EFFECT OF INCREASING THE RAMIFICATION ON PSEUDO-DEFORMATION RINGS

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Abstract. Given a continuous, odd, semi-simple 2-dimensional representation of $G_{\mathbb{Q},Np}$ over a finite field of odd characteristic p and a prime ℓ not dividing Np, we study the relation between the universal deformation rings of the corresponding pseudo-representation for the groups $G_{\mathbb{Q},N\ell p}$ and $G_{\mathbb{Q},Np}$. As a related problem, we investigate when the universal pseudo-representation arises from an actual representation over the universal deformation ring. Under some hypotheses, we prove that the reduced mod p universal deformation ring of the pseudo-representation is isomorphic to the reduced mod p universal deformation ring of a Borel representation and in some cases, we prove a stronger result. We prove analogues of theorems of Boston ([10]) and Böckle ([7]) in these cases. When the pseudo-representation is unobstructed and p divides $\ell+1$, we prove that the universal deformation rings in characteristic 0 and p of the pseudo-representation for $G_{\mathbb{Q},N\ell p}$ are not local complete intersection rings.

1. Introduction

In [10], Boston studied the effect of enlarging the set of primes that can ramify on the structure of the universal deformation ring of an odd, absolutely irreducible representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over a finite field which is attached to a modular eigenform of weight 2. His results were generalized by Böckle in [7] to any continuous 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over a finite field such that the centralizer of its image is exactly scalars. The aim of this paper is to study the same problem for pseudo-deformation rings i.e. universal deformation rings of pseudo-representations. Our interest in the problem mainly arises from its potential application in determining the structure of characteristic 0 and characteristic p Hecke algebras (as defined in [4] and [12]). But in this article, we stay on the deformation side.

To be more precise, let p be an odd prime. For an integer M, denote by $G_{\mathbb{Q},Mp}$ the Galois group of a maximal unramified extension of $\mathbb{Q}$ unramified outside $\{$primes q s.t. $q|Mp\} \cup \{\infty\}$ over $\mathbb{Q}$. Let N be a positive integer not divisible by p. Let $\mathbb{F}$ be a finite field of characteristic p and $W(\mathbb{F})$ be the ring of Witt vectors of $\mathbb{F}$. Let $\bar{\rho}_0 : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous, odd, reducible, semi-simple representation. So, $(\mathrm{tr}(\bar{\rho}_0), \det(\bar{\rho}_0)) : G_{\mathbb{Q},Np} \to \mathbb{F}$ is a continuous pseudo-representation of $G_{\mathbb{Q},Np}$ of dimension 2 (see [4, Section 1.4] for definition and properties of 2-dimensional pseudo-representations and [11] for general

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pseudo-representations). Let $\mathcal{R}^{\text{pd}}_{\bar\rho_0}$ and $\mathcal{R}^{\text{pd},\ell}_{\bar\rho_0}$ be the universal deformation rings of the pseudo-representation $(\text{tr}(\bar\rho_0), \det(\bar\rho_0))$ for the groups $G_{\mathbb{Q},Np}$ and $G_{\mathbb{Q},N\ell p}$, respectively (we will define the rings more precisely below).

Our aim is to compare $\mathcal{R}^{\text{pd},\ell}_{\bar\rho_0}$ with $\mathcal{R}^{\text{pd}}_{\bar\rho_0}$ and determine the structure of $\mathcal{R}^{\text{pd},\ell}_{\bar\rho_0}$ in terms of the structure of $\mathcal{R}^{\text{pd}}_{\bar\rho_0}$. In [10], Boston studied this problem for absolutely irreducible representations using the techniques and results of pro-p groups and the same techniques were used by Böckle in [7] to extend Boston's results to residually Borel representations (see [7, Theorem 4.7]). However, their method crucially depends on working with actual representations (and not just pseudo-representations). So, in order to use their techniques and results, we first investigate when a pseudo-representation comes from an actual representation which is also of an independent interest. As a consequence, under some hypotheses, we get an isomorphism between the universal pseudo-deformation ring of $\bar\rho_0$ and the universal deformation ring of a Borel representation which allows us to use results of [7] directly.

One of the main motivations to study this problem is the following: suppose $\bar\rho_0$ comes from a newform of level N. Then, we are interested in studying the relationship between the $\bar\rho_0$-components of characteristic 0 (resp. characteristic p) Hecke algebra of level $N\ell$ and the characteristic 0 (resp. characteristic p) Hecke algebra of level N (see [4] and [12] for the definitions of these Hecke algebras). In particular, it is natural to ask if the structure of $\mathbb{T}^{\Gamma_1(N\ell)}_{\bar\rho_0}$ (resp. $A^{\Gamma_1(N\ell)}_{\bar\rho_0}$), the $\bar\rho_0$-component of the characteristic 0 (resp. characteristic p) Hecke algebra of level $N\ell$, can be obtained from the structure of $\mathbb{T}^{\Gamma_1(N)}_{\bar\rho_0}$ (resp. $A^{\Gamma_1(N)}_{\bar\rho_0}$), the $\bar\rho_0$-component of the characteristic 0 (resp. characteristic p) Hecke algebra of level N. Note that, we have surjective maps $\mathcal{R}^{\text{pd},\ell}_{\bar\rho_0} \to \mathbb{T}^{\Gamma_1(N\ell)}_{\bar\rho_0}$ and $\mathcal{R}^{\text{pd}}_{\bar\rho_0} \to \mathbb{T}^{\Gamma_1(N)}_{\bar\rho_0}$. Thus, exploring this question for deformation rings serves as a good starting point for this study and it also gives us an idea of what to expect in the case of Hecke algebras. It would also be interesting to investigate if the methods developed in this article can be suitably modified to study the relationship between $\mathbb{T}^{\Gamma_1(N\ell)}_{\bar\rho_0}$ and $\mathbb{T}^{\Gamma_1(N)}_{\bar\rho_0}$. On the other hand, it might be possible to use some properties of $\mathbb{T}^{\Gamma_1(N\ell)}_{\bar\rho_0}$ to get more information about the structure of $\mathcal{R}^{\text{pd},\ell}_{\bar\rho_0}$. So, knowing the structure of $\mathcal{R}^{\text{pd},\ell}_{\bar\rho_0}$ can shed some light on the structure of $\mathbb{T}^{\Gamma_1(N\ell)}_{\bar\rho_0}$ and vice versa. We plan to address some of these questions in an upcoming work ([13]).

In [10], Boston also connects the increase in the space of deformations, after allowing ramification at an additional prime, to the level raising of modular forms. To be precise, he writes the bigger deformation space, obtained after allowing ramification at an additional prime ℓ, as a natural union of its closed subspaces with one of them being the original space that he started with. Then he shows, using the results of Ribet and Carayol, that each of the new closed subspaces contains a point corresponding to a modular eigenform which is new at ℓ.

Suppose $\bar{\rho}_0$ comes from a newform of level N. Then, our results, in which we determine the structure of $\mathcal{R}^{\text{pd},\ell}_{\bar{\rho}_0}$ in terms of the structure of $\mathcal{R}^{\text{pd}}_{\bar{\rho}_0}$, can also be connected to level raising of modular forms in the same way if the level raising is known those cases (see [5] and [21] for level raising results for reducible $\bar{\rho}_0$). In the cases where level raising is not known, those results can be treated as an evidence for level raising. Similar to [10], the case where $p|\ell+1$ turns out to be more interesting and difficult than the other cases.

Before proceeding further, let us first fix some more notation and conventions, in addition to the ones established above, which we will use throughout the paper.

1.1. **Notations and conventions.** Denote by $G_{\mathbb{Q}}$ the absolute Galois group of $\mathbb{Q}$. For a prime q, denote by $G_{\mathbb{Q}_q}$ the absolute Galois group of $\mathbb{Q}_q$ and by I_q, the inertia group at q. From now on, by a representation (resp. a pseudo-representation) of $G_{\mathbb{Q},Mp}$, we will mean a continuous representation (resp. a continuous pseudo-representation) of $G_{\mathbb{Q},Mp}$ unless otherwise mentioned. We will follow the same convention of $G_{\mathbb{Q}_q}$. All the cohomology groups and Ext^i groups of $G_{\mathbb{Q},Mp}$ and $G_{\mathbb{Q}_q}$ that we will work with are assumed to be continuous unless mentioned otherwise. For $G = G_{\mathbb{Q},Mp}$, $G_{\mathbb{Q}_q}$, given a representation ρ of G defined over $\mathbb{F}$, we denote by $\dim(H^i(G,\rho))$, the dimension of $H^i(G,\rho)$ as a vector space over $\mathbb{F}$.

For an integer M, fix an embedding $i_{q,M} : G_{\mathbb{Q}_q} \to G_{\mathbb{Q},Mp}$. For a fixed M, such an embedding is well defined upto conjugacy. For a representation ρ of $G_{\mathbb{Q},Mp}$ denote by $\rho|_{G_{\mathbb{Q}_q}}$ the representation $\rho \circ i_{q,M}$ of $G_{\mathbb{Q}_q}$. Moreover, for an element $g \in G_{\mathbb{Q}_q}$, we denote $\rho(i_{q,M}(g))$ by $\rho(g)$. If $\rho|I_q$ factors through the tame inertia quotient of I_q, then, given an element g in the tame inertia group at q, we write $\rho(g)$ for $\rho(i_{q,M}(g'))$ where g' is any lift of g in $G_{\mathbb{Q}_q}$. For a pseudo-representation (t,d) of $G_{\mathbb{Q},Mp}$ denote by $(t|_{G_{\mathbb{Q}_q}}, d|_{G_{\mathbb{Q}_q}})$ the pseudo-representation $(t \circ i_{q,M}, d \circ i_{q,M})$ of $G_{\mathbb{Q}_q}$.

1.2. **Deformation rings.** We now introduce the deformation rings with which we will be working for the rest of the article. Let $\mathcal{C}$ be the category whose objects are local complete noetherian rings with residue field $\mathbb{F}$ and the morphisms between the objects are local morphisms of $W(\mathbb{F})$-algebras. For an object R of $\mathcal{C}$, denote by $\tan(R)$ the tangent space of R and denote by $(R)^{\text{red}}$ its maximal reduced quotient i.e. $(R)^{\text{red}}$ is the quotient of R by the ideal of its nilpotent elements. We denote by $\dim(\tan(R))$, the dimension of $\tan(R)$ as a vector space over $\mathbb{F}$. Let $\mathcal{C}_0$ be the full sub-category of $\mathcal{C}$ consisting of local complete noetherian $\mathbb{F}$-algebras with residue field $\mathbb{F}$. Let $D_{\bar{\rho}_0}$ be the functor from $\mathcal{C}$ to the category of sets which sends an object R of $\mathcal{C}$ with maximal ideal m_R to the set of continuous pseudo-representations (t, d) of $G_{\mathbb{Q},Np}$ to R such that $t \pmod{m_R} = \mathrm{tr}(\bar\rho_0)$ and $d \pmod{m_R} = \det(\bar\rho_0)$. Let $\bar D_{\bar\rho_0}$ be the restriction of $D_{\bar\rho_0}$ to the sub-category $\mathcal{C}_0$.

From [11], it follows that the functors $D_{\bar\rho_0}$ and $\bar D_{\bar\rho_0}$ are representable by objects of $\mathcal{C}$ and $\mathcal{C}_0$, respectively. Let $R^{\mathrm{pd}}_{\bar\rho_0}$ and $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$ be the local complete Noetherian rings with residue field $\mathbb{F}$ representing $\bar D_{\bar\rho_0}$ and $D_{\bar\rho_0}$, respectively. So, we have $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}/(p) \simeq R^{\mathrm{pd}}_{\bar\rho_0}$. Let $\mathfrak{m}$ and $\mathfrak{m}'$ be the maximal ideals of $R^{\mathrm{pd}}_{\bar\rho_0}$ and $(R^{\mathrm{pd}}_{\bar\rho_0})^{\mathrm{red}}$, respectively. Let $\mathcal{M}$ and $\mathcal{M}'$ be the maximal ideals of $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$ and $(\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0})^{\mathrm{red}}$, respectively. Let $(t^{\mathrm{univ}}, d^{\mathrm{univ}})$ be the universal pseudo-representation of $G_{\mathbb{Q},Np}$ to $R^{\mathrm{pd}}_{\bar\rho_0}$ deforming $(\mathrm{tr}\,\bar\rho_0, \det\bar\rho_0)$. Let $(T^{\mathrm{univ}}, D^{\mathrm{univ}})$ be the universal pseudo-representation of $G_{\mathbb{Q},Np}$ to $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$ deforming $(\mathrm{tr}\,\bar\rho_0, \det\bar\rho_0)$.

As p is odd, it follows that a 2-dimensional pseudo-representation (t, d) of $G_{\mathbb{Q},Mp}$ to an object R of $\mathcal{C}$ is determined by t which is a pseudo-character of dimension 2 in the sense of Rouquier ([18]) (see [4, Section 1.4]). So, in this case, the theory of pseudo-representations is same as the theory of pseudo-characters. Therefore, we will be working with the residual pseudo-character $\mathrm{tr}(\bar\rho_0)$ and the universal pseudo-characters T^{univ} and t^{univ} deforming $\mathrm{tr}(\bar\rho_0)$ instead of working with the corresponding pseudo-representations.

Denote the pseudo-character obtained by composing t^{univ} with the surjective map $R^{\mathrm{pd}}_{\bar\rho_0} \to (R^{\mathrm{pd}}_{\bar\rho_0})^{\mathrm{red}}$ by $t^{\mathrm{univ,red}}$ and the pseudo-character obtained by composing T^{univ} with the surjective map $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0} \to (\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0})^{\mathrm{red}}$ by $T^{\mathrm{univ,red}}$.

Now, $\bar\rho_0 = \chi_1 \oplus \chi_2$ for some characters $\chi_1,\ \chi_2 : G_{\mathbb{Q},Np} \to \mathbb{F}^*$. Let $\chi = \chi_1/\chi_2$. Thus, $\chi : G_{\mathbb{Q},Np} \to \mathbb{F}^*$ is an odd character. For a non-zero element $x \in H^1(G_{\mathbb{Q},Np}, \chi)$, denote by $\bar\rho_x$ the corresponding representation of $G_{\mathbb{Q},Np}$. So $\bar\rho_x : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathbb{F})$ is such that $\bar\rho_x = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ where $*$ corresponds to x. Similarly, for a non-zero element $y \in H^1(G_{\mathbb{Q},Np}, \chi^{-1})$, denote by $\bar\rho_y$ the corresponding representation of $G_{\mathbb{Q},Np}$.

Let $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$ with $i \in \{1, -1\}$ be a non-zero element. Denote by $\mathcal{R}^{\mathrm{def}}_{\bar\rho_x}$ the universal deformation ring of $\bar\rho_x$ in the sense of Mazur ([16]). Note that, for a non-zero $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$ with $i \in \{1, -1\}$, the centralizer of the image of $\bar\rho_x$ is exactly the set of scalar matrices. Hence, the existence of $\mathcal{R}^{\mathrm{def}}_{\bar\rho_x}$ follows from [16] and [17]. Let $R^{\mathrm{def}}_{\bar\rho_x}$ be the universal deformation ring of $\bar\rho_x$ in characteristic p. So, we have $\mathcal{R}^{\mathrm{def}}_{\bar\rho_x}/(p) \simeq R^{\mathrm{def}}_{\bar\rho_x}$.

Let ℓ be a prime such that $\ell \nmid Np$. As $G_{\mathbb{Q},Np}$ is a quotient of $G_{\mathbb{Q},N\ell p}$, the representations $\bar\rho_0$ and $\bar\rho_x$ with $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$ with $i \in \{1, -1\}$ are also representations of $G_{\mathbb{Q},N\ell p}$. Thus, we can view $(\mathrm{tr}(\bar\rho_0), \det(\bar\rho_0))$ as a pseudo-representation of $G_{\mathbb{Q},N\ell p}$ as well. Let $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$ and $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ be the universal deformation rings of $(\mathrm{tr}(\bar\rho_0), \det(\bar\rho_0))$ considered as a pseudo-representation of $G_{\mathbb{Q},N\ell p}$ in the categories $\mathcal{C}$ and $\mathcal{C}_0$ respectively. Let $\mathfrak{m}^\ell$ and $\mathfrak{m}'^\ell$ be the maximal ideals of $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ and $(R^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}}$, respectively. Let $\mathcal{M}^\ell$ and $\mathcal{M}'^\ell$ be the maximal ideals of $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$ and $(\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}}$, respectively. For a non-zero $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$

with $i \in \{1, -1\}$, Let $\mathcal{R}^{\text{def},\ell}_{\bar{\rho}_x}$ and $R^{\text{def},\ell}_{\bar{\rho}_x}$ be the universal deformation rings of $\bar{\rho}_x$ considered as a representation of $G_{\mathbb{Q},N\ell p}$ in the categories $\mathcal{C}$ and $\mathcal{C}_0$, respectively.

1.3. **Main results.** The surjective group homomorphism $G_{\mathbb{Q},N\ell p} \to G_{\mathbb{Q},Np}$ induces the following local morphisms: $\mathcal{R}^{\text{pd},\ell}_{\bar{\rho}_0} \to \mathcal{R}^{\text{pd}}_{\bar{\rho}_0}$, $R^{\text{pd},\ell}_{\bar{\rho}_0} \to R^{\text{pd}}_{\bar{\rho}_0}$, $\mathcal{R}^{\text{def},\ell}_{\bar{\rho}_x} \to \mathcal{R}^{\text{def}}_{\bar{\rho}_x}$ and $R^{\text{def},\ell}_{\bar{\rho}_x} \to R^{\text{def}}_{\bar{\rho}_x}$. In [7], Böckle used the techniques of [10] to study the local morphism $\mathcal{R}^{\text{def},\ell}_{\bar{\rho}_x} \to \mathcal{R}^{\text{def}}_{\bar{\rho}_x}$ and found the structure of $\mathcal{R}^{\text{def},\ell}_{\bar{\rho}_x}$ in terms of the structure of $\mathcal{R}^{\text{def}}_{\bar{\rho}_x}$ (see [7, Theorem 4.7]). Our aim is to study the local morphisms $\mathcal{R}^{\text{pd},\ell}_{\bar{\rho}_0} \to \mathcal{R}^{\text{pd}}_{\bar{\rho}_0}$ and $R^{\text{pd},\ell}_{\bar{\rho}_0} \to R^{\text{pd}}_{\bar{\rho}_0}$ to determine the explicit structure of $\mathcal{R}^{\text{pd},\ell}_{\bar{\rho}_0}$ (respectively of $R^{\text{pd},\ell}_{\bar{\rho}_0}$) from the explicit structure of $\mathcal{R}^{\text{pd}}_{\bar{\rho}_0}$ (respectively of $R^{\text{pd}}_{\bar{\rho}_0}$).

As mentioned above, Boston uses the theory of pro-p groups in [10] to prove the main theorem and the same method is used by Böckle in [7] for residually Borel representations. So, if we know that there exists a representation of $G_{\mathbb{Q},N\ell p} \to \mathrm{GL}_2(\mathcal{R}^{\text{pd},\ell}_{\bar{\rho}_0})$ whose trace is the universal pseudo-character, then we can use their methods to determine the structure of $\mathcal{R}^{\text{pd},\ell}_{\bar{\rho}_0}$ in terms of the structure of $\mathcal{R}^{\text{pd}}_{\bar{\rho}_0}$.

So we first investigate the existence of representations $\rho : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(R^{\text{pd}}_{\bar{\rho}_0})$ and $\tau : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathcal{R}^{\text{pd}}_{\bar{\rho}_0})$ such that $\mathrm{tr}(\rho) = t^{\text{univ}}$ and $\mathrm{tr}(\tau) = T^{\text{univ}}$. We get the following results in this direction:

Given a non-zero $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$ with $i \in \{1, -1\}$, let $\rho^{\text{univ}}_x : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(R^{\text{def}}_{\bar{\rho}_x})$ be the universal deformation of $\bar{\rho}_x$ taking values in $R^{\text{def}}_{\bar{\rho}_x}$. So, the trace of ρ^{univ}_x is a pseudo-character of $G_{\mathbb{Q},Np}$ deforming $\mathrm{tr}(\bar{\rho}_0)$. Hence, it induces a map $\psi_x : R^{\text{pd}}_{\bar{\rho}_0} \to R^{\text{def}}_{\bar{\rho}_x}$.

Theorem 2.17. *Suppose $p \nmid \phi(N)$ and $\dim(H^1(G_{\mathbb{Q},Np}, \chi^i)) = 1$ for some $i \in \{1, -1\}$. Then there exists a representation $\rho^{red} : G_{\mathbb{Q},Np} \to \mathrm{GL}_2((R^{pd}_{\bar{\rho}_0})^{red})$ such that $\mathrm{tr}(\rho^{red}) = t^{univ,red}$. As a consequence, for a non-zero $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$, the map $\psi_x : R^{pd}_{\bar{\rho}_0} \to R^{def}_{\bar{\rho}_x}$ induces an isomorphism between $(R^{pd}_{\bar{\rho}_0})^{red}$ and $(R^{def}_{\bar{\rho}_x})^{red}$.*

Theorem 2.19. *Suppose $p \nmid \phi(N)$ and $\dim(H^1(G_{\mathbb{Q},Np}, \chi^i)) = 1$ for some $i \in \{1, -1\}$. Moreover, for such an i, assume that $\dim(H^1(G_{\mathbb{Q},Np}, \chi^{-i})) \in \{1, 2, 3\}$. Then, there exists a representation $\rho : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(R^{pd}_{\bar{\rho}_0})$ such that $\mathrm{tr}(\rho) = t^{univ}$. As a consequence, for a non-zero $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$, $R^{pd}_{\bar{\rho}_0} \simeq R^{def}_{\bar{\rho}_x}$.*

Denote by ω_p the mod p cyclotomic character of $G_{\mathbb{Q}}$. Because of the isomorphisms $(R^{\text{pd}}_{\bar{\rho}_0})^{\text{red}} \simeq (R^{\text{def}}_{\bar{\rho}_x})^{\text{red}}$ and $R^{\text{pd}}_{\bar{\rho}_0} \simeq R^{\text{def}}_{\bar{\rho}_x}$ found above, we directly use [7, Theorem 4.7] after some analysis to get the following results:

Theorem 4.4. *Suppose $p \nmid \phi(N)$, $\dim(H^1(G_{\mathbb{Q},Np}, \chi^i)) = 1$ and $\dim(H^1(G_{\mathbb{Q},Np}, \chi^{-i})) = m$ for some $i \in \{1, -1\}$. Let ℓ be a prime such that $p \nmid \ell^2 - 1$ and $\chi^{-i}|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$. Then:*

(1) *For any non-zero* $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$, $(R^{pd,\ell}_{\bar\rho_0})^{red} \simeq (R^{def,\ell}_{\bar\rho_x})^{red}$. *As a consequence, there exists* $r_1, \cdots, r_{n'}, \Phi \in \mathbb{F}[[X_1, \cdots, X_n, X]]$ *such that* $(R^{pd,\ell}_{\bar\rho_0})^{red} \simeq (\mathbb{F}[[X_1, \cdots, X_n, X]]/(r_1, \cdots, r_{n'}, X(\Phi - \ell)))^{red}$ *and* $(R^{pd}_{\bar\rho_0})^{red} \simeq (\mathbb{F}[[X_1, \cdots, X_n]]/(\bar r_1, \cdots, \bar r_{n'}))^{red}$, *where* $r_i \pmod X = \bar r_i$.

(2) *Suppose* $m = 1, 2$. *For any non-zero* $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$, $R^{pd,\ell}_{\bar\rho_0} \simeq R^{def,\ell}_{\bar\rho_x}$. *As a consequence, there exists* $r_1, \cdots, r_{n'}, \Phi \in \mathbb{F}[[X_1, \cdots, X_n, X]]$ *such that* $R^{pd,\ell}_{\bar\rho_0} \simeq \mathbb{F}[[X_1, \cdots, X_n, X]]/(r_1, \cdots, r_{n'}, X(\Phi - \ell))$ *and* $R^{pd}_{\bar\rho_0} \simeq \mathbb{F}[[X_1, \cdots, X_n]]/(\bar r_1, \cdots, \bar r_{n'})$, *where* $r_i \pmod X = \bar r_i$.

At the end, we turn our attention to the case when $\bar\rho_0$ is unobstructed to get more precise results. We call $\bar\rho_0$ unobstructed if $p \nmid \phi(N)$ and $\dim(H^1(G_{\mathbb{Q},Np}, \chi)) = \dim(H^1(G_{\mathbb{Q},Np}, \chi^{-1})) = 1$. Note that, using [1, Theorem 2], we see that this definition coincides with the ones given in [4] and [12]. So, in this case, we have $\mathcal{R}^{\rm pd}_{\bar\rho_0} \simeq W(\mathbb{F})[[X, Y, Z]]$. We know that if $N = 1$ and p is odd, then the Vandiver's conjecture implies that $\bar\rho_0$ is always unobstructed (see [4, Theorem 22] for more details and examples of unobstructed $\bar\rho_0$ for $N = 1$). In this case, we get the following results:

Let $\tilde\ell$ be the Teichmuller lift of $\ell \pmod p$.

Theorem 4.6. *Suppose* $\bar\rho_0$ *is unobstructed. Let* ℓ *be a prime such that* $p \nmid \ell^2 - 1$, $\chi^i|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ *for some* $i \in \{1, -1\}$. *Then,* $\mathcal{R}^{pd,\ell}_{\bar\rho_0} \simeq W(\mathbb{F})[[X_1, X_2, X_3, X_4]]/(X_4 f)$ *for some* $f \in W(\mathbb{F})[[X_1, X_2, X_3, X_4]]$. *Moreover, if* $\ell/\tilde\ell$ *is a topological generator of* $1 + p\mathbb{Z}_p$, *then* $\mathcal{R}^{pd,\ell}_{\bar\rho_0} \simeq W(\mathbb{F})[[X_1, X_2, X_3, X_4]]/(X_4(p + X_2))$.

The isomorphism between $\mathcal{R}^{\rm pd,\ell}_{\bar\rho_0}$ (resp. $\mathcal{R}^{\rm pd,\ell}_{\bar\rho_0}$) and $W(\mathbb{F})[[X_1, X_2, X_3, X_4]]/(X_4 f)$ (resp. $W(\mathbb{F})[[X_1, X_2, X_3, X_4]]/(X_4(p + X_2))$ when $\ell/\tilde\ell$ is a topological generator of $1 + p\mathbb{Z}_p$) that we find in the proof of Theorem 4.6 is such that the kernel of the map $\mathcal{R}^{\rm pd,\ell}_{\bar\rho_0} \to \mathcal{R}^{\rm pd}_{\bar\rho_0}$, under the isomorphism, is the ideal (X_4).

Note that, if $\bar\rho_0$ is unobstructed, $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ and $p | \ell + 1$, then $\dim(H^1(G_{\mathbb{Q},N\ell p}, \chi)) = \dim(H^1(G_{\mathbb{Q},N\ell p}, \chi^{-1})) = 2$. So, this case is different from the cases that we have dealt with so far and none of the results mentioned above apply to this case. In this setting, we prove the following results:

Theorem 4.10. *Suppose* $\bar\rho_0$ *is unobstructed. Let* ℓ *be a prime such that* $\ell \equiv -1 \pmod p$, $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ *and* $-\ell$ *is a topological generator of* $1 + p\mathbb{Z}_p$. *Then,* $(R^{pd,\ell}_{\bar\rho_0})^{red} \simeq \mathbb{F}[[X, Y, Z, T_1, T_2]]/(T_1T_2, T_1Z, T_2Z)$.

As $\bar\rho_0$ is unobstructed, we have $R^{\rm pd}_{\bar\rho_0} \simeq \mathbb{F}[[X, Y, Z]]$. So, the surjective map $R^{\rm pd,\ell}_{\bar\rho_0} \to R^{\rm pd}_{\bar\rho_0}$ factors through $(R^{\rm pd,\ell}_{\bar\rho_0})^{\rm red}$. The isomorphism between $(R^{\rm pd,\ell}_{\bar\rho_0})^{\rm red}$ and $\mathbb{F}[[X, Y, Z, T_1, T_2]]/(T_1T_2, T_1Z, T_2Z)$

that we find in the proof of Theorem 4.10 is such that the kernel of the map $(R^{\text{pd},\ell}_{\bar\rho_0})^{\text{red}} \to R^{\text{pd}}_{\bar\rho_0}$, under the isomorphism, is the ideal (T_1, T_2).

Theorem 4.18. *Suppose $\bar\rho_0$ is unobstructed. Let ℓ be a prime such that $\ell \equiv -1 \pmod p$, $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ and $-\ell$ is a topological generator of $1 + p\mathbb{Z}_p$. Then $R^{\text{pd},\ell}_{\bar\rho_0}$ is not a local complete intersection ring.*

Using Theorem 4.18, we also prove that $\mathcal{R}^{\text{pd},\ell}_{\bar\rho_0}$, under the hypotheses of Theorem 4.18, is not a local complete intersection ring. If $N = 1$, then $\chi = \omega_p^k$ for some odd k as $\bar\rho_0$ is odd. Therefore, in this case, if $\ell \equiv -1 \pmod p$, then $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ will always hold. Thus, when $N = 1$, using [4, Theorem 22], we get many examples of $\bar\rho_0$ and ℓ satisfying the hypotheses of Theorem 4.10 and Theorem 4.18. Moreover, [4, Theorem 22] implies that if Vandiver's conjecture is true, then, given a $\bar\rho_0$ with $N = 1$, any prime ℓ which is $-1 \pmod p$ but not $-1 \pmod{p^2}$ satisfies the hypotheses of Theorem 4.10 and Theorem 4.18.

Suppose $\bar\rho_0$ comes from a newform of level N and $p \geq 5$. For $M = N$, $N\ell$, $N\ell^2$, let $\mathbb{T}^{\Gamma_1(M)}_{\bar\rho_0}$ (resp. $A^{\Gamma_1(M)}_{\bar\rho_0}$) be the $\bar\rho_0$-component of the characteristic 0 (resp. characteristic p) Hecke algebra of level M (notation is borrowed from [12]). The proof of Theorem 4.6, along with results of [12] ([12, Section 2], proof of [12, Lemma 23] and [12, Theorem 1]), shows that in the cases considered in the Theorem, $\mathbb{T}^{\Gamma_1(N\ell)}_{\bar\rho_0}$ has Krull dimension 4 while $A^{\Gamma_1(N\ell)}_{\bar\rho_0}$ has Krull dimension 2. When $p|\ell+1$, the correct Hecke algebra to compare with $\mathcal{R}^{\text{pd},\ell}_{\bar\rho_0}$ would be $\mathbb{T}^{\Gamma_1(N\ell^2)}_{\bar\rho_0}$. The proof of Theorem 4.10, along with results of [12], shows that in the cases considered in the Theorem, $\mathbb{T}^{\Gamma_1(N\ell^2)}_{\bar\rho_0}$ has Krull dimension 4 while $A^{\Gamma_1(N\ell^2)}_{\bar\rho_0}$ has Krull dimension 2. Previously, only lower bounds on the Krull dimensions of Hecke algebras were known in all these cases. We have a surjective map $\mathcal{R}^{\text{pd},\ell}_{\bar\rho_0} \to \mathbb{T}^{\Gamma_1(N\ell^2)}_{\bar\rho_0}$ and both the rings have Krull dimension 4. So, this suggests that $\mathbb{T}^{\Gamma_1(N\ell^2)}_{\bar\rho_0}$ may well not be a local complete intersection ring.

