NEWFORMS MOD p IN SQUAREFREE LEVEL
WITH APPLICATIONS TO MONSKY'S HECKE-STABLE FILTRATION

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WITH AN APPENDIX BY ALEXANDRU GHITZA

ABSTRACT. We propose an algebraic definition of the space of ℓ-new mod-p modular forms for $\Gamma_0(N\ell)$ in the case that ℓ is prime to N, which naturally generalizes to a notion of newforms modulo p in squarefree level. We use this notion of newforms to interpret the Hecke algebras on the graded pieces of the space of mod-2 level-3 modular forms described by Paul Monsky. Along the way, we describe a renormalized version of the Atkin-Lehner involution: no longer an involution, it is an automorphism of the algebra of modular forms, even in characteristic p.

CONTENTS

1. OVERVIEW

This note is inspired by an explicit filtration on the space of modular forms modulo 2 of levels 3 and 5 described by Paul Monsky in [16, 17], and our search for a conceptual description thereof. The goals of the present text are three-fold:

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(1) Develop an algebraic theory of spaces of ℓ-new modular forms modulo p, consistent with the classical characteristic-zero definitions.
(2) Introduce a modified Atkin-Lehner "involution" that descends to an finite-order algebra automorphism of the space of modular forms modulo p. The appendix, written by Alex Ghitza, justifies this modification geometrically by viewing modular forms modulo p as regular functions on the Igusa curve with poles only at supersingular points, and interpreting the Atkin-Lehner operator moduli-theoretically.
(3) Construct a three-term Hecke-invariant filtration of the space of modular forms modulo p. On an old local component satisfying the level-raising condition at ℓ, the Hecke algebras on the graded pieces of the filtration may be identified with two copies of the ℓ-old Hecke algebra and one copy of the ℓ-new Hecke algebra. We compare this filtration and its Hecke algebras to those found by Monsky in the case $\ell \equiv -1 \bmod p$.

We now discuss each goal in detail. Throughout this section N is an integer level, and ℓ is a prime dividing N exactly once. The ring B is a commutative $\mathbb{Z}[\frac{1}{\ell}]$-algebra.

1.1. **Spaces of ℓ-new forms in characteristic p.** The theory of newforms in characteristic zero, developed by Atkin and Lehner [1], traditionally casts new eigenforms as eigenforms that are *not* old (i.e., do not come from lower level) and the space of newforms as a *complement* (under the Petersson inner product) to the space of old forms. Alternatively, one can define what it means to be a new eigenform — again, not old — and then the newforms are those expressible as linear combinations of new eigenforms. Viewed from both perspectives, newforms are classically identified by what they are not rather than what they are: in a sense, a quotient space rather than a subspace.

This "anti"-property of newforms creates problems as soon as we move into characteristic p. On one hand, there is no Petersson inner product, so no obvious way to find a complement of the old forms. On the other hand, in fixed level, there are infinitely many forms modulo p, but only finitely many eigenforms, so we cannot rely on eigenforms alone to characterize the newforms.

We propose two different algebraic notions of newness, both based on properties of presence rather than absence. The first is based on the Atkin-Lehner result that an eigenform of level N and weight k that is new at a prime ℓ exactly dividing the level has its U_ℓ-eigenvalue equal to $\pm\ell^{\frac{k-2}{2}}$ [1, Theorem 3]. The second is inspired by an observation of Serre from [23, §3.1(d)]: in the same setup, the ℓ-new forms of level N are exactly those forms f that satisfy both $\operatorname{Tr}_\ell f = 0$ and $\operatorname{Tr}_\ell w_\ell f = 0$. Here Tr_ℓ is the trace map from forms of level N to forms of level N/ℓ (see section 4), and w_ℓ is the Atkin-Lehner involution at ℓ (see section 3).

More precisely, we define two submodules of $S_k(N, B)$, the module of cuspforms of weight k and level N over B: let $S_k(N, B)^{U_\ell\text{-new}}$ be the kernel of the Hecke operator $U_\ell^2 - \ell^{k-2}$, and let $S_k(N, B)^{\operatorname{Tr}_\ell\text{-new}}$ be the intersection of the kernels of Tr_ℓ and $\operatorname{Tr}_\ell w_\ell$. Our first result is that these submodules coincide, and agree with the usual notion of ℓ-newforms for characteristic-zero B:

Theorem A (see Theorem 1, Proposition 6.1, Proposition 6.3)**.** *For any $\mathbb{Z}[\frac{1}{\ell}]$-domain B, we have $S_k(N, B)^{U_\ell\text{-new}} = S_k(N, B)^{\operatorname{Tr}_\ell\text{-new}}$. If $B \subset \mathbb{C}$, then they both coincide with $S_k(N, B)^{\ell\text{-new}}$.*

We give similar results for $S(N, B)$, the space of cuspforms of level N and all weights over B, viewed as q-expansions (see subsection 2.1 for definitions), if B is a domain. Theorem A allows us to define a robust notion of the module of ℓ-new forms in characteristic p, and hence a notion of a module of newforms in characteristic p for squarefree levels. In characteristic p the spaces of ℓ-new and ℓ-old forms need not be disjoint; the description of their intersection in section 7 matches the level-raising results of Ribet and Diamond [22, 5], supporting our definitions.

1.2. Atkin-Lehner operators as algebra automorphisms on forms mod p.

It is well known that Atkin-Lehner operator w_ℓ (see section 3) is an involution on $M_k(N, \mathbb{Z}[\frac{1}{\ell}])$, the space of modular forms of level N and weight k over $B = \mathbb{Z}[\frac{1}{\ell}]$, and descends to an involution on $M_k(N, \mathbb{F}_p)$ as well. Less popular is the (easy) fact that w_ℓ is an *algebra* involution of $M(N, \mathbb{Z}[\frac{1}{\ell}])$, the algebra of modular forms of level N and all weights at once (here viewed as q-expansions; see subsection 2.1 for definitions). Moreover, because of congruences between forms whose weights differ by an odd multiple of $p-1$, the Atkin-Lehner operator w_ℓ is not in general well-defined on $M(N, \mathbb{F}_p)$, essentially because of the factor of $\ell^{\frac{k}{2}}$ that appears in its definition. In section 3 we discuss this difficulty in detail, and propose a renormalization W_ℓ of w_ℓ that does descend to an algebra automorphism of $M(N, \mathbb{F}_p)$, with the property that W_ℓ^2 acts on forms of weight k by multiplication by ℓ^k.

In Appendix A, Alex Ghitza gives a geometric interpretation of the operator W_ℓ on $M(N, \mathbb{F}_p)$, constructing it from an automorphism of the Igusa curve covering the modular curve $X_0(N\ell)_{\mathbb{F}_p}$.

1.3. Hecke-stable filtrations of generalized eigenspaces modulo p.

In the last part of the paper, we focus on using the space of ℓ-new mod-p cuspforms to get information about the structure of the mod-p Hecke algebra of level N. We define a Hecke-stable filtration of $K(N, \mathbb{F}_p)$, the subspace of $S(N, \mathbb{F}_p)$ annihilated by the U_p operator (see (8.2)):

$$0 \subset K(N, \mathbb{F})_t^{\ell\text{-new}} \subset (\ker \mathrm{Tr}_\ell)_t \subset K(N, \mathbb{F})_t. \tag{1.1}$$

Here the t indicates that we've restricted to a generalized Hecke eigencomponent for the eigensystem carried by a pseudorepresentation t landing in a finite extension $\mathbb{F}$ of $\mathbb{F}_p$ (see subsection 7.1 for definitions). If t is ℓ-old but satisfies the level-raising condition, then under certain regularity conditions on the Hecke algebra at level N/ℓ, we show that the Hecke algebra on the graded pieces of this filtration are exactly $A(N, \mathbb{F})_t^{\ell\text{-new}}$, $A(N/\ell, \mathbb{F})_t$, $A(N/\ell, \mathbb{F})_t$, the shallow Hecke algebras acting faithfully on $K(N, \mathbb{F})_t^{\ell\text{-new}}$, $K(N/\ell, \mathbb{F})_t$, and $K(N/\ell, \mathbb{F})_t$, respectively. See Proposition 8.1.

Finally, we compare this filtration to the filtration given in the case $\ell \equiv -1 \bmod p$ by Paul Monsky in [16, 17] (see (8.4)):

$$0 \subset K(N/\ell, \mathbb{F})_t \subset (\ker \mathrm{Tr}_\ell)_t \subset K(N, \mathbb{F})_t. \tag{1.2}$$

Here again t marks an ℓ-old component satisfying the level-raising condition. It is not difficult to see that the Hecke algebras on the first and third graded pieces are both $A(N/\ell, \mathbb{F})_t$. Under similar regularity conditions on $A(N/\ell, \mathbb{F})_t$, we show that the Hecke algebra on the middle graded piece is once again $A(N, \mathbb{F})_t^{\ell\text{-new}}$. See Proposition 8.4.

Wayfinding: In section 2 we set the notation for the various spaces of modular forms that we consider. In section 3, we discuss problems with the Atkin-Lehner operator in characteristic p (when considering all weights at once) and introduce a modified version. In section 4 we discuss the trace-at-ℓ operator. In section 5 we discuss ℓ-old forms. In section 6 we discuss and propose a space of ℓ-new forms over rings that are not subrings of $\mathbb{C}$. Intersections between spaces of ℓ-old and ℓ-new spaces, especially restricted to local components of the Hecke algebra (defined in subsection 7.1) are discussed in section 7. Finally in section 8, we discuss two Hecke-stable filtrations and compare the Hecke algebras on the corresponding graded pieces.

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2. Notation and setup

2.1. **The space of modular forms with coefficients in B.** Fix $N \geq 1$. Let $M_k(N, \mathbb{Z}) \subset \mathbb{Z}[\![q]\!]$ be the space of q-expansions of modular forms of level $\Gamma_0(N)$ and weight k whose Fourier coefficients at infinity are integral. We define $M_k(N, B)$ for any commutative ring B as $M_k(N, \mathbb{Z}) \otimes_{\mathbb{Z}} B$. By the q-expansion principle [6, 12.3.4], the map $M_k(N, B) \to B[\![q]\!]$ is injective, so that we may view $M_k(N, B)$ as a submodule of $B[\![q]\!]$. Similarly, we let $S_k(N, \mathbb{Z}) \subset M_k(N, \mathbb{Z})$ be the q-expansions at infinity of cuspidal modular forms of level $\Gamma_0(N)$ and weight k, and let $S_k(N, B) := S_k(N, \mathbb{Z}) \otimes_{\mathbb{Z}} B$, which we again view as a submodule of $B[\![q]\!]$. Note that $S_k(N, B) \subset M_k(N, B)$.

Let $M(N, B) := \sum_{k=0}^{\infty} M_k(N, B) \subset B[\![q]\!]$, the algebra of all modular forms of level N over B. If B is a domain of characteristic zero, then this sum is direct and $M(N, B) = \bigoplus_{k=0}^{\infty} M_k(N, B)$ (for $B \subset \mathbb{C}$, this is [15, Lemma 2.1.1]; otherwise use the fact that B is flat over $\mathbb{Z}$). On the other hand, if B is a domain of characteristic p, then this sum is never direct: indeed, if $p \geq 5$, then a suitable multiple of the Eisenstein form $E_{p-1} \in M_{p-1}(1, B)$ has q-expansion 1, and therefore $M_k(N, B) \subset M_{k+p-1}(N, B)$. This is essentially the only wrinkle: for $i \in 2\mathbb{Z}/(p-1)\mathbb{Z}$, set

$$M(N, B)^i := \bigcup_{k \equiv i \bmod p-1} M_k(N, B);$$

then by [8, Theorem 5.4]

$$M(N, B) = \bigoplus_{i \in 2\mathbb{Z}/(p-1)\mathbb{Z}} M(N, B)^i, \tag{2.1}$$

making $M(N, B)$ into a $2\mathbb{Z}/(p-1)\mathbb{Z}$-graded algebra. If $p = 2, 3$ (and still B has characteristic p) then multiples of both E_4 and E_6 have q-expansion 1; certainly $M_k(N, B) \subset M_{k+12}(N, B)$.

Similarly, let $S(N, B) := \sum_{k=0}^{\infty} S_k(N, B) \subset B[\![q]\!]$, the space of all cuspidal forms of level N. This is a graded ideal of the graded algebra $M(N, B)$; let $S(N, B)^i$ be the i^{th} graded part, where $i \in \mathbb{Z}_{\geq 0}$ if B has characteristic zero, $i \in 2\mathbb{Z}/(p-1)\mathbb{Z}$ if $p \geq 3$ and $i = 0$ if $p = 2$.

For any $f \in B[\![q]\!]$ and $n \geq 0$, write $a_n(f)$ for the coefficient of q^n: that is, $f = \sum_{n \geq 0} a_n(f) q^n$. If $f \in M_k(N, B)$ or $M(N, B)$, then $a_n(f)$ is the n^{th} Fourier coefficient of f. For $m \geq 0$, write U_m for the formal B-linear operator $B[\![q]\!] \to B[\![q]\!]$ given by $a_n(U_m f) = a_{mn}(f)$.

2.2. **Hecke operators on $M_k(N, B)$ and $M(N, B)$.** The spaces $M_k(N, B)$ carry actions of the Hecke operators T_m, for all positive m if $k \geq 2$ and for m invertible in B if $k = 0$. These Hecke operators satisfy $T_1 = 1$ and $T_m T_{m'} = T_{mm'}$ if $(m, m') = 1$, so that it suffices to define them for prime power m only. If $f \in M_k(N, B)$ and r is a prime not dividing N (and again either $k \geq 2$ or $\frac{1}{r} \in B$), then the action of T_{r^s} is determined by the definition of T_r on q-expansions

$$a_n(T_r f) = a_{rn}(f) + r^{k-1} a_{n/r}(f), \tag{2.2}$$

where we interpret $a_{n/r}(f)$ to be zero if $r \nmid n$, and the recurrence

$$T_{r^s} = T_r T_{r^{s-1}} - r^{k-1} T_{r^{s-2}} \tag{2.3}$$

for all $s \geq 0$. On the other hand, if m divides N, then the action of T_m on $f \in M_k(N, B)$ is given by $a_n(T_m f) = a_{mn}(f)$, so that T_m coincides with the formal U_m operator defined earlier. Finally, if the characteristic of B is $c > 0$, and m divides c, then the action of T_m on $M_k(N, \mathbb{Z})$ coincides with the action of U_m so long as $k \geq 2$. We always work with and write U_m instead of T_m for m dividing N or the (positive) characteristic of B.

All of these classical Hecke operators commute with each other. Moreover, if B is a domain, then all of them extend to the algebra of modular forms $M(N, B)$. Indeed, this is immediate if B has characteristic zero (as $M(N, B)$ is the direct sum of the $M_k(N, B)$). If B has characteristic p and r is a prime not dividing Np, then T_r is well-defined on $M(N, B)$ from the q-expansion formula (2.2) because $M(N, B)$ is a direct sum of weight-modulo-$(p-1)$ spaces (2.1) and r^{k-1} is well-defined in characteristic p for k modulo $p - 1$. The action of T_m on $M(N, B)$ for prime power m relatively prime to Np follows from the recurrence (2.3). The action of U_m for m dividing Np is independent of the weight and hence always well defined.

We can streamline these arguments by introducing a weight-separating operator. If B is a domain and m is invertible in B, we define the operator $\mathcal{S}_m : M_k(N, B) \to M_k(N, B)$ by $\mathcal{S}_m f := m^k f$.[(i)] Note that $\mathcal{S}_m$ extends to an *algebra automorphism* of $M(N, B)$. If every m prime to N and the (positive) characteristic of B is invertible in B (for example, if B is a $\mathbb{Q}$-algebra or a finite extension of $\mathbb{F}_p$), then the action of all the T_m is generated by the action of the T_r and $\mathcal{S}_r$ for *primes* r not dividing N or the (positive) characteristic of B.

