}}_{ijk\ell} = \nabla_i u \nabla_j u g_{k\ell} + \nabla_k u \nabla_j u g_{i\ell} + \nabla_j \nabla_k u g_{i\ell} - \nabla_i \nabla_k u g_{j\ell}, \tag{5.7}$$

$$R^{\mathbf{WY}}_{ijk\ell} - R^{\mathbf{WY}}_{jik\ell} = 2\left(\nabla_j u \nabla_k u g_{i\ell} - \nabla_i \nabla_k u g_{j\ell} + \nabla_j u \nabla_k u g_{i\ell} - \nabla_i u \nabla_k u g_{j\ell}\right), \tag{5.8}$$

$$\begin{aligned} R^{\mathbf{WY}}_{ijk\ell} - R^{\mathbf{WY}}_{ij\ell k} &= \nabla_i \nabla_\ell u g_{jk} + \nabla_j \nabla_k u g_{i\ell} - \nabla_i \nabla_k u g_{j\ell} - \nabla_j \nabla_\ell u g_{ik} \\ &\quad + \nabla_i u \nabla_\ell u g_{jk} + \nabla_j u \nabla_k u g_{i\ell} - \nabla_i u \nabla_k u g_{j\ell} - \nabla_j u \nabla_\ell u g_{ik}, \end{aligned} \tag{5.9}$$

$$R^{\mathbf{WY}}_{ijk\ell} - R^{\mathbf{WY}}_{k\ell ij} = \nabla_j \nabla_k u g_{i\ell} - \nabla_\ell \nabla_i u g_{jk} + \nabla_j u \nabla_k u g_{i\ell} - \nabla_\ell u \nabla_i u g_{jk}, \tag{5.10}$$

$$R^{\mathbf{L}}_{ijk\ell} - R^{\mathbf{L}}_{jik\ell} = \nabla_j u \nabla_k u g_{i\ell} - \nabla_i u \nabla_k u g_{j\ell}, \tag{5.11}$$

$$R^{\mathbf{L}}_{ijk\ell} - R^{\mathbf{L}}_{ij\ell k} = \nabla_i u \nabla_\ell u g_{jk} - \nabla_i u \nabla_k u g_{j\ell}, \tag{5.12}$$

$$R^{\mathbf{L}}_{ijk\ell} - R^{\mathbf{L}}_{k\ell ij} = \nabla_k u \nabla_\ell u g_{ij} - \nabla_i u \nabla_j u g_{k\ell}, \tag{5.13}$$

$$\frac{1}{2}\left(R^{\mathbf{WY}}_{ijk\ell} - R^{\mathbf{WY}}_{jik\ell}\right) = \left(R^{\mathbf{L}}_{ijk\ell} - R^{\mathbf{L}}_{jik\ell}\right) + \nabla_j \nabla_k u g_{i\ell} - \nabla_i \nabla_k u g_{j\ell}, \tag{5.14}$$

$$\widehat{\mathrm{Ric}}^{\mathbf{WY}} - \mathrm{Ric}^{\mathbf{WY}} = \left(\Delta u + |\nabla u|^2\right) g - n\left(\nabla^2 u + du \otimes du\right). \tag{5.15}$$

5.1. **Integral inequalities for scalar and Ricci curvatures.** We now have four different types of Ricci curvatures, Ric, $\mathrm{Ric}^{\mathbf{L}}$, $\mathrm{Ric}^{\mathbf{WY}}$, and $\widehat{\mathrm{Ric}}^{\mathbf{WY}}$, and three different types of scalar curvatures, R, $R^{\mathbf{L}}$, and $R^{\mathbf{WY}}$. In order to compare those quantities, we introduce a notation $\mathcal{P} \leq_{\mathbf{I},\mu} \mathcal{Q}$, which is an integral inequality with respect to the measure μ.

Definition 5.3. *Given two scalar quantities $\mathcal{P}, \mathcal{Q}$ on (M, g), and a measure μ, we write $\mathcal{P} \leq_{\mathbf{I},\mu} \mathcal{Q}$ if the following inequality*

$$\int_M \mathcal{P}\, d\mu \leq \int_M \mathcal{Q}\, d\mu \tag{5.16}$$

holds. When $d\mu$ is the volume form dV, we simply write (5.16) as $\mathcal{P} \leq_{\mathbf{I}} \mathcal{Q}$. When $d\mu$ is the measured volume form $e^f dV$, we write (5.16) as $\mathcal{P}_{\leq_{\mathbf{I},f}}\mathcal{Q}$. Similarly, we can define $\mathcal{P}_{\mathbf{I},\mu}\mathcal{Q}$.

Proposition 5.4. *For any measure μ on M and smooth function u on M, we have*

$$R^{\mathbf{L}} \leq_{\mathbf{I},\mu} R, \quad R \leq_{\mathbf{I}} R^{\mathbf{WY}}, \quad R =_{\mathbf{I},u} R^{\mathbf{WY}}. \tag{5.17}$$

Proof. It follows from the definitions

$$R^{\mathbf{L}} = R - 2|\nabla u|^2, \quad R^{\mathbf{WY}} = R + (n-1)(|\nabla u|^2 + \Delta u)$$

and the fact that integral of $(\Delta u + |\nabla u|^2)$ with respect to $e^u dV$ is zero. □

This proposition shows that $R^{\mathbf{L}} \leq_{\mathbf{I}} R \leq_{\mathbf{I}} R^{\mathbf{WY}}$ and $R^{\mathbf{L}} \leq_{\mathbf{I},u} R =_{\mathbf{I},u} R^{\mathbf{WY}}$. Thus, in the sense of integrals, $R_{\mathbf{L}}$ is weaker and $R^{\mathbf{WY}}$ is stronger than R, respectively.

Next we consider the similar question on Ricci curvatures.

Definition 5.5. *Let (M, g) be a closed Riemannian manifold with a smooth function u, and μ be a given measure on M. Given two Ricci curvatures $\mathrm{Ric}^{\clubsuit}, \mathrm{Ric}^{\diamondsuit} \in \mathfrak{Ric}_4 := \{\mathrm{Ric}, \mathrm{Ric}^{\mathbf{L}}, \mathrm{Ric}^{\mathbf{WY}}, \widehat{\mathrm{Ric}}^{\mathbf{WY}}\}$, we say*

$$\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{I},\mu} \mathrm{Ric}^{\diamondsuit} \tag{5.18}$$

if $\mathrm{Ric}^{\clubsuit}(X, X) \leq_{\mathbf{I},\mu} \mathrm{Ric}^{\diamondsuit}(X, X)$ in the sense of Definition 5.3 for all vector fields $X \in \mathfrak{X}(M)$. Similarly we can define $\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{I}} \mathrm{Ric}^{\diamondsuit}$ and $\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{I},f} \mathrm{Ric}^{\diamondsuit}$.

We say

$$\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{IK},\mu} \mathrm{Ric}^{\diamondsuit} \tag{5.19}$$

if $\mathrm{Ric}^{\clubsuit}(X, X) \leq_{\mathbf{I},\mu} \mathrm{Ric}^{\diamondsuit}(X, X)$ in the sense of Definition 5.3 for all Killing vector fields $X \in \mathfrak{X}_{\mathbf{K}}(M)$, where $\mathfrak{X}_{\mathbf{K}}(M)$ is the space of all Killing vector fields on M. Similarly we can define $\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{IK}} \mathrm{Ric}^{\diamondsuit}$ and $\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{IK},f} \mathrm{Ric}^{\diamondsuit}$.

Consider the subset $\mathfrak{X}_{\mathbf{KC}}(M)$ of $\mathfrak{X}_{\mathbf{K}}(M)$, which consists of Killing vector fields on M with constant norm. we say

$$\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{IKC},\mu} \mathrm{Ric}^{\diamondsuit} \tag{5.20}$$

if $\mathrm{Ric}^{\clubsuit}(X, X) \leq_{\mathbf{I},\mu} \mathrm{Ric}^{\diamondsuit}(X, X)$ in the sense of Definition 5.3 for all $X \in \mathfrak{X}_{\mathbf{KC}}(M)$. Similarly we can define $\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{IKC}} \mathrm{Ric}^{\diamondsuit}$ and $\mathrm{Ric}^{\clubsuit} \leq_{\mathbf{IKC},f} \mathrm{Ric}^{\diamondsuit}$.

Theorem 5.6. *Let (M, g) be a closed Riemannian manifold with a smooth function u and μ be a given measure on M. Then we have*

(i) $\mathrm{Ric}^{\mathbf{L}} \leq_{\mathbf{I},\mu} \mathrm{Ric}$.
(ii) $\mathrm{Ric} \leq_{\mathbf{IKC}} \mathrm{Ric}^{\mathbf{WY}}$.
(iii) $\mathrm{Ric} \leq_{\mathbf{IKC}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}$.

Proof. From (5.5), we have $\mathrm{Ric}^{\mathbf{L}}(X,X) = \mathrm{Ric}(X,X) - 2\langle X, \nabla u\rangle^2$ and then the first part follows. To prove the last result, we compute

$$\int_M \left[\mathrm{Ric}^{\mathbf{WY}}(X,X) - \mathrm{Ric}(X,X)\right] dV = (n-1)\int_M \left[\langle X, \nabla u\rangle^2 + X^i X^j \nabla_i \nabla_j u\right] dV.$$

Then we suffice to verify that the last integral is nonnegative for any Killing vector field X. According to (5.23), we can prove (ii) and (iii) immediately. Indeed, for (iii),

$$\begin{aligned}\int_M \left[\widehat{\mathrm{Ric}}^{\mathbf{WY}}(X,X) - \mathrm{Ric}(X,X)\right] dV &= \frac{3}{2}\int_M u\Delta|X|^2 dV \\ &\quad + \int_M [|X|^2|\nabla u|^2 - \langle X, \nabla u\rangle^2] dV \\ &\geq \frac{3}{2}\int_M u\Delta|X|^2 dV\end{aligned}$$

for any Killing vector field X. □

To prove Lemma 5.7, we first recall Yano's formula (see [69, 70, 71] and, [45] for related topics)

$$\int_M |\mathscr{L}_X g|^2 dV = \int_M [|\nabla X|^2 + |\mathrm{div}(X)|^2 - \mathrm{Ric}(X,X)] dV, \quad X \in \mathfrak{X}(M), \tag{5.21}$$

gives a necessary and sufficient condition to X being Killing or not. Namely,

$$X \in \mathfrak{X}_{\mathbf{K}}(M) \Longleftrightarrow \Delta X + \nabla \mathrm{div} X + \mathrm{Ric}(X) = 0 = \mathrm{div} X. \tag{5.22}$$

Lemma 5.7. *For any vector field X and any smooth function u, we have*

$$\begin{aligned}\int_M X^i X^j \nabla_i \nabla_j u\, dV &= \int_M u\langle X, \Delta X + \nabla \mathrm{div} X + \mathrm{Ric}(X)\rangle dV \\ &\quad + \int_M \frac{1}{2} u[|\mathscr{L}_X g|^2 - \Delta|X|^2] dV - \int_M \langle X, \nabla u\rangle \mathrm{div} X\, dV.\end{aligned} \tag{5.23}$$

In particular, when $X \in \mathfrak{X}_{\mathbf{K}}(M)$, we get

$$\int_M X^i X^j \nabla_i \nabla_j u\, dV = -\frac{1}{2}\int_M u\Delta|X|^2 dV = -\frac{1}{2}\int_M |X|^2 \Delta u\, dV. \tag{5.24}$$

Proof. We start from the computation:

$$\begin{aligned}\int_M X^i X^j \nabla_i \nabla_j u\, dV &= -\int_M \nabla_j u \nabla_i (X^i X^j) dV \\ &= -\int_M \nabla_j u \left(X^j \mathrm{div} X + X^i \nabla_i X^j\right) dV \\ &= -\int_M \langle X, \nabla u\rangle \mathrm{div} X\, dV + \int_M u \nabla_j (X^i \nabla_i X^j) dV.\end{aligned}$$

The last integral, denoted by I_u, as a function of u, is equal to

$$
\begin{aligned}
I_u &= \int_M u(\nabla_j X^i \nabla_i X^j) dV + \int_M u X^i \nabla_j \nabla_i X^j dV \\
&= \int_M u(\nabla_j X^i \nabla_i X^j) dV + \int_M u X^i \left(\nabla_i \mathrm{div} X + R_{ij} X^j \right) dV \\
&= \int_M u \left[\nabla_j X^i \nabla_i X^j + \mathrm{Ric}(X,X) \right] dV - \int_M \mathrm{div} X \left(\langle X, \nabla u \rangle + u \mathrm{div} X \right) dV \\
&= \int_M u \left[\nabla_j X^i \nabla_i X^j + \mathrm{Ric}(X,X) - |\mathrm{div}(X)|^2 \right] dV - \int_M \langle X, \nabla u \rangle \mathrm{div} X dV.
\end{aligned}
$$

Using the following identities:

$$
\begin{aligned}
\frac{1}{2} |\mathscr{L}_X g|^2 &= |\nabla X|^2 + \nabla_j X^i \nabla_i X^j, \\
\int_M u |\mathrm{div}(X)|^2 dV &= \int_M (u \mathrm{div} X) \nabla_i X^i dV \\
&= -\int_M X^i \left(\nabla_i u \mathrm{div} X + u \nabla_i \mathrm{div} X \right) dV \\
&= -\int_M \langle X, \nabla u \rangle \mathrm{div} X dV - \int_M u \langle X, \nabla \mathrm{div} X \rangle dV, \\
\int_M u |\nabla X|^2 dV &= \int_M u \nabla^i X_j \nabla_i X^j dV \quad = \quad -\int_M X^j \left(\nabla_i u \nabla^i X_j + u \Delta X_j \right) dV \\
&= -\int_M u \langle X, \Delta X \rangle dV - \frac{1}{2} \int_M \nabla_i u \nabla^i |X|^2 dV,
\end{aligned}
$$

we obtain

$$
I_u = \int_M u \left[\langle X, \Delta X + \nabla \mathrm{div} X + \mathrm{Ric}(X) \rangle dV + \frac{1}{2} |\mathscr{L}_X g|^2 - \frac{1}{2} \Delta |X|^2 \right] dV. \tag{5.25}
$$

Then (5.23) follows from (5.25). □

A consequence of Theorem 5.6 indicates

$$
\mathrm{Ric}^{\mathbf{L}} \leq_{\mathbf{IKC}} \mathrm{Ric} \leq_{\mathbf{IKC}} \mathrm{Ric}^{\mathbf{WY}} \quad \text{and} \quad \mathrm{Ric}^{\mathbf{L}} \leq_{\mathbf{IKC}} \mathrm{Ric} \leq_{\mathbf{IKC}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}. \tag{5.26}
$$

To prove an analogous result in Theorem 5.6 between Ric and $\mathrm{Ric}^{\mathbf{WY}}, \widehat{\mathrm{Ric}}^{\mathbf{WY}}$ along constant Killing vector fields, for some measure μ other than the volume form, we first do the following computation. For the measure $e^f dV$, where $f \equiv f(u)$ is a smooth function depends only on u, we have

$$
\int_M \left[\mathrm{Ric}^{\mathbf{WY}}(X,X) - \mathrm{Ric}(X,X) \right] e^f dV = (n-1) \int_M \left[\langle X, \nabla u \rangle^2 + X^i X^j \nabla_i \nabla_j u \right] e^f dV. \tag{5.27}
$$

As above, the last integral is equal to

$$\begin{aligned}
\int_M X^i X^j \nabla_i \nabla_j u e^f dV &= -\int_M \nabla_j u \nabla_i (e^f X^i X^j) dV \\
&= -\int_M \nabla_j u \left(f' \nabla_i u X^i X^j + X^j \mathrm{div} X + X^i \nabla_i X^j\right) e^f dV \\
&= -\int_M \langle X, \nabla u\rangle \mathrm{div} X e^f dV - \int_M f' \langle \nabla u, X\rangle^2 e^f dV \\
&\quad + \int_M u e^f \nabla_j (X^i \nabla_i X^j) dV + \int_M u f' e^f \nabla_j u X^i \nabla_i X^j \\
&= -\int_M \langle X, \nabla u\rangle \mathrm{div} X e^f dV - \int_M f' \langle \nabla u, X\rangle^2 e^f dV \\
&\quad + I_{ue^f} - I_{u^2 f' e^f} - \int_M u \nabla_j \left(u f' e^f\right) X^i \nabla_i X^j dV
\end{aligned} \tag{5.28}$$

where we used the notion I_u that is explicitly given in (5.25). Next we require

$$u f' e^f = 1 \Longrightarrow \text{we can take } f(u) = \ln \ln u.$$

However, in the above function $f(u)$, we should assume that u is strictly positive. It motivates us to consider the modified function associated with u.