Note that, when $p|\ell+1$, the deformation ring appropriate for comparison with $\mathbb{T}^{\Gamma_1(N\ell)}_{\bar\rho_0}$, is the universal deformation ring parameterizing all deformations t of $\text{tr}(\bar\rho_0)$ such that $t|_{G_{\mathbb{Q}_\ell}}$ is reducible. Call this ring $S^{\text{pd},\ell}_{\bar\rho_0}$. Indeed, the surjective map $\mathcal{R}^{\text{pd},\ell}_{\bar\rho_0} \to \mathbb{T}^{\Gamma_1(N\ell^2)}_{\bar\rho_0}$ factors through $S^{\text{pd},\ell}_{\bar\rho_0}$. The proof of Theorem 4.10 also implies that $(S^{\text{pd},\ell}_{\bar\rho_0})^{\text{red}} \simeq W(\mathbb{F})[[X,Y,Z,T_1,T_2]]/(T_1T_2, T_1Z, T_2Z')$ for some $Z' \equiv Z \pmod p$. Since the characteristic 0 Hecke algebras are reduced, this gives a potential candidate for the structure of $\mathbb{T}^{\Gamma_1(N\ell)}_{\bar\rho_0}$.

Recall that, Mazur's conjecture ([16]) predicts that the mod p universal deformation ring of an absolutely irreducible 2-dimensional representation of $G_{\mathbb{Q},Mp}$ over some finite extension of $\mathbb{F}_p$ has Krull dimension 3. This also implies that the mod p universal deformation ring is always a local complete intersection ring. Combining Theorem 4.10 and Theorem 4.18, we find examples of mod p universal pseudo-deformation rings of Krull dimension 3 which are not local complete intersection rings. On the other hand, in [6], Bleher and Chinburg found examples of absolutely irreducible representations of profinite groups such that the corresponding universal deformation rings (in the sense of Mazur) are not locally complete intersection rings.

1.4. **Organization of the paper and overview of the proofs.** Let us now give an overview of the proofs of the results above and the organization of the paper. In Section 2.1, we do some Galois cohomology calculations and find some conditions on $H^1(G_{\mathbb{Q},Np},\chi)$ and $H^1(G_{\mathbb{Q},Np},\chi^{-1})$ for t^{univ} to arise from an actual representation.. In Section 2.2, we recall some standard definitions and results about Generalized Matrix algebras (GMA) which will be used throughout the paper.

To prove the first part of Theorem 2.17, we first use the results of [19] which give, under the hypotheses of the proposition, a presentation of $R^{\text{pd}}_{\bar\rho_0}$ as a quotient of a power series ring similar to the one given in [8, Theorem 2.4]. After this, we use the properties of reducibility ideal of t^{univ} along with the generalization of Krull's principal ideal theorem to prove that $t^{\text{univ}} \pmod P$ is not reducible for any minimal prime ideal P of $R^{\text{pd}}_{\bar\rho_0}$. Then, we use the arguments of the proof of [3, Proposition 1.7.4] to get the first part of Theorem 2.17. Using [15, Corollary 1.4.4(2)], we get the second part of Theorem 2.17.

To prove Theorem 2.19, we first use results of [3] (presented in [2] in an alternate form) to get a Generalized Matrix Algebra (GMA) over $R^{\text{pd}}_{\bar\rho_0}$ which is a quotient of $R^{\text{pd}}_{\bar\rho_0}[G_{\mathbb{Q},Np}]$ and whose trace restricted to the image of $G_{\mathbb{Q},Np}$ is t^{univ}. Then, we do a case by case analysis which uses the presentation of $R^{\text{pd}}_{\bar\rho_0}$ mentioned above and some basic commutative algebra, to prove that this GMA is isomorphic to a $R^{\text{pd}}_{\bar\rho_0}$-sub-GMA of $M_2(R^{\text{pd}}_{\bar\rho_0})$ from which the theorem follows.

In Section 3, we analyze when the analogues of Theorem 2.17 and Theorem 2.19 can hold for characteristic 0 deformation rings and prove them under certain conditions. In Section 4.1, we do some Galois cohomology computations to see how the dimensions of certain Galois cohomology groups change after enlarging the set of primes that can ramify. Note that, Theorem 4.4 immediately follows from the Galois cohomology computations of Section 4.1, Theorem 2.17 and [7, Theorem 4.7]. To prove Theorem 4.6, we first use the results of Section 3, Galois cohomology computations of Section 4.1 and Theorem 2.19 to prove $\mathcal{R}^{\text{pd},\ell}_{\bar\rho_0} \simeq \mathcal{R}^{\text{def},\ell}_{\bar\rho_x}$ for a suitable $\bar\rho_x$ when $\bar\rho_0$ is unobstructed. Then, we use the hypotheses to find a set of generators of the co-tangent space of $\mathcal{R}^{\text{pd},\ell}_{\bar\rho_0}$. Finally, we use the relations coming from $G_{\mathbb{Q}_\ell}$ between these generators, [8, Theorem 2.4], and $\mathcal{R}^{\text{pd},\ell}_{\bar\rho_0} \simeq \mathcal{R}^{\text{def},\ell}_{\bar\rho_x}$ to prove the Theorem.

The proof of Theorem 4.10 is split up in many steps. Keeping the hypothesis of Theorem 4.10, we first use the results of [7] to find the explicit structure of $R^{\text{def},\ell}_{\bar\rho_x}$ for a non-zero $x \in H^1(G_{\mathbb{Q},Np}\chi^i)$ with $i \in \{1,-1\}$. Using the explicit structure of $R^{\text{def},\ell}_{\bar\rho_x}$ and the hypothesis that $\bar\rho_0$ is unobstructed, we find 3 distinct prime ideals of $R^{\text{pd},\ell}_{\bar\rho_0}$ such that the quotient of $R^{\text{pd},\ell}_{\bar\rho_0}$ by each of them is isomorphic to $\mathbb{F}[[Z_1,Z_2,Z_3]]$. We find a set of generators of the co-tangent space of $(R^{\text{pd},\ell}_{\bar\rho_0})^{\text{red}}$ and find relations between them using GMA and [3, Theorem 1.4.4]. Combining all this with some basic commutative algebra, we prove the theorem.

Finally to prove Theorem 4.18, we find a set of generators of the cotangent space of $R^{\text{pd},\ell}_{\bar\rho_0}$ and use the GMA approach to get relations between them coming from $G_{\mathbb{Q}_\ell}$. We then use the hypotheses, Theorem 4.10 and some basic commutative algebra to conclude that if $R^{\text{pd},\ell}_{\bar\rho_0}$ is a local complete intersection ring, then a specific subset of these relations should generate all the relations in $R^{\text{pd},\ell}_{\bar\rho_0}$. But the description of this specific subset implies that the Krull dimension of $R^{\text{pd},\ell}_{\bar\rho_0}$ is at least 4 giving us the contradiction to prove the theorem.

The method of Boston and Böckle crucially depends on working with actual representations. To be precise, they first find a minimal set of generators and relations of certain appropriate pro-p groups. Then they send the generators to appropriate general matrices. Using the relations between them, we get relations between the general variables appearing in the entries of the matrices where the generators are getting mapped. This gives the relations in the deformation ring. Thus, it does not work when the universal pseudo-representation does not arise from an actual representation. However, we still know the existence of a GMA and a representation from $G_{\mathbb{Q},N\ell p}$ to that GMA whose trace is the universal pseudo-character. So, if one knows the exact structure of this GMA, it might be possible to modify their method to get results in those cases. Note that, we get most of the results by determining the cases where an actual representation gives rise to the universal pseudo-representation and then using the results of [7]. So, our methods do not apply to a general case. However, it might be possible to modify the proof of Theorem 4.10 to get results in some cases which are not dealt in this paper.

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2. Comparison between $R^{pd}_{\bar\rho_0}$ and $R^{def}_{\bar\rho_x}$

We are interested in determining when does the universal pseudo-character T^{univ} comes from a representation defined over $\mathcal{R}^{\text{pd}}_{\bar\rho_0}$. In this section, we will study when the universal mod p pseudo-character t^{univ} arises from a representation defined over $R^{\mathrm{pd}}_{\bar\rho_0}$. We do this by first assuming the existence of such a representation to study its implications. Then, we will explore if the necessary conditions found this way are sufficient for the existence of such a representation and its consequences on the relationship between $R^{\mathrm{pd}}_{\bar\rho_0}$ and $R^{\mathrm{def}}_{\bar\rho_x}$.

In this and the next section, we are going to take a slightly more general approach. To be precise, instead of $G_{\mathbb{Q},Np}$, we are going to consider a profinite group G which satisfies the finiteness condition Φ_p given by Mazur in [16, Section 1.1] and a continuous representation $\bar\rho_0 : G \to \mathrm{GL}_2(\mathbb{F})$ such that $\bar\rho_0 = \chi_1 \oplus \chi_2$ with $\chi_1 \neq \chi_2$ and $\chi = \chi_1/\chi_2$. Moreover, we are going to assume, throughout this section and the next, that $H^1(G, 1) \neq 0$, $H^1(G, \chi) \neq 0$ and $H^1(G, \chi^{-1}) \neq 0$. Note that, the pair $(G_{\mathbb{Q},Np}, \bar\rho_0)$ which was established in the introduction, satisfies these hypotheses. Indeed, $H^1(G_{\mathbb{Q},Np}, 1) \neq 0$ is clear and as $\bar\rho_0$ is assumed to be odd, by global Euler characteristic formula, we get that $H^1(G_{\mathbb{Q},Np}, \chi^i) \neq 0$ for $i \in \{1, -1\}$.

However, for this general set-up introduced above, we are going to retain the same notation for all the deformation rings and universal deformations that was established in the introduction. The reason for introducing the general set-up is to have clarity about the proofs and the exact properties of various cohomology groups that we need for our results. All the cohomology groups that we consider in this section and the next are also assumed to be continuous without its mention.

2.1. **Necessary condition for the existence of a representation with trace t^{univ}.** In this section, we will assume the existence of a representation over $R^{\mathrm{pd}}_{\bar\rho_0}$ with trace t^{univ} to relate the rings $R^{\mathrm{pd}}_{\bar\rho_0}$ and $R^{\mathrm{def}}_{\bar\rho_x}$. Specifically, we will compare the dimensions of their tangent spaces to get the necessary conditions for the existence of the required representation. We begin by doing a tangent space computation:

Lemma 2.1. *Let $x \in H^1(G, \chi^i)$, with $i \in \{1, -1\}$, be a non-zero element. Let $\dim(H^1(G, \chi^i)) = m$, $\dim(H^1(G, \chi^{-i})) = n$ and $\dim(H^1(G, 1)) = k$. Then $\dim(tan(R^{def}_{\bar\rho_x})) \leq m + n + 2k - 1$*

Proof. Recall that, $\dim(\tan(R^{\mathrm{def}}_{\bar\rho_x})) = \dim H^1(G, \mathrm{ad}(\bar\rho_x))$ ([16]). As p is odd, $\mathrm{ad}(\bar\rho_x) = 1 \oplus \mathrm{ad}^0(\bar\rho_x)$. We have the following two exact sequences of G-modules:

(1) $0 \to \chi^i \to \mathrm{ad}^0(\bar\rho_x) \to V \to 0,$
(2) $0 \to 1 \to V \to \chi^{-i} \to 0.$

So, from the second short exact sequence, we get the following exact sequence of cohomology groups:

$$0 \to H^0(G,1) \to H^0(G,V) \to H^0(G,\chi^{-i}) \to$$
$$\to H^1(G,1) \to H^1(G,V) \to H^1(G,\chi^{-i}) \to H^2(G,1)$$

We have $H^0(G,\chi^{-i}) = 0$. Hence, we get $\dim(H^1(G,V)) \leq \dim(H^1(G,1)) + \dim(H^1(G,\chi^{-i})) = k+n$. From the first short exact sequence, we get the following exact sequence of cohmology groups:

$$0 \to H^0(G,\chi^{i}) \to H^0(G,\mathrm{ad}^0(\bar\rho_x)) \to H^0(G,V) \to$$
$$\to H^1(G,\chi^{i}) \to H^1(G,\mathrm{ad}^0(\bar\rho_x)) \to H^1(G,V) \to H^2(G,\chi^{i})$$

We have $H^0(G,\chi^i) = 0$,$H^0(G,\mathrm{ad}^0(\bar\rho_x)) = 0$ and $\dim(H^0(G,V)) = 1$. Hence, we get $\dim(H^1(G,\mathrm{ad}^0(\bar\rho_x))) \leq \dim(H^1(G,V)) + \dim(H^1(G,\chi^i)) - 1$. Combining this with the inequality $\dim(H^1(G,V)) \leq k+n$, we get that $\dim(H^1(G,\mathrm{ad}^0(\bar\rho_x))) \leq k+m+n-1$ and hence, $\dim(H^1(G,\mathrm{ad}(\bar\rho_x))) = \dim(H^1(G,\mathrm{ad}^0(\bar\rho_x))) + \dim(H^1(G,1)) = \dim(H^1(G,\mathrm{ad}^0(\bar\rho_x))) + k \leq 2k+m+n-1$. □

We now prove a refinement of Lemma 2.1 for $G_{\mathbb{Q},Np}$ as it will be useful later.

Lemma 2.2. *Let $x \in H^1(G_{\mathbb{Q},Np},\chi^i)$, with $i \in \{1,-1\}$, be a non-zero element. Let $\dim(H^1(G_{\mathbb{Q},Np},\chi^i)) = m$, $\dim(H^1(G_{\mathbb{Q},Np},\chi^{-i})) = n$ and $\dim(H^1(G_{\mathbb{Q},Np},1)) = k$. Then $max\{m-1+k, 2k, n+k+1\} \leq \dim(tan(R^{def}_{\bar\rho_x})) \leq m+n+2k-1$. So, in particular, if $p \nmid \phi(N)$ and $m=1$, then $\dim(tan(R^{def}_{\bar\rho_x})) = 2+n$.*

Proof. The inequality $\dim(\tan(R^{\mathrm{def}}_{\bar\rho_x})) \leq m+n+2k-1$ follows from Lemma 2.1. Now, we have the following:

(1) $\dim(H^0(G_{\mathbb{Q},Np},1)) = \dim(H^0(G_{\mathbb{Q},Np},V)) = 1$ and $H^0(G_{\mathbb{Q},Np},\chi^{-i}) = 0$,
(2) It follows, from the global Euler characteristic formula, that $\dim(H^2(G_{\mathbb{Q},Np},1)) = k-1$.

Thus, from the long exact sequence coming from the short exact sequence $0 \to 1 \to V \to \chi^{-i} \to 0$, we see that $\max\{k, n+1\} \leq \dim(H^1(G_{\mathbb{Q},Np},V))$. We have the following:

(1) $H^0(G_{\mathbb{Q},Np},\chi^i) = H^0(G_{\mathbb{Q},Np},\mathrm{ad}^0(\bar\rho_x)) = 0$ and $\dim(H^0(G_{\mathbb{Q},Np},V)) = 1$,
(2) It follows, from the global Euler characteristic formula, that $\dim(H^2(G_{\mathbb{Q},Np},\chi^i)) = m-1$.

Therefore, from the long exact sequence coming from the short exact sequence $0 \to \chi^i \to \mathrm{ad}^0(\bar\rho_x) \to V \to 0$, we get $\max\{m-1, \dim(H^1(G_{\mathbb{Q},Np},V))\} \leq \dim(H^1(G_{\mathbb{Q},Np},\mathrm{ad}^0(\bar\rho_x))) \leq m-1+\dim(H^1(G_{\mathbb{Q},Np},V))$. This, combined with the inequality for $\dim(H^1(G_{\mathbb{Q},Np},V))$, implies $\max\{m-1,k,n+1\} \leq \dim(H^1(G_{\mathbb{Q},Np},\mathrm{ad}^0(\bar\rho_x)))$. Finally, $\dim(H^1(G_{\mathbb{Q},Np},\mathrm{ad}(\bar\rho_x)) = \dim(H^1(G_{\mathbb{Q},Np},1)) + \dim(H^1(G_{\mathbb{Q},Np},\mathrm{ad}^0(\bar\rho_x))) = k + \dim(H^1(G_{\mathbb{Q},Np},\mathrm{ad}^0(\bar\rho_x)))$. Therefore, we get $\max\{m+k-1, 2k, n+k+1\} \leq \dim(H^1(G_{\mathbb{Q},Np},\mathrm{ad}(\bar\rho_x)))$. This proves the first assertion of the lemma.

For the second assertion, note that $\dim(H^1(G_{\mathbb{Q},Np},1)) = 1+k'$, where k' is the number of prime divisors q of N such that $p|q-1$. Therefore, if $p \nmid \phi(N)$, then $k'=0$ and hence, $k=1$. Moreover, if $m=1$, then $\max\{m+k-1, 2k, n+k+1\} = \max\{1,2,n+2\} = n+2$ and $m+n+2k-1 = n+2$. This, along with the inequality for $\dim(H^1(G_{\mathbb{Q},Np},\mathrm{ad}(\bar\rho_x)))$, implies that $\dim(H^1(G_{\mathbb{Q},Np},\mathrm{ad}(\bar\rho_x))) = n+2$. □

Before proceeding further, we first analyze some reducibility properties of t^{univ}. Let R an object of $\mathcal{C}$. Note that, a continuous pseudo-character $t : G \to R$ of dimension n gives rise to a continuous pseudo-character $R[G] \to R$ of the group algebra $R[G_{\mathbb{Q},Np}]$ of dimension n and vice versa (see [2, Lemma 2.1.1] and [3, Section 1.2.1] for the definition of the pseudo-character of an algebra). We will denote by $\tilde t$ the pseudo-character of $R[G]$ induced by t. So, for $i=1,2$, χ_i induces a pseudo-character $\tilde\chi_i : \mathbb{F}[G] \to \mathbb{F}$ of $\mathbb{F}[G]$ of dimension 1.

Now suppose that t is a pseudo-character deforming $\mathrm{tr}(\bar\rho_0)$. Let I be the total reducibility ideal of $\tilde t$ (see [3, Definition 1.5.2] and [3, Proposition 1.5.1] for the definition and existence). This means that for an ideal J of R, the pseudo-character $\tilde t \otimes_R R/J$ of the algebra $R/J[G]$ is a sum of two one-dimensional pseudo-characters $\tilde t_1$, $\tilde t_2 : R/J[G] \to R/J$ such that $\tilde t_i \otimes_{R/J} \mathbb{F} = \tilde\chi_i$ for $i=1,2$ if and only if $I \subset J$.

Let R be an object of $\mathcal{C}$ and $t : G \to R$ be a pseudo-character deforming $\mathrm{tr}(\bar\rho_0)$. We will say that t is reducible if there exists characters η_1, $\eta_2 : G \to R^*$ such that $t = \eta_1 + \eta_2$ and η_i is a deformation of χ_i for $i=1,2$.

Now, the pseudo-character t^{univ} gives rise to a pseudo-character $\tilde t^{\mathrm{univ}} : R^{\mathrm{pd}}_{\bar\rho_0}[G] \to R^{\mathrm{pd}}_{\bar\rho_0}$ of the group algebra $R^{\mathrm{pd}}_{\bar\rho_0}[G]$. Let $I_{\bar\rho_0}$ be the total reducibility ideal of $\tilde t^{\mathrm{univ}}$. Observe that, for an ideal J of $R^{\mathrm{pd}}_{\bar\rho_0}$, $t^{\mathrm{univ}} \pmod J$ is a sum of two characters of G taking values in $R^{\mathrm{pd}}_{\bar\rho_0}/J$ and deforming χ_1 and χ_2 if and only if the pseudo-character $\tilde t^{\mathrm{univ}} \otimes_{R^{\mathrm{pd}}_{\bar\rho_0}} R^{\mathrm{pd}}_{\bar\rho_0}/J : R^{\mathrm{pd}}_{\bar\rho_0}/J[G] \to R^{\mathrm{pd}}_{\bar\rho_0}/J$ is a sum of two one dimensional pseudo-characters $\tilde t_1$, $\tilde t_2 : R^{\mathrm{pd}}_{\bar\rho_0}/J[G] \to R^{\mathrm{pd}}_{\bar\rho_0}/J$ of the algebra $R^{\mathrm{pd}}_{\bar\rho_0}/J[G]$ such that $\tilde t_i \otimes_{R^{\mathrm{pd}}_{\bar\rho_0}/J} \mathbb{F} = \tilde\chi_i$ for $i=1,2$. Therefore, by [3, Proposition 1.5.1], we conclude that, for an ideal J of $R^{\mathrm{pd}}_{\bar\rho_0}$, $t^{\mathrm{univ}} \pmod J$ is reducible if and only if $I_{\bar\rho_0} \subset J$.

Before proceeding further, let G^{ab} denote the continuous abelianization of G. Recall that $H^2(G,1)=0$ means that the abelianized p-completion of G i.e. $\varprojlim_i G^{\mathrm{ab}}/(G^{\mathrm{ab}})^{p^i}$ is a torsion-free pro-p abelian group which is a finitely generated $\mathbb{Z}_p$-module.

Lemma 2.3. *If* $H^2(G,1) = 0$ *and* $\dim(H^1(G,1)) = k$*, then* $R^{pd}_{\bar\rho_0}/I_{\bar\rho_0} \simeq \mathbb{F}[[X_1, \cdots, X_{2k}]]$.

Proof. As $I_{\bar\rho_0}$ is the total reducibility ideal of $\tilde{t}^{\mathrm{univ}}$, it follows, from [3, Proposition 1.5.1], that there exists two characters $\widehat{\chi}_1$, $\widehat{\chi}_2 : G \to (R^{\mathrm{pd}}_{\bar\rho_0}/I_{\bar\rho_0})^*$ such that

(1) The characters $\widehat{\chi}_1$ and $\widehat{\chi}_2$ are deformations of χ_1 and χ_2, respectively,
(2) $t^{\mathrm{univ}}(g) = \widehat{\chi}_1(g) + \widehat{\chi}_2(g)$ for all $g \in G$.

It follows that the maximal ideal $\mathfrak{m}/I_{\bar\rho_0}$ of $R^{\mathrm{pd}}_{\bar\rho_0}/I_{\bar\rho_0}$ is generated by the set $\{\widehat{\chi}_1(g) - \chi_1(g) | g \in G\} \cup \{\widehat{\chi}_2(g) - \chi_2(g) | g \in G\}$ ([11, Remark 3.5]). As $H^2(G,1) = 0$ and $\dim(H^1(G,1)) = k$, we have $\varprojlim_i G^{\mathrm{ab}}/(G^{\mathrm{ab}})^{p^i} \simeq \prod_{i=1}^{k} \mathbb{Z}_p$. It follows, from [16, Section 1.4], that $\mathfrak{m}/I_{\bar\rho_0}$ is generated by at most $2k$ elements. Thus, we have a surjective map $f : \mathbb{F}[[X_1, \cdots, X_{2k}]] \to R^{\mathrm{pd}}_{\bar\rho_0}/I_{\bar\rho_0}$.

On the other hand, by [16, Section 1.4], we know that the mod p universal deformation ring of χ_1 and χ_2 for G is $\mathbb{F}[[T_1, \cdots, T_k]]$. Let χ_1^{univ}, $\chi_2^{\mathrm{univ}} : G \to (\mathbb{F}[[T_1, \cdots, T_k]])^*$ be the universal deformations of χ_1 and χ_2, respectively. Let $\tilde\chi_1^{\mathrm{univ}} : G \to (\mathbb{F}[[X_1, \cdots, X_{2k}]])^*$ be the character obtained by composing χ_1^{univ} with the continuous $\mathbb{F}$-algebra homomorphism $\mathbb{F}[[T_1, \cdots, T_k]] \to \mathbb{F}[[X_1, \cdots, X_{2k}]]$ sending T_i to X_i. Similarly, let $\tilde\chi_2^{\mathrm{univ}} : G \to (\mathbb{F}[[X_1, \cdots, X_{2k}]])^*$ be the character obtained by composing χ_2^{univ} with the continuous $\mathbb{F}$-algebra homomorphism $\mathbb{F}[[T_1, \cdots, T_k]] \to \mathbb{F}[[X_1, \cdots, X_{2k}]]$ sending T_i to X_{k+i}. Thus, the pseudo-character $\tilde\chi_1^{\mathrm{univ}} + \tilde\chi_2^{\mathrm{univ}}$ of G is a deformation of $\mathrm{tr}(\bar\rho_0)$ to $\mathbb{F}[[X_1, \cdots, X_{2k}]]$.

So, this induces a map $R^{\mathrm{pd}}_{\bar\rho_0} \to \mathbb{F}[[X_1, \cdots, X_{2k}]]$ which factors through $R^{\mathrm{pd}}_{\bar\rho_0}/I_{\bar\rho_0}$ by [3, Proposition 1.5.1]. Let $f' : R^{\mathrm{pd}}_{\bar\rho_0}/I_{\bar\rho_0} \to \mathbb{F}[[X_1, \cdots, X_{2k}]]$ be this map. From the description of χ_1^{univ} and χ_2^{univ} ([16, Section 1.4]), it follows that $(1 + X_i) + (1 + X_{k+i})$ is in the image of f' for every $1 \leq i \leq k$. As $\chi_1 \neq \chi_2$, there exists a $g_0 \in G$ such that $\chi_1(g_0) \neq \chi_2(g_0)$. So, $\chi_1(g_0)(1 + X_i) + \chi_2(g_0)(1 + X_{k+i})$ is also in the image of f' for every $1 \leq i \leq k$. Therefore, both $1 + X_i$ and $1 + X_{k+i}$ are in the image of f' for every $1 \leq i \leq k$ and hence, f' is surjective. Thus, the homomorphism $f' \circ f : \mathbb{F}[[X_1, \cdots, X_{2k}]] \to \mathbb{F}[[X_1, \cdots, X_{2k}]]$ is surjective and hence, injective as well. As f is surjective, this implies that f' injective and hence, an isomorphism. So, we get that $R^{\mathrm{pd}}_{\bar\rho_0}/I_{\bar\rho_0} \simeq \mathbb{F}[[X_1, \cdots, X_{2k}]]$ □

Note that, one can also prove an analogue of Lemma 2.3 in the case when $H^2(G,1) \neq 0$ but we don't prove it here as we will mostly restrict ourselves to the case $H^2(G,1) = 0$ in what follows.

We are now ready to analyze when t^{univ} is the trace of a representation defined over $R^{\mathrm{pd}}_{\bar\rho_0}$.

Proposition 2.4. *(1) Suppose $H^2(G,1)=0$. If there exists a continuous representation $\rho : G \to \mathrm{GL}_2(R^{pd}_{\bar\rho_0})$ such that $\mathrm{tr}\,\rho = t^{univ}$, then either $\dim(H^1(G,\chi))=1$ or $\dim(H^1(G,\chi^{-1}))=1$.*

(2) Suppose $H^2(G,1)\neq 0$. If there exists a continuous representation $\rho : G \to \mathrm{GL}_2(R^{pd}_{\bar\rho_0})$ such that $\mathrm{tr}\,\rho = t^{univ}$, then:
$\dim(tan(R^{pd}_{\bar\rho_0})) \leq 2\dim(H^1(G,1)) + max\{\dim(H^1(G,\chi)), \dim(H^1(G,\chi^{-1}))\}$.

Proof. (1) Let $k=\dim(H^1(G,1))$, $m=\dim(H^1(G,\chi))$ and $n=\dim(H^1(G,\chi^{-1}))$. Recall that, $\mathrm{Ext}^1_G(\eta,\delta)\simeq H^1(G,\delta/\eta)$ and $\mathrm{Ext}^2_G(\eta,\eta)\simeq H^2(G,1)$ for any continuous characters η, $\delta : G \to \mathbb{F}^*$. Hence, from [1, Theorem 2], it follows that $\dim(\tan(R^{\mathrm{pd}}_{\bar\rho_0})) = 2k+mn$ (see also [4, Proposition 20]). As $m\neq 0$ and $n\neq 0$, $\dim(\tan(R^{\mathrm{pd}}_{\bar\rho_0})) > 2k$.

Suppose there exists a continuous representation $\rho : G \to \mathrm{GL}_2(R^{\mathrm{pd}}_{\bar\rho_0})$ such that $\mathrm{tr}\,\rho = t^{\mathrm{univ}}$. Let $\bar\rho$ be its reduction modulo $\mathfrak{m}$. As $\mathrm{tr}\,\bar\rho = \mathrm{tr}\,\bar\rho_0$, it follows, from the Brauer-Nesbitt theorem, that $\bar\rho$ is isomorphic over $\mathbb{F}$ to either $\bar\rho_0$ or $\bar\rho_x$ for some $x\in H^1(G,\chi)$ or $H^1(G,\chi^{-1})$ with $x\neq 0$.

Suppose $\bar\rho\simeq\bar\rho_0$. So, by changing the basis if necessary, we can assume that $\bar\rho=\bar\rho_0$. For $g\in G$, let $\rho(g) = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$. Therefore, we see that b_g, $c_g \in \mathfrak{m}$. Moreover, we can change the basis such that $a_g \equiv \chi_1(g) \pmod{\mathfrak{m}}$ and $d_g \equiv \chi_2(g) \pmod{\mathfrak{m}}$. Thus, we get two characters $\tilde\chi_1$, $\tilde\chi_2 : G \to (R^{\mathrm{pd}}_{\bar\rho_0}/\mathfrak{m}^2)^*$ sending g to $a_g \pmod{\mathfrak{m}^2}$ and $d_g \pmod{\mathfrak{m}^2}$, respectively. Moreover, the pseudo-character $t^{\mathrm{univ}} \pmod{\mathfrak{m}^2} = \mathrm{tr}(\rho) \pmod{\mathfrak{m}^2}$ is the sum of characters $\tilde\chi_1$ and $\tilde\chi_2$.

Thus, we see, from [3, Proposition 1.5.1], that the quotient map $R^{\mathrm{pd}}_{\bar\rho_0} \to R^{\mathrm{pd}}_{\bar\rho_0}/\mathfrak{m}^2$ factors through $R^{\mathrm{pd}}_{\bar\rho_0}/I_{\bar\rho_0}$. By Lemma 2.3, it follows that $\dim(\tan(R^{\mathrm{pd}}_{\bar\rho_0}/\mathfrak{m}^2)) \leq 2k$. But this contradicts the fact that $\dim(\tan(R^{\mathrm{pd}}_{\bar\rho_0})) > 2k$. So, we conclude that $\bar\rho \not\simeq \bar\rho_0$.

Thus, $\bar\rho\simeq\bar\rho_x$ for some $x\in H^1(G,\chi^i)$ with $i\in\{1,-1\}$ and $x\neq 0$. So, by changing the basis if necessary, we can assume that $\bar\rho=\bar\rho_x$. This means that ρ is a deformation of $\bar\rho_x$ and hence, there exists a continuous morphism $\phi_x : R^{\mathrm{def}}_{\bar\rho_x} \to R^{\mathrm{pd}}_{\bar\rho_0}$. Moreover, ϕ_x is surjective as the elements $t^{\mathrm{univ}}(g) = \mathrm{tr}(\rho(g))$ with $g\in G$ are topological generators of $R^{\mathrm{pd}}_{\bar\rho_0}$ as a local complete $\mathbb{F}$-algebra ([11, Remark 3.5]). So, in particular, $\dim(\tan(R^{\mathrm{def}}_{\bar\rho_x})) \geq \dim(\tan(R^{\mathrm{pd}}_{\bar\rho_0}))$.