3. The Atkin-Lehner involution at ℓ

We now fix an additional prime ℓ not dividing N. From now on, we assume that B is a $\mathbb{Z}[\frac{1}{\ell}]$-domain. Our eventual goal is to meaningfully compare the Hecke action on the algebras $M(N\ell, B)$ and $M(N, B)$. In this section, we discuss how to extend the Atkin-Lehner involution on $M_k(N\ell, B)$ to an *algebra involution* on $M(N\ell, B)$.

3.1. The Atkin-Lehner involution at ℓ in weight k.

For $k \in 2\mathbb{Z}_{\geq 0}$, we recall the definition and properties of the Atkin-Lehner involution on $M_k(N\ell, B)$ as in [1].

Let $\mathcal{H}$ be the complex upper half plane. We extend the weight-k right action of $\mathrm{SL}_2(\mathbb{Z})$ on functions $f : \mathcal{H} \to \mathbb{C}$ given by $f|_k \gamma = j(\gamma, z)^{-k} f(\gamma z)$ to $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+$ via

$$(f|_k \gamma)(z) = (\det \gamma)^{\frac{k}{2}} j(\gamma, z)^{-k} f(\gamma z). \tag{3.1}$$

Here, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})^+$, we write γz for $\frac{az+b}{cz+d}$ (this is the usual conformal action of $\mathrm{GL}_2(\mathbb{Q})^+$ on $\mathcal{H}^+ = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ leaving $\mathbb{P}^1(\mathbb{Q})$ invariant); and $j(\gamma, z) := cz + d$ is the usual automorphy factor. The normalization of $(\det \gamma)^{\frac{k}{2}}$ is chosen so that the scalars $\mathrm{GL}_2(\mathbb{Q})^+$ act trivially.

(i) Caution: For m prime to N and the (positive) characteristic of B, many authors have historically worked with the weight-separating operator $S_m := m^{k-2}\langle m \rangle$ on $M_k(\Gamma_1(N), B)$, where $\langle \cdot \rangle$ is the diamond operator. We use a different normalization here so that $\mathcal{S}_m$ extends to an algebra automorphism on $M(N, B)$. We will eventually work with $\mathcal{S}_\ell$ for ℓ is a prime exactly dividing the level.

Let $\gamma_\ell \in \mathrm{GL}_2(\mathbb{Q})^+$ be any matrix of the form $\left(\begin{smallmatrix} \ell & a \\ N\ell & \ell b \end{smallmatrix}\right)$, where a and b are integers such that $\ell b - aN = 1$, which can be found as we've assumed that $\ell \nmid N$. Let w_ℓ be the operator on functions $f : \mathcal{H} \to \mathbb{C}$ sending f to $f|_k \gamma_\ell$. One can check that

(1) the matrix γ_ℓ normalizes $\Gamma_0(N\ell)$, so that w_ℓ maps $M_k(N\ell, \mathbb{C})$ to $M_k(N\ell, \mathbb{C})$;
(2) any two choices of γ_ℓ differ by an element of $\Gamma_0(N\ell)$, so that the action of w_ℓ on $M_k(N\ell, \mathbb{C})$ is defined without ambiguity;
(3) the matrix $\gamma_\ell^2 \left(\begin{smallmatrix} \ell & 0 \\ 0 & \ell \end{smallmatrix}\right)^{-1}$ is in $\Gamma_0(N\ell)$, and therefore w_ℓ is an involution, called the *Atkin-Lehner involution* on $M_k(N\ell, \mathbb{C})$;
(4) for $N = 1$, the involution w_ℓ coincides with the *Fricke involution* $f \mapsto f|_k \left(\begin{smallmatrix} 0 & -1 \\ \ell & 0 \end{smallmatrix}\right)$;
(5) if $f \in M_k(N, \mathbb{C}) \subset M_k(N\ell, \mathbb{C})$, then $w_\ell f = \ell^{\frac{k}{2}} f(q^\ell)$;
(6) w_ℓ is $\mathbb{Z}[\frac{1}{\ell}]$-integral, and is therefore defined as an involution on any $M_k(N\ell, B)$ so long as B is a $\mathbb{Z}[\frac{1}{\ell}]$-algebra. (This statement relies on the geometric perspective of Atkin-Lehner induced on forms by a geometric involution of the modular curve $X_0(N\ell)$. See [23, §3.1(d)] for $N = 1$ and, for example, [21, Theorem A.1] for the general case.)

3.2. **Atkin-Lehner as an algebra involution in characteristic zero.** If B has characteristic zero, then it is clear from the definitions above and the direct sum property of $M(N\ell, B)$ that w_ℓ extends to an *algebra* involution on $M(N\ell, B)$. However, if B has characteristic p and ℓ is not a square modulo p, then we incur a sign ambiguity, essentially because of the factor of $\ell^{\frac{k}{2}}$ coming from the determinant term in (3.1).

In the next section, we discuss the extent to which the Atkin-Lehner involutions on $M_k(N\ell, B)$ patch together to an algebra involution on $M(N\ell, B)$ when B has characteristic p.

3.3. **Atkin-Lehner as an algebra involution in characteristic p: difficulties.** In this section we work with $B = \mathbb{F}_p$ and finite extensions. We also assume the theory of oldforms and newforms in characteristic zero [1], which will be reviewed in section 5 and section 6 below. From item (6) above, we know that if f and f' are characteristic-zero modular forms of the same weight and level $N\ell$ that are congruent modulo p, then $w_\ell f$ and $w_\ell f'$ are congruent modulo p as well. Indeed, this is what it means for w_ℓ to descend to an involution on $M_k(N\ell, B)$. However, if f and f' appear in weights that differ by an *odd* multiple of $p-1$, then $w_\ell f$ will be congruent to $w_\ell f'$ up to a factor of $\left(\frac{\ell}{p}\right)$ only.

3.3.1. *Some bad examples.* There are examples in both newforms and oldforms.

(1) **Newform example:** Let p be an odd prime. If $f \in M_k(N\ell, \mathbb{Z}_p)$ is a new eigenform, then f is an eigenform for w_ℓ as well, so that $w_\ell f = \varepsilon(f) f$ for some $\varepsilon = \pm 1$. Moreover, $a_\ell(f) = -\varepsilon(f) \ell^{\frac{k-2}{2}}$ [1]. Suppose now that $f' \in M_{k'}(N\ell, \mathbb{Z}_p)$ is another new eigenform congruent to f, so that, in particular $a_\ell(f) \equiv a_\ell(g)$ modulo p. Now, $\varepsilon(f) = -a_\ell(f) \ell^{\frac{2-k}{2}}$ and $\varepsilon(f') = -a_\ell(f') \ell^{\frac{2-k'}{2}}$. So $\varepsilon(f')$ will not be congruent to $\varepsilon(f)$ modulo p unless $\ell^{\frac{k-k'}{2}} \equiv 1 \pmod{p}$. In particular, if p is odd and $k - k'$ is an odd multiple of $p - 1$, then $\varepsilon(f) \equiv \varepsilon(f') \bmod p$ if and only if ℓ is a square modulo p.

For example, write $S_k(\ell, \mathbb{Q})^{\mathrm{new}, \pm}$ for the new subspace on which w_ℓ acts by ± 1. For $\ell = 3$ the spaces $S_{12}(3, \mathbb{Q})^+$ and $S_{16}(3, \mathbb{Q})^-$ are one-dimensional, spanned by

$$f_{12}^+ = q + 78q^2 - 243q^3 + 4036q^4 - 5370q^5 + O(q^6) \in S_{12}(3, \mathbb{Q})^{\mathrm{new},+}$$
$$f_{16}^- = q - 72q^2 + 2187q^3 - 27584q^4 - 221490q^5 + O(q^6) \in S_{16}(3, \mathbb{Q})^{\mathrm{new},-}.$$

Then f_{12}^{+} and f_{16}^{-} are congruent $\mod 5$, but $w_3 f_{12}^{+} = f_{12}^{+}$ and $w_3 f_{16}^{-} = -f_{16}^{-}$ are not.

(2) **Oldform example:** Let $f \in M_k(N, \mathbb{Z}_p)$ be any form, not necessarily eigen. Then $w_\ell f = \ell^{\frac{k}{2}} f(q^\ell)$. Suppose $f' \in M_{k'}(N, \mathbb{Z}_p)$ is congruent to f. Then we similarly see that $w_\ell f \equiv w_\ell f' \mod p$ if and only if either ℓ is a square modulo p or $k - k'$ is a multiple of $2(p-1)$. Indeed, for any $p \geq 5$, compare $f = E_{p-1} \in M_{p-1}(1, \mathbb{Z}_p)$ and the constant form $f' = 1 \in M_0(1, \mathbb{Z}_p)$. Then $w_\ell E_{p-1} = \ell^{\frac{p-1}{2}} E_{p-1}(q^\ell)$ and $w_\ell(1) = 1$; these are congruent modulo p exactly when $\left(\frac{\ell}{p}\right) = 1$.

3.3.2. *Sometimes we get an algebra involution compatible with reduction.* In light of these examples, it is not true in general that w_ℓ descends to an algebra involution of $M(N\ell, \mathbb{F})$. However it does work in certain cases:

(1) **If ℓ is a square modulo** p, then there is no sign ambiguity, and w_ℓ is an algebra involution of $M(N\ell, \mathbb{F}_p)$. This is easy to show by moving around different weights by multiplying by E_{p-1} and using the fact that $w_\ell(E_{p-1}) = \left(\frac{\ell}{p}\right) E_{p-1}(q^\ell)$. (Use E_4 and E_6 in place of E_{p-1} if $p = 2$ or 3.) In particular, $p = 2$ never poses a problem.

(2) **Restricting to** $M(N\ell, \mathbb{F}_p)^0$ **and** $p \geq 3$, we can define w_ℓ as an algebra involution compatible with reduction of *some* lift. Namely, $f \in M(N\ell, \mathbb{F}_p)^0$ is the reduction of some $\tilde{f} \in M_k(N\ell, \mathbb{Z}_p)$ with k divisible by $2(p-1)$; define $w_\ell f$ as the reduction of $w_\ell \tilde{f}$. Since any two such $\tilde{f}$s differ (multiplicatively) by a power of E_{p-1}^2, this construction is independent of the choice of $\tilde{f}$.

For $p \geq 5$, this construction is equivalent to the following geometric definition (see [24, Corollaire 2] for level one). By dividing $f \in M_{(p-1)k}(N\ell, \mathbb{F}_p)$ by E_{p-1}^k, we can identify $M(N\ell, \mathbb{F}_p)^0$ with the algebra of regular functions on the affine curve obtained by removing the supersingular points from $X_0(N\ell)_{\mathbb{F}_p}$. The geometric Atkin-Lehner involution on $X_0(N\ell)_{\mathbb{F}_p}$ preserves the supersingular locus and hence induces an algebra involution on $M(N\ell, \mathbb{F}_p)^0$.

3.3.3. *Sometimes no algebra involution compatible with reduction is possible.* However, it is not always possible to see w_ℓ as an algebra involution on $M(N\ell, \mathbb{F}_p)$ compatible with reduction.

Proposition 3.1. *If $p \equiv 1$ modulo 4 and ℓ is not a square modulo p, then there is no algebra involution on $M(N\ell, \mathbb{F}_p)$ extending the involution on $M(N\ell, \mathbb{F}_p)^0$ described in subsubsection 3.3.2 (2) with the property that every $f \in M(N\ell, \mathbb{F}_p)$ is sent to a reduction of $w_\ell \tilde{f}$ for some lift $\tilde{f} \in M(N\ell, \mathbb{Z}_p)$ of f.*

Proof. Suppose such an extension W exists. Choose a form $f \in M_k(N, \mathbb{F}_p) \subset M_k(N\ell, \mathbb{F}_p)$, where, to fix ideas, $k \equiv 2 \mod (p-1)$. By assumption, Wf is the reduction of $w_\ell \tilde{f}$ for some lift $\tilde{f}$ of f of weight k' with $k' \equiv k \mod (p-1)$.

By subsection 3.1 (5), we know that $w_\ell \tilde{f} = \ell^{k'/2} \tilde{f}(q^\ell)$, so that $Wf = \varepsilon \ell f(q^\ell)$ for some $\varepsilon = \pm 1$. Consider $g = f^{\frac{p-1}{2}}$. On one hand, we're assuming that W is an algebra involution, so that $Wg = (Wf)^{\frac{p-1}{2}} = \varepsilon^{\frac{p-1}{2}} \ell^{\frac{p-1}{2}} g(q^\ell)$. On the other hand, $g \in M(\ell, \mathbb{F}_p)^0$ and so that $Wg = w_\ell g = g(q^\ell)$ by the recipe in subsubsection 3.3.2 (2) above. Therefore $\varepsilon^{\frac{p-1}{2}} = (\frac{\ell}{p})$. Hence if $p \equiv 1 \mod 4$ but $(\frac{\ell}{p}) = -1$, we have a contradiction. $\square$

Question 1. For $p \equiv 3 \mod 4$ (if ℓ is not a square modulo p) the argument above fundamentally fails: one can indeed "extend" the definition of w_ℓ on $M(N\ell, \mathbb{F}_p)^0$ as in subsubsection 3.3.2 (2) to an algebra involution W on the ℓ-old forms in $M(N\ell, \mathbb{F}_p)$ in a reduction-compatible manner by setting $Wf = (-\ell)^{k/2} f(q^\ell)$ for $f \in M_k(N, \mathbb{F}_p)$, well-defined as $-\ell$ is now a square modulo p. But can this be extended in an algebra-involution way to all of $M(N\ell, \mathbb{F}_p)$ compatible with reductions? And can one show that any algebra involution on $M(N\ell, \mathbb{F}_p)$ compatible with some reduction of w_ℓ restricts to the construction from subsubsection 3.3.2 (2) on $M(N\ell, \mathbb{F}_p)^0$?

3.4. **Modified Atkin-Lehner as an algebra automorphism in characteristic p.** To fix this difficulty, we will renormalize w_ℓ to be compatible with algebra structures.

For any $m \in \mathbb{Z}$, possibly depending on k, the weight-k right action of $\mathrm{SL}_2(\mathbb{Z})$ on functions $f : \mathcal{H} \to \mathbb{C}$ can be extended to $\mathrm{GL}_2(\mathbb{Q})^+$ via the formula, for $z \in \mathcal{H}$,

$$(f|_{k,m}\gamma)(z) = (\det \gamma)^m j(\gamma, z)^{-k} f(\gamma z).$$

Scalar matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ then act via multiplication by a^{2m-k}. The usual choice in the definition of the Atkin-Lehner operator is $m = \frac{k}{2}$ (scalars act trivially; see, for example, [1, p.135]); another possibility that appears in the literature is $m = k - 1$ (used to define Hecke operators; see, for example, [7, Exercise 1.2.11]). For our renormalized Atkin-Lehner operator, we adopt $m = k$, so that scalars act through their k^{th} power.

We define a new map

$$\begin{aligned} W_\ell : M_k(N\ell, \mathbb{Z}[\tfrac{1}{\ell}]) &\to M_k(N\ell, \mathbb{Z}[\tfrac{1}{\ell}]) \\ f &\mapsto f|_{k,k}\gamma_\ell. \end{aligned}$$

Here γ_ℓ is again a matrix of the form $\left(\begin{smallmatrix} \ell & a \\ N\ell & \ell b \end{smallmatrix}\right)$, where a and b are integers such that $\ell b - aN = 1$, as in subsection 3.1. Since $W_\ell = \ell^{\frac{k}{2}} w_\ell$, it is clear that this map is well-defined independent of the choice of γ_ℓ. Moreover, W_ℓ satisfies the following properties.

Proposition 3.2.

(1) W_ℓ extends to an automorphism of $M_k(N\ell, B)$ for any $\mathbb{Z}[\frac{1}{\ell}]$-algebra B, satisfying
  - *(a) $W_\ell^2 = \mathcal{S}_\ell$;*
  - *(b) $W_\ell f = \mathcal{S}_\ell\, f(q^\ell)$ and $W_\ell f(q^\ell) = f$ for $f \in M_k(N, B)$.*

(2) W_ℓ extends to an algebra automorphism of $M(N\ell, B)$ for any characteristic-zero $\mathbb{Z}[\frac{1}{\ell}]$-domain B. This algebra automorphism preserves the ideal $S(N\ell, B)$.

