On biselective operations

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Motivation

Let X be a nonempty set

Definition

$F\colon X^2 \to X$ is said to be *transitive* if

$$F(F(x,z),F(y,z)) \;=\; F(x,y) \qquad x,y,z \in X$$

We select 2 variables among 3

What happens if we select 2 variables among 4?

Biselectiveness

Definition

Let $i, j \in \{1, 2, 3, 4\}$. We say that $F\colon X^2 \to X$ is *(i, j)-selective* if

$$F(F(x_1, x_2), F(x_3, x_4)) \;=\; F(x_i, x_j) \qquad x_1, x_2, x_3, x_4 \in X$$

Biselectiveness

$$F(F(x_1, x_2), F(x_3, x_4)) \quad = \quad F(x_i, x_j)$$

(1,1)	(1,2)	(1,3)	(1,4)
(2,1)	(2,2)	(2,3)	(2,4)
(3,1)	(3,2)	(3,3)	(3,4)
(4,1)	(4,2)	(4,3)	(4,4)

Biselectiveness

$$F(F(x_1, x_2), F(x_3, x_4)) \quad = \quad F(x_i, x_j)$$

(1,1)	(1,2)	(1,3)	(1,4)
(2,1)	(2,2)	(2,3)	(2,4)
(3,1)	(3,2)	(3,3)	(3,4)
(4,1)	(4,2)	(4,3)	(4,4)

Proposition

The following assertions are equivalent.

(i) F is (i, j)-selective with $j < i$.

(ii) F is $(2, 3)$-selective.

(iii) F is constant.

Biselectiveness

$$F(F(x_1, x_2), F(x_3, x_4)) \quad = \quad F(x_i, x_j)$$

(1,1)	(1,2)	(1,3)	(1,4)
~~(2,1)~~	(2,2)	~~(2,3)~~	(2,4)
~~(3,1)~~	~~(3,2)~~	(3,3)	(3,4)
~~(4,1)~~	~~(4,2)~~	~~(4,3)~~	(4,4)

Proposition

The following assertions are equivalent.

(i) F is $(2,2)$-selective.

(ii) F is $(3,3)$-selective.

(iii) $F|_{\mathrm{ran}(F)}$ is constant and $\forall x\colon F(F(x,x),F(x,x)) = F(x,x)$.

Biselectiveness

$$F(F(x_1, x_2), F(x_3, x_4)) \quad = \quad F(x_i, x_j)$$

(1,1)	(1,2)	(1,3)	(1,4)
~~(2,1)~~	~~(2,2)~~	~~(2,3)~~	(2,4)
~~(3,1)~~	~~(3,2)~~	~~(3,3)~~	(3,4)
~~(4,1)~~	~~(4,2)~~	~~(4,3)~~	(4,4)

Lemma

$$F(x, y) \text{ is } (1,1)\text{-selective} \quad \Leftrightarrow \quad F(y, x) \text{ is } (4,4)\text{-selective}$$
$$F(x, y) \text{ is } (1,2)\text{-selective} \quad \Leftrightarrow \quad F(y, x) \text{ is } (3,4)\text{-selective}$$
$$\vdots$$

Biselectiveness

$$F(F(x_1, x_2), F(x_3, x_4)) \quad = \quad F(x_i, x_j)$$

(1,1)	(1,2)	(1,3)	(1,4)
$\cancel{(2,1)}$	$\cancel{(2,2)}$	$\cancel{(2,3)}$	$\cancel{(2,4)}$
$\cancel{(3,1)}$	$\cancel{(3,2)}$	$\cancel{(3,3)}$	$\cancel{(3,4)}$
$\cancel{(4,1)}$	$\cancel{(4,2)}$	$\cancel{(4,3)}$	$\cancel{(4,4)}$

Proposition

F is $(1,2)$-selective $\quad \Leftrightarrow \quad F|_{\mathrm{ran}(F)} = \pi_1|_{\mathrm{ran}(F)}$

Biselectiveness

$$F(F(x_1, x_2), F(x_3, x_4)) \quad = \quad F(x_i, x_j)$$

(1,1)	~~(1,2)~~	(1,3)	(1,4)
~~(2,1)~~	~~(2,2)~~	~~(2,3)~~	~~(2,4)~~
~~(3,1)~~	~~(3,2)~~	~~(3,3)~~	~~(3,4)~~
~~(4,1)~~	~~(4,2)~~	~~(4,3)~~	~~(4,4)~~

Some definitions

Definition

- $x \in X$ is said to be *idempotent for F* if $F(x, x) = x$.
- We denote *the set of all idempotent elements of F* by $\mathrm{id}(F)$, that is,

$$\mathrm{id}(F) \ = \ \{x \in X \mid F(x, x) = x\}.$$

Some definitions

Given $F\colon X^2 \to X$ we define the equivalence relation $\sim_F$ on X by

$$x \sim_F y \qquad \Leftrightarrow \qquad F(x,x) = F(y,y), \qquad x,y \in X.$$

Also, we denote by $[x]_{\sim_F}$ the equivalence class of $x \in X$ for $\sim_F$.