Given a smooth function u on the closed Riemannian manifold (M, g), set

$$\widetilde{u} := u - u_{\min} + c_0 \geq c_0 \geq \frac{1}{e}, \quad \widetilde{f} := \ln \ln \widetilde{u}, \tag{5.29}$$

where $u_{\min} := \min_M u$ and $c_0 \geq e^{-1}$ is the unique constant such that $c_0 \ln c_0 = 1$. Replacing (u, f) by $(\widetilde{u}, \widetilde{f})$ in (5.27) (since $\nabla \widetilde{u} = \nabla u$), we immediately obtain

$$\int_M \left[\mathrm{Ric}^{\mathbf{WY}}(X, X) - \mathrm{Ric}(X, X)\right] e^{\widetilde{f}} dV = (n-1) \int_M \left[\langle X, \nabla \widetilde{u}\rangle^2 + X^i X^j \nabla_i \nabla_j \widetilde{u}\right] e^{\widetilde{f}} dV$$

where

$$\begin{aligned}
\int_M X^i X^j \nabla_i \nabla_j \widetilde{u} e^{\widetilde{f}} dV &= -\int_M \langle X, \nabla \widetilde{u}\rangle \mathrm{div} X e^f dV - \int_M \widetilde{f}' \langle \nabla \widetilde{u}, X\rangle^2 e^{\widetilde{f}} dV \\
&\quad + I_{\widetilde{u} e^{\widetilde{f}}} - I_{\widetilde{u}^2 \widetilde{f}' e^{\widetilde{f}}} - \int_M \widetilde{u} \nabla_j \left(\widetilde{u} \widetilde{f}' e^{\widetilde{f}}\right) X^i \nabla_i X^j dV \\
&= -\int_M \langle X, \nabla \widetilde{u}\rangle \mathrm{div} X e^f dV - \int_M \frac{1}{\widetilde{u} \ln \widetilde{u}} \langle X, \nabla \widetilde{u}\rangle^2 e^{\widetilde{f}} dV \\
&\quad + I_{\widetilde{u} e^{\widetilde{f}}} - I_{\widetilde{u}^2 \widetilde{f}' e^{\widetilde{f}}}.
\end{aligned}$$

When $X \in \mathfrak{X}_{\mathbf{KC}}(M)$, we can simplify the above integral into

$$\int_M X^i X^j \nabla_i \nabla_j \widetilde{u} e^{\widetilde{f}} dV = -\int_M \frac{1}{\widetilde{u} \ln \widetilde{u}} \langle X, \nabla \widetilde{u}\rangle^2 e^{\widetilde{f}} dV$$

and hence

$$\int_M \left[\mathrm{Ric}^{\mathbf{WY}}(X, X) - \mathrm{Ric}(X, X)\right] e^{\widetilde{f}} dV = (n-1) \int_M \left(1 - \frac{1}{\widetilde{u} \ln \widetilde{u}}\right) \langle X, \nabla \widetilde{u}\rangle^2 e^{\widetilde{f}} dV.$$

Since the function $t \ln t$, $t \geq e^{-1}$, is increasing, it follows that $\widetilde{u} \ln \widetilde{u} \geq c_0 \ln c_0 = 1$. Therefore, we obtain the first part of the following theorem.

Theorem 5.8. *Let (M, g) be a closed Riemannian manifold with a smooth function u and μ be a given measure on M. Then we have*

(i) $\mathrm{Ric} \leq_{\mathbf{IKC},\widetilde{f}} \mathrm{Ric}^{\mathbf{WY}}$, *and*

(ii) $\mathrm{Ric} \leq_{\mathbf{IKC},\widetilde{f}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}$.

where $\widetilde{f} := u - u_{\min} + c_0$ *and* $c_0 \geq 1/e$.

Proof. The proof of the second part is precisely the same as that of (iii) in Theorem 5.6. □

5.2. **Killing vector fields with constant length.** In Theorem 5.6 and Theorem 5.8, we proved integral inequalities along Killing vector fields with constant length. In this subsection we review some existence results on such vector fields.

Eisenhart [16] proved that a unit vector field X on a (connected) complete Riemannian manifold (M,g) is the unit Killing vector field if and only if the angles between X and tangent vectors to each geodesic in (M,g) are constant along this geodesic. As earlier, Bianchi [5] proved that A Killing vector field X on a complete Riemannian manifold (M,g) has constant length if and only if every integral curve of X is a geodesic in (M,g). For a proof, we refer to [2].

There are two necessary conditions for the existence of Killing vector fields of constant length on a given Riemannian manifold (M,g), one is $\chi(M) = 0$ by Hopf's theorem while another one, by Bott's theorem [6], is that all the Pontrjagin numbers of the oriented cover of M are zero

The existence of Killing vector fields with constant length on a complete Riemannian manifold (M,g) is connected with *Clifford-Wolf translations* or *Clifford translations*, which is an isometry $s^{\mathbf{CW}}$ on (M,g) such that $d(x, s^{\mathbf{CW}}(x)) \equiv$ constant for all $x \in M$.

- If a one-parameter isometry group generated by a Killing vector field X consists of Clifford-Wolf translations, then X had constant length.
- If X is a Killing vector field of constant length on a compact Riemannian manifold (M,g), then the isometries $\gamma(t)$, generated by X, are Clifford-Wolf translations for sufficiently small $|t|$.

The first fact is obvious by definition. The compactness condition in the second fact can be generalized to the condition that (M,g) has the injectivity radius, bounded from below by some positive constant. A proof can be found in [2].

Proposition 5.9. *On each of* 28 *homotopical seven-dimensional spheres M, there exist a Riemannian metric g and a nonzero vector field X, such that*

- $\mathrm{Ric}^{\mathbf{L}}(X,X) \leq_{\mathbf{I}} \mathrm{Ric}(X,X) \leq_{\mathbf{I}} \mathrm{Ric}^{\mathbf{WY}}(X,X)$ *and* $\mathrm{Ric}^{\mathbf{L}}(X,X) \leq_{\mathbf{I}} \mathrm{Ric}(X,X) \leq_{\mathbf{I}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}(X,X)$ *hold.*
- *for any smooth function u on M,* $\mathrm{Ric}^{\mathbf{L}}(X,X) \leq_{\mathbf{I},\widetilde{f}} \mathrm{Ric}(X,X) \leq_{\mathbf{I},\widetilde{f}} \mathrm{Ric}^{\mathbf{WY}}(X,X)$ *and* $\mathrm{Ric}^{\mathbf{L}}(X,X) \leq \mathrm{Ric}(X,X) \leq_{\mathbf{I},\widetilde{f}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}(X,X)$ *hold, where* $\widetilde{f} := u - u_{\min} + c_0$ *with* $c_0 \geq 1/e$.

Proof. According to [2] (see Corollary 11) and [48] we can take X to be a Killing vector field of constant length with respect to g. Then the results follow from Theorem 5.6 and Theorem 5.8. □

We say that a Riemannian metric g on M is of *cohomogeneity 1* if some compact Lie group G acts smoothly and isometrically on M and the space of orbits M/G with respect to this action is one-dimensional.

Proposition 5.10. *Let $n \geq 2$ and $\epsilon > 0$. On the sphere $\mathbb{S}^{2n-1}$, there are a (real-analytic) Riemannian metric g_ϵ, of cohomogeneity 1, with the property that all section curvatures of g_ϵ differ from 1 at most by ϵ, and a (real-analytic) nonzero vector field X_ϵ, such that*

- $\mathrm{Ric}^{\mathbf{L}}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq_{\mathbf{I}} \mathrm{Ric}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq_{\mathbf{I}} \mathrm{Ric}^{\mathbf{WY}}_{g_\epsilon}(X_\epsilon, X_\epsilon)$ *and* $\mathrm{Ric}^{\mathbf{L}}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq_{\mathbf{I}} \mathrm{Ric}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq_{\mathbf{I}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}_{g_\epsilon}(X_\epsilon, X_\epsilon)$ *hold.*
- *for any smooth function u on M,* $\mathrm{Ric}^{\mathbf{L}}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq_{\mathbf{I},\tilde{f}} \mathrm{Ric}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq_{\mathbf{I},\tilde{f}} \mathrm{Ric}^{\mathbf{WY}}_{g_\epsilon}(X_\epsilon, X_\epsilon)$ *and* $\mathrm{Ric}^{\mathbf{L}}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq \mathrm{Ric}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq_{\mathbf{I},\tilde{f}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}_{g_\epsilon}(X_\epsilon, X_\epsilon)$ *hold, where* $\tilde{f} := u - u_{\min} + c_0$ *with* $c_0 \geq 1/e$.

Proof. According to [2] (see Theorem 21), we can take X to be a Killing vector field X of unit length with respect to g. Then the results follow from Theorem 5.6 and Theorem 5.8. □

Remark 5.11. *Fix $n \geq 2$ and $D > 0$. Let $\mathfrak{S}_{n,D}$ denote the class of simply-connected n-dimensional Riemannian manifolds (M, g) with the sectional curvature $|\mathrm{Sec}_g|_g \leq 1$ and with $\mathrm{diam}(M, g) \leq D$. The class $\mathfrak{S}_{n,\pi/\sqrt{\delta}}$ contains a subsclass $\mathfrak{P}\mathfrak{S}_{n,\delta}$, which consists of all simply-connected n-dimensional Riemannian manifold (M, g) with the sectional curvature $0 < \delta < \mathrm{Sec}_g \leq 1$.*

Tuschmann [61] *proved that there is a positive number $v := v(n, D)$ with the following property: if $(M, g) \in \mathfrak{S}_{n,D}$ satisfies $\mathrm{vol}(M, g) < v$, then*

(i) *there is a smooth locally free action of the $\mathbb{S}^1$-action on M, and*
(ii) *for every $\epsilon > 0$ there exists a $\mathbb{S}^1$-invariant metric g_ϵ on M such that*

$$e^{-\epsilon} g \leq g_\epsilon < e^{\epsilon} g, \quad |\nabla_g - \nabla_{g_\epsilon}|_g < \epsilon, \quad |\nabla^i_{g_\epsilon} \mathrm{Rm}_{g_\epsilon}|_g < C(n, i, \epsilon).$$

If the $\mathbb{S}^1$-action in (i) is free, Berestovskii and Nikonorov [2] *observed that we can find a Killing vector field X_ϵ of unit length with respect to g_ϵ. Consequently, in this case,*

- $\mathrm{Ric}^{\mathbf{L}}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq_{\mathbf{I}} \mathrm{Ric}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq_{\mathbf{I}} \mathrm{Ric}^{\mathbf{WY}}_{g_\epsilon}(X_\epsilon, X_\epsilon)$ *and* $\mathrm{Ric}^{\mathbf{L}}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq_{\mathbf{I}} \mathrm{Ric}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq_{\mathbf{I}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}_{g_\epsilon}(X_\epsilon, X_\epsilon)$ *hold.*
- *for any smooth function u on M,* $\mathrm{Ric}^{\mathbf{L}}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq_{\mathbf{I},\tilde{f}} \mathrm{Ric}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq_{\mathbf{I},\tilde{f}} \mathrm{Ric}^{\mathbf{WY}}_{g_\epsilon}(X_\epsilon, X_\epsilon)$ *and* $\mathrm{Ric}^{\mathbf{L}}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq \mathrm{Ric}_{g_\epsilon}(X_\epsilon, X_\epsilon) \leq_{\mathbf{I},\tilde{f}} \widehat{\mathrm{Ric}}^{\mathbf{WY}}_{g_\epsilon}(X_\epsilon, X_\epsilon)$ *hold, where* $\tilde{f} := u - u_{\min} + c_0$ *with* $c_0 \geq 1/e$.

5.3. **Remark on $\mathrm{Rm}^{\mathbf{L}}$ and $\mathrm{Rm}^{\mathbf{WY}}$.** The nonnegativity of $R^{\mathbf{L}}_{ijk\ell}$ was used in [66] to prove the compactness for gradient shrinking Ricci harmonic solitons.

There is no useful relation between $\mathrm{Rm}^{\mathbf{L}}$ and $\mathrm{Rm}^{\mathbf{WY}}$. More precisely, we can find a Riemannian manifold (M, g) so that $\mathrm{Rm}^{\mathbf{L}}(X, Y, Y, X) < \mathrm{Rm}^{\mathbf{WY}}(X, Y, Y, X)$ for some triple (X, Y, u) of smooth vector fields X, Y and smooth function u, and $\mathrm{Rm}^{\mathbf{L}}(X, Y, Y, X) > \mathrm{Rm}^{\mathbf{WY}}(X, Y, Y, X)$ for another such triple (X', Y', u').

Example 5.12. *Consider the Euclidean space $(\mathbb{R}^n, g_{\mathbb{R}^n})$ with the flat Riemannian metric $g_{\mathbb{R}^n}$. Consider a smooth function $u(x) = \varphi(r)$ with $r = |x|^2$, where $\varphi(r)$ is a smooth function of variable r. Let $T = x^i \partial/\partial x^i$ denote the position vector field on $\mathbb{R}^n$. Then*

$$\nabla_i u = 2T_i \varphi', \quad \nabla_i \nabla_j u = 4T_i T_j \varphi'' + 2\delta_{ij} \varphi'.$$

According to (5.5), we have

$$\begin{aligned} R^{\mathbf{L}}_{ijk\ell} &= -4\varphi'^2 T_i T_k \delta_{j\ell} - 4\varphi'^2 T_i T_j \delta_{k\ell}, \\ R^{\mathbf{WY}}_{ijk\ell} &= \left(4\varphi'' T_j T_k + 2\varphi' \delta_{jk}\right)\delta_{i\ell} - \left(4\varphi'' T_i T_k + 2\varphi'\delta_{ik}\right)\delta_{j\ell} \\ &\quad + 4\varphi'^2 \delta_{i\ell} T_j T_k - 4\varphi'^2 \delta_{j\ell} T_i T_k, \end{aligned}$$

so that

$$\begin{aligned} \mathrm{Rm}^{\mathbf{L}}(X,Y,Y,X) &= -8\varphi'^2 \langle X,T\rangle\langle Y,T\rangle\langle X,Y\rangle, \\ \mathrm{Rm}^{\mathbf{WY}}(X,Y,Y,X) &= 4(\varphi'' + \varphi'^2)\left(|X|^2\langle Y,T\rangle^2 - \langle X,T\rangle\langle Y,T\rangle\langle X,Y\rangle\right) \\ &\quad + 2\varphi'(|X|^2|Y|^2 - \langle X,Y\rangle^2) \end{aligned}$$

For any vector fields X, Y. Choosing $X = T$ and Y with the property that $\langle Y, T\rangle = 0$ yields

$$\mathrm{Rm}^{\mathbf{L}}(T,Y,Y,T) = 0, \quad \mathrm{Rm}^{\mathbf{WY}}(T,Y,Y,T) = 2\varphi'|T|^2|Y|^2.$$

When $\varphi(r) = r$, we have $\mathrm{Rm}^{\mathbf{L}}(T,Y,Y,T) \le \mathrm{Rm}^{\mathbf{WY}}(T,Y,Y,T)$. On the other hand, for $\varphi(r) = -r$, we have $\mathrm{Rm}^{\mathbf{L}}(T,Y,Y,T) \ge \mathrm{Rm}^{\mathbf{WY}}(T,Y,Y,T)$.

There is also no useful pointwise relation between Rm and $\mathrm{Rm}^{\mathbf{L}}$. By definition,

$$\mathrm{Rm}^{\mathbf{L}}(X,Y,Y,X) = \mathrm{Rm}(X,Y,Y,X) - 2\langle X,\nabla u\rangle\langle Y,\nabla u\rangle\langle X,Y\rangle$$

for any vector fields $X, Y \in \mathfrak{X}(M)$.

Remark 5.13. *(1) For $n \ge 2$, take $X, Y \in \mathfrak{X}(M)$ so that $\langle X, Y\rangle = 0$ at a point. Then, at this point, we get $\mathrm{Rm}^{\mathbf{L}}(X,Y,Y,X) = \mathrm{Rm}(X,Y,Y,X)$.*

(2) When $X = \nabla u$, we have

$$\mathrm{Rm}^{\mathbf{L}}(\nabla u,Y,Y,\nabla u) = \mathrm{Rm}(\nabla u,Y,Y,\nabla u) - 2|\nabla u|^2\langle Y,\nabla u\rangle^2 \le \mathrm{Rm}(\nabla u,Y,Y,\nabla u).$$

(3) For $n \ge 3$, take $X, Y \in \mathfrak{X}(M)$ so that $\langle X,\nabla u\rangle, \langle Y,\nabla u\rangle, \langle X,Y\rangle < 0$ at a point. Then, at this point, $\mathrm{Rm}^{\mathbf{L}}(X,Y,Y,X) > \mathrm{Rm}(X,Y,Y,X)$.