(3) W_ℓ descends to an algebra automorphism for any characteristic-p domain B. This algebra automorphism restricts to the involution on $M(N\ell, \mathbb{F}_p)^0$ defined in subsubsection 3.3.2 (2). For $p \geq 3$, the order of W_ℓ divides $p - 1$; for $p = 2$, W_ℓ coincides with w_ℓ and hence has order 2.

Only the last item requires justification. It relies on the following:

Lemma 3.3. *If $f, g \in M(N\ell, \mathbb{Z}_p)$ are congruent modulo p, then so are $W_\ell(f)$ and $W_\ell(g)$.*

Proof. It suffices to consider f, g appearing in single weights, so let these be $k(f), k(g)$, respectively. Since w_ℓ already has this property for $k(f) = k(g)$, so does W_ℓ. It therefore suffices prove the case $k(f) < k(g)$. By a theorem of Serre (see equation (2.1)) $k(g) - k(f) = n(p-1)$ for some $n \in \mathbb{Z}^+$. But then $E_{p-1}^n f$ and g are congruent in the same weight, so $W_\ell(E_{p-1})^n W_\ell(f) \equiv W_\ell(g) \pmod{p}$. The observation that $W_\ell(E_{p-1}) = \ell^{p-1} E_{p-1}(q^\ell) \equiv 1$ modulo p completes the proof. □

Appendix A shows that the renormalized Atkin-Lehner operator W_ℓ in characteristic p is induced geometrically on modular forms by an automorphism of the Igusa curve.

4. The trace from level $N\ell$ to level N

For any characteristic-zero $\mathbb{Z}[\frac{1}{\ell}]$-domain B, there is a B-linear trace operator

$$\mathrm{Tr}_\ell : M_k(N\ell, B) \to M_k(N, B)$$

given, for $B = \mathbb{C}$, by

$$\mathrm{Tr}_\ell(f) = \sum_{\gamma \in \Gamma_0(N\ell)\backslash\Gamma_0(N)} f|_k \gamma, \tag{4.1}$$

first studied for $N = 1$ by Serre in [23, §3.1.(c)].

One can show ([11, Lemma 2.2], or [23, §3.1.(c)] for $N = 1$) that $\mathrm{Tr}_\ell f = f + \ell^{1-\frac{k}{2}} U_\ell w_\ell f$. Equivalently,

$$\mathrm{Tr}_\ell f = f + \mathcal{S}_\ell^{-1} \ell U_\ell W_\ell f. \tag{4.2}$$

Equation (4.2) shows immediately that Tr_ℓ extends to a B-linear operator $M(N\ell, B) \to M(N, B)$ for any $\mathbb{Z}[\frac{1}{\ell}]$-domain B. The following identities are adjusted from [23, §3.1.(c)]. They are valid for any $\mathbb{Z}[\frac{1}{\ell}]$-domain B. In fixed weight k, they are valid for any $\mathbb{Z}[\frac{1}{\ell}]$-algebra B.

(1) For $f \in S(N\ell, B)$, we have $\mathrm{Tr}_\ell f \in S(N, B)$.
(2) For $f \in M(N\ell, B)$, we have $\mathrm{Tr}_\ell \mathcal{S}_\ell f = \mathcal{S}_\ell f + \ell U_\ell W_\ell f$.
(3) For $f \in M(N\ell, B)$, we have $\mathrm{Tr}_\ell W_\ell f = W_\ell f + \ell U_\ell f$.
(4) For $f \in M(N, B)$, we have $\mathrm{Tr}_\ell f = (\ell + 1)f$.
(5) For $f \in M(N, B)$, we have $\ell T_\ell f = \ell U_\ell f + W_\ell f$.
(6) For $f \in M(N, B)$, we have $\mathrm{Tr}_\ell W_\ell f = \ell T_\ell f$.

The shape of these equations suggest that it might be more natural to renormalize T_ℓ and U_ℓ by scaling them by ℓ, so that the Hecke operators are true "trace" rather than a scaled trace and stay integral even in weight 0. In fact, this renormalization would amount to using the $|_{k,k}$-action discussed in subsection 3.4 to define the Hecke operators, which we are already using to define W_ℓ. But we will not do so here.

5. The space of ℓ-old forms

5.1. Two copies of $M(N, B)$ in $M(N\ell, B)$.

There are two embeddings of $M(N, B)$ into $M(N\ell, B)$: the identity and W_ℓ. First we study their intersection.

Proposition 5.1. *For any $\mathbb{Z}[\frac{1}{\ell}]$-algebra B, if $f \in M_k(N, B) \cap W_\ell M_k(N, B)$, then f is constant.*

Proof. Let $g \in M_k(N, B)$ be such that $f = W_\ell(g)$. We use Proposition 3.2 (1a) and (1b) to see that $\ell^k g = W_\ell^2(g) = W_\ell(f) = \ell^k f(q^\ell)$, so that

$$f = \ell^k g(q^\ell) = \ell^k f(q^{\ell^2}).$$

But this means that f has to be a constant! Indeed, suppose $n > 0$ is the least integer such that $a_n(f) \neq 0$. Since the right-hand side is in $B[\![q^{\ell^2}]\!]$, we must have $n = m\ell^2$ for some $m < n$. But the q^n-coefficient on the right-hand side is $\ell^k a_m(f)$, which must be zero as n was the least index of a nonzero coefficient of f. □

Alternatively, we can deduce Proposition 5.1 in characteristic zero from [1, Theorem 1] and in characteristic p from the following more recent theorem of Ono-Ramsey.

Theorem 5.2 (Ono-Ramsey, [20, Theorem 1.1])**.** *Let p be a prime, and f a form in $M_k(N,\mathbb{Z})$ with $\bar{f} = \sum a_n q^n \in M_k(N,\mathbb{F}_p)$ its mod-p image. Suppose that there exists an m prime to Np and a power series $g \in \mathbb{F}_p[\![q]\!]$ so that $\bar{f} = g(q^m)$. Then $\bar{f} = a_0$.*

Corollary 5.3. *If B is a $\mathbb{Z}[\frac{1}{\ell}]$-domain, then $M(N,B) \cap W_\ell M(N,B) = B \subset B[\![q]\!]$.*

Proof. Let $f, g \in M(N,B)$ be forms so that $f = W_\ell(g) \in B[\![q^\ell]\!]$. In light of Proposition 5.1, it suffices to show that we may assume that both f and g appear in a fixed weight k. As a $\mathbb{Z}[\frac{1}{\ell}]$-domain, B is flat over either $\mathbb{Z}[\frac{1}{\ell}]$ or over $\mathbb{F}_p$ for some p prime to ℓN. In either case, from subsection 2.1, we know that we can express both f and g as finite sums of forms $f = \sum f_i$ and $g = \sum g_i$ with $f_i, g_i \in M_{k_i}(N,B)$ for some weights k_i, with $M_{k_i}(N\ell, B)$ linearly independent inside $M(N\ell,B)$. Then $\sum f_i = \sum W_\ell(g_i)$ forces $f_i = W_\ell(g_i)$ in a single weight k_i. □

5.2. **ℓ-Old forms.** Following Atkin-Lehner [1] and others, define the ℓ-old forms in $M_k(N\ell,\mathbb{Q})$ as the span of $M_k(N,\mathbb{Q})$ and $W_\ell M_k(N,\mathbb{Q})$:

$$M_k(N\ell,\mathbb{Q})^{\ell\text{-old}} := M_k(N,\mathbb{Q}) + W_\ell M_k(N,\mathbb{Q}) \subset M_k(N\ell,\mathbb{Q}). \tag{5.1}$$

Note that both $M_k(N,\mathbb{Q})$ and $W_\ell M_k(N,\mathbb{Q})$ have bases in $\mathbb{Z}[\![q]\!]$; therefore $M_k(N\ell,\mathbb{Q})^{\ell\text{-old}}$ does as well. Let $M_k(N\ell,\mathbb{Z})^{\ell\text{-old}}$ be the forms in $M_k(N\ell,\mathbb{Q})^{\ell\text{-old}}$ whose q-expansions are integral:

$$M_k(N\ell,\mathbb{Z})^{\ell\text{-old}} := M_k(N\ell,\mathbb{Q})^{\ell\text{-old}} \cap \mathbb{Z}[\![q]\!],$$

and let $S_k(N\ell,\mathbb{Z})^{\ell\text{-old}} := S_k(N\ell,\mathbb{Z}) \cap M_k(N\ell,\mathbb{Z})^{\ell\text{-old}}$ be the cuspidal submodule. Finally, for any ring B, let $M_k(N\ell,B)^{\ell\text{-old}}$ (respectively, $S_k(N\ell,B)^{\ell\text{-old}}$) be the image of $M_k(N\ell,\mathbb{Z})^{\ell\text{-old}} \otimes_{\mathbb{Z}} B$ (respectively, $S_k(N\ell,\mathbb{Z})^{\ell\text{-old}} \otimes_{\mathbb{Z}} B$) inside $B[\![q]\!]$. Our definitions are not self-contradictory: for $B = \mathbb{Q}$ the definition of $M_k(N\ell,\mathbb{Q})^{\ell\text{-old}}$ coincides with (5.1) because of its $\mathbb{Z}$-structure. For the same reason, $S_k(N\ell,B)^{\ell\text{-old}} = S(N\ell,B) \cap M_k(N\ell,B)^{\ell\text{-old}}$ for any B.

Note that $M_k(N\ell,B)^{\ell\text{-old}}$ may a priori be bigger than $M_k(N,B) + W_\ell M_k(N,B)$. For example, if E_k is the *normalized* (i.e., with $a_1 = 1$) weight-k level-one Eisenstein series and $B = \mathbb{Z}_p$, then

$$E_{p-1}^{\ell\text{-crit}} := E_{p-1}(q) - E_{p-1}(q^\ell)$$

is in $M_k(\ell,B)^{\ell\text{-old}}$ but not in $M_k(1,B) + W_\ell M_k(1,B)$, since E_{p-1} is has p in the denominator of its constant term.[ii] For our purposes, the following will suffice:

Proposition 5.4. *If B is a $\mathbb{Z}[\frac{1}{\ell}]$-algebra, then $S_k(N\ell,B)^{\ell\text{-old}} = S_k(N,B) \oplus W_\ell S_k(N,B)$.*

Proof. Since we are in a single weight, it suffices to consider $B = \mathbb{Z}[\frac{1}{\ell}]$.

Certainly $S_k(N,B) \oplus W_\ell S_k(N,B)$ is contained in $S_k(N\ell,B)^{\ell\text{-old}}$, and by Corollary 5.3 this sum is direct. For the other containment, any element of $S_k(N\ell,B)^{\ell\text{-old}}$ looks like $f = M^{-1}\big(g + W_\ell(h)\big)$ for some $g, h \in S_k(N,B)$ and $M \in B$. Then the fact that $\frac{1}{M}\big(g + W_\ell(h)\big)$ is B-integral means that $g \equiv -W_\ell(h) \bmod MB$. But by Proposition 5.1 applied to B/MB, we must have

$$W_\ell(h) \equiv g \equiv a_0(g) \equiv 0 \bmod MB,$$

so that both $\frac{1}{M} g$ and $\frac{1}{M} W_\ell(h)$ are in fact B-integral. □

Finally, let $M(N\ell,B)^{\ell\text{-old}} := \sum_k M_k(N\ell,B)^{\ell\text{-old}} \subset B[\![q]\!]$, the space of ℓ-old forms of any weight. Similarly, $S(N\ell,B)^{\ell\text{-old}} := \sum_k S_k(N\ell,B)^{\ell\text{-old}}$ is the submodule of ℓ-old cuspforms.

[ii] In fact for $N = 1$ and $B = \mathbb{Z}_p$ or $\mathbb{F}_p$ one can show that this is essentially the only such exception.

6. The space of ℓ-new forms

6.1. ℓ-New forms in characteristic zero.

6.1.1. *Analytic notion.* For $B = \mathbb{C}$ one can follow Atkin-Lehner's characterization of newforms to define the space $S_k(N\ell, \mathbb{C})^{\ell\text{-new}}$ of cuspidal ℓ-new forms of level $N\ell$ and weight k as the orthogonal complement to the space of ℓ-old forms under the Petersson inner product [1, p. 145]. Alternatively, the space of ℓ-new cuspforms is the $\mathbb{C}$-span of the *ℓ-new eigenforms*: those eigenforms that are not in $S_k(N\ell, \mathbb{C})^{\ell\text{-old}}$ [1, Lemma 18]. This latter definition can be extended to Eisenstein forms as well, to obtain well-defined spaces $M_k(N\ell, \mathbb{C})^{\ell\text{-new}}$ and $S_k(N\ell, \mathbb{C})^{\ell\text{-new}}$, which we here identify with their q-expansions.

One can show that $M_k(N\ell, \mathbb{C})^{\ell\text{-new}}$ has a basis in $\mathbb{Z}[\![q]\!]$ (since Galois conjugates of ℓ-new eigenforms are ℓ-new [6, Corollary 12.4.5], one can mimic the argument in [7, Corollary 6.5.6]; see also Brunault's answer to MathOverflow question 109871). Therefore, the definitions

$$M_k(N\ell, \mathbb{Z})^{\ell\text{-new}} := M_k(N\ell, \mathbb{C})^{\ell\text{-new}} \cap \mathbb{Z}[\![q]\!]$$

and, for any characteristic-zero domain B,

$$M_k(N\ell, B)^{\ell\text{-new}} := M_k(N\ell, \mathbb{Z})^{\ell\text{-new}} \otimes_{\mathbb{Z}} B \subset B[\![q]\!]$$

are compatible with the definition of $M_k(N\ell, \mathbb{C})^{\ell\text{-new}}$ above. Finally, set

$$M(N\ell, B)^{\ell\text{-new}} := \sum_k M_k(N\ell, B)^{\ell\text{-new}} \subset B[\![q]\!]$$

as usual. In characteristic zero, of course, this sum is direct.

6.1.2. *First algebraic notion: U_ℓ-eigenvalue.* Combining the Atkin-Lehner computations of ℓ-new eigenvalues together with the Weil bound, one can obtain a purely algebraic characterization of the space of newforms. Define two operators $\mathcal{D}_\ell{}^{\pm} : M_k(N\ell, B) \to M_k(N\ell, B)$ via $\mathcal{D}_\ell{}^{+} := \ell U_\ell - \ell^{\frac{k}{2}}$ and $\mathcal{D}_\ell{}^{-} := \ell U_\ell + \ell^{\frac{k}{2}}$, and let $\mathcal{D}_\ell := \mathcal{D}_\ell{}^{+} \mathcal{D}_\ell{}^{-}$. Since $\mathcal{D}_\ell = \ell^2 U_\ell^2 - \mathcal{S}_\ell$, the operator $\mathcal{D}_\ell$ defines B-linear grading-preserving operator $M(N\ell, B) \to M(N\ell, B)$ and $S(N\ell, B) \to S(N\ell, B)$.

Proposition 6.1. *If B is a domain of characteristic zero, then*

$$M(N\ell, B)^{\ell\text{-new}} = \ker \mathcal{D}_\ell\,.$$

We sketch a proof below, starting with a lemma that relies on the Ramanujan-Petersson Conjecture ("Weil bound"), implied by the Weil Conjectures, proved by Deligne.

