Functional equations that characterize higher order derivations

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Initiation of the problem

Let K be a subfield of $\mathbb{C}$. A function $f : K \to \mathbb{C}$ is called additive if

$$f(x+y) = f(x) + f(y) \;\; \forall x, y \in K.$$

Now let f_i be some unknown additive functions that satisfies

$$\sum_{i=1}^{n} x^{p_i} f_i(x^{q_i}) = 0 \;\; \forall x \in K, \tag{1}$$

where p_i and q_i are nonnegative integers.

Question

How can we characterize the solutions of (1)*?*

This question was studied first systematically by B. Ebanks in [1, 2].

Special additive functions on fields

Let $f : K \to \mathbb{C}$ be an additive function.

- f is a (field) homomorphism if

$$f(xy) = f(x)f(y) \quad \forall x, y \in K.$$

- f is a derivation if

$$f(xy) = xf(y) + yf(x) \quad \forall x, y \in K.$$

Origin of the problem

F. Halter-Koch [4] characterize a pair of additive functions $f, g : K \to \mathbb{C}$ that satisfies

$$g(x^{ln}) = Ax^{ln} + x^{l(n-l)}f(x^l) \quad \forall x \in K,$$

where K is a subfield of $\mathbb{C}$, $n \in \mathbb{Z} \setminus \{0, 1\}$, $l \in \mathbb{N}$ and $A \in K$.

Solution: Let $e = f(1)$ and $F, G : K \to \mathbb{C}$ are defined by

$$F(x) = f(x) - ex \text{ and } G(x) = g(x) - (A + e)x,$$

then F and G are derivations such that $F = nG$.

Higher order derivations

Higher order derivation by Unger and Reich [5]:
The identically zero map is the only derivation of order 0.
An additive function f is a derivation of order n if there exists $B : K \times K \to \mathbb{C}$ such that B is a derivation of order $n-1$ in each variables and

$$f(xy) - xf(y) - f(x)y = B(x, y).$$

Every derivation is a derivation of order 1.

Claim

For derivations $d_1, \ldots, d_n$ the function $d_1 \circ \cdots \circ d_n$ is a derivation of order n.

Clearly, this holds for every linear combination as well.

Composition of derivations

Let $\mathcal{D}_n$ denote the set of (complex) linear combination of maps

$$d_1 \circ \cdots \circ d_k,$$

where $d_1, \ldots, d_k : K \to K$ are derivations for every $1 \le k \le n$. Also $\mathcal{D}$ denote the $\cup_{n=1}^{\infty} \mathcal{D}_n$.

Let $F \subset \mathbb{C}$ and $\mathbb{Q}(F)$ denote the field generated by F over $\mathbb{Q}$.

Theorem

Let $K \subset \mathbb{C}$ be a field and $f : K \to \mathbb{C}$ be a derivation of order n. Then for every finite set $F \subset K$ there exists $g \in \mathcal{D}_n$ such that $f|_{\mathbb{Q}(F)} = g|_{\mathbb{Q}(F)}$.

Thus we can concentrate on finitely generated fields over $\mathbb{Q}$.

Results

We denote the maximal cardinality of algebraically independent elements in K by $\deg_{\mathbb{Q}} K$.

Theorem

Let K be a field with $\deg_{\mathbb{Q}}(K) < \infty$ and f_i be additive solutions of (1). Let $e_i = f_i(1)$. Then

$$f_i = e_i x + D_i$$

and $D_i \in \mathcal{D}_n$ for all $i = 1, \ldots, n$. Moreover $\sum_{i=1}^{n} e_i = 0$.

Corollary

Suppose that $f_i : \mathbb{C} \to \mathbb{C}$ are additive solutions of (1). Then $f_i = f_i(1)x + D_i$, where D_i are derivations of order n and $\sum_{i=1}^{n} f_i(1) = 0$.

Scope of our proof

1. Reformulation of the problem by rational homogeneity.
2. We introduce the symmetrization process which makes it possible to distinguish several variables in equation of type (1).
3. We use a discrete spectral theory to characterize the solutions of (1).

Homogeneity argument

Fact

For every additive function f we have

$$f(rx) = rf(x) \;\; \forall x \in K, r \in \mathbb{Q}.$$

If we substitute x by rx in (1), then we get

$$P_x(r) = \sum_{i=1}^{N} \Big(x^{p_i} f_i(x^{q_i}) \Big) r^{p_i+q_i} = 0. \qquad (2)$$

This implies that every coefficient of $P_x(r)$ is 0 for all $x \in K$. Thus we can separate the terms of (1) according to the value $p_i + q_i$.

Homogeneity argument

Now we fix that $p_i + q_i = n \geq 2$, then we can reformulate (1) as follows.

$$\sum_{i=0}^{n-1} x^i f_{n-i}\left(x^{n-i}\right) = 0 \qquad (x \in K)\,. \tag{3}$$

Symmetrization process

Theorem (Polarization formula)

Let $n \in \mathbb{N}$, $n \geq 2$ and $A\colon K^n \to \mathbb{C}$ be a symmetric, n-additive function, then for all $x, y_1, \ldots, y_m \in K$ we have

$$\Delta_{y_1} \cdots \Delta_{y_m} A(x, \ldots, x) = \begin{cases} 0 & \text{if} \quad m > n \\ n!A(y_1, \ldots, y_m) & \text{if} \quad m = n. \end{cases}$$

Then it follows that $A(y_1, \ldots, y_n) = 0$ for all $y_1, \ldots, y_n \in K$ if and only if $A(x, \ldots, x) = 0$ for all $x \in K$.