(4) In general, for any vector fields $X, Y \in \mathfrak{X}(M)$, we have

$$\mathrm{Rm}^{\mathbf{L}}(X,X,X,X) = -2\langle X,\nabla u\rangle^2|X|^2 \le 0 = \mathrm{Rm}(X,X,X,X)$$

and the equality holds at a point if and only if $\langle X,\nabla u\rangle = 0$ or $X = 0$ at this point.

To give a relation between $\mathbf{Rm}^{\mathbf{L}}(X,Y,Y,X)$ and $\mathbf{Rm}(X,Y,Y,X)$ in the sense of **IKC**, we let

$$J(X,Y) := \int_M \langle X,\nabla u\rangle\langle Y,\nabla u\rangle\langle X,Y\rangle dV, \quad X,Y \in \mathfrak{X}(M). \tag{5.30}$$

It is clear that $J(X,Y) = J(Y,X)$. Compute

$$\begin{aligned} \int_M \langle X,\nabla u\rangle\langle Y,\nabla u\rangle\langle X,Y\rangle dV &= \int_X \nabla_i u\left[X^i\langle Y,\nabla u\rangle\langle X,Y\rangle\right] dV \\ &= -\int_M u\left[\mathrm{div}(X)\langle Y,\nabla u\rangle\langle X,Y\rangle + X^i\nabla_i(\langle Y,\nabla u\rangle\langle X,Y\rangle)\right] dV \\ &= -\int_M u\left[\mathrm{div}(X)\langle Y,\nabla u\rangle\langle X,Y\rangle + X^i\nabla_i(Y^j\nabla_j u X^k Y_k)\right] dV \\ &= -\int_M u\Big[\mathrm{div}(X)\langle Y,\nabla u\rangle\langle X,Y\rangle + X^i\Big(\nabla_i Y^j\nabla_j u X^k Y_k \end{aligned}$$

$$+ Y^j\nabla_i\nabla_j u X^k Y_k + Y^j\nabla_j u\nabla_i X^k Y_k + Y^j\nabla_j u X^k\nabla_i Y_k\Big)\Big]dV$$

$$= -\int_M u\mathrm{div}(X)\langle Y,\nabla u\rangle\langle X,Y\rangle dV - \int_M uX^i\nabla_i Y^j\nabla_j u X^k Y_k dV$$

$$- \int_M uX^iY^j\nabla_i\nabla_j u X^k Y_k dV - \int_M uX^iY^j\nabla_j u\nabla_i X^k Y_k dV - \int_M uX^iY^j\nabla_j u X^k\nabla_i Y_k dV$$

$$=: -\int_M u\mathrm{div}(X)\langle Y,\nabla u\rangle\langle X,Y\rangle dV - J_1 - J_2 - J_3 - J_4.$$

Using the definition of Lie derivative that $(\mathscr{L}_Y g)_{ik} = \nabla_i Y_k + \nabla_k Y_i$, we have

$$\begin{aligned} J_4 &= \int_M uX^iY^j\nabla_j u X^k\nabla_i Y_k dV = -\int_M u\nabla_j(uX^iY^jX^k\nabla_i Y_k)dV \\ &= -\int_M u\Big[\nabla_j u X^iY^jX^k\nabla_i Y_k + u\nabla_j X^iY^jX^k\nabla_i Y_k \\ &\quad + uX^i\mathrm{div}(Y)X^k\nabla_i Y_k + uX^iY^j\nabla_j X^k\nabla_i Y_k + uX^iY^jX^k\nabla_j\nabla_i Y_k\Big]dV \\ &= -J_4 - \int_M u^2\Big[\mathrm{div}(Y)X^iX^k\nabla_i Y_k + X^iY^jX^k\nabla_j\nabla_i Y_k \\ &\quad + X^iY^j\nabla_j X^k\nabla_i Y_k + X^kY^j\nabla_j X^i\nabla_i Y_k\Big]dV \\ &= -J_4 - \int_M u^2\left[\mathrm{div}(Y)X^iX^k\nabla_i Y_k + X^iY^jX^k\nabla_j\nabla_i Y_k + X^iY^j\nabla_j X^k(\mathscr{L}_Y g)_{ik}\right]dV; \end{aligned}$$

hence

(5.31)
$$J_4 = -\frac{1}{2}\int_M u^2\left[\mathrm{div}(Y)X^iX^k\nabla_i Y_k + X^iY^jX^k\nabla_j\nabla_i Y_k + X^iY^j\nabla_j X^k(\mathscr{L}_Y g)_{ik}\right]dV.$$

In particular

(5.32)
$$J_4 = -\frac{1}{2}\int_M u^2(X^iY^jX^k\nabla_j\nabla_i Y_k)dV = -\frac{1}{2}\int_M u^2(X^iX^jY^k\nabla_k\nabla_i Y_j)dV, \quad Y\in\mathfrak{X}_{\mathbf{K}}(M).$$

Similarly,

$$\begin{aligned} J_3 &= \int_M uX^iY^j\nabla_j u Y^k\nabla_i X_k dV = -\int_M u\nabla_j(uX^iY^jY^k\nabla_i X_k)dV \\ &= -\int_M u\Big[\nabla_j u X^iY^jY^k\nabla_i X_k + u\nabla_j X^iY^jY^k\nabla_i X_k \\ &\quad + uX^i\mathrm{div}(Y)Y^k\nabla_i X_k + uX^iY^j\nabla_j Y^k\nabla_i X_k + uX^iY^jY^k\nabla_j\nabla_i X_k\Big]dV \\ &= -J_3 - \int_M u^2\Big[\mathrm{div}(Y)X^iY^k\nabla_i X_k + X^iY^jY^k\nabla_j\nabla_i X_k \\ &\quad + \nabla_j X^i\nabla_i X_k Y^jY^k + \nabla_j Y^k\nabla_i X_k X^iY^j\Big]dV. \end{aligned}$$

Hence

$$
\begin{aligned}
J_3 &= -\frac{1}{2}\int_M u^2\Big[\operatorname{div}(Y)X^iY^k\nabla_iX_k + X^iY^jY^k\nabla_j\nabla_iX_k \\
&\quad + \nabla_jX^i\nabla_iX_kY^jY^k + \langle\nabla_XX,\nabla_YY\rangle\Big]dV. \qquad (5.33)
\end{aligned}
$$

From (5.32) and (5.33), we obtain

$$
\begin{aligned}
J(X,Y) &= -\int_M u\nabla_ju\nabla_XY^j\langle X,Y\rangle dV - \int_M uX^iY^j\nabla_i\nabla_ju\langle X,Y\rangle dV \\
&\quad + \frac{1}{2}\int_M u^2\left[X^iY^jY^k\nabla_j\nabla_iX_k + \nabla_YX^i\nabla_iX_kY^k + \langle\nabla_XX,\nabla_YY\rangle\right]dV \\
&\quad + \frac{1}{2}\int_M u^2\left(X^iX^jY^k\nabla_k\nabla_iY_j\right)dV \\
&= -\int_M u\langle X,Y\rangle\left[\langle\nabla u,\nabla_XY\rangle + X^iY^j\nabla_i\nabla_ju\right]dV \qquad (5.34)\\
&\quad + \frac{1}{2}\int_M u^2\Big[\langle\nabla_XX,\nabla_YY\rangle + \langle\nabla_{\nabla_YX}X,Y\rangle \\
&\quad + X^iY^k(Y^j\nabla_j\nabla_iX_k + X^j\nabla_k\nabla_iY_j)\Big]dV.
\end{aligned}
$$

According to the following identities

$$
\begin{aligned}
X^iY^kY^j\nabla_j\nabla_iX_k &= Y^kY^j\nabla_j(X^i\nabla_iX_k) - Y^kY^j\nabla_jX^i\nabla_iX_k \\
&= Y^kY^j\nabla_j\nabla_XX_k - \nabla_YX^i\nabla_iX_kY^k \\
&= \langle Y,\nabla_Y\nabla_XX\rangle - \langle Y,\nabla_{\nabla_YX}X\rangle, \\
X^iY^kX^j\nabla_k\nabla_iY_j &= Y^kX^j\nabla_k(X^i\nabla_iY_j) - Y^kX^j\nabla_kX^i\nabla_iY_j \\
&= Y^kX^j\nabla_k\nabla_XY_j - \nabla_YX^i\nabla_iY_jX^j \\
&= \langle X,\nabla_Y\nabla_XY\rangle - \langle X,\nabla_{\nabla_YX}Y\rangle,
\end{aligned}
$$

we find that

$$
\begin{aligned}
&\langle\nabla_XX,\nabla_YY\rangle + \langle\nabla_{\nabla_YX}X,Y\rangle + X^iY^k(Y^j\nabla_j\nabla_iX_k + X^j\nabla_k\nabla_iY_j) \\
&= \langle\nabla_XX,\nabla_YY\rangle + \langle Y,\nabla_{\nabla_YX}X\rangle + \langle X,\nabla_Y\nabla_XY\rangle \\
&\quad + \langle Y,\nabla_Y\nabla_XX\rangle - \langle X,\nabla_{\nabla_YX}X\rangle - \langle Y\nabla_{\nabla_YX}X\rangle \\
&= \langle X,\nabla_Y\nabla_XY - \nabla_{\nabla_YX}X\rangle + \langle\nabla_XX,\nabla_YY\rangle + \langle Y,\nabla_Y\nabla_XX\rangle \\
&= \operatorname{Rm}(Y,X,Y,X) + \langle\nabla_X\nabla_YY - \nabla_{\nabla_XY}Y,X\rangle + Y\langle Y,\nabla_XX\rangle.
\end{aligned}
$$

Plugging it into (5.34) and using $\nabla_XX = 0$ for any $X \in \mathfrak{X}_{\mathbf{KC}}(M)$ yields

$$
\begin{aligned}
J(X,Y) &= -\int_M u\langle X,Y\rangle[\langle\nabla u,\nabla_XY\rangle + X^iY^j\nabla_i\nabla_ju]dV \\
&\quad + \frac{1}{2}\int_M u^2\left[-\operatorname{Rm}(X,Y,Y,X) - \langle X,\nabla_{\nabla_XY}Y\rangle\right]dV, \quad X,Y\in\mathfrak{X}_{\mathbf{KC}}(M). \qquad (5.35)
\end{aligned}
$$

Since J is symmetric, we can rewrite (5.35) in a symmetric form. Changing X and Y in (5.35) we have

$$\begin{aligned} J(X,Y) &= -\int_M u\langle X,Y\rangle[\langle\nabla u,\nabla_Y X\rangle + X^iY^j\nabla_i\nabla_j u]dV \\ &\quad + \frac{1}{2}\int_M u^2\left[-\mathrm{Rm}(X,Y,Y,X) - \langle Y,\nabla_{\nabla_Y X}YX\rangle\right]dV, \quad X,Y\in\mathfrak{X}_{\mathbf{KC}}(M). \end{aligned}$$

Combining it with (5.35) we arrive at

$$\begin{aligned} J(X,Y) &= -\int_M u\langle X,Y\rangle\left[\frac{1}{2}\langle\nabla u,\nabla_X Y+\nabla_Y X\rangle + X^iY^j\nabla_i\nabla_j u\right]dV \\ &\quad + \frac{1}{2}\int_M u^2\Lambda dV, \quad X,Y\in\mathfrak{X}_{\mathbf{KC}}(M) \end{aligned} \tag{5.36}$$

where

$$\Lambda := -\mathrm{Rm}(X,Y,Y,X) - \frac{1}{2}\langle X,\nabla_{\nabla_X Y}Y\rangle - \frac{1}{2}\langle Y,\nabla_{\nabla_Y X}X\rangle. \tag{5.37}$$

The following obvious identities

$$-\mathrm{Rm}(X,Y,Y,X) - \langle X,\nabla_{\nabla_X Y}Y\rangle = -\mathrm{Rm}(X,Y,Y,X) + \langle Y,\nabla_{\nabla_X Y}X\rangle - \nabla_X Y\langle X,Y\rangle,$$

$$-\mathrm{Rm}(X,Y,Y,X) - \langle Y,\nabla_{\nabla_Y X}X\rangle = -\mathrm{Rm}(X,Y,Y,X) + \langle X,\nabla_{\nabla_Y X}Y\rangle - \nabla_Y X\langle Y,X\rangle,$$

imply

$$\begin{aligned} \Lambda &= -\mathrm{Rm}(X,Y,Y,X) - \frac{1}{4}\langle X,Y\rangle(\nabla_X Y+\nabla_Y X) \\ &\quad + \frac{1}{4}\left[\langle Y,\nabla_{[X,Y]}X\rangle + \langle X,\nabla_{[Y,X]}Y\rangle\right]. \end{aligned} \tag{5.38}$$

In summary, we obtain

$$\begin{aligned} \int_M \mathbf{Rm}^{\mathbf{L}}(X,Y,Y,X)dV &= \int_M u\langle X,Y\rangle\left[\langle\nabla u,\nabla_X Y+\nabla_Y X\rangle + 2X^iY^j\nabla_i\nabla_j u\right]dV \\ &\quad + \int_M \frac{u^2}{2}\left[\langle X,\nabla_{\nabla_X Y}Y\rangle + \langle Y,\nabla_{\nabla_Y X}X\rangle\right]dV \\ &\quad + \int_M (1+u^2)\mathrm{Rm}(X,Y,Y,X)dV, \quad X,Y\in\mathfrak{X}_{\mathbf{KC}}(M). \end{aligned} \tag{5.39}$$

or

$$\begin{aligned} \int_M \mathbf{Rm}^{\mathbf{L}}(X,Y,Y,X)dV &= \int_M u\langle X,Y\rangle\left[\langle\nabla u,\nabla_X Y+\nabla_Y X\rangle + 2X^iY^j\nabla_i\nabla_j u\right]dV \\ &\quad + \int_M \frac{u^2}{4}\langle X,Y\rangle(\nabla_X Y+\nabla_Y X)dV \\ &\quad - \int_M \frac{u^2}{4}\left[\langle Y,\nabla_{[X,Y]}X\rangle + \langle X,\nabla_{[Y,X]}Y\rangle\right]dV \\ &\quad + \int_M (1+u^2)\mathrm{Rm}(X,Y,Y,X)dV, \quad X,Y\in\mathfrak{X}_{\mathbf{KC}}(M). \end{aligned} \tag{5.40}$$

6. Uniqueness for the Ricci-harmonic flow

In the section, we prove the forward and backward uniqueness of solutions for the Ricci-harmonic flow. Suppose that (M, g_0) is a complete Riemannian manifold of dimension n and u_0 is a smooth function on M. Consider the Ricci-harmonic flow

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)} + 4\nabla_{g(t)}u(t) \otimes \nabla_{g(t)}u(t), \quad \partial_t u(t) = \Delta_{g(t)}u(t), \tag{6.1}$$

on $M \times [0, T]$.

6.1. Forward uniqueness.

We now use the idea in [29] to prove the following

Theorem 6.1. (Forward uniqueness of the Ricci-harmonic flow) *Suppose that $(g(t), u(t))$ and $(\tilde{g}(t), \tilde{u}(t))$ are two smooth complete solutions of (6.1) with*

$$\sup_{M\times[0,T]} \left(|\mathrm{Rm}_{g(t)}|_{g(t)} + |\mathrm{Rm}_{\tilde{g}(t)}|_{\tilde{g}(t)}\right) \leq K, \tag{6.2}$$

for some uniform constant K. If $(g(0), u(0)) = (\tilde{g}(0), \tilde{u}(0)) = (g_0, u_0)$, then $g(t) \equiv \tilde{g}(t)$ for each $t \in [0, T]$.