Lemma 6.2 (Ramanujan-Petersson, Weil, Deligne)**.** *If $g = \sum a_n q^n \in S_k(N, \mathbb{C})$ is a normalized Hecke eigenform, and ℓ is any prime, then $|a_\ell(g)| < (\ell+1)\ell^{\frac{k-2}{2}}$.*

Proof. The negation of the inequality violates the the Weil bound $|a_\ell(g)| \leq 2\ell^{\frac{k-1}{2}}$. Indeed, $(\ell+1)\ell^{\frac{k-2}{2}} \leq 2\ell^{\frac{k-1}{2}}$ is equivalent to $(\ell+1)^2 \leq 4\ell$, which cannot happen for $\ell > 1$. □

Proof of Proposition 6.1. It suffices to prove that the kernel of $\mathcal{D}_\ell|_{M_k(N\ell,B)}$ is $M_k(N\ell, B)^{\ell\text{-new}}$ in a single weight k. Moreover, since B is flat over $\mathbb{Z}$ it suffices to prove the statement for $B = \mathbb{Z}$; and since $M_k(N\ell, \mathbb{C})^{\ell\text{-new}}$ has a basis over $\mathbb{Z}$, it suffices to take $B = \mathbb{C}$.

The module $M_k(N\ell,\mathbb{C})$ is a direct sum of $\mathbb{C}$-spans of eigenforms ℓ-new and ℓ-old. Since $\mathcal{D}_\ell$ preserves away-from-ℓ Hecke eigenspaces, it suffices to see that $\mathcal{D}_\ell$ annihilates all ℓ-new eigenforms and never annihilates ℓ-old eigenforms. If $f \in M_k(N\ell,\mathbb{C})$ is Eisenstein, then it must be old at ℓ, the ℓ-stabilization of a form $g \in M_k(N,\mathbb{C})$ with $a_\ell(g) = \chi(\ell)\ell^{k-1} + \chi(\ell)^{-1}$ for some Dirichlet character χ of modulus M with $M^2 \mid N$ (see, for example, [7, Theorem 4.5.2]). Hence the absolute value of the U_ℓ-eigenvalue of f is either ℓ^{k-1} or 1. If $f \in M_k(N\ell,\mathbb{C})$ is a cuspidal ℓ-new form, then by [1, Theorem 5], its U_ℓ-eigenvalue is $\pm\ell^{\frac{k}{2}-1}$, so that $\mathcal{D}_\ell f = 0$. Finally, if $f \in M_k(N\ell,\mathbb{C})$ is a cuspidal ℓ-old eigenform, then f is the ℓ-stabilization of some normalized eigenform $g \in M_k(N,\mathbb{C})$, and the U_ℓ-eigenvalue of f is a root of the polynomial $P_{\ell,g}(X) = X^2 - a_\ell(g)X + \ell^{k-1}$. If one root of $P_{\ell,g}$ is $\pm\ell^{\frac{k}{2}-1}$, then the other root must be $\pm\ell^{\frac{k}{2}}$, so that $a_\ell(g) = \pm(\ell+1)\ell^{\frac{k-2}{2}}$, which impossible by Lemma 6.2. □

6.1.3. *Second algebraic notion: kernel of trace.* On the other hand, if B is additionally a $\mathbb{Z}[\frac{1}{\ell}]$-algebra, then Serre suggests an alternate description of the space of newforms of level ℓ.

Proposition 6.3 (Serre [23, §3.1(c), remarque (3)])**.** *If B is a characteristic-zero $\mathbb{Z}[\frac{1}{\ell}]$-domain,*

$$M(N\ell,B)^{\ell\text{-new}} = \ker \operatorname{Tr}_\ell \cap \ker \operatorname{Tr}_\ell W_\ell.$$

Proof. Since B is flat over $\mathbb{Z}[\frac{1}{\ell}]$, we may replace $\mathbb{Z}$ by $\mathbb{Z}[\frac{1}{\ell}]$ in the beginning of the proof of Proposition 6.1 to see that it suffices to establish this in a single weight k for $B = \mathbb{C}$. Since both Tr_ℓ and W_ℓ commute with Hecke operators prime to ℓ, it suffices to consider separately the one-dimensional eigenspaces spanned by ℓ-new eigenforms and the two-dimensional ℓ-old eigenspaces coming from eigenforms of level N. If $f \in M_k(N\ell,\mathbb{C})$ is ℓ-new eigen, then both $\operatorname{Tr}_\ell f$ and $\operatorname{Tr}_\ell W_\ell f$ are forms of level N with the same eigenvalues away from ℓ as f, which is impossible by [1, Lemma 23]. Therefore both $\operatorname{Tr}_\ell f = 0$ and $\operatorname{Tr}_\ell W_\ell f = 0$, so that $\ker \operatorname{Tr}_\ell \cap \ker \operatorname{Tr}_\ell W_\ell$ does indeed contain $M(N\ell,B)^{\ell\text{-new}}$. For the reverse containment, if f is in $M_k(N\ell,\mathbb{C})^{\ell\text{-old}}$, then it suffices to consider to f contained in the two-dimensional span of g and $W_\ell(g)$ for some eigenform $g \in M_k(N,\mathbb{C})$. From the identities in section 4, the operators Tr_ℓ and $\operatorname{Tr}_\ell W_\ell$, on the ordered basis $\{g, W_\ell(g)\}$ of the ℓ-old subspace of $M_k(N\ell,\mathbb{C})$ associated to g, have matrix form

$$\operatorname{Tr}_\ell = \begin{pmatrix} \ell+1 & \ell\, a_\ell(g) \\ 0 & 0 \end{pmatrix} \qquad \operatorname{Tr}_\ell W_\ell = \begin{pmatrix} \ell\, a_\ell(g) & (\ell+1)\ell^k \\ 0 & 0 \end{pmatrix}.$$

The kernels of matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} c & d \\ 0 & 0 \end{smallmatrix}\right)$ have a nontrivial intersection if and only if $ad = bc$. In our case that would mean that $a_\ell(g)^2 = (\ell+1)^2\ell^{k-2}$, which is again impossible by the Weil bounds (Lemma 6.2). □

6.2. **Newforms over any domain: a proposal.** Inspired by the algebraic characterisations of Proposition 6.1 and Proposition 6.3 of newforms in characteristic zero, we make the following two definitions.

Definition 1. *For any ring B and any Hecke-invariant submodule $C \subset M(N\ell,B)$, let*

$$C^{U_\ell\text{-new}} := \ker \mathcal{D}_\ell|_C \quad \text{and} \quad C^{\operatorname{Tr}_\ell\text{-new}} := (\ker \operatorname{Tr}_\ell|_C) \cap (\ker \operatorname{Tr}_\ell W_\ell|_C).$$

Proposition 6.1 already establishes that if B is a characteristic zero $\mathbb{Z}[\frac{1}{\ell}]$-domain and C is $M(N\ell,B)$, then both of these "ℓ-new" spaces coincide with $M(N\ell,B)^{\ell\text{-new}}$. In other words, these definitions both extend the Atkin-Lehner analytic notion of ℓ-new forms. The main result of this section is to show that on cuspforms, these two definitions coincide for more general B as well.

Theorem 1. *For any $\mathbb{Z}[\frac{1}{\ell}]$-domain B, we have*

$$S(N\ell, B)^{U_\ell\text{-new}} = S(N\ell, B)^{\mathrm{Tr}_\ell\text{-new}}.$$

To prove Theorem 1, we first establish $\left(S(N\ell, B)^{\ell\text{-old}}\right)^{U_\ell\text{-new}} = \left(S(N\ell, B)^{\ell\text{-old}}\right)^{\mathrm{Tr}_\ell\text{-new}}$:

Proposition 6.4. *Let B be a $\mathbb{Z}[\frac{1}{\ell}]$-algebra. Suppose f, g in $S_k(N, B)$ for some weight k. Then the following are equivalent.*

(1) $f + W_\ell(g) \in \ker \mathcal{D}_\ell$
(2) $f + W_\ell(g) \in \ker \mathrm{Tr}_\ell \cap \ker \mathrm{Tr}_\ell W_\ell$
(3) $\ell\, T_\ell\, f = -(\ell+1)\, \mathcal{S}_\ell\, g$ *and* $\ell\, T_\ell\, g = -(\ell+1) f$

Remark.

(1) Proposition 6.4 may be rewritten more symmetrically in terms of w_ℓ, the involution-normalized Atkin-Lehner operator on $S_k(N\ell, B)$. Namely, let $\lambda_k = -(\ell+1)\ell^{\frac{k-2}{2}}$. Then the claim of the proposition is that
$$\begin{aligned}\left(S_k(N\ell, B)^{\ell\text{-old}}\right)^{U_\ell\text{-new}} &= \left(S_k(N\ell, B)^{\ell\text{-old}}\right)^{\mathrm{Tr}_\ell\text{-new}} \\ &= \{f + w_\ell\, g : f, g \in S_k(N, B) \text{ s.t. } T_\ell f = \lambda_k g \text{ and } T_\ell g = \lambda_k f\}.\end{aligned}$$
The constant λ_k appears in connection with level-raising theorems of Ribet [22] and Diamond [5]. See also subsection 7.2 for more details.

(2) From the proof Proposition 6.4 below, it is clear that the conclusions hold for any $f, g \in S(N, B)$ as long as we assume that B is $\mathbb{Z}[\frac{1}{\ell}]$-domain.

(3) Proposition 6.4 does not hold as stated for $M_k(N, B)$ if B has characteristic p. For example, if $\ell^{k-2} \equiv 1 \bmod p$ but $\ell \not\equiv -1 \bmod p$ (say, if $\ell \equiv 1 \bmod p$ but $p \neq 2$), then $f := 1 \in M_{p-1}(N, B)$ is in the kernel of $\mathcal{D}_\ell$ but is not in the kernel of Tr_ℓ.

Proof of Proposition 6.4. We use the identities from section 4 repeatedly, including the fact that for $f \in M(N, B)$, we have $\ell\, U_\ell f = \ell\, T_\ell f - W_\ell f$ and $U_\ell\, W_\ell f = U_\ell\big(\mathcal{S}_\ell\, f(q^\ell)\big) = \mathcal{S}_\ell\, f$. We first show that (1) $\iff$ (3). Let $f \in S_k(N, B)$. On one hand we have

$$\begin{aligned}\mathcal{D}_\ell\, f &= \ell^2\, U_\ell^2 f - \mathcal{S}_\ell\, f = \ell\, U_\ell(\ell\, T_\ell f - W_\ell f) - \mathcal{S}_\ell\, f \\ &= \ell\, T_\ell(\ell\, T_\ell f) - W_\ell(\ell\, T_\ell f) - \ell\, U_\ell W_\ell f - \mathcal{S}_\ell\, f \\ &= \ell^2\, T_\ell^2 f - (\ell+1)\, \mathcal{S}_\ell\, f - W_\ell\, \ell\, T_\ell f\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_\ell\; W_\ell\, g &= \ell^2\, U_\ell^2 W_\ell\, g - \mathcal{S}_\ell\, W_\ell\, g = \ell^2 U_\ell\, \mathcal{S}_\ell\, g - W_\ell\, \mathcal{S}_\ell\, g \\ &= \ell^2\, T_\ell\, \mathcal{S}_\ell\, g - \ell\, W_\ell\, \mathcal{S}_\ell\; g - W_\ell\, \mathcal{S}_\ell\; g = \ell^2\, T_\ell\, \mathcal{S}_\ell\, g - (\ell+1) W_\ell\, \mathcal{S}_\ell\, g.\end{aligned}$$

From Proposition 5.1, the intersection of $S_k(N, B)$ and $W_\ell S_k(N, B)$ inside $M_k(N\ell, B)$ is trivial. So $\mathcal{D}_\ell(f + W_\ell g) = 0$ if and only if

$$\begin{aligned}0 &= \mathcal{D}_\ell(f + W_\ell g) \\ &= \left(\ell^2\, T_\ell^2 f - (\ell+1)\, \mathcal{S}_\ell\, f + \ell^2 T_\ell\, \mathcal{S}_\ell\, g\right) - W_\ell\left(\ell\, T_\ell f + (\ell+1)\, \mathcal{S}_\ell\, g\right),\end{aligned}$$

which holds if and only if

$$\ell^2\, T_\ell^2 f - (\ell+1)\, \mathcal{S}_\ell\, f + \ell^2 T_\ell\, \mathcal{S}_\ell\, g = 0 \qquad \text{and} \qquad \ell\, T_\ell f + (\ell+1)\, \mathcal{S}_\ell\, g = 0.$$

The second equation reduces to

$$\ell\, T_\ell f = -(\ell+1)\, \mathcal{S}_\ell\, g. \tag{6.1}$$

Inserting this into the first equation, combining like terms, and eliminating $\mathcal{S}_\ell$ reveals

$$\ell\, T_\ell g = -(\ell+1) f, \tag{6.2}$$

as required.

For (2) $\iff$ (3), we recall that for $f \in M_k(N,B)$,

$$\mathrm{Tr}_\ell\, f = (\ell+1) f \qquad \text{and} \qquad \mathrm{Tr}_\ell\, W_\ell f = \ell\, T_\ell f.$$

Therefore $\mathrm{Tr}_\ell(f + W_\ell g) = 0 \iff (\ell+1)f + \ell\, T_\ell g = 0 \iff \ell\, T_\ell\, g = -(\ell+1)f$. Symmetrically, $0 = \mathrm{Tr}_\ell\, W_\ell(f + W_\ell g) = \mathrm{Tr}_\ell(\mathcal{S}_\ell\, g + W_\ell f) \iff \ell\, T_\ell\, f = -(\ell+1)\, \mathcal{S}_\ell\, g$. □

Proof of Theorem 1. If B has characteristic zero, then this statement is already known (Proposition 6.1 & Proposition 6.3), but we prove it again without using the Weil bound. As in the proof of Proposition 6.1, we may assume that we are in a single weight k and that $B = \mathbb{C}$, and note that each one-dimensional ℓ-new eigenspace is annihilated by all three operators $\mathcal{D}_\ell$, Tr_ℓ, and $\mathrm{Tr}_\ell\, W_\ell$. Now Proposition 6.4 establishes the desired statement for each two-dimensional ℓ-old away-from-ℓ Hecke eigenspace and completes the proof.

If B has characteristic p, then we may assume that $B = \mathbb{F}_p$ and again as in the proof of Corollary 5.3 work in a single weight k. We will have to distinguish between coefficients in $\mathbb{Z}_p$ and quotients, so for any ring B, write X^B for the operator X acting on $S_k(N\ell, B)$.

Take $f \in S_k(N\ell, \mathbb{F}_p)$. Then there exist integral forms $\tilde{f}^{\ell\text{-new}}$ and $\tilde{f}^{\ell\text{-old}}$ in $S_k(N\ell, \mathbb{Z}_p)^{\ell\text{-new}}$ and $S_k(N\ell, \mathbb{Z}_p)^{\ell\text{-old}}$, respectively, and a $b \in \mathbb{Z}_{\geq 0}$ so that f is the mod-p reduction of

$$\tilde{f} = p^{-b}(\tilde{f}^{\text{new}} + \tilde{f}^{\text{old}}) \in S_k(N\ell, \mathbb{Z}_p).$$

Suppose now that $f \in \ker \mathcal{D}_\ell{}^{\mathbb{F}_p}$, so that $\mathcal{D}_\ell{}^{\mathbb{Z}_p}(\tilde{f})$ is in $p\mathbb{Z}_p[\![q]\!]$. Since $\mathcal{D}_\ell{}^{\mathbb{Z}_p}(\tilde{f}^{\text{new}}) = 0$ we have $\mathcal{D}_\ell{}^{\mathbb{Z}_p}(\tilde{f}) = p^{-b}\mathcal{D}_\ell{}^{\mathbb{Z}_p}(\tilde{f}^{\text{old}})$. In other words, the form f^{old} is in $\ker \mathcal{D}_\ell{}^{\mathbb{Z}/p^{b+1}\mathbb{Z}}$, where $f^{\text{old}} \in S_k(N\ell, \mathbb{Z}/p^{b+1}\mathbb{Z})$ is the image of $\tilde{f}^{\text{old}}$ under the reduction-mod-p^{b+1} map. By Proposition 6.4, f^{old} is in $\ker(\mathrm{Tr}_\ell)^{\mathbb{Z}/p^{b+1}\mathbb{Z}} \cap \ker(\mathrm{Tr}_\ell\, W_\ell)^{\mathbb{Z}/p^{b+1}\mathbb{Z}}$. By lifting back up to characteristic zero, we see that both $\mathrm{Tr}_\ell{}^{\mathbb{Z}_p}(\tilde{f}^{\text{old}})$ and $(\mathrm{Tr}_\ell\, W_\ell)^{\mathbb{Z}_p}(\tilde{f}^{\text{old}})$ are in $p^{b+1}\mathbb{Z}_p[\![q]\!]$.