From Lemma 2.1, we know that $\dim(\tan(R^{\mathrm{def}}_{\bar\rho_x})) \leq 2k+m+n-1$. So, we get that $2k+m+n-1 \geq 2k+mn$ which implies that $0\geq (m-1)(n-1)$. Therefore, we conclude that either $m=1$ or $n=1$.

(2) Let $k=\dim(H^1(G,1))$, $m=\dim(H^1(G,\chi))$ and $n=\dim(H^1(G,\chi^{-1}))$. By [1, Theorem 2], we have the following exact sequence:

$0 \to \mathrm{Ext}^1_G(\chi_1,\chi_1) \oplus \mathrm{Ext}^1_G(\chi_2,\chi_2) \to \tan(R^{\mathrm{pd}}_{\bar\rho_0}) \xrightarrow{i} \mathrm{Ext}^1_G(\chi_1,\chi_2) \otimes \mathrm{Ext}^1_G(\chi_2,\chi_1) \to \mathrm{Ext}^2_G(\chi_1,\chi_1) \oplus \mathrm{Ext}^2_G(\chi_2,\chi_2)$.

If there exists a continuous representation $\rho : G \to \mathrm{GL}_2(R^{\mathrm{pd}}_{\bar\rho_0})$ such that $\operatorname{tr}\rho = t^{\mathrm{univ}}$, then every pseudo-character of G to $\mathbb{F}[\epsilon]/(\epsilon^2)$ deforming $\operatorname{tr}\bar\rho_0$ is the trace of a representation defined over $\mathbb{F}[\epsilon]/(\epsilon^2)$. So, from [1, Theorem 4], it follows that Image(i) consists only of pure tensors. Hence, Image(i) is a subspace of either $\mathrm{Ext}^1_G(\chi_1,\chi_2) \otimes V_0$ with V_0 a subspace of $\mathrm{Ext}^1_G(\chi_2,\chi_1)$ of dimension at most 1 or $W_0 \otimes \mathrm{Ext}^1_G(\chi_2,\chi_1)$ with W_0 a subspace of $\mathrm{Ext}^1_G(\chi_1,\chi_2)$ of dimension at most 1. So, it follows that $\dim(\mathrm{Image}(i)) \leq \max\{m,n\}$. Therefore, it follows that $\dim(\tan(R^{\mathrm{pd}}_{\bar\rho_0})) = 2k + \dim(\mathrm{Image}(i)) \leq 2k + \max\{m,n\}$.

□

Remark 2.5. *Suppose $G = G_{\mathbb{Q},Np}$ and $\bar\rho_0$ is an odd representation of $G_{\mathbb{Q},Np}$. So both χ and χ^{-1} will be odd characters of $G_{\mathbb{Q},Np}$. Thus, it follows from the global Euler characteristic formula, that $m \neq 0$ and $n \neq 0$. If $p \nmid \phi(N)$, we have $\dim(H^1(G_{\mathbb{Q},Np},1)) = 1$. Therefore, by Tate's global Euler characteristic formula, $H^2(G_{\mathbb{Q},Np},1) = 0$. If $p|\phi(N)$, we have $\dim(H^1(G_{\mathbb{Q},Np},1)) > 1$. Thus, by Tate's global Euler characteristic formula, $\dim(H^2(G_{\mathbb{Q},Np},1)) = \dim(H^1(G_{\mathbb{Q},Np},1)) - 1 > 0$ and hence, $H^2(G_{\mathbb{Q},Np},1) \neq 0$. Therefore, when $\bar\rho_0$ is odd, $G_{\mathbb{Q},Np}$ satisfies hypotheses of the first part of Proposition 2.4 if $p \nmid \phi(N)$ and $G_{\mathbb{Q},Np}$ satisfies hypotheses of the second part of Proposition 2.4 if $p|\phi(N)$.*

Remark 2.6. *The first part of the proposition also follows from* [1, Theorem 4].

Remark 2.7. *Let us assume $p|\phi(N)$. Let $k = \dim H^1(G_{\mathbb{Q},Np},1)$, $m = \dim(H^1(G_{\mathbb{Q},Np},\chi))$ and $n = \dim(H^1(G_{\mathbb{Q},Np},\chi^{-1}))$. It follows, from Lemma 2.2, that $\dim(tan(R^{def}_{\bar\rho_x})) \leq m + n + 2k - 1$. On the other hand, from* [1, Theorem 2] *and the global Euler characteristic formula, it follows that $\dim(tan(R^{pd}_{\bar\rho_0})) \geq 2 + mn$. Therefore, from the arguments used in the proof of the part 1 of the proposition above, we can conclude that there does not exist a continuous $\rho : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(R^{pd}_{\bar\rho_0})$ such that $\operatorname{tr}\rho = t^{univ}$ if $2 + mn > m + n + 2k - 1$ i.e. if $(m-1)(n-1) > 2k - 2$.*

2.2. **Reminder on Generalized Matrix Algebras (GMAs).** In this subsection, we recall some standard definitions and results about Generalized Matrix Algebras which will be used frequently in the rest of the article. From now on, we will use the abbreviation GMA for Generalized Matrix Algebra.

We first recall the definition of a topological Generalized Matrix Algebra of type $(1,1)$. Let R be a complete Noetherian local ring with maximal ideal m_R and residue field $\mathbb{F}$. So, R is a topological ring under the m_R-adic topology which we fix from now on. Let $A = \begin{pmatrix} R & B \\ C & R \end{pmatrix}$ be a topological GMA of type $(1,1)$ over R. This means the following:

(1) B and C are topological R-modules,
(2) An element of A is of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, d \in R$, $b \in B$ and $c \in C$,
(3) There exists a continuous morphism $m' : B \otimes_R C \to R$ of R-modules such that for all b_1, $b_2 \in B$ and c_1, $c_2 \in C$, $m'(b_1 \otimes c_1)b_2 = m'(b_2 \otimes c_1)b_1$ and $m'(b_1 \otimes c_1)c_2 = m'(b_1 \otimes c_2)c_1$.

So, A is a topological R-algebra with the addition given by $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$, the multiplication given by $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + m'(b_1 \otimes c_2) & a_1b_2 + d_2b_1 \\ d_1c_2 + a_2c_1 & d_1d_2 + m'(b_2 \otimes c_1) \end{pmatrix}$ and the topology given by the topology on R, B and C. For more information, we refer the reader to [2, Section 2.2] (for GMAs of type $(1, 1)$), [2, Section 2.3] (for topological GMAs) and [3, Chapter 1] (for the general theory of GMAs).

For the rest of this article, GMA means topological GMA unless mentioned otherwise. By abuse of notation, we will always denote by m' the multiplication map $B \otimes_R C \to R$ for any GMA and any R. From now on, given a profinite group G and a GMA A, a representation $\rho : G \to A^*$ means a continuous homomorphism from G to A^* unless mentioned otherwise. For a topological R-module B, we denote by $\mathrm{Hom}_R(B/m_RB, \mathbb{F})$ the set of all continuous R-module homomorphisms from B/m_RB to $\mathbb{F}$.

Let $A = \begin{pmatrix} R & B \\ C & R \end{pmatrix}$ be a GMA with the map $m : B \otimes_R C \to R$ giving the multiplication in A. We say that A is faithful if the following conditions hold:

(1) If $b \in B$ and $m'(b \otimes c) = 0$ for all $c \in C$, then $b = 0$,
(2) If $c \in C$ and $m'(b \otimes c) = 0$ for all $b \in B$, then $c = 0$.

We say that A' is an R-sub-GMA of A if there exists an R-submodule B' of B and an R-submodule C' of C such that $A' = \begin{pmatrix} R & B' \\ C' & R \end{pmatrix}$ i.e. $A' = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A | b \in B', c \in C'\}$ (see [2, Section 2.1] for the definitions of sub-GMA and R-sub-GMA). Note that, A' is a sub-algebra of A and hence, a GMA over R.

Lemma 2.8. *Let R be a complete Noetherian local ring with maximal ideal m_R and residue field $\mathbb{F}$. Let $t : G \to R$ be a pseudo-character deforming $\mathrm{tr}(\bar\rho_0)$. Then, there exists a faithful GMA $A = \begin{pmatrix} R & B \\ C & R \end{pmatrix}$ and a representation $\rho : G \to A^*$ such that*

(1) $t = \mathrm{tr}(\rho)$,
(2) For $g \in G$, if $\rho(g) = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$, then $a_g \equiv \chi_1(g) \pmod{m_R}$ and $d_g \equiv \chi_2(g) \pmod{m_R}$,
(3) $m'(B \otimes_R C) \subset m_R$, where m is the map giving the multiplication in A and the total reducibility ideal of $\tilde t$ is $m'(B \otimes_R C)$,

(4) $R[\rho(G)] = A$*,*

(5) B *and* C *are finitely generated* R*-modules,*

(6) Number of generators of B *as an* R*-module* $\leq \dim(H^1(G,\chi))$ *and number of generators of* C *as an* R*-module* $\leq \dim(H^1(G,\chi^{-1}))$

Proof. The existence A and ρ with most of the given properties follows from parts (i), (v), (vii) of [2, Proposition 2.4.2]. The only claims not implied by [2, Proposition 2.4.2] are the assertions about the reducibility ideal of $\tilde{t}$ and the number of generators of B and C. Note that, $R[G]/\ker(\tilde{t})$ is a Cayley-Hamilton quotient of $(R[G],\tilde{t})$. Hence, from [3, Proposition 1.5.1], we get that the total reducibility ideal of $\tilde{t}$ is $m'(B \otimes_R C)$.

The proof of the last assertion is same as that of [3, Theorem 1.5.5]. We only give a brief summary here. Note that, $R[G] = A$. Given $f \in \mathrm{Hom}_R(B/m_RB,\mathbb{F})$, we get a morphism of R-algebras $f^* : A \to M_2(\mathbb{F})$, sending an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of A to $\begin{pmatrix} a \pmod{m} & f(b) \\ 0 & d \pmod{m} \end{pmatrix}$. Thus restricting f^* to G gives us an extension of χ_2 by χ_1 and hence, an element $\tilde{f}^*$ of $H^1(G,\chi)$ (see proof of [3, Theorem 1.5.5] for more details). So, we get a linear map $j : \mathrm{Hom}_R(B/m_RB,\mathbb{F}) \to H^1(G,\chi)$ sending f to $\tilde{f}^*$. From the proof of [3, Theorem 1.5.5], we get that the map j is injective. Hence, Nakayama's lemma gives the assertion about the number of generators of B. The assertion about the number of generators of C follows similarly. □

Remark 2.9. *It follows, from Lemma 2.8, that if* $H^1(G,\chi^i) = 0$ *for some* $i \in \{1,-1\}$*, then* t^{univ} *is reducible and hence, there exists a* 2*-dimensional* G*-representation over* $R^{pd}_{\bar{\rho}_0}$ *whose trace is* t^{univ}*.*

We now turn our attention to the case when $G = G_{\mathbb{Q},Np}$ and state some results which will be used later.

Lemma 2.10. *Let* R *be a complete Noetherian local ring with maximal ideal* m_R *and residue field* $\mathbb{F}$*. Let* ℓ *be a prime such that* $\ell \nmid Np$ *and* $\chi|_{G_{\mathbb{Q}_\ell}} \neq 1$*. Let* $t : G_{\mathbb{Q},N\ell p} \to R$ *be a pseudo-character deforming* $\mathrm{tr}(\bar{\rho}_0)$*. Let* g_ℓ *be a lift of* $Frob_\ell$ *in* $G_{\mathbb{Q}_\ell}$*. Then, there exists a faithful GMA* $A = \begin{pmatrix} R & B \\ C & R \end{pmatrix}$ *and a representation* $\rho : G_{\mathbb{Q},N\ell p} \to A^*$ *such that*

(1) $t = \mathrm{tr}(\rho)$*,*

(2) $m'(B \otimes_R C) \subset m_R$*, where* m' *is the map giving the multiplication in* A*,*

(3) $R[\rho(G_{\mathbb{Q},N\ell p})] = A$*,*

(4) B *and* C *are finitely generated* R*-modules,*

(5) $\rho(g_\ell) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$*, where* $a \equiv \chi_1(Frob_\ell) \pmod{m_R}$ *and* $d \equiv \chi_2(Frob_\ell) \pmod{m_R}$*.*

Moreover, $R[\rho(G_{\mathbb{Q}_\ell})]$ *is a sub* R*-GMA of* A*.*

Proof. The existence A and ρ with given properties follows from parts (i), (iii), (v), (vi) of [2, Proposition 2.4.2]. The rest of the lemma follows from [2, Lemma 2.4.5]. $\square$

Lemma 2.11. *Let R be a complete Noetherian local ring with maximal ideal m_R and residue field $\mathbb{F}$. Let ℓ be a prime such that $\ell \nmid Np$ and $\chi|_{G_{\mathbb{Q}_\ell}} \neq 1$. Let g_ℓ be a lift of Frob$_\ell$ in $G_{\mathbb{Q}_\ell}$. Let $t : G_{\mathbb{Q},N\ell p} \to R$ be a pseudo-character deforming $\mathrm{tr}(\bar\rho_0)$. Let A be the GMA associated to the tuple (R, ℓ, t, g_ℓ) in Lemma 2.10 and let $\rho : G_{\mathbb{Q},N\ell p} \to A^*$ be the corresponding representation found in Lemma 2.10. Then, $\rho|_{I_\ell}$ factors through the $\mathbb{Z}_p$-quotient of the tame inertia group at ℓ.*

Proof. Let K_0 be the maximal extension of $\mathbb{Q}$ unramified outside the set of primes dividing $N\ell p$ and ∞. So, $G_{\mathbb{Q},N\ell p} = \mathrm{Gal}(K_0/\mathbb{Q})$. Let K be the extension of $\mathbb{Q}$ fixed by $\ker(\bar\rho_0)$. So, K is a sub-extension of K_0 and K is unramified at ℓ. By [11, Lemma 3.8], the pseudo-character $\tilde t$ factors through $R[G_{\mathbb{Q},Np}/H]$, where $H \in \mathrm{Gal}(K_0/K)$ is the smallest closed normal subgroup such that $\mathrm{Gal}(K_0/K)/H$ is a pro-p quotient of $\mathrm{Gal}(K_0/K)$.

Since A is faithful, $\rho|_{\mathrm{Gal}(K_0/K)}$ factors through $\mathrm{Gal}(K_0/K)/H$. Indeed, let g be an element of H. As $\tilde t$ factors through $R[G_{\mathbb{Q},Np}/H]$, we get $\tilde t(xg) = \tilde t(x)$ for all $x \in R[G_{\mathbb{Q},Nellp}]$. So, in particular $\mathrm{tr}(\rho(g'g)) = \mathrm{tr}(\rho(g'))$ for all $g' \in G_{\mathbb{Q},N\ell p}$. So, $\mathrm{tr}(\rho(g)) = 2$ and hence, $\rho(g) = \begin{pmatrix} 1+a & b \\ d & 1-a \end{pmatrix}$. Now, $\mathrm{tr}(\rho(g_\ell g)) = \mathrm{tr}(\rho(g_\ell))$ implies that $a = 0$. Let $A = \begin{pmatrix} R & B \\ C & R \end{pmatrix}$. As $R[\rho(G_{\mathbb{Q},N\ell p})] = A$, we get $\mathrm{tr}(\begin{pmatrix} 1 & b \\ c & 1 \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}) = \mathrm{tr}(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix})$ for all $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in A$. Putting $b' = a' = d' = 0$, we get $m'(b \otimes c') = 0$ for all $c' \in C$. So faithfulness of A implies $b = 0$. Similarly, putting $c' = a' = d' = 0$ gives us $c = 0$ which proves the claim.

As K is unramified at ℓ, we see that $\rho|_{I_\ell}$ factors through the $\mathbb{Z}_p$-quotient of the tame inertia group at ℓ. $\square$

2.3. **Existence of the representation over $(R^{pd}_{\bar\rho_0})^{\mathrm{red}}$.** It follows, from Proposition 2.4, that if $H^2(G, 1) = 0$, then $\dim(H^1(G, \chi^i)) = 1$ for some $i \in \{1, -1\}$ is a necessary condition for t^{univ} to be the trace of a representation defined over $R^{\mathrm{pd}}_{\bar\rho_0}$. Now, we explore if this condition is sufficient for t^{univ} to be the trace of a representation defined over $R^{\mathrm{pd}}_{\bar\rho_0}$. Let $t^{\mathrm{univ,red}}$ be the pseudo-character obtained by composing t^{univ} with the surjective map $R^{\mathrm{pd}}_{\bar\rho_0} \to (R^{\mathrm{pd}}_{\bar\rho_0})^{\mathrm{red}}$. We will now prove the existence of such a representation for $t^{\mathrm{univ,red}}$.

We first prove a lemma about the structure of $R^{\mathrm{pd}}_{\bar\rho_0}$

Lemma 2.12. *Suppose $H^2(G, 1) = 0$, $\dim(H^1(G, 1)) = k$ and $\dim(H^1(G, \chi^i)) = 1$ for some $i \in \{1, -1\}$. For such an i, let $\dim(H^1(G, \chi^{-i})) = m$, $\dim(H^2(G, \chi^{-i})) = m'$*

and $\dim(H^2(G,\chi^i)) = n'$. *Then,* $R^{pd}_{\bar\rho_0} \simeq \mathbb{F}[[X_1,\cdots,X_{m+2k}]]/I$ *where* I *is an ideal of* $\mathbb{F}[[X_1,\cdots,X_{m+2k}]]$ *generated by at most* $m'+mn'$ *elements.*

Proof. Since we are assuming that $\dim(H^1(G,\chi^i)) = 1$ for some $i \in \{1,-1\}$, it follows, from [19], that no relations in $R^{\text{pd}}_{\bar\rho_0}$ arise from invariant theory and so, all of them come from H^2 (see [19] for more details). As a consequence, we see that the map given in Theorem 2.3.1 of summary of [19] is injective. Now, $\dim(\text{Ext}^1_G(\chi_1,\chi_1)) = \dim(\text{Ext}^1_G(\chi_2,\chi_2)) = k$. Hence, by [19, Theorem 2.3.2 of summary], it follows that $R^{\text{pd}}_{\bar\rho_0} \simeq \mathbb{F}[[X_1,\cdots,X_{m+2k}]]/I$ where I is an ideal of $\mathbb{F}[[X_1,\cdots,X_{m+2}]]$ generated by at most k_0 elements, where $k_0 = \sum_{j=1}^{2}\dim(\text{Ext}^2_G(\chi_j,\chi_j)) + \dim(\text{Ext}^2_G(\chi_1,\chi_2)).\dim(\text{Ext}^1_G(\chi_2,\chi_1)) + \dim(\text{Ext}^2_G(\chi_2,\chi_1)).\dim(\text{Ext}^1_G(\chi_1,\chi_2))$.

Recall that, $\text{Ext}^2_G(\eta,\delta) \simeq H^2(G,\delta/\eta)$ for any characters η, $\delta : G \to \mathbb{F}^*$ and we have assumed $H^2(G,1) = 0$. Therefore, we see that $k = \sum_{j=1}^{2} 0 + (m').1 + m.n' = m' + mn'$. □

Theorem 2.13. *Suppose* $H^2(G,1) = 0$. *Suppose there exists an* $i \in \{1,-1\}$ *such that* $\dim(H^1(G,\chi^i)) = 1$, $H^2(G,\chi^i) = 0$ *and* $\dim(H^2(G,\chi^{-i})) < \dim(H^1(G,\chi^{-i}))$. *Then there exists a representation* $\rho^{red} : G \to \text{GL}_2((R^{pd}_{\bar\rho_0})^{red})$ *such that* $\text{tr}(\rho^{red}) = t^{univ,red}$. *As a consequence, for a non-zero* $x \in H^1(G,\chi^i)$, *the map* $\psi_x : R^{pd}_{\bar\rho_0} \to R^{def}_{\bar\rho_x}$ *induces an isomorphism between* $(R^{pd}_{\bar\rho_0})^{red}$ *and* $(R^{def}_{\bar\rho_x})^{red}$.

Before proving the theorem, we prove a couple of lemmas which will be used in the proof of the Proposition. Let us set up the notation first. Let $K_{\bar\rho_0}$ be the total fraction field of $(R^{\text{pd}}_{\bar\rho_0})^{\text{red}}$. So, $K_{\bar\rho_0} = \prod_{P\in S} K_P$, where S is the set of minimal primes of $(R^{\text{pd}}_{\bar\rho_0})^{\text{red}}$ and K_P is the fraction field of $(R^{\text{pd}}_{\bar\rho_0})^{\text{red}}/P$. As $(R^{\text{pd}}_{\bar\rho_0})^{\text{red}}$ is Noetherian, S is a finite set.

Lemma 2.14. *Suppose* $H^2(G,1) = 0$. *Suppose there exists an* $i \in \{1,-1\}$ *such that* $\dim(H^1(G,\chi^i)) = 1$, $H^2(G,\chi^i) = 0$ *and* $\dim(H^2(G,\chi^{-i})) < \dim(H^1(G,\chi^{-i}))$. *If* $P \in S$, *then the pseudo-character* $t^{univ,red} \pmod{P}$ *is not reducible.*

Proof. Without loss of generality, assume $\dim(H^1(G,\chi)) = 1$. Let $\dim(H^1(G,1)) = k$, $\dim(H^1(G,\chi^{-1})) = m$ and $\dim(H^2(G,\chi^{-1})) = m'$. As we are assuming that $H^2(G,1) = H^2(G,\chi) = 0$, we get, by Lemma 2.12, that $R^{\text{pd}}_{\bar\rho_0} \simeq \mathbb{F}[[X_1,\cdots,X_{m+2k}]]/I$ where I is an ideal generated by at most m' elements. Let Q be a prime ideal of $\mathbb{F}[[X_1,\cdots,X_{m+2k}]]$ which is minimal over I. As I is an ideal of $\mathbb{F}[[X_1,\cdots,X_{m+2k}]]$ generated by at most m' elements, it follows, from the generalization of Krull's principal ideal theorem ([14, Theorem 10.2]), that the height of Q is at most m'. Since $\mathbb{F}[[X_1,\cdots,X_{m+2k}]]$ is universally catenary, we have that the Krull dimension of $\mathbb{F}[[X_1,\cdots,X_{m+2k}]]/Q$ is at least $2k+m-m'$. Since we are assuming $m > m'$, we see that the Krull dimension of $\mathbb{F}[[X_1,\cdots,X_{m+2k}]]/Q$ is greater than $2k$.

Let Q' be the image of Q in $R^{\mathrm{pd}}_{\bar{\rho}_0}$ under the surjective map $\mathbb{F}[[X_1, \cdots, X_{m+2k}]] \to R^{\mathrm{pd}}_{\bar{\rho}_0}$. So, Q' is a minimal prime of $R^{\mathrm{pd}}_{\bar{\rho}_0}$ and $R^{\mathrm{pd}}_{\bar{\rho}_0}/Q' \simeq \mathbb{F}[[X_1, \cdots, X_{m+2k}]]/Q$. If $I_{\bar{\rho}_0} \subset Q'$, then Lemma 2.3 implies that $\dim(R^{\mathrm{pd}}_{\bar{\rho}_0}/Q') \leq 2k$. But, from the previous paragraph, we know that $\dim(R^{\mathrm{pd}}_{\bar{\rho}_0}/Q') > 2k$. Hence, we get that $I_{\bar{\rho}_0} \not\subset Q'$.

Therefore, we get, from [3, Proposition 1.5.1], that the pseudo-character $\tilde{t}^{\mathrm{univ}} \otimes_{R^{\mathrm{pd}}_{\bar{\rho}_0}} R^{\mathrm{pd}}_{\bar{\rho}_0}/Q'$ is not a sum of two one dimensional pseudo-characters $\tilde{\eta}_1$ and $\tilde{\eta}_2$ of $R^{\mathrm{pd}}_{\bar{\rho}_0}/Q'[G]$ taking values in $R^{\mathrm{pd}}_{\bar{\rho}_0}/Q'$ such that $\tilde{\eta}_i \otimes_{R^{\mathrm{pd}}_{\bar{\rho}_0}/Q'} \mathbb{F} = \tilde{\chi}_i$ for $i = 1, 2$. Hence, it follows that $t^{\mathrm{univ}} \pmod{Q'}$ is not reducible which proves the lemma. □

Note that, given any non-zero element $x \in H^1(G, \chi^i)$ with $i \in \{1, -1\}$, one has a map $\psi_x : R^{\mathrm{pd}}_{\bar{\rho}_0} \to R^{\mathrm{def}}_{\bar{\rho}_x}$ induced by the trace of ρ^{univ}_x. We now recall a result due to Kisin ([15, Corollary 1.4.4(2)]) on the nature of the map ψ_x:

Lemma 2.15. *If* $\dim(H^1(G, \chi^i)) = 1$ *for some* $i \in \{1, -1\}$ *and* $x \in H^1(G, \chi^i)$ *is a non-zero element, then the map* $\psi_x : R^{pd}_{\bar{\rho}_0} \to R^{def}_{\bar{\rho}_x}$ *is surjective.*

Proof of Theorem 2.13. Without loss of generality, assume $\dim(H^1(G, \chi)) = 1$. Let $A^{\mathrm{red}} = \begin{pmatrix} (R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}} & B^{\mathrm{red}} \\ C^{\mathrm{red}} & (R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}} \end{pmatrix}$ be the GMA obtained for the pseudo-character $t^{\mathrm{univ,red}} : G \to (R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}}$ in Lemma 2.8 and ρ be the corresponding representation. By [3, Theorem 1.4.4, Part(ii)], we can assume that B^{red} and C^{red} are fractional ideals of $K_{\bar{\rho}_0}$ and the multiplication map $m'(B^{\mathrm{red}} \otimes_{(R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}}} C^{\mathrm{red}})$ is given by the multiplication in $K_{\bar{\rho}_0}$. We keep this assumption for rest of the proof.

By Lemma 2.8, we see that B^{red} is generated by at most 1 element. If $B^{\mathrm{red}} = 0$, then we get that $t^{\mathrm{univ,red}}$ is a sum of two characters of $G_{\mathbb{Q},Np}$ taking values in $((R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}})^*$ and deforming χ_1 and χ_2. By Lemma 2.14, this is not possible. Hence, we get $B^{\mathrm{red}} \neq 0$ and it is generated by 1 element.

Let b be a generator of B^{red}. Viewing B^{red} as a submodule of $K_{\bar{\rho}_0} = \prod_{P \in S} K_P$, we can write $b = (b_P)_{P \in S}$ with $b_P \in K_P$ for all $P \in S$. Suppose $b_{P_0} = 0$ for some $P_0 \in S$. In this case, $b.C^{\mathrm{red}} \subset P_0$ as $b.C^{\mathrm{red}} \subset (R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}}$. Therefore, we have $B^{\mathrm{red}}.C^{\mathrm{red}} \subset P_0$. For $g \in G$, suppose $\rho(g) = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$. Then, $B^{\mathrm{red}}.C^{\mathrm{red}} \subset P_0$ and Lemma 2.8 together imply that the maps $G \to ((R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}}/P_0)^*$ sending g to $a_g \pmod{P_0}$ and $d_g \pmod{P_0}$ are characters of G deforming χ_1 and χ_2, respectively. But we know that this is not possible by Lemma 2.14. Hence, we get that $b_P \neq 0$ for all $P \in S$.

Thus, we see that b is a unit in $K_{\bar{\rho}_0}$. If $Q = \begin{pmatrix} b^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(K_{\bar{\rho}_0})$, then $QA^{\mathrm{red}}Q^{-1} = \begin{pmatrix} (R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}} & (R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}} \\ b.C^{\mathrm{red}} & (R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}} \end{pmatrix}$. As $b.C^{\mathrm{red}} \subset (R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}}$, we see that $QA^{\mathrm{red}}Q^{-1}$ is a $(R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}}$-sub-GMA of $M_2((R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}})$. Hence, we see that $Q\rho Q^{-1}$ is the required representation. This finishes the proof of the first part of the theorem.

We will now prove the second part of the Theorem. Note that, if $\dim(H^1(G,\chi^i)) = 1$ for some $i \in \{1,-1\}$, then $\bar{\rho}_x \simeq \bar{\rho}_{x'}$ for all non-zero x, $x' \in H^1(G,\chi^i)$. Indeed, if x, $x' \in H^1(G,\chi^i)$ are both non-zero, then $x' = ax$ for some non-zero $a \in \mathbb{F}$. Therefore, by conjugating $\bar{\rho}_x$ by the matrix $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, we get $\bar{\rho}_{x'}$. Therefore, we see that $R^{\mathrm{def}}_{\bar{\rho}_x} \simeq R^{\mathrm{def}}_{\bar{\rho}_{x'}}$ for any two non-zero x, $x' \in H^1(G,\chi^i)$ with $\dim(H^1(G,\chi^i)) = 1$.

Without loss of generality, assume $\dim(H^1(G,\chi)) = 1$. Now, we have found a representation $\rho^{\mathrm{red}} : G \to \mathrm{GL}_2((R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}})$ such that $\mathrm{tr}(\rho^{\mathrm{red}}) = t^{\mathrm{univ,red}}$. Moreover, we have also shown that the reduction of ρ^{red} modulo $\mathfrak{m}'$ is a non-split extension of χ_2 by χ_1. Let x_0 be the non-zero element of $H^1(G,\chi)$ such that $\rho^{\mathrm{red}} \pmod{\mathfrak{m}'} = \bar{\rho}_{x_0}$. Thus, ρ^{red} induces a map $R^{\mathrm{def}}_{\bar{\rho}_{x_0}} \to (R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}}$ which is surjective as $\{\mathrm{tr}(\rho^{\mathrm{red}}(g)) | g \in G\}$ is a set of topological generators of $(R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}}$ over $\mathbb{F}$ as a local complete $\mathbb{F}$-algebra. So, this induces a surjective map $(\psi')^{\mathrm{red}}_{x_0} : (R^{\mathrm{def}}_{\bar{\rho}_{x_0}})^{\mathrm{red}} \to (R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}}$.

On the other hand, we have the surjective map $\psi^{\mathrm{red}}_{x_0} : (R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}} \to (R^{\mathrm{def}}_{\bar{\rho}_{x_0}})^{\mathrm{red}}$ induced by ψ_{x_0}. So, the composition $(\psi')^{\mathrm{red}}_{x_0} \circ \psi^{\mathrm{red}}_{x_0}$ gives us a surjective map from $(R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}}$ to itself. Now if $g \in G$, then $(\psi')^{\mathrm{red}}_{x_0} \circ \psi^{\mathrm{red}}_{x_0}(t^{\mathrm{univ,red}}(g)) = (\psi')^{\mathrm{red}}_{x_0}(\mathrm{tr}(\rho^{\mathrm{univ}}_{x_0}(g))) = \mathrm{tr}(\rho^{\mathrm{red}}(g)) = t^{\mathrm{univ,red}}(g)$. Hence, $(\psi')^{\mathrm{red}}_{x_0} \circ \psi^{\mathrm{red}}_{x_0}$ is identity and as a consequence, $\psi^{\mathrm{red}}_{x_0} : R^{\mathrm{pd}}_{\bar{\rho}_0} \to R^{\mathrm{def}}_{\bar{\rho}_{x_0}}$ is an isomorphism. □

Remark 2.16. *The first part of Theorem 2.17 can also be proved using* [3, Proposition 1.7.4]*. Indeed our argument is based on the proof of* [3, Proposition 1.7.4].