We now write

$$g := g(t),\ \mathrm{Rm} := \mathrm{Rm}_{g(t)},\ \Gamma := \Gamma_{g(t)},\ u := u(t),\ \nabla := \nabla_{g(t)},\ dV := dV_{g(t)},$$

and

$$\tilde{g} := \tilde{g}(t),\ \widetilde{\mathrm{Rm}} := \mathrm{Rm}_{\tilde{g}(t)},\ \widetilde{\Gamma} := \Gamma_{\tilde{g}(t)},\ \tilde{u} := \tilde{u}(t),\ \widetilde{\nabla} := \nabla_{\tilde{g}(t)},\ d\widetilde{V} := dV_{\tilde{g}(t)}.$$

We further fix a norm $|\cdot| := |\cdot|_{g(t)}$. The (p, q)-tensor fields $T = (T^{i_1\cdots i_p}{}_{j_1\cdots j_q})$ are smooth sections of the (p, q)-tensor bundle $\mathcal{T}^p_q(M)$ over M. For $(p, q) = (0, 0)$, we write $\mathcal{T}^0_0(M)$ as $C^\infty(M)$. Introduce

$$(h, A, T, v, w, y) \in \mathcal{T}^0_2(M) \oplus \mathcal{T}^1_2(M) \oplus \mathcal{T}^1_3(M) \oplus \mathcal{T}^0_0(M) \oplus \mathcal{T}^0_1(M) \oplus \mathcal{T}^0_2(M)$$

according to the following definitions:

$$h \equiv h(t) := g - \tilde{g}, \quad h_{ij} := g_{ij} - \tilde{g}_{ij}, \tag{6.3}$$

$$A \equiv A(t) := \nabla - \widetilde{\nabla}, \quad A^k_{ij} = \Gamma^k_{ij} - \widetilde{\Gamma}^k_{ij}, \tag{6.4}$$

$$T \equiv T(t) = \mathrm{Rm} - \widetilde{\mathrm{Rm}}, \quad T^\ell_{ijk} = R^\ell_{ijk} - \widetilde{R}^\ell_{ijk}, \tag{6.5}$$

$$v \equiv v(t) := u - \tilde{u}, \tag{6.6}$$

$$w \equiv w(t) := \nabla u - \widetilde{\nabla}\tilde{u}, \quad w_i = \nabla_i v, \tag{6.7}$$

$$y \equiv y(t) := \nabla^2 u - \widetilde{\nabla}^2\tilde{u}, \quad y_{ij} = \nabla_i\nabla_j u - \widetilde{\nabla}_i\widetilde{\nabla}_j\tilde{u}. \tag{6.8}$$

From the definitions, we have

$$\begin{aligned} y_{ij} &= \partial^2_{ij}u - \partial^2_{ij}\tilde{u} - \left(\Gamma^k_{ij} - \widetilde{\Gamma}^k_{ij}\right)\partial_k\tilde{u} + \Gamma^k_{ij}\left(\partial_k\tilde{u} - \partial_k u\right) \\ &= \left[\left(\partial^2_{ij}u - \Gamma^k_{ij}\partial_k u\right) - \left(\partial^2_{ij}\tilde{u} - \Gamma^k_{ij}\partial_k\tilde{u}\right)\right] - A^k_{ij}\partial_k\tilde{u} \\ &= \nabla_i w_j - A^k_{ij}\partial_k\tilde{u} \ = \ \frac{1}{2}\left(\nabla_i w_j + \nabla_j w_i\right) - A^k_{ij}\partial_k\tilde{u} \end{aligned}$$

so that

$$y = \nabla w + A * \widetilde{\nabla}\tilde{u}. \tag{6.9}$$

Here $V * W$ means the linear combination of contractions of tensors V and W with respect to $g(t)$. From [29], we have

$$\tilde{g}^{ij} - g^{ij} = \tilde{g}^{ia} g^{jb} h_{ab}, \quad g^{-1} - \tilde{g}^{-1} = \tilde{g}^{-1} * g * h, \tag{6.10}$$

$$\nabla_k \tilde{g}^{ij} = \tilde{g}^{\ell j} A^i_{k\ell} + \tilde{g}^{i\ell} A^j_{ka}, \quad \nabla \tilde{g}^{-1} = \tilde{g}^{-1} * A. \tag{6.11}$$

Further,

$$\partial_t h_{ij} = -2T^{\ell}_{\ell ij} + 4\left(\partial_i u \partial_j u - \partial_i \tilde{u} \partial_j \tilde{u}\right)$$

so that

$$\partial_t h_{ij} = -2T^{\ell}_{\ell ij} + w_i w_j + w_i \partial_j \tilde{u} + w_j \partial_i \tilde{u} = -2T^{\ell}_{\ell ij} + w * w + w * \widetilde{\nabla} \tilde{u}. \tag{6.12}$$

The evolution of $\partial_t A^k_{ij}$ can be derived similarly as that in [29]. Indeed,

$$\begin{aligned}
\partial_t A^k_{ij} &= \tilde{g}^{mk}\left(\widetilde{\nabla}_i \widetilde{R}_{jm} + \widetilde{\nabla}_j \widetilde{R}_{im} - \widetilde{\nabla}_m \widetilde{R}_{ij}\right) - g^{mk}\left(\nabla_i R_{jm} + \nabla_j R_{im} - \nabla_m R_{ij}\right) \\
&\quad + 4\left(g^{mk} \nabla_m u \nabla_i \nabla_j u - \tilde{g}^{mk} \partial_m \tilde{u} \widetilde{\nabla}_i \widetilde{\nabla}_j \tilde{u}\right) \\
&= \tilde{g}^{-1} * h * \widetilde{\nabla} \widetilde{\mathrm{Rm}} + A * \widetilde{\mathrm{Rm}} + 1 * \nabla T \\
&\quad + 4\left[\left(g^{mk} - \tilde{g}^{mk}\right) \widetilde{\nabla}_m \tilde{u} \widetilde{\nabla}_i \widetilde{\nabla}_j \tilde{u} + g^{mk}\left(\nabla_m u \nabla_i \nabla_j u - \widetilde{\nabla}_m \tilde{u} \widetilde{\nabla}_i \widetilde{\nabla}_j \tilde{u}\right)\right]
\end{aligned}$$

where the third line comes from the computation in [29]. In the last line, writing

$$\nabla_m u \nabla_i \nabla_j u - \widetilde{\nabla}_m \tilde{u} \widetilde{\nabla}_i \widetilde{\nabla}_j \tilde{u} = \left(\nabla_m u - \widetilde{\nabla}_m \tilde{u}\right) \widetilde{\nabla}_i \widetilde{\nabla}_j \tilde{u} + \nabla_m u\left(\nabla_i \nabla_j u - \widetilde{\nabla}_i \widetilde{\nabla}_j \tilde{u}\right)$$

we can conclude

$$\begin{aligned}
\partial_t A &= \tilde{g}^{-1} * h * \widetilde{\nabla} \widetilde{\mathrm{Rm}} + A * \widetilde{\mathrm{Rm}} + 1 * \nabla T \\
&\quad + \tilde{g}^{-1} * h * \widetilde{\nabla} u * \widetilde{\nabla}^2 \tilde{u} + w * \widetilde{\nabla}^2 \tilde{u} + \nabla u * y.
\end{aligned} \tag{6.13}$$

The same argument gives

$$\begin{aligned}
\partial_t T^{\ell}_{ijk} &= \nabla_a\left(g^{ab} \nabla_b R^{\ell}_{ijk} - \tilde{g}^{ab} \widetilde{\nabla}_b \widetilde{R}^{\ell}_{ijk}\right) + \tilde{g}^{-1} * A * \widetilde{\nabla} \widetilde{\mathrm{Rm}} + \tilde{g}^{-1} * h * \widetilde{\mathrm{Rm}} * \widetilde{\mathrm{Rm}} \\
&\quad + T * \mathrm{Rm} + T * \widetilde{\mathrm{Rm}} + g^{\ell m}\left(\nabla_i \nabla_m u \nabla_k \nabla_j u - \nabla_i \nabla_k u \nabla_j \nabla_m u\right) \\
&\quad - \tilde{g}^{\ell m}\left(\widetilde{\nabla}_i \widetilde{\nabla}_m \tilde{u} \widetilde{\nabla}_k \widetilde{\nabla}_j \tilde{u} - \widetilde{\nabla}_i \widetilde{\nabla}_k \tilde{u} \widetilde{\nabla}_j \widetilde{\nabla}_m \tilde{u}\right).
\end{aligned}$$

Since

$$\begin{aligned}
& g^{\ell m} \nabla_i \nabla_m u \nabla_k \nabla_j u - \tilde{g}^{\ell m} \widetilde{\nabla}_i \widetilde{\nabla}_m \tilde{u} \widetilde{\nabla}_k \widetilde{\nabla}_j \tilde{u} \\
&= \left(g^{\ell m} - \tilde{g}^{\ell m}\right) \widetilde{\nabla}_i \widetilde{\nabla}_m \tilde{u} \widetilde{\nabla}_k \widetilde{\nabla}_j \tilde{u} + g^{\ell m}\left(\nabla_i \nabla_m u \nabla_k \nabla_j u - \widetilde{\nabla}_i \widetilde{\nabla}_m \tilde{u} \widetilde{\nabla}_k \widetilde{\nabla}_j \tilde{u}\right) \\
&= \tilde{g}^{-1} * h * \widetilde{\nabla}^2 \tilde{u} * \widetilde{\nabla}^2 \tilde{u} + g^{\ell m}\Big[\left(\nabla_i \nabla_m u - \widetilde{\nabla}_i \widetilde{\nabla}_m \tilde{u}\right) \widetilde{\nabla}_k \widetilde{\nabla}_j \tilde{u} \\
&\quad + \nabla_i \nabla_m u\left(\nabla_k \nabla_j u - \widetilde{\nabla}_k \widetilde{\nabla}_j \tilde{u}\right)\Big] = \tilde{g}^{-1} * h * \widetilde{\nabla}^2 \tilde{u} * \widetilde{\nabla}^2 \tilde{u} + y * \widetilde{\nabla}^2 \tilde{u} + y * \nabla^2 u,
\end{aligned}$$

it follows that

$$\begin{aligned}
\partial_t T^{\ell}_{ijk} &= \nabla_a\left(g^{ab} \nabla_b R^{\ell}_{ijk} - \tilde{g}^{ab} \widetilde{\nabla}_b \widetilde{R}^{\ell}_{ijk}\right) + \tilde{g}^{-1} * A * \widetilde{\nabla} \widetilde{\mathrm{Rm}} + \tilde{g}^{-1} * h * \widetilde{\mathrm{Rm}} * \widetilde{\mathrm{Rm}} \\
&\quad + T * \mathrm{Rm} + T * \widetilde{\mathrm{Rm}} + \tilde{g}^{-1} * h * \widetilde{\nabla}^2 \tilde{u} * \widetilde{\nabla}^2 \tilde{u} + y * \nabla^2 u + y * \widetilde{\nabla}^2 \tilde{u}.
\end{aligned} \tag{6.14}$$

Finally, we compute the evolution equation of v. Because

$$\begin{aligned}
\partial_t v &= \Delta u - \widetilde{\Delta}\tilde{u} = \Delta(u-\tilde{u}) + \left(g^{ij}\nabla_i\nabla_j - \tilde{g}^{ij}\widetilde{\nabla}_i\widetilde{\nabla}_j\right)\tilde{u}\\
&= \Delta v + \left(g^{ij}-\tilde{g}^{ij}\right)\widetilde{\nabla}_i\widetilde{\nabla}_j\tilde{u} + g^{ij}\left(\nabla_i\nabla_j - \widetilde{\nabla}_i\widetilde{\nabla}_j\right)\tilde{u}\\
&= \Delta v + \left(g^{ij}-\tilde{g}^{ij}\right)\widetilde{\nabla}_i\widetilde{\nabla}_j\tilde{u} + g^{ij}\left(\widetilde{\Gamma}^k_{ij} - \Gamma^k_{ij}\right)\partial_k\tilde{u}
\end{aligned}$$

we get

$$\partial_t v = \Delta v + \tilde{g}^{-1} * h * \widetilde{\nabla}^2\tilde{u} + A * \widetilde{\nabla}\tilde{u}. \tag{6.15}$$

From (6.12) – (6.15), we arrive at

$$|\partial_t h| \lesssim |T| + |w|^2 + |w||\widetilde{\nabla}\tilde{u}|, \tag{6.16}$$

$$\begin{aligned}
|\partial_t A| &\lesssim |\tilde{g}^{-1}||h||\widetilde{\nabla}\widetilde{\mathrm{Rm}}| + |A||\widetilde{\mathrm{Rm}}| + |\nabla T|\\
&\quad + |\tilde{g}^{-1}||h||\widetilde{\nabla}^2\tilde{u}| + |w||\widetilde{\nabla}^2\tilde{u}| + |\nabla u||y|,
\end{aligned} \tag{6.17}$$

$$\begin{aligned}
|\partial_t T - \Delta T - \mathrm{div}\, S| &\lesssim |\tilde{g}^{-1}||A||\widetilde{\nabla}\widetilde{\mathrm{Rm}}| + |\tilde{g}^{-1}||h||\widetilde{\mathrm{Rm}}|^2 + |T|(|\mathrm{Rm}| + |\widetilde{\mathrm{Rm}}|)\\
&\quad + |\tilde{g}^{-1}||h||\widetilde{\nabla}^2\tilde{u}|^2 + |y|(|\nabla^2 u| + |\widetilde{\nabla}^2\tilde{u}|),
\end{aligned} \tag{6.18}$$

$$|\partial_t v - \Delta v| \lesssim |\tilde{g}^{-1}||h||\widetilde{\nabla}^2\tilde{u}| + |A||\widetilde{\nabla}\tilde{u}|. \tag{6.19}$$

Here the tensor $S = (S^{a\ell}_{ijk})$ is defined as

$$S^{a\ell}_{ijk} := g^{ab}\nabla_b\widetilde{R}^\ell_{ijk} - \tilde{g}^{ab}\widetilde{\nabla}_b\widetilde{R}^\ell_{ijk} = \tilde{g}^{-1} * h * \widetilde{\nabla}\widetilde{\mathrm{Rm}} + A * \widetilde{\mathrm{Rm}}$$

and satisfies

$$|S| \lesssim |\tilde{g}^{-1}||h||\widetilde{\nabla}\widetilde{\mathrm{Rm}}| + |A||\widetilde{\mathrm{Rm}}|. \tag{6.20}$$

To further study, we need a version of Bernstein-Bando-Shi (BBS) estimate on higher derivatives of Rm and u. Under the curvature condition (6.2), according to Theorem B.2, we have

$$|\nabla u| \lesssim 1, \quad |\widetilde{\nabla}\tilde{u}| \lesssim 1, \tag{6.21}$$

and

$$|\nabla \mathrm{Rm}| + |\nabla^2 u| + |\widetilde{\nabla}\widetilde{\mathrm{Rm}}| + |\widetilde{\nabla}^2\tilde{u}| \lesssim 1. \tag{6.22}$$

Proposition 6.2. *Assume the curvature condition (6.2). We first prove*

$$|h(t)| \le K_0 t, \quad |A(t)| \le K_0 t^{1/2}, \quad |v(t)| \le K_0 t, \tag{6.23}$$

on $M \times [0, T]$ for some uniform constant $K_0 = K_0(n, K, T, g_0, u_0)$.

Proof. Since $g(0) = \tilde{g}(0) = g_0$, the inequality (6.23) is trivial for $t = 0$. In the following we may without loss of generality assume that $t \in (0, T]$. Given a space-time point $(p, t) \in M \times (0, T]$. The first can be proved by using (6.21)

$$\begin{aligned}
|h(p,t)| &\lesssim |h(p,t)|_{g_0} \lesssim \int_0^t |\partial_s h(p,s)|_{g_0} ds \lesssim \int_0^t |\partial_s h(p,s)| ds\\
&\lesssim \int_0^t \left[|T| + |w|^2 + |w||\widetilde{\nabla}\tilde{u}|\right](p,s) ds \lesssim \int_0^t ds \lesssim t.
\end{aligned}$$

For second one, using , one has, by (6.22),

$$\begin{aligned}|A(p,t)| &\lesssim |A(p,t)|_{g_0} \lesssim \int_0^t |\partial_s A(p,s)|_{g_0} ds \lesssim \int_0^t |\partial_s A(p,s)| ds\\ &\lesssim \int_0^t \left[|\widetilde{\nabla}\widetilde{\mathrm{Rm}}| + |\nabla \mathrm{Rm}| + |\nabla u||\nabla^2 u| + |\widetilde{\nabla}\tilde{u}||\widetilde{\nabla}^2\tilde{u}|\right] ds\\ &\lesssim \int_0^t ds \lesssim t \lesssim t^{1/2}.\end{aligned}$$

The last one follows from $|\partial_t v| \lesssim |\nabla^2 u| + |\widetilde{\nabla}^2 \tilde{u}|$. □

Notice that the above proposition gives an explicit bound for $\nabla^2 u$ and hence for Δu, provided the condition (6.2) holds. However, Theorem 1.1 or Theorem 2.6 gives an explicit bound for Δu under a weaker condition (1.2).