Fact : If F is (i,j)-selective, then

$$\mathrm{id}(F) \cap [x]_{\sim_F} \;=\; \{F(x,x)\}, \qquad x \in X.$$

Indeed, by (i,j)-selectiveness, we have

$$F(F(x,x),F(x,x)) \;=\; F(x,x), \qquad x \in X.$$

Also, if $y \in \mathrm{id}(F) \cap [x]_{\sim_F}$, then $y = F(y,y) = F(x,x)$.

$(1,1)$-selectiveness

$$F(F(x_1,x_2),F(x_3,x_4)) \quad = \quad F(x_1,x_1)$$

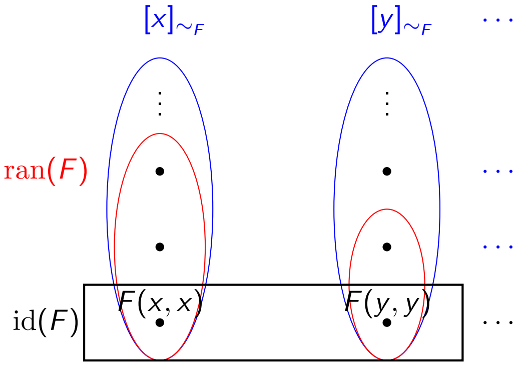
Proposition

F is $(1,1)$-selective iff the following hold.

(a) $F(x,y) = F(x,x)$ for any $x, y \in \mathrm{ran}(F)$.

(b) $F(x,y) \in [x]_{\sim_F}$ for any $x, y \in X$.

$(1,1)$-selectiveness



$(1,3)$-selectiveness

$$F(F(x_1,x_2),F(x_3,x_4)) \quad = \quad F(x_1,x_3)$$

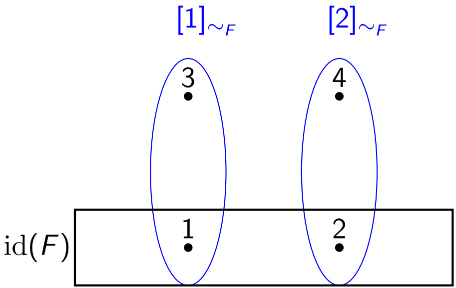
Example.

On $X = \{1,2,3,4\}$, define F by

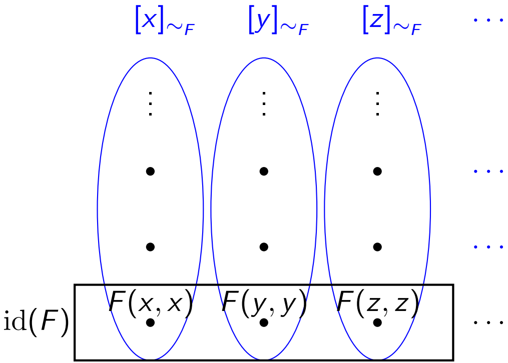
$$\begin{aligned}
1 &= F(1,1) = F(1,3) = F(3,1) = F(3,3)\\
2 &= F(2,2) = F(2,4) = F(4,2) = F(4,4)\\
3 &= F(1,2) = F(1,4) = F(3,2) = F(3,4)\\
4 &= F(2,1) = F(2,3) = F(4,1) = F(4,3)
\end{aligned}$$

(1, 3)-selectiveness

$$
\begin{array}{lllllllll}
1 & = & F(1,1) & = & F(1,3) & = & F(3,1) & = & F(3,3) \\
2 & = & F(2,2) & = & F(2,4) & = & F(4,2) & = & F(4,4) \\
3 & = & F(1,2) & = & F(1,4) & = & F(3,2) & = & F(3,4) \\
4 & = & F(2,1) & = & F(2,3) & = & F(4,1) & = & F(4,3)
\end{array}
$$



$(1, 3)$-selectiveness



(1, 3)-selectiveness

$$F(F(x_1, x_2), F(x_3, x_4)) \quad = \quad F(x_1, x_3)$$

Theorem

The following assertions are equivalent.

(i) F is $(1, 3)$-selective

(ii) For any $x, y \in X$, $u \in [x]_{\sim_F}$, and $v \in [y]_{\sim_F}$, we have

$$F(x, y) = F(u, v) \in [x]_{\sim_F}$$

(iii) For any $x, y, z \in X$, we have

$$F(F(x, y), z) = F(x, z) \quad \text{and} \quad F(x, F(y, z)) = F(x, y)$$

Some definitions

Definition

$F\colon X^2 \to X$ is said to be

- *anticommutative* if $\forall x, y \in X$:

$$F(x,y) = F(y,x) \quad \Rightarrow \quad x = y$$

- *bisymmetric* if

$$F(F(x,y),F(u,v)) = F(F(x,u),F(y,v)) \quad x,y,u,v \in X$$

- *idempotent* if

$$F(x,x) = x \qquad x \in X$$

Some definitions

We define the binary relations $\sim_{F,1}$ and $\sim_{F,2}$ on X by

$$x \sim_{F,1} y \qquad \Leftrightarrow \qquad F(x,y) = x$$

and

$$x \sim_{F,2} y \qquad \Leftrightarrow \qquad F(x,y) = y$$

$(1,4)$-selectiveness

$$F(F(x_1,x_2),F(x_3,x_4)) \quad = \quad F(x_1,x_4)$$

Proposition

Let F be $(1,4)$-selective. The following are equivalent.

