# An *n*-ary generalization of the concept of distance

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# Motivation

During AGOP 2009 in Palma, Gaspar Mayor began his talk by:


What is the distance among Santander, Valladolid, Barcelona, Madrid, Seville, and Palma?

# An answer: multidistance

**Definition (Martín, Mayor).**

A map $d\colon \bigcup_{n\geq 1} X^n \to \mathbb{R}^+$ is a *multidistance* if

- $d$ is symmetric
- $d(x_1, \dots, x_n) = 0$ iff $x_1 = \cdots = x_n$
- $d(x_1, \dots, x_n) \leq \sum_{i=1}^{n} d(x_i, z)$ for every $z \in X$

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This definition

- requires an existing distance
- is made for variadic functions

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$n \geq 2$

$\mathbf{x}_i^z$ is obtained from $\mathbf{x} \in X^n$ by replacing its $i$-th element by $z$.

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For $n = 3$,

$$d(x_1, x_2, x_3) \leq d(z, x_2, x_3) + d(x_1, z, x_3) + d(x_1, x_2, z)$$

$$d(x_1, \ldots, x_n) \leq \sum_{i=1}^{n} d(x_1, \ldots, x_n)_i^z$$

**Definition.**

The *best constant* $K^*$ of $d$ is the *infimum* of the $K > 0$ such that

$$d(x_1, \ldots, x_n) \leq K \sum_{i=1}^{n} d(x_1, \ldots, x_n)_i^z$$

for every $x_1, \ldots, x_n, z \in X$

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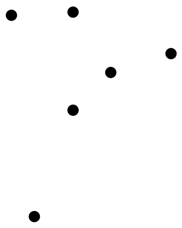
# Basic examples of $n$-distances

## Example (Drastic $n$-distance)

$$d(x_1, \ldots, x_n) \ := \ \begin{cases} 0 & \text{if } x_1 = \cdots = x_n \\ 1 & \text{otherwize} \end{cases}$$



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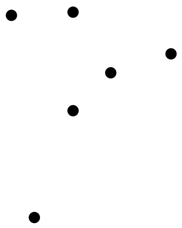
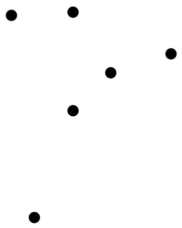
It has best constant $K^* = \frac{1}{n-1}$.



## Example (Cardinality based *n*-distance)

$$d(x_1, \ldots, x_n) \ := \ |\{x_1, \ldots, x_n\}| - 1$$



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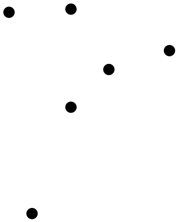
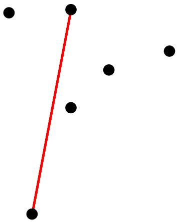


## Example (Diameter in a metric space)

$$d_{\max}(x_1, \ldots, x_n) \;:=\; \max\{d(x_i, x_j) \mid i, j \leq n\}$$



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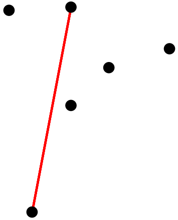
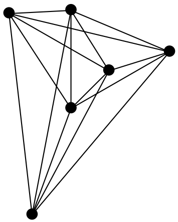


## Example (Sum based $n$-distance in a metric space)

$$d_{\max}(x_1, \dots, x_n) \;:=\; \sum_{\{i,j\}\subseteq X} d(x_i, x_j)$$



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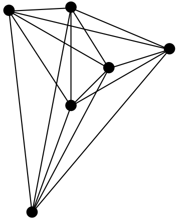


# Fermat points based $n$-distances

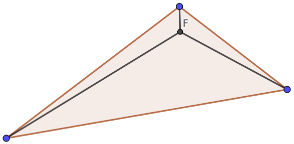
# Fermat point of a triangle

$F$ is a *Fermat point* of $ABC$ if

$$d(A,F)+d(B,F)+d(C,F)$$

is minimum.

Fermat points of triangles are unique.




The function $(A,B,C)\mapsto d(A,F)+d(B,F)+d(C,F)$ is a 3-distance.

## Example (Fermat points based $n$-distance)

$$d_F(x_1, \ldots, x_n) \;:=\; \min_{x \in X} \sum_{i=1}^{n} d(x_i, x)$$

It is defined in any metric space in which closed balls are compact.

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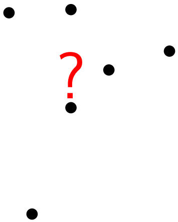
Computation is hard.




**Theorem.** We have $\frac{1}{n-1} \leq K^* \leq \frac{4n-4}{3n^2-4n}$

# Fermat distance in median graphs

**Definition.** A connected graph is *median* if for any three vertices $a, b, c$, there is only one vertex $m(a, b, c)$ that is at the interesection of shortest paths between any two of $\{a, b, c\}$

Cubes are examples of median graphs.

**Proposition.** In a median graph,

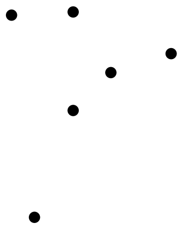
- $a, b, c$ have a unique Fermat point $m := m(a, b, c)$
- $d_F(a, b, c) = d(a, m) + d(b, m) + d(c, m)$
- $K^* = \frac{1}{2} = \frac{1}{n-1}$

# $n$-distances based on geometric constructions

# Smallest enclosing sphere

$x_1, \ldots, x_n \in \mathbb{R}^k$

$S(x_1, \ldots, x_n) :=$ the smallest $(k-1)$-sphere enclosing $x_1, \ldots, x_n$.