To prove the uniqueness, we need the following

Lemma 6.3. *Consider a smooth family $(g(t))_{t\in[0,T]}$ of complete metrics on M with $g(t) \geq \gamma^{-1} g$ for some uniform constant γ, where $g := g(0)$. Choose any given point $x_0 \in M$ and set $r(x) := d_g(x, x_0)$. Then the following statement is true: For any given constants $L_1, L_2 > 0$, there exist a constant $T' := T'(n, \gamma, L_1, L_2, T) > 0$ and a function $\eta : M \times [0, T'] \to \mathbb{R}$ smooth in t, Lipschitz on $M \times \{t\}$, and satisfying*

$$\partial_t \eta \geq L_1 |\eta|^2_{g(t)}, \quad e^{-\eta} \leq e^{-L_2 r^2} \tag{6.24}$$

on $M \times [0, \tau]$ for all $\tau \in (0, T']$.

Proof. See [29]. Actually, we can pick $T' = \min\{T, 1/4(\gamma L_1 L_2)^{1/2}\}$. □

Under the condition (6.2), Lemma 6.3 implies that for any given (sufficiently small and independent of T) constant $B > 0$ there exist a constant $T' = T'(n, K, B, T) > 0$ and such a function $\eta : M \times [0, T'] \to \mathbb{R}$ satisfy

$$e^{-\eta} \leq e^{-Br_0^2/T}, \quad \partial_t \eta \geq B|\nabla \eta|^2 \tag{6.25}$$

on $M \times [0, \tau]$ for all $\tau \in (0, T']$; hence $\partial_t \eta, |\nabla \eta|^2$ are $e^{-\eta} dV$-integrable for all $t \in (0, T']$. For detail, see [29].

There are three claims about the integrability of $\mathcal{E}(t)$ defined below. The proof is similar to that in [29], except for some extra terms.

(a) $t^{-1}|h|^2, t^{-\beta}|A|^2, |T|^2, |v|^2$, and $|w|^2$ are uniformly bounded. It has been proved in Proposition 6.2.

(b) $\partial_t|h|^2, \partial_t|A|^2, \partial_t|T|^2, \partial_t|v|^2$, and $\partial_t|w|^2$ are uniformly bounded on $M \times [0, T]$ and consequently are $e^{-\eta} dV$-integrable for all $t \in (0, T']$. Compute

$$\begin{aligned}\left|\partial_t |h|^2\right| &= \left|\partial_t (g^{-1} * g^{-1} * h * h)\right|\\ &= \left|g^{-1} * g^{-1} * g^{-1} * \partial_t g * h * h + g^{-1} * g^{-1} * h * \partial_t h\right|\\ &\lesssim |\mathrm{Ric}||h|^2 + |h||\partial_t h| \lesssim |h|^2 + |h|\left(|T| + |w|^2 + |w|\right) \lesssim 1,\end{aligned}$$

by (6.16) and (6.22). For $\partial_t|A|^2$ one has

$$\begin{aligned}\left|\partial_t|A|^2\right| &= \partial_t(g^{-1}*g^{-1}*g*A*A)\\ &= g^{-1}*g^{-1}*g^{-1}*g*\partial_t g*A*A+g^{-1}*g^{-1}*g*A*\partial_t A\\ &\lesssim |\mathrm{Ric}||A|^2+|A||\partial_t A|\\ &\lesssim |A|^2+|A|\left(|\widetilde{\nabla}\widetilde{\mathrm{Rm}}|+|\nabla\mathrm{Rm}|+|\nabla u||\nabla^2 u|+|\widetilde{\nabla}\tilde{u}||\widetilde{\nabla}^2\tilde{u}|\right) \lesssim 1,\end{aligned}$$

by (6.21), (6.22), (6.23), and the proof of Proposition 6.2. Similarly, we can deal with $\partial_t|S|^2$, $\partial_t|v|^2$, and $\partial_t|w|^2$. For example,

$$\left|\partial_t|v|^2\right| \lesssim |v|\left[|\nabla^2 u|+|\widetilde{\nabla}^2\tilde{u}|\right] \lesssim |v| \lesssim 1$$

using again Proposition 6.2.

Fixed $\beta\in(0,1)$, introduce the quantity

$$\mathcal{E}(t) := \int_M \left(t^{-1}|h|^2+t^{-\beta}|A|^2+|T|^2+|v|^2+|w|^2\right)e^{-\eta}dV. \tag{6.26}$$

Here the function η is determined by (6.25).

(c) $\mathcal{E}(t)$ is differentiable on $[0,T']$ and $\lim_{t\to 0}\mathcal{E}(t)=0$. This follows from Proposition 6.2 and Lebesgue's dominated convergence theorem.

The above claims (a) – (c) allow us frequently to take time derivatives inside the integrals.

Proposition 6.4. *Assume that the curvature condition (6.2) is satisfied. There exist uniform constants $N=N(n,K,T,g_0,u_0)>0$ and $T_0=T_0(n,K,T,g_0,u_0,\beta)\in(0,T]$ such that*

$$\mathcal{E}'(t)\le N\mathcal{E}(t) \tag{6.27}$$

for $t\in[0,T_0]$. Hence $\mathcal{E}(t)\equiv 0$ for all $t\in[0,T_0]$.

Proof. As in [29], for any $t\in(0,T]$ and $\alpha\in(0,1)$, define

$$\mathcal{G}(t) := \int_M |T|^2 e^{-\eta}dV, \quad \mathcal{H}(t) := \int_M t^{-1}|h|^2e^{-\eta}dV, \tag{6.28}$$

$$\mathcal{I}(t) := \int_M t^{-\beta}|A|^2e^{-\eta}dV, \quad \mathcal{J}(t) := \int_M |\nabla T|^2e^{-\eta}dV, \tag{6.29}$$

$$\mathcal{V}(t) := \int_M |v|^2e^{-\eta}dV,\ \mathcal{D}(t) := \int_M |y|^2e^{-\eta}dV,\ \mathcal{B}(t) := \int_M |w|^2e^{-\eta}dV. \tag{6.30}$$

Then

$$\mathcal{E}(t)=\mathcal{G}(t)+\mathcal{H}(t)+\mathcal{I}(t)+\mathcal{V}(t)+\mathcal{B}(t). \tag{6.31}$$

We denote by C any constant depending only on n, and by N any constant depending on n,K,T,g_0,u_0,β.

Since $g(t)$ are all uniformly equivalent to $g=g(0)$, we can replace the norm $|\cdot|$ in (6.26) by $|\cdot|_g$. Hence we may regard the norm $|\cdot|$ in (6.26) is independent of time.

(1) Estimate for $\mathcal{G}'$. Start with

$$\begin{aligned}\mathcal{G}' &= \int_M \left[\partial_t |T|^2 - |T|^2 \partial_t \eta + \left(-R + 2|\nabla u|^2\right)|T|^2\right] e^{-\eta} dV \\ &\leq N\mathcal{G} + \int_M \left[2\langle \partial_t T, T\rangle - |T|^2 \partial_t \eta\right] e^{-\eta} dV.\end{aligned}$$

Using (6.18), (6.20), (6.2), (6.29), and (6.31), we have

$$\begin{aligned}\mathcal{G}' &\leq N\mathcal{G} + \int_M \Big[2\langle \Delta T + \operatorname{div} S, T\rangle + C|\tilde{g}^{-1}||\widetilde{\nabla}\widetilde{\mathrm{Rm}}||A||T| \\ &\quad + C|\tilde{g}^{-1}||\widetilde{\mathrm{Rm}}|^2|h||T| + C\left(|\mathrm{Rm}| + |\widetilde{\mathrm{Rm}}|\right)|T|^2 - |T|^2\partial_t\eta \\ &\quad + C|\tilde{g}^{-1}||\widetilde{\nabla}^2\tilde{u}|^2|h||T| + \left(|\nabla^2 u| + |\widetilde{\nabla}^2\tilde{u}|\right)|y||T|\Big] e^{-\eta} dV \\ &\leq N\mathcal{G} + \int_M \Big[\left(2\langle \Delta T + \operatorname{div} S, T\rangle - |T|^2\partial_t\eta\right) \\ &\quad + Nt^{-1/2}|A||T| + N|h||T| + N|T|^2 + N|y||T|\Big] e^{-\eta} dV\end{aligned}$$

The same argument in [29] implies

$$\int_M \left[2\langle \Delta T + \operatorname{div} S, T\rangle - |T|^2 \partial_t \eta\right] e^{-\eta} dV \leq -\mathcal{J} + N\mathcal{H} + Nt^{\beta}\mathcal{I}$$

by choosing an appropriate constant B in (6.25). Therefore

$$\mathcal{G}' \leq N\mathcal{G} + N\mathcal{H} + \left(N + t^{\beta-1}\right)\mathcal{I} - \mathcal{J} + \frac{1}{4}\mathcal{D}. \tag{6.32}$$

(2) Estimate for $\mathcal{H}'$. Since $\partial_t \eta \geq 0$, we get

$$\begin{aligned}\mathcal{H}' &= -t^{-1}\mathcal{H} + t^{-1}\int_M \left[2\langle \partial_t h, h\rangle - |h|^2\partial_t\eta + |h|^2\left(-R + 2|\nabla u|^2\right)\right] e^{-\eta} dV \\ &\leq (N - t^{-1})\mathcal{H} + Ct^{-1}\int_M |h|\left(|T|^2 + |w|^2 + |w||\widetilde{\nabla}\tilde{u}|\right) e^{-\eta} dV \\ &\leq (N - t^{-1})\mathcal{H} + Ct^{-1}\int_M |h||T| e^{-\eta} dV + Nt^{-1}\int_M |h|w|e^{-\eta} dV \\ &\leq \left(N - \frac{1}{2}t^{-1}\right)\mathcal{H} + C\mathcal{G} + N\int_M |w|^2 e^{-\eta} dV.\end{aligned}$$

Thus

$$\mathcal{H}' \leq \left(N - \frac{1}{2}t^{-1}\right)\mathcal{H} + C\mathcal{G} + N\mathcal{B}. \tag{6.33}$$

(3) Estimate for $\mathcal{I}'$. As in the estimation of $\mathcal{H}'$, we have

$$\begin{aligned}
\mathcal{I}' &\leq (N-\beta t^{-1})\mathcal{I} + Ct^{-\beta}\int_M |A|\Big(|\tilde{g}^{-1}||\widetilde{\nabla}\widetilde{\mathrm{Rm}}||h| + |A||\widetilde{\mathrm{Rm}}| \\
&\quad + |\nabla T| + |\tilde{g}^{-1}||\widetilde{\nabla}^2\tilde{u}||h| + |w|\widetilde{\nabla}^2\tilde{u}| + |\nabla u||y|\Big)e^{-\eta}dV \\
&\leq (N-\beta t^{-1})\mathcal{I} + \int_M\Big(Ct^{-\beta}|\nabla T||A| + Nt^{-\frac{1}{2}-\beta}|h||A| \\
&\quad + Nt^{-\beta}|h||A| + Nt^{-\beta}|w||A| + Nt^{-\beta}|A||y|\Big)e^{-\eta}dV \\
&\leq (N-\beta t^{-1})\mathcal{I} + \Big(Ct^{-\beta}\mathcal{I} + \mathcal{J}\Big) + \Big(N\mathcal{H} + Ct^{-\beta}\mathcal{I}\Big) \\
&\quad + (N\mathcal{H} + t\mathcal{I}) + \Big(Ct^{-\beta}\mathcal{I} + N\mathcal{B}\Big) + \Big(Nt^{-\beta}\mathcal{I} + \frac{1}{4}\mathcal{D}\Big).
\end{aligned}$$

Consequently

$$\mathcal{I}' \leq N\mathcal{H} + \Big(N - \beta t^{-1} + Nt^{-\beta}\Big)\mathcal{I} + \mathcal{J} + N\mathcal{B} + \frac{1}{4}\mathcal{D}. \tag{6.34}$$

(4) Estimate for $\mathcal{V}'$. This estimate is similar to (1),

$$\begin{aligned}
\mathcal{V}' &\leq N\mathcal{V} + \int_M \Big[2\langle\partial_t v, v\rangle - |v|^2\partial_t\eta\Big]e^{-\eta}dV \\
&\leq N\mathcal{V} + \int_M \Big[2\langle\Delta v, v\rangle - |v|^2\partial_t\eta + C|v|\Big(|\tilde{g}^{-1}||h|\widetilde{\nabla}^2\tilde{u}| + |A||\widetilde{\nabla}\tilde{u}|\Big)\Big]e^{-\eta}dV \\
&\leq N\mathcal{V} + \int_M \Big[2\langle\Delta v, v\rangle - |v|^2\partial_t\eta\Big]e^{-\eta}dV + N\int_M (|v||h| + |v||A|)e^{-\eta}dV \\
&\leq N\mathcal{V} + t\mathcal{H} + t^\beta\mathcal{I} + \int_M \Big[-2|\nabla v|^2 + 2|v||\nabla\eta||\nabla v| - |v|^2\partial_t\eta\Big]e^{-\eta}dV.
\end{aligned}$$

Thus, by choosing an appropriate constant B in (6.25),

$$\mathcal{V}' \leq N\mathcal{V} + t\mathcal{H} + t^\beta\mathcal{I} - \mathcal{B} \tag{6.35}$$

(5) Estimate for $\mathcal{B}'$. Because the evolution equation (A.4) is linear in $\nabla_i u$, we conclude that

$$\partial_t w = \Delta w + \mathrm{Rm} * w \tag{6.36}$$

and hence

$$\begin{aligned}
\mathcal{B}' &\leq N\mathcal{B} + \int_M \Big[2\langle\partial_t w, w\rangle - |w|^2\partial_t\eta\Big]e^{-\eta}dV \\
&\leq N\mathcal{B} + \int_M \Big[2\langle\Delta w, w\rangle - |w|^2\partial_t\eta\Big]e^{-\eta}dV \\
&\leq N\mathcal{B} - \int_M |\nabla w|^2 e^{-\eta}dV
\end{aligned} \tag{6.37}$$

by choosing an appropriate constant B in (6.25). From (6.32) – (6.37), we arrive at

$$\mathcal{E}' \leq N\mathcal{E} - t^{-1}\Big(\beta - t^\beta - Nt^{1-\beta}\Big)\mathcal{I} + \frac{1}{2}\mathcal{D} - \int_M |\nabla w|^2 e^{-\eta}dV. \tag{6.38}$$

On the other hand, the equation (6.9) yields

$$\begin{aligned}\frac{1}{2}\mathcal{D} &= \frac{1}{2}\int_M |y|^2 e^{-\eta} dV \leq \int_M \left[|\nabla w|^2 + N|A|^2|\widetilde{\nabla}\tilde{u}|^2\right] e^{-\eta} dV \\ &\leq \int_M |\nabla w|^2 e^{-\eta} dV + Nt^{\beta}\mathcal{I}.\end{aligned}$$

Therefore, the inequality (6.38) can be rewritten as

$$\mathcal{E}' \leq N\mathcal{E} - t^{-1}\left(\beta - t^{\beta} - Nt^{1-\beta}\right)\mathcal{I}. \tag{6.39}$$

Now we can choose appropriate constants T_0 and N such that the term $\beta - t^{\beta} - Nt^{1-\beta}$ is nonnegative for any $t \in [0, T_0]$. In this case, the inequality (6.43) gives us the desired estimate $\mathcal{E}'(t) \leq N\mathcal{E}(t)$ on $[0, T_0]$. □

The proof of Theorem 6.1. Now the proof immediately follows from the above Proposition 6.4.

