Derivations and differential operators on rings and fields

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Basic definition

Let $R = (R, +, \cdot)$ be a (commutative) ring.

- R has characteristic 0, if $n \cdot x \neq 0$ for every $x \in R \setminus \{0\}$ and for every positive integer n.
- R has characteristic $n \in \mathbb{N}$ if n is the smallest positive integer such that $n \cdot x = 0$ for some $x \in R \setminus \{0\}$.

A commutative ring R (with unit element $1 \neq 0$) is an integral domain if $x, y \in R \setminus \{0\}$ implies $xy \neq 0$ (no zero-divisors other than 0).

In this talk we assume that R is an integral domain.

Additive function and derivation

A function $f\colon R \to R$ is called

- an additive function if

$$f(x+y) = f(x) + f(y) \quad \forall x, y \in R.$$

- a derivation if f is additive and satisfies the Leibniz rule, i.e

$$f(xy) = xf(y) + yf(x) \quad \forall x, y \in R.$$

Higher order derivations

Higher order derivation by Unger and Reich [3]:
The identically zero map is the only derivation of order 0.
An additive function $f\colon R \to R$ is a derivation of order at most n ($n \in \mathbb{N}$) if there exists $B\colon R \times R \to R$ such that B is a derivation of order $n-1$ in each of its variables and

$$f(xy) - xf(y) - f(x)y = B(x,y).$$

Let $\mathcal{D}_n (= \mathcal{D}_n(R))$ denote the set of derivations of order at most n defined on R.

Claim

A function $d : R \to R$ is a derivation if and only if $d \in \mathcal{D}_1$.

We say that D is a derivation of order n if $D \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$.

Differential operators

We say that the map $D : R \to R$ is a differential operator of degree at most n if D is the linear combination (with coefficients from R) of maps of the form

$$d_1 \circ \cdots \circ d_k,$$

where $d_1, \ldots, d_k : R \to R$ are derivations and $0 \leq k \leq n$.
Note: If $k = 0$, then we interpret $d_1 \circ \cdots \circ d_k$ as the identity function on R.
Let $\mathcal{O}_n (= \mathcal{O}_n(R))$ denote the set of differential operators of order at most n defined on R.
We say that D is a differential operator of order n if $D \in \mathcal{O}_n \setminus \mathcal{O}_{n-1}$.

Differential operators on fields

Let $K = \mathbb{Q}(t_1, \ldots, t_k)$, where $t_1, \ldots, t_k$ are algebraically independent over $\mathbb{Q}$.
Then K is a field of all rational function of $t_1, \ldots, t_k$ with rational coefficients.

Claim

- *The function* $\frac{\partial}{\partial t_i} : K \to K$ *is a derivation (on* K*) for every* $i = 1, \ldots, k$.
- *Every derivation* $d \colon K \to K$ *can be written as*

$$d = \sum_{i=1}^{k} c_i \frac{\partial}{\partial t_i},$$

for some $c_i \in K$.

Differential operators on fields II.

Let L be a field containing algebraically independent elements $t_1, \ldots, t_n$. A function $f : L \to L$ of the following form

$$f = \sum_{i_1+\ldots+i_k \le n} c_{i_1,\ldots,i_k} \cdot \frac{\partial^{i_1+\cdots+i_k}}{\partial t_1^{i_1} \cdots \partial t_k^{i_k}} \tag{1}$$

where the coefficients $c_{i_1,\ldots,i_k}$ belong to L, is a *differential operator* of degree at most n on the field L.
The converse is also true if $K = \mathbb{Q}(t_1, \ldots, t_k)$:

Proposition

$D : K \to K$ is a differential operator of degree at most n if and only if D is of the form (1).

Connection I.

Now we go back to the case when R is an integral domain.

Claim

Let $d_1, \ldots, d_n \in \mathcal{D}_1(R)$. Then $d_1 \circ \cdots \circ d_n \in \mathcal{D}_n(R)$.

Clearly, this holds for every linear combination of compositions of length at most n. Thus

Claim

If $D \in \mathcal{O}_n$ such that $D(1) = 0$, then $D \in \mathcal{D}_n$.

We denote by $\mathcal{O}_n^0$ the set of differential operators D of degree at most n satisfying $D(1) = 0$.

Questions

In the sequel we investigate two basic questions.

Question

Is there any converse of the previous claim?