As a consequence, we get the following theorem for $G = G_{\mathbb{Q},Np}$:

Theorem 2.17. *Suppose $p \nmid \phi(N)$ and $\dim(H^1(G_{\mathbb{Q},Np},\chi^i)) = 1$ for some $i \in \{1,-1\}$. Then there exists a representation $\rho^{red} : G_{\mathbb{Q},Np} \to \mathrm{GL}_2((R^{pd}_{\bar{\rho}_0})^{red})$ such that $\mathrm{tr}(\rho^{red}) = t^{univ,red}$. As a consequence, for a non-zero $x \in H^1(G_{\mathbb{Q},Np},\chi^i)$, the map $\psi_x : R^{pd}_{\bar{\rho}_0} \to R^{def}_{\bar{\rho}_x}$ induces an isomorphism between $(R^{pd}_{\bar{\rho}_0})^{red}$ and $(R^{def}_{\bar{\rho}_x})^{red}$.*

Proof. It suffices to check that the hypotheses of Theorem 2.13 hold in the case considered here. Without loss of assume $\dim(H^1(G_{\mathbb{Q},Np},\chi)) = 1$. So, we have $\dim(H^1(G_{\mathbb{Q},Np},\chi^{-1})) = m$. As we have assumed that $p \nmid \phi(N)$ and $\bar{\rho}_0$ is odd, it follows, from the global Euler characteristic formula, that:

(1) $m \neq 0$,
(2) $H^2(G_{\mathbb{Q},Np}, 1) = 0$,
(3) $H^2(G_{\mathbb{Q},Np}, \chi) = 0$ and $\dim(H^2(G_{\mathbb{Q},Np}, \chi^{-1})) = m - 1$.

Therefore, we see that the hypotheses of Theorem 2.13 are satisfied. Hence, the theorem follows. ☐

2.4. **Existence of the representation over $R^{pd}_{\bar\rho_0}$.** We will now present an improvement of Theorem 2.17 in certain cases. We do so by improving Theorem 2.13.

Theorem 2.18. *Suppose $H^2(G, 1) = 0$. Suppose there exists an $i \in \{1, -1\}$ such that $\dim(H^1(G, \chi^i)) = 1$, $H^2(G, \chi^i) = 0$, $\dim(H^1(G, \chi^{-i})) \in \{1, 2, 3\}$ and $\dim(H^2(G, \chi^{-i})) < \dim(H^1(G, \chi^{-i}))$. Then, there exists a representation $\rho : G \to \mathrm{GL}_2(R^{pd}_{\bar\rho_0})$ such that $\mathrm{tr}(\rho) = t^{univ}$. As a consequence, for a non-zero $x \in H^1(G, \chi^i)$, $R^{pd}_{\bar\rho_0} \simeq R^{def}_{\bar\rho_x}$.*

Proof. Without loss of generality, assume $\dim(H^1(G, \chi)) = 1$. So, we have $\dim(H^1(G, \chi^{-1})) \in \{1, 2, 3\}$. Let $A = \begin{pmatrix} R^{\mathrm{pd}}_{\bar\rho_0} & B \\ C & R^{\mathrm{pd}}_{\bar\rho_0} \end{pmatrix}$ be the GMA attached to the pseudo-character $t^{\mathrm{univ}} : G_{\mathbb{Q},Np} \to R^{\mathrm{pd}}_{\bar\rho_0}$ in Lemma 2.8 and let ρ be the corresponding representation. Note that, A is a faithful GMA. This means that if $y \in R^{\mathrm{pd}}_{\bar\rho_0}$ annihilates B i.e. $yB = 0$ then $y.m'(B \otimes_{R^{\mathrm{pd}}_{\bar\rho_0}} C) = 0$. From Lemma 2.8, we get that $I_{\bar\rho_0} = m'(B \otimes_{R^{\mathrm{pd}}_{\bar\rho_0}} C)$. So, we have for $y \in R^{\mathrm{pd}}_{\bar\rho_0}$, if $y.B = 0$ then $y.I_{\bar\rho_0} = 0$.

By Lemma 2.12, $R^{\mathrm{pd}}_{\bar\rho_0} \simeq \mathbb{F}[[X_1, X_2, \cdots, X_{m+2k}]]/I$, where $k = \dim(H^1(G, 1))$, $m = \dim(H^1(G, \chi^{-1}))$ and I is an ideal of $\mathbb{F}[[X_1, X_2, \cdots, X_{m+2k}]]$ generated by at most $\dim(H^2(G, \chi^{-1})) - 1$ elements. By Lemma 2.3, it follows that $X_1, X_2, \cdots, X_{m+2k}$ can be chosen such that the ideal $I_{\bar\rho_0}$ of $R^{\mathrm{pd}}_{\bar\rho_0}$ is generated by the images of the elements $X_{2k+1}, \cdots, X_{m+2k}$ in $R^{\mathrm{pd}}_{\bar\rho_0}$. Hence, we see that $I \subset (X_{2k+1}, \cdots, X_{m+2k})$ in $\mathbb{F}[[X_1, X_2, \cdots, X_{m+2k}]]$. As $I_{\bar\rho_0} \neq 0$, we see that $m'(B \otimes_{R^{\mathrm{pd}}_{\bar\rho_0}} C)$ is not zero and hence, B and C are non-zero.

Let $y \in R^{\mathrm{pd}}_{\bar\rho_0}$ be such that $y.B = 0$. So, we have $y.I_{\bar\rho_0} = 0$ in $R^{\mathrm{pd}}_{\bar\rho_0}$. Let $\tilde y$ be a lift of y in $\mathbb{F}[[X_1, X_2, \cdots, X_{m+2k}]]$. So we have $\tilde y.X_i \in I$ for all $2k+1 \leq i \leq m+2k$. Now, we will do a case by case analysis.

Suppose $\dim(H^1(G, \chi^{-1})) = 1$. In this case, $R^{\mathrm{pd}}_{\bar\rho_0} \simeq \mathbb{F}[[X_1, \cdots, X_{2k+1}]]$. In this case, we have $\tilde y.X_{2k+1} = 0$. This implies $\tilde y = 0$ and hence, $y = 0$.

Suppose $\dim(H^1(G, \chi^{-1})) = 2$. In this case, $R^{\mathrm{pd}}_{\bar\rho_0} \simeq \mathbb{F}[[X_1, \cdots, X_{2k+2}]]/I$, where I is either (0) or (α) for some non-zero $\alpha \in \mathbb{F}[[X_1, \cdots, X_{2k+2}]]$. We have $\tilde y.X_{2k+1}, \tilde y.X_{2k+2} \in I$. If $I = (0)$, then this implies $\tilde y = 0$ and hence, $y = 0$.

Suppose $I = (\alpha)$ for some non-zero $\alpha \in \mathbb{F}[[X_1, \cdots, X_{2k+2}]]$. So we have $\alpha | \tilde y.X_{2k+1}$ and $\alpha | \tilde y.X_{2k+2}$. As $\mathbb{F}[[X_1, \cdots, X_{2k+2}]]$ is a regular local ring, it is also a UFD ([14, Theorem

19.19]). Note that, X_{2k+1} and X_{2k+2} are irreducible elements in it. So, $\alpha|\tilde{y}.X_{2k+1}$ implies that if f is an irreducible element of $\mathbb{F}[[X_1, \cdots, X_{2k+2}]]$ such that $f \neq X_{2k+1}$, then the highest power of f dividing $\alpha \leq$ the highest power of f dividing $\tilde{y}$. Similarly, $\alpha|\tilde{y}.X_{2k+2}$ implies that if f is an irreducible element of $\mathbb{F}[[X_1, \cdots, X_{2k+2}]]$ such that $f \neq X_{2k+2}$, then the highest power of f dividing $\alpha \leq$ the highest power of f dividing $\tilde{y}$. So, we conclude that for any irreducible $f \in \mathbb{F}[[X_1, \cdots, X_{2k+2}]]$, the highest power of f dividing $\alpha \leq$ the highest power of f dividing $\tilde{y}$ and hence, $\alpha|\tilde{y}$. So, we get $y = 0$.

Suppose $\dim(H^1(G_{\mathbb{Q},Np}, \chi^{-1})) = 3$. In this case, $R^{\mathrm{pd}}_{\bar{\rho}_0} \simeq \mathbb{F}[[X_1, \cdots, X_{2k+3}]]/I$, where I is generated by at most 2 elements and we have $\tilde{y}.X_{2k+1}$, $\tilde{y}.X_{2k+2}$, $\tilde{y}.X_{2k+3} \in I$. If $I = (0)$, then this implies $\tilde{y} = 0$ and hence, $y = 0$. If $I = (\alpha)$ for some non-zero α, then the argument given in the previous case implies that $\tilde{y} \in (\alpha)$ and hence, $y = 0$. Let us denote $\mathbb{F}[[X_1, \cdots, X_{2k+2}]]$ by R for the rest of the proof.

Suppose $I = (\alpha, \beta)$ with $\alpha \nmid \beta$ and $\beta \nmid \alpha$. Note that, as $I \subset (X_{2k+1}, X_{2k+2}, X_{2k+3})$, the image of the prime ideal $(X_{2k+1}, X_{2k+2}, X_{2k+3})$ in R/I is also a prime ideal. Now, R is regular local ring and hence, a UFD. So, we can find f, α' and $\beta' \in R$ such that $f.\alpha' = \alpha$, $f.\beta' = \beta$ and α' and β' are co-prime. By the argument given in the previous case, we get $f|\tilde{y}$. Let $\tilde{y}' = \tilde{y}/f$. So, $\tilde{y}' \in R$ and $\tilde{y}'.X_{2k+1}$, $\tilde{y}'.X_{2k+2}$, $\tilde{y}'.X_{2k+3} \in (\alpha', \beta')$.

Suppose $\tilde{y}' \notin (\alpha', \beta')$. Then, it follows that the ideal $(X_{2k+1}, X_{2k+2}, X_{2k+3})$ of R consists of zero-divisors for $R/(\alpha', \beta')$. Hence, it is contained in the union of primes associated to the ideal (α', β'). As $(X_{2k+1}, X_{2k+2}, X_{2k+3})$ is a prime ideal, it follows, from the prime avoidance lemma ([14, Lemma 3.3]), that $(X_{2k+1}, X_{2k+2}, X_{2k+3})$ is contained in some prime associated to (α', β').

Note that, R is a regular local ring and hence, a Cohen-Macaulay ring ([14, Corollary 18.17]). As α' and β' are co-prime, it follows that α', β' is a regular sequence in R. Therefore, every prime associated to (α', β') is minimal over it and hence, has height 2 ([14, Corollary 18.14]). As the height of $(X_{2k+1}, X_{2k+2}, X_{2k+3})$ is 3, it can not be contained in any prime associated to (α', β'). Therefore, we get a contradiction to our assumption that $\tilde{y}' \notin (\alpha', \beta')$. Hence, we have $\tilde{y}' \in (\alpha', \beta')$ and $\tilde{y} \in (f\alpha', f\beta') = (\alpha, \beta) = I$. So, we get $y = 0$.

So, in both cases, we have $y = 0$ which means the annihilator ideal of B is (0).

As we are assuming $\dim(H^1(G, \chi)) = 1$, it follows, from Lemma 2.8, that $\dim_{\mathbb{F}}(\mathrm{Hom}_{R^{\mathrm{pd}}_{\bar{\rho}_0}}(B/\mathfrak{m}B, \mathbb{F})) \leq 1$. On the other hand, we know B is non-zero which means $\dim_{\mathbb{F}}(\mathrm{Hom}_{R^{\mathrm{pd}}_{\bar{\rho}_0}}(B/\mathfrak{m}B, \mathbb{F})) \geq 1$. Therefore, we get that $\dim_{\mathbb{F}}(\mathrm{Hom}_{R^{\mathrm{pd}}_{\bar{\rho}_0}}(B/\mathfrak{m}B, \mathbb{F})) = 1$ which means B is generated by one element over $R^{\mathrm{pd}}_{\bar{\rho}_0}$. This, combined with the fact that annihilator of B is (0), implies that B is a free $R^{\mathrm{pd}}_{\bar{\rho}_0}$-module of rank 1.

Let A' be the $R^{\mathrm{pd}}_{\bar\rho_0}$-sub-GMA of $M_2(R^{\mathrm{pd}}_{\bar\rho_0})$ given by $\begin{pmatrix} R^{\mathrm{pd}}_{\bar\rho_0} & R^{\mathrm{pd}}_{\bar\rho_0} \\ I_{\bar\rho_0} & R^{\mathrm{pd}}_{\bar\rho_0} \end{pmatrix}$. Let γ be a generator of B. Consider the map $\tilde f : A \to A'$ which sends $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A$ to $\begin{pmatrix} a & b' \\ m'(\gamma \otimes c) & d \end{pmatrix}$, where $b' \in R^{\mathrm{pd}}_{\bar\rho_0}$ is such that $b = b'.\gamma$. It is easy to check, using the fact that the multiplication map $m' : B \otimes_{R^{\mathrm{pd}}_{\bar\rho_0}} C \to R^{\mathrm{pd}}_{\bar\rho_0}$ is $R^{\mathrm{pd}}_{\bar\rho_0}$-linear, that $\tilde f$ is a continuous homomorphism of $R^{\mathrm{pd}}_{\bar\rho_0}$-algebras. Now, as $B = R^{\mathrm{pd}}_{\bar\rho_0}\gamma$, $m'(\gamma \otimes c) = 0$ implies that $m'(b \otimes c) = 0$ for all $b \in B$ and hence, $c = 0$. Therefore, it follows that the map $\tilde f$ is injective. Moreover, as $I_{\bar\rho_0} = m'(B \otimes_{R^{\mathrm{pd}}_{\bar\rho_0}} C)$, $B = R^{\mathrm{pd}}_{\bar\rho_0}\gamma$ and m' is $R^{\mathrm{pd}}_{\bar\rho_0}$-linear, we get that $I_{\bar\rho_0} = m'(\gamma \otimes C)$. Hence, $\tilde f$ is also surjective which means that $\tilde f$ is an isomorphism of $R^{\mathrm{pd}}_{\bar\rho_0}$-algebras. Note that, if $a \in A$, then $\mathrm{tr}(a) = \mathrm{tr}(\tilde f(a))$.

Composing the representation $\rho : G \to A^*$ with the map $\tilde f$, we get a representation $\rho' : G \to \mathrm{GL}_2(R^{\mathrm{pd}}_{\bar\rho_0})$. As $\mathrm{tr}(a) = \mathrm{tr}(\tilde f(a))$, we see that $\mathrm{tr}(\rho'(g)) = t^{\mathrm{univ}}(g)$. Therefore, ρ' is the required representation.

Moreover, from the description of A', we see that $\rho' \pmod{\mathfrak{m}} = \bar\rho_x$ for some non-zero $x \in H^1(G, \chi)$. Therefore, ρ' is a deformation of $\bar\rho_x$ and hence, it induces a map $\psi'_x : R^{\mathrm{def}}_{\bar\rho_x} \to R^{\mathrm{pd}}_{\bar\rho_0}$. This map is surjective as the set $\{\mathrm{tr}(\rho'(g)) | g \in G\}$ is a set of topological generators of $R^{\mathrm{pd}}_{\bar\rho_0}$ over $\mathbb{F}$. So, we get a surjective map $\psi'_x \circ \psi_x : R^{\mathrm{pd}}_{\bar\rho_0} \to R^{\mathrm{pd}}_{\bar\rho_0}$. Now for all $g \in G$, $\psi'_x \circ \psi_x(t^{\mathrm{univ}}(g)) = \psi'_x(\mathrm{tr}(\rho^{univ}_x(g))) = \mathrm{tr}(\rho'(g)) = t^{\mathrm{univ}}(g)$. Therefore, we see that $\psi'_x \circ \psi_x$ is just the identity map and hence, ψ_x is injective which means ψ_x is an isomorphism. From the proof of Theorem 2.17, it follows that $R^{\mathrm{pd}}_{\bar\rho_0} \simeq R^{\mathrm{def}}_{\bar\rho_x}$ for any non-zero $x \in H^1(G, \chi)$ which completes the proof. □

Theorem 2.19. *Suppose $p \nmid \phi(N)$ and $\dim(H^1(G_{\mathbb{Q},Np}, \chi^i)) = 1$ for some $i \in \{1, -1\}$. Moreover, for such an i, assume that $\dim(H^1(G_{\mathbb{Q},Np}, \chi^{-i})) \in \{1, 2, 3\}$. Then, there exists a representation $\rho : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(R^{pd}_{\bar\rho_0})$ such that $\mathrm{tr}(\rho) = t^{univ}$. As a consequence, for a non-zero $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$, $R^{pd}_{\bar\rho_0} \simeq R^{def}_{\bar\rho_x}$.*

Proof. It suffices to check that the hypotheses of Theorem 2.18 hold in the case considered here. Without loss of assume $\dim(H^1(G_{\mathbb{Q},Np}, \chi)) = 1$. So, we have $1 \leq \dim(H^1(G_{\mathbb{Q},Np}, \chi^{-1})) \leq 3$. As we have assumed that $p \nmid \phi(N)$ and $\bar\rho_0$ is odd, it follows, from the global Euler characteristic formula, that:

(1) $H^2(G_{\mathbb{Q},Np}, 1) = 0$,
(2) $H^2(G_{\mathbb{Q},Np}, \chi) = 0$ and $\dim(H^2(G_{\mathbb{Q},Np}, \chi^{-1})) = \dim(H^1(G_{\mathbb{Q},Np}, \chi^{-1})) - 1$.

Therefore, we see that the hypotheses of Theorem 2.18 are satisfied. Hence, the theorem follows. □

Remark 2.20. *It follows, from the work done so far, that if* $p \nmid \phi(N)$ *then the following are equivalent:*

(1) *There exists a representation* $\rho : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(R^{pd}_{\bar{\rho}_0})$ *such that* $\mathrm{tr}(\rho) = t^{univ}$,
(2) $\dim(H^1(G_{\mathbb{Q},Np}, \chi^i)) = 1$ *for some* $i \in \{1, -1\}$ *and for such an* i, *the map* ψ_x : $R^{pd}_{\bar{\rho}_0} \to R^{def}_{\bar{\rho}_x}$ *is an isomorphism for any non-zero* $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$.

Remark 2.21. *More generally, if we remove the assumption* $\dim(H^1(G_{\mathbb{Q},Np}, \chi^{-i})) \in \{1, 2, 3\}$, *the proof of Theorem 2.19 still works if we know that* $R^{pd}_{\bar{\rho}_0}$ *is isomorphic to a quotient of* $\mathbb{F}[[X_1, \cdots, X_m]]$ *by an ideal* I *such that the prime* $(X_3, \cdots, X_m)$ *is not a prime associated to* I. *In particular, the proof works if* I *is generated by at most* 2 *elements. Note that, if* $m \geq 6$ *and* I *is generated by at most* 2 *elements, then the Krull dimension of* $R^{pd}_{\bar{\rho}_0}$ *is* ≥ 4. *In* [9, Section 4], *there are examples of* $R^{def}_{\bar{\rho}_x}$ *having arbitrary large Krull dimension. So, it is indeed possible to have* I *to be generated by* 2 *elements even when* $m \geq 6$.

Remark 2.22. *Without the assumption* $\dim(H^1(G_{\mathbb{Q},Np}, \chi^{-i})) \in \{1, 2, 3\}$, *we know that* $R^{pd}_{\bar{\rho}_0} \simeq \mathbb{F}[[X_1, \cdots, X_{m+2}]]/I$, *where* $m = \dim(H^1(G_{\mathbb{Q},Np}, \chi^{-i}))$ *and* I *is an ideal generated by at most* $m-1$ *elements. If* I *is generated by at least* 3 *elements and we do not know that* $(X_3, \cdots, X_{m+2})$ *is not a prime ideal associated to* I, *then we can not use the method of the proof of Theorem 2.19. To be precise, the analysis of the annihilator of* B *breaks down. The main reason of this breakdown is the following: If the minimal number of generators of an ideal* I *of the ring* $\mathbb{F}[[X_1, \cdots, X_m]]$ *with* $m \geq 6$ *is* $m-3$, *then for* $y \in \mathbb{F}[[X_1, \cdots, X_m]]$, $y.X_i \in I$ *for all* $3 \leq i \leq m$ *does not necessarily imply that* $y \in I$. *For example, consider the ideal* $I = (xu^2, yv^2, x^2u - y^2v)$ *in* $\mathbb{F}[[x, y, u, v, z, w]]$. *Now,* $xyuv \notin I$ *but* $\{xyuv.x, xyuv.y, xyuv.u, xyuv.v\} \subset I$. *However, if we can prove that the annihilator of* B *is* (0), *then the proof of Theorem 2.19 would imply the existence of such a representation.*

3. Comparison between $\mathcal{R}^{pd}_{\bar{\rho}_0}$ and $\mathcal{R}^{def}_{\bar{\rho}_x}$

In this section, we will turn our attention to characteristic 0 deformation rings $\mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0}$ and $\mathcal{R}^{\mathrm{def}}_{\bar{\rho}_x}$. Let $T^{\mathrm{univ}} : G_{\mathbb{Q},Np} \to \mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0}$ be the universal pseudo-character deforming $\mathrm{tr}(\bar{\rho}_0)$. We would like to explore what information the techniques and results of the previous section can give about the relationship between $\mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0}$ and $\mathcal{R}^{\mathrm{def}}_{\bar{\rho}_x}$ and the existence of a representation $\tau : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0})$ with $\mathrm{tr}(\tau) = T^{\mathrm{univ}}$.

As in the previous section, instead of $G_{\mathbb{Q},Np}$, we are going to consider a profinite group G which satisfies the finiteness condition Φ_p given by Mazur in [16, Section 1.1] and a representation $\bar{\rho}_0 : G \to \mathrm{GL}_2(\mathbb{F})$ such that $\bar{\rho}_0 = \chi_1 \oplus \chi_2$ with $\chi_1 \neq \chi_2$ and $\chi = \chi_1/\chi_2$

with the hypotheses, that $H^1(G,1) \neq 0$, $H^1(G,\chi) \neq 0$ and $H^1(G,\chi^{-1}) \neq 0$. As in the previous section, we are going to retain the notation for the deformation rings and universal deformations from the introduction.

Note that, $\mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0}/(p) \simeq R^{\mathrm{pd}}_{\bar{\rho}_0}$ and $\mathcal{R}^{\mathrm{def}}_{\bar{\rho}_x}/(p) \simeq R^{\mathrm{def}}_{\bar{\rho}_x}$. Thus, if $d = \dim(\tan(R^{\mathrm{pd}}_{\bar{\rho}_0}))$, then $\mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0} \simeq W(\mathbb{F})[[X_1,\cdots,X_d]]/I$, where I is an ideal contained in $(p,(X_1,\cdots,X_d)^2)$. Similarly, if $d' = \dim(H^1(G,ad(\bar{\rho}_x)))$, then $\mathcal{R}^{\mathrm{def}}_{\bar{\rho}_x} \simeq W(\mathbb{F})[[X_1,\cdots,X_{d'}]]/J$, where J is an ideal contained in $(p,(X_1,\cdots,X_d)^2)$ and it is generated by at most $\dim(H^2(G,ad(\bar{\rho}_x)))$ elements ([8, Theorem 2.4]). Most of the initial results of the previous section carry over to the characteristic 0 deformation rings. We briefly give a list of such results first.

Given any non-zero element $x \in H^1(G,\chi^i)$ with $i \in \{1,-1\}$, one has a map $\Psi_x : \mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0} \to \mathcal{R}^{\mathrm{def}}_{\bar{\rho}_x}$ induced by the trace of the universal deformation of $\bar{\rho}_x$. We have an analogue of Lemma 2.15 for Ψ_x ([15, Corollary 1.4.4(2)]):

Lemma 3.1. *If* $\dim(H^1(G,\chi^i)) = 1$ *for some* $i \in \{1,-1\}$ *and* $x \in H^1(G,\chi^i)$ *is a non-zero element, then the map* $\Psi_x : \mathcal{R}^{pd}_{\bar{\rho}_0} \to \mathcal{R}^{def}_{\bar{\rho}_x}$ *is surjective.*

Thus, it follows, from the previous section, that if $H^2(G,1) = 0$ and there exists a representation $\tau : G \to \mathrm{GL}_2(\mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0})$ such that $\mathrm{tr}(\tau) = T^{\mathrm{univ}}$, then the following holds:

(1) $\dim(H^1(G,\chi^i)) = 1$ for some $i \in \{1,-1\}$,
(2) $\tau \pmod{\mathcal{M}} \simeq \bar{\rho}_x$ for some non-zero $x \in H^1(G,\chi^i)$, where $i \in \{1,-1\}$ is such that $\dim(H^1(G,\chi^i)) = 1$.

Moreover, using the arguments of the previous section, we see that the existence of such a representation implies that the morphism $\Psi_x : \mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0} \to \mathcal{R}^{\mathrm{def}}_{\bar{\rho}_x}$ is an isomorphism for all non-zero $x \in H^1(G,\chi^i)$ where $i \in \{1,-1\}$ is such that $\dim(H^1(G,\chi^i)) = 1$.

Now, the pseudo-character T^{univ} gives rise to a pseudo-character $\tilde{T^{\mathrm{univ}}} : \mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0}[G] \to \mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0}$ of the group algebra $\mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0}[G]$. Let $\mathcal{I}_{\bar{\rho}_0}$ be the total reducibility ideal of $\tilde{T}^{\mathrm{univ}}$. From the arguments used in the case of t^{univ}, we get the following: For an ideal J of $\mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0}$, $T^{\mathrm{univ}} \pmod{J}$ is a sum of two characters of G taking values in $\mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0}/J$ and deforming χ_1 and χ_2 if and only if $\mathcal{I}_{\bar{\rho}_0} \subset J$. As $I_{\bar{\rho}_0} \neq (0)$, $\mathcal{I}_{\bar{\rho}_0} \neq (0)$.

Lemma 3.2. *If* $H^2(G,1) = 0$ *and* $\dim(H^1(G,1)) = k$*, then* $\mathcal{R}^{pd}_{\bar{\rho}_0}/\mathcal{I}_{\bar{\rho}_0} \simeq W(\mathbb{F})[[X_1,\cdots,X_{2k}]]$.

Proof. Same as that of Lemma 2.3 after making appropriate changes to account for the fact that we are working with $W(\mathbb{F})$-algebras instead of $\mathbb{F}$-algebras. □

On the other hand, the techniques of the proof of Lemma 2.12 can not be used to prove a similar statement for $\mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0}$. This is because the results of [19] play a key role in the proof of Lemma 2.12 and they are only known for the equi-characteristic pseudo-deformation rings. But we can prove the analogue of Lemma 2.12 under an additional hypothesis.

Lemma 3.3. *Suppose* $H^2(G,1) = 0$*,* $\dim(H^1(G,1)) = k$ *and* $\dim(H^1(G,\chi^i)) = 1$ *for some* $i \in \{1,-1\}$*. For such an* i*, let* $\dim(H^1(G,\chi^{-i})) = m$*,* $\dim(H^2(G,\chi^{-i})) = m'$ *and* $\dim(H^2(G,\chi^i)) = n'$*. If* p *is not a zero-divisor in* $\mathcal{R}^{pd}_{\bar{\rho}_0}$*, then* $\mathcal{R}^{pd}_{\bar{\rho}_0} \simeq W(\mathbb{F})[[X_1,\cdots,X_{m+2k}]]/\mathcal{I}$ *where* $\mathcal{I}$ *is an ideal of* $W(\mathbb{F})[[X_1,\cdots,X_{m+2k}]]$ *generated by at most* $mn'+m'$ *elements.*

Proof. In this case, $\dim(\tan(R^{\mathrm{pd}}_{\bar{\rho}_0})) = m+2k$. So $\mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0} \simeq W(\mathbb{F})[[X_1,\cdots,X_{m+2k}]]/\mathcal{I}$. Suppose the minimal number of generators of $\mathcal{I}$ is greater than $mn'+m'$. Let j be the minimal number of generators of $\mathcal{I}$. So $j \geq mn'+m'+1$. Let $\{f_1,\cdots,f_j\}$ be a minimal set of generators of $\mathcal{I}$. Let $\bar{f}_i$ be the image of f_i in $\mathbb{F}[[X_1,\cdots,X_{m+2k}]]$.

Now, $\mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0}/(p) \simeq R^{\mathrm{pd}}_{\bar{\rho}_0}$. By Lemma 2.12, $R^{\mathrm{pd}}_{\bar{\rho}_0} \simeq \mathbb{F}[[X_1,\cdots,X_{m+2k}]]/I$ with minimal number of generators of the ideal I at most $mn'+m'$. Therefore, we see that the minimal number of generators of the ideal $\bar{\mathcal{I}}$ of $\mathbb{F}[[X_1,\cdots,X_{m+2k}]]$ generated by the set $\{\bar{f}_1,\cdots,\bar{f}_j\}$ is less than $mn'+m'+1$. Let k_0 be the minimal number of generators of $\bar{\mathcal{I}}$. So, $k_0 < mn'+m'+1 \leq j$. Let $\{\bar{g}_1,\cdots,\bar{g}_{k_0}\}$ be a minimal set of generators of $\bar{\mathcal{I}}$. So, for every $1 \leq i \leq j$, we have $\bar{f}_i = \sum_{l=1}^{k_0} \bar{h}_{i,l}\bar{g}_l$ with $\bar{h}_{i,l} \in \mathbb{F}[[X_1,\cdots,X_{m+2k}]]$ for all $1 \leq i \leq j$ and $1 \leq l \leq k_0$.

For $1 \leq i \leq k_0$, let g_i be a lift of $\bar{g}_i$ in $W(\mathbb{F})[[X_1,\cdots,X_{m+2k}]]$ belonging to $\mathcal{I}$. As $\bar{\mathcal{I}}$ is the image of $\mathcal{I}$ in $\mathbb{F}[[X_1,\cdots,X_{m+2k}]]$ we can choose such lifts. For $1 \leq i \leq j$ and $1 \leq l \leq k_0$, choose a lift $h_{i,l}$ of $\bar{h}_{i,l}$ in $W(\mathbb{F})[[X_1,\cdots,X_{m+2k}]]$. So, for every $1 \leq i \leq j$, there exists a $f'_i \in W(\mathbb{F})[[X_1,\cdots,X_{m+2k}]]$ such that $f_i = \sum_{l=1}^{k_0} h_{i,l}g_l + pf'_i$. This implies that $pf'_i \in \mathcal{I}$ for all $1 \leq i \leq j$. As p is not a zero-divisor in $\mathcal{R}^{\mathrm{pd}}_{\bar{\rho}_0}$, it follows that $f'_i \in \mathcal{I}$ for $1 \leq i \leq j$.