As $(\mathrm{Tr}_\ell)^{\mathbb{Z}_p}(\tilde{f}^{\text{new}}) = (\mathrm{Tr}_\ell\, W_\ell)^{\mathbb{Z}_p}(\tilde{f}^{\text{new}}) = 0$, we get that both $\mathrm{Tr}_\ell{}^{\mathbb{Z}_p}(\tilde{f})$ and $(\mathrm{Tr}_\ell\, W_\ell)^{\mathbb{Z}_p}(\tilde{f})$ are in $p\mathbb{Z}_p[\![q]\!]$. Therefore, $\mathrm{Tr}_\ell{}^{\mathbb{F}_p}(f) \equiv \mathrm{Tr}_\ell{}^{\mathbb{Z}_p}(\tilde{f}) \equiv 0 \bmod p$ and

$$(\mathrm{Tr}_\ell\, W_\ell)^{\mathbb{F}_p}(f) \equiv (\mathrm{Tr}_\ell\, W_\ell)^{\mathbb{Z}_p}(\tilde{f}) \equiv 0 \bmod p.$$

Hence f is in $\ker(\mathrm{Tr}_\ell)^{\mathbb{F}_p} \cap \ker(\mathrm{Tr}_\ell\, W)^{\mathbb{F}_p}$. Reverse all steps for the reverse containment.

□

In light of Theorem 1, we introduce the following definition:

Definition 2. *If B is any $\mathbb{Z}[\frac{1}{\ell}]$-algebra, the submodule of ℓ-new cuspforms of weight k is*

$$S_k(N\ell, B)^{\ell\text{-new}} := S_k(N\ell, B)^{U_\ell\text{-new}} = S_k(N\ell, B)^{\mathrm{Tr}_\ell\text{-new}}.$$

If B is any $\mathbb{Z}[\frac{1}{\ell}]$-domain, the submodule of ℓ-new cuspforms of all weights is

$$S(N\ell, B)^{\ell\text{-new}} := S(N\ell, B)^{U_\ell\text{-new}} = S(N\ell, B)^{\mathrm{Tr}_\ell\text{-new}}.$$

We will also use the notation $M(N\ell, B)^{\ell\text{-new}} := M(N\ell, B)^{\mathrm{Tr}_\ell\text{-new}}$. Observe that the space of ℓ-new forms is stable under W_ℓ.

7. Interactions between ℓ-old and ℓ-new spaces mod p

In characteristic zero, spaces of ℓ-new and ℓ-old forms are disjoint. This fails in characteristic p because of congruences between ℓ-new and ℓ-old forms. A related phenomenon: over a field of characteristic zero, ℓ-new and ℓ-old forms together span the space of forms of level $N\ell$. This already fails over a ring like $\mathbb{Z}_p$, again because of congruences between ℓ-new and ℓ-old forms. A guiding scenario: if $f \in S_k(N\ell, \mathbb{Z}_p)^{\ell\text{-new}}$ is nonzero modulo p but congruent to $g \in S_k(N\ell, \mathbb{Z}_p)^{\ell\text{-old}}$ modulo p but not modulo p^2, then $\frac{1}{p}(f-g)$ is in $S_k(N\ell, \mathbb{Z}_p)$ but not in $S_k(N\ell, \mathbb{Z}_p)^{\ell\text{-new}} \oplus S_k(N\ell, \mathbb{Z}_p)^{\ell\text{-old}}$, and the (nonzero) reduction $\bar{f}$ of f modulo p is in $S_k(N\ell, \mathbb{F}_p)^{\ell\text{-new}} \cap S_k(N\ell, \mathbb{F}_p)^{\ell\text{-old}}$.

Example 1. Take $N=5$, $\ell=3$, $p=7$, $k=4$. There is only one cuspform at level N, namely, $f = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 - 8q^6 + 6q^7 - 23q^9 + O(q^{10}) \in S_4(5, \mathbb{Z}_7)$. In level $N\ell$, there are two newforms, forming a basis of $S_4(15, \mathbb{Z}_7)$ (but not over $\mathbb{Z}$, as they are congruent modulo 2):

$$a = q + q^2 + 3q^3 - 7q^4 + 5q^5 + 3q^6 - 24q^7 - 15q^8 + 9q^9 + O(q^{10})$$
$$b = q + 3q^2 - 3q^3 + q^4 - 5q^5 - 9q^6 + 20q^7 - 21q^8 + 9q^9 - O(q^{10})$$

One can check that $b \equiv f + 2f(q^3)$ modulo 7[(iii)] and that $\frac{1}{7}(f + \frac{2}{81}W_3 f - b)$ is in $S_4(15, \mathbb{Z}_7)$ but not in $S_4(15, \mathbb{Z}_7)^{3\text{-old}} \oplus S_4(15, \mathbb{Z}_7)^{3\text{-new}}$. Modulo 7, we likewise find $\bar{b}$ in $S_4(15, \mathbb{F}_7)^{3\text{-old}} \cap S_4(15, \mathbb{F}_7)^{3\text{-new}}$.

In this section, we describe the intersection of the ℓ-old and the ℓ-new subspaces modulo p and comment on the failure of these to span the whole level-$N\ell$ space. We will fix a prime p and work with $B = \mathbb{F}_p$ or a finite extension, suppressing B from notation. We start with the following corollary to Proposition 6.4 and the first remark following:

Corollary 7.1.

(1) $S(N\ell)^{\ell\text{-old}} \cap S(N\ell)^{\ell\text{-new}}$

$$= \{ f + W_\ell(g) : \ f, g \in S(N), \ \ell T_\ell f = -(\ell+1)\, \mathcal{S}_\ell\, g, \ \ell T_\ell g = -(\ell+1) f \}.$$

(2) If $p \neq 2$, then in fixed weight k with $\lambda_k = -(\ell+1)\ell^{\frac{k-2}{2}}$, we have

$$S_k(N\ell)^{\ell\text{-old}} \cap S_k(N\ell)^{\ell\text{-new}} = V^+_{\lambda_k} \oplus V^-_{-\lambda_k},$$

where, for $\alpha \in \mathbb{F}_p$, we write $V^{\pm}_\alpha := \left\{ f \pm w_\ell f : f \in \ker\, (T_\ell - \alpha)|_{S_k(N)} \right\}$.

To offer a more detailed analysis, we will pass to generalized Hecke eigenspaces. In subsection 7.1 we recall definitions and notations for mod-p big Hecke algebras. And in subsection 7.2 we state our conclusions on the intersection of ℓ-old and ℓ-new subspaces in characteristic p.

7.1. The Hecke algebra acting on modular forms mod p.

In this section, we briefly recall the construction of the big mod-p Hecke algebra acting on $M(N) = M(N, \mathbb{F})$. For more details, see [2, 1.2] or [12, 2.3–2.5] for the construction for $N=1$, [4, Section 1] for general N.

We work over $B = \mathbb{F}$, a finite extension of $\mathbb{F}_p$. For any level N, let $A(N) = A(N, \mathbb{F})$ be the closed Hecke algebra topologically generated inside $\mathrm{End}_{\mathbb{F}}\big(M(N)\big)$ by the action of Hecke operators T_n for n prime to Np under the compact-open topology on $\mathrm{End}_{\mathbb{F}}\big(M(N)\big)$ induced by the discrete topology on $M(N)$. We write $A(N) = \underline{\mathrm{Hecke}}(M(N))$ for this contruction. This is the *big*

(iii) Indeed, the level-raising condition for f at 3 modulo 7 is satisfied, so that the existence of such a congruence is guaranteed by Diamond [5]. See also subsection 7.2.

shallow Hecke algebra acting on the space of modular forms of level N modulo p, the only kind of Hecke algebra we study here.(iv)

One can show that $A(N)$ is a complete noetherian semilocal ring that factors into a product of its localizations at its maximal ideals, which by Deligne and Serre reciprocity (formerly Serre's conjecture) correspond to Galois orbits of odd dimension-2 Chenevier pseudorepresentations $(t,d) : G_{\mathbb{Q},Np} \to \bar{\mathbb{F}}_p$, where $d = \omega_p^{\kappa-1}$ for some $\kappa \in \mathbb{Z}/(p-1)\mathbb{Z}$. Here ω_p is the mod-p cyclotomic character, and $G_{\mathbb{Q},Np}$ is the Galois group $\mathrm{Gal}(\mathbb{Q}_{Np}/\mathbb{Q})$, where $\mathbb{Q}_{Np}$ is the maximal extension of $\mathbb{Q}$ unramified outside the support of $Np\infty$. Since the d in each pseudorepresentation is entirely determined by t in this $\Gamma_0(N)$ setting (indeed, if $p > 2$ we have $d(g) = \frac{t(g)^2 - t(g^2)}{2}$ for any $g \in G_{\mathbb{Q},Np}$; and if $p = 2$ then $d = 1$), we will frequently suppress it from notation. For more on Chenevier pseudorepresentations see [3] or [2, 1.4]. If we assume that $\mathbb{F}$ is large enough to contain all the finitely many Hecke eigenvalue systems appearing in $M(N)$, then the Galois orbits become trivial; from now on we assume that this is done.

Let $K(N) \subset M(N)$ be the kernel of the U_p operator. Since U_p in characteristic p is a left inverse of the raising to the p^{th} power operator V_p, given any form $f \in M(N\ell)$ the form $g = (1 - V_pU_p)f$ has the property that $a_n(g) = a_n(f)$ unless $p \mid n$, in which case $a_n(g) = 0$. Therefore $K(N)$ is a nontrivial subspace of $M(N)$. Further, since U_p preserves the grading from (2.1), we can set $K(N)^k := K(N) \cap M(N)^k$ for $k \in \mathbb{Z}/(p-1)\mathbb{Z}$ and then $K(N) = \bigoplus_k K(N)^k$. One can show that $A(N)$ acts faithfully on $K(N)$, so that $A(N)$ is also $\underline{\mathrm{Hecke}}(K(N))$. Studying this smaller space eliminates minor complications caused by the behavior of our Hecke eigensystems at p.

For $\kappa \in \mathbb{Z}/(p-1)\mathbb{Z}$, let

$$\mathrm{PS}_\kappa(N) := \{(t,d) : G_{\mathbb{Q},Np} \to \bar{\mathbb{F}}_p \text{ odd Chenevier pseudorepresentation with } d = \omega_p^{\kappa-1}\},$$

and let $\mathrm{PS}(N) = \bigcup_{\kappa \in \mathbb{Z}/(p-1)\mathbb{Z}} \mathrm{PS}_\kappa(N)$.

By the remarks above, $\mathrm{PS}(N)$ corresponds to the set of maximal ideals of $A(N)$. Let $A(N)_t$ be the localization of $A(N)$ at the maximal ideal corresponding to $t \in \mathrm{PS}(N)$. This is a complete local noetherian ring, and we have a decomposition

$$A(N) = \prod_{t \in \mathrm{PS}(N)} A(N)_t.$$

The factorization of $A(N)$ leads to a splitting of $M(N)$ and $K(N)$ into generalized eigenspaces for $t \in \mathrm{PS}(N)$, refining the gradings on $M(N)$ and $K(N)$:

$$M(N)^\kappa = \bigoplus_{t \in \mathrm{PS}_\kappa(N)} M(N)_t \quad \text{and} \quad K(N)^\kappa = \bigoplus_{t \in \mathrm{PS}_\kappa(N)} K(N)_t.$$

7.2. **ℓ-old and ℓ-new forms restricted to eigencomponents.** We now return to working with modular forms of level $N\ell$, where ℓ is a prime not dividing Np. Recall that we work over $B = \mathbb{F}$, an extension of $\mathbb{F}_p$ containing all of the Hecke eigensystems appearing in $M(N\ell) = M(N\ell, \mathbb{F})$.

Since the operators Tr_ℓ, W_ℓ, $\mathcal{D}_\ell$ used to define the ℓ-old and ℓ-new subspaces of $M(N\ell)$, commute with Hecke operators away from ℓ, the spaces $M(N\ell)^{\ell\text{-new}}$ and $M(N\ell)^{\ell\text{-old}}$ also decompose into

(iv)One can also consider the *big partially full* Hecke algebra $A(N)^{\mathrm{pf}}$, topologically generated inside $\mathrm{End}_{\mathbb{F}}(M(N))$ by the action of T_n for all $(n, Np) = 1$ as well as U_ℓ for $\ell \mid N$, and the *big full* Hecke algebra $A(N)^{\mathrm{full}}$, which also includes the action of U_p. Many authors also consider the "smaller" algebras $A_k(N)$, $A_k(N)^{\mathrm{pf}}$, $A_k(N)^{\mathrm{full}}$ acting on forms in a single weight.

generalized eigenspaces for the various $t \in \mathrm{PS}(N\ell)$. For a Hecke module $C \subset M(N\ell)$, write $C_t := C \cap M(N\ell)_t$, so that we define $S(N\ell)_t$, $S(N\ell)_t^{\ell\text{-old}}$ and $S(N\ell)_t^{\ell\text{-new}}$.

Theorem 2. *Fix $\kappa \in 2\mathbb{Z}/(p-1)\mathbb{Z}$ (or $\kappa = 0$ if $p = 2$) and $t \in \mathrm{PS}_\kappa(N\ell)$. For k even with $k \equiv \kappa \bmod (p-1)$, let λ_k be the image of $-(\ell+1)\ell^{\frac{k-2}{2}}$ in $\mathbb{F}_p$. Note that the set $\{\pm\lambda_k\}$ depends only on κ.*

(1) If $t \in \mathrm{PS}_\kappa(N\ell) - \mathrm{PS}_\kappa(N)$ (that is, any representation carrying t is ramified at ℓ), then no forms of level N carry this eigensystem. Therefore $M(N\ell)_t^{\ell\text{-old}} = 0$ and hence $M(N\ell)_t = M(N\ell)_t^{\ell\text{-new}}$.

(2) Otherwise, $t \in \mathrm{PS}_\kappa(N)$, and we are in one of two situations:

(a) If $t(\mathrm{Frob}_\ell) \neq \pm\lambda_k$, then $M(N\ell)_t^{\ell\text{-new}} = 0$, and therefore $M(N\ell)_t = M(N\ell)_t^{\ell\text{-old}}$.

(b) If $t(\mathrm{Frob}_\ell) = \pm\lambda_k$, then all three of $M(N\ell)_t^{\ell\text{-old}}$, $M(N\ell)_t^{\ell\text{-new}}$, and $M(N\ell)_t^{\ell\text{-new}} \cap M(N\ell)_t^{\ell\text{-old}}$ are nonzero. Moreover:

(i) If $\lambda_k = 0$, then, writing $\ker T_\ell$ for $\ker T_\ell|_{S(N)_t}$, we have

$$S(N\ell)_t^{\ell\text{-old}} \cap S(N\ell)_t^{\ell\text{-new}} = \big(\ker T_\ell\big) \oplus W_\ell\big(\ker T_\ell\big).$$

(ii) If $\lambda_k \neq 0$, let $\varepsilon_k = \pm 1$ be determined by $t(\mathrm{Frob}_\ell) = \varepsilon_k\lambda_k$. Then

$$S(N\ell)_t^{\ell\text{-old}} \cap S(N\ell)_t^{\ell\text{-new}} = \big\{f - \varepsilon_k w_\ell f : f \in \ker\,(T_\ell - \varepsilon_k\lambda_k)|_{S(N)_t}\big\}.$$

In part (2(b)ii), note that $\varepsilon_k w_\ell$ depends only on κ, not on k (in other words, $\varepsilon_k w_\ell$ is well defined on $S(N\ell)_t$). It also straightforward to see that $\varepsilon_k w_\ell = (\varepsilon_k\lambda_k)\ell(\ell+1)^{-1}\,\mathcal{S}_\ell^{-1}\,W_\ell$. The statements of Theorem 2 dovetail nicely with the level-raising results [22, 5]: if f is an integral eigenform of level N and weight k whose mod-p representation is absolutely irreducible, then there is another eigenform of level $N\ell$ congruent modulo p to f (away from $N\ell p$) if and only if $a_\ell(f)^2 \equiv \lambda_k^2$ modulo p. For a level-N pseudorepresentation t mod p, we will say that the *level-raising condition is satisfied for* (t, ℓ) if $t(\mathrm{Frob}_\ell) = \pm\lambda_k$.