Let $d_1, \ldots, d_n \in D_1(R)$.

Question

How can we guarantee that $d_1 \circ \cdots \circ d_n$ is a derivation of order exactly n?

Discrete topology

Let X and Y be nonempty sets. Then Y^X denotes the set of all maps $f\colon X \to Y$.

We endow the space Y with the discrete topology, and Y^X with the product topology. The closure of a set $\mathcal{A} \subseteq Y^X$ w. r. t. the product topology is denoted by $\mathrm{cl}\mathcal{A}$.

A function $f\colon X \to Y$ belongs to $\mathrm{cl}\mathcal{A}$ if and only if, for every finite set $F \subseteq X$ there is a function $g \in \mathcal{A}$ such that $f(x) = g(x)$ for every $x \in F$.

Results I.

Theorem

Let R be an integral domain of characteristic zero and let n be a positive integer. Then for every $D\colon R \to R$, the following are equivalent.

1. $D \in \mathcal{D}_n(R)$,
2. $D \in \mathrm{cl}\mathcal{O}_n^0(R)$.

Corollary

Let R be an integral domain of characteristic zero and let n be a positive integer. Then for every $D\colon R \to R$, the following are equivalent.

1. $D \in \mathcal{D}_n(R) \setminus \mathcal{D}_{n-1}(R)$,
2. $D \in \mathrm{cl}\mathcal{O}_n^0(R) \setminus \mathrm{cl}\mathcal{O}_{n-1}^0(R)$.

Easy part $(2 \Rightarrow 1)$

Lemma

For every ring R and for every nonnegative integer n, the set $\mathcal{D}_n$ is closed in R^R.

Lemma

For every ring R we have $\mathrm{cl}\mathcal{O}_n^0 \subseteq \mathcal{D}_n$.

Generalized polynomial

Let G be an Abelian semigroup, and let H be an Abelian group.

- Difference operator Δ_g $(g \in G)$:

$$\Delta_g f(x) = f(x+g) - f(x)$$

  for every $f\colon G \to H$ and $x \in G$.

- A function $f : G \to H$ is a generalized polynomial, if there is a k such that

$$\Delta_{g_1} \dots \Delta_{g_{k+1}} f = 0 \tag{2}$$

  for every $g_1, \dots, g_{k+1} \in G$.

- The degree of the generalized polynomial f: the smallest k such that (2) holds for for every $g_1, \dots, g_{k+1} \in G$. We denote it by $\deg f$.

Extended theorem

Let $j : R \to R$ denote the identity function on R.

Theorem

Let R be an integral domain of characteristic zero and K its field of fractions and let n be a positive integer. Then for every $D\colon R \to R$, the following are equivalent

1. $D \in \mathcal{D}_n(R)$,
2. $D \in \mathrm{cl}\mathcal{O}_n^0(R)$.
3. *D is additive on R, $D(1) = 0$, and D/j, as a map from the semigroup R^* to K, is a generalized polynomial of degree at most n.*

$1 \Rightarrow 3$ and $3 \Rightarrow 2$

Lemma

Let R be a subring of $\mathbb{C}$, let $K \subseteq \mathbb{C}$ be its field of fractions, and suppose that the transcendence degree of K over $\mathbb{Q}$ is finite. Let the map $D : R \to R$ be additive. If D/j, as a map from the semigroup R^* to $\mathbb{C}$ is a generalized polynomial of degree at most n, then $D \in \mathcal{O}_n$.

Lemma

Let R be an integral domain and K be its field of fractions. If $D \in \mathcal{D}_n$, then $p = D/j$, as a map from the semigroup R^* to K is a generalized polynomial of degree at most n.

Easy inductive argument: Using $p(xy) - p(x) - p(y) = B(x,y)/xy$ for every $x, y \in K^*$, we have

$$\Delta_y p(x) = p(y) + \frac{1}{y} \cdot \frac{B(x,y)}{x}. \tag{3}$$

$3 \Rightarrow 1$

Lemma

Let R be an integral domain, and let K be its field of fractions. If $d_1, \ldots, d_n$ are nonzero derivations on R and $D = d_1 \circ \ldots \circ d_n$, then D/j, as a map from the semigroup R^* to K, is a generalized polynomial of degree at most n.
If R is of characteristic zero, then $\deg D/j = n$.