In our situation:
If
$$A(x, \ldots, x) = \sum_{i=0}^{n-1} x^i f_{n-i}\left(x^{n-i}\right) = 0 \qquad (x \in K),$$
then for all $x_1, \ldots, x_n \in K$
$$A(x_1, \ldots, x_n) =$$
$$\sum_{i=0}^{n-1} \frac{1}{\binom{n-1}{i}} \sum_{\mathrm{card}(I)=i} \left(\prod_{j \in I} x_j\right) \cdot f_{n-i}\left(\prod_{k \in \{1,\ldots,n\} \setminus I} x_k\right) = 0.$$

Higher order derivations are solutions

Using the definition of derivations of order $n-1$ we get the following.

Proposition

Let $K \subset \mathbb{C}$ be a field. For every derivation D of order n

$$\sum_{i=0}^{n-1}(-1)^i \sum_{\mathrm{card}(I)=i} \left(\prod_{j\in I} x_j\right) \cdot D\left(\prod_{k\in\{1,\dots,n\}\setminus I} x_k\right) = 0.$$

In particular,

$$\sum_{i=0}^{n-1}(-1)^i \binom{n-1}{i} x^i D(x^{n-i}) = 0. \quad (4)$$

Thus $f_{n-i} = (-1)^i\binom{n-1}{i} D$ is a solution of (3).

Spectral theory

Let $(G, +)$ be an Abelian group with the discrete topology and the set $(\mathbb{C}^k)^G$ of all functions from G to $\mathbb{C}^k$ endowed with the product topology.
The elements of $(\mathbb{C}^k)^G$ can be identified with the set of k-tuples $(f_1, \ldots, f_k)$ where $f_i : G \to \mathbb{C}$. A set $V \subseteq (\mathbb{C}^k)^G$ is called a *variety* if V is a

- translation invariant
- linear space
- closed w. r. t. the product topology.

Let K^* denote the multiplicative group of the field K.

Theorem

Let $K \subset \mathbb{C}$ be a field with $\deg_{\mathbb{Q}}(K) < \infty$. Spectral analysis and synthesis holds in every variety V of $(\mathbb{C}^k)^{K^*}$ containing functions that are additive in each coordinate.

- Spectral analysis: V contains a function of the form

$$(c_1\phi, \ldots, c_k\phi),$$

where $\phi : K \to \mathbb{C}$ is a field homomorphism and not all c_i are zero $(1 \le i \le k)$.

- Spectral synthesis: V is spanned by the functions of the form

$$(\phi_1 \circ D_1, \ldots, \phi_k \circ D_k),$$

where $\phi_i : K \to \mathbb{C}$ are field homomorphisms and all $D_i \in \mathcal{D}$.

Solution space

Proposition

The space of solutions of (3) *is a variety if* $\sum_{i=0}^{n-1} f_{n-i}(1) = 0$.

Linearity, closedness are clear.

Translation invariance is tricky: By symmetrization process for an $c \in K^*$ we have

$$0 = A(x, \dots, x, cx) - xA(x, \dots, x, c) = \sum_{i=0}^{n-1} x^i g_{n-i}\left(cx^{n-i}\right)\big) + x^n \cdot h(c),$$

where

$$g_{n-i} = \left(\frac{n-i}{n} f_{n-i} - \frac{n-(i-1)}{n} f_{n-(i-1)}\right)$$

holds for all $1 \leq i \leq n-1$ and

$$g_n = f_n \text{ and } h = \sum_{i=0}^{n-1} f_{n-i}.$$

Application

Proposition

All solutions of (3) *of the form* $f_{n-i} = c_{n-i}\phi$ *satisfies* $\phi(x) = x$, $c_{n-i} = f_{n-i}(0)$ *and* $\sum_{i=0}^{n-1} c_{n-i} = 0$.

Now we may assume that $f_{n-i}(1) = 0$ since the functions $\tilde{f}_{n-i}(x) = f_{n-i}(x) - f_{n-i}(1)x$ provide also a solution of (3).

Corollary

All solutions of (3) *with* $f_{n-i}(1) = 0$ *satisfies that* $f_i \in \mathcal{D}$.

Using a recursive method we can reduce the number of unknown functions in (3).

Proposition

All solutions of (1) *with* $f_{n-i}(1) = 0$ *satisfies that* $f_{n-i} \in \mathcal{D}_{n-1}$ *for all* $0 \leq i \leq n-1$.

Better (best) upper bound

Theorem (Ebanks-Sahoo-Reidel)

Let k denote the number of nonzero terms in (3). Then all solutions of (3) with $f_i(1) = 0$ satisfies that $f_i \in \mathcal{D}_{k-1}$.

Using the symmetrization process we can write

$$0 = A(\underbrace{x, \ldots, x}_{k-1}, 1, \ldots, 1) = \sum_{j=0}^{k-1} x^j f_j'(x^{n-j}),$$

where each f_j' is a positive linear combination of f_i with rational coefficients. Thus $f_j' \neq 0$.

It can be shown that this is the best upper bound.

Happy birthday,
Professor Zdun!

Thank you for your kind attention.

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