Now, $S_0 = \{g_1,\cdots,g_{k_0},pf'_1,\cdots,pf'_j\}$ is a set of generators of $\mathcal{I}$. Let $S' \subset S_0$ be a minimal set of generators of $\mathcal{I}$ i.e. the set S' generates $\mathcal{I}$ but no proper subset of it generates $\mathcal{I}$. As j is the minimal number of generators of I and $k_0 < j$, it follows that there exists some i_0 such that $pf'_{i_0} \in S'$. This means that $pf'_{i_0} \neq 0$ and hence, $f'_{i_0} \neq 0$. As $f'_{i_0} \in \mathcal{I}$, we have $f'_{i_0} = a(pf'_{i_0}) + \sum_{\alpha \in S' \setminus \{pf'_{i_0}\}} a_\alpha.\alpha$, where a and $a_\alpha \in W(\mathbb{F})[[X_1,\cdots,X_{m+2k}]]$.

Since S' is a minimal set of generators of $\mathcal{I}$, we see that $a \neq 0$ as otherwise we would get that $S' \setminus \{pf'_{i_0}\}$ generates $\mathcal{I}$. However, this means $(1-ap)f'_{i_0} = \sum_{\alpha \in S' \setminus \{pf'_{i_0}\}} a_\alpha.\alpha$. But $(1-ap)$ is a unit in $W(\mathbb{F})[[X_1,\cdots,X_{m+2k}]]$. Therefore, we see that f'_{i_0} is in the ideal generated by $S' \setminus \{pf'_{i_0}\}$. But this implies that the set $S' \setminus \{pf'_{i_0}\}$ generates $\mathcal{I}$ contradicting the minimality of S'. Hence, it follows that $\mathcal{I}$ is generated by at most $mn'+m'$ elements. ∎

Let $T^{\mathrm{univ,red}}$ be the pseudo-character of G obtained by composing T^{univ} with the surjective map $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0} \to (\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0})^{\mathrm{red}}$. Let $\mathcal{K}_{\bar\rho_0}$ be the total fraction field of $(\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0})^{\mathrm{red}}$. So, $\mathcal{K}_{\bar\rho_0} = \prod_{P\in\mathcal{S}} \mathcal{K}_P$, where $\mathcal{S}$ is the set of minimal primes of $(\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0})^{\mathrm{red}}$ and $\mathcal{K}_P$ is the fraction field of $(\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0})^{\mathrm{red}}/P$. As $(\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0})^{\mathrm{red}}$ is Noetherian, $\mathcal{S}$ is a finite set.

Lemma 3.4. *Suppose $H^2(G,1) = 0$. Suppose there exists an $i \in \{1,-1\}$ such that $\dim(H^1(G,\chi^i)) = 1$, $H^2(G,\chi^i) = 0$ and $\dim(H^2(G,\chi^{-i})) < \dim(H^1(G,\chi^{-i}))$, and p is not a zero-divisor in $\mathcal{R}^{pd}_{\bar\rho_0}$. Then, the pseudo-character $T^{univ,red} \pmod{P}$ is not reducible for all $P \in \mathcal{S}$.*

Proof. The proof is exactly same as that of Lemma 2.14. We only give a summary of key points here. It follows, from Lemma 3.3, generalization of Krull's principal ideal theorem ([14, Theorem 10.2]) and the proof of Lemma 2.14, that the Krull dimension of $(\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0})^{\mathrm{red}}/P$ is at least $2k+2$ for all $P \in \mathcal{S}$. Now, Lemma 3.2 implies that if $\mathcal{J}_{\bar\rho_0}$ is the image of $\mathcal{I}_{\bar\rho_0}$ in $(\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0})^{\mathrm{red}}$, then $\mathcal{J}_{\bar\rho_0} \not\subset P$ for all $P \in \mathcal{S}$. This proves the lemma. □

Proposition 3.5. *Suppose $H^2(G,1) = 0$. Suppose there exists an $i \in \{1,-1\}$ such that $\dim(H^1(G,\chi^i)) = 1$, $H^2(G,\chi^i) = 0$ and $\dim(H^2(G,\chi^{-i})) < \dim(H^1(G,\chi^{-i}))$. For such an i, let $x \in H^1(G,\chi^i)$ be a non-zero element. Suppose p is not a zero-divisor in $\mathcal{R}^{pd}_{\bar\rho_0}$. Then:*

1. *There exists a representation $\tau^{red} : G \to \mathrm{GL}_2((\mathcal{R}^{pd}_{\bar\rho_0})^{red})$ such that $\mathrm{tr}(\tau^{red}) = T^{univ,red}$. As a consequence, $\Psi_x : \mathcal{R}^{pd}_{\bar\rho_0} \to \mathcal{R}^{def}_{\bar\rho_x}$ induces an isomorphism between $(\mathcal{R}^{pd}_{\bar\rho_0})^{red}$ and $(\mathcal{R}^{def}_{\bar\rho_x})^{red}$.*
2. *Moreover, for such an i, if $\dim(H^1(G,\chi^{-i})) \in \{1,2,3\}$, then there exists a representation $\tau : G \to \mathrm{GL}_2(\mathcal{R}^{pd}_{\bar\rho_0})$ such that $\mathrm{tr}(\tau) = T^{univ}$. As a consequence, the map $\Psi_x : \mathcal{R}^{pd}_{\bar\rho_0} \to \mathcal{R}^{def}_{\bar\rho_x}$ is an isomorphism.*

Proof.

(1) The proof is exactly same as that of Theorem 2.17. We only give a summary of key points here. As p is not a zero-divisor in $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$, we know, by Lemma 3.4, that the pseudo-character $T^{\mathrm{univ,red}} \pmod{P}$ is not reducible for all $P \in \mathcal{S}$. Following the proof of Theorem 2.17 for the GMA attached to $T^{\mathrm{univ,red}} : G \to (\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0})^{\mathrm{red}}$ gives the existence of τ^{red}. Following the proof of Theorem 2.17 gives us the rest of the assertion.

(2) The proof is exactly same as that of Theorem 2.19. We only give a summary of key points here. Without loss of generality, assume $\dim(H^1(G,\chi)) = 1$. Let $A = \begin{pmatrix} \mathcal{R}^{\mathrm{pd}}_{\bar\rho_0} & \mathcal{B} \\ \mathcal{C} & \mathcal{R}^{\mathrm{pd}}_{\bar\rho_0} \end{pmatrix}$ be the GMA attached to the pseudo-character $T^{\mathrm{univ}} : G \to \mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$ in Lemma 2.8.

By [3, Proposition 1.5.1], we have $m(\mathcal{B}\otimes_{\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}}\mathcal{C}) = \mathcal{I}_{\bar\rho_0}$. By Lemma 3.2, $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}/\mathcal{I}_{\bar\rho_0} \simeq W(\mathbb{F})[[X_1, \cdots, X_{2k}]]$ where $k = \dim(H^1(G, 1))$. As $W(\mathbb{F})[[X_1, \cdots, X_d]]$ is a regular local ring, it is a UFD and a Cohen-Macaulay ring ([14, Theorem 19.19] and [14, Corollary 18.17]). Therefore, we can imitate the case by case analysis done in the proof of Theorem 2.19 to conclude that the annihilator of $\mathcal{B}$ is (0). This, along with the facts $\mathcal{I}_{\bar\rho_0} \neq (0)$ and $\dim(H^1(G,\chi)) = 1$, implies that $\mathcal{B}$ is free $R^{\mathrm{pd}}_{\bar\rho_0}$-module of rank 1. Following the proof of Theorem 2.19 from here, we get the representation with trace T^{univ} and see that Ψ_x is an isomorphism for all non-zero $x \in H^1(G,\chi)$.

□

Proposition 3.6. *Suppose $p \nmid \phi(N)$ and $\dim(H^1(G_{\mathbb{Q},Np},\chi^i)) = 1$ for some $i \in \{1,-1\}$. For such an i, let $x \in H^1(G_{\mathbb{Q},Np},\chi^i)$ be a non-zero element. Suppose p is not a zero-divisor in $\mathcal{R}^{pd}_{\bar\rho_0}$. Then:*

(1) There exists a representation $\tau^{red} : G_{\mathbb{Q},Np} \to \mathrm{GL}_2((\mathcal{R}^{pd}_{\bar\rho_0})^{red})$ such that $\mathrm{tr}(\tau^{red}) = T^{univ,red}$. As a consequence, $\Psi_x : \mathcal{R}^{pd}_{\bar\rho_0} \to \mathcal{R}^{def}_{\bar\rho_x}$ induces an isomorphism between $(\mathcal{R}^{pd}_{\bar\rho_0})^{red}$ and $(\mathcal{R}^{def}_{\bar\rho_x})^{red}$.

(2) Moreover, for such an i, if $\dim(H^1(G_{\mathbb{Q},Np},\chi^{-i})) \in \{1,2,3\}$, then there exists a representation $\tau : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathcal{R}^{pd}_{\bar\rho_0})$ such that $\mathrm{tr}(\tau) = T^{univ}$. As a consequence, the map $\Psi_x : \mathcal{R}^{pd}_{\bar\rho_0} \to \mathcal{R}^{def}_{\bar\rho_x}$ is an isomorphism.

Note that, if we can prove Lemma 3.3 without the assumption that p is not a zero divisor in $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$, then we can prove the analogues of Theorem 2.17 and Theorem 2.19 for $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$ without the assumption that p is not a zero divisor in $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$. Finally, we now give a result which will be used in the next section.

Proposition 3.7. *Suppose $H^2(G,1) = 0$. Suppose there exists an $i \in \{1,-1\}$ such that $\dim(H^1(G,\chi^i)) = 1$, $H^2(G,\chi^i) = 0$, $\dim(H^1(G,\chi^{-i})) \in \{1,2,3\}$ and $\dim(H^2(G,\chi^{-i})) < \dim(H^1(G,\chi^{-i}))$. Let $x \in H^1(G,\chi^i)$ be a non-zero element. If p is not a zero-divisor in $\mathcal{R}^{def}_{\bar\rho_x}$, then the map $\Psi_x : \mathcal{R}^{pd}_{\bar\rho_0} \to \mathcal{R}^{def}_{\bar\rho_x}$ is an isomorphism.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{R}^{\mathrm{pd}}_{\bar\rho_0} & \xrightarrow{\Psi_x} & \mathcal{R}^{\mathrm{def}}_{\bar\rho_x} \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ R^{\mathrm{pd}}_{\bar\rho_0} & \xrightarrow{\psi_x} & R^{\mathrm{def}}_{\bar\rho_x} \end{array}$$

Here the vertical maps f_1 and f_2 are the morphisms induced by t^{univ} and ρ^{univ}_x, respectively. Now, $\ker(f_1)$ is the ideal generated by p in $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$, while $\ker(f_2)$ is the ideal generated by p in $\mathcal{R}^{\mathrm{def}}_{\bar\rho_x}$. By Theorem 2.19, ψ_x is an isomorphism. So, $\ker(\psi_x \circ f_1) = \ker(f_1) = (p)$. As $\psi_x \circ f_1 = f_2 \circ \Psi_x$, it follows that $\ker(f_2 \circ \Psi_x) = (p)$. Thus $\ker(\Psi_x) \subset (p)$.

Let $h \in \ker(\Psi_x)$. So, $h \in (p)$. Suppose $h \neq 0$. As $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$ is a complete local ring, $\cap_{n\geq 1}(p^n) = (0)$. Therefore, we have $h = p^{n_0}h'$ where $n_0 \geq 1$ is an integer, $h' \in \mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$ and $h' \notin (p)$. Thus, $h' \notin \ker(\Psi_x)$ and hence, $\Psi_x(h') \neq 0$. But $\Psi_x(h) = 0$. So, we get $\Psi_x(h) = \Psi_x(p^{n_0}.h') = p^{n_0}.\Psi_x(h') = 0$. Thus, we get that p is a zero-divisor in $\mathcal{R}^{\mathrm{def}}_{\bar\rho_x}$ which contradicts our assumption. Therefore, it follows that $\ker(\Psi_x) = (0)$. From Lemma 3.1, we know that Ψ_x is surjective. Hence, it follows that Ψ_x is an isomorphism. □

After putting $G = G_{\mathbb{Q},Np}$ in the proposition above, we get the following result:

Proposition 3.8. *Suppose $p \nmid \phi(N)$ and $\dim(H^1(G_{\mathbb{Q},Np}, \chi^i)) = 1$ for some $i \in \{1, -1\}$. For such an i, assume that $\dim(H^1(G_{\mathbb{Q},Np}, \chi^{-i})) \in \{1, 2, 3\}$. Let $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$ be a non-zero element. If p is not a zero-divisor in $\mathcal{R}^{def}_{\bar\rho_x}$, then the map $\Psi_x : \mathcal{R}^{pd}_{\bar\rho_0} \to \mathcal{R}^{def}_{\bar\rho_x}$ is an isomorphism.*

4. Increasing the ramification

Let us summarize what we have done so far. We fixed an odd prime p, a natural number N such that $p \nmid N$ and an odd, semi-simple reducible representation $\bar\rho_0 : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathbb{F})$. After fixing such a data, we studied the relationship between universal deformation rings of the pseudo-representation $(\mathrm{tr}(\bar\rho_0), \det(\bar\rho_0)) : G_{\mathbb{Q},Np} \to \mathbb{F}$ and the universal deformation rings of the representations $\bar\rho_x : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathbb{F})$ such that the semi-simplification of $\bar\rho_x$ is $\bar\rho_0$. In this section, we will study, for a prime $\ell \nmid Np$, the relationship between the universal deformation rings of the pseudo-representation $(\mathrm{tr}(\bar\rho_0), \det(\bar\rho_0))$ for the groups $G_{\mathbb{Q},Np}$ and $G_{\mathbb{Q},N\ell p}$, respectively.

We keep the notation from the introduction in this section. So, we are interested in studying the relationship between $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$ (resp. $R^{\mathrm{pd},\ell}_{\bar\rho_0}$) and $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$ (resp. $R^{\mathrm{pd}}_{\bar\rho_0}$). In particular, we want to know what information one can deduce about the structure of $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$ (resp. $R^{\mathrm{pd},\ell}_{\bar\rho_0}$) from the structure of $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$ (resp. $R^{\mathrm{pd}}_{\bar\rho_0}$). Let $t^{\mathrm{univ},\ell}$ be the universal pseudo-character from $G_{\mathbb{Q},N\ell p}$ to $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ deforming $\mathrm{tr}(\bar\rho_0)$ and $T^{\mathrm{univ},\ell}$ be the universal pseudo-character from $G_{\mathbb{Q},N\ell p}$ to $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$ deforming $\mathrm{tr}(\bar\rho_0)$. Denote the pseudo-character obtained by composing $t^{\mathrm{univ},\ell}$ with the surjective map $R^{\mathrm{pd},\ell}_{\bar\rho_0} \to (R^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}}$ by $(t^{\mathrm{univ},\ell})^{\mathrm{red}}$. Denote the pseudo-character obtained by composing $T^{\mathrm{univ},\ell}$ with the surjective map $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0} \to (\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}}$ by $(T^{\mathrm{univ},\ell})^{\mathrm{red}}$.

Let $\bar\rho : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathbb{F})$ be a representation which is either irreducible or Borel (i.e. reducible but not semi-simple). In this case, the relationship between the universal deformation rings $\mathcal{R}^{\mathrm{def},\ell}_{\bar\rho}$ and $\mathcal{R}^{\mathrm{def},\ell}_{\bar\rho}$ was studied in [7]. So, from [7], we know how to determine the structure of $\mathcal{R}_{\bar\rho}^{\mathrm{def},\ell}$ from the structure of $\mathcal{R}_{\bar\rho}^{\mathrm{def}}$. We will use the results from [7] for the Borel case, along with the results obtained in the previous sections to compare $\mathcal{R}_{\bar\rho_0}^{\mathrm{pd},\ell}$ (resp. $R_{\bar\rho_0}^{\mathrm{pd},\ell}$) with $\mathcal{R}_{\bar\rho_0}^{\mathrm{pd}}$ (resp. $R_{\bar\rho_0}^{\mathrm{pd}}$). Throughout this section, we will assume that $p \nmid \phi(N)$ unless otherwise mentioned.

4.1. **Effect of increasing the ramification on the tangent spaces.** We first turn our attention to tangent spaces. As $p \nmid \phi(N)$, it follows from [1, Theorem 2] that $\dim(\tan(R_{\bar\rho_0}^{\mathrm{pd}})) = 2 + mn$, where $m = \dim(H^1(G_{\mathbb{Q},Np},\chi))$ and $n = \dim(H^1(G_{\mathbb{Q},Np},\chi^{-1}))$. Let $m_1 = \dim(H^1(G_{\mathbb{Q},N\ell p},\chi))$ and $n_1 = \dim(H^1(G_{\mathbb{Q},N\ell p},\chi^{-1}))$. Now, from[1, Theorem 2], it follows that if $p \nmid \ell - 1$, then $\dim(\tan(R_{\bar\rho_0}^{\mathrm{pd},\ell})) = 2 + m_1 n_1$, while if $p | \ell - 1$, then $3 + m_1 n_1 \leq \dim(\tan(R_{\bar\rho_0}^{\mathrm{pd},\ell})) \leq 4 + m_1 n_1$.

So, we now analyze how addition of the prime ℓ changes the cohomology groups of χ and χ^{-1}. Let ω_p be the mod p cyclotomic character. If $\chi = \omega_p$, then by Kummer theory, $\dim(H^1(G_{\mathbb{Q},Np},\omega_p)) = 1+$ number of distinct primes dividing N (see the proof of [12, Proposition 24] and the remark after it). Thus, $\dim(H^1(G_{\mathbb{Q},N\ell p},\omega_p)) = 1 + \dim(H^1(G_{\mathbb{Q},Np},\omega_p))$. If $\chi \neq 1, \omega_p$ and χ is odd, then, by the Greenberg-Wiles version of the Poitou-Tate duality ([20, Theorem 2]), we see that $\dim(H^1(G_{\mathbb{Q},Np},\chi)) = \dim(H^1_0(G_{\mathbb{Q},Np},\chi^{-1}\omega_p)) + 1 + \sum_{q|Np} \dim(H^0(G_{\mathbb{Q}_q},\chi^{-1}\omega_p|_{G_{\mathbb{Q}_q}}))$, where $H^1_0(G_{\mathbb{Q},Np},\chi^{-1}\omega_p) = \ker(H^1(G_{\mathbb{Q},Np},\chi^{-1}\omega_p) \to \prod_{q|Np} H^1(G_{\mathbb{Q}_q},\chi^{-1}\omega_p|_{G_{\mathbb{Q}_q}}))$. Thus, if $\chi \neq 1, \omega_p$, then $\dim(H^1(G_{\mathbb{Q},Np},\chi)) \leq \dim(H^1(G_{\mathbb{Q},N\ell p},\chi)) \leq 1 + \dim(H^1(G_{\mathbb{Q},Np},\chi))$.

So, if $\dim(H^1(G_{\mathbb{Q},N\ell p},\chi)) = 1 + \dim(H^1(G_{\mathbb{Q},Np},\chi))$, then $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$. On the other hand, if $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ and $H^1_0(G_{\mathbb{Q},Np},\chi^{-1}\omega_p) = 0$, then $\dim(H^1(G_{\mathbb{Q},N\ell p},\chi)) = 1 + \dim(H^1(G_{\mathbb{Q},Np},\chi))$. However, it is not clear if $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ is sufficient for $\dim(H^1(G_{\mathbb{Q},N\ell p},\chi)) = 1 + \dim(H^1(G_{\mathbb{Q},Np},\chi))$ as the group $H^1_0(G_{\mathbb{Q},N\ell p},\chi^{-1}\omega_p)$ might be strictly smaller than $H^1_0(G_{\mathbb{Q},Np},\chi^{-1}\omega_p)$. If it is indeed smaller, then the difference between their dimension would be 1 and hence, we would get $\dim(H^1(G_{\mathbb{Q},N\ell p},\chi)) = \dim(H^1(G_{\mathbb{Q},Np},\chi))$.

To summarize, if $\dim(H^1(G_{\mathbb{Q},Np},\chi)) = m$, $\dim(H^1(G_{\mathbb{Q},Np},\chi^{-1})) = n$ and ℓ is a prime such that $\ell \nmid Np$ and $p \nmid \ell - 1$, then we have:

(1) If $\chi|_{G_{\mathbb{Q}_\ell}} \neq \omega_p|_{G_{\mathbb{Q}_\ell}}, \omega_p^{-1}|_{G_{\mathbb{Q}_\ell}}$, then $\dim(H^1(G_{\mathbb{Q},N\ell p},\chi)) = m$, $\dim(H^1(G_{\mathbb{Q},N\ell p},\chi^{-1})) = n$,
(2) If $p \nmid \ell + 1$ and $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$, then $m \leq \dim(H^1(G_{\mathbb{Q},N\ell p},\chi)) \leq m + 1$, and $\dim(H^1(G_{\mathbb{Q},N\ell p},\chi^{-1})) = n$. Moreover, $\dim(H^1(G_{\mathbb{Q},N\ell p},\chi)) = m + 1$ if $H^1_0(G_{\mathbb{Q},Np},\chi^{-1}\omega_p) = 0$,

(3) If $p \nmid \ell+1$ and $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p^{-1}|_{G_{\mathbb{Q}_\ell}}$, then $m = \dim(H^1(G_{\mathbb{Q},N\ell p},\chi))$, and $n \leq \dim(H^1(G_{\mathbb{Q},N\ell p},\chi^{-1})) \leq n+1$. Moreover, $\dim(H^1(G_{\mathbb{Q},N\ell p},\chi^{-1})) = n+1$ if $H^1_0(G_{\mathbb{Q},Np},\chi\omega_p)=0$,

(4) If $p|\ell+1$ and $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$, then $m \leq \dim(H^1(G_{\mathbb{Q},N\ell p},\chi)) \leq m+1$, and $n \leq \dim(H^1(G_{\mathbb{Q},N\ell p},\chi^{-1})) \leq n+1$. Moreover, $\dim(H^1(G_{\mathbb{Q},N\ell p},\chi)) = m+1$ if $H^1_0(G_{\mathbb{Q},Np},\chi^{-1}\omega_p)=0$ and $\dim(H^1(G_{\mathbb{Q},N\ell p},\chi^{-1})) = n+1$ if $H^1_0(G_{\mathbb{Q},Np},\chi\omega_p)=0$.

Suppose $\bar\rho_0$ is unobstructed i.e. $p \nmid \phi(N)$ and $\dim(H^1(G_{\mathbb{Q},Np},\chi)) = \dim(H^1(G_{\mathbb{Q},Np},\chi^{-1})) = 1$. Note that, Vandiver's conjecture implies that $\bar\rho_0$ is unobstructed if $N=1$ (see [4]). In this case, by Lemma 2.2, we know that $\dim(H^1(G_{\mathbb{Q},Np},\mathrm{ad}^0(\bar\rho_x))) = 2$ for any non-zero $x \in H^1(G_{\mathbb{Q},Np},\chi^i)$ with $i \in \{1,-1\}$. In this case, we know the following result:

Lemma 4.1. *Suppose $p \nmid \phi(N)$ and $\bar\rho_0$ is unobstructed. Then, for a non-zero $x \in H^1(G_{\mathbb{Q},Np},\chi^i)$ with $i \in \{1,-1\}$, the map $\Psi_x : \mathcal{R}^{pd}_{\bar\rho_0} \to \mathcal{R}^{def}_{\bar\rho_x}$ is an isomorphism and both are isomorphic to $W(\mathbb{F})[[X,Y,Z]]$.*

Proof. Since $\bar\rho_0$ is odd and $p \nmid \phi(N)$, we get, by the global Euler characteristic formula, that $H^2(G_{\mathbb{Q},Np},1) = H^2(G_{\mathbb{Q},Np},\chi) = H^2(G_{\mathbb{Q},Np},\chi^{-1}) = H^2(G_{\mathbb{Q},Np},\mathrm{ad}(\bar\rho_x)) = 0$. Therefore, we get, from [8, Theorem 2.4], that $\mathcal{R}^{\mathrm{def}}_{\bar\rho_x} \simeq W(\mathbb{F})[[X,Y,Z]]$. The result now follows from Proposition 3.8. □

Let us now analyze dimensions of some Galois cohomology groups.

For $i \in \{1,-1\}$, we have:

(1) As χ^i is odd, $H^0(G_{\mathbb{Q}},\chi^i) = H^0(G_\infty,\chi^i) = 0$, where G_∞ is the subgroup of order 2 generated by a complex conjugation c. As $|G_\infty| = 2$ and $p > 2$, we have $H^1(G_\infty,\chi^i)=0$,

(2) For all primes $q|N$, by the local Euler characteristic formula, $\dim(H^1(G_{\mathbb{Q}_q},\chi^i|_{G_{\mathbb{Q}_q}})) - \dim(H^0(G_{\mathbb{Q}_q},\chi^i|_{G_{\mathbb{Q}_q}})) = \dim(H^0(G_{\mathbb{Q}_q},\chi^{-i}\omega_p|_{G_{\mathbb{Q}_q}})) \geq 0$,

(3) Suppose $\dim(H^0(G_{\mathbb{Q}},\chi^{-i}\omega_p)) = k$. By the local Euler characteristic formula, $\dim(H^1(G_{\mathbb{Q}_p},\chi^i|_{G_{\mathbb{Q}_p}})) - \dim(H^0(G_{\mathbb{Q}_p},\chi^i|_{G_{\mathbb{Q}_p}})) = 1 + \dim(H^0(G_{\mathbb{Q}_p},\chi^{-i}\omega_p|_{G_{\mathbb{Q}_p}})) \geq 1+k$.

Now, by the Greenberg-Wiles version of the Poitou-Tate duality ([20, Theorem 2]), we get that $\dim(H^1(G_{\mathbb{Q},Np},\chi^i)) \geq \dim(H^1_0(G_{\mathbb{Q},Np},\chi^i)) + \dim(H^1(G_{\mathbb{Q}_p},\chi^i|_{G_{\mathbb{Q}_p}})) - \dim(H^0(G_{\mathbb{Q}_p},\chi^i|_{G_{\mathbb{Q}_p}})) + \dim(H^0(G_{\mathbb{Q}},\chi^i)) - \dim(H^0(G_{\mathbb{Q}},\chi^{-i}\omega_p)) \geq 1 + \dim(H^1_0(G_{\mathbb{Q},Np},\chi^i))$. As $\dim(H^1(G_{\mathbb{Q},Np},\chi^i)) = 1$, we get that $H^1_0(G_{\mathbb{Q},Np},\chi^i) = 0$ for $i \in \{1,-1\}$.

For $i \in \{1,-1\}$ and non-zero $x \in H^1(G_{\mathbb{Q},Np},\chi^i)$, we have:

(1) Note that, $H^0(G_{\mathbb{Q}}, \mathrm{ad}^0(\bar\rho_x)) = 0$. As $\bar\rho_x$ is odd, $\dim(H^0(G_\infty, \mathrm{ad}^0(\bar\rho_x))) = 1$. As $|G_\infty| = 2$ and $p > 2$, we have $H^1(G_\infty, \mathrm{ad}^0(\bar\rho_x)) = 0$,
(2) For all primes $q|N$, by the local Euler characteristic formula, $\dim(H^1(G_{\mathbb{Q}_q}, \mathrm{ad}^0(\bar\rho_x)|_{G_{\mathbb{Q}_q}})) - \dim(H^0(G_{\mathbb{Q}_q}, \mathrm{ad}^0(\bar\rho_x)|_{G_{\mathbb{Q}_q}})) = \dim(H^0(G_{\mathbb{Q}_q}, (\mathrm{ad}^0(\bar\rho_x))^* \otimes \omega_p|_{G_{\mathbb{Q}_q}})) \geq 0$,
(3) Suppose $\dim(H^0(G_{\mathbb{Q}}, (\mathrm{ad}^0(\bar\rho_x))^* \otimes \omega_p)) = k'$. By the local Euler characteristic formula, $\dim(H^1(G_{\mathbb{Q}_p}, \mathrm{ad}^0(\bar\rho_x)|_{G_{\mathbb{Q}_p}})) - \dim(H^0(G_{\mathbb{Q}_p}, \mathrm{ad}^0(\bar\rho_x)|_{G_{\mathbb{Q}_p}})) = 3 + \dim(H^0(G_{\mathbb{Q}_p}, (\mathrm{ad}^0(\bar\rho_x))^* \otimes \omega_p|_{G_{\mathbb{Q}_p}})) \geq 3 + k'$.

Now, by the Greenberg-Wiles version of the Poitou-Tate duality ([20, Theorem 2]), we get that $\dim(H^1(G_{\mathbb{Q},Np}, \mathrm{ad}^0(\bar\rho_x))) \geq \dim(H^1_0(G_{\mathbb{Q},Np}, \mathrm{ad}^0(\bar\rho_x))) + \dim(H^1(G_{\mathbb{Q}_p}, \mathrm{ad}^0(\bar\rho_x)|_{G_{\mathbb{Q}_p}})) - \dim(H^0(G_{\mathbb{Q}_p}, \mathrm{ad}^0(\bar\rho_x)|_{G_{\mathbb{Q}_p}})) + \dim(H^0(G_{\mathbb{Q}}, \mathrm{ad}^0(\bar\rho_x))) - \dim(H^0(G_{\mathbb{Q}}, (\mathrm{ad}^0(\bar\rho_x))^* \otimes \omega_p)) + \dim(H^1(G_\infty, \mathrm{ad}^0(\bar\rho_x))) - \dim(H^0(G_\infty, \mathrm{ad}^0(\bar\rho_x))) \geq 3 + k' - 1 - k' + \dim(H^1_0(G_{\mathbb{Q},Np}, \mathrm{ad}^0(\bar\rho_x))) = 2 + \dim(H^1_0(G_{\mathbb{Q},Np}, \mathrm{ad}^0(\bar\rho_x)))$. As $\dim(H^1(G_{\mathbb{Q},Np}, \mathrm{ad}^0(\bar\rho_x))) = 2$, we get that $H^1_0(G_{\mathbb{Q},Np}, \mathrm{ad}^0(\bar\rho_x)) = 0$ for $i \in \{1, -1\}$ and non-zero $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$.