Corollary

Let R be an integral domain, and let K be its field of fractions. If $D \in \mathrm{cl}\mathcal{O}_n^0(R)$, then D/j, as a map from the semigroup R^* to K, is a generalized polynomial of degree at most n.

On fields of finite transcendence degree

Theorem

Let K be field of fractions with finite transcendence degree and let n be a positive integer. Then for every $D\colon R \to R$, the following are equivalent:

1. $D \in \mathcal{D}_n(R) \setminus \mathcal{D}_{n-1}(R)$,
2. $D \in \mathrm{cl}\mathcal{O}_n^0(R) \setminus \mathrm{cl}\mathcal{O}_{n-1}^0(R)$.
3. *D is additive on K, $D(1) = 0$, and $D/j : K^* \to K$ is a generalized polynomial of degree n.*
4. *D is additive on K, $D(1) = 0$, and $D/j : K^* \to K$ is a polynomial of degree n.*

An example

The previous Theorem and Corollary do not necessarily hold without assuming that R is of characteristic zero.
Let F_2 denote the field having two elements, and let $R = F_2[x]$ be the ring of polynomials with coefficients from F_2. We put

$$D\left(\sum_{i=0}^{n} a_i \cdot x^i\right) = \sum_{i=2}^{n} \frac{i(i-1)}{2} \cdot a_i \cdot x^{i-2}$$

for every $n \geq 0$ and $a_0, \ldots, a_n \in F_2$. Then

- $D \in \mathcal{D}_2(R)$.
- $D(x) = 0$ and $D(x^2) = 1$, thus $D \in \mathcal{D}_2 \setminus \mathcal{D}_1$.

An example (cont.)

Recall $R = F_2[x]$. Let d_1 and d_2 be arbitrary derivations on R. Then $d_1 \circ d_2$ is also a derivation, i.e,

$$(d_1 \circ d_2)(x^k) = k \cdot x^{k-1} \cdot d_1(d_2(x)), \qquad (\forall k \in \mathbb{N} \cup \{0\}) \tag{4}$$

Indeed, for $k \geq 2$

$$\begin{aligned} d_1(d_2(x^k)) &= d_1(k \cdot x^{k-1} \cdot d_2(x)) \\ &= k(k-1) \cdot x^{k-2} \cdot d_1(x) \cdot d_2(x) + k \cdot x^{k-1} \cdot d_1(d_2(x)). \end{aligned}$$

Let $a = d_1(d_2(x)) \in R$, then $d_1(d_2(p)) = a \cdot \frac{\partial p}{\partial x}$ for every $p \in R$. This implies that $\mathcal{O}_2^0 = \mathcal{O}_1^0$, and thus $\mathcal{O}_2^0 \subsetneq \mathcal{D}_2$.

Nonzero Characteristic

Theorem

Let R be an integral domain of characteristic zero, and let n be a positive integer. If $d_1, \ldots, d_n$ are nonzero derivations on R, then $d_1 \circ \ldots \circ d_n \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$.

In previous example we show $d(p) = \frac{\partial p}{\partial x}$ $(p \in F_2[x])$ is derivation and $d \circ \cdots \circ d$ is also.

Theorem (B. Ebanks '18)

Let m and n be a positive integers. Let R be an integral domain of characteristic m and $d_1, \ldots, d_n$ be nonzero derivations on R. Then $d_1 \circ \ldots \circ d_n \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$ if and only if $n! < m$.

Analogue of the main result

Theorem

Let n be a positive integer, m be a prime. Let R be an integral domain of characteristic m and K its field of fractions. Then for every $D\colon R \to R$, the following are equivalent if and only if $n! < m$

1. $D \in \mathcal{D}_n(R)$,
2. $D \in \mathrm{cl}\mathcal{O}_n^0(R)$.
3. *D is additive on R, $D(1) = 0$, and D/j, as a map from the semigroup R^* to K, is a generalized polynomial of degree at most n.*

R is not a integral domain

None of the previous results holds if the ring is not an integral domain. Not even for rings of characteristic zero.
Let $R = \mathbb{Q}[x] \times \mathbb{Q}[x]$, and for every $(p, q) \in R$ we put
$d_1(p, q) = (\frac{\partial p}{\partial x}, 0)$,
$d_2(p, q) = (0, \frac{\partial q}{\partial x})$.
Then d_1 and d_2 are nonzero derivations on R, but $d_1 \circ d_2 = 0$.

Thank you for your kind attention.

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