6.2. **Backward uniqueness.** In this subsection we use the main idea in [28] to prove the backward uniqueness of the Ricci-harmonic flow. Recall

Theorem 6.5. (Kotschwar [28]) *Consider a smooth family of complete Riemannian metrics $(g(t))_{t\in[0,T]}$ on a smooth n-dimensional manifold M, satisfying the evolution equation*

$$\partial_t g(t) = b(t), \tag{6.40}$$

and a symmetric, positive-definite family of $(2,0)$-tensor fields $\Lambda(t)$, $t \in [0, T]$, with

$$\square := \partial_t - \Delta_{\Lambda(t),g(t)}, \quad \Delta_{\Lambda,g(t)} := \mathrm{tr}_{\Lambda(t)}\nabla^2_{g(t)}. \tag{6.41}$$

Let $\mathscr{X}, \mathscr{Y}$ be finite direct sums of the (p,q)-bundle $\mathcal{T}^p_q(M)$ over M, and $\mathbf{X} \in C^\infty(\mathscr{X} \times [0,T])$, $\mathbf{Y} \in C^\infty(\mathscr{Y} \times [0,T])$. Suppose the following assumptions hold:

(1) *there exist positive constants P, Q, α_1, α_2 such that*

$$\begin{aligned}|b(t)|^2_{g(t)} + |\nabla_{g(t)} b(t)|^2_{g(t)} &\leq P, \\ |\partial_t\Lambda(t)|^2_{g(t)} + |\nabla_{g(t)}\Lambda(t)|^2_{g(t)} &\leq Q, \\ \alpha_1 g^{-1}(t) \leq \Lambda(t) &\leq \alpha_2 g^{-1}(t),\end{aligned}$$

(2) *there exists a nonnegative constant K such that $\mathrm{Ric}_{g(t)} \geq -Kg(t)$,*

(3) *there exist positive constants a, A and some point $x_0 \in M$ such that*

$$|\mathbf{X}(x,t)|^2_{g(t)} + |\nabla_{g(t)}\mathbf{X}(x,t)|^2_{g(t)} + |\mathbf{Y}(x,t)|^2_{g(t)} \leq Ae^{ad_{g(t)}(x_0,x)}, \tag{6.42}$$

(4) *$\mathbf{X}$ and $\mathbf{Y}$ satisfy the inequality*

$$|\square\mathbf{X}|^2_{g(t)} \leq C\left(|\mathbf{X}|^2_{g(t)} + |\nabla_{g(t)}\mathbf{X}|^2_{g(t)} + |\mathbf{Y}|^2_{g(t)}\right), \tag{6.43}$$

$$|\partial_t\mathbf{Y}|^2_{g(t)} \leq C\left(|\mathbf{X}|^2_{g(t)} + |\nabla_{g(t)}\mathbf{Y}|^2_{g(t)} + |\mathbf{Y}|^2_{g(t)}\right) \tag{6.44}$$

for some positive constant C.

If $\mathbf{X}(T) = \mathbf{Y}(T) = 0$, then $\mathbf{X} = \mathbf{Y} \equiv 0$ on $M \times [0, T]$.

Theorem 6.6. (Backward uniqueness of the Ricci-harmonic flow) *Suppose that $(g(t), u(t))$ and $(\tilde{g}(t), \tilde{u}(t))$ are two smooth complete solutions of (6.1) satisfying (6.2) for some uniform constant K. If $(g(T), u(T)) = (\tilde{g}(T), \tilde{u}(T))$, then $(g(t), u(t)) \equiv (\tilde{g}(t), \tilde{u}(t))$ for each $t \in [0, T]$.*

Recall notions in (6.3) – (6.8),

$$\begin{aligned} h &:= g-\tilde{g}, \quad h_{ij} = g_{ij}-\tilde{g}_{ij}, \quad A := \nabla - \widetilde{\nabla}, \quad A^k_{ij} = \Gamma^k_{ij} - \widetilde{\Gamma}^k_{ij}, \\ T &:= \mathrm{Rm} - \widetilde{\mathrm{Rm}}, \quad T^\ell_{ijk} = R^\ell_{ijk} - \widetilde{R}^\ell_{ijk}, \quad v := u - \tilde{u}, \quad w := \nabla u - \widetilde{\nabla}\tilde{u}, \\ w_i &:= \nabla_i v, \quad y := \nabla^2 u - \widetilde{\nabla}^2 \tilde{u}, \quad y_{ij} = \nabla_i\nabla_j u - \widetilde{\nabla}_i\widetilde{\nabla}_j \tilde{u}. \end{aligned}$$

Define new tensor fields

$$B := \nabla A, \tag{6.45}$$

$$U := \nabla \mathrm{Rm} - \widetilde{\nabla}\widetilde{\mathrm{Rm}}, \tag{6.46}$$

$$x := \nabla w, \tag{6.47}$$

$$z := \nabla^3 u - \widetilde{\nabla}^3 \tilde{u}. \tag{6.48}$$

Consider direct sums of tensor fields

$$\mathbf{X} := (T \oplus U) \oplus (y \oplus z), \quad \mathbf{Y} := (h \oplus A \oplus B) \oplus (v \oplus w \oplus x).$$

Using Lemma A.2, we obtain

$$\begin{aligned} \partial_t \Gamma &= g^{-1} * \nabla \mathrm{Rm} + g^{-1} * \nabla u * \nabla^2 u, \\ \partial_t \mathrm{Rm} &= \Delta \mathrm{Rm} + g^{-1} * \mathrm{Rm} * \mathrm{Rm} + g^{-1} * \mathrm{Rm} * \nabla u * \nabla u + g^{-1} * \nabla^2 u * \nabla^2 u, \\ \partial_t \nabla \mathrm{Rm} &= \Delta \nabla \mathrm{Rm} + g^{-1} * \mathrm{Rm} * \nabla \mathrm{Rm} + g^{-1} * \nabla \mathrm{Rm} * \nabla u * \nabla u \\ &\quad + g^{-1} * \mathrm{Rm} * \nabla u * \nabla^2 u + g^{-1} * \nabla^2 u * \nabla^3 u. \end{aligned}$$

We also recall

$$\tilde{g}^{ij} - g^{ij} = \tilde{g}^{ia} g^{jb} h_{ab} \quad \text{or} \quad \tilde{g}^{-1} - g^{-1} = \tilde{g}^{-1} * g^{-1} * h$$

and

$$\nabla_c h_{ab} = A^p_{ca}\tilde{g}_{pb} + A^p_{cb}\tilde{g}_{ap} \quad \text{or} \quad \nabla h = A * \tilde{g}. \tag{6.49}$$

The last identity follows from

$$\begin{aligned} \nabla_v h_{ab} &= \partial_v h_{ab} - \Gamma^p_{ca} h_{pb} - \Gamma^p_{cb} h_{ap} \\ &= \partial_c g_{ab} - \partial_c \tilde{g}_{ab} - \Gamma^p_{ca} g_{pb} + \Gamma^p_{ca}\tilde{g}_{pb} - \Gamma^p_{cb} g_{ap} + \Gamma^p_{cb}\tilde{g}_{ap} \\ &= \nabla_c g_{ab} - \partial_c \tilde{g}_{ab} + \left(A^p_{ca} + \widetilde{\Gamma}^p_{ca}\right)\tilde{g}_{pb} + \left(A^p_{cb} + \widetilde{\Gamma}^p_{cb}\right)\tilde{g}_{ap} \\ &= \nabla_c h_{ab} - \widetilde{\nabla}_c \tilde{g}_{ab} + A^p_{ca}\tilde{g}_{pb} + A^p_{cb}\tilde{g}_{ap}. \end{aligned}$$

Lemma 6.7. *One has*

$$\begin{aligned}\partial_t h_{ij} &= -2T^{\ell}_{\ell ij}\\ &\quad + 4\left(w_iw_j + w_i\widetilde{\nabla}_j\tilde{u} + w_j\widetilde{\nabla}_i\tilde{u}\right),\end{aligned} \tag{6.50}$$

$$\begin{aligned}\partial_t A^k_{ij} &= \Big[- g^{mk}\left(U^p_{ipjm} + U^p_{jpim} - U^p_{mpij}\right)\\ &\quad + g^{kb}\tilde{g}^{ma}h_{ab}\left(\widetilde{\nabla}_i\widetilde{R}_{jm} + \widetilde{\nabla}_j\widetilde{R}_{im} - \widetilde{\nabla}_m\widetilde{R}_{ij}\right)\Big] + 4g^{mk}\nabla_m u y_{ij}\\ &\quad + 4g^{mk}w_m\widetilde{\nabla}_i\widetilde{\nabla}_j\tilde{u} - 4\tilde{g}^{ma}g^{kb}h_{ab}\widetilde{\nabla}_m\tilde{u}\widetilde{\nabla}_i\widetilde{\nabla}_j\tilde{u},\end{aligned} \tag{6.51}$$

$$\begin{aligned}\partial_t B &= \Big[\nabla U + h * A * \tilde{g}^{-1} * \widetilde{\nabla}\widetilde{\mathrm{Rm}} + A * \tilde{g}^{-1} * \widetilde{\nabla}\widetilde{\mathrm{Rm}}\\ &\quad + h * \tilde{g}^{-1} * \widetilde{\nabla}^2\widetilde{\mathrm{Rm}} + A * U + A * \widetilde{\nabla}\widetilde{\mathrm{Rm}}\Big] + \tilde{g}^{-1} * h * \widetilde{\nabla}\tilde{u} * \widetilde{\nabla}^2\tilde{u}\\ &\quad + \tilde{g}^{-1} * A * \widetilde{\nabla}\tilde{u} * \widetilde{\nabla}^2\tilde{u} + \tilde{g}^{-1} * h * \widetilde{\nabla}^2\tilde{u} * \widetilde{\nabla}^2\tilde{u} + \nabla u * \nabla^2 u * A\\ &\quad + \tilde{g}^{-1} * h * A * \widetilde{\nabla}\tilde{u} * \widetilde{\nabla}^2\tilde{u} + \tilde{g}^{-1} * h * \widetilde{\nabla}\tilde{u} * \widetilde{\nabla}^3\tilde{u}\\ &\quad + \nabla^2 u * y + \nabla u * \nabla y + x * \widetilde{\nabla}^2\tilde{u} + w * \widetilde{\nabla}^3\tilde{u} + A * w * \widetilde{\nabla}^2\tilde{u},\end{aligned} \tag{6.52}$$

$$\begin{aligned}\Box T &= \Big[h * \tilde{g}^{-1} * \widetilde{\nabla}^2\widetilde{\mathrm{Rm}} + A * \widetilde{\nabla}\widetilde{\mathrm{Rm}} + T * \widetilde{\mathrm{Rm}} + T * T + A * A * \widetilde{\mathrm{Rm}}\\ &\quad + B * \widetilde{\mathrm{Rm}} + h * \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} * \widetilde{\mathrm{Rm}}\Big] + h * \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} * \widetilde{\nabla}\tilde{u} * \widetilde{\nabla}\tilde{u}\\ &\quad + \widetilde{\mathrm{Rm}} * w * w + \widetilde{\mathrm{Rm}} * w * \widetilde{\nabla}\tilde{u} + T * \widetilde{\nabla}\tilde{u} * \widetilde{\nabla}\tilde{u} + T * w * \widetilde{\nabla}\tilde{u}\\ &\quad + T * w * w + y * y + y * \widetilde{\nabla}^2\tilde{u} + h * g^{-1} * \widetilde{\nabla}^2\tilde{u} * \widetilde{\nabla}^2\tilde{u},\end{aligned} \tag{6.53}$$

$$\begin{aligned}\Box U &= \Big[h * \tilde{g}^{-1} * \widetilde{\nabla}^3\widetilde{\mathrm{Rm}} + A * \widetilde{\nabla}^2\widetilde{\mathrm{Rm}} + A * A * \widetilde{\nabla}\widetilde{\mathrm{Rm}} + B * \widetilde{\nabla}\widetilde{\mathrm{Rm}}\\ &\quad + h * \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} * \widetilde{\nabla}\widetilde{\mathrm{Rm}} + U * \widetilde{\mathrm{Rm}} + T * \widetilde{\nabla}\widetilde{\mathrm{Rm}} + T * U\Big]\\ &\quad + h * \tilde{g}^{-1} * \widetilde{\nabla}\widetilde{\mathrm{Rm}} * \widetilde{\nabla}\tilde{u} * \widetilde{\nabla}\tilde{u} + U * w * w + w * w * \widetilde{\nabla}\widetilde{\mathrm{Rm}}\\ &\quad + U * w * \widetilde{\nabla}\tilde{u} + \widetilde{\nabla}\widetilde{\mathrm{Rm}} * w * \widetilde{\nabla}\tilde{u} + U * \widetilde{\nabla}\tilde{u} * \widetilde{\nabla}\tilde{u}\\ &\quad + h * \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} * \widetilde{\nabla}\tilde{u} * \widetilde{\nabla}^2\tilde{u} + T * \widetilde{\nabla}\tilde{u} * y + \widetilde{\mathrm{Rm}} * \widetilde{\nabla}\tilde{u} * y\\ &\quad + T * w * \widetilde{\nabla}^2\tilde{u} + \widetilde{\mathrm{Rm}} * w * \widetilde{\nabla}^2\tilde{u} + T * \widetilde{\nabla}\tilde{u} * \widetilde{\nabla}^2\tilde{u} + T * w * y\\ &\quad + \widetilde{\mathrm{Rm}} * w * y + h * \tilde{g}^{-1} * \widetilde{\nabla}^2\tilde{u} * \widetilde{\nabla}^3\tilde{u} + y * z + y * \widetilde{\nabla}^3\tilde{u} + z * \widetilde{\nabla}^2\tilde{u}.\end{aligned} \tag{6.54}$$

Proof. For h_{ij}, one has

$$\begin{aligned}\partial_t h_{ij} &= \partial_t g_{ij} - \partial_t\tilde{g}_{ij} = -2R_{ij} + 4\nabla_i u\nabla_j u + 2\widetilde{R}_{ij} - 4\widetilde{\nabla}_i\tilde{u}\widetilde{\nabla}_j\tilde{u}\\ &= -2T^{\ell}_{\ell ij} + 4\left(w_i + \widetilde{\nabla}_i\tilde{u}\right)\left(w_j + \widetilde{\nabla}_j\tilde{u}\right) - 4\widetilde{\nabla}_i\tilde{u}\widetilde{\nabla}_j\tilde{u}\end{aligned}$$

which implies (6.50).

In (6.51), the terms enclosed in the bracket were derived in [28]. The remaining terms are

$$4g^{mk}\nabla_m u\nabla_i\nabla_j u - 4\tilde{g}^{mk}\widetilde{\nabla}_m\tilde{u}\widetilde{\nabla}_i\widetilde{\nabla}_j\tilde{u}$$

which, using $y_{ij} = \nabla_i \nabla_j u - \widetilde{\nabla}_i \widetilde{\nabla}_j \tilde{u}$, $w_m = \nabla_m y - \widetilde{\nabla}_m \tilde{u}$, and $\tilde{g}^{mk} - g^{mk} = \tilde{g}^{ma} g^{kb} h_{ab}$, is equal to

$$\begin{aligned} & g^{mk} \nabla_m u \left(y_{ij} + \widetilde{\nabla}_i \widetilde{\nabla}_j \tilde{u} \right) - \tilde{g}^{mk} \widetilde{\nabla}_m \tilde{u} \widetilde{\nabla}_i \widetilde{\nabla}_j \tilde{u} \\ = \ & g^{mk} \nabla_m u y_{ij} + \left(g^{mk} \nabla_m u - \tilde{g}^{mk} \widetilde{\nabla}_m \tilde{u} \right) \widetilde{\nabla}_i \widetilde{\nabla}_j \tilde{u} \\ = \ & g^{mk} \nabla_m u y_{ij} + \left[g^{mk} \left(\nabla_m u - \widetilde{\nabla}_m \tilde{u} \right) - \left(\tilde{g}^{mk} - g^{mk} \right) \widetilde{\nabla}_m \tilde{u} \right] \widetilde{\nabla}_i \widetilde{\nabla}_j \tilde{u} \\ = \ & g^{mk} \nabla_m u y_{ij} + g^{mk} w_m \widetilde{\nabla}_i \widetilde{\nabla}_j \tilde{u} - \tilde{g}^{ma} g^{kb} h_{ab} \widetilde{\nabla}_m \tilde{u} \widetilde{\nabla}_i \widetilde{\nabla}_j \tilde{u}. \end{aligned}$$

In particular, (6.51) implies

$$\partial_t A \ = \ \left(g^{-1} * U + g^{-1} * \tilde{g}^{-1} * h * \widetilde{\nabla} \widetilde{\mathrm{Rm}} \right).$$

According to the relation $\partial_t B = \partial_t \nabla A = \nabla \partial_t A + \partial_t \Gamma * A$, we obtain (6.52) where the bracket follows from [28] and the remaining terms are

$$\begin{aligned} & g^{-1} * \nabla u * \nabla^2 u * A + g^{-1} * \nabla^2 u * y + g^{-1} * \nabla u * \nabla y + g^{-1} * \nabla w * \widetilde{\nabla}^2 \tilde{u} \\ & + \tilde{g}^{-1} * w * \nabla \widetilde{\nabla}^2 \tilde{u} + g^{-1} * \nabla \tilde{g}^{-1} * h * \widetilde{\nabla} \tilde{u} * \widetilde{\nabla}^2 \tilde{u} + g^{-1} * \tilde{g}^{-1} * \nabla h * \widetilde{\nabla} \tilde{u} * \widetilde{\nabla}^2 \tilde{u} \\ & + g^{-1} * \tilde{g}^{-1} * h * \nabla \widetilde{\nabla} \tilde{u} * \widetilde{\nabla}^2 \tilde{u} + g^{-1} * \tilde{g}^{-1} * h * \widetilde{\nabla} \tilde{u} * \nabla \widetilde{\nabla}^2 \tilde{u}. \end{aligned}$$

Applying the formula

$$\nabla W = \widetilde{\nabla} W + A * W \tag{6.55}$$

for any tensor field W, $\nabla h = \tilde{g} * A$, and $\nabla \tilde{g}^{-1} = \tilde{g}^{-1} * A$, which follows from

$$\nabla_k \tilde{g}^{ij} = \tilde{g}^{ia} \tilde{g}^{jb} \nabla_k h_{ab} = \tilde{g}^{ia} \tilde{g}^{jb} \left(A^p_{ka} \tilde{g}^{pb} + A^p_{kb} \tilde{g}_{ap} \right) = \tilde{g}^{ia} A^j_{ka} + \tilde{g}^{jb} A^i_{kb},$$

we complete the proof of (6.52).