Now let ℓ be a prime such that $p \nmid \ell^2 - 1$ and $\chi|_{G_{\mathbb{Q}_\ell}}$ is $\omega_p^i|_{G_{\mathbb{Q}_\ell}}$ with $i \in \{1, -1\}$. In this case, $\omega_p|_{G_{\mathbb{Q}_\ell}} \neq \omega_p^{-1}|_{G_{\mathbb{Q}_\ell}}$. So, from the discussion so far, we get $\dim(H^1(G_{\mathbb{Q},N\ell p}, \chi)) = 2$ and $\dim(H^1(G_{\mathbb{Q},N\ell p}, \chi^{-1})) = 1$ if $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$, while $\dim(H^1(G_{\mathbb{Q},N\ell p}, \chi)) = 1$ and $\dim(H^1(G_{\mathbb{Q},N\ell p}, \chi^{-1})) = 2$ if $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p^{-1}|_{G_{\mathbb{Q}_\ell}}$. Let $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$ be non-zero with $i \in \{1, -1\}$. From the discussion above, $H^1_0(G_{\mathbb{Q},Np}, \mathrm{ad}^0(\bar\rho_x)) = 0$. Therefore, we have $H^1_0(G_{\mathbb{Q},N\ell p}, \mathrm{ad}^0(\bar\rho_x)) = 0$. By the Greenberg-Wiles version of the Poitou-Tate duality ([20, Theorem 2]), we get that $\dim(H^1(G_{\mathbb{Q},N\ell p}, \mathrm{ad}^0(\bar\rho_x)))$
$= \dim(H^1(G_{\mathbb{Q},Np}, \mathrm{ad}^0(\bar\rho_x))) + \dim(H^1(G_{\mathbb{Q}_\ell}, \mathrm{ad}^0(\bar\rho_x)|_{G_{\mathbb{Q}_\ell}})) - \dim(H^0(G_{\mathbb{Q}_\ell}, \mathrm{ad}^0(\bar\rho_x)|_{G_{\mathbb{Q}_\ell}}))$.
By the local Euler characteristic formula, $\dim(H^1(G_{\mathbb{Q}_\ell}, \mathrm{ad}^0(\bar\rho_x)|_{G_{\mathbb{Q}_\ell}})) - \dim(H^0(G_{\mathbb{Q}_\ell}, \mathrm{ad}^0(\bar\rho_x)|_{G_{\mathbb{Q}_\ell}}))$
$= \dim(H^0(G_{\mathbb{Q}_\ell}, (\mathrm{ad}^0(\bar\rho_x))^* \otimes \omega_p|_{G_{\mathbb{Q}_\ell}}))$.

Now, $\mathrm{ad}^0(\bar\rho_x)|_{G_{\mathbb{Q}_\ell}} \simeq 1 \oplus \chi|_{G_{\mathbb{Q}_\ell}} \oplus \chi^{-1}|_{G_{\mathbb{Q}_\ell}}$. As $p \nmid \ell^2 - 1$ and $\chi^i|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ for some $i \in \{1, -1\}$, it follows that $\dim(H^0(G_{\mathbb{Q}_\ell}, (\mathrm{ad}^0(\bar\rho_x))^* \otimes \omega_p|_{G_{\mathbb{Q}_\ell}})) = 1$. Thus, we get $\dim(H^1(G_{\mathbb{Q},N\ell p}, \mathrm{ad}^0(\bar\rho_x))) = \dim(H^1(G_{\mathbb{Q},Np}, \mathrm{ad}^0(\bar\rho_x))) + 1 = 2 + 1 = 3$. As $p \nmid \phi(N\ell)$, $\dim(H^1(G_{\mathbb{Q},N\ell p}, \mathrm{ad}(\bar\rho_x))) = 3 + 1 = 4$.

Now let ℓ be a prime such that $\ell \equiv -1 \pmod p$ and $\chi|_{G_{\mathbb{Q}_\ell}}$ is $\omega_p|_{G_{\mathbb{Q}_\ell}}$. In this case $\omega_p|_{G_{\mathbb{Q}_\ell}} = \omega_p^{-1}|_{G_{\mathbb{Q}_\ell}}$ and hence, $\chi^{-1}|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$. From the discussion so far, we get $\dim(H^1(G_{\mathbb{Q},N\ell p}, \chi)) = \dim(H^1(G_{\mathbb{Q},N\ell p}, \chi^{-1})) = 2$. On the other hand, for $i \in \{1, -1\}$, let $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$ be a non-zero element. Then, we get that $\dim(H^1(G_{\mathbb{Q},N\ell p}, \mathrm{ad}^0(\bar\rho_x)))$
$= \dim(H^1(G_{\mathbb{Q},Np}, \mathrm{ad}^0(\bar\rho_x))) + \dim(H^0(G_{\mathbb{Q}_\ell}, (\mathrm{ad}^0(\bar\rho_x))^* \otimes \omega_p|_{G_{\mathbb{Q}_\ell}})$. As $\mathrm{ad}^0(\bar\rho_x)|_{G_{\mathbb{Q}_\ell}} \simeq 1 \oplus \omega_p|_{G_{\mathbb{Q}_\ell}} \oplus \omega_p|_{G_{\mathbb{Q}_\ell}}$, we get that $\dim(H^1(G_{\mathbb{Q},N\ell p}, \mathrm{ad}^0(\bar\rho_x))) = \dim(H^1(G_{\mathbb{Q},Np}, \mathrm{ad}^0(\bar\rho_x))) + 2$
$= 2 + 2 = 4$. Hence, we have $\dim(H^1(G_{\mathbb{Q},N\ell p}, \mathrm{ad}(\bar\rho_x))) = 5$.

4.2. **Comparison between $\mathcal{R}^{\mathbf{pd},\ell}_{\bar\rho_0}$ and $\mathcal{R}^{\mathbf{def},\ell}_{\bar\rho_x}$.** For a prime ℓ, denote by $\tilde\ell$ be the Teichmuller lift of $\ell \pmod p$ in $\mathbb{Z}_p$. So, $\ell/\tilde\ell \in 1+p\mathbb{Z}_p$. For $\alpha \in \mathbb{F}$, denote by $\hat\alpha$ its Teichmuller lift in $W(\mathbb{F})$. Let ℓ be a prime such that $\ell \nmid Np$ and $\chi|_{G_{\mathbb{Q}_\ell}} \neq 1$. Fix a lift g_ℓ of Frob_ℓ in $G_{\mathbb{Q}_\ell}$. Let $A^{\mathrm{pd},\ell}$ be the GMA found in Lemma 2.10 for the tuple $(\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}, \ell, T^{\mathrm{univ},\ell}, g_\ell)$ and $\rho^{\mathrm{pd},\ell}$ be the corresponding representation. So we have $\rho^{\mathrm{pd},\ell}(g_\ell) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $a = \chi_1\widehat{(\mathrm{Frob}_\ell)}(1+a_\ell)$ and $d = \chi_2\widehat{(\mathrm{Frob}_\ell)}(1+d_\ell)$ for some a_ℓ, $d_\ell \in \mathcal{M}^\ell$. Let $\mathcal{I}^\ell_{\bar\rho_0}$ be the total reducibility ideal of $T^{\mathrm{univ},\ell}$. We first give a set of generators of the maximal ideal of $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$ which will be used later.

Lemma 4.2. *Let ℓ be a prime such that $\ell \nmid Np$, $p \nmid \ell-1$ and $\chi^i|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ for some $i \in \{1,-1\}$. Moreover, assume that $\ell/\tilde\ell$ is a topological generator of $1+p\mathbb{Z}_p$. Then, in the notation as above, the ideal of $\mathcal{R}^{pd,\ell}_{\bar\rho_0}$ generated by p, a_ℓ, d_ℓ and $\mathcal{I}^\ell_{\bar\rho_0}$ is $\mathcal{M}^\ell$.*

Proof. Let I be the ideal of $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$ generated by p, a_ℓ, d_ℓ and $\mathcal{I}^\ell_{\bar\rho_0}$. Therefore, from [3, Proposition 1.5.1], we see that $T^{\mathrm{univ},\ell} \pmod I$ is sum of two characters ψ_1 and ψ_2 of $G_{\mathbb{Q},N\ell p}$ such that ψ_i is a deformation of χ_i for $i=1,2$. As $p \nmid \ell-1$, we get $G^{\mathrm{ab}}_{\mathbb{Q},N\ell p}/(G^{\mathrm{ab}}_{\mathbb{Q},N\ell p})^p \simeq G^{\mathrm{ab}}_{\mathbb{Q},Np}/(G^{\mathrm{ab}}_{\mathbb{Q},Np})^p$. Therefore, it follows, from [16, Section 1.4], that both ψ_1 and ψ_2 factor through $G_{\mathbb{Q},Np}$ and hence, are unramified at ℓ.

Since $a_\ell, d_\ell \in I$, we have $\psi_1(\mathrm{Frob}_\ell)+\psi_2(\mathrm{Frob}_\ell) = \chi_1(\mathrm{Frob}_\ell)+\chi_2(\mathrm{Frob}_\ell)$ and $\psi_1(\mathrm{Frob}_\ell)\psi_2(\mathrm{Frob}_\ell) = \chi_1(\mathrm{Frob}_\ell)\chi_2(\mathrm{Frob}_\ell)$. Suppose $\psi_i(\mathrm{Frob}_\ell) = \chi_i(\mathrm{Frob}_\ell)(1+a_i)$ with $a_i \in \mathcal{M}^\ell/I$ for $i=1,2$. So, the equalities above imply that $\sum_{i=1}^2 \chi_i(\mathrm{Frob}_\ell)a_i = 0$ and $(1+a_1)(1+a_2)=1$. This implies that $\ell a_1 = -a_2$ and $a_1(1-\ell-\ell a_1) = 0$. As $p \nmid \ell-1$ and $a_1 \in \mathcal{M}^\ell/I$, $(1-\ell-\ell a_1)$ is a unit in $\mathcal{R}^{\mathrm{pd},\ell}/I$. Hence, it follows that $a_1 = 0$ and $a_2 = 0$.

Thus, for $i=1,2$, ψ_i is a deformation of χ_i with $\psi_i(\mathrm{Frob}_\ell) = \chi_i(\mathrm{Frob}_\ell)$. As $p \nmid \phi(N)$, $G^{\mathrm{ab}}_{\mathbb{Q},Np}/(G^{\mathrm{ab}}_{\mathbb{Q},Np})^p \simeq \mathbb{Z}_p$. Moreover, as $\ell/\tilde\ell$ is a topological generator of $1+p\mathbb{Z}_p$, it follows that the image of Frob_ℓ in $G^{\mathrm{ab}}_{\mathbb{Q},Np}/(G^{\mathrm{ab}}_{\mathbb{Q},Np})^p$ is a topological generator of $G^{\mathrm{ab}}_{\mathbb{Q},Np}/(G^{\mathrm{ab}}_{\mathbb{Q},Np})^p$. Therefore, it follows, from [16, Section 1.4], that $\psi_1 = \chi_1$ and $\psi_2 = \chi_2$. Therefore, we have $T^{\mathrm{univ},\ell} \pmod I = \mathrm{tr}(\bar\rho_0)$, and hence, from ([11, Remark 3.5]), we get that $I = \mathcal{M}^\ell$. □

Lemma 4.3. *If $p \nmid \ell-1$ and $\chi|_{G_{\mathbb{Q}_\ell}} \neq \omega_p|_{G_{\mathbb{Q}_\ell}}, \omega_p^{-1}|_{G_{\mathbb{Q}_\ell}}, 1$, then $\mathcal{R}^{pd,\ell}_{\bar\rho_0} \simeq \mathcal{R}^{pd}_{\bar\rho_0}$.*

Proof. From Lemma 2.10, there exists a faithful GMA A^{univ} over $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$ and a representation $\rho : G_{\mathbb{Q},N\ell p} \to A^{\mathrm{univ}}$ such that $\mathrm{tr}(\rho) = T^{\mathrm{univ},\ell}$, $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}[\rho(G_{\mathbb{Q},N\ell p})] = A^{\mathrm{univ}}$ and $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}[\rho(G_{\mathbb{Q}_\ell})]$ is a sub $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$-GMA of A^{univ}. So, $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}[\rho(G_{\mathbb{Q}_\ell})] = \begin{pmatrix} \mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0} & B_\ell \\ C_\ell & \mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0} \end{pmatrix}$, where B_ℓ and C_ℓ are $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$-submodules of B and C, respectively and hence, both of them are finitely generated $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$-modules.

As A^{univ} is a faithful quotient of $(\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}[G_{\mathbb{Q},N\ell p}], \widetilde{T^{\mathrm{univ},\ell}})$, it follows that $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}[\rho(G_{\mathbb{Q}_\ell})]$ is a Cayley-Hamilton quotient of $(\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}[G_{\mathbb{Q}_\ell}], \widetilde{T^{\mathrm{univ},\ell}}|_{\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}[G_{\mathbb{Q}_\ell}]})$.

Therefore, by repeating the proof of Lemma 2.10 for $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}[\rho(G_{\mathbb{Q}_\ell})]$, we get injective homomorphisms
$\mathrm{Hom}_{\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}}(B_\ell/\mathcal{M}^\ell B_\ell, \mathbb{F}) \to H^1(G_{\mathbb{Q}_\ell}, \chi|_{G_{\mathbb{Q}_\ell}})$ and $\mathrm{Hom}_{\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}}(C_\ell/\mathcal{M}^\ell C_\ell, \mathbb{F}) \to H^1(G_{\mathbb{Q}_\ell}, \chi^{-1}|_{G_{\mathbb{Q}_\ell}})$ (see proof of [3, Theorem 1.5.5] as well). But as $\chi|_{G_{\mathbb{Q}_\ell}} \neq \omega_p|_{G_{\mathbb{Q}_\ell}}, \omega_p^{-1}|_{G_{\mathbb{Q}_\ell}}, 1$, by local Euler characteristic formula, we get that $H^1(G_{\mathbb{Q}_\ell}, \chi|_{G_{\mathbb{Q}_\ell}}) = H^1(G_{\mathbb{Q}_\ell}, \chi^{-1}|_{G_{\mathbb{Q}_\ell}}) = 0$. Therefore, we get, by Nakayama's lemma, that $B_\ell = C_\ell = 0$.

Thus, we get characters $\tilde\chi_1, \tilde\chi_2 : G_{\mathbb{Q}_\ell} \to (\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0})^*$ sending $g \in G_{\mathbb{Q}_\ell}$ to the upper and lower diagonal entries of $\rho(g)$, respectively. As $p \nmid \ell - 1$, $\mathbb{Z}_\ell^*$ does not admit any non-trivial pro-p quotient. Hence, $\tilde\chi_1(I_\ell) = \tilde\chi_2(I_\ell) = 1$. So, the pseudo-character $t^{\mathrm{univ},\ell}$ factors through $G_{\mathbb{Q},Np}$. Hence, this induces a surjective map $f : \mathcal{R}^{\mathrm{pd}}_{\bar\rho_0} \to \mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$. Viewing T^{univ} as a pseudo-character of $G_{\mathbb{Q},N\ell p}$ gives us a surjective map $f' : \mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0} \to \mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$. Now, for $g \in G_{\mathbb{Q},Np}$, $f(T^{\mathrm{univ}}(g)) = T^{\mathrm{univ},\ell}(g')$ for any lift g' of g in $G_{\mathbb{Q},N\ell p}$. Thus, $f' \circ f(T^{\mathrm{univ}}(g)) = f'(T^{\mathrm{univ},\ell}(g')) = T^{\mathrm{univ}}(g)$ for all $g \in G_{\mathbb{Q},Np}$. Therefore, $f' \circ f$ is the identity map and hence, f is an isomorphism. Thus, we get $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0} \simeq \mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$. □

Theorem 4.4. *Suppose $p \nmid \phi(N)$, $\dim(H^1(G_{\mathbb{Q},Np}, \chi^i)) = 1$ and $\dim(H^1(G_{\mathbb{Q},Np}, \chi^{-i})) = m$ for some $i \in \{1, -1\}$. Let ℓ be a prime such that $p \nmid \ell^2 - 1$ and $\chi^{-i}|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$. Then:*

1. *For any non-zero $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$, $(R^{pd,\ell}_{\bar\rho_0})^{red} \simeq (R^{def,\ell}_{\bar\rho_x})^{red}$. As a consequence, there exists $r_1, \cdots, r_{n'}, \Phi \in \mathbb{F}[[X_1, \cdots, X_n, X]]$ such that $(R^{pd,\ell}_{\bar\rho_0})^{red} \simeq (\mathbb{F}[[X_1, \cdots, X_n, X]]/(r_1, \cdots, r_{n'}, X(\Phi - \ell)))^{red}$ and $(R^{pd}_{\bar\rho_0})^{red} \simeq (\mathbb{F}[[X_1, \cdots, X_n]]/(\bar r_1, \cdots, \bar r_{n'}))^{red}$, where $r_i \pmod{X} = \bar r_i$.*
2. *Suppose $m = 1, 2$. For any non-zero $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$, $R^{pd,\ell}_{\bar\rho_0} \simeq R^{def,\ell}_{\bar\rho_x}$. As a consequence, there exists $r_1, \cdots, r_{n'}, \Phi \in \mathbb{F}[[X_1, \cdots, X_n, X]]$ such that $R^{pd,\ell}_{\bar\rho_0} \simeq \mathbb{F}[[X_1, \cdots, X_n, X]]/(r_1, \cdots, r_{n'}, X(\Phi-\ell))$ and $R^{pd}_{\bar\rho_0} \simeq \mathbb{F}[[X_1, \cdots, X_n]]/(\bar r_1, \cdots, \bar r_{n'})$, where $r_i \pmod{X} = \bar r_i$.*

Proof. Under the hypotheses of the lemma, we see, from the calculations done in the previous sub-section, that $\dim(H^1(G_{\mathbb{Q},N\ell p}, \chi^i)) = 1$ and $m \leq \dim(H^1(G_{\mathbb{Q},N\ell p}, \chi^{-i})) \leq m + 1$. So, $\dim(H^1(G_{\mathbb{Q},N\ell p}, \chi^i)) = \dim(H^1(G_{\mathbb{Q},Np}, \chi^i)) = 1$. Therefore, by Theorem 2.17, we have for any non-zero $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$, $(R^{\mathrm{pd}}_{\bar\rho_0})^{\mathrm{red}} \simeq (R^{\mathrm{def}}_{\bar\rho_x})^{\mathrm{red}}$ and $(R^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}} \simeq (R^{\mathrm{def},\ell}_{\bar\rho_x})^{\mathrm{red}}$. Moreover, if $m = 1, 2$, then by Theorem 2.19, for any non-zero $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$, $R^{\mathrm{pd}}_{\bar\rho_0} \simeq R^{\mathrm{def}}_{\bar\rho_x}$ and $R^{\mathrm{pd},\ell}_{\bar\rho_0} \simeq R^{\mathrm{def},\ell}_{\bar\rho_x}$. It follows, from [7, Theorem 4.7], that if $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$ is non-zero, then, under the hypotheses of the lemma, there exists $r_i, \cdots, r_n, \Phi \in \mathbb{F}[[X_1, \cdots, X_n, X]]$

such that $R^{\text{def},\ell}_{\bar\rho_x} \simeq \mathbb{F}[[X_1,\cdots,X_n,X]]/(r_1,\cdots,r_{n'},X(\Phi-\ell))$ and $R^{\text{def}}_{\bar\rho_x} \simeq \mathbb{F}[[X_1,\cdots,X_n]]/(\bar r_1,\cdots,\bar r_{n'})$, where $r_i \pmod X = \bar r_i$. Combining all these observations, we get the lemma. $\square$

4.3. **Structure of $\mathcal{R}^{\mathbf{pd},\ell}_{\bar\rho_0}$ with unobstructed $\bar\rho_0$ and $p \nmid \ell^2-1$.** We now turn our attention to unobstructed $\bar\rho_0$. Recall that, we say $\bar\rho_0$ is unobstructed if $p \nmid \phi(N)$ and $\dim(H^1(G_{\mathbb{Q},Np},\chi)) = \dim(H^1(G_{\mathbb{Q},Np},\chi^{-1})) = 1$. In this sub-section, we are going to work with unobstructed $\bar\rho_0$ and a prime ℓ such that $\ell \nmid Np$ and $p \nmid \ell^2-1$. For a non-zero $x \in H^1(G_{\mathbb{Q},Np},\chi^i)$ with $i \in \{1,-1\}$, let $\rho^{\text{univ},\ell}_x : G_{\mathbb{Q},N\ell p} \to \mathrm{GL}_2(R^{\text{def},\ell}_{\bar\rho_x})$ be the universal deformation of $\bar\rho_x$ over $R^{\text{def},\ell}_{\bar\rho_x}$.

Proposition 4.5. *Suppose $\bar\rho_0$ is unobstructed. Let ℓ be a prime such that $p \nmid \ell^2-1$, $\chi^i|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ for some $i \in \{1,-1\}$. Then, for any non-zero $x \in H^1(G_{\mathbb{Q},Np},\chi^{-i})$, $\mathcal{R}^{pd,\ell}_{\bar\rho_0} \simeq \mathcal{R}^{def,\ell}_{\bar\rho_x}$.*

Proof. Without loss of generality, suppose $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$. We have already seen that, in this case, $\dim(H^1(G_{\mathbb{Q},N\ell p},\mathrm{ad}(\bar\rho_x))) = 4$ for any non-zero $x \in H^1(G_{\mathbb{Q},Np},\chi^{-1})$. By the global Euler characteristic formula, this means that $\dim(H^2(G_{\mathbb{Q},N\ell p},\mathrm{ad}(\bar\rho_x))) = 1$. Therefore, by [8, Theorem 2.4], $\mathcal{R}^{\text{def},\ell}_{\bar\rho_x} \simeq W(\mathbb{F})[[X,Y,Z,W]]/I$ where I is either (0) or a principal ideal of $W(\mathbb{F})[[X,Y,Z,W]]$.

Suppose p is a zero divisor in $\mathcal{R}^{\text{def},\ell}_{\bar\rho_x}$. As $W(\mathbb{F})[[X,Y,Z,W]]$ is a UFD, this means that $I=(pf)$ for some $f \in W(\mathbb{F})[[X,Y,Z,W]]$. Thus, we get $R^{\text{def},\ell}_{\bar\rho_x} \simeq \mathbb{F}[[X,Y,Z,W]]$.

From [7, Lemma 4.8] and [7, Lemma 4.9], it follows that $\rho^{\text{univ},\ell}_x|_{I_\ell}$ factors through the $\mathbb{Z}_p$-quotient of the tame inertia group at ℓ and if i_ℓ is a generator of this $\mathbb{Z}_p$-quotient, then $\rho^{\text{univ},\ell}_x(i_\ell) = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$ for some $w \in R^{\text{def},\ell}_{\bar\rho_x}$. Moreover, there exists a lift z of Frob_ℓ in $G_{\mathbb{Q}_\ell}$ such that $\rho^{\text{univ},\ell}_x(z) = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}$. From the action of Frob_ℓ on the tame inertia group at ℓ, we see that $(\phi_1/\phi_2 - \ell)w = 0$.

If $w=0$, then the universal deformation $\rho^{\text{univ},\ell}_x$ factors through $G_{\mathbb{Q},Np}$. So, all the infinitesimal deformations of $\bar\rho_x$ for $G_{\mathbb{Q},N\ell p}$ factor through $G_{\mathbb{Q},Np}$. But this would imply that $\dim(\tan(R^{\text{def},\ell}_{\bar\rho_x})) \leq \dim(H^1(G_{\mathbb{Q},Np},\mathrm{ad}(\bar\rho_x))) = 3$ which is not true as we know $\dim(\tan(R^{\text{def},\ell}_{\bar\rho_x})) = 4$. Therefore, we see that $w \neq 0$. As $R^{\text{def},\ell}_{\bar\rho_x}$ is an integral domain, we get that $\phi_1/\phi_2 = \ell$.

Let $f_1 : R^{\text{def},\ell}_{\bar\rho_x} \to R^{\text{def}}_{\bar\rho_x}$ obtained by considering $\rho^{\text{univ},\ell}_x$ as a representation of $G_{\mathbb{Q},N\ell p}$. As $f_1 \circ \rho^{\text{univ},\ell}_x = \rho^{\text{univ}}_x$, we see that $\rho^{\text{univ}}_x(\mathrm{Frob}_\ell) = \begin{pmatrix} \phi'_1 & 0 \\ 0 & \phi'_2 \end{pmatrix}$ with $\phi'_1/\phi'_2 = \ell$. This means $\ell(\frac{\mathrm{tr}(\rho^{\text{univ}}_x(\mathrm{Frob}_\ell))}{(\ell+1)})^2 = \det(\rho^{\text{univ}}_x(\mathrm{Frob}_\ell))$. Now $\psi_x : R^{\text{pd}}_{\bar\rho_0} \to R^{\text{def}}_{\bar\rho_x}$ is an isomorphism. Therefore, we see that $\ell(\frac{t^{\text{univ}}(\mathrm{Frob}_\ell)}{(\ell+1)})^2 = d^{\text{univ}}(\mathrm{Frob}_\ell)$.

Consider the pseudo-character $\tilde{\chi}_1^{\text{univ}} + \tilde{\chi}_2^{\text{univ}} : G_{\mathbb{Q},Np} \to \mathbb{F}[[X_1, X_2]]$ constructed in the proof of Lemma 2.3. It is a deformation of $\text{tr}(\bar{\rho}_0)$. Therefore, we should have $\ell(\frac{\tilde{\chi}_1^{\text{univ}}(\text{Frob}_\ell)+\tilde{\chi}_2^{\text{univ}}(\text{Frob}_\ell)}{\ell+1})^2 = \tilde{\chi}_1^{\text{univ}}(\text{Frob}_\ell).\tilde{\chi}_2^{\text{univ}}(\text{Frob}_\ell)$. This equality simplifies to give $(\ell\tilde{\chi}_1^{\text{univ}}(\text{Frob}_\ell) - \tilde{\chi}_2^{\text{univ}}(\text{Frob}_\ell))(\tilde{\chi}_1^{\text{univ}}(\text{Frob}_\ell) - \ell\tilde{\chi}_2^{\text{univ}}(\text{Frob}_\ell)) = 0$. But $\tilde{\chi}_1^{\text{univ}}(\text{Frob}_\ell)$ and $\tilde{\chi}_2^{\text{univ}}(\text{Frob}_\ell)$ are in the subrings $\mathbb{F}[[X_1]]$ and $\mathbb{F}[[X_2]]$ of $\mathbb{F}[[X_1, X_2]]$, respectively. Therefore, $(\ell\tilde{\chi}_1^{\text{univ}}(\text{Frob}_\ell) - \tilde{\chi}_2^{\text{univ}}(\text{Frob}_\ell)) \neq 0$, $(\tilde{\chi}_1^{\text{univ}}(\text{Frob}_\ell) - \ell\tilde{\chi}_2^{\text{univ}}(\text{Frob}_\ell)) \neq 0$ and hence, $(\ell\tilde{\chi}_1^{\text{univ}}(\text{Frob}_\ell) - \tilde{\chi}_2^{\text{univ}}(\text{Frob}_\ell))(\tilde{\chi}_1^{\text{univ}}(\text{Frob}_\ell) - \ell\tilde{\chi}_2^{\text{univ}}(\text{Frob}_\ell)) \neq 0$, giving us a contradiction.

Hence, $R_{\bar{\rho}_x}^{\text{def},\ell} \not\simeq \mathbb{F}[[X, Y, Z, W]]$ and p is not a zero-divisor in $\mathcal{R}_{\bar{\rho}_x}^{\text{def},\ell}$. From the discussion before, we know that $\dim(H^1(G_{\mathbb{Q},N\ell p}, \chi)) = 2$ and $\dim(H^1(G_{\mathbb{Q},N\ell p}, \chi^{-1})) = 1$. Therefore, by Proposition 3.8 and Theorem 2.19, we conclude that $\mathcal{R}_{\bar{\rho}_0}^{\text{pd},\ell} \simeq \mathcal{R}_{\bar{\rho}_x}^{\text{def},\ell}$ for any non-zero $x \in H^1(G_{\mathbb{Q},Np}, \chi^{-1})$. □

Theorem 4.6. *Suppose $\bar{\rho}_0$ is unobstructed. Let ℓ be a prime such that $p \nmid \ell^2 - 1$, $\chi^i|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ for some $i \in \{1, -1\}$. Then, $\mathcal{R}_{\bar{\rho}_0}^{pd,\ell} \simeq W(\mathbb{F})[[X_1, X_2, X_3, X_4]]/(X_4 f)$ for some $f \in W(\mathbb{F})[[X_1, X_2, X_3, X_4]]$. Moreover, if $\ell/\tilde{\ell}$ is a topological generator of $1 + p\mathbb{Z}_p$, then $\mathcal{R}_{\bar{\rho}_0}^{pd,\ell} \simeq W(\mathbb{F})[[X_1, X_2, X_3, X_4]]/(X_4(p + X_2))$.*

Proof. Without loss of generality assume $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$. By Proposition 4.5, we have $\mathcal{R}_{\bar{\rho}_0}^{\text{pd},\ell} \simeq \mathcal{R}_{\bar{\rho}_x}^{\text{def},\ell}$ for any non-zero $x \in H^1(G_{\mathbb{Q},N\ell p}, \chi^{-1})$. Therefore, there exists a representation $\rho : G_{\mathbb{Q},Np} \to \text{GL}_2(\mathcal{R}_{\bar{\rho}_0}^{\text{pd},\ell})$ such that $\text{tr}(\rho) = T^{\text{univ},\ell}$. Moreover, from [7, Lemma 4.8] and [7, Lemma 4.9], we see that $\rho(G_{\mathbb{Q}_\ell})$ is generated by two elements and we can choose them to be $\begin{pmatrix}\phi_1 & 0\\ 0 & \phi_2\end{pmatrix}$ and $\begin{pmatrix}1 & w\\ 0 & 1\end{pmatrix}$. Note that, the generators are chosen such that $\rho(I_\ell)$ is the group generated by $\begin{pmatrix}1 & w\\ 0 & 1\end{pmatrix}$ and $\rho(g_\ell) = \begin{pmatrix}\phi_1 & 0\\ 0 & \phi_2\end{pmatrix}$ for some lift g_ℓ of Frob_ℓ. From the proof of Proposition 4.5, we also get that $w \neq 0$ and $w(\phi_2/\phi_1 - \ell) = 0$.

Note that, the kernel of the map $f_1 : \mathcal{R}_{\bar{\rho}_0}^{\text{pd},\ell} \to \mathcal{R}_{\bar{\rho}_0}^{\text{pd}}$ is just (w), the ideal generated by w in $\mathcal{R}_{\bar{\rho}_0}^{\text{pd},\ell}$. Indeed, we have $f_1(w(\phi_1/\phi_2 - \ell)) = 0$ but, from the proof of Proposition 4.5, $f_1(\phi_1/\phi_2 - \ell) \neq 0$. As $\mathcal{R}_{\bar{\rho}_0}^{\text{pd}}$ is an integral domain, this means that $f_1(w) = 0$. On the other hand, $T^{\text{univ},\ell} \pmod{(w)}$ factors through $G_{\mathbb{Q},Np}$ and hence, the natural surjective map $\mathcal{R}_{\bar{\rho}_0}^{\text{pd},\ell} \to \mathcal{R}_{\bar{\rho}_0}^{\text{pd},\ell}/(w)$ factors through $\mathcal{R}_{\bar{\rho}_0}^{\text{pd}}$. This implies that $\ker(f_1) \subset (w)$. From Proposition 4.5 and [8, Theorem 2.4], we know that $\mathcal{R}_{\bar{\rho}_0}^{\text{pd},\ell} \simeq W(\mathbb{F})[[X, Y, Z, W]]/I$, where I is a non-zero principal ideal of $W(\mathbb{F})[[X, Y, Z, W]]$. As $\ker(f_1)$ is a principal ideal and $\mathcal{R}_{\bar{\rho}_0}^{\text{pd}} \simeq W(\mathbb{F})[[X_1, X_2, X_3]]$, we get that $\mathcal{R}_{\bar{\rho}_0}^{\text{pd},\ell} \simeq W(\mathbb{F})[[X_1, X_2, X_3, X_4]]/(X_4 f)$ for some $f \in W(\mathbb{F})[[X_1, X_2, X_3, X_4]]$.