(i) F is anticommutative

(ii) F is idempotent

(iii) F is onto

(iv) $\sim_{F,1}$ and $\sim_{F,2}$ are equivalence relations

$(1,4)$-selectiveness

$$F(F(x_1,x_2),F(x_3,x_4)) \quad = \quad F(x_1,x_4)$$

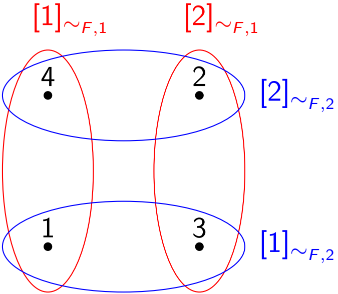
Example.

On $X=\{1,2,3,4\}$, define F by

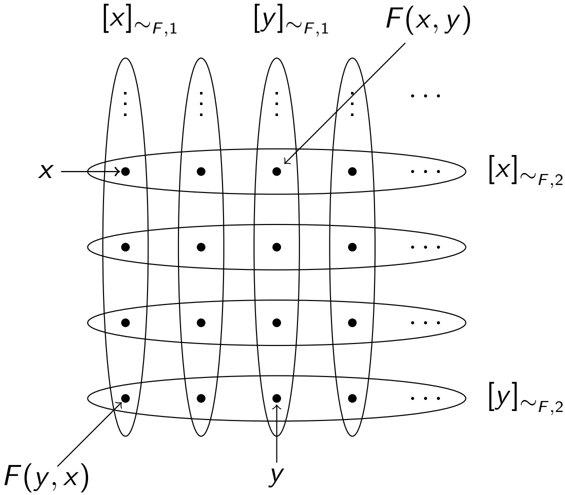
$$\begin{aligned}
1 &= F(1,1) = F(1,4) = F(3,1) = F(3,4)\\
2 &= F(2,2) = F(2,3) = F(4,2) = F(4,3)\\
3 &= F(1,2) = F(1,3) = F(3,2) = F(3,3)\\
4 &= F(2,1) = F(2,4) = F(4,1) = F(4,4)
\end{aligned}$$

(1, 4)-selectiveness

$$\begin{aligned}
1 &= F(1,1) = F(1,4) = F(3,1) = F(3,4)\\
2 &= F(2,2) = F(2,3) = F(4,2) = F(4,3)\\
3 &= F(1,2) = F(1,3) = F(3,2) = F(3,3)\\
4 &= F(2,1) = F(2,4) = F(4,1) = F(4,4)
\end{aligned}$$



$(1,4)$-selectiveness



$(1, 4)$-selectiveness

$$F(F(x_1, x_2), F(x_3, x_4)) \quad = \quad F(x_1, x_4)$$

Theorem

F is $(1, 4)$-selective and idempotent (resp. anticommutative, onto) iff the following hold.

(a) $\sim_{F,1}$ and $\sim_{F,2}$ are equivalence relations

(b) For any $u \in [x]_{\sim_{F,1}}$, there exists a unique $v \in [y]_{\sim_{F,1}}$ such that $u \sim_{F,2} v$

(c) For any $x, y, z \in X$ such that $y \sim_{F,1} z$, we have

$$F(x, y) \ = \ F(x, z)$$

$(1,4)$-selectiveness

$$F(F(x_1, x_2), F(x_3, x_4)) \quad = \quad F(x_1, x_4)$$

Theorem

F is $(1,4)$-selective and idempotent (resp. anticommutative, onto) iff $\sim_{F,1}$ and $\sim_{F,2}$ are equivalence relations such that

$$[x]_{\sim_{F,2}} \cap [y]_{\sim_{F,1}} \quad = \quad \{F(x,y)\} \qquad x, y \in X$$

$(1,4)$-selectiveness

$$F(F(x_1,x_2),F(x_3,x_4)) \quad = \quad F(x_1,x_4)$$

Lemma

If F is $(1,4)$-selective, then $\mathrm{id}(F)=\mathrm{ran}(F)$.

Theorem

The following assertions are equivalent.

(i) F is $(1,4)$-selective

(ii) $F|_{\mathrm{ran}(F)}$ is $(1,4)$-selective and $F(x,y)=F(F(x,x),F(y,y))$ for any $x,y\in X$

(iii) $F(F(x,y),z) \;=\; F(x,F(y,z)) \;=\; F(x,z)$ for any $x,y,z\in X$

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