To prove the last two identities, recall from [28] that

$$\widetilde{\nabla}^2 W = \nabla^2 W + A * \widetilde{\nabla} W + B * W + A * A * W \tag{6.56}$$

for any tensor field W. In particular,

$$\begin{aligned} \widetilde{\Delta} \widetilde{\mathrm{Rm}} \ &= \ \tilde{g}^{-1} * h * \widetilde{\nabla}^2 \widetilde{\mathrm{Rm}} + \Delta \widetilde{\mathrm{Rm}} + A * \widetilde{\nabla} \widetilde{\mathrm{Rm}} + B * \widetilde{\mathrm{Rm}} + A * A * \widetilde{\mathrm{Rm}}, \\ \widetilde{\Delta} \widetilde{\nabla} \widetilde{\mathrm{Rm}} \ &= \ \tilde{g}^{-1} * h * \widetilde{\nabla}^3 \widetilde{\mathrm{Rm}} + \Delta \widetilde{\nabla} \widetilde{\mathrm{Rm}} + A * \widetilde{\nabla}^2 \widetilde{\mathrm{Rm}} + B * \widetilde{\nabla} \widetilde{\mathrm{Rm}} + A * A * \widetilde{\nabla} \widetilde{\mathrm{Rm}}. \end{aligned}$$

For (6.53), we have

$$\begin{aligned} \square T \ &= \ (\partial_t - \Delta) \left(\mathrm{Rm} - \widetilde{\mathrm{Rm}} \right) \\ = \ & g^{-1} * \mathrm{Rm} * \mathrm{Rm} + g^{-1} * \mathrm{Rm} * \nabla u * \nabla u + g^{-1} * \nabla^2 u * \nabla^2 u - \partial_t \widetilde{\mathrm{Rm}} + \Delta \widetilde{\mathrm{Rm}} \\ = \ & g^{-1} * \mathrm{Rm} * \mathrm{Rm} + g^{-1} * \mathrm{Rm} * \nabla u * \nabla u + g^{-1} * \nabla^2 u * \nabla^2 u \\ & - \left[\widetilde{\Delta} \widetilde{\mathrm{Rm}} + \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} * \widetilde{\mathrm{Rm}} + \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} * \widetilde{\nabla} \tilde{u} * \widetilde{\nabla} \tilde{u} + \tilde{g}^{-1} * \widetilde{\nabla}^2 \tilde{u} * \widetilde{\nabla}^2 \tilde{u} \right] \\ & + \left[\widetilde{\Delta} \widetilde{\mathrm{Rm}} + h * \tilde{g}^{-1} * \widetilde{\nabla}^2 \widetilde{\mathrm{Rm}} + A * \widetilde{\nabla} \widetilde{\mathrm{Rm}} + B * \widetilde{\mathrm{Rm}} + A * A * \widetilde{\mathrm{Rm}} \right] \\ = \ & h * \tilde{g}^{-1} * \widetilde{\nabla}^2 \widetilde{\mathrm{Rm}} + A * \widetilde{\nabla} \widetilde{\mathrm{Rm}} + B * \widetilde{\mathrm{Rm}} + A * A * \widetilde{\mathrm{Rm}} \\ & + g^{-1} * \left(T + \widetilde{\mathrm{Rm}} \right) * \left(T + \widetilde{\mathrm{Rm}} \right) - \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} * \widetilde{\mathrm{Rm}} \\ & + g^{-1} * \left(T + \widetilde{\mathrm{Rm}} \right) * \left(w + \widetilde{\nabla} \tilde{u} \right) * \left(w + \widetilde{\nabla} \tilde{u} \right) - \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} * \widetilde{\nabla} \tilde{u} * \widetilde{\nabla} \tilde{u} \\ & + \tilde{g}^{-1} * \left(y + \widetilde{\nabla}^2 \tilde{u} \right) * \left(y + \widetilde{\nabla}^2 \tilde{u} \right) - \tilde{g}^{-1} * \widetilde{\nabla}^2 \tilde{u} * \widetilde{\nabla}^2 \tilde{u}. \end{aligned}$$

Simplifying terms gives (6.53). Similarly,

$$\begin{aligned}\square U &= h * \tilde{g}^{-1} * \widetilde{\nabla}^3\widetilde{\mathrm{Rm}} + A * \widetilde{\nabla}^2\widetilde{\mathrm{Rm}} + B * \widetilde{\nabla}\widetilde{\mathrm{Rm}} + A * A * \widetilde{\nabla}\widetilde{\mathrm{Rm}} \\ &+ \left[g^{-1} * \left(T + \widetilde{\mathrm{Rm}}\right) * \left(U + \widetilde{\nabla}\widetilde{\mathrm{Rm}}\right) - \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} * \widetilde{\nabla}\widetilde{\mathrm{Rm}}\right] \\ &+ \left[g^{-1} * \left(U + \widetilde{\nabla}\widetilde{\mathrm{Rm}}\right) * \left(w + \widetilde{\nabla}\tilde{u}\right) * \left(w + \widetilde{\nabla}\tilde{u}\right) - \tilde{g}^{-1} * \widetilde{\nabla}\widetilde{\mathrm{Rm}} * \widetilde{\nabla}\tilde{u} * \widetilde{\nabla}\tilde{u}\right] \\ &+ \left[g^{-1} * \left(T + \widetilde{\mathrm{Rm}}\right) * \left(w + \widetilde{\nabla}\tilde{u}\right) * \left(y + \widetilde{\nabla}^2\tilde{u}\right) - \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} * \widetilde{\nabla}\tilde{u} * \widetilde{\nabla}^2\tilde{u}\right] \\ &+ \left[g^{-1} * \left(y + \widetilde{\nabla}^2\tilde{u}\right) * \left(z + \widetilde{\nabla}^3\tilde{u}\right) - \tilde{g}^{-1} * \widetilde{\nabla}^2\tilde{u} * \widetilde{\nabla}^3\tilde{u}\right].\end{aligned}$$

Simplifying terms gives (6.54). □

According to Theorem B.2, the condition $|\mathrm{Rm}|_{g(t)} + |\widetilde{\mathrm{Rm}}|_{\tilde{g}(t)} \leq K$ implies that, for all $m \geq 0$, there exist constants $C_m = C_m(\delta, K, n, T) > 0$ such that

$$|\nabla^m \mathrm{Rm}| + \left|\nabla^{m+1}u\right| + \left|\widetilde{\nabla}^m\widetilde{\mathrm{Rm}}\right|_{\tilde{g}(t)} + \left|\widetilde{\nabla}^{m+1}\tilde{u}\right|_{\tilde{g}(t)} \leq C_m \tag{6.57}$$

on $M \times [0, T]$. We also have

$$\frac{1}{\gamma} g(t) \leq \tilde{g}(t) \leq \gamma g(t)$$

on $M \times [0, T]$, for some positive constant $\gamma = \gamma(K, T)$. Hence $\nabla^m\mathrm{Rm}, \nabla^{m+1}u$, $\widetilde{\nabla}^m\widetilde{\mathrm{Rm}}, \widetilde{\nabla}^{m+1}\tilde{u}$, $m \geq 0$, and $\tilde{g}^{-1}$ are uniformly bounded with respect to $g(t)$ on $[0, T]$ so that we can replace the norm $|\cdot|_{\tilde{g}(t)}$ by $|\cdot| := |\cdot|_{g(t)}$.

Lemma 6.8. *h, A, B, T, U are uniformly bounded with respect to $g(t)$ on $[0, T]$. Moreover, v, w, x, y, z are also uniformly bounded with respect to $g(t)$ on $[0, T]$.*

Proof. The first part was proved in [28] in the exact manner. The second part follows immediately from Lemma A.1. □

Lemma 6.9. *Using the above lemma, one has*

$$\begin{aligned} |\partial_t h|^2 &\lesssim |T|^2 + |w|^2, \\ |\partial_t A|^2 &\lesssim |U|^2 + |h|^2 + |y|^2 + |w|^2, \\ |\partial_t B|^2 &\lesssim |\nabla U|^2 + |h|^2 + |A|^2 + |U|^2 + |y|^2 + |\nabla y|^2 + |x|^2 + |w|^2, \\ |\partial_t v|^2 &\lesssim |y|^2 + |h|^2, \\ |\partial_t w|^2 &\lesssim |z|^2 + |A|^2 + |h|^2 + |v|^2, \\ |\partial_t x|^2 &\lesssim |\nabla z|^2 + |B|^2 + |A|^2 + |h|^2 + |v|^2 + |w|^2, \\ |\square y|^2 &\lesssim |h|^2 + |A|^2 + |B|^2 + |T|^2 + |w|^2 + |y|^2, \\ |\square z|^2 &\lesssim |h|^2 + |A|^2 + |B|^2 + |T|^2 + |z|^2 + |U|^2 + |y|^2 + |w|^2. \end{aligned}$$

Proof. The first four inequality follows from Lemma 6.9. For the next three inequalities, we verify only the inequality for $|\partial_t v|^2$. By definition,

$$\begin{aligned}\partial_t v &= \Delta u - \widetilde{\Delta}\tilde{u} = g^{ij}\nabla_i\nabla_j u - \tilde{g}^{ij}\widetilde{\nabla}_i\widetilde{\nabla}_j\tilde{u} \\ &= g^{ij}\left(\nabla_i\nabla_j u - \widetilde{\nabla}_i\widetilde{\nabla}_j\tilde{u}\right) + (g^{ij} - \tilde{g}^{ij})\widetilde{\nabla}_i\widetilde{\nabla}_j\tilde{u} = g^{ij}y_{ij} - \tilde{g}^{ia}g^{jb}h_{ab}\widetilde{\nabla}_i\widetilde{\nabla}_j\tilde{u}\end{aligned}$$

so that $\partial_t v = g^{-1} * y + \tilde{g}^{-1} * g^{-1} * h * \widetilde{\nabla}^2 \tilde{u}$. Similarly

$$\begin{aligned}\partial_t w &= g^{-1} * \nabla y + \tilde{g}^{-1} * A * h * \widetilde{\nabla}^2 \tilde{u} + \tilde{g}^{-1} * \tilde{g} * A * \widetilde{\nabla}^2 \tilde{u} \\ &\quad + \tilde{g}^{-1} * h * \left(\widetilde{\nabla}^3 \tilde{u} + A * \widetilde{\nabla}^2 \tilde{u} \right)\end{aligned}$$

with $\nabla y = \nabla(\nabla^2 u - \widetilde{\nabla}^2 \tilde{u}) = z + A * \widetilde{\nabla}^2 \tilde{u}$. For the final two inequalities we only verify the inequality for $|\Box y|^2$. From the identity

$$\begin{aligned}\widetilde{\Delta} \widetilde{\nabla}^2 \tilde{u} &= \tilde{g}^{-1} \widetilde{\nabla}^2 \widetilde{\nabla}^2 \tilde{u} \\ &= \Delta \widetilde{\nabla}^2 \tilde{u} + h * \tilde{g}^{-1} * \widetilde{\nabla}^4 \tilde{u} + A * \widetilde{\nabla}^3 \tilde{u} + B * \widetilde{\nabla}^2 \tilde{u} + A * A * \widetilde{\nabla}^2 \tilde{u},\end{aligned}$$

and

$$\partial_t \nabla^2 u = \Delta \nabla^2 u + \mathrm{Rm} * \nabla^2 u + g^{-1} * \nabla u * \nabla u * \nabla^2 u,$$

we obtain

$$\begin{aligned}\Box y &= \left[\mathrm{Rm} * \nabla^2 u + g^{-1} * \nabla u * \nabla u * \nabla^2 u \right] - \partial_t \widetilde{\nabla}^2 \tilde{u} + \Delta \widetilde{\nabla}^2 \tilde{u} \\ &= h * \tilde{g}^{-1} * \widetilde{\nabla}^4 \tilde{u} + A * \widetilde{\nabla}^3 \tilde{u} + B * \widetilde{\nabla}^2 \tilde{u} + A * A * \widetilde{\nabla}^2 \tilde{u} \\ &\quad + \left(T + \widetilde{\mathrm{Rm}} \right) * \nabla^2 u - \widetilde{\mathrm{Rm}} * \widetilde{\nabla}^2 \tilde{u} \\ &\quad + g^{-1} * \left(w + \widetilde{\nabla} \tilde{u} \right) * \left(w + \widetilde{\nabla} \tilde{u} \right) * \left(y + \widetilde{\nabla}^2 \tilde{u} \right) - \tilde{g}^{-1} * \widetilde{\nabla} \tilde{u} * \widetilde{\nabla} \tilde{u} * \widetilde{\nabla}^2 \tilde{u} \\ &= h * \tilde{g}^{-1} * \widetilde{\nabla}^4 \tilde{u} + A * \widetilde{\nabla}^3 \tilde{u} + B * \widetilde{\nabla}^2 \tilde{u} + A * A * A * \widetilde{\nabla}^2 \tilde{u} + T * \nabla^2 u \\ &\quad + w * w * y + \widetilde{\nabla} \tilde{u} * w * y + \widetilde{\nabla} \tilde{u} * \widetilde{\nabla} \tilde{u} * y + w * w * \widetilde{\nabla}^2 \tilde{u} + w * \widetilde{\nabla} \tilde{u} * \widetilde{\nabla}^2 \tilde{u} \\ &\quad + h * \tilde{g}^{-1} * \widetilde{\nabla} \tilde{u} * \widetilde{\nabla} \tilde{u} * \widetilde{\nabla}^2 \tilde{u}.\end{aligned}$$

For $|\Box z|^2$, we need the evolution equation

$$\begin{aligned}\partial_t \nabla^3 u &= \Delta \nabla^3 u + g^{-1} * \mathrm{Rm} * \nabla^3 u + g^{-1} * \nabla \mathrm{Rm} * \nabla^2 u \\ &\quad + g^{-1} * \nabla u * \nabla^2 u * \nabla^2 u + g^{-1} * \nabla u * \nabla u * \nabla^3 u\end{aligned}$$

and the identity

$$\widetilde{\Delta} \widetilde{\nabla}^3 \tilde{u} = \Delta \widetilde{\nabla}^3 u + h * \tilde{g}^{-2} * \widetilde{\nabla}^5 u + A * \widetilde{\nabla}^4 u + B * \widetilde{\nabla}^3 u + A * A * \widetilde{\nabla}^3 u.$$

Thus we prove the results. □

The proof of Theorem 6.6. The above two lemmas, Lemma 6.8 and Lemma 6.9, imply

$$\begin{aligned}|\Box T|^2 &\lesssim |h|^2 + |A|^2 + |T|^2 + |B|^2 + |w|^2 + |y|^2, \\ |\Box U|^2 &\lesssim |h|^2 + |A|^2 + |B|^2 + |U|^2 + |T|^2 + |w|^2 + |y|^2 + |z|^2.\end{aligned}$$

Together Lemma 6.9, we have

$$\begin{aligned}|\Box \mathbf{X}|^2_{g(t)} &\lesssim |\mathbf{X}|^2_{g(t)} + |\mathbf{Y}|^2_{g(t)}, \\ |\partial_t \mathbf{Y}|^2_{g(t)} &\lesssim |\mathbf{X}|^2_{g(t)} + |\mathbf{Y}|^2_{g(t)} + |\nabla \mathbf{X}|^2_{g(t)}.\end{aligned}$$

To apply Theorem 6.5, we need to check the boundedness of $\mathbf{X}$, $\nabla \mathbf{X}$ and $\mathbf{Y}$ on $[0, T]$, which are however, followed from Lemma A.1. Therefore, $\mathbf{X} = \mathbf{Y} \equiv 0$ on $[0, T]$. Thus, $(g(t), u(t)) = (\tilde{g}(t), \tilde{u}(t)) \equiv 0$ on $[0, T]$.