Now assume $\ell/\tilde{\ell}$ is a topological generator of $1+p\mathbb{Z}_p$. Recall that, $\mathcal{R}_{\bar{\rho}_0}^{\text{pd}} \simeq W(\mathbb{F})[[X_1, X_2, X_3]]$ and $\mathcal{R}_{\bar{\rho}_0}^{\text{pd}}/\mathcal{I}_{\bar{\rho}_0} \simeq W(\mathbb{F})[[X_1, X_2]]$. Thus, $\mathcal{I}_{\bar{\rho}_0}$ is generated by one element. Let $\tilde{x}$ be a lift of a generator of $\mathcal{I}_{\bar{\rho}_0}$ in $\mathcal{R}_{\bar{\rho}_0}^{\text{pd},\ell}$.

Therefore, $\phi_1 = \chi_1\widehat{(\mathrm{Frob}_\ell)}(1+y)$ and $\phi_2 = \chi_2\widehat{(\mathrm{Frob}_\ell)}(1+z)$ for some y, $z \in \mathcal{M}^\ell$. Let I be the ideal of $\mathcal{R}^{\mathrm{pd},\ell}_{\bar{\rho}_0}$ generated by the set $\{p, \tilde{x}, y, z, w\}$. Therefore, from [3, Proposition 1.5.1], we see that $T^{\mathrm{univ},\ell} \pmod{I}$ is a sum of two characters deforming χ_1 and χ_2. Thus, I contains $\mathcal{I}^\ell_{\bar{\rho}_0}$. Hence, it follows, from Lemma 4.2, that $I = \mathcal{M}^\ell$.

Hence, we have a surjective local homomorphism $f : W(\mathbb{F})[[X, Y, Z, W]] \to \mathcal{R}^{\mathrm{pd},\ell}_{\bar{\rho}_0}$ of $W(\mathbb{F})$-algebras sending X, Y, Z, W to $\tilde{x}$, y, z, w, respectively. Now, from Proposition 4.5, we know that $\mathcal{R}^{\mathrm{pd},\ell}_{\bar{\rho}_0} \simeq W(\mathbb{F})[[X_1, X_2, X_3, X_4]]/J$ where J is a non-zero principal ideal. Therefore, $\ker(f)$ is a non-zero principal ideal. As $w(\phi_1/\phi_2 - \ell) = 0$, we see that $\ker(f)$ contains $W(\widehat{\chi}_1(\mathrm{Frob}_\ell)(1+X) - \ell\widehat{\chi}_2(\mathrm{Frob}_\ell)(1+Y))$. Note that, $\widehat{\chi}_1(\mathrm{Frob}_\ell) = \tilde{\ell}\widehat{\chi}_2(\mathrm{Frob}_\ell)$. Therefore, $\ker(f)$ contains $W(\tilde{\ell} - \ell + \tilde{\ell}X - \ell Y)$. As $\ell/\tilde{\ell}$ is a topological generator of $1 + p\mathbb{Z}_p$, we get that $\tilde{\ell} - \ell = p.u$ for some $u \in \mathbb{Z}_p^*$. Therefore, by the Eisenstein criteria, $\tilde{\ell} - \ell + \tilde{\ell}X - \ell Y$ is an irreducible element of the UFD $W(\mathbb{F})[[X, Y, Z, W]]$.

Since $\ker(f)$ is principal ideal, we see that it is either (W), $(\tilde{\ell} - \ell + \tilde{\ell}X - \ell Y)$ or $(W(\tilde{\ell} - \ell + \tilde{\ell}X - \ell Y))$. But we know $w \neq 0$. If $\tilde{\ell} - \ell + \tilde{\ell}\tilde{x} - \ell y = 0$, then it would imply that $\phi_1/\phi_2 - \ell = 0$ which is not true. Therefore, we see that $\ker(f)$ is not (W) and $(\tilde{\ell} - \ell + \tilde{\ell}X - \ell Y)$. Hence, $\ker(f) = (W(\tilde{\ell} - \ell + \tilde{\ell}X - \ell Y))$. Taking $X_1 = X$, $X_2 = u^{-1}(\tilde{\ell}X - \ell Y)$, $X_3 = Z$, $X_4 = W$ gives us the proposition. $\square$

Remark 4.7. *The structure of $R^{def,\ell}_{\bar{\rho}_x}$ under the hypotheses of Theorem 4.6 was also found in* [7, Theorem 4.7]. *But it is not clear how to get the explicit structure of $R^{def,\ell}_{\bar{\rho}_x}$ as given in Theorem 4.6 directly from* [7, Theorem 4.7] *or its proof.*

4.4. **Structure of $R^{\mathbf{pd},\ell}_{\bar{\rho}_0}$ with unobstructed $\bar{\rho}_0$ and $p|\ell+1$.** We now turn to the case where $\bar{\rho}_0$ is unobstructed and ℓ is a prime such that $\ell \nmid Np$ and $p|\ell+1$. As we will see, this case is a bit more complicated than the previous case. Even in the cases considered in [10] and [7], the structure of the deformation ring obtained after allowing ramification at a prime ℓ which is $-1 \pmod{\ell}$ was different and more complicated than the structure of the deformation ring obtained after allowing ramification at a prime ℓ which is not 1 or $-1 \pmod{p}$. We begin by determining the explicit structure of $R^{\mathrm{def},\ell}_{\bar{\rho}_x}$ under certain hypotheses. Before proceeding further, we need a piece of notation. Let $\{h_i | i \in \mathbb{Z}, i \geq 0\}$ be the set of polynomials in $\mathbb{F}[\sqrt{1+UV}]$ satisfying the recurrence relation $b_{i+1} - 2(\sqrt{1+UV})b_i + b_{i-1} = 0$ with $h_0 = 0$ and $h_1 = 1$ (see [10] for more details). So, $\{h_i | i \in \mathbb{Z}, i \geq 0\} \subset W(\mathbb{F})[[U, V]]$. Note that, $h_\ell \equiv \ell \pmod{(UV)}$. For a non-zero $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$ with $i \in \{1, -1\}$, let $\tau_x^{\mathrm{univ},\ell} : G_{\mathbb{Q},N\ell p} \to \mathrm{GL}_2(\mathcal{R}^{\mathrm{def},\ell}_{\bar{\rho}_x})$ be the universal deformation of $\bar{\rho}_x$.

Lemma 4.8. *Suppose $\bar{\rho}_0$ is unobstructed. Let ℓ be a prime such that $\ell \equiv -1 \pmod{p}$, $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ and $-\ell$ is a topological generator of $1 + p\mathbb{Z}_p$. Let $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$ be* *a non-zero element for* $i \in \{1, -1\}$. *Then,* $\mathcal{R}^{def,\ell}_{\bar\rho_x} \simeq W(\mathbb{F})[[X,Y,Z,U,V]]/(U((1+X)+h_\ell(1+Y)), V((1+Y)+h_\ell(1+X)))$.

Proof. From [7, Lemma 4.8] and [7, Lemma 4.9], it follows that $\tau_x^{\mathrm{univ},\ell}|_{I_\ell}$ factors through the $\mathbb{Z}_p$-quotient of the tame inertia group at ℓ and if i_ℓ is a generator of this $\mathbb{Z}_p$-quotient, then $\tau_x^{\mathrm{univ},\ell}(i_\ell) = \begin{pmatrix} \sqrt{1+uv} & u \\ v & \sqrt{1+uv} \end{pmatrix}$ for some $u, v \in \mathcal{R}^{\mathrm{def},\ell}_{\bar\rho_x}$. Moreover, there exists a lift w of Frob_ℓ in $G_{\mathbb{Q}_\ell}$ such that $\tau_x^{\mathrm{univ},\ell}(w) = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}$.

From Lemma 4.1, we know that $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0} \simeq \mathcal{R}^{\mathrm{def}}_{\bar\rho_x}$. Hence, we have a representation $\rho : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0})$ such that $\mathrm{tr}(\rho) = T^{\mathrm{univ}}$ and $\rho \pmod{\mathcal{M}} = \bar\rho_x$. Considering ρ as a representation of $G_{\mathbb{Q},N\ell p}$, we get a map $f_1 : \mathcal{R}^{\mathrm{def},\ell}_{\bar\rho_x} \to \mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$. As $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$ is topologically generated over $W(\mathbb{F})$ by $T^{\mathrm{univ}}(g)$ with $g \in G_{\mathbb{Q},Np}$, we see that f_1 is surjective. Note that, $\ker(f_1) = (u, v)$.

We know that $\mathcal{I}_{\bar\rho_0}$ is generated by one element. Let z be a lift of a generator of $\mathcal{I}_{\bar\rho_0}$ in $\mathcal{R}^{\mathrm{def},\ell}_{\bar\rho_0}$. Let $\mathcal{N}$ be the maximal ideal of $\mathcal{R}^{\mathrm{def},\ell}_{\bar\rho_x}$. Now, $\phi_1 = \widehat{\chi_1(\mathrm{Frob}_\ell)}(1+\tilde{x})$ and $\phi_2 = \widehat{\chi_2(\mathrm{Frob}_\ell)}(1+y)$, with $\tilde{x}, y \in \mathcal{N}$. Let I be the ideal of $\mathcal{R}^{\mathrm{def},\ell}_{\bar\rho_x}$ generated by the set $\{p, \tilde{x}, y, z, u, v\}$. So, there exists two characters $\eta_1, \eta_2 : G_{\mathbb{Q},Np} \to (\mathcal{R}^{\mathrm{univ},\ell}_{\bar\rho_x}/I)^*$ such that $\mathrm{tr}(\rho_x^{\mathrm{univ},\ell}) \pmod{I} = \eta_1 + \eta_2$, η_i is unramified at ℓ for $i = 1, 2$, η_i is a deformation of χ_i for $i = 1, 2$, $\prod_{i=1}^2 \eta_i(\mathrm{Frob}_\ell) = \prod_{i=1}^2 \chi_i(\mathrm{Frob}_\ell)$ and $\sum_{i=1}^2 \eta_i(\mathrm{Frob}_\ell) = \sum_{i=1}^2 \chi_i(\mathrm{Frob}_\ell)$. Thus, from the proof of Lemma 4.2, we get that $\eta_i = \chi_i$ for $i = 1, 2$. Therefore, we have $I = \mathcal{N}$.

Thus, we have a surjective map $h_0 : W(\mathbb{F})[[X,Y,Z,U,V]] \to \mathcal{R}^{\mathrm{def},\ell}_{\bar\rho_x}$ of $W(\mathbb{F})$-algebras sending X to $\tilde{x}$, Y to y, Z to z, U to u and V to v. Let $J_0 = \ker(h_0)$. As $\mathcal{R}^{\mathrm{def},\ell}_{\bar\rho_x}/(u,v) \simeq \mathcal{R}^{\mathrm{pd}}_{\bar\rho_0} \simeq W(\mathbb{F})[[X_1, X_2, X_3]]$, we get that $J_0 \subset (U, V)$. From the action of Frob_ℓ on the tame inertia group at ℓ, we see that $(\phi_1/\phi_2 - h_\ell)u = 0$ and $(\phi_2/\phi_1 - h_\ell)v = 0$. Note that, as $p | \ell + 1$ and $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$, we have $\widehat{\chi_1(\mathrm{Frob}_\ell)} = -\widehat{\chi_2(\mathrm{Frob}_\ell)}$. Therefore, we have $((1+\tilde{x}) + h_\ell(1+y))u = 0$ and $((1+y) + h_\ell(1+\tilde{x}))v = 0$. So, $((1+X) + h_\ell(1+Y))U$, $((1+Y) + h_\ell(1+X))V \in J_0$. In this case, we know that $\dim(H^1(G_{\mathbb{Q},N\ell p}, \mathrm{ad}(\bar\rho_x))) = 5$. By [8, Theorem 2.4], $\mathcal{R}^{\mathrm{def},\ell}_{\bar\rho_x} \simeq W(\mathbb{F})[[X_1, X_2, X_3, X_4, X_5]]/J$, where J is generated by at most 2 elements. Therefore, J_0 is generated by at most 2 elements.

Denote $W(\mathbb{F})[[X,Y,Z,U,V]]$ by R and its maximal ideal (p, X, Y, Z, U, V) by m_0. Note that, $h_\ell \equiv \ell \pmod{(UV)}$. Thus, $((1+X) + h_\ell(1+Y)) \equiv (\ell + 1 + X + \ell Y) \pmod{(UV)}$ and $((1+Y) + h_\ell(1+X)) \equiv (\ell + 1 + Y + \ell X) \pmod{(UV)}$. So, $(1+X) + h_\ell(1+Y)$, $(1+Y) + h_\ell(1+X) \in m_0 \setminus m_0^2$. So, both of them are irreducible elements of the UFD R. If J_0 is generated by one element, say α, then $\alpha | ((1+X) + h_\ell(1+Y))U$ and $\alpha | ((1+Y) + h_\ell(1+X))V$. This means that either $\alpha | ((1+X) + h_\ell(1+Y))$ or $\alpha | ((1+Y) + h_\ell(1+X))$. As both $((1+X) + h_\ell(1+Y))$ and $((1+Y) + h_\ell(1+X))$ are irreducible, we see that (α) is either

$((1+X)+h_\ell(1+Y))$ or $((1+Y)+h_\ell(1+X))$. Hence, either $J_0 = ((1+X)+h_\ell(1+Y))$ or $J_0 = ((1+Y)+h_\ell(1+X))$. But this is not true as $J_0 \subset (U,V)$. So, J_0 is generated by two elements.

Suppose $J_0 = (h_1,h_2)$. Recall that, $\mathcal{R}^{\mathrm{def},\ell}_{\bar\rho_x}/(p) \simeq R^{\mathrm{def},\ell}_{\bar\rho_x}$ and $\dim(\tan(R^{\mathrm{def},\ell}_{\bar\rho_x})) = \dim(H^1(G_{\mathbb{Q},N\ell p},\mathrm{ad}(\bar\rho_x))) = 5$. As $J_0 \subset (U,V)$, it follows that $h_1,h_2 \in U.m_0 + V.m_0 \subset m_0^2$. It follows from Nakayama's lemma that J_0/m_0J_0 is a vector space over $\mathbb{F}$ of dimension 2. As $h_\ell \equiv \ell \pmod{(UV)}$ and $m_0J_0 \subset m_0^3$, we see that the images of the elements $((1+Y)+h_\ell(1+X))V$ and $((1+X)+h_\ell(1+Y))U)$ in J_0/m_0J_0 are linearly independent over $\mathbb{F}$. Hence, they form an $\mathbb{F}$ basis of the vector space J_0/m_0J_0. Therefore, by Nakayama's lemma we get that $J_0 = (((1+Y)+h_\ell(1+X))V, ((1+X)+h_\ell(1+Y))U)$. □

Remark 4.9. *The structure of* $R^{def,\ell}_{\bar\rho_x}$ *under the hypotheses of Lemma 4.8 was also found in* [7, Theorem 4.7]*. But it is not clear how to get the explicit structure of* $R^{def,\ell}_{\bar\rho_x}$ *as given in Lemma 4.8 directly from* [7, Theorem 4.7] *or its proof.*

We now turn our attention to the problem of finding the structure of $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ when $\bar\rho_0$ is unobstructed and $p|\ell+1$. Note that, in this case, we have $\dim(H^1(G_{\mathbb{Q},N\ell p},\chi)) = \dim(H^1(G_{\mathbb{Q},N\ell p},\chi^{-1})) = 2$. So this case is different from the cases we have dealt with so far. So, we can not use the results obtained so far. However, we can still use the technique of comparing $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ with the universal deformation rings of residually Borel representations. However, we need to make a small change in our approach. So far, we were comparing $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ with the universal deformation ring of a specific Borel representation coming from the situation. But in this case, we have to compare $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ with the universal deformation rings of multiple Borel representations. It turns out that using this approach, we can find the structure of $(R^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}}$

Theorem 4.10. *Suppose* $\bar\rho_0$ *is unobstructed. Let* ℓ *be a prime such that* $\ell \equiv -1 \pmod{p}$*,* $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ *and* $-\ell$ *is a topological generator of* $1+p\mathbb{Z}_p$*. Then,* $(R^{pd,\ell}_{\bar\rho_0})^{red} \simeq \mathbb{F}[[X,Y,Z,T_1,T_2]]/(T_1T_2,T_1Z,T_2Z)$*.*

We will first prove a series of lemmas which will be used to prove Theorem 4.10.

Lemma 4.11. *Suppose* $\bar\rho_0$ *is unobstructed. Let* ℓ *be a prime such that* $\ell \equiv -1 \pmod{p}$ *and* $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$*. Let* R *be a complete Noetherian local ring with maximal ideal* m_R *and residue field* $\mathbb{F}$*. Fix a lift* g_ℓ *of* $Frob_\ell$ *in* $G_{\mathbb{Q}_\ell}$*. Let* $t : G_{\mathbb{Q},N\ell p} \to R$ *be a pseudo-character deforming* $\mathrm{tr}(\bar\rho_0)$*. Let* $A = \begin{pmatrix} R & B \\ C & R \end{pmatrix}$ *be the GMA associated to the tuple* (R,ℓ,t,g_ℓ) *in Lemma 2.10. Let* $\rho : G_{\mathbb{Q},N\ell p} \to A^*$ *be the corresponding representation found in Lemma 2.10 and* i_ℓ *be a generator of the* $\mathbb{Z}_p$*-quotient of the tame inertia group at* ℓ*. Suppose* $\rho(i_\ell) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$*.*

(1) *Then, both B and C are generated by at most 2 elements.*
(2) *There exist $b', c' \in R$ such that B and C are generated by $\{b, b'\}$ and $\{c, c'\}$, respectively.*

Proof. (1) Imitating the proof of the last part of Lemma 2.8, which uses the proof of [3, Theorem 1.5.5], we get the following injective maps of $\mathbb{F}$-vector spaces: $j_1 : \mathrm{Hom}_R(B/m_RB, \mathbb{F}) \to H^1(G_{\mathbb{Q},N\ell p}, \chi)$ and $j_2 : \mathrm{Hom}_R(C/m_RC, \mathbb{F}) \to H^1(G_{\mathbb{Q},N\ell p}, \chi^{-1})$. We have already seen that under the conditions on ℓ and $\bar{\rho}_0$, $\dim(H^1(G_{\mathbb{Q},N\ell p}, \chi)) = \dim(H^1(G_{\mathbb{Q},N\ell p}, \chi^{-1})) = 2$. So, by Nakayama's lemma, we see that both B and C are generated by at most 2 elements.

(2) By Lemma 2.11, $\rho(i_\ell)$ is well defined and $\rho(I_\ell)$ is generated by $\rho(i_\ell)$. Let x be an element of the subspace $\mathrm{Hom}_R(B/R.b+m_RB, \mathbb{F})$ of $\mathrm{Hom}_R(B/m_RB, \mathbb{F})$. So, $j_1(x)$ is an element of $H^1(G_{\mathbb{Q},N\ell p}, \chi)$ such that $j_1(x)(I_\ell) = 0$ i.e. $j_1(x)$ is unramified at ℓ. Thus, $j_1(x)$ lies in the image of the injective map $j_1' : H^1(G_{\mathbb{Q},Np}, \chi) \to H^1(G_{\mathbb{Q},N\ell p}, \chi)$. So, $j_1(\mathrm{Hom}_R(B/R.b + m_RB, \mathbb{F})) \subset j_1'(H^1(G_{\mathbb{Q},Np}, \chi))$. Hence, $\dim(\mathrm{Hom}_R(B/R.b+m_RB, \mathbb{F})) \leq \dim(H^1(G_{\mathbb{Q},Np}, \chi)) = 1$, Therefore, by Nakayama's lemma, $B/R.b$ is generated by at most 1 element. By the same logic, we also get that $C/R.c$ is generated by at most 1 element. So, if $B = R.b$, then we can take $b' = 0$. Otherwise, $B/R.b$ is generated by one element and let b' be a lift of the generator in B. Thus, $\{b, b'\}$ generates B in both the cases. The lemma for C and c follows similarly.

□

Let P be a prime $R^{\mathrm{pd},\ell}_{\bar{\rho}_0}$. Fix a lift g_ℓ of Frob_ℓ in $G_{\mathbb{Q}_\ell}$. Let A_P be the GMA obtained in Lemma 2.10 for the tuple $(R^{\mathrm{pd},\ell}_{\bar{\rho}_0}/P, \ell, t^{\mathrm{univ},\ell} \pmod{P}, g_\ell)$. Let $A_P = \begin{pmatrix} R^{\mathrm{pd},\ell}_{\bar{\rho}_0}/P & B_P \\ C_P & R^{\mathrm{pd},\ell}_{\bar{\rho}_0}/P \end{pmatrix}$ and $\rho_P : G_{\mathbb{Q},N\ell p} \to A_P^*$ be the corresponding representation. By Lemma 2.11, we see that $\rho_P|_{I_\ell}$ factors through the $\mathbb{Z}_p$-quotient of the tame inertia group at ℓ. Fix a generator i_ℓ of this $\mathbb{Z}_p$-quotient. We will now use this notation throughout the paper.

Lemma 4.12. *Suppose ℓ is a prime such that $\ell \nmid Np$, $p \nmid \ell - 1$ and $\chi|_{G_{\mathbb{Q}_\ell}} \neq 1$. If P is a prime of $R^{pd,\ell}_{\bar{\rho}_0}$, then $t^{univ,\ell} \pmod{P}|_{G_{\mathbb{Q}_\ell}}$ is reducible.*

Proof. By [2, Lemma 2.2.2], we can choose B_P and C_P to be fractional ideals of $K_P = \mathrm{Frac}(R^{\mathrm{pd},\ell}_{\bar{\rho}_0}/P)$ such that the multiplication map $m' : B_P \otimes_{R^{\mathrm{pd},\ell}_{\bar{\rho}_0}/P} C_P \to R^{\mathrm{pd},\ell}_{\bar{\rho}_0}/P$ coincides with the multiplication map in K_P.

From Lemma 2.11, we see that $\rho_P|_{I_\ell}$ factors through the $\mathbb{Z}_p$-quotient of the tame inertia group at ℓ. By the action of Frob_ℓ on the tame inertia group by conjugation, we see that $\rho_P(i_\ell)$ is conjugate to $\rho_P(i_\ell)^\ell$. So, if $a \in \bar{K}_P$ is an eigenvalue of $\rho_P(i_\ell)$, then a^ℓ is also an eigenvalue of $\rho_P(i_\ell)$. As $p \nmid \ell - 1$, $\det(\rho_P(I_\ell)) = 1$. Hence, we get that either $a^\ell = a$ or $a^\ell = a^{-1}$. In particular, a is an n-th root of unity for some $n \in \mathbb{N}$. As K_P has characteristic p, it follows that $\rho_P(i_\ell)$ has finite order. As i_ℓ is a generator of the $\mathbb{Z}_p$-quotient of I_ℓ, we see that $\rho_P(i_\ell)$ has order p^n for some $n \in \mathbb{N}$. Since K_P has characteristic p, it follows that 1 is the only eigenvalue of $\rho_P(i_\ell)$.

So, there exists some $Q \in \mathrm{GL}_2(K_P)$ such that $Q\rho_P(i_\ell)Q^{-1} = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$ for some $w \in K_P$. Thus, $Q\rho_P(I_\ell)Q^{-1} = \{\begin{pmatrix} 1 & n.w \\ 0 & 1 \end{pmatrix} | 0 \leq n \leq p-1\}$. As I_ℓ is normal in $G_{\mathbb{Q}_\ell}$, we see that $Q\rho_P(G_{\mathbb{Q}_\ell})Q^{-1}$ is a subgroup of the group of upper triangular matrices in $\mathrm{GL}_2(K_P)$. By Lemma 2.10, we know that $\rho_P(g_\ell) = \begin{pmatrix} a_P & 0 \\ 0 & d_P \end{pmatrix} \in \mathrm{GL}_2(R^{\mathrm{pd},\ell}_{\bar\rho_0}/P)$ such that a_P and d_P are congruent modulo $\mathfrak{m}^\ell/P$ to $\chi_1(\mathrm{Frob}_\ell)$ and $\chi_2(\mathrm{Frob}_\ell)$, respectively. So there exists $w' \in K_P$ such that $Q\rho_P(g_\ell)Q^{-1}$ is either $\begin{pmatrix} a_P & w' \\ 0 & d_P \end{pmatrix}$ or $\begin{pmatrix} d_P & w' \\ 0 & a_P \end{pmatrix}$.

Note that, $\rho_P(G_{\mathbb{Q}_\ell})$ is generated by $\rho_P(g_\ell)$ and $\rho_P(i_\ell)$. So, from the description of $Q\rho_P(I_\ell)Q^{-1}$ and $Q\rho_P(g_\ell)Q^{-1}$, it follows that $t^{\mathrm{univ},\ell} \pmod P|_{G_{\mathbb{Q}_\ell}} = \mathrm{tr}(\rho_P)|_{G_{\mathbb{Q}_\ell}} = \theta_1 + \theta_2$, where θ_1, $\theta_2 : G_{\mathbb{Q}_\ell} \to (R^{\mathrm{pd},\ell}_{\bar\rho_0}/P)^*$ are unramified characters such that $\theta_1(\mathrm{Frob}_\ell) = a_P$ and $\theta_2(\mathrm{Frob}_\ell) = d_P$. As $a_p \pmod{\mathfrak{m}^\ell/P} = \chi_1(\mathrm{Frob}_\ell)$ and $d_p \pmod{\mathfrak{m}^\ell/P} = \chi_2(\mathrm{Frob}_\ell)$, it follows that θ_1 and θ_2 are deformations of χ_1 and χ_2, respectively. So, in particular, $t^{\mathrm{univ},\ell} \pmod P|_{G_{\mathbb{Q}_\ell}}$ is reducible. □

For a non-zero $x \in H^1(G_{\mathbb{Q},Np}, \chi)$, $\mathrm{tr}(\rho_x^{\mathrm{univ},\ell})$ induces a map $f_{1,x} : R^{\mathrm{pd},\ell}_{\bar\rho_0} \to R^{\mathrm{def},\ell}_{\bar\rho_x}$. For a non-zero $y \in H^1(G_{\mathbb{Q},Np}, \chi^{-1})$, $\mathrm{tr}(\rho_y^{\mathrm{univ},\ell})$ induces a map $f_{2,y} : R^{\mathrm{pd},\ell}_{\bar\rho_0} \to R^{\mathrm{def},\ell}_{\bar\rho_y}$. Note that, these maps are not surjective. Composing $f_{2,y}$ with the surjective map $R^{\mathrm{def},\ell}_{\bar\rho_y} \to R^{\mathrm{def},\ell}_{\bar\rho_y}/(V, X-Y)$, we get a map $f'_{2,y} : R^{\mathrm{pd},\ell}_{\bar\rho_0} \to R^{\mathrm{def},\ell}_{\bar\rho_y}/(V, X-Y)$ and composing $f_{1,x}$ with the surjective map $R^{\mathrm{def},\ell}_{\bar\rho_x} \to R^{\mathrm{def},\ell}_{\bar\rho_x}/(U, X-Y)$, we get a map $f'_{1,x} : R^{\mathrm{pd},\ell}_{\bar\rho_0} \to R^{\mathrm{def},\ell}_{\bar\rho_y}/(U, X-Y)$.

Lemma 4.13. *Suppose $\bar\rho_0$ is unobstructed. Let ℓ be a prime such that $\ell \equiv -1 \pmod p$, $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ and $-\ell$ is a topological generator of $1 + p\mathbb{Z}_p$. Then, the maps $f'_{1,x}$ and $f'_{2,y}$ are surjective and both $\ker(f'_{1,x})$ and $\ker(f'_{2,y})$ are prime ideals of $R^{pd,\ell}_{\bar\rho_0}$.*

Proof. Let $y \in H^1(G_{\mathbb{Q},Np}, \chi^{-1})$ be a non-zero element. By Lemma 4.8, we know that $R^{\mathrm{def},\ell}_{\bar\rho_y} \simeq \mathbb{F}[[X,Y,Z,U,V]]/(U(1+X+h_\ell(1+Y)), V(1+Y+h_\ell(1+X)))$. Note that, $h_\ell \equiv \ell \equiv -1 \pmod{(UV)}$. So, we have $R^{\mathrm{def},\ell}_{\bar\rho_y}/(V, X-Y) \simeq \mathbb{F}[[X', Z, U]]$.

On the other hand, by composing $f_{2,y} : R^{\mathrm{pd},\ell}_{\bar\rho_0} \to R^{\mathrm{def},\ell}_{\bar\rho_y}$ with the surjective map $R^{\mathrm{def},\ell}_{\bar\rho_y} \to R^{\mathrm{pd}}_{\bar\rho_0}$, we get the surjective map $f_0 : R^{\mathrm{pd},\ell}_{\bar\rho_0} \to R^{\mathrm{pd}}_{\bar\rho_0}$. Note that, $\widetilde{\rho}_y^{\mathrm{univ},\ell} = \rho_y^{\mathrm{univ},\ell} \pmod{(V)} : G_{\mathbb{Q},N\ell p} \to \mathrm{GL}_2(R^{\mathrm{def},\ell}_{\bar\rho_y}/(V))$ is a representation such that $\widetilde{\rho}_y^{\mathrm{univ},\ell}(i_\ell) = \begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix}$. Observe that, $\mathrm{tr}(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix}) - \mathrm{tr}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = c.U$. As $\widetilde{\rho}_y^{\mathrm{univ},\ell}$ is a deformation of $\bar\rho_y$, there exists some $g \in G_{\mathbb{Q},N\ell p}$ such that $\mathrm{tr}(\widetilde{\rho}_y^{\mathrm{univ},\ell}(g.i_\ell)) - \mathrm{tr}(\widetilde{\rho}_y^{\mathrm{univ},\ell}(g)) = s.U$ for some $s \in (R^{\mathrm{def},\ell}_{\bar\rho_y}/(V))^*$. Note that, $(R^{\mathrm{def},\ell}_{\bar\rho_y}/(V))/(U) \simeq R^{\mathrm{pd}}_{\bar\rho_0}$.

Combining all this, we see that composing $f_{2,y}$ with the surjective map $R^{\mathrm{def},\ell}_{\bar\rho_y} \to R^{\mathrm{def},\ell}_{\bar\rho_y}/(V)$, we get a map $j_1 : R^{\mathrm{pd},\ell}_{\bar\rho_0} \to R^{\mathrm{def},\ell}_{\bar\rho_y}/(V)$ which is surjective on the corresponding co-tangent spaces. Therefore, the map j_1 is surjective. Hence, the map $f'_{2,y}$ is surjective.

Now, as $h_\ell \equiv \ell \equiv -1 \pmod{(UV)}$, we have $R^{\mathrm{def},\ell}_{\bar\rho_x}/(U, X-Y) \simeq \mathbb{F}[[X', Z, V]]$. By the logic used in the case of the map $f'_{2,y}$, we see that $f'_{1,x}$ is surjective. As the images of both $f'_{1,x}$ and $f'_{2,y}$ are power series rings, both $\ker(f'_{1,x})$ and $\ker(f'_{2,y})$ are prime ideals of $R^{\mathrm{pd},\ell}_{\bar\rho_0}$. □

Under the hypotheses of Lemma 4.13, let $P_1 = \ker(f'_{1,x})$ and $P_2 = \ker(f'_{2,y})$. From what we have already seen, we know that P_1 and P_2 do not depend on the choice of x and y, respectively.