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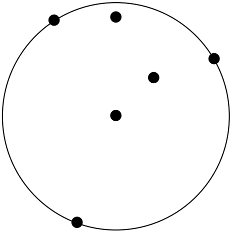
This sphere can be computed in linear time.

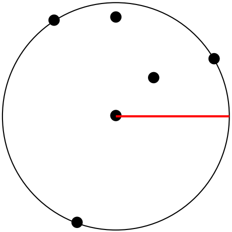


## Example (radius of $S(x_1, \ldots, x_n)$ in $\mathbb{R}^2$)

$n \geq 2$

$x_1, \ldots, x_n \in \mathbb{R}^2$

$$d_r(x_1, \ldots, x_n) \ := \ \text{radius of } S(x_1, \ldots, x_n)$$



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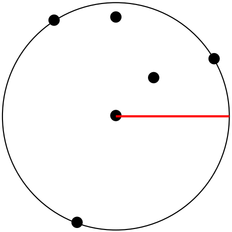
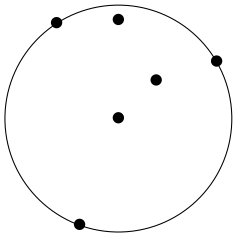


## Example (area of $S(x_1, \ldots, x_n)$ in $\mathbb{R}^2$)

$n \geq 3$

$x_1, \ldots, x_n \in \mathbb{R}^2$

$$d_a(x_1, \ldots, x_n) \; := \; \text{area bounded by } S(x_1, \ldots, x_n)$$



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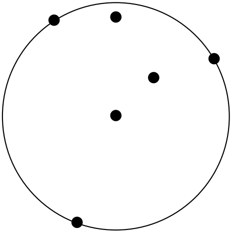
It has best constant $K^* = (n - \frac{3}{2})^{-1}$.



## Example (number of directions in $\mathbb{R}^2$)

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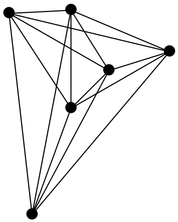
$$d_{\Delta}(x_1, \ldots, x_n) \ := \ \#\text{ directions given by pairs of } x_1, \ldots, x_n$$



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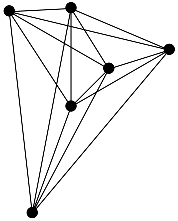
Its best constant $K^*$ satisfies $(n - 2 + \frac{2}{n})^{-1} \leq K^* < (n-2)^{-1}$.



# Homogeneity degree

**Definition.** An $n$-distance on $\mathbb{R}^k$ is *homogeneous of degree $q \geq 0$* if

$$d(tx_1, \ldots, tx_n) = t^q d(x_1, \ldots, x_n)$$

for every $x_1, \ldots, x_n \in \mathbb{R}^n$ and every $t > 0$.

# Summary

| $n$-distance | $K^*$ | q |
|---|---|---|
| Drastic $n$-distance | $1/(n-1)$ | 0 |
| Cardinality based $n$-distance | $1/(n-1)$ | 0 |
| Diameter | $1/(n-1)$ | 1 |
| Sum based $n$-distance | $1/(n-1)$ | 1 |
| Fermat $n$-distance | $1/(n-1) \leq K^* \leq \frac{4n-4}{3n^2-4n}$ | 1 |
| Median Fermat 3-distance | $1/(n-1)$ | 1 |
| Radius of $S(x_1, \ldots, x_n)$ in $R^2$ | $1/(n-1)$ | 1 |
| *Area of $S(x_1, \ldots, x_n)$ in $R^2$ | $(n-3/2)^{-1}$ | 2 |
| *Number of directions in $\mathbb{R}^2$ | $(n-2+\frac{2}{n})^{-1} \leq K^* < \frac{1}{n-2}$ | 0 |

* These $n$-distances are not multidistances.

# Generation Theorems

# Two classical constructions

**Proposition.** Let $d, d'$ be $n$-distances on $X$, and $\lambda > 0$.

(1) $d + d'$ and $\lambda d$ are $n$-distances

(2) $\frac{d}{1+d}$ is an $n$-distance valued in $[0, 1]$.

# Constructing $n$-distances from $n$-hemimetric

**Definition.** A map $d\colon X^n \to \mathbb{R}^+$ is an $(n-1)$-*hemimetric* if

- $d$ is symmetric
- $d(x_1, \ldots, x_n) = 0$ iff there are $i \neq j$ such that $x_i = x_j$
- $d(x_1, \ldots, x_n) \leq \sum_{i=1}^{n} d(x_1, \ldots, x_n)_i^z$ (simplex inequality)

**Proposition.** Let

- $d$ be an $n$-distance on $X$,
- $d'$ be an $(n-1)$-hemimetric on X,

then $d + d'$ is an $n$-distance on $X$.

# Topics of further research

I. Improve the bounds for Fermat $n$-distances.

II. Is the radius of $S(x_1, \dots, x_n)$ an $n$-distance on $\mathbb{R}^k$ for $k > 2$?

III. Is the volume of the region bounded by $S(x_1, \dots, x_n)$ an $n$-distance with $K^* = (n - 2 + 2^{1-k})^{-1}$ for $k > 2$?

IV. Find classification/clustering applications.

V. Characterize the $n$-distances for which $K^* = 1/(n - 1)$.

G. Kiss, J.-L. Marichal, and B. Teheux. A generalization of the concept of distance based on the simplex inequality. *Contributions to Algebra and Geometry*, 59(2):247 - 266, 2018.