APPENDIX A. EVOLUTION EQUATIONS OF THE RICCI-HARMONIC FLOW

We review some basic evolution equations of the Ricci-harmonic flow. Consider a Ricci-harmonic flow

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)} + 4du(t)\otimes du(t), \quad \partial_t u(t) = \Delta_{g(t)} u(t) \tag{A.1}$$

on a smooth manifold M. As before, we follow the convention in Section 1.

Lemma A.1. *Under the flow (A.1), we have*

$$\begin{aligned} \Box R_{ij} &= -2R_{ik}R^k{}_j + 2R_{pijq}R^{pq} - 4R_{pijq}\nabla^p u\nabla^q u \\ &\quad + 4\Delta u\,\nabla_i\nabla_j u - 4\nabla_i\nabla_k u\nabla^k\nabla_j u, \end{aligned} \tag{A.2}$$

$$\partial_t\Gamma^k_{ij} = g^{k\ell}\left(-\nabla_i R_{j\ell} - \nabla_j R_{i\ell} + \nabla_\ell R_{ij} + 4\nabla_i\nabla_j u\nabla_\ell u\right), \tag{A.3}$$

$$\Box\partial_i u = -R_{ij}\nabla^j u, \tag{A.4}$$

$$\Box\nabla_i\nabla_j u = 2R_{pijq}\nabla^p\nabla^q u - R_{ip}\nabla_j\nabla^p u - R_{jp}\nabla_i\nabla^p u - 4|\nabla u|^2\nabla_i\nabla_j u, \tag{A.5}$$

$$\begin{aligned} \Box R_{ijk\ell} &= 2\left(B_{ijk\ell} - B_{ij\ell k} - B_{i\ell jk} + B_{ikj\ell}\right) \\ &\quad - \left(R_i{}^p R_{pjk\ell} + R_j{}^p R_{ijp\ell} + R_\ell{}^p R_{ijkp}\right) \\ &\quad + 4\left(\nabla_i\nabla_\ell u\nabla_j\nabla_k u - \nabla_i\nabla_k u\nabla_j\nabla_\ell u\right), \end{aligned} \tag{A.6}$$

where $B_{ijk\ell} := -g^{pr}g^{qs}R_{ipjq}R_{kr\ell s}$, *and*

$$\Box R = 2|\mathrm{Ric}|^2 + 4|\Delta u|^2 - 4|\nabla^2 u|^2 - 8\mathrm{Ric}(\nabla u,\nabla u), \tag{A.7}$$

$$\Box|\nabla u|^2 = -2|\nabla^2 u|^2 - 4|\nabla u|^4, \tag{A.8}$$

$$\Box\left(R - 2|\nabla u|^2\right) = 2\left|\mathrm{Ric} - 2\nabla u\otimes\nabla u\right|^2 + 4|\Delta u|^2, \tag{A.9}$$

$$\partial_t dV = -\left(R - 2|\nabla u|^2\right)dV. \tag{A.10}$$

Proof. See for example [42, 46, 47, 49, 51]. Note that in our notation for R^ℓ_{ijk} defined by

$$R^\ell_{ijk} = \partial_i\Gamma^\ell_{jk} - \partial_j\Gamma^\ell_{ik} + \Gamma^p_{jk}\Gamma^\ell_{ip} - \Gamma^p_{ik}\Gamma^\ell_{jp},$$

the tensors R^ℓ_{ijk} is the minus of those used in [46]. $\Box$

Lemma A.2. *As* $(1,3)$*-tensor, we have*

$$\partial_t\mathrm{Rm} = g^{-1}\nabla^2\mathrm{Ric} + g^{-1}\mathrm{Ric}*\mathrm{Rm} + g^{-1}\nabla^2 u*\nabla^2 u + g^{-1}\mathrm{Rm}*\nabla u*\nabla u, \tag{A.11}$$

and

$$\begin{aligned}\partial_t R^\ell_{ijk} &= \Big[\Delta R^\ell_{ijk} + g^{pq}\left(R^r_{ijp}R^\ell_{rqk} - 2R^r_{pik}R^\ell_{jqr} + 2R^\ell_{pir}R^r_{jqk}\right) - g^{pq}\Big(R_{ip}R^\ell_{qjk} \\ &+ R_{jp}R^\ell_{iqk} + R_{kp}R^\ell_{ijq}\Big) + g^{p\ell}R_{pq}R^q_{ijk}\Big] - 4g^{p\ell}R^q_{ijk}\nabla_q u\nabla_p u \\ &+ 4g^{p\ell}\left(\nabla_i\nabla_p u\nabla_k\nabla_j u - \nabla_i\nabla_k u\nabla_j\nabla_p u\right),\end{aligned} \tag{A.12}$$

$$\begin{aligned}\partial_t\nabla_a R^\ell_{ijk} &= \Big[\Delta\nabla_a R^\ell_{ijk} + g^{pq}\nabla_a\left(R^r_{ijp}R^\ell_{rqk} - 2R^r_{pik}R^\ell_{jqr} + 2R^\ell_{pir}R^r_{jqk}\right) \\ &- g^{pq}\left(R_{ip}\nabla_a R^\ell_{qjk} - R_{jp}\nabla_a R^\ell_{iqk} - R_{kp}\nabla_a R^\ell_{ijq}\right) + g^{p\ell}R_{pq}\nabla_a R^q_{ijk}\Big] \\ &- 4g^{p\ell}\nabla_a R^q_{ijk}\nabla_q u\nabla_p u - 4g^{p\ell}R^p_{ijk}\nabla_a\nabla_q u\nabla_p u - 4g^{p\ell}R^q_{ijk}\nabla_q u\nabla_a\nabla_p u \\ &+ 4g^{p\ell}\nabla_a\nabla_i\nabla_p u\nabla_k\nabla_j u + 4g^{p\ell}\nabla_i\nabla_p u\nabla_a\nabla_k\nabla_j u \\ &- 4g^{p\ell}\nabla_a\nabla_i\nabla_k u\nabla_j\nabla_p u - 4g^{p\ell}\nabla_i\nabla_k u\nabla_a\nabla_j\nabla_p u \\ &+ 4g^{\ell k}R^b_{ijk}\nabla_a\nabla_b u\nabla_k u - 4g^{nk}\nabla_a\nabla_i u\nabla_k u R^\ell_{bjk} \\ &- 4g^{bk}\nabla_a\nabla_i u\nabla_k u R^\ell_{ibk} - 4g^{b\ell}\nabla_a\nabla_k u\nabla_\ell u R^\ell_{ijb}.\end{aligned} \tag{A.13}$$

We also need the existence result for the Ricci-harmonic flow.

Theorem A.3. (List, 2005) *Let (M, g_0) be a smooth complete n-dimensional Riemannian manifold with bounded curvature $|\mathrm{Rm}_{g_0}|_{g_0} \le K_0$. Consider a smooth function u_0 on M satisfying*

$$|u_0|^2_{g_0} + |\nabla_{g_0}u_0|^2_{g_0} \le C_0, \quad |\nabla^2_{g_0}u_0|^2_{g_0} \le C_1. \tag{A.14}$$

Here K_0, C_0, C_1 are some positive constants. Then there exists a positive constant $T := T(n, K_-, C_0)$ such that the initial value problem (A.1) with $(g(0), u(0)) = (g_0, u_0)$ has a smooth solution $(g(t), u(t))$ on $M \times [0, T]$. Moreover, the solution satisfies

$$\frac{1}{C}g_0 \le g(t) \le Cg_0, \quad t \in [0, T], \tag{A.15}$$

for some constant $C := C(n, K_0, C_0, C_1)$, and on $M \times [0, T]$ there is a bound

$$|\mathrm{Rm}_{g(t)}|^2_{g(t)} + |u(t)|^2 + |du(t)|^2_{g(t)} + |\nabla^2_{g(t)}u(t)|^2_{g(t)} \le C' \tag{A.16}$$

for another constant $C' := C'(n, K_0, C_0, C_1)$.

Proof. See [42, 46, 47, 49, 51]. □

Appendix B. Some estimates of the Ricci-harmonic flow

In this section we assume that $(g(t), u(t))$ is a solution to (A.1) on $[0, T]$ where M is a complete n-dimensional smooth manifold. Consider a geodesic ball $B_{g(T)}(x_0, r)$ centered at a fixed point $x_0 \in M$ with radius $R > 0$.

Theorem B.1. (Interior estimates) *Under the above hypotheses, we have*

(i) *If the following estimate*

$$\sup_{B_{g(T)}(x_0,R)} |\mathrm{Ric}_{g(t)}|_{g(t)} \le \frac{C}{R^2} \tag{B.1}$$

holds for some positive constant C, then, for all $t \in (0, T]$, there exists a constant C_n, depending only on n, such that

$$\sup_{B_{g(t)}(x_0, R/2)} |\nabla_{g(t)} u(t)|^2_{g(t)} \leq C C_n \left(\frac{1}{R^2} + \frac{1}{t} \right). \tag{B.2}$$

(ii) *If the following estimate*

$$\sup_{B_{g(T)}(x_0, R)} |\mathrm{Rm}_{g(t)}|_{g(t)} \leq \frac{C^2}{R^4} \tag{B.3}$$

holds for some positive constant C, then, for all $t \in (0, T]$ and all $m \geq 0$, there exists a constant $C_{n,m}$, depending only on n and m, such that

$$\sup_{B_{g(t)}(x_0, R/2)} \left[\left| \nabla^m_{g(t)} \mathrm{Rm}_{g(t)} \right|^2_{g(t)} + \left| \nabla^{m+2}_{g(t)} u(t) \right|^2_{g(t)} \right] \leq C^{m+2} C_{n,m} \left(\frac{1}{R^2} + \frac{1}{t} \right)^{m+2}. \tag{B.4}$$

(iii) *If in addition $(g(t), u(t))$ is constructed in Theorem A.3, then*

$$\inf_M u_0 \leq u(t) \leq \sup_M u_0 \tag{B.5}$$

for all $t \in (0, T]$ as long as the constructed solution exists, where (g_0, u_0) is the initial data.

Proof. See [46, 47]. □

Theorem B.2. *Suppose that $(g(t), u(t))$ is a solution to (A.1) on $M \times [0, T]$, where M is a complete n-dimensional smooth manifold and $T \in (0, \infty)$.*

(i) *If the following estimate*

$$\sup_{M \times [0,T]} |\mathrm{Ric}_{g(t)}|_{g(t)} \leq K \tag{B.6}$$

holds for some positive constant K, then

$$|\nabla_{g(t)} u(t)|^2_{g(t)} \leq 2 K C_n \tag{B.7}$$

on $M \times [0, T]$, for some positive constant C_n depending only on n.

(ii) *If the following estimate*

$$\sup_{M \times [0,T]} |\mathrm{Rm}_{g(t)}|_{g(t)} \leq K \tag{B.8}$$

holds for some positive constant K, then

$$\left| \nabla^m_{g(t)} \mathrm{Rm}_{g(t)} \right|^2_{g(t)} + \left| \nabla^{m+2}_{g(t)} u(t) \right|^2_{g(t)} \leq C_{n,m} (4K)^{1 + \frac{m}{2}}, \quad m \geq 0, \tag{B.9}$$

on $M \times [0, T]$, for some positive constant $C_{n,m}$ depending only on n and m.

(iii) *If $(g(t), u(t))$ is constructed in Theorem A.3 with the initial data (g_0, u_0) satisfying the condition (A.14), then*

$$\left| \nabla^m_{g(t)} \mathrm{Rm}_{g(t)} \right|^2_{g(t)} + \left| \nabla^{m+2}_{g(t)} u(t) \right|^2_{g(t)} \leq C'_{n,m}, \quad m \geq 0, \tag{B.10}$$

on $M \times [0, T]$, for some positive constant $C'_{n,m}$ depending only on n, m, K_0, C_0, C_1.

Proof. Given a space-time point $(x_0, t) \in M \times (0, T]$ and consider the geodesic ball $B_{g(t)}(x_0, \sqrt{t}/2)$. Since

$$\sup_{B_{g(T)}(x_0,\sqrt{t})} (\sqrt{t})^2 |\mathrm{Ric}_{g(t)}|_{g(t)} \le (\sqrt{t})^2 K$$

by (B.1) and (B.6), it follows that

$$\sup_{B_{g(t)}(x_0,\sqrt{t}/2)} |\nabla_{g(t)} u(t)|^2_{g(t)} \le (\sqrt{t})^2 K C_n \left(\frac{1}{(\sqrt{t})^2} + \frac{1}{t} \right) = 2KC_n.$$

In particular, $|\nabla_{g(t)} u(t)|^2_{g(t)}(x_0) \le 2KC_n$.

For (ii), consider the same geodesic ball $B_{g(t)}(x_0, \sqrt{t}/2)$ and apply (B.4). The part follows from Theorem A.3 and the second one. $\square$

Appendix C. Evolution equations of the Ricci-harmonic flow

Consider the $(\alpha_1, 0, \beta_1, \beta_2)$-Ricci flow:

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)} + 2\alpha_1 \nabla_{g(t)} u(t) \otimes \nabla_{g(t)} u(t), \tag{C.1}$$

$$\partial_t u(t) = \Delta_{g(t)} u(t) + \beta_1 |\nabla_{g(t)} u(t)|^2_{g(t)} + \beta_2 u(t) \tag{C.2}$$

on a smooth manifold M, where $\alpha_1, \beta_1, \beta_2$ are given constants.

Lemma C.1. *Under the flow (C.1) – (C.2), we have*

$$\begin{aligned} \Box R_{ij} &= -2R_{ik}R^k{}_j + 2R_{pijq}R^{pq} - 2\alpha_1 R_{pijq}\nabla^p u \nabla^q u \\ &\quad + 2\alpha_1 \Delta u \nabla_i \nabla_j u - 2\alpha_1 \nabla_i \nabla_k u \nabla^k \nabla_j u, \end{aligned} \tag{C.3}$$

$$\Box R = 2|\mathrm{Ric}|^2 + 2\alpha_1 |\Delta u|^2 - 2\alpha_1 |\nabla^2 u|^2 - 4\alpha_1 \langle \mathrm{Ric}, \nabla u \otimes \nabla u \rangle, \tag{C.4}$$

$$\begin{aligned} \Box R_{ijk\ell} &= 2(B_{ijk\ell} - B_{ij\ell k} + B_{ikj\ell} - B_{i\ell jk}) - (R_i{}^p R_{pjk\ell} + R_j{}^p R_{ipk\ell} \\ &\quad + R_k{}^p R_{ijp\ell} + R_\ell{}^p R_{ijkp}) + 2\alpha_1(\nabla_i \nabla_\ell u \nabla_j \nabla_k u - \nabla_i \nabla_k u \nabla_j \nabla_\ell u), \end{aligned} \tag{C.5}$$

$$\Box |\nabla u|^2 = 2\beta_2 |\nabla u|^2 - 2|\nabla^2 u|^2 - 2\alpha_1 |\nabla u|^4 + 4\beta_1 \langle \nabla u \otimes \nabla u, \nabla^2 u \rangle, \tag{C.6}$$

$$\begin{aligned} \Box \nabla_i \nabla_j u &= 2R_{pijq}\nabla^p \nabla^q u + \beta_2 \nabla_i \nabla_j u - R_{ip}\nabla^p \nabla_j u - R_{jp}\nabla^p \nabla_i u \\ &\quad - 2\alpha_1 |\nabla u|^2 \nabla_i \nabla_j u + 2\beta_1 \nabla^k u \nabla_k \nabla_i \nabla_j u \\ &\quad + 2\beta_1 \nabla_i \nabla^k u \nabla_j \nabla_k u + 2\beta_1 R_{pijq} \nabla^o u \nabla^q u, \end{aligned} \tag{C.7}$$

$$\begin{aligned} \Box(\nabla_i u \nabla_j u) &= -\nabla^k u (R_{ik}\nabla_j u + R_{jk}\nabla_i u) - 2\nabla_i \nabla^k u \nabla_j \nabla_k u + 2\beta_2 \nabla_i u \nabla_j u \\ &\quad + 2\beta_1 \nabla^k u (\nabla_i u \nabla_j \nabla_k u + \nabla_j u \nabla_i \nabla_k u), \end{aligned} \tag{C.8}$$

where $B_{ijk\ell} := -g^{pr} g^{qs} R_{ipjq} R_{kr\ell s}$.

Proof. See [42]. $\square$

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FACULTY OF SCIENCE, TECHNOLOGY AND COMMUNICATION (FSTC), MATHEMATIC RESEARCH UNIT, CAMPUS BELVAL, UNIVERSITE DU LUXEMBOURG, MAISON DU NOMBRE, 6, AVENUE DE LA FONTE, L-4364, ESCH-SUR-ALZETTE, GRAND-DUCHY OF LUXEMBOURG

E-mail address: `yilicms@gmail.com`