Proof of Theorem 2. If t does not factor through $G_{\mathbb{Q},Np}$, then there are no ℓ-old eigenforms and every form is ℓ-new: this will be true mod p because it is true over $\bar{\mathbb{Z}}_p$. So assume $t \in \mathrm{PS}_\kappa(N)$, carried by some eigenform $f \in S(N)$. If $M(N\ell)_t^{\ell\text{-new}} = \ker \mathcal{D}_\ell|_{M(N\ell)_t}$ is nonzero, then it contains an eigenform g, cuspidal after twisting by θ^{p-1} if necessary, which by assumption is also an eigenform for U_ℓ with eigenvalue $\pm\ell^{\frac{k-2}{2}}$. Since g is ℓ-old (more precisely, since g can be lifted to an ℓ-old eigenform in characteristic zero by the Deligne-Serre lifting lemma), it is the ℓ-refinement of some eigenform $f \in M_k(N)$ for some weight k, and its U_ℓ-eigenvalue is a root of $X^2 - a_\ell(f)X + \ell^{k-1} = X^2 - t(\mathrm{Frob}_\ell)X + \ell^{k-1}$. Since one root is $\pm\ell^{\frac{k-2}{2}}$, the other root is $\pm\ell(\ell^{\frac{k-2}{2}})$, so that $t(\mathrm{Frob}_\ell) = \pm(\ell+1)\ell^{\frac{k-2}{2}} = \mp\lambda_k$. This proves (2a).

For (2b): if $\lambda_k = 0$, then remark (1) after Proposition 6.4 restricted to $S(N\ell)_t$ gives us $f + W_\ell g \in S(N\ell)_t^{\ell\text{-old}} \cap S(N\ell)_t^{\ell\text{-new}}$ if and only if f and g are in $S(N)_t$ and killed by T_ℓ. If λ_k is nonzero (so $p \neq 2$), then only one of $\pm\lambda_k$, namely $\varepsilon_k\lambda_k$, appears as a T_ℓ-eigenvalue in $S(N)_t$. In particular, from the formulation in Corollary 7.1, we see that $f + w_\ell g \in S_k(N\ell)_t^{\ell\text{-old}} \cap S_k(N\ell)_t^{\ell\text{-new}}$ if and only if f is in the kernel of $T_\ell - \varepsilon_k\lambda_k$ and $g = \varepsilon_k f$. But any f and g in $S(N)_t$ appear together in some weight k. □

7.3. **The span of ℓ-old and ℓ-new forms.** If B is a field of characteristic zero, then we always have $S(N\ell, B)^{\ell\text{-new}} \oplus S(N\ell, B)^{\ell\text{-old}} = S(N\ell, B)$. But the analogous statement fails already for $B = \mathbb{Z}_p$, as $S(N\ell, B)^{\ell\text{-new}} \oplus S(N\ell, B)^{\ell\text{-old}}$ may miss congruences between ℓ-old and ℓ-new forms.

For $B = \mathbb{F}_p$ and extensions, we no longer expect a direct sum in general, but we may still ask whether ℓ-old and ℓ-new forms together span all cuspforms. To illuminate the behavior most effectively, we restrict to a generalized eigenspace for some $t \in \mathrm{PS}(N\ell)$.

To this end, fix t, let $\mathbb{F}$ be an extension of $\mathbb{F}_p$ containing its values, and let $\mathcal{O} := W(\mathbb{F})$, the unique unramified extension of $\mathbb{Z}_p$ with residue field $\mathbb{F}$. We have defined $S(N\ell, \mathbb{F})_t$ as the set of generalized eigenforms in $S(N\ell, \mathbb{F})$ for the (shallow) Hecke eigensystem carried by t. We define $S(N\ell, \mathcal{O})_t$ as the subspace of $S(N\ell, \mathcal{O})$ consisting of linear combinations of eigenforms whose corresponding shallow Hecke eigensystem is a lift of t. Unlike in characteristic p, it will no longer be true that every eigensystem is defined over $\mathcal{O}$, but if $\mathbb{F}$ is large enough to contain the values of all the elements of $\mathrm{PS}(N\ell)$, then it is still true that $S(N\ell, \mathcal{O})$ splits as a direct sum of all its generalized t-eigenspaces $S(N\ell, \mathcal{O}) = \bigoplus_{t \in \mathrm{PS}(N\ell)} S(N\ell, \mathcal{O})_t$. See [4, Section 1] for details. Similarly, we define $S(N\ell, \mathcal{O})_t^{\ell\text{-old}}$ and $S(N\ell, \mathcal{O})_t^{\ell\text{-new}}$.

Proposition 7.2. *With t, $\mathbb{F}$, $\mathcal{O}$ as above, the following are equivalent:*

(1) The action of $\mathcal{D}_\ell$ on $S(N\ell, \mathcal{O})_t^{\ell\text{-old}}$ is surjective.
(2) The intersection $S(N\ell, \mathbb{F})_t^{\ell\text{-new}} \cap S(N\ell, \mathbb{F})_t^{\ell\text{-old}}$ is trivial.
(3) $S(N\ell, \mathbb{F})_t = S(N\ell, \mathbb{F})_t^{\ell\text{-new}} \oplus S(N\ell, \mathbb{F})_t^{\ell\text{-old}}$.
(4) Either t is new at ℓ, or (t, ℓ) does not satisfy the level-raising condition.

If these equivalent conditions hold, then we additionally have

(5) $S(N\ell, \mathcal{O})_t = S(N\ell, \mathcal{O})_t^{\ell\text{-new}} \oplus S(N\ell, \mathcal{O})_t^{\ell\text{-old}}$.

Finally, if t is absolutely irreducible[(v)]*, then (1), (2), (4), (3) and (5) are all equivalent.*

Proof. The equivalence of (2), (4) and (3) follows from Theorem 2.

We demonstrate (1) $\iff$ (2): Since $S(N\ell, \mathcal{O})_t^{\ell\text{-old}}$ breaks up into a graded sum of its fixed-weight pieces, and since $\mathcal{D}_\ell$ is weight-preserving, surjectivity on $S(N\ell, \mathcal{O})_t^{\ell\text{-old}}$ is equivalent to surjectivity on $S_k(N\ell, \mathcal{O})_t^{\ell\text{-old}}$. By right-exactness of tensoring or Nakayama's lemma (depending on the direction) this last is equivalent to surjectivity on $S_k(N\ell, \mathbb{F})_t^{\ell\text{-old}}$. This space is a finite-dimensional vector space, so $\mathcal{D}_\ell$ acts surjectively if and only if it has trivial kernel, which is equivalent by definition to $S_k(N\ell, \mathbb{F})_t^{\ell\text{-old}} \cap S_k(N\ell, \mathbb{F})_t^{\ell\text{-new}} = \{0\}$. Finally trivial intersection in all finite weights k is equivalent to trivial intersection of $S(N\ell, \mathbb{F})_t^{\ell\text{-new}}$ and $S(N\ell, \mathbb{F})_t^{\ell\text{-old}}$.

Now (1) $\implies$ (5): The surjectivity on $S(N\ell, \mathcal{O})_t^{\ell\text{-old}}$ implies the that for both $B = \mathcal{O}$ and $B = \mathbb{F}$, the following sequence is split exact.

$$0 \to S(N\ell, B)_t^{\ell\text{-new}} \to S(N\ell, B)_t \xrightarrow{\mathcal{D}_\ell} S(N\ell, B_t)^{\ell\text{-old}} \to 0,$$

which means that $S(N\ell, B)_t = S(N\ell, B)_t^{\ell\text{-old}} \oplus S(N\ell, B)_t^{\ell\text{-new}}$.

Finally, if t is absolutely irreducible, then the level-raising theorems [22, 5] hold. Therefore if $t \in \mathrm{PS}(N)$ and (t, ℓ) satisfies the level-raising condition, then there exists an ℓ-new form congruent to an ℓ-old form (over some extension of $\mathcal{O}$), which implies that

$$S(N\ell, \mathcal{O})_t \supsetneq S(N\ell, \mathcal{O})_t^{\ell\text{-new}} \oplus S(N\ell, \mathcal{O})_t^{\ell\text{-old}}.$$

□

[(v)]That is, t is not the sum of two characters $G_{\mathbb{Q}, N\ell p} \to \bar{\mathbb{F}}_p$.

Question 2. Is it always true that $S(N\ell,\mathbb{F}_p)_t^{\ell\text{-new}} + S(N\ell,\mathbb{F}_p)_t^{\ell\text{-old}} = S(N\ell,\mathbb{F}_p)_t$? A positive answer would furnish additional support for the present definition of ℓ-new forms.

8. Hecke-stable filtrations mod p

In this section we describe a filtration for the space of modular forms of level $N\ell$ modulo p, and compare it to the filtration described by Monsky in [16, 17], which appears if $\ell \equiv -1$ modulo p. We assume that $B = \mathbb{F}$, a finite extension of $\mathbb{F}_p$ big enough to contain all mod-p eigensystems, throughout, and suppress B from notation.

8.1. **The standard filtration (after Paul Monsky).** For simplicity, we will restrict to the kernel of the U_p operator $K(N\ell) \subset M(N\ell)$, where formulas are simpler but no Hecke eigensystem information is lost. See also subsection 7.1 and subsection 7.2 for additional notation. Then $K(N\ell)$ contains two subspaces

$$K(N\ell)^{\ell\text{-old}} = K(N) \oplus W_\ell K(N) \quad \text{and} \quad K(N\ell)^{\ell\text{-new}} := \ker \mathcal{D}_\ell = \ker \mathrm{Tr}_\ell \cap \mathrm{Tr}_\ell\, W_\ell.$$

Here the action of all operators is restricted to $K(N\ell)$, so that $\ker \mathcal{D}_\ell = \ker \mathcal{D}_\ell|_{K(N\ell)}$, etc.

The Hecke algebra $A(N\ell) = \underline{\mathrm{Hecke}}\left(K(N\ell)\right)$ has quotients $A(N\ell)^{\ell\text{-new}} := \underline{\mathrm{Hecke}}\left(K(N\ell)^{\ell\text{-new}}\right)$ and

$$A(N)^{\ell\text{-old}} := \underline{\mathrm{Hecke}}\left(K(N\ell)^{\ell\text{-old}}\right) \cong \underline{\mathrm{Hecke}}\left(K(N)\right) = A(N).$$

To study the Hecke structure on $K(N\ell)$ more closely, we consider the following filtration by Hecke-invariant submodules, which we'll call the *standard filtration*:

$$0 \subset K(N\ell)^{\ell\text{-new}} \subset \ker \mathrm{Tr}_\ell \subset K(N\ell). \tag{8.1}$$

For any $t \in \mathrm{PS}(N\ell)$, we can pass to the sequence on the t-eigenspace:

$$0 \subset K(N\ell)_t^{\ell\text{-new}} \subset (\ker \mathrm{Tr}_\ell)_t \subset K(N\ell)_t. \tag{8.2}$$

We also consider the following two conditions relative to a pseudorepresentation $t \in \mathrm{PS}(N)$ and a Hecke operator $T \in A(N)_t$.

Condition *Surj*(t,T): Operator $T \in A(N)_t$ acts surjectively on $K(N)_t$.
Condition *NZDiv*(t,T): Element $0 \neq T \in A(N)_t$ is not a zero divisor on $K(N)_t$.

Note that *Surj*(t,T) implies *NZDiv*(t,T): suppose $TK(N)_t = K(N)_t$, and suppose there exists $T' \in A(N)_t$ with $T'T = 0$. Then T' annihilates $K(N)_t$; since the action of $A(N)_t$ is faithful, we must have $T' = 0$. Both conditions are satisfied if $A(N)_t$ is a regular local $\mathbb{F}$-algebra of dimension 2.[vi] See section subsection 8.3 below for more details.

We are now ready to analyze the standard filtration (8.2).

If $t \in \mathrm{PS}(N\ell)\setminus \mathrm{PS}(N)$, then $K(N\ell)_t = K(N\ell)_t^{\ell\text{-new}}$, so that the filtration stabilizes; clearly then $A(N\ell)_t = A(N\ell)_t^{\ell\text{-new}}$. For the rest of this section, assume that $t \in \mathrm{PS}_\kappa(N)$. Recall that λ_k is the image of $-(\ell+1)\ell^{\frac{k-2}{2}}$ in $\mathbb{F}_p$.

Proposition 8.1. *Suppose that* $t \in \mathrm{PS}_\kappa(N)$.

[vi] It's not unreasonable to expect that this is always the case for $N = 1$. No counterexamples are known; for reducible $t \in \mathrm{PS}(1)$, Vandiver's conjecture implies that $A(1)_t$ is a regular local ring of dimension 2: see [2, §10].

(1) *If* $\left\{\begin{array}{l}\textit{EITHER } \ell \not\equiv -1 \textit{ modulo } p, \\ \textit{OR } \ell \equiv -1 \textit{ modulo } p \textit{ and Surj}(t, T_\ell) \textit{ holds}\end{array}\right\}$, *then*

$$K(N\ell)_t/(\ker \mathrm{Tr}_\ell) \cong K(N)_t.$$

(2) *If* $\left\{\begin{array}{l}\textit{EITHER } \ell \not\equiv -1 \textit{ mod } p \textit{ and Surj}(t, T_\ell^2 - \lambda_k^2) \textit{ holds} \\ \textit{OR } \ell \equiv -1 \textit{ mod } p \textit{ and Surj}(t, T_\ell) \textit{ holds}\end{array}\right\}$, *then*

$$(\ker \mathrm{Tr}_\ell)/K(N\ell)_t^{\ell\text{-new}} \cong K(N)_t.$$

In other words, under regularity conditions on $A(N)_t$, the Hecke algebras acting on the graded pieces of the standard filtration are one copy of $A(N\ell)_t^{\ell\text{-new}}$ and two copies of $A(N\ell)_t^{\ell\text{-old}}$. Note that $K(N\ell)_t^{\ell\text{-new}}$ and $A(N\ell)_t^{\ell\text{-new}}$ will be zero if the level-raising condition for (t, ℓ) is not satisfied.

Proof. For part (1), we show that under the given conditions, the sequence

$$0 \to (\ker \mathrm{Tr}_\ell)_t \to K(N\ell)_t \xrightarrow{\mathrm{Tr}_\ell} K(N)_t \to 0$$

is exact. On the left, exactness is by definition. On the right, if $\ell \neq -1$ modulo p then for any $f \in K(N)$ we have $\mathrm{Tr}_\ell(f) = (\ell+1)f$, which spans $\langle f \rangle_{\mathbb{F}}$. Otherwise, $\mathrm{Tr}_\ell W_\ell f = \ell T_\ell(f)$, so condition *Surj*$(t, T_\ell)$ suffices.