Lemma 4.14. *Suppose $\bar\rho_0$ is unobstructed. Let ℓ be a prime such that $\ell \equiv -1 \pmod{p}$, $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ and $-\ell$ is a topological generator of $1 + p\mathbb{Z}_p$. Then, P_0, P_1 and P_2 are distinct prime ideals of $R^{pd,\ell}_{\bar\rho_0}$.*

Proof. Now clearly, $P_0 \neq P_1$ and $P_0 \neq P_2$ as $t^{\mathrm{univ},\ell} \pmod{P_0}$ factors through $G_{\mathbb{Q},Np}$ but $t^{\mathrm{univ},\ell} \pmod{P_1}$ and $t^{\mathrm{univ},\ell} \pmod{P_2}$ do not. Suppose $P_1 = P_2$. So we have, $R^{\mathrm{pd},\ell}_{\bar\rho_0}/P_1 = R^{\mathrm{pd},\ell}_{\bar\rho_0}/P_2$. Let us call this ring R, its maximal ideal by m_0 and its fraction field by K. Note that $R \simeq \mathbb{F}[[Z_1, Z_2, Z_3]]$.

Let $y \in H^1(G_{\mathbb{Q},Np}, \chi^{-1})$ be a non-zero element and ρ'_{P_2} be the representation obtained by composing $\rho_y^{\mathrm{univ},\ell} \pmod{(V, X-Y)}$ with the isomorphism $R^{\mathrm{def},\ell}_{\bar\rho_y}/(V, X-Y) \simeq R$ considered in the proof of Lemma 4.13. From the proof of Lemma 4.8, we know that there exists a lift g_ℓ of Frob_ℓ in $G_{\mathbb{Q},N\ell p}$ such that $\rho'_{P_2}(g_\ell)$ is diagonal with diagonal entries distinct modulo m_0. Hence, from [2, Lemma 2.4.5], it follows that $R[\rho'_{P_2}(G_{\mathbb{Q},N\ell p})]$ is a sub-R-GMA of $M_2(R)$. As ρ'_{P_2} is a deformation of $\bar\rho_y$, it follows that $R[\rho'_{P_2}(G_{\mathbb{Q},N\ell p})]$ is the GMA $\begin{pmatrix} R & I \\ R & R \end{pmatrix}$ for some ideal I of R. Note that, $\mathrm{tr}(\rho'_{P_2}) = t^{\mathrm{univ},\ell} \pmod{P_2}$.

Let $x \in H^1(G_{\mathbb{Q},Np}, \chi)$ be a non-zero element and ρ'_{P_1} be the representation obtained by composing $\rho_x^{\mathrm{univ},\ell} \pmod{(U, X-Y)}$ with the isomorphism $R^{\mathrm{def},\ell}_{\bar\rho_x}/(U, X-Y) \simeq R$ considered in the proof of Lemma 4.13. Note that, ρ'_{P_1} is a deformation of $\bar\rho_x$. From the logic given in the previous paragraph, we get that $\mathrm{tr}(\rho'_{P_1}) = t^{\mathrm{univ},\ell} \pmod{P_1}$ and $R[\rho'_{P_1}(G_{\mathbb{Q},N\ell p})]$ is the GMA $\begin{pmatrix} R & R \\ I' & R \end{pmatrix}$ for some ideal I' of R. As $P_1 = P_2$, $\mathrm{tr}(\rho'_{P_1}) = \mathrm{tr}(\rho'_{P_2})$.

From the proof of Lemma 4.8, we know that the upper triangular entry of $\rho'_{P_2}(i_\ell)$ and the lower triangular entry of $\rho'_{P_1}(i_\ell)$ are both not zero. Hence, $I \neq 0$ and $I' \neq 0$. Thus, we have $K[\rho'_{P_1}(G_{\mathbb{Q},N\ell p})] = K[\rho'_{P_2}(G_{\mathbb{Q},N\ell p})] = M_2(K)$. Thus, ρ'_{P_1} and ρ'_{P_2} are absolutely irreducible representations over K with the same trace. So, by Brauer-Nesbitt Theorem, there exists $Q \in GL_2(\bar{K})$ such that $Q\rho'_{P_1}(g)Q^{-1} = \rho'_{P_2}(g)$ for all $g \in G_{\mathbb{Q},N\ell p}$. Note that, for $i = 1, 2$, $\rho'_{P_i}(g_\ell) = \begin{pmatrix} a_{P_i} & 0 \\ 0 & d_{P_i} \end{pmatrix}$ with $a_{P_i} \equiv \chi_1(\mathrm{Frob}_\ell) \pmod{m_0}$ and $d_{P_i} \equiv \chi_2(\mathrm{Frob}_\ell) \pmod{m_0}$. Recall that, $\chi_1|_{G_{\mathbb{Q}_\ell}}$ and $\chi_2|_{G_{\mathbb{Q}_\ell}}$ are distinct unramified characters of $G_{\mathbb{Q}_\ell}$. Therefore, we must have Q to be diagonal matrix, $a_{P_1} = a_{P_2}$ and $d_{P_1} = d_{P_2}$.

Note that, conjugation by Q induces an isomorphism of R-algebras between $R[\rho'_{P_1}(G_{\mathbb{Q},N\ell p})]$ and $R[\rho'_{P_2}(G_{\mathbb{Q},N\ell p})]$. As Q is diagonal, it follows that, under this isomorphism, the R submodule $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$ of $R[\rho'_{P_2}(G_{\mathbb{Q},N\ell p})]$ onto the R submodule $\begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$ of $R[\rho'_{P_1}(G_{\mathbb{Q},N\ell p})]$. Thus, it follows that $I \simeq R$ as R-module and in particular, I is generated by one element.

From the proof of Lemma 4.8 and Lemma 4.13, we know that $a_{P_2} = \chi_1(\mathrm{Frob}_\ell)(1+a)$ and $d_{P_2} = \chi_2(\mathrm{Frob}_\ell)(1+a)$ for some $a \in m_0$. Let J be the ideal of R generated by a and I. Thus, J is an ideal generated by two elements. Now, by [3, Proposition 1.5.1], I is the total reducibility ideal of $t^{\mathrm{univ},\ell} \pmod{P_2}$. So, we get, by Lemma 4.2, that $J = m_0$. But the minimal number of generators of m_0 is 3, while J is generated by two elements. Hence, we get a contradiction. So, $P_1 \neq P_2$. □

We are now ready to prove Theorem 4.10.

Proof of Theorem 4.10. Our startegy to prove the theorem is the following: We first find a set of generators of the co-tangent space of $(R^{\mathrm{pd},\ell}_{\bar{\rho}_0})^{\mathrm{red}}$ and then find the relations between them using GMA. Using the series of lemmas that we proved above, we will show that the relations we find generate all the relations in $(R^{\mathrm{pd},\ell}_{\bar{\rho}_0})^{\mathrm{red}}$.

Fix a lift g_ℓ of Frob_ℓ in $G_{\mathbb{Q}_\ell}$. Let $A^{\mathrm{red}} = \begin{pmatrix} (R^{\mathrm{pd},\ell}_{\bar{\rho}_0})^{\mathrm{red}} & B^{\mathrm{red}} \\ C^{\mathrm{red}} & (R^{\mathrm{pd},\ell}_{\bar{\rho}_0})^{\mathrm{red}} \end{pmatrix}$ be the GMA for the tuple $((R^{\mathrm{pd},\ell}_{\bar{\rho}_0})^{\mathrm{red}}, \ell, (t^{\mathrm{univ},\ell})^{\mathrm{red}}, g_\ell)$ and ρ^{red} be the corresponding representation. Let K_0 be the field of total fractions of $(R^{\mathrm{pd},\ell}_{\bar{\rho}_0})^{\mathrm{red}}$. By part (ii) of [3, Theorem 1.4.4], we can take B^{red} and C^{red} to be the fractional ideals of K_0 such that the map $m'(B^{\mathrm{red}} \otimes_{(R^{\mathrm{pd},\ell}_{\bar{\rho}_0})^{\mathrm{red}}} C^{\mathrm{red}})$ coincides with the multiplication in K_0. Now, $A^{\mathrm{red},\ell} = (R^{\mathrm{pd},\ell}_{\bar{\rho}_0})^{\mathrm{red}}[\rho^{\mathrm{red}}(G_{\mathbb{Q}_\ell})]$ is a $(R^{\mathrm{pd},\ell}_{\bar{\rho}_0})^{\mathrm{red}}$-sub-GMA of A^{red}. Let $A^{\mathrm{red},\ell} = \begin{pmatrix} (R^{\mathrm{pd},\ell}_{\bar{\rho}_0})^{\mathrm{red}} & B^{\mathrm{red},\ell} \\ C^{\mathrm{red},\ell} & (R^{\mathrm{pd},\ell}_{\bar{\rho}_0})^{\mathrm{red}} \end{pmatrix}$.

Note that, $A^{\mathrm{red},\ell}$ is a Cayley-Hamilton quotient of $((R^{\mathrm{pd},\ell}_{\bar{\rho}_0})^{\mathrm{red}}[G_{\mathbb{Q}_\ell}], \widetilde{(t^{\mathrm{univ},\ell})^{\mathrm{red}}})|_{(R^{\mathrm{pd},\ell}_{\bar{\rho}_0})^{\mathrm{red}}[G_{\mathbb{Q}_\ell}]})$. From Lemma 4.12, we get that $(t^{\mathrm{univ},\ell})^{\mathrm{red}}|_{G_{\mathbb{Q}_\ell}}$ is reducible modulo every prime ideal of $(R^{\mathrm{pd},\ell}_{\bar{\rho}_0})^{\mathrm{red}}$. Hence, by [3, Proposition 1.5.1], we see that $B^{\mathrm{red},\ell}.C^{\mathrm{red},\ell}$ is contained in every prime ideal of $(R^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}}$. As $(R^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}}$ is a reduced ring, we have $B^{\mathrm{red},\ell}.C^{\mathrm{red},\ell} = 0$. Therefore, for $g \in G_{\mathbb{Q}_\ell}$, if $\rho^{\mathrm{red}}(g) = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$, then the maps $G_{\mathbb{Q}_\ell} \to ((R^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}})^*$ sending g to a_g and d_g give two characters deforming χ_1 and χ_2, respectively.

From Lemma 2.10, we know that $\rho^{\mathrm{red}}(g_\ell) = \begin{pmatrix} a^{\mathrm{red}} & 0 \\ 0 & d^{\mathrm{red}} \end{pmatrix}$. From Lemma 2.11, it follows that $\rho^{\mathrm{red}}|_{I_\ell}$ factors through the $\mathbb{Z}_p$-quotient of the tame inertia group at ℓ. As $p \nmid \ell - 1$ and the diagonal entries of $\rho^{\mathrm{red}}(G_{\mathbb{Q}_\ell})$ give characters deforming χ_1 and χ_2, we see that $\rho^{\mathrm{red}}(i_\ell) = \begin{pmatrix} 1 & b \\ c & 1 \end{pmatrix}$ and $bc = 0$. So, $\rho^{\mathrm{red}}(G_{\mathbb{Q}_\ell})$ is generated by $\rho^{\mathrm{red}}(g_\ell)$ and $\rho^{\mathrm{red}}(i_\ell)$.

By Lemma 4.11, we see that C^{red} and B^{red} are generated by at most two elements and there exists $b', c' \in (R^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}}$ such that $\{b, b'\}$ is a set of generators of B^{red}, while $\{c, c'\}$ is a set of generators of C^{red}. Let $z = b'c'$, $x_1 = bc'$ and $x_2 = b'c$. Now, $a^{\mathrm{red}} = \chi_1(\mathrm{Frob}_\ell)(1+a_0)$ and $d^{\mathrm{red}} = \chi_2(\mathrm{Frob}_\ell)(1 + d_0)$ for some $a_0, d_0 \in \mathfrak{m}'^{\ell}$.

Let J be the ideal generated by the set $\{a_0, d_0, z, x_1, x_2\}$. By [3, Proposition 1.5.1], we know that the ideal generated by $\{z, x_1, x_2\}$ is the total reducibility ideal of $(t^{\mathrm{univ},\ell})^{\mathrm{red}}$. Thus, by Lemma 4.2, we see that $J = \mathfrak{m}'^{\ell}$. Thus, we get a surjective local morphism of $\mathbb{F}$-algebras $g_0 : \mathbb{F}[[X, Y, Z, X_1, X_2]] \to (R^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}}$ such that $g_0(X) = a_0 + d_0$, $g_0(Y) = a_0 - d_0$, $g_0(Z) = z$, $g_0(X_1) = x_1$ and $g_0(X_2) = x_2$.

Let $I_0 = \ker(g_0)$. We will now analyze I_0. As $bc = 0$, we get $x_1.x_2 = bc'.b'c = 0$. So, $X_1X_2 \in I_0$. Note that, from the action of Frob_ℓ on the tame inertia group, we get $\rho^{\mathrm{red}}(g_\ell i_\ell g_\ell^{-1}) = \rho^{\mathrm{red}}(i_\ell)^\ell$. Now, $\rho^{\mathrm{red}}(g_\ell i_\ell g_\ell^{-1}) = \begin{pmatrix} 1 & (a^{\mathrm{red}}/d^{\mathrm{red}})b \\ (d^{\mathrm{red}}/a^{\mathrm{red}})c & 1 \end{pmatrix}$. As $bc = 0$, we have $\rho^{\mathrm{red}}(i_\ell^\ell) = \begin{pmatrix} 1 & \ell.b \\ \ell.c & 1 \end{pmatrix}$. Thus, we have $(a^{\mathrm{red}}/d^{\mathrm{red}} - \ell)b = 0$ i.e. $(a^{\mathrm{red}} - \ell.d^{\mathrm{red}})b = 0$ and $(d^{\mathrm{red}}/a^{\mathrm{red}} - \ell)c = 0$ i.e. $(d^{\mathrm{red}} - \ell.a^{\mathrm{red}})c = 0$. As $\chi_1(\mathrm{Frob}_\ell)/\chi_2(\mathrm{Frob}_\ell) = \omega_p(\mathrm{Frob}_\ell) = \ell$, we get $(a_0 - d_0)b = 0$ and $(d_0 - a_0)c = 0$. Thus, $(a_0 - d_0)x_1 = (a_0 - d_0)x_2 = 0$ and hence, $YX_1, YX_2 \in I_0$.

Therefore, the surjective map $g_0 : \mathbb{F}[[X, Y, Z, X_1, X_2]] \to (R^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}}$ factors through $\mathbb{F}[[X, Y, Z, X_1, X_2]]/(X_1X_2, YX_1, YX_2)$. For $i = 0, 1, 2$, let P'_i be the kernel of the map $g_i : \mathbb{F}[[X, Y, Z, X_1, X_2]] \to R^{\mathrm{pd},\ell}_{\bar\rho_0}/P_i$ obtained by composing g_0 with the surjective map $(R^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}} \to R^{\mathrm{pd},\ell}_{\bar\rho_0}/P_i$. Here, the primes $P - i$ are the ones appearing in Lemma 4.14. As each g_i is surjective, each P'_i is a prime of $\mathbb{F}[[X, Y, Z, X_1, X_2]]$ containing I_0 and in particular, $(X_1X_2, YX_1, YX_2) \subset P'_i$ for $i = 0, 1, 2$. So each P'_i contains one of the (Y, X_1), (Y, X_2) or (X_1, X_2). Now, $R^{\mathrm{pd},\ell}_{\bar\rho_0}/P_i$ and hence, $\mathbb{F}[[X, Y, Z, X_1, X_2]]/P'_i$ is isomorphic to $\mathbb{F}[[Z_1, Z_2, Z_3]]$ for $i = 0, 1, 2$. Therefore, every P'_i is either (Y, X_1), (Y, X_2) or (X_1, X_2). Since P_0, P_1 and P_2 are distinct prime ideals of $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ (by Lemma 4.14), P'_0, P'_1 and P'_2 are distinct prime ideals of $\mathbb{F}[[X, Y, Z, X_1, X_2]]$. Hence, we have $\{P'_0, P'_1, P'_2\} = \{(Y, X_1), (Y, X_2), (X_1, X_2)\}$. So, $I_0 \subset P'_0 \cap P'_1 \cap P'_2 = (Y, X_1) \cap (Y, X_2) \cap (X_1, X_2)$.

Note that, $(Y, X_2) \cap (Y, X_1) = (Y, X_1X_2)$. If $Yf \in (X_1, X_2)$, then $f \in (X_1, X_2)$ and hence, $Yf \in (YX_1, YX_2)$. Therefore, $(Y, X_1X_2) \cap (X_1, X_2) = (YX_1, YX_2, X_1X_2)$. Hence, $I_0 \subset (YX_1, YX_2, X_1X_2)$. This implies that $I_0 = (YX_1, YX_2, X_1X_2)$ and hence, $(R^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}} \simeq \mathbb{F}[[X, Y, Z, X_1, X_2]]/(YX_1, YX_2, X_1X_2)$. □

Remark 4.15. *The proof of Theorem 4.10, description of the GMA A^{red}, and* [3, Proposition 1.7.4] *together imply that there does not exists a representation $\rho : G_{\mathbb{Q},N\ell p} \to \mathrm{GL}_2((R^{pd,\ell}_{\bar\rho_0})^{red})$ such that $\mathrm{tr}(\rho) = (t^{univ,\ell})^{red}$.*

Remark 4.16. *As mentioned in the introduction, let $S^{pd,\ell}_{\bar\rho_0}$ be the universal deformation ring parameterizing all pseudo-characters t of $G_{\mathbb{Q},N\ell p}$ deforming $\mathrm{tr}(\bar\rho_0)$ such that $t|_{G_{\mathbb{Q}_\ell}}$ is reducible. The proof of Theorem 4.10 can be used to prove that $(S^{pd,\ell}_{\bar\rho_0})^{red} \simeq W(\mathbb{F})[[X, Y, Z, T_1, T_2]]/(T_1T_2, T_1Z, T_2Z')$ for some $Z' \equiv Z \pmod p$.*

It is natural to ask if the same approach can give us the structure of $(\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0})^{\mathrm{red}}$ as well. But the method does not work. More specifically, Lemma 4.12 is not true for $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$. Indeed, let $x \in H^1(G_{\mathbb{Q},Np}, \chi^i)$ be a non-zero element with $i \in \{1, -1\}$ and $\mathcal{O}$ be the ring of integers in the finite extension of $\mathbb{Q}_p$ obtained by attaching all the p-th roots of unity to $\mathbb{Q}_p$. Let ζ_p be a primitive p-th root of unity. It can be checked that there exists a $W(\mathbb{F})$-algebra morphism $\mathcal{R}^{\mathrm{def},\ell}_{\bar\rho_x} = W(\mathbb{F})[[X, Y, Z, U, V]]/(U((1+X)+h_\ell(1+Y)), V((1+Y)+h_\ell(1+X))) \to \mathcal{O}[[Z]]$ sending both U and V to $\frac{\zeta_p - \zeta_p^{-1}}{2}$, X and Y to 0 and Z to Z. Composing this map with the map $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0} \to \mathcal{R}^{\mathrm{def},\ell}_{\bar\rho_x}$, we get a map $f : \mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0} \to \mathcal{O}[[Z]]$. Observe that $f \circ T^{\mathrm{univ},\ell}|_{G_{\mathbb{Q}_\ell}}$ is not reducible and $\ker(f)$ is a prime ideal. See [10, Section 3] for a similar analysis. Thus, the ring $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$ has more than 3 minimal primes and probably has a more complicated structure.

Corollary 4.17. *Suppose $\bar\rho_0$ is unobstructed. Let ℓ be a prime such that $\ell \equiv -1 \pmod p$, $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ and $-\ell$ is a topological generator of $1 + p\mathbb{Z}_p$. Then $R^{pd,\ell}_{\bar\rho_0}$ is not reduced ring.*

Proof. This follows directly from Theorem 4.10 and the fact that $\dim(\tan(R^{\mathrm{pd},\ell}_{\bar\rho_0})) = 6$. □

Though we do not determine the explicit structure of $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ in this case, we can still prove the following theorem:

Theorem 4.18. *Suppose $\bar\rho_0$ is unobstructed. Let ℓ be a prime such that $\ell \equiv -1 \pmod p$, $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ and $-\ell$ is a topological generator of $1 + p\mathbb{Z}_p$. Then $R^{pd,\ell}_{\bar\rho_0}$ is not a local complete intersection ring.*

Proof. We use a strategy similar to the one used in the proof of Theorem 4.10. Namely, we first find a set of generators of the co-tangent space of $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ and then find the relations between the using GMA. After assuming that $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ is a local complete intersection ring, we will find a subset of these relations which will generate all the relations in $R^{\mathrm{pd},\ell}_{\bar\rho_0}$. But the description of this subset will give a contradiction to Theorem 4.10 which will complete the proof.

Fix a lift g_ℓ of Frob_ℓ in $G_{\mathbb{Q}_\ell}$. Let $A^{\mathrm{pd}} = \begin{pmatrix} R^{\mathrm{pd},\ell}_{\bar\rho_0} & B^{\mathrm{pd}} \\ C^{\mathrm{pd}} & \mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0} \end{pmatrix}$ be the GMA associated to the tuple $(R^{\mathrm{pd},\ell}_{\bar\rho_0}, \ell, t^{\mathrm{univ},\ell}, g_\ell)$ in Lemma 2.10 and $\rho : G_{\mathbb{Q},N\ell p} \to (A^{\mathrm{pd}})^*$ be the corresponding representation. By Lemma 2.11, $\rho|_{I_\ell}$ factors through the $\mathbb{Z}_p$ quotient of the tame inertia group at ℓ. Suppose $\rho^{\mathrm{pd},\ell}(i_\ell) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By Lemma 2.10, we know that $\rho^{\mathrm{pd},\ell}(g_\ell) = \begin{pmatrix} a_0 & 0 \\ 0 & d_0 \end{pmatrix}$.

By [3, Proposition 1.5.1], $I^{\ell}_{\bar\rho_0}$ is $m(B^{\mathrm{pd}} \otimes_{R^{\mathrm{pd},\ell}_{\bar\rho_0}} C^{\mathrm{pd}})$. From Lemma 4.11, it follows that there exists $b', c' \in R^{\mathrm{pd},\ell}_{\bar\rho_0}$ such that $\{b, b'\}$ is a set of generators of B^{pd}, while $\{c, c'\}$ is a set of generators of C^{pd}. Thus, the ideal $I^{\ell}_{\bar\rho_0}$ is generated by the set $\{m'(b \otimes c), m'(b' \otimes c), m'(b \otimes c'), m'(b' \otimes c')\}$. Let $z = m'(b' \otimes c')$, $x_1 = m'(b \otimes c')$, $x_2 = m'(b' \otimes c)$ and $x_3 = m'(b \otimes c)$.

Now, $a_0 = \chi_1(\mathrm{Frob}_\ell)(1 + a'_0)$ and $d_0 = \chi_2(\mathrm{Frob}_\ell)(1 + d'_0)$ for some $a'_0, d'_0 \in \mathfrak{m}^\ell$. Let J be the ideal generated by the set $\{a'_0, d'_0, z, x_1, x_2, x_3\}$. Thus, from Lemma 4.2, we see that $J = \mathfrak{m}^\ell$. Thus, we get a surjective local morphism of $\mathbb{F}$-algebras $g_0 : \mathbb{F}[[X, Y, Z, X_1, X_2, X_3]] \to R^{\mathrm{pd},\ell}_{\bar\rho_0}$ such that $g_0(X) = a'_0 + d'_0$, $g_0(Y) = a'_0 - d'_0$, $g_0(Z) = z$, $g_0(X_1) = x_1$, $g_0(X_2) = x_2$ and $g_0(X_3) = x_3$. Let $J_0 = \ker(g_0)$. Denote the maximal ideal (X, Y, Z, X_1, X_2, X_3) by m_0 and $\mathbb{F}[[X, Y, Z, X_1, X_2, X_3]]$ by R_0. We know that $\dim(\tan(R^{\mathrm{pd},\ell}_{\bar\rho_0})) = 6$. Hence, $J_0 \subset m_0^2$. Suppose $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ is a local complete intersection ring. The Krull dimension of $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ is 3 by Theorem 4.10. This means that J_0 is generated by 3 elements.

Note that, $R^{\mathrm{pd},\ell}_{\bar\rho_0}[\rho^{\mathrm{pd},\ell}(G_{\mathbb{Q}_\ell})]$ is a Cayley-Hamilton quotient of $(R^{\mathrm{pd},\ell}_{\bar\rho_0}[G_{\mathbb{Q}_\ell}], \widetilde{t}^{\mathrm{univ},\ell}|_{G_{\mathbb{Q}_\ell}})$. As $\rho(G_{\mathbb{Q}_\ell})$ is generated by $\rho(z)$ and $\rho(i_\ell)$, it follows, from [3, Proposition 1.5.1], that the ideal (x_3) is the reducibility ideal of $\widetilde{t}^{\mathrm{univ},\ell}|_{R^{\mathrm{pd},\ell}_{\bar\rho_0}[G_{\mathbb{Q}_\ell}]}$. Thus, if $g \in G_{\mathbb{Q}_\ell}$ and $\rho(g) = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$, then we get two characters c_1, $c_2 : G_{\mathbb{Q}_\ell} \to (R^{\mathrm{pd},\ell}_{\bar\rho_0}/(x_3))^*$ sending g to $a_g \pmod{(x_3)}$ and $d_g \pmod{(x_3)}$, respectively. Moreover, c_1 and c_2 are deformations of $\chi_1|_{G_{\mathbb{Q}_\ell}}$ and $\chi_2|_{G_{\mathbb{Q}_\ell}}$, respectively. As $p \nmid \ell - 1$, this means that $c_1(I_\ell) = c_2(I_\ell) = 1$. So, we have $a = 1 + x_3 a'$ and $d = 1 + x_3 d'$.

From the action of the Frobenius on the tame inertia, we get that $\rho(z i_\ell z^{-1}) = \rho(i_\ell)^\ell$. As $x_3 = m'(b \otimes c)$, we see, by induction, that for a positive integer n, $\rho(i_\ell)^n = \begin{pmatrix} 1 + x_3 a'_n & b(n + x_3 b'_n) \\ c(n + x_3 c'_n) & 1 + x_3 d'_n \end{pmatrix}$ for some $a'_n, b'_n, c'_n, d'_n \in R^{\mathrm{pd},\ell}_{\bar\rho_0}$. Therefore, we get that $\begin{pmatrix} a & (a_0/d_0) b \\ (d_0/a_0) c & d \end{pmatrix} = \begin{pmatrix} 1 + x_3 a'_\ell & b(\ell + x_3 b'_\ell) \\ c(\ell + x_3 c'_\ell) & 1 + x_3 d'_\ell \end{pmatrix}$.

Thus, $(a_0/d_0)b = b(\ell + x_3b'_\ell)$ implies that $m'((a_0/d_0 - \ell - x_3b'_\ell)b \otimes C^{\mathrm{pd}}) = 0$ and $(d_0/a_0)c = c(\ell + x_3c'_\ell)$ implies that $m'((d_0/a_0 - \ell - x_3c'_\ell)c \otimes B^{\mathrm{pd}}) = 0$. Therefore, we have $x_3(a_0/d_0 - \ell - x_3b'_\ell) = 0$, $x_1(a_0/d_0 - \ell - x_3b'_\ell) = 0$, $x_3(d_0/a_0 - \ell - x_3c'_\ell) = 0$ and $x_2(d_0/a_0 - \ell - x_3b'_\ell) = 0$. As $p|\ell + 1$ and $\chi_1(\mathrm{Frob}_\ell) = \ell\chi_2(\mathrm{Frob}_\ell)$, we get the following relations from the relations above: there exists $b'', c'' \in R^{\mathrm{pd},\ell}_{\bar\rho_0}$ such that $x_3(a'_0 - d'_0 + x_3b'') = 0$, $x_1(a'_0 - d'_0 + x_3b'') = 0$, $x_3(d'_0 - a'_0 + x_3c'') = 0$ and $x_2(d'_0 - a'_0 + x_3c'') = 0$.

Thus, J_0 contains the elements $X_3Y + X_3^2q_1$, $X_1Y + X_1X_3q_2$ and $-X_2Y + X_2X_3q_3$ for some q_1, q_2, $q_3 \in R_0$. As minimum number of generators of J_0 is 3, it follows, by Nakayama's lemma, that J_0/m_0J_0 is an $\mathbb{F}$ vector space of dimension 3. As $m_0J_0 \in m_0^3$, we see that the images of $X_3Y + X_3^2q_1$, $X_1Y + X_1X_3q_2$ and $-X_2Y + X_2X_3q_3$ inside J_0/m_0J_0 are linearly independent over $\mathbb{F}$. Therefore, they form an $\mathbb{F}$-basis of the vector space J_0/m_0J_0. Hence, by Nakayama's lemma, we get that $J_0 = (X_3Y + X_3^2q_1, X_1Y + X_1X_3q_2, -X_2Y + X_2X_3q_3)$.

In particular, $J_0 \subset (X_3, Y)$. This implies that the Krull dimension of $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ is 4. However, we know that the Krull dimension of $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ is 3. Hence, we get a contradiction to the hypothesis that J_0 is generated by 3 elements. Therefore, $R^{\mathrm{pd},\ell}_{\bar\rho}$ is not a local complete intersection ring. □

Corollary 4.19. *Suppose $\bar\rho_0$ is unobstructed. Let ℓ be a prime such that $\ell \equiv -1 \pmod p$, $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$ and $-\ell$ is a topological generator of $1 + p\mathbb{Z}_p$. Then $\mathcal{R}^{pd,\ell}_{\bar\rho_0}$ is not a local complete intersection ring.*

Proof. Since $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}/(p) \simeq R^{\mathrm{pd},\ell}_{\bar\rho_0}$, we see, from Theorem 4.10, that the Krull dimension of $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$ is either 3 or 4. As $\bar\rho_0$ is unobstructed, we know that $\mathcal{R}^{\mathrm{pd}}_{\bar\rho_0} \simeq W(\mathbb{F})[[X,Y,Z]]$. We have surjective map $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0} \to \mathcal{R}^{\mathrm{pd}}_{\bar\rho_0}$ induced from the surjection $G_{\mathbb{Q},N\ell p} \to G_{\mathbb{Q},Np}$. Hence, the Krull dimension of $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$ is 4. As $\dim(\tan(R^{\mathrm{pd},\ell}_{\bar\rho_0})) = 6$, we know that $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0} \simeq W(\mathbb{F})[[X,Y,Z,X_1,X_2,X_3]]/J$ for some ideal J of $W(\mathbb{F})[[X,Y,Z,X_1,X_2,X_3]]$. If $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$ is a local complete intersection ring, then J is generated by 3 elements. But this would imply that $R^{\mathrm{pd},\ell}_{\bar\rho_0}$ is a local complete intersection ring which is not true by Theorem 4.18. Hence, we see that $\mathcal{R}^{\mathrm{pd},\ell}_{\bar\rho_0}$ is not a local complete intersection ring. □

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