For part (2), we establish the exactness of

$$0 \to K(N\ell)_t^{\ell\text{-new}} \to (\ker \mathrm{Tr}_\ell)_t \xrightarrow{\mathrm{Tr}_\ell W_\ell} K(N)_t \to 0. \tag{8.3}$$

Again, left exactness holds since $K(N\ell)^{\ell\text{-new}} = \ker \mathrm{Tr}_\ell \cap \ker \mathrm{Tr}_\ell W_\ell$. For right exactness, if $\ell \equiv -1 \mod p$, then $K(N)_t \subset \ker \mathrm{Tr}_\ell$, and then $\mathrm{Tr}_\ell W_\ell f = \ell T_\ell f$ for any $f \in K(N)_t$. Otherwise use the computations of Proposition 6.4 to see that $g = T_\ell f - (\ell+1)/\ell W_\ell f$ is in $\ker \mathrm{Tr}_\ell$, and then $\mathrm{Tr}_\ell W_\ell(\ell^{-1} g) = (T_\ell^2 - \lambda_k^2) f$. ☐

Corollary 8.2. *If $t \in \mathrm{PS}_\kappa(N)$ and both Surj(t, T_ℓ) and Surj$(t, T_\ell^2 - \lambda_k^2)$ hold, then the graded pieces associated to the standard filtration of $K(N\ell)_t$ are isomorphic to two copies of $K(N)_t$ and one copy of $K(N\ell)_t^{\ell\text{-new}}$. The corresponding Hecke algebras are $A(N)_t$, $A(N)_t$, and $A(N\ell)_t^{\ell\text{-new}}$.*

Note that if the level-raising condition for (t, ℓ) is not satisfied, then both $K(N\ell)_t^{\ell\text{-new}} = 0$ and $A(N\ell)_t^{\ell\text{-new}} = 0$; both Proposition 8.1 and Corollary 8.2 hold.

We can in fact slightly relax the assumptions of Corollary 8.2:

Proposition 8.3. *If $t \in \mathrm{PS}_\kappa(N)$, and both NZDiv$(t, T_\ell)$ and NZDiv$(t, T_\ell^2 - \lambda_k^2)$ hold, then the Hecke algebras on graded pieces of the standard filtration are two copies of $A(N)_t$ and one copy of $A(N\ell)_t^{\ell\text{-new}}$.*

Proof. From the proof of Proposition 8.1, we see that $K(N\ell)_t/(\ker \mathrm{Tr}_\ell)_t$ is isomorphic to a Hecke module that sits between $T_\ell K(N)_t$ and $K(N)_t$. If T_ℓ is not a zero divisor on $K(N)_t$, then $A(N)_t$ acts faithfully on $T_\ell K(N)_t$: indeed, if any $T \in A(N)_t$ annihilates $T_\ell K(N)_t$, then TT_ℓ annihilates $K(N)_t$. Therefore the Hecke algebra on $T_\ell K(N)_t$, and hence on $K(N\ell)_t/(\ker \mathrm{Tr}_\ell)_t$, is still $A(N)_t$. The reasoning for the Hecke algebra on $(\ker \mathrm{Tr}_\ell)_t/K(N\ell)_t^{\ell\text{-new}}$ is analogous. ☐

8.2. **Connection to the Monsky filtration.** In [16] and [17], Monsky studies $K(N\ell)$ and related Hecke algebras in the case $p=2$, $N=1$ and $\ell=3,5$. For $p=2$, there is only one $t\in \mathrm{PS}(1)$, namely $t=0$, the trace of the trivial representation. Monsky describes a different filtration of $K(\ell)=K(\ell)_0$ by Hecke-invariant subspaces, and proves that the Hecke algebras on the graded pieces are two copies of $A(1)$ plus a third "new" Hecke algebra. The goal of this section is to compare the Monsky filtration to the standard filtration from subsection 8.1, and to establish that the "new" Monsky Hecke algebra coincides with $A(\ell)^{\mathrm{new}}$ defined here. The Monsky filtration exists more generally, so long as the level ℓ is congruent to -1 modulo p. As in the previous section, we will assume regularity conditions on t (namely, $\mathit{Surj}(t,T_\ell)$), guaranteed in Monsky's $p=2$ case by work of Nicolas and Serre [19] (via Lemma 8.5).

Fix a $t\in \mathrm{PS}(N)$, and let $\mathbb{F}/\mathbb{F}_p$ be an extension containing the image of t. Fix a prime ℓ congruent to -1 modulo p. Then we have the following filtration of $K(N\ell)_t$ by Hecke-invariant subspaces, due to Monsky [16, remark p. 5][(vii)]:

$$0\subset K(N)_t\subset (\ker \mathrm{Tr}_\ell)_t\subset K(N\ell)_t. \tag{8.4}$$

Indeed, if $\ell+1=0$ in $\mathbb{F}_p$, then $\mathrm{Tr}_\ell\big(K(N)\big)=0$, so that $(\ker \mathrm{Tr}_\ell)_t$ contains $K(N)_t$.

As in Proposition 8.1(1), if $\mathit{Surj}(t,T_\ell)$ holds, then the sequence

$$0\to \ker \mathrm{Tr}_\ell\to K(N\ell)_t\xrightarrow{\mathrm{Tr}_\ell} K(N)_t\to 0$$

is exact. Therefore, the Hecke algebra on $K(N\ell)_t/(\ker \mathrm{Tr}_\ell)_t$ is isomorphic to $A(N)_t$.[(viii)] Clearly, the Hecke algebra on $K(N)_t$ is $A(N)_t$ as well.

Let $K(N\ell)_t^{\mathrm{Monsky}}$ be the Hecke module $(\ker \mathrm{Tr}_\ell)_t/K(N)_t$, and $A(N\ell)_t^{\mathrm{Monsky}}$ be the Hecke algebra on $K(N\ell)_t^{\mathrm{Monsky}}$.

Proposition 8.4. *Suppose $\ell\equiv -1 \pmod p$ and $\mathit{Surj}(t,T_\ell)$ holds. The sequence*

$$0\to \ker T_\ell|_{K(N)_t}\to K(N\ell)_t^{\ell\text{-new}}\to K(N\ell)_t^{\mathrm{Monsky}}\to 0$$

is exact, and induces an isomorphism of Hecke algebras $A(N\ell)_t^{\ell\text{-new}}\cong A(N\ell)_t^{\mathrm{Monsky}}$.

Proof. Denote $\ker T_\ell|_{K(N)_t}$ by $(\ker T_\ell)_{N,t}$ below. We compare the exact sequences of the middle-graded piece of the Monsky filtration to the same from the standard filtration:

(vii) The filtration that appers in Monsky's work is actually conjugated by W_ℓ, namely:

$$0\subset W_\ell K(1)\subset \ker W_\ell \mathrm{Tr}_\ell W_\ell\subset K(\ell),$$

where the second-to-last term is the kernel of the map $W_\ell \mathrm{Tr}_\ell W_\ell : K(\ell)\to W_\ell K(1)$.

(viii) As in Proposition 8.3, condition $\mathit{NZDiv}(t,T_\ell)$ suffices for the Hecke algebra conclusion.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \dashrightarrow & (\ker T_\ell)_{N,t} & \dashrightarrow & K(N)_t & \overset{\ell T_\ell}{\dashrightarrow} & K(N)_t & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K(N\ell)_t^{\ell\text{-new}} & \longrightarrow & (\ker \mathrm{Tr}_\ell)_t & \xrightarrow{\mathrm{Tr}_\ell W_\ell} & K(N)_t & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & K(N\ell)_t^{\ell\text{-new}}/(\ker T_\ell)_{N,t} & & K(N\ell)_t^{\mathrm{Monsky}} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Here the Monsky sequence is vertical with solid arrows and the standard sequence (8.3) is horizontal with solid arrows. The inclusion $K(N)_t \hookrightarrow (\ker \mathrm{Tr}_\ell)_t$ from the Monsky sequence induces the upper horizontal exact sequence; note that the map $\mathrm{Tr}_\ell W_\ell$ restricted to $K(N)_t$ coincides with ℓT_ℓ. Finally, the snake lemma on the resulting two horizontal short exact sequences gives us a natural isomorphism that we unpack as a short exact sequence below:

$$0 \to (\ker T_\ell)_{N,t} \to K(N\ell)_t^{\ell\text{-new}} \to K(N\ell)_t^{\mathrm{Monsky}} \to 0.$$

The first map is natural inclusion; the second map is the composition

$$K(N\ell)^{\ell\text{-new}} \hookrightarrow (\ker \mathrm{Tr}_\ell)_t \to (\ker \mathrm{Tr}_\ell)_t / K(N)_t = K(N\ell)_t^{\mathrm{Monsky}}.$$

To see that the induced surjection on Hecke algebras $A(N\ell)_t^{\ell\text{-new}} \overset{\alpha}{\twoheadrightarrow} A(N\ell)_t^{\mathrm{Monsky}}$ is an isomorphism, we have to see that $A(N\ell)_t^{\ell\text{-new}}$ acts faithfully on $K(N\ell)_t^{\mathrm{Monsky}}$. If $T = 0$ in $A(N\ell)_t^{\mathrm{Monsky}}$, then T sends $K(N\ell)_t^{\ell\text{-new}}$ to $(\ker T_\ell)_{N,t}$. Since T commutes with W_ℓ, it must also send $K(N\ell)_t^{\ell\text{-new}} = W_\ell K(N\ell)_t^{\ell\text{-new}}$ to $W_\ell(\ker T_\ell)_t$. Since $K(N)$ and $W_\ell K(N)$ are disjoint, T must in fact annihilate all of $K(N\ell)_t^{\ell\text{-new}}$: that is, $T = 0$ in $A(N\ell)_t^{\ell\text{-new}}$. □

8.3. **Regularity conditions on the Hecke algebra $A(1)_t$.** In this section we prove:

Lemma 8.5. *If $A(1)_t \cong \mathbb{F}[\![x,y]\!]$, the action of any nonzero $T \in A(1)_t$ is surjective on $K(1)_t$.*

Note that $A(1)_t \cong \mathbb{F}[\![x,y]\!]$ if t is *unobstructed* in the sense of deformation theory. See [19] for $p = 2$, [2] for $p \geq 5$, [13] for $p = 3$, and [12] for more discussion of $p = 2, 3, 5, 7, 13$.

Proof of Lemma 8.5. In level one, we have a perfect continuous duality between $A(1)$ and $K(1)$ as $A(1)$-modules under the pairing $A(1)_t \times K(1)_t \to \mathbb{F}$ given by $\langle T, f\rangle := a_1(Tf)$. Therefore, we may choose a basis $\{m(a,b)\}_{a\geq 0, b\geq 0}$ of $K(1)$ dual to the "Hilbert basis" $\{x^a y^b\}$: more precisely, one which satisfies $x \cdot m(0,b) = y \cdot m(a,0) = 0$ for all a, b, and $x \cdot m(a,b) = m(a-1,b)$ for $a \geq 1$ and $y \cdot m(a,b) = m(a, b-1)$ if $b \geq 1$.

We introduce a total order on pairs of nonnegative integers: we'll say that $(a,b) \prec (c,d)$ if $a + b < c + d$, or if $a + b = c + d$ and $b < d$. (In fact any total order will do.) Suppose

$$T = \sum_{a+b=k} c_{a,b} x^a y^b + O\big((x,y)^{k+1}\big) \in \mathbb{F}[\![x,y]\!]$$

for some $k \geq 0$. Let (a_0, b_0) be the $\prec$-minimal pair among all the pairs (a,b) with $c_{a,b}$ nonzero; by scaling T if necessary, we may assume that $c_{a_0,b_0} = 1$. For example, if $T_\ell = 5x^2y - y^3 + O\big((x,y)^4\big)$, then $(a_0, b_0) = (2, 1)$. We induct on $\prec$ to show that $m(a, b)$ is in the image of T for any pair (a, b). It's clear that $T \cdot m(a_0, b_0) = m(0, 0)$: base case. For the inductive step, suppose that the vector space $V_{a,b} = \langle m(c, d) : (c, d) \prec (a, b)\rangle_{\mathbb{F}}$ is in the image of T already. Since $T \cdot m(a + a_0, b + b_0)$ is in $m(a, b) + V_{a,b}$, in fact $m(a, b)$ is in the image of T as well. □

Question 3. Can one prove a similar statement for $A(N)_t$ if it is not a power series ring? At the very least, can one show that condition $NZDiv(t, T)$ is satisfied?

Appendix A. The Atkin-Lehner automorphism mod p geometrically (Alexandru Ghitza)

Our aim is to describe a geometric construction of the modified Atkin-Lehner automorphism W_ℓ on the algebra of modular forms $M(N\ell, \mathbb{F}_p)$. This will be an intrinsic characteristic p construction, stemming from an automorphism of the Igusa curve.

A.1. **Classical Atkin-Lehner via geometry.** Let's start by recalling the geometric construction of the Atkin-Lehner operator w_ℓ, following Conrad [21].

Let ℓ be a prime and N a positive integer coprime to ℓ. The noncuspidal points on the modular curve $X_0(N\ell)$ have the moduli interpretation

$$(E; C_\ell, C_N) \quad \text{with } E \text{ an elliptic curve, } C_j \text{ cyclic subgroup of order } j.$$

We define an involution $w_\ell \colon Y_0(N\ell) \to Y_0(N\ell)$ by

$$(E; C_\ell, C_N) \mapsto (\phi(E); \phi(E[\ell]), \phi(C_\ell + C_N)),$$

where $\phi\colon E \to E/C_\ell$ is the quotient isogeny.

Conrad explains in what sense this involution can be extended to the cusps of $Y_0(N\ell)$, and shows that over $\mathbb{C}$, this construction yields the classical Atkin-Lehner involution on $M_k(N\ell)$. He also proves that, if $f(q) \in \mathbb{Z}[\frac{1}{\ell}][\![q]\!]$, then $(w_\ell f)(q) \in \mathbb{Z}[\frac{1}{\ell}][\![q]\!]$, from which we get the Atkin-Lehner involution w_ℓ on modular forms mod p for any prime $p \neq \ell$.

As our setup is simpler (having the extra assumption that $p \nmid N$), we think of the classical mod p Atkin-Lehner involution as coming directly from the map $w_\ell \colon Y_0(N\ell)_{\mathbb{F}_p} \to Y_0(N\ell)_{\mathbb{F}_p}$:

$$(E; C_\ell, C_N) \mapsto (\phi(E); \phi(E[\ell]), \phi(C_\ell + C_N)),$$

where E is an elliptic curve in characteristic p and $\phi\colon E \to E/C_\ell$ is the quotient isogeny. More explicitly, if $f \in M_k(N\ell, \mathbb{F}_p)$ and ω is a nonzero invariant differential on E, we have

$$(w_\ell f)(E; C_\ell, C_N, \omega) = f\left(\phi(E); \phi(E[\ell]), \phi(C_\ell + C_N), \widehat{\phi}^*\omega\right),$$

where $\widehat{\phi}\colon E/C_\ell \to E$ is the dual isogeny to $\phi\colon E \to E/C_\ell$.

A.2. **The Igusa curve $I_0(N\ell)$.** We summarize the features of Igusa curves that are essential to our construction. We follow mainly Gross's exposition in [9, Section 5], which develops the theory for $\Gamma_1(N)$-structure; this can be adapted to our $\Gamma_0(N)$ situation with minor changes, as summarized in [9, Section 10]. A thorough study of Igusa curves appears in [10, Chapter 12], however without treatment of modular forms. The $\Gamma_0(1)$ case is described briefly by Serre in [24, end of p. 416-05]; see also the discussion in MathOverflow question 93059.

Note that when $p = 2$ we have $W_\ell = w_\ell$, the classical Atkin-Lehner automorphism. We will henceforth assume that $p \geq 3$.

Consider a prime $p \neq \ell$ and coprime to N. Given an elliptic curve E in characteristic p, there are morphisms Frobenius $F\colon E \to E^{(p)}$ and Verschiebung $V\colon E^{(p)} \to E$ such that $V \circ F = [p]\colon E \to E$ and a canonical short exact sequence of group schemes

$$0 \to \ker F \to E[p] \xrightarrow{F} \ker V \to 0.$$

An Igusa structure of level p on E is a choice of generator of (the Cartier divisor) $\ker V$. This is equivalent to choosing a surjective morphism of group schemes $E[p] \to \ker V$, or (by Cartier duality) to choosing an embedding of group schemes $(\ker V)^* \hookrightarrow E[p]$. We can be more precise by distinguishing the two cases:

- If E is ordinary, then $\ker V \cong \mathbb{Z}/p\mathbb{Z}$ and $(\ker V)^* \cong \mu_p$ so an Igusa structure is an embedding $\mu_p \hookrightarrow E[p]$.
- If E is supersingular, then $\ker V \cong \alpha_p$ and $(\ker V)^* \cong \alpha_p$ so an Igusa structure is an embedding $\alpha_p \hookrightarrow E[p]$. In fact, there is a unique such embedding (see [8, Example 3.14]).

If we restrict our attention to ordinary elliptic curves E, the moduli problem defined by the data

$$\left(E; C_\ell, C_N, \mu_p \overset{i_p}{\hookrightarrow} E[p]\right)$$

is representable (as we assume $p \geq 3$) by an affine curve $I_0(N\ell)^{\mathrm{ord}}$ whose coordinate ring we denote $S(N\ell)$. It has a natural smooth compactification $I_0(N\ell)$ with a canonical map $\pi\colon I_0(N\ell) \to X_0(N\ell)_{\mathbb{F}_p}$ that is totally ramified over the supersingular points. It can be thought of as quotienting by the automorphism group $(\mathbb{Z}/p\mathbb{Z})^\times/(\pm 1)$, which acts freely on $I_0(N\ell)^{\mathrm{ord}}$ via

$$\langle d\rangle_p \left(E; C_\ell, C_N, \mu_p \overset{i_p}{\hookrightarrow} E[p]\right) = \left(E; C_\ell, C_N, \mu_p \overset{i_p}{\hookrightarrow} E[p] \xrightarrow{[d]} E[p]\right).$$

This defines a grading on the algebra of functions

$$S(N\ell) = \bigoplus_\alpha S_\alpha(N\ell),$$

where $S_\alpha(N\ell)$ consists of the functions on $I_0(N\ell)^{\mathrm{ord}}$ that satisfy $\langle d\rangle_p g = d^\alpha g$ for all $d \in (\mathbb{Z}/p\mathbb{Z})^\times$.

The line bundle $\underline{\omega}^{\otimes 2} := \Omega^1(\text{cusps})$ on $X_0(N\ell)_{\mathbb{F}_p}$ pulls back to a line bundle $\pi^*\underline{\omega}^{\otimes 2}$ on $I_0(N\ell)$. It is equipped with a canonical section a^2 with the following properties[(ix)]:

(1) a^2 is non-vanishing on $I_0(N\ell)^{\mathrm{ord}}$, and has simple zeros at the supersingular points;
(2) $a^{p-1} = \pi^* A$, where $A \in M_{p-1}(1, \mathbb{F}_p)$ is the Hasse invariant;
(3) a^2 has q-expansion $a^2(q) = 1 \in \mathbb{F}_p[\![q]\!]$;
(4) $\langle d\rangle_p(a^2) = d^{-2}a^2$ for all $d \in (\mathbb{Z}/p\mathbb{Z})^\times$.

(This is the Γ_0-analogue of the Γ_1 result in [9, Proposition 5.2], see also [9, Section 10].)

We use the section a^2 to trivialize the line bundle $\pi^*\underline{\omega}^{\otimes 2}$. This allows us to treat *sections* of $\underline{\omega}^{\otimes k}$ on $X_0(N\ell)_{\mathbb{F}_p}$ as *functions* on the ordinary locus $I_0(N\ell)^{\mathrm{ord}}$. More precisely, the q-expansion

(ix) We abuse notation by writing a^2 even though there is no a itself for Γ_0-structures; so whenever we write a^k we implicitly assume that k is even and we set $a^k := (a^2)^{(k/2)}$.

map gives an isomorphism of graded $\mathbb{F}_p$-algebras

$$\Phi\colon S(N\ell) \xrightarrow{\cong} M(N\ell, \mathbb{F}_p) \subset \mathbb{F}_p[\![q]\!].$$

To see that the image of Φ is contained in $M(N\ell, \mathbb{F}_p)$, let $g \in S_\alpha(N\ell)$ and let $k \equiv \alpha \pmod{p-1}$ be such that $a^k g$ is regular on $I_0(N\ell)$. Since

$$\langle d\rangle_p(a^k g) = d^{-k}a^k d^\alpha g = d^{\alpha-k}(a^k g) = a^k g,$$

we see that $a^k g$ descends to a global section $f \in M_k(N\ell, \mathbb{F}_p)$, and $g(q) = f(q) \in M(N\ell, \mathbb{F}_p)^\alpha$.

For the inverse map: given $f(q) \in M(N\ell, \mathbb{F}_p)^\alpha$, let $f \in M_k(N\ell, \mathbb{F}_p)$ be any modular form with q-expansion $f(q)$, and let

$$g = \frac{\pi^* f}{a^k}$$

Then g is a function on $I_0(N\ell)^{\mathrm{ord}}$ with $\langle d\rangle_p g = d^\alpha g$ and $g(q) = f(q)$.

A.3. **From maps on the Igusa curve to operators on modular forms mod p.** A morphism $\psi\colon I_0(N\ell)^{\mathrm{ord}} \to I_0(N\ell)^{\mathrm{ord}}$ on the ordinary locus of the Igusa curve determines a homomorphism of graded $\mathbb{F}_p$-algebras $\Psi\colon M(N\ell, \mathbb{F}_p) \to M(N\ell, \mathbb{F}_p)$ by setting

$$\Psi = \Phi \circ \psi^* \circ \Phi^{-1},$$

where, given $g \in S(N\ell)$, $\psi^* g = g \circ \psi \in S(N\ell)$.

For example, if we take $\psi = \langle \ell\rangle_p$ then Ψ is the weight-separating automorphism $\mathcal{S}_\ell$ defined in subsection 2.2. To see this, let $f(q) \in M(N\ell, \mathbb{F}_p)^\alpha$ and let f be a modular form of weight $k \equiv \alpha \pmod{p-1}$ with q-expansion $f(q)$; we have

$$\Psi(f(q)) = \Phi\left(\langle \ell\rangle_p\left(\frac{\pi^* f}{a^k}\right)\right) = \Phi\left(\frac{\pi^* f}{(\ell^{-2}a^2)^{k/2}}\right) = \ell^k \Phi\left(\frac{\pi^* f}{a^k}\right) = \ell^\alpha f(q) = \mathcal{S}_\ell\, f(q).$$

In order to recover the modified Atkin-Lehner automorphism W_ℓ defined in subsection 3.4, we start with the map $\widetilde{w}_\ell\colon I_0(N\ell)^{\mathrm{ord}} \to I_0(N\ell)^{\mathrm{ord}}$ given by

$$\widetilde{w}_\ell\,(E; C_\ell, C_N, i_p) = (\phi(E); \phi(E[\ell]), \phi(C_\ell + C_N), \phi \circ i_p)\,,$$

where $\phi\colon E \to E/C_\ell$ is the quotient isogeny. Since

$$\widetilde{w}_\ell^2\,(E; C_\ell, C_N, i_p) = \left(E; C_\ell, C_N, \widehat{\phi} \circ \phi \circ i_p\right),$$

we conclude that $\widetilde{w}_\ell^2 = \langle \ell\rangle_p$.

We denote the corresponding algebra homomorphism Ψ resulting from $\psi = \widetilde{w}_\ell$ by $\widetilde{W}_\ell$, so that

$$\widetilde{W}_\ell = \Phi \circ \widetilde{w}_\ell^* \circ \Phi^{-1}.$$

Lemma A.1. *For any modular form f of level $\Gamma_0(N\ell)$ we have $\widetilde{w}_\ell^*(\pi^* f) = \pi^*(w_\ell f)$.*

Proof. This is a simple calculation on the moduli:

$$\begin{aligned}
\widetilde{w}_\ell^*(\pi^* f)(E; C_\ell, C_N, i_p, \omega) &= (\pi^* f)(\phi(E); \phi(E[\ell]), \phi(C_\ell + C_N), \phi \circ i_p, \widehat{\phi}^*\omega) \\
&= f(\phi(E); \phi(E[\ell]), \phi(C_\ell + C_N), \widehat{\phi}^*\omega) \\
&= (w_\ell f)(E; C_\ell, C_N, \omega) \\
&= \pi^*(w_\ell f)(E; C_\ell, C_N, i_p, \omega)
\end{aligned}$$

□

Lemma A.2. $\widetilde{w}_\ell^*(a^2) = \ell^{-1}a^2$ *as elements of* $H^0(I_0(N\ell), \pi^*\underline{\omega}^{\otimes 2})$.

Proof. We temporarily pass to $\Gamma_1(N\ell)$-structures and work with the corresponding Igusa covering $\pi\colon I_1(N\ell) \to X_1(N\ell)_{\mathbb{F}_p}$ of degree $p-1$. This setting has the advantage that we obtain a $(p-1)$-st root a_1 of the Hasse invariant A, as a canonical section of the line bundle $\pi^*\underline{\omega}$, as detailed in [9, Proposition 5.2]. Given a choice of ℓ-th root of unity ζ, [9, Section 6] defines an automorphism w_ζ of $X_1(N\ell)$ by giving a modular recipe

$$w_\zeta(E;\beta_\ell,\alpha_N) = (\phi(E);\phi(\beta_\ell),\phi(\alpha_N)),$$

where $\phi(\alpha_N) = \phi\circ\alpha_N\colon \mu_N \hookrightarrow \phi(E)[N]$. The definition of $\phi(\beta_\ell)$ is more intricate, and involves the choice of ℓ-th root of unity ζ. If $e\colon E[\ell]\times E[\ell] \to \mu_\ell$ denote the Weil pairing, there is a unique $P_\beta \in E[\ell]/\beta_\ell(\mu_\ell) = \phi(E[\ell])$ such that

$$e(\beta_\ell(z), P_\beta) = z \qquad \text{for all } z\in\mu_\ell.$$

Let $\phi(\beta_\ell)\colon \mu_\ell \to \phi(E)[\ell]$ be defined by $\phi(\beta_\ell)(\zeta) = \phi(P_\beta)$. Note that $\phi(\beta_\ell)(\mu_\ell) = \phi(E[\ell])$.

We can adapt this into an automorphism $\widetilde{w}_\zeta$ of $I_1(N\ell)^{\mathrm{ord}}$ by setting

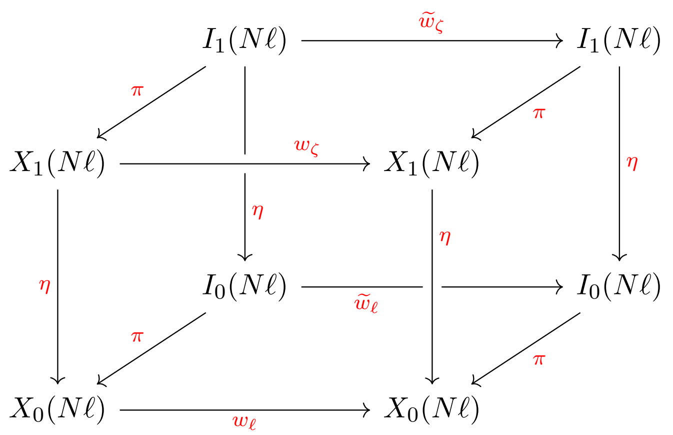
$$\widetilde{w}_\zeta(E;\beta_\ell,\alpha_N,i_p) = (\phi(E);\phi(\beta_\ell),\phi(\alpha_N),\phi(i_p))$$

where $\phi(i_p) = \phi\circ i_p\colon \mu_p \hookrightarrow \phi(E)[p]$.

We illustrate the various spaces and maps in the following cube diagram whose commutativity is readily checked via calculations similar to that in Lemma A.1, using the moduli interpretation of the covering maps $\eta\colon I_1(N\ell)\to I_0(N\ell)$:

$$\eta(E;\beta_\ell,\alpha_N,i_p) = (E;\beta_\ell(\mu_\ell),\alpha_N(\mu_N),i_p)$$

and similarly for $\eta\colon X_1(N\ell)\to X_0(N\ell)$.



Our immediate interest is in the back face of the cube, so we spell out its commutativity:

$$\begin{aligned}\eta\circ\widetilde{w}_\zeta(E;\beta_\ell,\alpha_N,i_p) &= \big(\phi(E);\phi(\beta_\ell)(\mu_\ell),\phi(\alpha_N)(\mu_N),\phi(i_p)\big)\\ \widetilde{w}_\ell\circ\eta(E;\beta_\ell,\alpha_N,i_p) &= \big(\phi(E);\phi(E[\ell]),\phi\big(\beta_\ell(\mu_\ell)+\alpha_N(\mu_N)\big),\phi(i_p)\big),\end{aligned}$$

where $\phi\colon E\to E/\beta_\ell(\mu_\ell)$ is the quotient isogeny. We observed above that $\phi(\beta_\ell)(\mu_\ell) = \phi(E[\ell])$ from the definition of $\phi(\beta_\ell)$; it remains to see that $\phi(\alpha_N)(\mu_N) = \phi\big(\beta_\ell(\mu_\ell)+\alpha_N(\mu_N)\big)$, which simply follows from $\beta_\ell(\mu_\ell)$ being killed by ϕ.

So $\widetilde{w}_\ell\circ\eta = \eta\circ\widetilde{w}_\zeta$, which combined with the surjectivity of η and the following Lemma, yields the claim $\widetilde{w}_\ell^*(a^2) = \ell^{-1}a^2$. □

Lemma A.3. $\widetilde{w}_\zeta^*(a_1^2) = \ell^{-1}a_1^2$ *as elements of* $H^0(I_1(N\ell),\pi^*\underline{\omega}^{\otimes 2})$.

Proof. Let $g = \widetilde{w}_\zeta^*(a_1)$, then

$$g^{p-1} = \widetilde{w}_\zeta^*(a_1^{p-1}) = \widetilde{w}_\zeta^*(\pi^* A) = \pi^*(w_\zeta A).$$

Passing to q-expansions and recalling that $(w_\zeta A)(q) = \ell^{(p-1)/2} A(q^\ell) = \ell^{(p-1)/2}$, we get that $g(q)^{p-1} = \ell^{(p-1)/2}$. The crucial point is that the q-expansion $g(q)^{p-1}$ is a constant, which implies that $g(q)$ itself is a constant, which we will call γ for short. The space $H^0(I_1(N\ell), \pi^*\underline{\omega})$ has another element whose q-expansion is γ, namely γa_1. So by the q-expansion principle, we conclude that $\widetilde{w}_\zeta^*(a_1) = \gamma a_1$. Upon iterating, we get

$$\ell^{-1} a_1 = \langle \ell \rangle_p (a_1) = (\widetilde{w}_\zeta^*)^2 (a_1) = \widetilde{w}_\zeta^*(\gamma a_1) = \gamma^2 a_1,$$

so that $\gamma^2 = \ell^{-1}$. We conclude that

$$\widetilde{w}_\zeta^*(a_1^2) = (\gamma a_1)^2 = \ell^{-1} a_1^2.$$

□

Remark. The reader is perhaps wondering why we had to involve Γ_1-structures. It is indeed possible to apply the argument in Lemma A.3 directly to the trivializing section a^2 on $I_0(N\ell)$, but that only allows us to conclude that $\widetilde{w}_\ell^*(a^2) = \pm \ell^{-1} a^2$, and we are unable to rule out the possible negative sign when $p \equiv 1 \pmod 4$. The Γ_1 setting provides us with a square root of a_1^2, which strenghens the argument enough to rule out the unwanted -1. It is possible that working with the moduli *stack* $\mathcal{X}_0(N\ell)$ instead of the *coarse moduli space* $X_0(N\ell)$ could also provide the needed flexibility, without the artifice of changing level structures.

Proposition A.4. *If f is a modular form of weight k and q-expansion $f(q)$, we have*

$$\widetilde{W}_\ell f(q) = W_\ell f(q).$$

Proof. This is just a matter of combining Lemma A.1 and Lemma A.2:

$$\widetilde{W}_\ell f(q) = \Phi\left(\widetilde{w}_\ell^*\left(\frac{\pi^* f}{a^k}\right)\right) = \Phi\left(\frac{\widetilde{w}_\ell^*(\pi^* f)}{\widetilde{w}_\ell^*(a^k)}\right) = \Phi\left(\frac{\pi^*(w_\ell f)}{\ell^{-k/2} a^k}\right) = \ell^{k/2} w_\ell f(q) = W_\ell f(q).$$

□

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